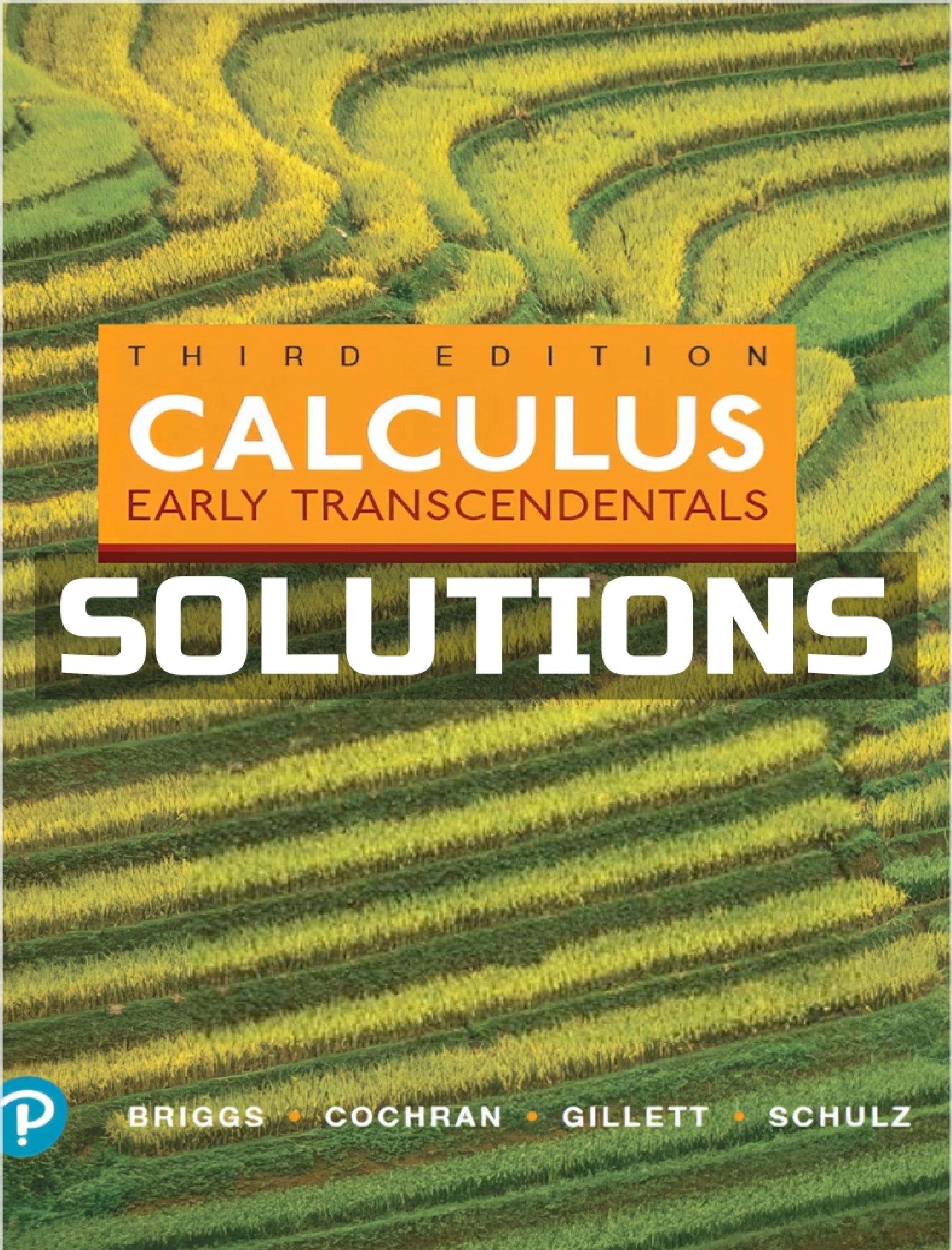

THIRD EDITION

CALCULUS

EARLY TRANSCENDENTALS

SOLUTIONS

BRIGGS • COCHRAN • GILLETT • SCHULZ

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Chapter 1

Functions

1.1 Review of Functions

1.1.1 A function is a rule that assigns each to each value of the independent variable in the domain a unique value of the dependent variable in the range.

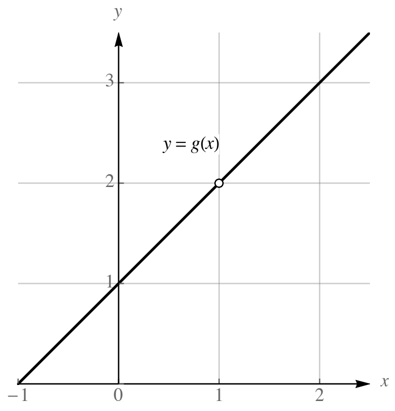
1.1.2 The independent variable belongs to the domain, while the dependent variable belongs to the range.

1.1.3 Graph A does not represent a function, while graph B does. Note that graph A fails the vertical line test, while graph B passes it.

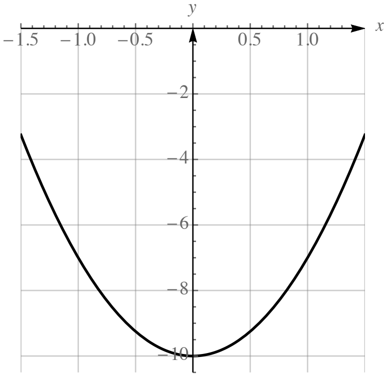
1.1.4 The domain of f is $[1,4)$, while the range of f is $(1,5]$. Note that the domain is the "shadow" of the graph on the x-axis, while the range is the "shadow" of the graph on the y-axis.

1.1.5 Item i. is true while item ii. isn't necessarily true. In the definition of function, item i. is stipulated. However, item ii. need not be true – for example, the function $f(x) = x^2$ has two different domain values associated with the one range value 4, because $f(2) = f(-2) = 4$.

1.1.6 $g(x) = \frac{x^2+1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1$, $x \neq 1$. The domain is $\{x : x \neq 1\}$ and the range is $\{x : x \neq 2\}$.



1.1.7 The domain of this function is the set of a real numbers. The range is $[-10, \infty)$.



1.1.8 The independent variable t is elapsed time and the dependent variable d is distance above the ground. The domain in context is $[0, 8]$

1.1.9 The independent variable h is the height of the water in the tank and the dependent variable V is the volume of water in the tank. The domain in context is $[0, 50]$

1.1.10 $f(2) = \dfrac{1}{2^3+1} = \dfrac{1}{9}$. $f(y^2) = \frac{1}{(y^2)^3+1} = \frac{1}{y^6+1}$.

1.1.11 $f(g(1/2)) = f(-2) = -3$; $g(f(4)) = g(9) = \frac{1}{8}$; $g(f(x)) = g(2x+1) = \dfrac{1}{(2x+1)-1} = \dfrac{1}{2x}$.

1.1.12 One possible answer is $g(x) = x^2 + 1$ and $f(x) = x^5$, because then $f(g(x)) = f(x^2+1) = (x^2+1)^5$. Another possible answer is $g(x) = x^2$ and $f(x) = (x+1)^5$, because then $f(g(x)) = f(x^2) = (x^2+1)^5$.

1.1.13 The domain of $f \circ g$ consists of all x in the domain of g such that $g(x)$ is in the domain of f.

1.1.14 $(f \circ g)(3) = f(g(3)) = f(25) = \sqrt{25} = 5$.
$(f \circ f)(64) = f(\sqrt{64}) = f(8) = \sqrt{8} = 2\sqrt{2}$.
$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = x^{3/2} - 2$.
$(f \circ g)(x) = f(g(x)) = f(x^3 - 2) = \sqrt{x^3 - 2}$

1.1.15

a. $(f \circ g)(2) = f(g(2)) = f(2) = 4$.
b. $g(f(2)) = g(4) = 1$.
c. $f(g(4)) = f(1) = 3$.
d. $g(f(5)) = g(6) = 3$.
e. $f(f(8)) = f(8) = 8$.
f. $g(f(g(5))) = g(f(2)) = g(4) = 1$.

1.1.16

a. $h(g(0)) = h(0) = -1$.
b. $g(f(4)) = g(-1) = -1$.
c. $h(h(0)) = h(-1) = 0$.
d. $g(h(f(4))) = g(h(-1)) = g(0) = 0$.
e. $f(f(f(1))) = f(f(0)) = f(1) = 0$.
f. $h(h(h(0))) = h(h(-1)) = h(0) = -1$.
g. $f(h(g(2))) = f(h(3)) = f(0) = 1$.
h. $g(f(h(4))) = g(f(4)) = g(-1) = -1$.
i. $g(g(g(1))) = g(g(2)) = g(3) = 4$.
j. $f(f(h(3))) = f(f(0)) = f(1) = 0$.

1.1.17 $\dfrac{f(5) - f(0)}{5 - 0} = \dfrac{83 - 6}{5} = 15.4$; the radiosonde rises at an average rate of 15.4 ft/s during the first 5 seconds after it is released.

1.1.18 $f(0) = 0$. $f(34) = 127852.4 - 109731 = 18121.4$. $f(64) = 127852.4 - 75330.4 = 52522$.

$$\frac{f(64) - f(34)}{64 - 34} = \frac{52522 - 18121.4}{30} \approx 1146.69\,\text{ft/s}.$$

1.1.19 $f(-2) = f(2) = 2$; $g(-2) = -g(2) = -(-2) = 2$; $f(g(2)) = f(-2) = f(2) = 2$; $g(f(-2)) = g(f(2)) = g(2) = -2$.

1.1.20 The graph would be the result of leaving the portion of the graph in the first quadrant, and then also obtaining a portion in the third quadrant which would be the result of reflecting the portion in the first quadrant around the y-axis and then the x-axis.

1.1.21 Function A is symmetric about the y-axis, so is even. Function B is symmetric about the origin so is odd. Function C is symmetric about the y-axis, so is even.

1.1.22 Function A is symmetric about the y-axis, so is even. Function B is symmetric about the origin, so is odd. Function C is also symmetric about the origin, so is odd.

1.1.23 $f(x) = \dfrac{x^2 - 5x + 6}{x - 2} = \dfrac{(x-2)(x-3)}{x-2} = x - 3$, $x \neq 2$. The domain of f is $\{x : x \neq 2\}$. The range is $\{y : y \neq -1\}$.

1.1.24 $f(x) = \frac{x-2}{2-x} = \frac{x-2}{-(x-2)} = -1$, $x \neq 2$. The domain is $\{x : x \neq 2\}$. The range is $\{-1\}$.

1.1.25 The domain of the function is the set of numbers x which satisfy $7 - x^2 \geq 0$. This is the interval $[-\sqrt{7}, \sqrt{7}]$. Note that $f(\sqrt{7}) = 0$ and $f(0) = \sqrt{7}$. The range is $[0, \sqrt{7}]$.

1.1.26 The domain of the function is the set of numbers x which satisfy $25 - x^2 \geq 0$. This is the interval $[-5, 5]$. Note that $f(0) = -5$ and $f(5) = 0$. The range is $[-5, 0]$.

1.1.27 Because the cube root function is defined for all real numbrs, the domain is $\mathbb{R}$, the set of all real numbers.

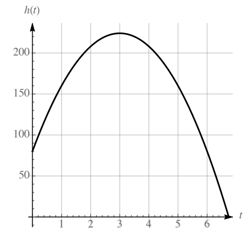
1.1.28 The domain consists of the set of numbers w for which $2 - w \geq 0$, so the interval $(-\infty, 2]$.

1.1.29 The domain consists of the set of numbers x for which $9 - x^2 \geq 0$, so the interval $[-3, 3]$.

1.1.30 Because $1 + t^2$ is never zero for any real numbered value of t, the domain of this function is $\mathbb{R}$, the set of all real numbers.

1.1.31

a. The formula for the height of the rocket is valid from $t = 0$ until the rocket hits the ground, which is the positive solution to $-16t^2 + 96t + 80 = 0$, which the quadratic formula reveals is $t = 3 + \sqrt{14}$. Thus, the domain is $[0, 3 + \sqrt{14}]$.

b. 

The maximum appears to occur at $t = 3$. The height at that time would be 224.

1.1.32

a. $d(0) = (10 - (2.2) \cdot 0)^2 = 100$.

b. The tank is first empty when $d(t) = 0$, which is when $10 - (2.2)t = 0$, or $t = 50/11$.

c. An appropriate domain would $[0, 50/11]$.

1.1.33 $g(1/z) = (1/z)^3 = \frac{1}{z^3}$

1.1.34 $F(y^4) = \frac{1}{y^4-3}$

1.1.35 $F(g(y)) = F(y^3) = \frac{1}{y^3-3}$

1.1.36 $f(g(w)) = f(w^3) = (w^3)^2 - 4 = w^6 - 4$

1.1.37 $g(f(u)) = g(u^2-4) = (u^2-4)^3$

1.1.38 $\dfrac{f(2+h)-f(2)}{h} = \dfrac{(2+h)^2-4-0}{h} = \dfrac{4+4h+h^2-4}{h} = \dfrac{4h+h^2}{h} = 4+h$

1.1.39 $F(F(x)) = F\left(\dfrac{1}{x-3}\right) = \dfrac{1}{\frac{1}{x-3}-3} = \dfrac{1}{\frac{1}{x-3}-\frac{3(x-3)}{x-3}} = \dfrac{1}{\frac{10-3x}{x-3}} = \dfrac{x-3}{10-3x}$

1.1.40 $g(F(f(x))) = g(F(x^2-4)) = g\left(\dfrac{1}{x^2-4-3}\right) = \left(\dfrac{1}{x^2-7}\right)^3$

1.1.41 $f(\sqrt{x+4}) = (\sqrt{x+4})^2 - 4 = x+4-4 = x.$

1.1.42 $F((3x+1)/x) = \dfrac{1}{\frac{3x+1}{x}-3} = \dfrac{1}{\frac{3x+1-3x}{x}} = \dfrac{x}{3x+1-3x} = x.$

1.1.43 $g(x) = x^3-5$ and $f(x) = x^{10}$.

1.1.44 $g(x) = x^6+x^2+1$ and $f(x) = \frac{2}{x^2}$.

1.1.45 $g(x) = x^4+2$ and $f(x) = \sqrt{x}$.

1.1.46 $g(x) = x^3-1$ and $f(x) = \frac{1}{\sqrt{x}}$.

1.1.47 $(f \circ g)(x) = f(g(x)) = f(x^2-4) = |x^2-4|$. The domain of this function is the set of all real numbers.

1.1.48 $(g \circ f)(x) = g(f(x)) = g(|x|) = |x|^2 - 4 = x^2 - 4$. The domain of this function is the set of all real numbers.

1.1.49 $(f \circ G)(x) = f(G(x)) = f\left(\frac{1}{x-2}\right) = \left|\frac{1}{x-2}\right| = \frac{1}{|x-2|}$. The domain of this function is the set of all real numbers except for the number 2.

1.1.50 $(f \circ g \circ G)(x) = f(g(G(x))) = f\left(g\left(\frac{1}{x-2}\right)\right) = f\left(\left(\frac{1}{x-2}\right)^2 - 4\right) = \left|\left(\frac{1}{x-2}\right)^2 - 4\right|$. The domain of this function is the set of all real numbers except for the number 2.

1.1.51 $(G \circ g \circ f)(x) = G(g(f(x))) = G(g(|x|)) = G(x^2-4) = \frac{1}{x^2-4-2} = \frac{1}{x^2-6}$. The domain of this function is the set of all real numbers except for the numbers $\pm\sqrt{6}$.

1.1.52 $(g \circ F \circ F)(x) = g(F(F(x))) = g(F(\sqrt{x})) = g(\sqrt{\sqrt{x}}) = \sqrt{x} - 4$. The domain is $[0, \infty)$.

1.1.53 $(g \circ g)(x) = g(g(x)) = g(x^2-4) = (x^2-4)^2 - 4 = x^4 - 8x^2 + 16 - 4 = x^4 - 8x^2 + 12$. The domain is the set of all real numbers.

1.1.54 $(G \circ G)(x) = G(G(x)) = G(1/(x-2)) = \dfrac{1}{\frac{1}{x-2}-2} = \dfrac{1}{\frac{1-2(x-2)}{x-2}} = \dfrac{x-2}{1-2x+4} = \dfrac{x-2}{5-2x}$. Then $G \circ G$ is defined except where the denominator vanishes, so its domain is the set of all real numbers except for $x = \frac{5}{2}$.

1.1.55 Because $(x^2+3) - 3 = x^2$, we may choose $f(x) = x - 3$.

1.1.56 Because the reciprocal of x^2+3 is $\frac{1}{x^2+3}$, we may choose $f(x) = \frac{1}{x}$.

1.1.57 Because $(x^2+3)^2 = x^4 + 6x^2 + 9$, we may choose $f(x) = x^2$.

1.1.58 Because $(x^2+3)^2 = x^4+6x^2+9$, and the given expression is 11 more than this, we may choose $f(x) = x^2+11$.

1.1.59 Because $(x^2)^2+3 = x^4+3$, this expression results from squaring x^2 and adding 3 to it. Thus we may choose $f(x) = x^2$.

1.1.60 Because $x^{2/3}+3 = (\sqrt[3]{x})^2+3$, we may choose $f(x) = \sqrt[3]{x}$.

1.1.61

a. True. A real number z corresponds to the domain element $z/2+19$, because $f(z/2+19) = 2(z/2+19)-38 = z+38-38 = z$.

b. False. The definition of function does not require that each range element comes from a unique domain element, rather that each domain element is paired with a unique range element.

c. True. $f(1/x) = \frac{1}{1/x} = x$, and $\frac{1}{f(x)} = \frac{1}{1/x} = x$.

d. False. For example, suppose that f is the straight line through the origin with slope 1, so that $f(x) = x$. Then $f(f(x)) = f(x) = x$, while $(f(x))^2 = x^2$.

e. False. For example, let $f(x) = x+2$ and $g(x) = 2x-1$. Then $f(g(x)) = f(2x-1) = 2x-1+2 = 2x+1$, while $g(f(x)) = g(x+2) = 2(x+2)-1 = 2x+3$.

f. True. This is the definition of $f \circ g$.

g. True. If f is even, then $f(-z) = f(z)$ for all z, so this is true in particular for $z = ax$. So if $g(x) = cf(ax)$, then $g(-x) = cf(-ax) = cf(ax) = g(x)$, so g is even.

h. False. For example, $f(x) = x$ is an odd function, but $h(x) = x+1$ isn't, because $h(2) = 3$, while $h(-2) = -1$ which isn't $-h(2)$.

i. True. If $f(-x) = -f(x) = f(x)$, then in particular $-f(x) = f(x)$, so $0 = 2f(x)$, so $f(x) = 0$ for all x.

1.1.62 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{10-10}{h} = \dfrac{0}{h} = 0.$

1.1.63 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{3(x+h)-3x}{h} = \dfrac{3x+3h-3x}{h} = \dfrac{3h}{h} = 3.$

1.1.64 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{4(x+h)-3-(4x-3)}{h} = \dfrac{4x+4h-3-4x+3}{h} = \dfrac{4h}{h} = 4.$

1.1.65 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{(x+h)^2-x^2}{h} = \dfrac{(x^2+2hx+h^2)-x^2}{h} = \dfrac{h(2x+h)}{h} = 2x+h.$

1.1.66 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{2(x+h)^2-3(x+h)+1-(2x^2-3x+1)}{h} =$
$\dfrac{2x^2+4xh+2h^2-3x-3h+1-2x^2+3x-1}{h} = \dfrac{4xh+2h^2-3h}{h} = \dfrac{h(4x+2h-3)}{h} = 4x+2h-3.$

1.1.67 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\frac{2}{x+h}-\frac{2}{x}}{h} = \dfrac{\frac{2x-2(x+h)}{x(x+h)}}{h} = \dfrac{2x-2x-2h}{hx(x+h)} = -\dfrac{2h}{hx(x+h)} = -\dfrac{2}{x(x+h)}.$

1.1.68 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\frac{x+h}{x+h+1}-\frac{x}{x+1}}{h} = \dfrac{\frac{(x+h)(x+1)-x(x+h+1)}{(x+1)(x+h+1)}}{h} = \dfrac{x^2+x+hx+h-x^2-xh-x}{h(x+1)(x+h+1)} =$
$\dfrac{h}{h(x+1)(x+h+1)} = \dfrac{1}{(x+1)(x+h+1)}$

1.1.69 $\dfrac{f(x)-f(a)}{x-a}=\dfrac{x^2+x-(a^2+a)}{x-a}=\dfrac{(x^2-a^2)+(x-a)}{x-a}=\dfrac{(x-a)(x+a)+(x-a)}{x-a}=$ $\dfrac{(x-a)(x+a+1)}{x-a}=x+a+1.$

1.1.70

$$\frac{f(x)-f(a)}{x-a}=\frac{4-4x-x^2-(4-4a-a^2)}{x-a}=\frac{-4(x-a)-(x^2-a^2)}{x-a}=\frac{-4(x-a)-(x-a)(x+a)}{x-a}$$
$$=\frac{(x-a)(-4-(x+a))}{x-a}=-4-x-a.$$

1.1.71 $\dfrac{f(x)-f(a)}{x-a}=\dfrac{x^3-2x-(a^3-2a)}{x-a}=\dfrac{(x^3-a^3)-2(x-a)}{x-a}=\dfrac{(x-a)(x^2+ax+a^2)-2(x-a)}{x-a}=$ $\dfrac{(x-a)(x^2+ax+a^2-2)}{x-a}=x^2+ax+a^2-2.$

1.1.72 $\dfrac{f(x)-f(a)}{x-a}=\dfrac{x^4-a^4}{x-a}=\dfrac{(x^2-a^2)(x^2+a^2)}{x-a}=\dfrac{(x-a)(x+a)(x^2+a^2)}{x-a}=(x+a)(x^2+a^2).$

1.1.73 $\dfrac{f(x)-f(a)}{x-a}=\dfrac{\frac{-4}{x^2}-\frac{-4}{a^2}}{x-a}=\dfrac{\frac{-4a^2+4x^2}{a^2x^2}}{x-a}=\dfrac{4(x^2-a^2)}{(x-a)a^2x^2}=\dfrac{4(x-a)(x+a)}{(x-a)a^2x^2}=\dfrac{4(x+a)}{a^2x^2}.$

1.1.74 $\dfrac{f(x)-f(a)}{x-a}=\dfrac{\frac{1}{x}-x^2-(\frac{1}{a}-a^2)}{x-a}=\dfrac{\frac{1}{x}-\frac{1}{a}}{x-a}-\dfrac{x^2-a^2}{x-a}=\dfrac{\frac{a-x}{ax}}{x-a}-\dfrac{(x-a)(x+a)}{x-a}=-\dfrac{1}{ax}-(x+a).$

1.1.75

a. The slope is $\frac{12227-10499}{3-1}=864$ ft/h. The hiker's elevation increases at an average rate of 874 feet per hour.

b. The slope is $\frac{12144-12631}{5-4}=-487$ ft/h. The hiker's elevation decreases at an average rate of 487 feet per hour.

c. The hiker might have stopped to rest during this interval of time or the trail is level on this portion of the hike.

1.1.76

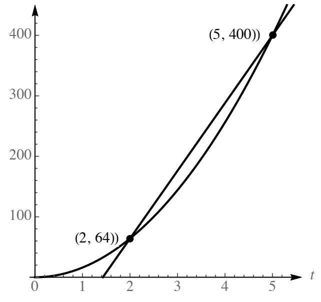
a. The slope is $\frac{11302-9954}{3-1}=674$ ft/m. The elevation of the trail increases by an average of 674 feet per mile for $1\le d\le 3$.

b. The slope is $\frac{12237-12357}{6-5}=-120$ ft/m. The elevation of the trail decreases by an average of 120 feet per mile for $5\le d\le 6$.

c. The elevation of the trail doesn't change much for $4.5\le d\le 5$.

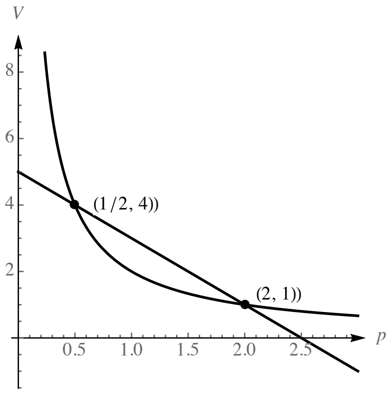
1.1.77

a.



b. The slope of the secant line is given by $\frac{400-64}{5-2}=\frac{336}{3}=112$ feet per second. The object falls at an average rate of 112 feet per second over the interval $2\le t\le 5$.

1.1.78

a. 

b. The slope of the secant line is given by $\frac{4-1}{.5-2} = \frac{3}{-1.5} = -2$ cubic centimeters per atmosphere. The volume decreases by an average of 2 cubic centimeters per atmosphere over the interval $0.5 \leq p \leq 2$.

1.1.79 This function is symmetric about the y-axis, because $f(-x) = (-x)^4 + 5(-x)^2 - 12 = x^4 + 5x^2 - 12 = f(x)$.

1.1.80 This function is symmetric about the origin, because $f(-x) = 3(-x)^5 + 2(-x)^3 - (-x) = -3x^5 - 2x^3 + x = -(3x^5 + 2x^3 - x) = f(x)$.

1.1.81 This function has none of the indicated symmetries. For example, note that $f(-2) = -26$, while $f(2) = 22$, so f is not symmetric about either the origin or about the y-axis, and is not symmetric about the x-axis because it is a function.

1.1.82 This function is symmetric about the y-axis. Note that $f(-x) = 2|-x| = 2|x| = f(x)$.

1.1.83 This curve (which is not a function) is symmetric about the x-axis, the y-axis, and the origin. Note that replacing either x by $-x$ or y by $-y$ (or both) yields the same equation. This is due to the fact that $(-x)^{2/3} = ((-x)^2)^{1/3} = (x^2)^{1/3} = x^{2/3}$, and a similar fact holds for the term involving y.

1.1.84 This function is symmetric about the origin. Writing the function as $y = f(x) = x^{3/5}$, we see that $f(-x) = (-x)^{3/5} = -(x)^{3/5} = -f(x)$.

1.1.85 This function is symmetric about the origin. Note that $f(-x) = (-x)|(-x)| = -x|x| = -f(x)$.

1.1.86 This curve (which is not a function) is symmetric about the x-axis, the y-axis, and the origin. Note that replacing either x by $-x$ or y by $-y$ (or both) yields the same equation. This is due to the fact that $|-x| = |x|$ and $|-y| = |y|$.

1.1.87

a. $f(g(-2)) = f(-g(2)) = f(-2) = 4$

b. $g(f(-2)) = g(f(2)) = g(4) = 1$

c. $f(g(-4)) = f(-g(4)) = f(-1) = 3$

d. $g(f(5) - 8) = g(-2) = -g(2) = -2$

e. $g(g(-7)) = g(-g(7)) = g(-4) = -1$

f. $f(1 - f(8)) = f(-7) = 7$

1.1.88

a. $f(g(-1)) = f(-g(1)) = f(3) = 3$

b. $g(f(-4)) = g(f(4)) = g(-4) = -g(4) = 2$

c. $f(g(-3)) = f(-g(3)) = f(4) = -4$

d. $f(g(-2)) = f(-g(2)) = f(1) = 2$

e. $g(g(-1)) = g(-g(1)) = g(3) = -4$

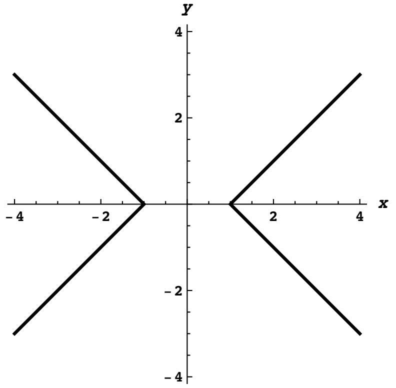
f. $f(g(0) - 1) = f(-1) = f(1) = 2$

g. $f(g(g(-2))) = f(g(-g(2))) = f(g(1)) = f(-3) = 3$

h. $g(f(f(-4))) = g(f(-4)) = g(-4) = 2$

i. $g(g(g(-1))) = g(g(-g(1))) = g(g(3)) = g(-4) = 2$

1.1.89 We will make heavy use of the fact that $|x|$ is x if $x > 0$, and is $-x$ if $x < 0$. In the first quadrant where x and y are both positive, this equation becomes $x - y = 1$ which is a straight line with slope 1 and y-intercept -1. In the second quadrant where x is negative and y is positive, this equation becomes $-x - y = 1$, which is a straight line with slope -1 and y-intercept -1. In the third quadrant where both x and y are negative, we obtain the equation $-x - (-y) = 1$, or $y = x + 1$, and in the fourth quadrant, we obtain $x + y = 1$. Graphing these lines and restricting them to the appropriate quadrants yields the following curve:



1.1.90 We have $y = 10 + \sqrt{-x^2 + 10x - 9}$, so by subtracting 10 from both sides and squaring we have $(y - 10)^2 = -x^2 + 10x - 9$, which can be written as

$$x^2 - 10x + (y - 10)^2 = -9.$$

To complete the square in x, we add 25 to both sides, yielding

$$x^2 - 10x + 25 + (y - 10)^2 = -9 + 25,$$

or

$$(x - 5)^2 + (y - 10)^2 = 16.$$

This is the equation of a circle of radius 4 centered at $(5, 10)$. Because $y \geq 10$, we see that the graph of f is the upper half of this circle. The domain of the function is $[1, 9]$ and the range is $[10, 14]$.

1.1.91 We have $y = 2 - \sqrt{-x^2 + 6x + 16}$, so by subtracting 2 from both sides and squaring we have $(y - 2)^2 = -x^2 + 6x + 16$, which can be written as

$$x^2 - 6x + (y - 2)^2 = 16.$$

To complete the square in x, we add 9 to both sides, yielding

$$x^2 - 6x + 9 + (y - 2)^2 = 16 + 9,$$

or

$$(x - 3)^2 + (y - 2)^2 = 25.$$

This is the equation of a circle of radius 5 centered at $(3, 2)$. Because $y \leq 2$, we see that the graph of f is the lower half of this circle. The domain of the function is $[-2, 8]$ and the range is $[-3, 2]$.

1.1.92

a. No. For example $f(x) = x^2 + 3$ is an even function, but $f(0)$ is not 0.

b. Yes. because $f(-x) = -f(x)$, and because $-0 = 0$, we must have $f(-0) = f(0) = -f(0)$, so $f(0) = -f(0)$, and the only number which is its own additive inverse is 0, so $f(0) = 0$.

1.1.93 Because the composition of f with itself has first degree, f has first degree as well, so let $f(x) = ax+b$. Then $(f \circ f)(x) = f(ax + b) = a(ax + b) + b = a^2x + (ab + b)$. Equating coefficients, we see that $a^2 = 9$ and $ab + b = -8$. If $a = 3$, we get that $b = -2$, while if $a = -3$ we have $b = 4$. So the two possible answers are $f(x) = 3x - 2$ and $f(x) = -3x + 4$.

1.1.94 Since the square of a linear function is a quadratic, we let $f(x) = ax+b$. Then $f(x)^2 = a^2x^2+2abx+b^2$. Equating coefficients yields that $a = \pm 3$ and $b = \pm 2$. However, a quick check shows that the middle term is correct only when one of these is positive and one is negative. So the two possible such functions f are $f(x) = 3x - 2$ and $f(x) = -3x + 2$.

1.1.95 Let $f(x) = ax^2 + bx + c$. Then $(f \circ f)(x) = f(ax^2 + bx + c) = a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c$. Expanding this expression yields $a^3x^4 + 2a^2bx^3 + 2a^2cx^2 + ab^2x^2 + 2abcx + ac^2 + abx^2 + b^2x + bc + c$, which simplifies to $a^3x^4 + 2a^2bx^3 + (2a^2c + ab^2 + ab)x^2 + (2abc + b^2)x + (ac^2 + bc + c)$. Equating coefficients yields $a^3 = 1$, so $a = 1$. Then $2a^2b = 0$, so $b = 0$. It then follows that $c = -6$, so the original function was $f(x) = x^2 - 6$.

1.1.96 Because the square of a quadratic is a quartic, we let $f(x) = ax^2 + bx + c$. Then the square of f is $c^2 + 2bcx + b^2x^2 + 2acx^2 + 2abx^3 + a^2x^4$. By equating coefficients, we see that $a^2 = 1$ and so $a = \pm 1$. Because the coefficient on x^3 must be 0, we have that $b = 0$. And the constant term reveals that $c = \pm 6$. A quick check shows that the only possible solutions are thus $f(x) = x^2 - 6$ and $f(x) = -x^2 + 6$.

1.1.97 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\sqrt{x+h}-\sqrt{x}}{h} = \dfrac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \dfrac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} =$
$\dfrac{1}{\sqrt{x+h}+\sqrt{x}}$.

$$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{x-a} \cdot \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} = \frac{1}{\sqrt{x}+\sqrt{a}}.$$

1.1.98 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\sqrt{1-2(x+h)}-\sqrt{1-2x}}{h} =$
$\dfrac{\sqrt{1-2(x+h)}-\sqrt{1-2x}}{h} \cdot \dfrac{\sqrt{1-2(x+h)}+\sqrt{1-2x}}{\sqrt{1-2(x+h)}+\sqrt{1-2x}} =$
$\dfrac{1-2(x+h)-(1-2x)}{h(\sqrt{1-2(x+h)}+\sqrt{1-2x})} =$
$-\dfrac{2}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}$.

$$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{1-2x}-\sqrt{1-2a}}{x-a} = \frac{\sqrt{1-2x}-\sqrt{1-2a}}{x-a} \cdot \frac{\sqrt{1-2x}+\sqrt{1-2a}}{\sqrt{1-2x}+\sqrt{1-2a}} =$$
$$\frac{(1-2x)-(1-2a)}{(x-a)(\sqrt{1-2x}+\sqrt{1-2a})} = \frac{(-2)(x-a)}{(x-a)(\sqrt{1-2x}+\sqrt{1-2a})} = -\frac{2}{(\sqrt{1-2x}+\sqrt{1-2a})}.$$

1.1.99 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\frac{-3}{\sqrt{x+h}} - \frac{-3}{\sqrt{x}}}{h} = \dfrac{-3(\sqrt{x}-\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}} = \dfrac{-3(\sqrt{x}-\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}} \cdot \dfrac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} =$
$\dfrac{-3(x-(x+h))}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \dfrac{3}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$.

$$\frac{f(x)-f(a)}{x-a} = \frac{\frac{-3}{\sqrt{x}} - \frac{-3}{\sqrt{a}}}{x-a} = \frac{-3\left(\frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}\sqrt{x}}\right)}{x-a} = \frac{(-3)(\sqrt{a}-\sqrt{x})}{(x-a)\sqrt{a}\sqrt{x}} \cdot \frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}+\sqrt{x}} =$$
$$\frac{(3)(x-a)}{(x-a)(\sqrt{a}\sqrt{x})(\sqrt{a}+\sqrt{x})} = \frac{3}{\sqrt{ax}(\sqrt{a}+\sqrt{x})}.$$

1.1.100 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h} =$
$\dfrac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h} \cdot \dfrac{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}} =$
$\dfrac{(x+h)^2+1-(x^2+1)}{h(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} = \dfrac{x^2+2hx+h^2-x^2}{h(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} = \dfrac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}$.

$$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a} = \frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a}\cdot\frac{\sqrt{x^2+1}+\sqrt{a^2+1}}{\sqrt{x^2+1}+\sqrt{a^2+1}} =$$
$$\frac{x^2+1-(a^2+1)}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})} = \frac{(x-a)(x+a)}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})} = \frac{x+a}{\sqrt{x^2+1}+\sqrt{a^2+1}}.$$

1.1.101 This would not necessarily have either kind of symmetry. For example, $f(x) = x^2$ is an even function and $g(x) = x^3$ is odd, but the sum of these two is neither even nor odd.

1.1.102 This would be an odd function, so it would be symmetric about the origin. Suppose f is even and g is odd. Then $(f\cdot g)(-x) = f(-x)g(-x) = f(x)\cdot(-g(x)) = -(f\cdot g)(x)$.

1.1.103 This would be an even function, so it would be symmetric about the y-axis. Suppose f is even and g is odd. Then $g(f(-x)) = g(f(x))$, because $f(-x) = f(x)$.

1.1.104 This would be an even function, so it would be symmetric about the y-axis. Suppose f is even and g is odd. Then $f(g(-x)) = f(-g(x)) = f(g(x))$.

1.2 Representing Functions

1.2.1 Functions can be defined and represented by a formula, through a graph, via a table, and by using words.

1.2.2 The domain of every polynomial is the set of all real numbers.

1.2.3 The slope of the line shown is $m = \frac{-3-(-1)}{3-0} = -2/3$. The y-intercept is $b = -1$. Thus the function is given by $f(x) = -\frac{2}{3}x - 1$.

1.2.4 Because it is to be parallel to a line with slope 2, it must also have slope 2. Using the point-slope form of the equation of the line, we have $y - 0 = 2(x-5)$, or $y = 2x - 10$.

1.2.5 The domain of a rational function $\frac{p(x)}{q(x)}$ is the set of all real numbers for which $q(x) \neq 0$.

1.2.6 A piecewise linear function is one which is linear over intervals in the domain.

1.2.7 For $x < 0$, the graph is a line with slope 1 and y- intercept 3, while for $x > 0$, it is a line with slope $-1/2$ and y-intercept 3. Note that both of these lines contain the point $(0, 3)$. The function shown can thus be written

$$f(x) = \begin{cases} x+3 & \text{if } x < 0; \\ -\frac{1}{2}x + 3 & \text{if } x \geq 0. \end{cases}$$

1.2.8 The transformed graph would have equation $y = \sqrt{x-2} + 3$.

1.2.9 Compared to the graph of $f(x)$, the graph of $f(x+2)$ will be shifted 2 units to the left.

1.2.10 Compared to the graph of $f(x)$, the graph of $-3f(x)$ will be scaled vertically by a factor of 3 and flipped about the x axis.

1.2.11 Compared to the graph of $f(x)$, the graph of $f(3x)$ will be compressed horizontally by a factor of $\frac{1}{3}$.

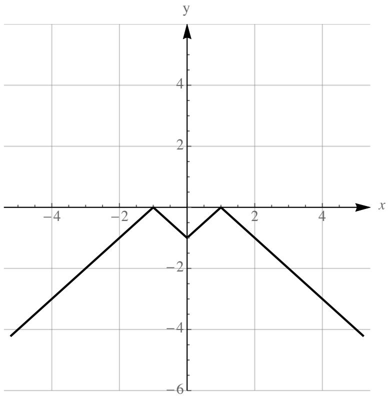
1.2.12 To produce the graph of $y = 4(x+3)^2 + 6$ from the graph of x^2, one must

1. shift the graph horizontally by 3 units to left
2. scale the graph vertically by a factor of 4
3. shift the graph vertically up 6 units.

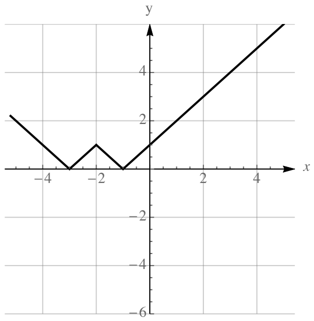
1.2.13 $f(x) = |x-2|+3$, because the graph of f is obtained from that of $|x|$ by shifting 2 units to the right and 3 units up.

$g(x) = -|x+2|-1$, because the graph of g is obtained from the graph of $|x|$ by shifting 2 units to the left, then reflecting about the x-axis, and then shifting 1 unit down.

1.2.14



a.



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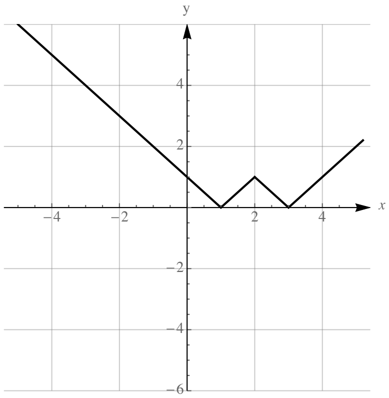
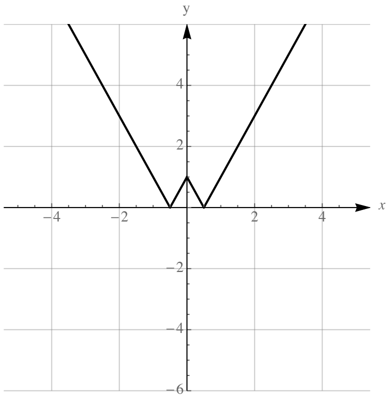
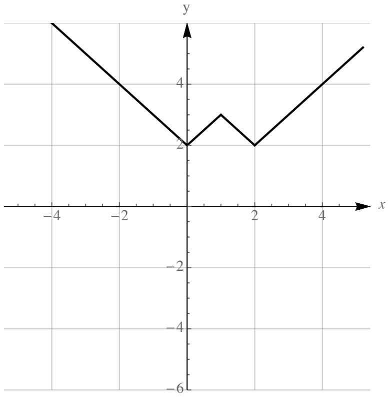


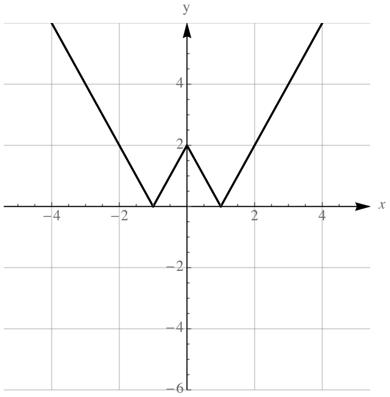
c.



d.



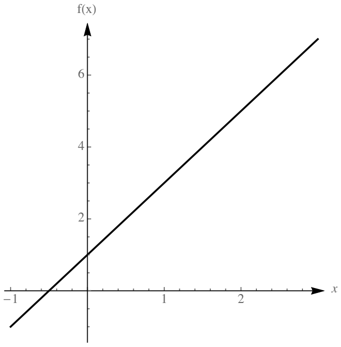
e.



f.

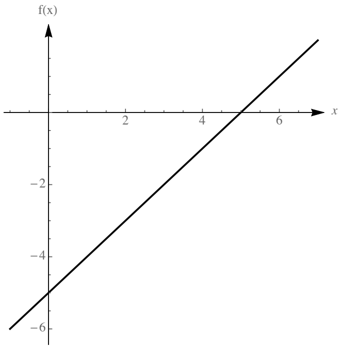
1.2.15

The slope is given by $\frac{5-3}{2-1} = 2$, so the equation of the line is $y - 3 = 2(x - 1)$, which can be written as $f(x) = 2x - 2 + 3$, or $f(x) = 2x + 1$.



1.2.16

The slope is given by $\frac{0-(-3)}{5-2} = 1$, so the equation of the line is $y - 0 = 1(x - 5)$, or $f(x) = x - 5$.



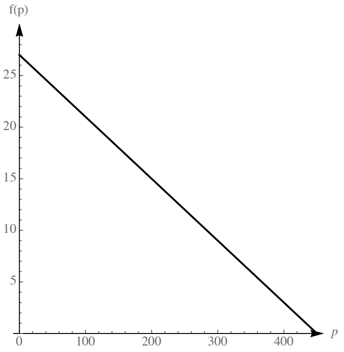
1.2.17 We are looking for the line with slope 3 that goes through the point $(3, 2)$. Using the point-slope form of the equation of a line, we have $y - 2 = 3(x - 3)$, which can be written as $y = 2 + 3x - 9$, or $y = 3x - 7$.

1.2.18 We are looking for the line with slope -4 which goes through the point $(-1, 4)$. Using the point-slope form of the equation of a line, we have $y - 4 = -4(x - (-1))$, which can be written as $y = 4 - 4x - 4$, or $y = -4x$.

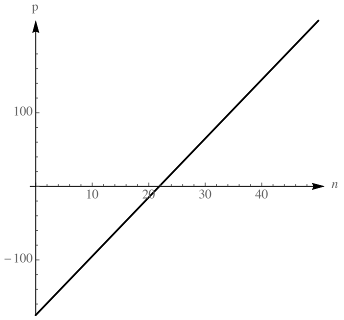
1.2.19 We have $571 = C_s(100)$, so $C_s = 5.71$. Therefore $N(150) = 5.71(150) = 856.5$ million.

1.2.20 We have $226 = C_s(100)$, so $C_s = 2.26$. Therefore $N(150) = 2.26(150) = 339$ million.

1.2.21 Using price as the independent variable p and the average number of units sold per day as the dependent variable d, we have the ordered pairs $(250, 12)$ and $(200, 15)$. The slope of the line determined by these points is $m = \frac{15-12}{200-250} = \frac{3}{-50}$. Thus the demand function has the form $d(p) = (-3/50)p + b$ for some constant b. Using the point $(200, 15)$, we find that $15 = (-3/50) \cdot 200 + b$, so $b = 27$. Thus the demand function is $d = (-3p/50) + 27$. While the domain of this linear function is the set of all real numbers, the formula is only likely to be valid for some subset of the interval $(0, 450)$, because outside of that interval either $p \leq 0$ or $d \leq 0$.



1.2.22 The profit is given by $p = f(n) = 8n - 175$. The break-even point is when $p = 0$, which occurs when $n = 175/8 = 21.875$, so they need to sell at least 22 tickets to not have a negative profit.

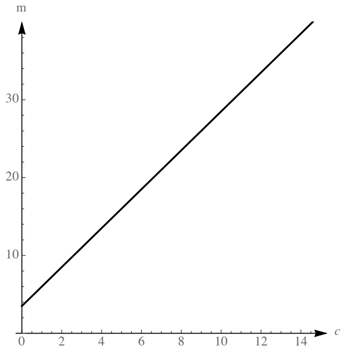


1.2.23

a. Using the points $(1986, 1875)$ and $(2000, 6471)$ we see that the slope is about 328.3. At $t = 0$, the value of p is 1875. Therefore a line which reasonably approximates the data is $p(t) = 328.3t + 1875$.

b. Using this line, we have that $p(9) = 4830$ breeding pairs.

1.2.24 The cost per mile is the slope of the desired line, and the intercept is the fixed cost of 3.5. Thus, the cost per mile is given by $c(m) = 2.5m + 3.5$. When $m = 9$, we have $c(9) = (2.5)(9) + 3.5 = 22.5 + 3.5 = 26$ dollars.



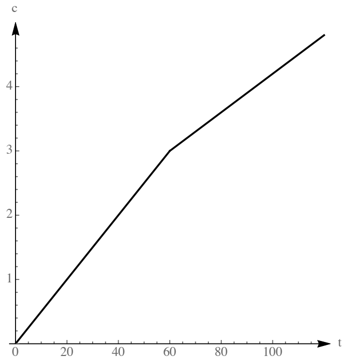
1.2.25 For $x \le 3$, we have the constant function 3. For $x \ge 3$, we have a straight line with slope 2 that contains the point $(3, 3)$. So its equation is $y - 3 = 2(x - 3)$, or $y = 2x - 3$. So the function can be written as $f(x) = \begin{cases} 3 & \text{if } x \le 3; \\ 2x - 3 & \text{if } x > 3 \end{cases}$

1.2.26 For $x < 3$ we have straight line with slope 1 and y-intercept 1, so the equation is $y = x+1$. For $x \geq 3$, we have a straight line with slope $-\frac{1}{3}$ which contains the point $(3, 2)$, so its equation is $y - 2 = -\frac{1}{3}(x - 3)$, or $y = -\frac{1}{3}x + 3$. Thus the function can be written as $f(x) = \begin{cases} x+1 & \text{if } x < 3; \\ -\frac{1}{3}x + 3 & \text{if } x \geq 3 \end{cases}$

1.2.27

The cost is given by

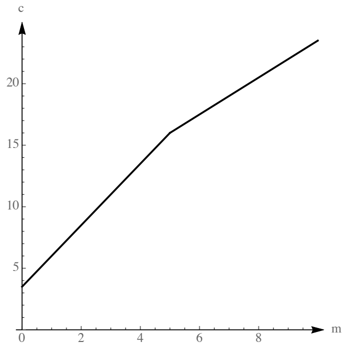
$$c(t) = \begin{cases} 0.05t & \text{for } 0 \leq t \leq 60 \\ 1.2 + 0.03t & \text{for } 60 < t \leq 120 \end{cases}.$$



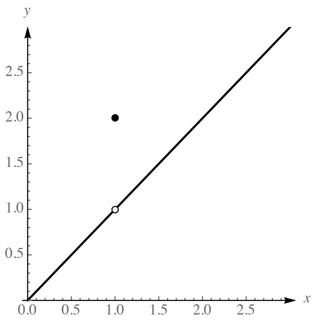
1.2.28

The cost is given by

$$c(m) = \begin{cases} 3.5 + 2.5m & \text{for } 0 \leq m \leq 5 \\ 8.5 + 1.5m & \text{for } m > 5 \end{cases}.$$



1.2.29



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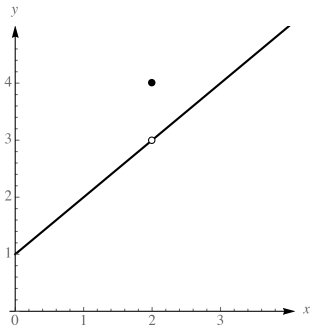

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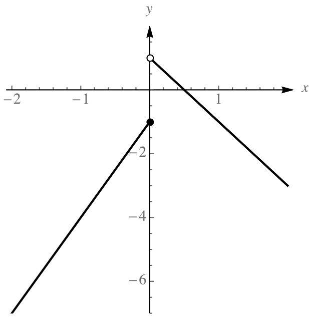
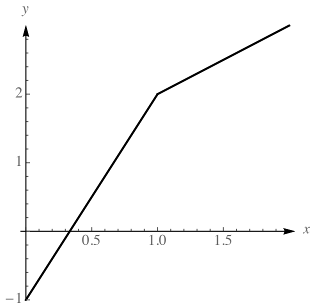
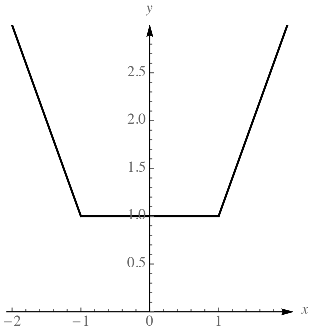


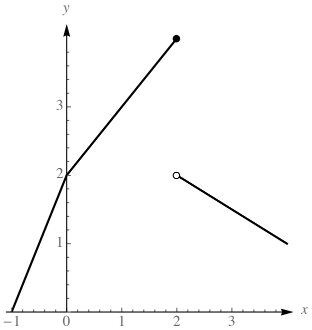
1.2.32



1.2.33

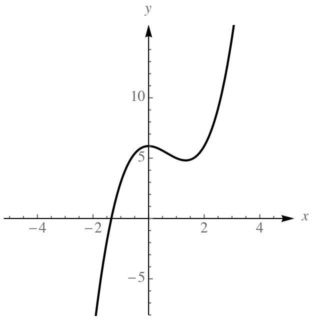


1.2.34



1.2.35

a.

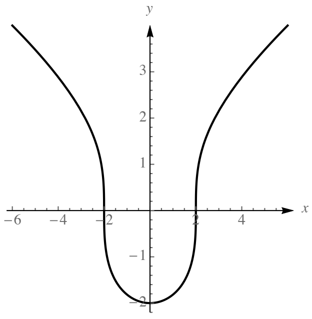


b. The function is a polynomial, so its domain is the set of all real numbers.

c. It has one peak near its y-intercept of $(0, 6)$ and one valley between $x = 1$ and $x = 2$. Its x-intercept is near $x = -4/3$.

1.2.36

a.

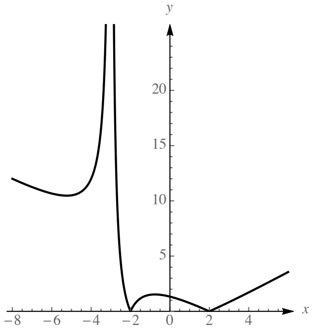


b. The function's domain is the set of all real numbers.

c. It has a valley at the y-intercept of $(0, -2)$, and is very steep at $x = -2$ and $x = 2$ which are the x-intercepts. It is symmetric about the y-axis.

1.2.37

a.

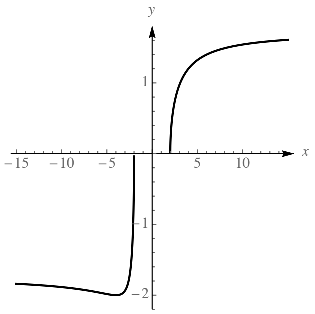


b. The domain of the function is the set of all real numbers except −3.

c. There is a valley near $x = -5.2$ and a peak near $x = -0.8$. The x-intercepts are at −2 and 2, where the curve does not appear to be smooth. There is a vertical asymptote at $x = -3$. The function is never below the x-axis. The y-intercept is $(0, 4/3)$.

1.2.38

a.

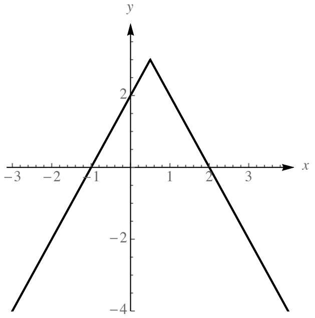


b. The domain of the function is $(-\infty, -2] \cup [2, \infty)$

c. x-intercepts are at −2 and 2. Because 0 isn't in the domain, there is no y-intercept. The function has a valley at $x = -4$.

1.2.39

a.

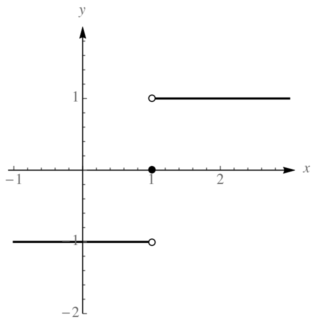


b. The domain of the function is $(-\infty, \infty)$

c. The function has a maximum of 3 at $x = 1/2$, and a y-intercept of 2.

1.2.40

a.



b. The domain of the function is $(-\infty, \infty)$

c. The function contains a jump at $x = 1$. The maximum value of the function is 1 and the minimum value is -1.

1.2.41

a. The zeros of f are the points where the graph crosses the x-axis, so these are points A, D, F, and I.

b. The only high point, or peak, of f occurs at point E, because it appears that the graph has larger and larger y values as x increases past point I and decreases past point A.

c. The only low points, or valleys, of f are at points B and H, again assuming that the graph of f continues its apparent behavior for larger values of x.

d. Past point H, the graph is rising, and is rising faster and faster as x increases. It is also rising between points B and E, but not as quickly as it is past point H. So the marked point at which it is rising most rapidly is I.

e. Before point B, the graph is falling, and falls more and more rapidly as x becomes more and more negative. It is also falling between points E and H, but not as rapidly as it is before point B. So the marked point at which it is falling most rapidly is A.

1.2.42

a. The zeros of g appear to be at $x = 0$, $x = 1$, $x = 1.6$, and $x \approx 3.15$.

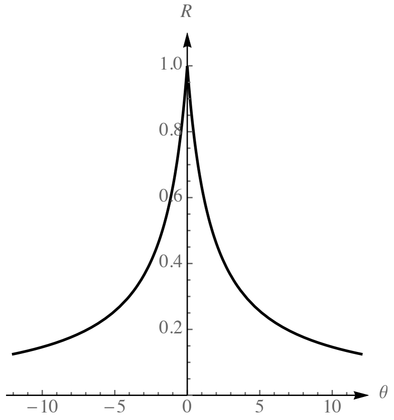
b. The two peaks of g appear to be at $x \approx 0.5$ and $x \approx 2.6$, with corresponding points $\approx (0.5, 0.4)$ and $\approx (2.6, 3.4)$.

c. The only valley of g is at $\approx (1.3, -0.2)$.

d. Moving right from $x \approx 1.3$, the graph is rising more and more rapidly until about $x = 2$, at which point it starts rising less rapidly (because, by $x \approx 2.6$, it is not rising at all). So the coordinates of the point at which it is rising most rapidly are approximately $(2.1, g(2)) \approx (2.1, 2)$. Note that while the curve is also rising between $x = 0$ and $x \approx 0.5$, it is not rising as rapidly as it is near $x = 2$.

e. To the right of $x \approx 2.6$, the curve is falling, and falling more and more rapidly as x increases. So the point at which it is falling most rapidly in the interval $[0, 3]$ is at $x = 3$, which has the approximate coordinates $(3, 1.4)$. Note that while the curve is also falling between $x \approx 0.5$ and $x \approx 1.3$, it is not falling as rapidly as it is near $x = 3$.

1.2.43

a. 

b. This appears to have a maximum when $\theta = 0$. Our vision is sharpest when we look straight ahead.

c. For $|\theta| \leq .19°$. We have an extremely narrow range where our eyesight is sharp.

1.2.44 Because the line is horizontal, the slope is constantly 0. So $S(x) = 0$.

1.2.45 The slope of this line is constantly 2, so the slope function is $S(x) = 2$.

1.2.46 The function can be written as $|x| = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$.

The slope function is $S(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$.

1.2.47 The slope function is given by $S(x) = \begin{cases} 1 & \text{if } x < 0; \\ -1/2 & \text{if } x > 0. \end{cases}$

1.2.48 The slope function is given by $s(x) = \begin{cases} 1 & \text{if } x < 3; \\ -1/3 & \text{if } x > 3. \end{cases}$

1.2.49

a. Because the area under consideration is that of a rectangle with base 2 and height 6, $A(2) = 12$.

b. Because the area under consideration is that of a rectangle with base 6 and height 6, $A(6) = 36$.

c. Because the area under consideration is that of a rectangle with base x and height 6, $A(x) = 6x$.

1.2.50

a. Because the area under consideration is that of a triangle with base 2 and height 1, $A(2) = 1$.

b. Because the area under consideration is that of a triangle with base 6 and height 3, the $A(6) = 9$.

c. Because $A(x)$ represents the area of a triangle with base x and height $(1/2)x$, the formula for $A(x)$ is $\frac{1}{2} \cdot x \cdot \frac{x}{2} = \frac{x^2}{4}$.

1.2.51

a. Because the area under consideration is that of a trapezoid with base 2 and heights 8 and 4, we have $A(2) = 2 \cdot \frac{8+4}{2} = 12$.

b. Note that $A(3)$ represents the area of a trapezoid with base 3 and heights 8 and 2, so $A(3) = 3 \cdot \frac{8+2}{2} = 15$. So $A(6) = 15 + (A(6) - A(3))$, and $A(6) - A(3)$ represents the area of a triangle with base 3 and height 2. Thus $A(6) = 15 + 6 = 21$.

c. For x between 0 and 3, $A(x)$ represents the area of a trapezoid with base x, and heights 8 and $8 - 2x$. Thus the area is $x \cdot \frac{8+8-2x}{2} = 8x - x^2$. For $x > 3$, $A(x) = A(3) + A(x) - A(3) = 15 + 2(x - 3) = 2x + 9$. Thus

$$A(x) = \begin{cases} 8x - x^2 & \text{if } 0 \le x \le 3; \\ 2x + 9 & \text{if } x > 3. \end{cases}$$

1.2.52

a. Because the area under consideration is that of trapezoid with base 2 and heights 3 and 1, we have $A(2) = 2 \cdot \frac{3+1}{2} = 4$.

b. Note that $A(6) = A(2) + (A(6) - A(2)$, and that $A(6) - A(2)$ represents a trapezoid with base $6 - 2 = 4$ and heights 1 and 5. The area is thus $4 + \left(4 \cdot \frac{1+5}{2}\right) = 4 + 12 = 16$.

c. For x between 0 and 2, $A(x)$ represents the area of a trapezoid with base x, and heights 3 and $3 - x$. Thus the area is $x \cdot \frac{3+3-x}{2} = 3x - \frac{x^2}{2}$. For $x > 2$, $A(x) = A(2) + A(x) - A(2) = 4 + (A(x) - A(2))$. Note that $A(x) - A(2)$ represents the area of a trapezoid with base $x - 2$ and heights 1 and $x - 1$. Thus $A(x) = 4 + (x - 2) \cdot \frac{1+x-1}{2} = 4 + (x - 2)\left(\frac{x}{2}\right) = \frac{x^2}{2} - x + 4$. Thus

$$A(x) = \begin{cases} 3x - \frac{x^2}{2} & \text{if } 0 \le x \le 2; \\ \frac{x^2}{2} - x + 4 & \text{if } x > 2. \end{cases}$$

1.2.53

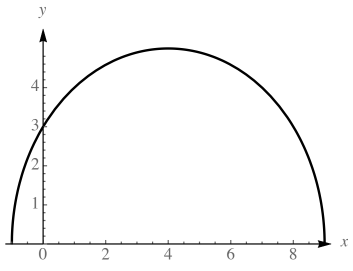
a. True. A polynomial $p(x)$ can be written as the ratio of polynomials $\frac{p(x)}{1}$, so it is a rational function. However, a rational function like $\frac{1}{x}$ is not a polynomial.

b. False. For example, if $f(x) = 2x$, then $(f \circ f)(x) = f(f(x)) = f(2x) = 4x$ is linear, not quadratic.

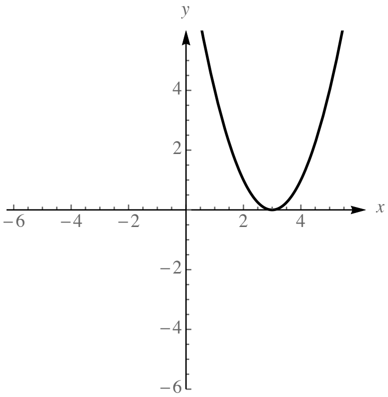
c. True. In fact, if f is degree m and g is degree n, then the degree of the composition of f and g is $m \cdot n$, regardless of the order they are composed.

d. False. The graph would be shifted two units to the left.

1.2.54 We complete the square for $-x^2 + 8x + 9$. Call this quantity z. Then $z = -(x^2 - 8x - 9)$, so $z = -(x^2 - 8x + 16 + (-16 - 9)) = -((x-4)^2 - 25 = 25 - (x-4)^2$. Thus $f(x)$ is obtained from the graph of $g(x) = \sqrt{25 - x^2}$ by shifting 4 units to the right. Thus the graph of f is the upper half of a circle of radius 5 centered at $(4, 0)$.

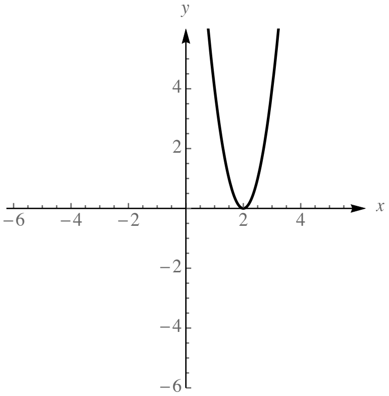


1.2.55



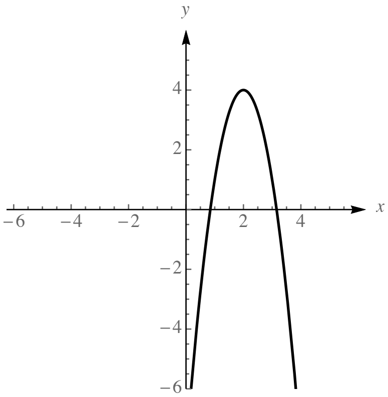
a.

Shift 3 units to the right.



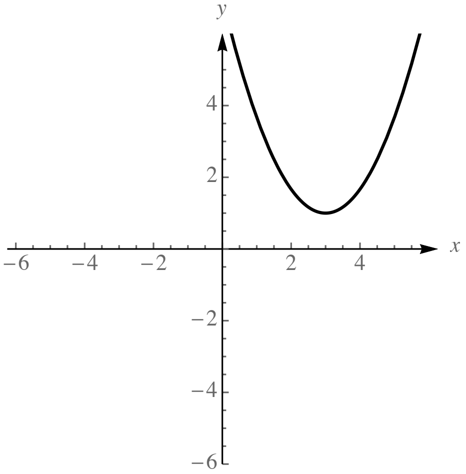
b.

Horizontal compression by a factor of $\frac{1}{2}$, then shift 2 units to the right.



c.

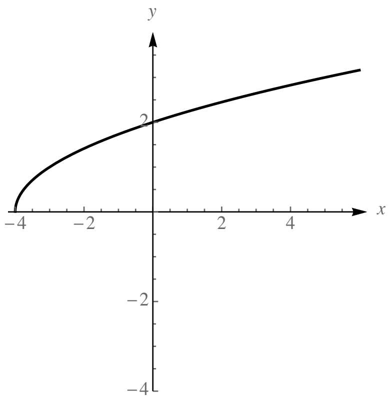
Shift to the right 2 units, vertically stretch by a factor of 3, reflect across the x-axis, and shift up 4 units.



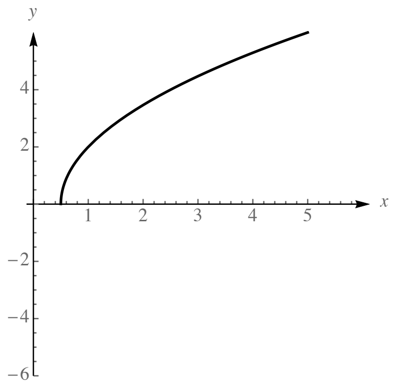
d.

Horizontal stretch by a factor of 3, horizontal shift right 2 units, vertical stretch by a factor of 6, and vertical shift up 1 unit.

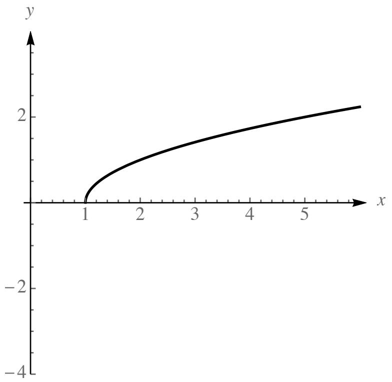
1.2.56



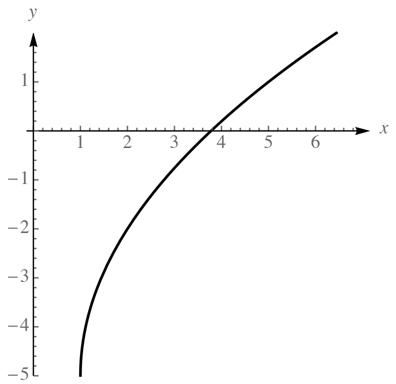
a. Shift 4 units to the left.



b. Horizontal compression by a factor of $\frac{1}{2}$, then shift $\frac{1}{2}$ units to the right. Then stretch vertically by a factor of 2.

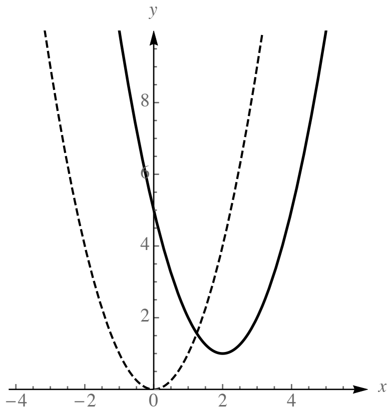


c. Shift 1 unit to the right.

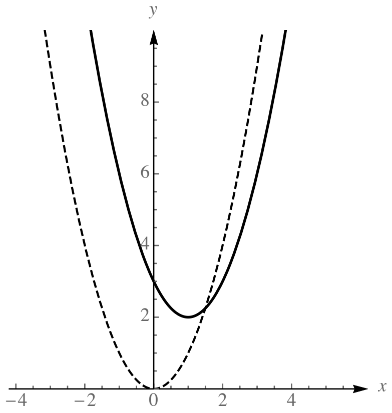


d. Shift 1 unit to the right, then stretch vertically by a factor of 3, then shift down 5 units.

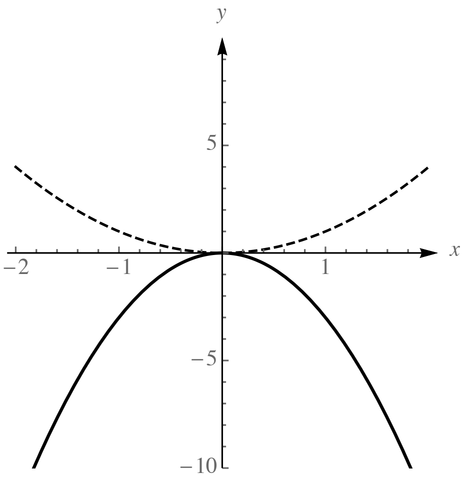
1.2.57 The graph is obtained by shifting the graph of x^2 two units to the right and one unit up.



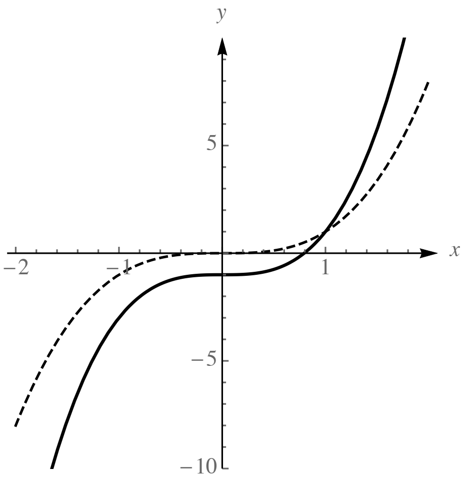
1.2.58 Write x^2-2x+3 as $(x^2-2x+1)+2=(x-1)^2+2$. The graph is obtained by shifting the graph of x^2 one unit to the right and two units up.



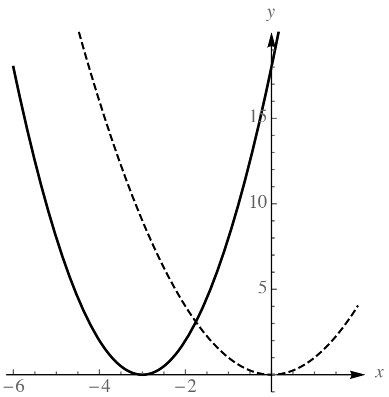
1.2.59 Stretch the graph of $y=x^2$ vertically by a factor of 3 and then reflect across the x-axis.



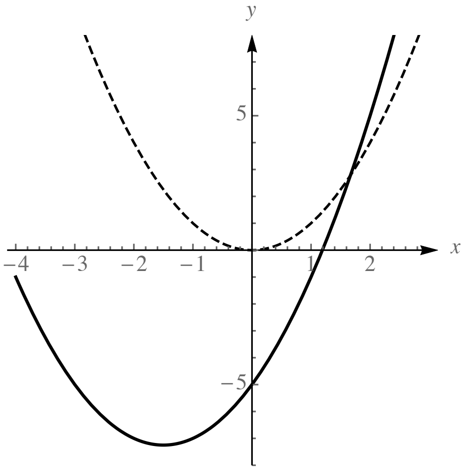
1.2.60 Scale the graph of $y=x^3$ vertically by a factor of 2, and then shift down 1 unit.



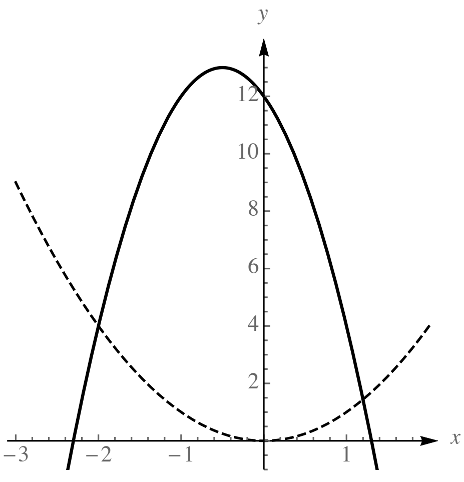
1.2.61 Shift the graph of $y = x^2$ left 3 units and stretch vertically by a factor of 2.



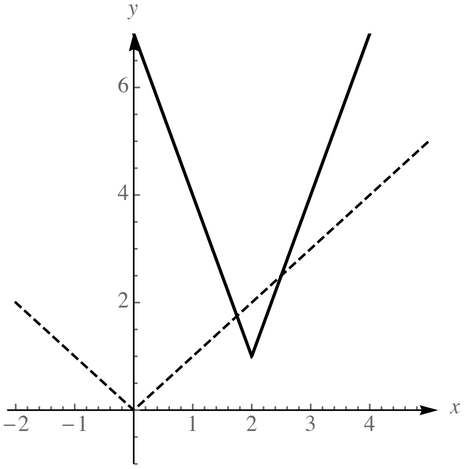
1.2.62 By completing the square, we have that $p(x) = x^2 + 3x - 5 = x^2 + 3x + \frac{9}{4} - 5 - \frac{9}{4} = (x + \frac{3}{2})^2 - \frac{29}{4}$. So it is $f(x + \frac{3}{2}) - (\frac{29}{4})$ where $f(x) = x^2$. The graph is shifted $\frac{3}{2}$ units to the left and then down $\frac{29}{4}$ units.



1.2.63 By completing the square, we have that $h(x) = -4(x^2 + x - 3) = -4\left(x^2 + x + \frac{1}{4} - \frac{1}{4} - 3\right) = -4(x + \frac{1}{2})^2 + 13$. So it is $-4f(x + (\frac{1}{2})) + 13$ where $f(x) = x^2$. The graph is shifted $\frac{1}{2}$ unit to the left, stretched vertically by a factor of 4, then reflected about the x-axis, then shifted up 13 units.



1.2.64 Because $|3x-6|+1 = 3|x-2|+1$, this is $3f(x-2)+1$ where $f(x) = |x|$. The graph is shifted 2 units to the right, then stretched vertically by a factor of 3, and then shifted up 1 unit.

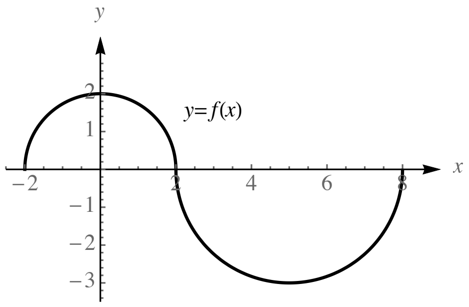


1.2.65 The curves intersect where $4\sqrt{2x} = 2x^2$. If we square both sides, we have $32x = 4x^4$, which can be written as $4x(8 - x^3) = 0$, which has solutions at $x = 0$ and $x = 2$. So the points of intersection are $(0, 0)$ and $(2, 8)$.

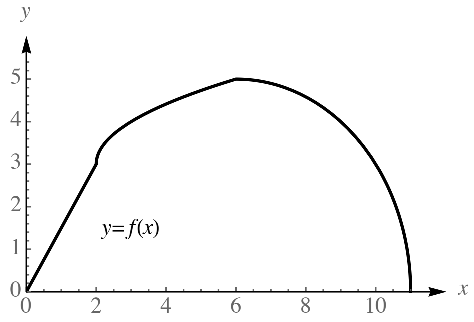
1.2.66 The points of intersection are found by solving $x^2 + 2 = x + 4$. This yields the quadratic equation $x^2 - x - 2 = 0$ or $(x - 2)(x + 1) = 0$. So the x-values of the points of intersection are 2 and -1. The actual points of intersection are $(2, 6)$ and $(-1, 3)$.

1.2.67 The points of intersection are found by solving $x^2 = -x^2 + 8x$. This yields the quadratic equation $2x^2 - 8x = 0$ or $(2x)(x - 4) = 0$. So the x-values of the points of intersection are 0 and 4. The actual points of intersection are $(0, 0)$ and $(4, 16)$.

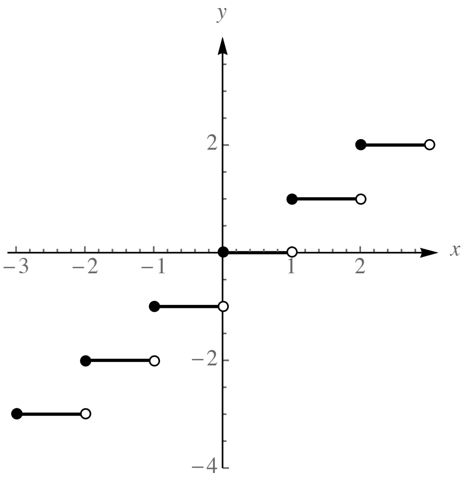
1.2.68 $$f(x) = \begin{cases} \sqrt{4 - x^2} & \text{if } -2 \le x \le 2 \\ -\sqrt{9 - (x-5)^2} & \text{if } 2 < x \le 6. \end{cases}.$$



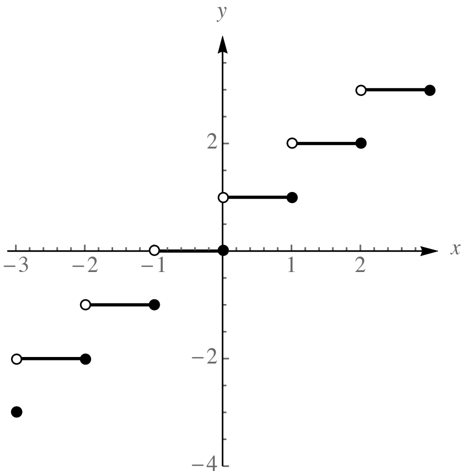
1.2.69



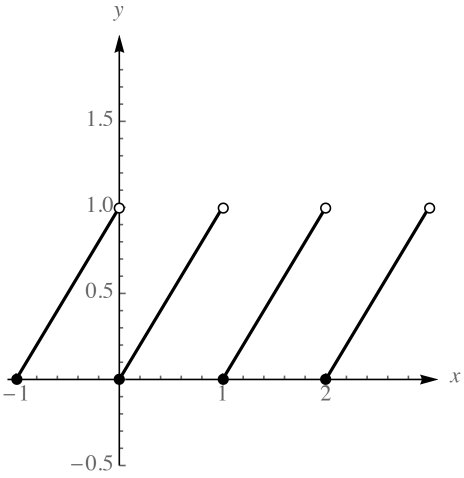
1.2.70



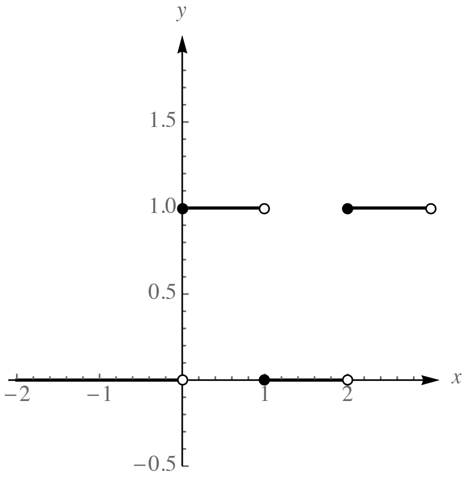
1.2.71



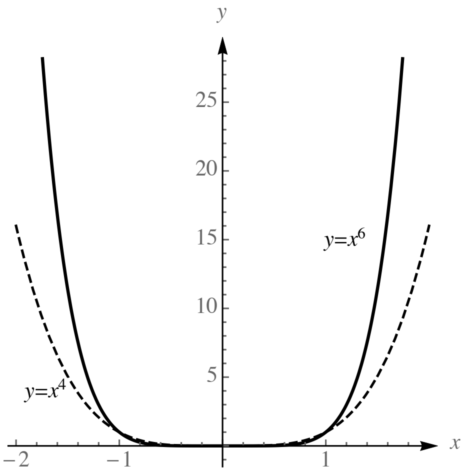
1.2.72



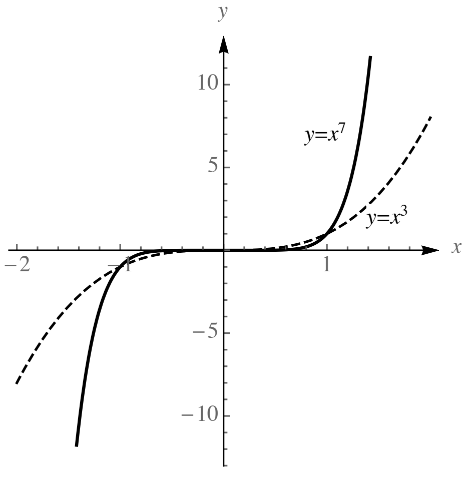
1.2.73



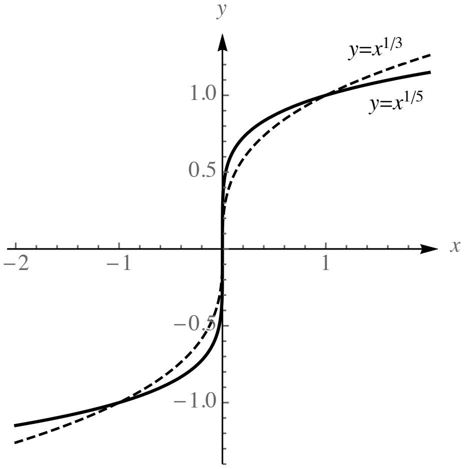
1.2.74



1.2.75



1.2.76



1.2.77

a. $f(0.75) = \frac{.75^2}{1-2(.75)(.25)} = .9$. There is a 90% chance that the server will win from deuce if they win 75% of their service points.

b. $f(0.25) = \frac{.25^2}{1-2(.25)(.75)} = .1$. There is a 10% chance that the server will win from deuce if they win 25% of their service points.

1.2.78

a. We know that the points $(32, 0)$ and $(212, 100)$ are on our line. The slope of our line is thus $\frac{100-0}{212-32} = \frac{100}{180} = \frac{5}{9}$. The function $f(F)$ thus has the form $C = (5/9)F + b$, and using the point $(32, 0)$ we see that $0 = (5/9)32 + b$, so $b = -(160/9)$. Thus $C = (5/9)F - (160/9)$

b. Solving the system of equations $C = (5/9)F - (160/9)$ and $C = F$, we have that $F = (5/9)F - (160/9)$, so $(4/9)F = -160/9$, so $F = -40$ when $C = -40$.

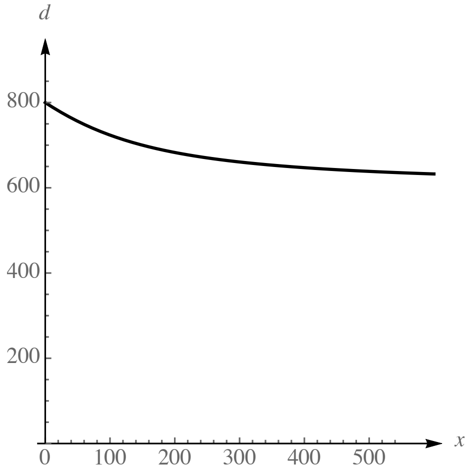
1.2.79

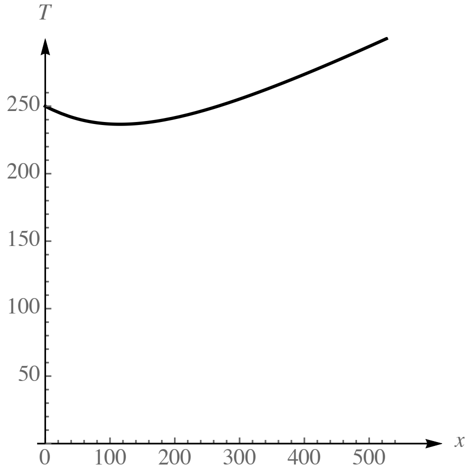
a. Because you are paying \$350 per month, the amount paid after m months is $y = 350m + 1200$.

b. After 4 years (48 months) you have paid $350 \cdot 48 + 1200 = 18000$ dollars. If you then buy the car for \$10,000, you will have paid a total of \$28,000 for the car instead of \$25,000. So you should buy the car instead of leasing it.

1.2.80

a. Note that the island, the point P on shore, and the point down shore x units from P form a right triangle. By the Pythagorean theorem, the length of the hypotenuse is $\sqrt{40000 + x^2}$. So Kelly must row this distance and then jog $600-x$ meters to get home. So her total distance $d(x) = \sqrt{40000 + x^2} + (600 - x)$.

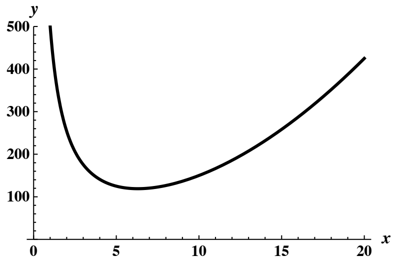




b. Because distance is rate times time, we have that time is distance divided by rate. Thus $T(x) = \frac{\sqrt{40000+x^2}}{2} + \frac{600-x}{4}$.

c. By inspection, it looks as though she should head to a point about 115 meters down shore from P. This would lead to a time of about 236.6 seconds.

1.2.81



a. The volume of the box is x^2h, but because the box has volume 125 cubic feet, we have that $x^2h = 125$, so $h = \frac{125}{x^2}$. The surface area of the box is given by x^2 (the area of the base) plus $4 \cdot hx$, because each side has area hx. Thus $S = x^2 + 4hx = x^2 + \frac{4\cdot 125\cdot x}{x^2} = x^2 + \frac{500}{x}$.

b. By inspection, it looks like the value of x which minimizes the surface area is about 6.3.

1.2.82 Let $f(x) = a_nx^n +$ smaller degree terms and let $g(x) = b_mx^m +$ some smaller degree terms.

a. The largest degree term in $f \cdot f$ is $a_nx^n \cdot a_nx^n = a_n^2x^{n+n}$, so the degree of this polynomial is $n+n = 2n$.

b. The largest degree term in $f \circ f$ is $a_n \cdot (a_nx^n)^n$, so the degree is n^2.

c. The largest degree term in $f \cdot g$ is $a_nb_mx^{m+n}$, so the degree of the product is $m + n$.

d. The largest degree term in $f \circ g$ is $a_n \cdot (b_mx^m)^n$, so the degree is mn.

1.2.83 Suppose that the parabola f crosses the x-axis at a and b, with $a < b$. Then a and b are roots of the polynomial, so $(x - a)$ and $(x - b)$ are factors. Thus the polynomial must be $f(x) = c(x - a)(x - b)$ for some non-zero real number c. So $f(x) = cx^2 - c(a + b)x + abc$. Because the vertex always occurs at the x value which is $\frac{-\text{coefficient on x}}{2\cdot\text{coefficient on } x^2}$ we have that the vertex occurs at $\frac{c(a+b)}{2c} = \frac{a+b}{2}$, which is halfway between a and b.

1.2.84

a. We complete the square to rewrite the function f. Write $f(x) = ax^2 + bx + c$ as $f(x) = a(x^2 + \frac{b}{a}x + \frac{c}{a})$. Completing the square yields

$$a\left(\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a}\right) + \left(\frac{c}{a} - \frac{b^2}{4a}\right)\right) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4}\right).$$

Thus the graph of f is obtained from the graph of x^2 by shifting $\frac{b}{2a}$ units horizontally (and then doing some scaling and vertical shifting) – moving the vertex from 0 to $-\frac{b}{2a}$. The vertex is therefore $\left(\frac{-b}{2a}, c - \frac{b^2}{4}\right)$.

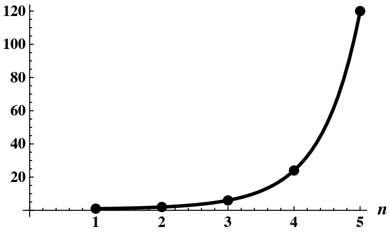
b. We know that the graph of f touches the x-axis twice if the equation $ax^2 + bx + c = 0$ has two real solutions. By the quadratic formula, we know that this occurs exactly when the discriminant $b^2 - 4ac$ is positive. So the condition we seek is for $b^2 - 4ac > 0$, or $b^2 > 4ac$.

1.2.85

a.

n	1	2	3	4	5
$n!$	1	2	6	24	120

b.



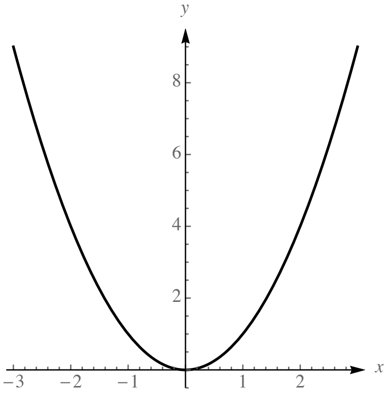
c. Using trial and error and a calculator yields that 10! is more than a million, but 9! isn't.

1.3 Inverse, Exponential and Logarithmic Functions

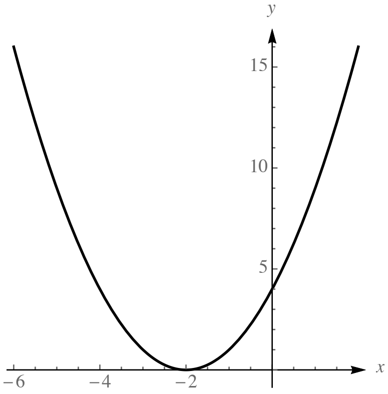
1.3.1 $D = \mathbb{R}, R = (0, \infty)$.

1.3.2 $f(x) = 2x + 1$ is one-to-one on all of $\mathbb{R}$. If $f(a) = f(b)$, then $2a + 1 = 2b + 1$, so it must follow that $a = b$.

1.3.3



1.3.4



1.3.5 f is one-to-one on $(-\infty, -1]$, on $[-1, 1]$, and on $[1, \infty)$.

1.3.6 f is one-to-one on $(-\infty, -2]$, on $[-2, 0]$, on $[0, 2]$, and on $[2, \infty)$.

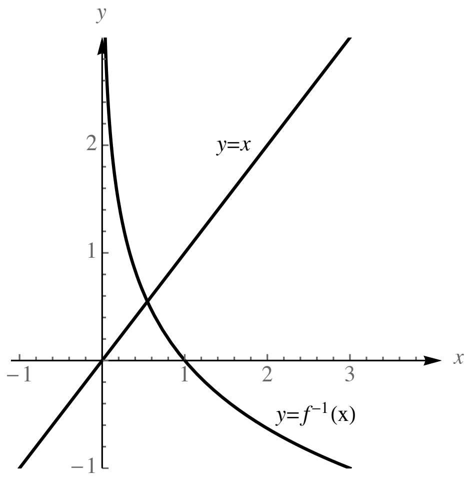
1.3.7 If a function f is not one-to-one, then there are domain values $x_1 \neq x_2$ with $f(x_1) = f(x_2)$. If f^{-1} were to exist, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2))$ which would imply that $x_1 = x_2$, a contradiction.

1.3.8 Because $f(1) = 2$, $f^{-1}(2) = 1$. Because $f(5) = 9$, $f^{-1}(9) = 5$. Because $f(7) = 12$, $f^{-1}(12) = 7$.

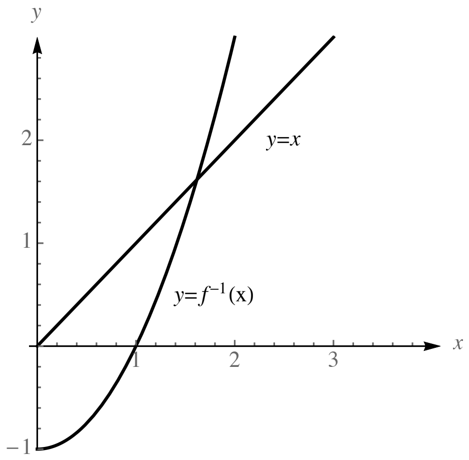
1.3.9 Suppose $x = 2y$, then $y = \frac{1}{2}x$, so the inverse of f is $f^{-1}(x) = \frac{1}{2}x$. Then $f(f^{-1}(x)) = f(\frac{1}{2}x) = 2 \cdot \frac{1}{2}x = x$. Also, $f^{-1}(f(x)) = f^{-1}(2x) = \frac{1}{2} \cdot 2x = x$.

1.3.10 Suppose $x = \sqrt{y}$. Then $y = x^2$. So the inverse of f is $f^{-1}(x) = x^2$, $x \geq 0$. Then $f(f^{-1}(x)) = f(x^2) = \sqrt{x^2} = |x| = x$ for $x \geq 0$. Also, $f^{-1}(f(x)) = f^{-1}(\sqrt{x}) = (\sqrt{x})^2 = x$.

1.3.11



1.3.12



1.3.13 $g_1(x)$ is the right side of the standard parabola shifted up one unit. So $g_1(x) = x^2 + 1$, $x \geq 0$. The domain for g_1 is $[0, \infty)$ and the range is $[1, \infty)$. The inverse of g_1 is therefore the square root function shifted one unit to the right. So $g_1^{-1}(x) = \sqrt{x-1}$, and its domain is $[1, \infty)$ and its range is $[0, \infty)$.

1.3.14 $g_2(x)$ is the left side of the standard parabola shifted up one unit. So $g_2(x) = x^2 + 1$, $x \leq 0$. The domain for g_1 is $(\infty, 0]$ and the range is $[1, \infty)$. The inverse of g_2 is therefore the square root function shifted one unit to the right and reflected across the x-axis. So $g_2^{-1}(x) = -\sqrt{x-1}$, and its domain is $[1, \infty)$ and its range is $(\infty, 0]$.

1.3.15 $\log_b x$ represents the power to which b must be raised in order to obtain x. So, $b^{\log_b x} = x$.

1.3.16 The properties are related in that each can be used to derive the other. Assume $b^{x+y} = b^x b^y$, for all real numbers x and y. Then applying this rule to the numbers $\log_b x$ and $\log_b y$ gives $b^{\log_b x + \log_b y} = b^{\log_b x} b^{\log_b y} = xy$. Taking logs of the leftmost and rightmost sides of this equation yields $\log_b x + \log_b y = \log_b(xy)$.

Now assume that $\log_b(xy) = \log_b x + \log_b y$ for all positive numbers x and y. Applying this rule to the product $b^x b^y$, we have $\log_b(b^x b^y) = \log_b b^x + \log_b b^y = x + y$. Now looking at the leftmost and rightmost sides of this equality and applying the definition of logarithm yields $b^{x+y} = b^x b^y$, as was desired.

1.3.17 Because the domain of b^x is $\mathbb{R}$ and the range of b^x is $(0, \infty)$, and because $\log_b x$ is the inverse of b^x, the domain of $\log_b x$ is $(0, \infty)$ and the range is $\mathbb{R}$.

1.3.18 Let $2^5 = z$. Then $\ln(2^5) = \ln(z)$, so $\ln(z) = 5\ln(2)$. Taking the exponential function of both sides gives $z = e^{5\ln(2)}$. Therefore, $2^5 = e^{5\ln(2)}$.

1.3.19

a. Because $10^3 = 1000$, $\log_{10} 1000 = 3$.

b. Because $2^4 = 16$, $\log_2 16 = 4$.

c. Because $10^{-2} = \frac{1}{100} = 0.01$, $\log_{10} 0.01 = -2$.

d. Because e^x and $\ln x$ are inverses, $\ln e^3 = 3$.

e. Because e^x and $\ln x$ are inverses, $\ln \sqrt{e} = \ln e^{1/2} = \frac{1}{2}$.

1.3.20 $\log_2 a = \frac{\ln a}{\ln 2} \approx 5.4923$, and $\log_a 2 = \frac{\ln 2}{\ln a} \approx 0.1821$.

1.3.21 f is one-to-one on $(-\infty, \infty)$, so it has an inverse on $(-\infty, \infty)$.

1.3.22 f is one-to-one on $[-1/2, \infty)$, so it has an inverse on that set. (Alternatively, it is one-to-one on the interval $(-\infty, -1/2]$, so that interval could be used as well.)

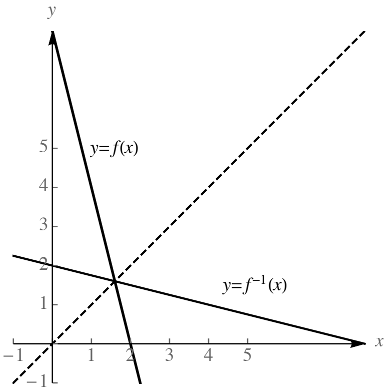
1.3.23 f is one-to-one on its domain, which is $(-\infty, 5) \cup (5, \infty)$, so it has an inverse on that set.

1.3.24 f is one-to-one on the set $(-\infty, 6]$, so it has an inverse on that set. (Alternatively, it is one-to-one on the interval $[6, \infty)$, so that interval could be used as well.)

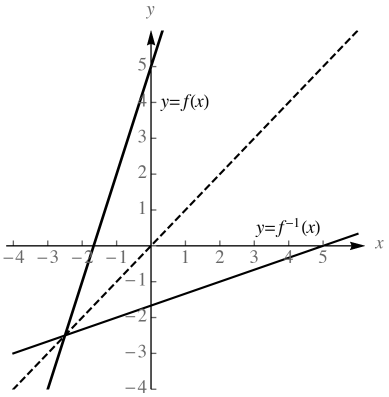
1.3.25 f is one-to-one on the interval $(0, \infty)$, so it has an inverse on that interval. (Alternatively, it is one-to-one on the interval $(-\infty, 0)$, so that interval could be used as well.)

1.3.26 Note that f can be written as $f(x) = x^2 - 2x + 8 = x^2 - 2x + 1 + 7 = (x-1)^2 + 7$. It is one-to-one on the interval $(1, \infty)$, so it has an inverse on that interval. (Alternatively, it is one-to-one on the interval $(-\infty, 1)$, so that interval could be used as well.)

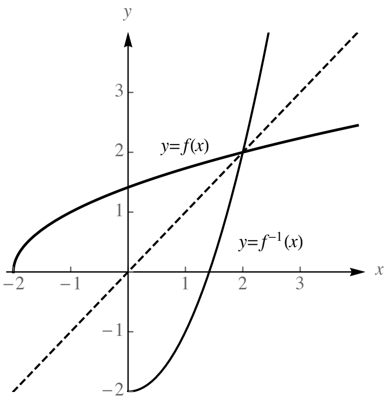
1.3.27 Switching x and y gives $x = 8 - 4y$. Solving this for y yields $y = f^{-1}(x) = \frac{8-x}{4}$.



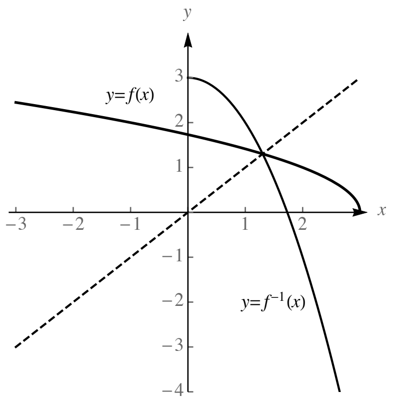
1.3.28 Switching x and y, we have $x = 3y+5$. Solving for y in terms of x we have $y = \frac{x-5}{3}$, so $y = f^{-1}(x) = \frac{x-5}{3}$.



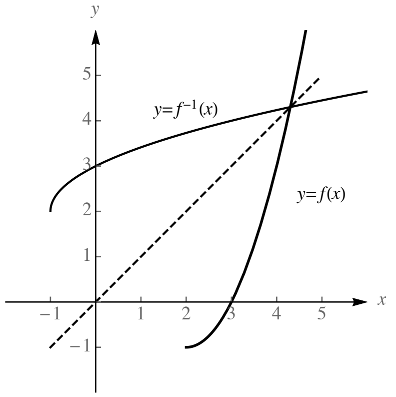
1.3.29 Switching x and y, we have $x = \sqrt{y+2}$. Solving for y in terms of x we have $y = f^{-1}(x) = x^2 - 2$. Note that because the range of f is $[0, \infty)$, that is also the domain of f^{-1}.



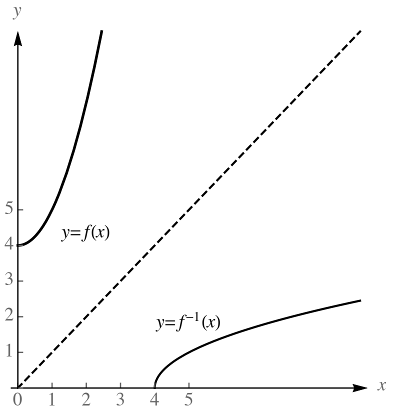
1.3.30 Switching x and y gives $x = \sqrt{3-y}$. Solving for y gives $3 - y = x^2$, or $y = 3 - x^2$. Since the domain of f is $(-\infty, 3]$ and the range of f is $[0, \infty)$, the domain of f^{-1} is $[0, \infty)$ and the range is $(-\infty, 3]$.



1.3.31 Switching x and y gives $x = (y-2)^2 - 1$. Then $x + 1 = (y-2)^2$, so $\sqrt{x+1} = |y-2|$, but because in the original function the variable is greater than or equal to 2, we choose the the positive portion of the graph of $|y-2|$. So we have $y = 2 + \sqrt{x+1}$.



1.3.32 Switching x and y, we have $x = y^2 + 4$. Solving for y in terms of x we have $y^2 = x - 4$, so $|y| = \sqrt{x-4}$. But because we are given that the domain of f is $\{x \colon x \geq 0\}$, we know that the range of f^{-1} is also non-negative. So $y = f^{-1}(x) = \sqrt{x-4}$.



1.3.33 Switching x and y, we have $x = \frac{2}{y^2+1}$. Then $y^2 + 1 = \frac{2}{x}$, so $y^2 = \frac{2}{x} - 1$. So $|y| = \sqrt{\frac{2}{x} - 1}$. We choose the positive portion, so that $y = \sqrt{\frac{2}{x} - 1}$. Note that the domain of f is $[0, \infty)$ while the range of f is $(0, 2]$. So the domain of f^{-1} is $(0, 2]$ and the range is $[0, \infty)$.

1.3.34 Switching x and y gives $x = \frac{6}{y^2-9}$. Solving yields $y^2 - 9 = \frac{6}{x}$, or $|y| = \sqrt{\frac{6}{x} + 9}$, but because the domain of f is positive, the range of f^{-1} must be positive as well, so we have $f^{-1}(x) = \sqrt{\frac{6}{x} + 9}$.

1.3.35 Switching x and y, we have $x = e^{2y+6}$. Then $\ln x = 2y + 6$, so $2y = \ln x - 6$, and $y = f^{-1}(x) = \frac{1}{2} \ln x - 3$.

1.3.36 Switching x and y, we have $x = 4e^{5y}$. Then $e^{5y} = \frac{x}{4}$, so $5y = \ln \frac{x}{4}$, and $y = f^{-1}(x) = .\frac{1}{5} \ln \frac{x}{4}$.

1.3.37 Switching x and y, we have $x = \ln(3y+1)$. Then $e^x = 3y+1$, so $3y = e^x - 1$ and $y = f^{-1}(x) = \frac{e^x-1}{3}$.

1.3.38 Switching x and y, we have $x = \log_{10} 4y$. Then $10^x = 4y$, so $y = f^{-1}(x) = \frac{10^x}{4}$.

1.3.39 Switching x and y, we have $x = 10^{-2y}$. Then $\log_{10} x = -2y$, so $y = f^{-1}(x) = -\frac{1}{2}\log_{10} x$.

1.3.40 Switching x and y, we have $x = \frac{1}{e^y+1}$. Then $e^y + 1 = \frac{1}{x}$. so $e^y = \frac{1}{x} - 1$, and $y = f^{-1}(x) = \ln\left(\frac{1}{x} - 1\right)$.

1.3.41 Switching x and y, we have $x = \frac{e^y}{e^y+2}$. Taking the reciprocal of both sides, we have $\frac{1}{x} = \frac{e^y+2}{e^y} = 1 + 2e^{-y}$. Then $2e^{-y} = \frac{1}{x} - 1$, and $e^{-y} = \frac{1}{2x} - \frac{1}{2}$. So $-y = \ln\left(\frac{1}{2x} - \frac{1}{2}\right)$, and $y = f^{-1}(x) = -\ln\left(\frac{1}{2x} - \frac{1}{2}\right) = -\ln\left(\frac{1-x}{2x}\right) = \ln\left(\frac{2x}{1-x}\right)$.

1.3.42 Switching x and y, we have $x = \frac{y}{y-2}$. Cross multiplying yields $xy - 2x = y$, so $xy - y = 2x$, and $y(x-1) = 2x$. Then $y = \frac{2x}{x-1}$. Note that the domain of f is given to be $(2, \infty)$, and the range is $(1, \infty)$. So the domain of f^{-1} must be restricted to be $(1, \infty)$.

1.3.43 First note that because the expression is symmetric, switching x and y doesn't change the expression. Solving for y gives $|y| = \sqrt{1-x^2}$. To get the four one-to-one functions, we restrict the domain and choose either the upper part or lower part of the circle as follows:

a. $f_1(x) = \sqrt{1-x^2},\ 0 \le x \le 1$
$f_2(x) = \sqrt{1-x^2},\ -1 \le x \le 0$
$f_3(x) = -\sqrt{1-x^2},\ -1 \le x \le 0$
$f_4(x) = -\sqrt{1-x^2},\ 0 \le x \le 1$

b. Reflecting these functions across the line $y = x$ yields the following:
$f_1^{-1}(x) = \sqrt{1-x^2},\ 0 \le x \le 1$
$f_2^{-1}(x) = -\sqrt{1-x^2},\ 0 \le x \le 1$
$f_3^{-1}(x) = -\sqrt{1-x^2},\ -1 \le x \le 0$
$f_4^{-1}(x) = \sqrt{1-x^2},\ -1 \le x \le 0$

1.3.44 First note that because the expression is symmetric, switching x and y doesn't change the expression. Solving for y gives $|y| = \sqrt{2|x|}$. To get the four one-to-one functions, we restrict the domain and choose either the upper part or lower part of the parabola as follows:

a. $f_1(x) = \sqrt{2x},\ x \ge 0$
$f_2(x) = \sqrt{-2x},\ x \le 0$
$f_3(x) = -\sqrt{-2x},\ x \le 0$
$f_4(x) = -\sqrt{2x},\ x \ge 0$

b. Reflecting these functions across the line $y = x$ yields the following:
$f_1^{-1}(x) = x^2/2,\ x \ge 0$
$f_2^{-1}(x) = -x^2/2,\ x \ge 0$
$f_3^{-1}(x) = -x^2/2,\ x \le 0$
$f_4^{-1}(x) = x^2/2,\ x \le 0$

1.3.45 $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y = 0.36 - .056 = -0.2$.

1.3.46 $\log_b x^2 = 2\log_b x = 2(0.36) = 0.72$.

1.3.47 $\log_b xz = \log_b x + \log_b z = 0.36 + 0.83 = 1.19$.

1.3.48 $\log_b \frac{\sqrt{xy}}{z} = \log_b(xy)^{1/2} - \log_b z = \frac{1}{2}(\log_b x + \log_b y) - \log_b z = (0.36)/2 + (0.56)/2 - 0.83 = -0.37.$

1.3.49 $\log_b \frac{\sqrt{x}}{\sqrt[3]{z}} = \log_b x^{1/2} - \log_b z^{1/3} = (1/2)\log_b x - (1/3)\log_b z = (0.36)/2 - (0.83)/3 = -.09\overline{6}.$

1.3.50 $\log_b \frac{b^2 x^{5/2}}{\sqrt{y}} = \log_b b^2 x^{5/2} - \log_b y^{1/2} = \log_b b^2 + (5/2)\log_b x - (1/2)\log_b y = 2 + (5/2)(0.36) - (1/2)(0.56) = 2.62.$

1.3.51 If $\log_{10} x = 3$, then $10^3 = x$, so $x = 1000$.

1.3.52 If $\log_5 x = -1$, then $5^{-1} = x$, so $x = 1/5$.

1.3.53 If $\log_8 x = 1/3$, then $x = 8^{1/3} = 2$.

1.3.54 If $\log_b 125 = 3$, then $b^3 = 125$, so $b = 5$ because $5^3 = 125$.

1.3.55 $\ln x = -1$, then $e^{-1} = x$, so $x = \frac{1}{e}$.

1.3.56 If $\ln y = 3$, then $y = e^3$.

1.3.57 Since $7^x = 21$, we have that $\ln 7^x = \ln 21$, so $x \ln 7 = \ln 21$, and $x = \frac{\ln 21}{\ln 7}$.

1.3.58 Since $2^x = 55$, we have that $\ln 2^x = \ln 55$, so $x \ln 2 = \ln 55$, and $x = \frac{\ln 55}{\ln 2}$.

1.3.59 Since $3^{3x-4} = 15$, we have that $\ln 3^{3x-4} = \ln 15$, so $(3x-4)\ln 3 = \ln 15$. Thus, $3x - 4 = \frac{\ln 15}{\ln 3}$, so $x = \frac{(\ln 15)/(\ln 3)+4}{3} = \frac{\ln 15 + 4\ln 3}{3\ln 3} = \frac{\ln 5 + \ln 3 + 4\ln 3}{3\ln 3} = \frac{\ln 5}{3\ln 3} + \frac{5}{3}$.

1.3.60 Since $5^{3x} = 29$, we have that $\ln 5^{3x} = \ln 29$, so $(3x)\ln 5 = \ln 29$. Solving for x gives $x = \frac{\ln 29}{3\ln 5}$.

1.3.61 We are seeking t so that $50 = 100e^{-t/650}$. This occurs when $e^{-t/650} = \frac{1}{2}$, which is when $-\frac{t}{650} = \ln(1/2)$, so $t = 650\ln 2 \approx 451$ years.

1.3.62 We need to solve $150 = 64e^{0.004t}$ for t. We have $\frac{150}{64} = e^{0.004t}$. So $0.004t = \ln\frac{150}{64}$, so $t = \frac{\ln\frac{150}{64}}{0.004} \approx 212.938$, so about 213 days.

1.3.63 We need to solve $1100 = 1000\left(1 + \dfrac{0.01}{12}\right)^{12t}$ for t. We have $1.1 = \left(1 + \dfrac{0.01}{12}\right)^{12t}$, so $\ln 1.1 = 12t\ln\left(1 + \dfrac{0.01}{12}\right)$. Then $t = \dfrac{\ln 1.1}{12\ln\left(1 + \frac{0.01}{12}\right)} \approx 9.53$ years.

1.3.64 We need to solve $22000 = 20000\left(1 + \dfrac{0.025}{12}\right)^{12t}$ for t. We have $1.1 = \left(1 + \dfrac{0.025}{12}\right)^{12t}$, so $\ln 1.1 = 12t\ln\left(1 + \dfrac{0.025}{12}\right)$. Then $t = \dfrac{\ln 1.1}{12\ln\left(1 + \frac{0.025}{12}\right)} \approx 3.82$ years.

1.3.65

a. No. The function takes on the values from 0 to 64 as t varies from 0 to 2, and then takes on the values from 64 to 0 as t varies from 2 to 4, so h is not one-to-one.

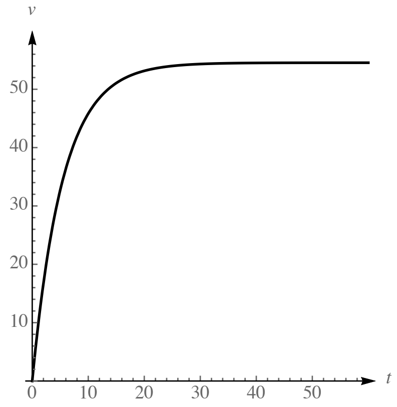
b. Solving for h in terms of t we have $h = 64t - 16t^2$, so (completing the square) we have $h - 64 = -16(t^2 - 4t + 4)$. Thus, $h - 64 = -16(t-2)^2$, and $(t-2)^2 = \frac{64-h}{16}$. Therefore $|t - 2| = \frac{\sqrt{64-h}}{4}$. When the ball is on the way up we know that $t < 2$, so the inverse of f is $f^{-1}(h) = 2 - \frac{\sqrt{64-h}}{4}$.

c. Using the work from the previous part of this problem, we have that when the ball is on the way down (when $t > 2$) we have that the inverse of f is $f^{-1}(h) = 2 + \frac{\sqrt{64-h}}{4}$.

d. On the way up, the ball is at a height of 30 ft at $2-\frac{\sqrt{64-30}}{4}\approx 0.542$ seconds.

e. On the way down, the ball is at a height of 10 ft at $2+\frac{\sqrt{64-10}}{4}\approx 3.837$ seconds.

1.3.66 The terminal velocity for $k=11$ is $\frac{600}{11}$.



1.3.67 $\log_2 15=\frac{\ln 15}{\ln 2}\approx 3.9069$.

1.3.68 $\log_3 30=\frac{\ln 30}{\ln 3}\approx 3.0959$.

1.3.69 $\log_4 40=\frac{\ln 40}{\ln 4}\approx 2.6610$.

1.3.70 $\log_6 60=\frac{\ln 60}{\ln 6}\approx 2.2851$.

1.3.71 Let $2^x=z$. Then $\ln 2^x=\ln z$, so $x\ln 2=\ln z$. Taking the exponential function of both sides gives $z=e^{x\ln 2}$.

1.3.72 Let $3^{\sin x}=z$. Then $\ln 3^{\sin x}=\ln z$, so $(\sin x)\ln 3=\ln z$. Taking the exponential function of both sides gives $z=e^{(\sin x)\ln 3}$.

1.3.73 Let $z=\ln|x|$. Then $e^z=|x|$. Taking logarithms with base 5 of both sides gives $\log_5 e^z=\log_5|x|$, so $z\cdot\log_5 e=\log_5|x|$, and thus $z=\frac{\log_5|x|}{\log_5 e}$.

1.3.74 Using the change of base formula, $\log_2(x^2+1)=\frac{\ln(x^2+1)}{\ln 2}$.

1.3.75 Let $z=a^{1/\ln a}$. Then $\ln z=\ln\left(a^{1/\ln a}\right)=\frac{1}{\ln a}\cdot\ln a=1$. Thus $z=e$.

1.3.76 Let $z=a^{1/\log a}$. Then $\log z=\log\left(a^{1/\log a}\right)=\frac{1}{\log a}\cdot\log a=1$. Thus $z=10$.

1.3.77

a. False. For example, $3=3^1$, but $1\neq\sqrt[3]{3}$.

b. False. For example, suppose $x=y=b=2$. Then the left-hand side of the equation is equal to 1, but the right-hand side is 0.

c. False. $\log_5 4^6=6\log_5 4>4\log_5 6$.

d. True. This follows because 10^x and $\log_{10}$ are inverses of each other.

e. False. $\ln 2^e=e\ln 2<2$.

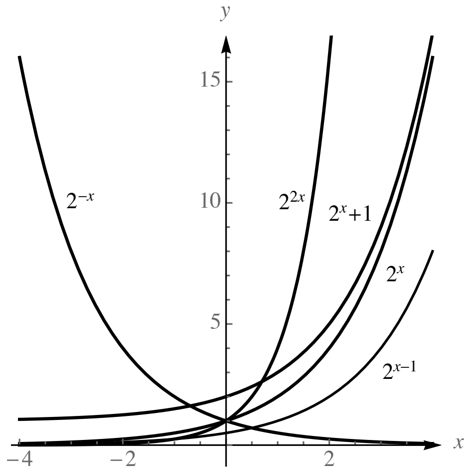
f. False. For example $f(0)=1$, but the alleged inverse function evaluated at 1 is not 0 (rather, it has value 1/2.)

g. True. f is its own inverse because $f(f(x))=f(1/x)=\frac{1}{1/x}=x$.

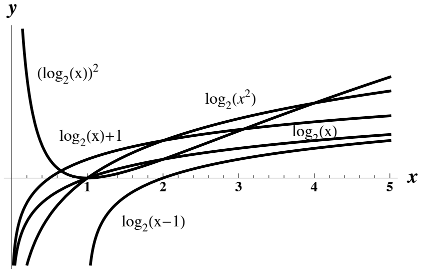
1.3.78 A is 2^{-x}, B is 3^{-x}, C is 3^x and D is 2^x.

1.3.79 A is $\log_2 x$, B is $\log_4 x$, C is $\log_{10} x$.

1.3.80



1.3.81



Note: need better pic from back of book

1.3.82

a. $p(0) = 150(2^{0/12}) = 150$.

b. At a given time t, let the population be $z = 150(2^{t/12})$. Then 12 hours later, the time is $12 + t$, and the population is $150(2^{(t+12)/12}) = 150(2^{(t/12)+1}) = 150(2^{t/12} \cdot 2) = 2z$.

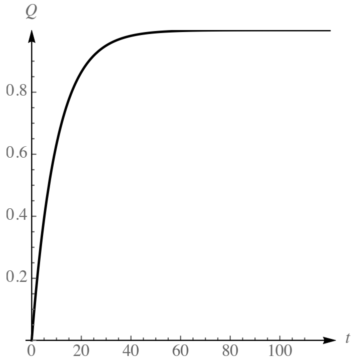
c. Since 4 days is 96 hours, we have $p(96) = 150(2^{96/12}) = 150(2^8) = 38{,}400$.

d. We can find the time to triple by solving $450 = 150(2^{t/12})$, which is equivalent to $3 = 2^{t/12}$. By taking logs of both sides we have $\ln 3 = \frac{t}{12} \cdot \ln 2$, so $t = \frac{12 \ln 3}{\ln 2} \approx 19.0$ hours.

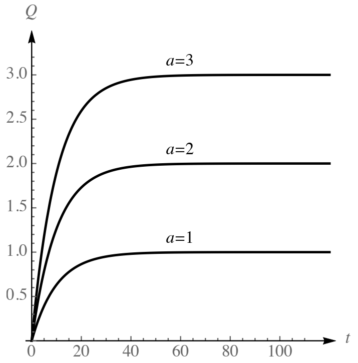
e. The population will reach 10,000 when $10,000 = 150(2^{12/t})$, which is equivalent to $\frac{200}{3} = 2^{t/12}$. By taking logs of both sides we have $\ln(200/3) = \frac{t}{12} \ln 2$, so $= \frac{12 \cdot \ln(200/3)}{\ln 2} \approx 72.7$ hours.

1.3.83

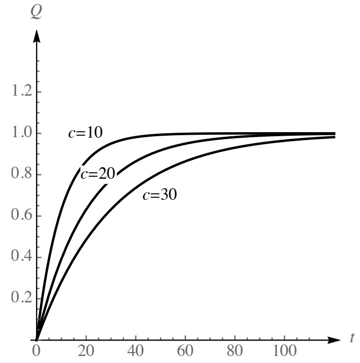
a. The relevant graph is:



b. Varying a while holding c constant scales the curve vertically. It appears that the steady-state charge is equal to a.



c. Varying c while holding a constant scales the curve horizontally. It appears that the steady-state charge does not vary with c.



d. As t grows large, the term $ae^{-t/c}$ approaches zero for any fixed c and a. So the steady-state charge for $a - ae^{-t/c}$ is a.

1.3.84 Since $e^x = x^{123}$, we have $x = \ln(x^{123})$, so $x = 123 \ln x$. Consider the function $f(x) = x - 123 \ln x$. Plotting this function using a computer or calculator reveals a graph which crosses the x axis twice, near $x = 1$ and near $x = 826$. (Try graphing it using the domain $(0, 900)$). Using a calculator and some trial and error reveals that the roots of f are approximately 1.0082 and 826.1659.

1.3.85 Begin by completing the square: $f(x) = x^2 - 2x + 6 = (x^2 - 2x + 1) + 5 = (x - 1)^2 + 5$. Switching x and y yields $x = (y - 1)^2 + 5$. Solving for y gives $|y - 1| = \sqrt{x - 5}$. Choosing the principal square root (because the original given interval has x positive) gives $y = f^{-1}(x) = \sqrt{x - 5} + 1$, $x \geq 5$.

1.3.86 Begin by completing the square: $f(x) = -x^2 - 4x - 3 = -(x^2 + 4x + 3) = -(x^2 + 4x + 4 - 1) = -((x + 2)^2 - 1) = 1 - (x + 2)^2$. Switching x and y yields $x = 1 - (y + 2)^2$, and solving for y gives $|y + 2| = \sqrt{1 - x}$. Since the given domain of f was negative, the range of f^{-1} must be negative, so we must have $y + 2 = -\sqrt{1 - x}$, so the inverse function is $f^{-1}(x) = -\sqrt{1 - x} - 2$.

1.3.87 Note that f is one-to-one, so there is only one inverse. Switching x and y gives $x = (y + 1)^3$. Then $\sqrt[3]{x} = y + 1$, so $y = f^{-1}(x) = \sqrt[3]{x} - 1$. The domain of f^{-1} is $\mathbb{R}$.

1.3.88 Note that to get a one-to-one function, we should restrict the domain to either $[4, \infty)$ or $(-\infty, 4]$. Switching x and y yields $x = (y - 4)^2$, so $\sqrt{x} = |y - 4|$. So $y = 4 \pm \sqrt{x}$. So the inverse of f when the domain of f is restricted to $[4, \infty)$ is $f^{-1}(x) = 4 + \sqrt{x}$, while if the domain of f is restricted to $(-\infty, 4]$ the inverse is $f^{-1}(x) = 4 - \sqrt{x}$. In either case, the domain of f^{-1} is $[0, \infty)$.

1.3.89 Note that to get a one-to-one function, we should restrict the domain to either $[0, \infty)$ or $(-\infty, 0]$. Switching x and y yields $x = \frac{2}{y^2+2}$, so $y^2 + 2 = (2/x)$. So $y = \pm\sqrt{(2/x) - 2}$. So the inverse of f when the domain of f is restricted to $[0, \infty)$ is $f^{-1}(x) = \sqrt{(2/x) - 2}$, while if the domain of f is restricted to $(-\infty, 0]$ the inverse is $f^{-1}(x) = -\sqrt{(2/x) - 2}$. In either case, the domain of f^{-1} is $(0, 1]$.

1.3.90 Note that f is one-to-one. Switching x and y yields $x = \frac{2y}{y+2}$, so $x(y + 2) = 2y$. Thus $xy + 2x = 2y$, so $2x = 2y - xy = y(2 - x)$. Thus, $y = \frac{2x}{2-x}$. The domain of $f^{-1}(x) = \frac{2x}{2-x}$ is $(-\infty, 2) \cup (2, \infty)$.

1.3.91 Using the change of base formula, we have $\log_{1/b} x = \frac{\ln x}{\ln 1/b} = \frac{\ln x}{\ln 1 - \ln b} = \frac{\ln x}{-\ln b} = -\frac{\ln x}{\ln b} = -\log_b x$.

1.3.92

a. Given $x = b^p$, we have $p = \log_b x$, and given $y = b^q$, we have $q = \log_b y$.

b. $xy = b^p b^q = b^{p+q}$.

c. $\log_b xy = \log_b b^{p+q} = p + q = \log_b x + \log_b y$.

1.3.93 Let $x = b^p$ and $y = b^q$. Then
$\frac{x}{y} = \frac{b^p}{b^q} = b^{p-q}$. Thus $\log_b \frac{x}{y} = \log_b b^{p-q} = p - q = \log_b x - \log_b y$.

1.3.94

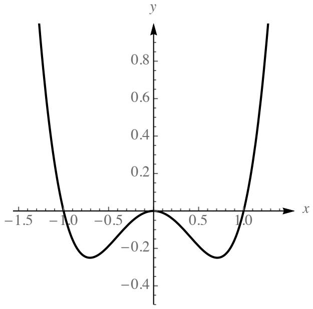
a. Given $x = b^p$, we have $p = \log_b x$.

b. $x^y = (b^p)^y = b^{yp}$.

c. $\log_b x^y = \log_b b^{yp} = yp = y \log_b x$.

1.3.95

a. f is one-to-one on $(-\infty, -\sqrt{2}/2]$, on $[-\sqrt{2}/2, 0]$, on $[0, \sqrt{2}/2]$, and on $[\sqrt{2}/2, \infty)$.



b. If $u = x^2$, then our function becomes $y = u^2 - u$. Completing the square gives $y + (1/4) = u^2 - u + (1/4) = (u - (1/2))^2$. Thus $|u - (1/2)| = \sqrt{y + (1/4)}$, so $u = (1/2) \pm \sqrt{y + (1/4)}$, with the "+" applying for $u = x^2 > (1/2)$ and the "−" applying when $u = x^2 < (1/2)$. Now letting $u = x^2$, we have $x^2 = (1/2) \pm \sqrt{y + (1/4)}$, so $x = \pm\sqrt{(1/2) \pm \sqrt{y + (1/4)}}$. Now switching the x and y gives the following inverses:

Domain of f	$(-\infty, -\sqrt{2}/2]$	$[-\sqrt{2}/2, 0]$	$[0, \sqrt{2}/2]$	$[\sqrt{2}/2, \infty)$
Range of f	$[-1/4, \infty)$	$[-1/4, 0]$	$[-1/4, 0]$	$[-1/4, \infty)$
Inverse of f	$-\sqrt{(1/2) + \sqrt{x + (1/4)}}$	$-\sqrt{(1/2) - \sqrt{x + (1/4)}}$	$\sqrt{(1/2) - \sqrt{x + (1/4)}}$	$\sqrt{(1/2) + \sqrt{x + (1/4)}}$

1.3.96

a. $f(x) = g(h(x)) = g(x^3) = 2x^3 + 3$. To find the inverse of f, we switch x and y to obtain $x = 2y^3 + 3$, so that $y^3 = \frac{x-3}{2}$, so $f^{-1}(x) = \sqrt[3]{\frac{x-3}{2}}$. Note that $g^{-1}(x) = \frac{x-3}{2}$, and $h^{-1}(x) = \sqrt[3]{x}$, and so $f^{-1}(x) = h^{-1}(g^{-1}(x))$.

b. $f(x) = g(h(x)) = g(\sqrt{x}) = (\sqrt{x})^2 + 1 = x + 1$. so the inverse of f is $f^{-1}(x) = x - 1$.
Note that $g^{-1}(x) = \sqrt{x - 1}$, and $h^{-1}(x) = x^2$, and so $f^{-1}(x) = h^{-1}(g^{-1}(x))$.

c. If h and g are one-to-one, then their inverses exist, and $f^{-1}(x) = h^{-1}(g^{-1}(x))$, because $f(f^{-1}(x)) = g(h(h^{-1}(g^{-1}(x))) = g(g^{-1}(x)) = x$ and likewise, $f^{-1}(f(x)) = h^{-1}(g^{-1}(g(h(x))) = h^{-1}(h(x)) = x$.

1.3.97 Using the change of base formulas $\log_b c = \frac{\ln c}{\ln b}$ and $\log_c b = \frac{\ln b}{\ln c}$ we have

$$(\log_b c) \cdot (\log_c b) = \frac{\ln c}{\ln b} \cdot \frac{\ln b}{\ln c} = 1.$$

1.4 Trigonometric Functions and Their Inverses

1.4.1 Let O be the length of the side opposite the angle x, let A be length of the side adjacent to the angle x, and let H be the length of the hypotenuse. Then $\sin x = \dfrac{O}{H}$, $\cos x = \dfrac{A}{H}$, $\tan x = \dfrac{O}{A}$, $\csc x = \dfrac{H}{O}$, $\sec x = \dfrac{H}{A}$, and $\cot x = \dfrac{A}{O}$.

1.4.2 Note that the distance from the origin to the point $(-4, -3)$ is $\sqrt{(-4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$. Then we have $\sin\theta = -\dfrac{3}{5}$, $\cos\theta = -\dfrac{4}{5}$, $\tan\theta = \dfrac{3}{4}$, $\cot\theta = \dfrac{4}{3}$, $\sec\theta = -\dfrac{5}{4}$, $\csc\theta = -\dfrac{5}{3}$.

1.4.3 We have $t = \dfrac{v\sin\theta}{16} = \dfrac{96\sin\dfrac{\pi}{6}}{16} = \dfrac{96/2}{16} = \dfrac{48}{16} = 3$ seconds.

1.4.4

a. Because $\tan\theta = \dfrac{50}{d}$, we have $d = \frac{50}{\tan\theta}$.

b. Because $\sin\theta = \dfrac{50}{L}$, we have $L = \frac{50}{\sin\theta}$.

1.4.5 The radian measure of an angle θ is the length of the arc s on the unit circle associated with θ.

1.4.6 The period of a function is the smallest positive real number k so that $f(x+k) = f(x)$ for all x in the domain of the function. The sine, cosine, secant, and cosecant function all have period 2π. The tangent and cotangent functions have period π.

1.4.7 $\sin^2 x + \cos^2 x = 1$, $1 + \cot^2 x = \csc^2 x$, and $\tan^2 x + 1 = \sec^2 x$.

1.4.8
$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1/\sqrt{5}}{-2/\sqrt{5}} = -\dfrac{1}{2}$.
$\cot\theta = \frac{1}{\tan\theta} = -2$.
$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-2/\sqrt{5}} = -\frac{\sqrt{5}}{2}$.
$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{1/\sqrt{5}} = \sqrt{5}$.

1.4.9 The only point on the unit circle whose second coordinate is -1 is the point $(0,-1)$, which is the point associated with $\theta = \frac{3\pi}{2}$. So that is the only solution for $0 \le \theta < 2\pi$.

1.4.10 Note that if $0 \le \theta < 2\pi$, then $0 \le 2\theta < 4\pi$. So we must consider "two trips" around the unit circle. The second coordinate on the unit circle is 1 at the point $(0,1)$, which is associated with $\frac{\pi}{2}$ and $\frac{5\pi}{2}$. When $2\theta = \frac{\pi}{2}$ we have $\theta = \frac{\pi}{4}$, and when $2\theta = \frac{5\pi}{2}$ we have $\theta = \frac{5\pi}{4}$.

1.4.11 The tangent function is undefined where $\cos x = 0$, which is at all real numbers of the form $\dfrac{\pi}{2} + k\pi$, k an integer.

1.4.12 $\sec x$ is defined wherever $\cos x \ne 0$, which is $\{x \colon x \ne \dfrac{\pi}{2} + k\pi, k \text{ an integer}\}$.

1.4.13 The sine function is not one-to-one over its whole domain, so in order to define an inverse, it must be restricted to an interval on which it is one-to-one.

1.4.14 In order to define an inverse for the cosine function, we restricted the domain to $[0,\pi]$ in order to get a one-to-one function. Because the range of the inverse of a function is the domain of the function, we have that the values of $\cos^{-1} x$ lie in the interval $[0,\pi]$.

1.4.15 $\cos\dfrac{5\pi}{4} = -\dfrac{\sqrt{2}}{2}$, so $\cos^{-1}\left(\cos\dfrac{5\pi}{4}\right) = \cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

1.4.16 $\sin\dfrac{11\pi}{6} = -\dfrac{1}{2}$, so $\sin^{-1}\left(\sin\dfrac{11\pi}{6}\right) = \sin^{-1}\left(-\dfrac{1}{2}\right) = -\dfrac{\pi}{6}$.

1.4.17 The numbers $\pm\pi/2$ are not in the range of $\tan^{-1} x$. The range is $(-\pi/2, \pi/2)$. However, it is true that as x increases without bound, the values of $\tan^{-1} x$ get close to $\pi/2$, and as x decreases without bound, the values of $\tan^{-1} x$ get close to $-\pi/2$.

1.4.18 The domain of $\sec^{-1} x$ is $\{x \colon |x| \ge 1\}$. The range is $[0,\pi/2) \cup (\pi/2, \pi]$.

1.4.19 The point on the unit circle associated with $2\pi/3$ is $(-1/2, \sqrt{3}/2)$, so $\cos(2\pi/3) = -1/2$.

1.4.20 The point on the unit circle associated with $2\pi/3$ is $(-1/2, \sqrt{3}/2)$, so $\sin(2\pi/3) = \sqrt{3}/2$.

1.4.21 The point on the unit circle associated with $-3\pi/4$ is $(-\sqrt{2}/2, -\sqrt{2}/2)$, so $\tan(-3\pi/4) = 1$.

1.4.22 The point on the unit circle associated with $15\pi/4$ is $(\sqrt{2}/2, -\sqrt{2}/2)$, so $\tan(15\pi/4) = -1$.

1.4.23 The point on the unit circle associated with $-13\pi/3$ is $(1/2, -\sqrt{3}/2)$, so $\cot(-13\pi/3) = -1/\sqrt{3} = -\sqrt{3}/3$.

1.4.24 The point on the unit circle associated with $7\pi/6$ is $(-\sqrt{3}/2, -1/2)$, so $\sec(7\pi/6) = -2/\sqrt{3} = -2\sqrt{3}/3$.

1.4.25 The point on the unit circle associated with $-17\pi/3$ is $(1/2, \sqrt{3}/2)$, so $\cot(-17\pi/3) = 1/\sqrt{3} = \sqrt{3}/3$.

1.4.26 The point on the unit circle associated with $16\pi/3$ is $(-1/2, -\sqrt{3}/2)$, so $\sin(16\pi/3) = -\sqrt{3}/2$.

1.4.27 Because the point on the unit circle associated with $\theta = 0$ is the point $(1, 0)$, we have $\cos 0 = 1$.

1.4.28 Because $-\pi/2$ corresponds to a quarter circle clockwise revolution, the point on the unit circle associated with $-\pi/2$ is the point $(0, -1)$. Thus $\sin(-\pi/2) = -1$.

1.4.29 Because $-\pi$ corresponds to a half circle clockwise revolution, the point on the unit circle associated with $-\pi$ is the point $(-1, 0)$. Thus $\cos(-\pi) = -1$.

1.4.30 Because 3π corresponds to one and a half counterclockwise revolutions, the point on the unit circle associated with 3π is $(-1, 0)$, so $\tan 3\pi = \frac{0}{-1} = 0$.

1.4.31 Because $5\pi/2$ corresponds to one and a quarter counterclockwise revolutions, the point on the unit circle associated with $5\pi/2$ is the same as the point associated with $\pi/2$, which is $(0, 1)$. Thus $\sec 5\pi/2$ is undefined.

1.4.32 Because π corresponds to one half circle counterclockwise revolution, the point on the unit circle associated with π is $(-1, 0)$. Thus $\cot \pi$ is undefined.

1.4.33 Using the fact that $\frac{\pi}{12} = \frac{\pi/6}{2}$ and the half-angle identity for cosine:

$$\cos^2(\pi/12) = \frac{1 + \cos(\pi/6)}{2} = \frac{1 + \sqrt{3}/2}{2} = \frac{2 + \sqrt{3}}{4}.$$

Thus, $\cos(\pi/12) = \sqrt{\frac{2 + \sqrt{3}}{4}}$.

1.4.34 Using the fact that $\frac{3\pi}{8} = \frac{3\pi/4}{2}$ and the half-angle identities for sine, we have:

$$\sin^2\left(\frac{3\pi}{8}\right) = \frac{1 - \cos(3\pi/4)}{2} = \frac{1 - (-\sqrt{2}/2)}{2} = \frac{2 + \sqrt{2}}{4},$$

and using the fact that $3\pi/8$ is in the first quadrant (and thus has positive value for sine) we deduce that $\sin\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2 + \sqrt{2}}}{2}$.

1.4.35 First note that $\tan x = 1$ when $\sin x = \cos x$. Using our knowledge of the values of the standard angles between 0 and 2π, we recognize that the sine function and the cosine function are equal at $\pi/4$. Then, because we recall that the period of the tangent function is π, we know that $\tan(\pi/4 + k\pi) = \tan(\pi/4) = 1$ for every integer value of k. Thus the solution set is $\{\pi/4 + k\pi, \text{where } k \text{ is an integer}\}$.

1.4.36 Given that $2\theta\cos(\theta)+\theta=0$, we have $\theta(2\cos(\theta)+1)=0$. Which means that either $\theta=0$, or $2\cos(\theta)+1=0$. The latter leads to the equation $\cos\theta=-1/2$, which occurs at $\theta=2\pi/3$ and $\theta=4\pi/3$. Using the fact that the cosine function has period 2π the entire solution set is thus

$$\{0\}\cup\{2\pi/3+2k\pi,\text{where } k \text{ is an integer}\}\cup\{4\pi/3+2l\pi,\text{where } l \text{ is an integer}\}.$$

1.4.37 Given that $\sin^2\theta=\dfrac{1}{4}$, we have $|\sin\theta|=\dfrac{1}{2}$, so $\sin\theta=\dfrac{1}{2}$ or $\sin\theta=-\dfrac{1}{2}$. It follows that $\theta=\pi/6, 5\pi/6, 7\pi/6, 11\pi/6$.

1.4.38 Given that $\cos^2\theta=\dfrac{1}{2}$, we have $|\cos\theta|=\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}$. Thus $\cos\theta=\dfrac{\sqrt{2}}{2}$ or $\cos\theta=-\dfrac{\sqrt{2}}{2}$. We have $\theta=\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

1.4.39 The equation $\sqrt{2}\sin(x)-1=0$ can be written as $\sin x=\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}$. Standard solutions to this equation occur at $x=\pi/4$ and $x=3\pi/4$. Because the sine function has period 2π the set of all solutions can be written as:

$$\{\pi/4+2k\pi,\text{where } k \text{ is an integer}\}\cup\{3\pi/4+2l\pi,\text{where } l \text{ is an integer}\}.$$

1.4.40 $\sin^2(\theta)-1=0$ wherever $\sin^2(\theta)=1$, which is wherever $\sin(\theta)=\pm1$. This occurs for $\theta=\pi/2+k\pi,$ where k is an integer.

1.4.41 If $\sin\theta\cos\theta=0$, then either $\sin\theta=0$ or $\cos\theta=0$. This occurs for $\theta=0,\pi/2,\pi,3\pi/2$.

1.4.42 Let $u=3x$. Note that because $0\le x<2\pi$, we have $0\le u<6\pi$. Because $\sin u=\sqrt{2}/2$ for $u=\pi/4$, $3\pi/4$, $9\pi/4$, $11\pi/4$, $17\pi/4$, and $19\pi/4$, we must have that $\sin 3x=\sqrt{2}/2$ for $3x=\pi/4$, $3\pi/4$, $9\pi/4$, $11\pi/4$, $17\pi/4$, and $19\pi/4$, which translates into

$$x=\pi/12,\pi/4,3\pi/4,11\pi/12,17\pi/12,\text{and }19\pi/12.$$

1.4.43 Let $u=3x$. Then we are interested in the solutions to $\cos u=\sin u$, for $0\le u<6\pi$. This would occur for $u=3x=\pi/4,5\pi/4,9\pi/4,13\pi/4,17\pi/4,\text{and }21\pi/4$. Thus there are solutions for the original equation at

$$x=\pi/12,5\pi/12,3\pi/4,13\pi/12,17\pi/12,\text{and }7\pi/4.$$

1.4.44 If $\tan^2 2\theta=1$, then $\sin^2 2\theta=\cos^2 2\theta$, so we have either $\sin 2\theta=\cos 2\theta$ or $\sin 2\theta=-\cos 2\theta$. This occurs for $2\theta=\pi/4,3\pi/4,5\pi/4,7\pi/4$ for $0\le 2\theta\le 2\pi$, so the corresponding values for θ are $\pi/8$, $3\pi/8$, $5\pi/8$, $7\pi/8$, $0\le\theta\le\pi$.

1.4.45 Using a computer algebra system or graphing calculator, we find that the roots are approximately 0.1007 and 1.4701.

1.4.46 Using a computer algebra system or graphing calculator, we find that the roots are approximately 0.375962, 1.71843, and 2.47036.

1.4.47 We are seeking solutions to the equation $400=\frac{150^2}{32}\sin 2\theta$, or $\sin 2\theta=0.56\overline{8}$. Using a computer algebra system or graphing calculator, we find that the solutions are about 0.30257 radians or about 17.3 degrees, and about 1.2682 radians which is about 72.7 degrees.

1.4.48 We are seeking solutions to the equation $350=\frac{160^2}{32}\sin 2\theta$, or $\sin 2\theta=0.4375$. Using a computer algebra system or graphing calculator, we find that the solutions are about 0.2264 radians which is about 13 degrees, and about 1.3444 radians which is about 77 degrees.

1.4.49 Let $z=\sin^{-1}1$. Then $\sin z=1$, and because $\sin\pi/2=1$, and $\pi/2$ is in the desired interval, $z=\pi/2$.

1.4.50 Let $z=\cos^{-1}(-1)$. Then $\cos z=-1$, and because $\cos\pi=-1$ and π is in the desired interval, $z=\pi$.

1.4.51 Let $z = \sin^{-1}(-1/2)$. Then $\sin z = 1/2$, and because $\sin(-\pi/6) = -1/2$, and $-\pi/6$ is in the desired interval, $z = -\pi/6$.

1.4.52 Let $z = \cos^{-1}(-\sqrt{2}/2)$. Then $\cos z = -\sqrt{2}/2$. Because $\cos 3\pi/4 = -\sqrt{2}/2$ and $3\pi/4$ is in the desired interval, we have $z = 3\pi/4$. (Note that $\cos(-\pi/4)$ is also equal to $-\sqrt{2}/2$, but $-\pi/4$ isn't in the desired interval $[0, \pi]$.)

1.4.53 $\sin^{-1}(\sqrt{3}/2) = \pi/3$, because $\sin(\pi/3) = \sqrt{3}/2$.

1.4.54 $\cos^{-1} 2$ does not exist, because 2 is not in the domain of the inverse cosine function (because 2 is not in the range of the cosine function.)

1.4.55 $\cos^{-1}(-1/2) = 2\pi/3$, because $\cos(2\pi/3) = -1/2$.

1.4.56 $\sin^{-1}(-1) = -\pi/2$, because $\sin(-\pi/2) = -1$.

1.4.57 $\cos(\cos^{-1}(-1)) = \cos(\pi) = -1$.

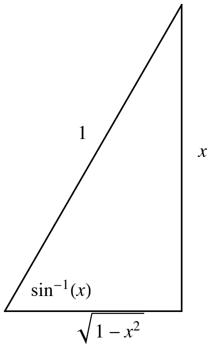
1.4.58 $\cos^{-1}(\cos(7\pi/6)) = \cos^{-1}(-\sqrt{3}/2) = 5\pi/6$. Note that the range of the inverse cosine function is $[0, \pi]$.

1.4.59 Because $\theta = \cos^{-1}(5/13)$, we know that $\cos\theta = 5/13$. The triangle in question has a leg of length 5 and a hypotenuse of length 13, so we can deduce using the Pythagorean theorem that the other leg as length 12. So $\sin\theta = 12/13$. Then $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{12/13}{5/13} = \frac{12}{5}$.

1.4.60 Because $\theta = \tan^{-1}(4/3)$, we know that $\tan\theta = 4/3$. The triangle in question has legs of length 3 and 4, so the hypotenuse has length 5 by the Pythagorean theorem. Then $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{3/5} = \dfrac{5}{3}$. And $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{4/5} = \frac{5}{4}$.

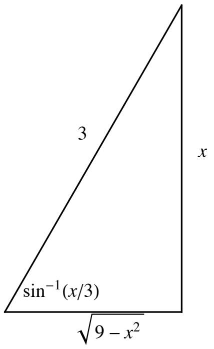
1.4.61

$$\cos(\sin^{-1}(x)) = \frac{\text{side adjacent to } \sin^{-1}(x)}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$



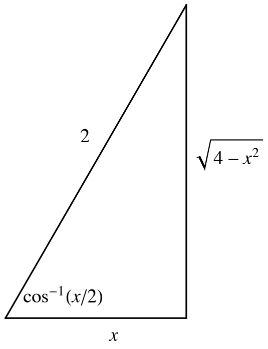
1.4.62

$$\cos(\sin^{-1}(x/3)) = \frac{\text{side adjacent to } \sin^{-1}(x/3)}{\text{hypotenuse}} = \frac{\sqrt{9-x^2}}{3}.$$



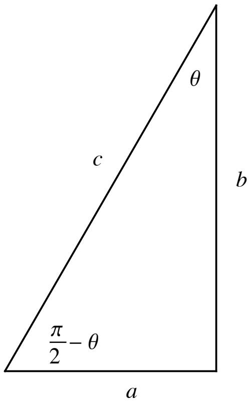
1.4.63

$$\sin(\cos^{-1}(x/2)) = \frac{\text{side opposite of } \cos^{-1}(x/2)}{\text{hypotenuse}} = \frac{\sqrt{4-x^2}}{2}.$$


2
$\sqrt{4-x^2}$
$\cos^{-1}(x/2)$
x

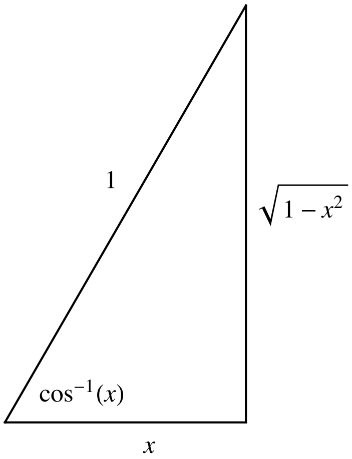
1.4.64

Note (from the triangle pictured) that $\cos\theta = \frac{b}{c} = \sin(\frac{\pi}{2} - \theta)$. Thus $\sin^{-1}(\cos\theta) = \sin^{-1}(\sin(\frac{\pi}{2} - \theta)) = \frac{\pi}{2} - \theta$.


θ
c
b
$\frac{\pi}{2} - \theta$
a

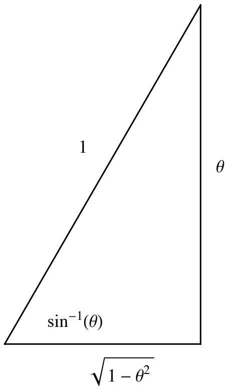
1.4.65

Using the identity given, we have $\sin(2\cos^{-1}(x)) = 2\sin(\cos^{-1}(x))\cos(\cos^{-1}(x)) = 2x\sin(\cos^{-1}(x)) = 2x\sqrt{1-x^2}$.


1
$\sqrt{1-x^2}$
$\cos^{-1}(x)$
x

1.4.66

First note that $\cos(\sin^{-1}(\theta)) = \sqrt{1-\theta^2}$, as indicated in the triangle shown.
Using the identity given, we have $\cos(2\sin^{-1}(x)) = \cos^2((\sin^{-1}(x)) - \sin^2(\sin^{-1}(x)) = (\sqrt{1-x^2})^2 - x^2 = 1 - 2x^2$.



1.4.67 From our definitions of the trigonometric functions via a point $P(x, y)$ on a circle of radius $r = \sqrt{x^2+y^2}$, we have $\sec\theta = \frac{r}{x} = \frac{1}{x/r} = \frac{1}{\cos\theta}$.

1.4.68 From our definitions of the trigonometric functions via a point $P(x, y)$ on a circle of radius $r = \sqrt{x^2+y^2}$, we have $\tan\theta = \frac{y}{x} = \frac{y/r}{x/r} = \frac{\sin\theta}{\cos\theta}$.

1.4.69 We have already established that $\sin^2\theta + \cos^2\theta = 1$. Dividing both sides by $\cos^2\theta$ gives $\tan^2\theta + 1 = \sec^2\theta$.

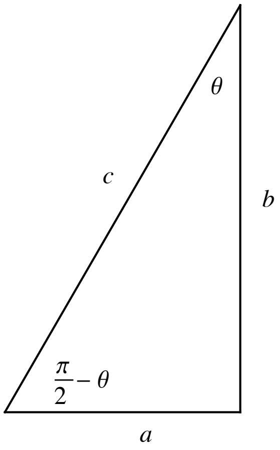
1.4.70 We have already established that $\sin^2\theta + \cos^2\theta = 1$. We can write this as $\frac{\sin\theta}{(1/\sin\theta)} + \frac{\cos\theta}{(1/\cos\theta)} = 1$, or $\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = 1$.

1.4.71

Using the triangle pictured, we see that

$$\sec(\pi/2 - \theta) = \frac{c}{a} = \csc\theta.$$

This also follows from the sum identity $\cos(a+b) = \cos a\cos b - \sin a \sin b$ as follows: $\sec(\pi/2 - \theta) = \frac{1}{\cos(\pi/2 + (-\theta))} = \frac{1}{\cos(\pi/2)\cos(-\theta) - \sin(\pi/2)\sin(-\theta)} = \frac{1}{0 - (-\sin(\theta))} = \csc(\theta)$.

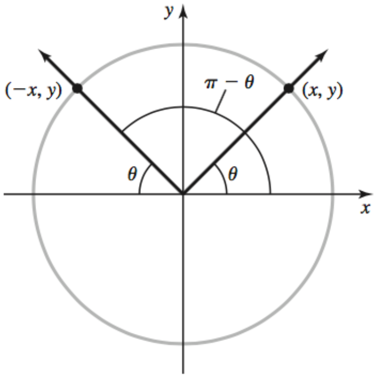


1.4.72 Using the trig identity for the cosine of a sum (mentioned in the previous solution) we have:

$$\sec(x+\pi) = \frac{1}{\cos(x+\pi)} = \frac{1}{\cos(x)\cos(\pi) - \sin(x)\sin(\pi)} = \frac{1}{\cos(x)\cdot(-1) - \sin(x)\cdot 0} = \frac{1}{-\cos(x)} = -\sec x.$$

1.4.73

Let $\theta = \cos^{-1}(x)$, and note from the diagram that it then follows that $\cos^{-1}(-x) = \pi - \theta$. So $\cos^{-1}(x) + \cos^{-1}(-x) = \theta + \pi - \theta = \pi$.



1.4.74 Let $\theta = \sin^{-1}(y)$. Then $\sin\theta = y$, and $\sin(-\theta) = -\sin(\theta) = -y$ (because the sine function is an odd function) and it then follows that $-\theta = \sin^{-1}(-y)$. Therefore, $\sin^{-1}(y) + \sin^{-1}(-y) = \theta + -\theta = 0$. It would be instructive for the reader to draw his or her own diagram like that in the previous solution.

1.4.75 $\tan^{-1}(\sqrt{3}) = \tan^{-1}\left(\dfrac{\sqrt{3}/2}{1/2}\right) = \pi/3$, because $\sin(\pi/3) = \sqrt{3}/2$ and $\cos(\pi/3) = 1/2$.

1.4.76 $\cot^{-1}(-1/\sqrt{3}) = \cot^{-1}\left(-\dfrac{1/2}{\sqrt{3}/2}\right) = 2\pi/3$, because $\sin(2\pi/3) = \sqrt{3}/2$ and $\cos(2\pi/3) = -1/2$.

1.4.77 $\sec^{-1}(2) = \sec^{-1}\left(\dfrac{1}{1/2}\right) = \pi/3$, because $\sec(\pi/3) = \dfrac{1}{\cos(\pi/3)} = \dfrac{1}{1/2} = 2$.

1.4.78 $\csc^{-1}(-1) = \sin^{-1}(-1) = -\pi/2$.

1.4.79 $\tan^{-1}(\tan(\pi/4)) = \tan^{-1}(1) = \pi/4$.

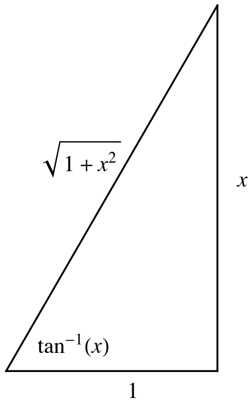
1.4.80 $\tan^{-1}(\tan(3\pi/4)) = \tan^{-1}(-1) = -\pi/4$.

1.4.81 Let $\csc^{-1}(\sec 2) = z$. Then $\csc z = \sec 2$, so $\sin z = \cos 2$. Now by applying the result of problem 64, we see that $z = \sin^{-1}(\cos 2) = \pi/2 - 2 = \dfrac{\pi - 4}{2}$.

1.4.82 $\tan(\tan^{-1}(1)) = \tan(\pi/4) = 1$.

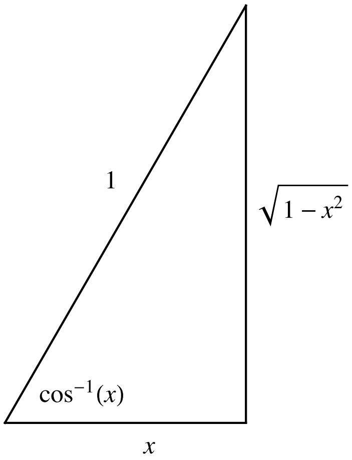
1.4.83

$$\cos(\tan^{-1}(x)) = \frac{\text{side adjacent to } \tan^{-1}(x)}{\text{hypotenuse}} = \frac{1}{\sqrt{1+x^2}}.$$



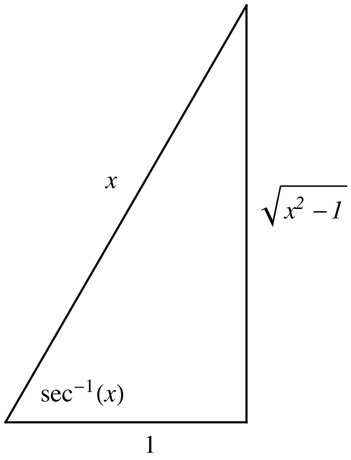
1.4.84

$$\tan(\cos^{-1}(x)) = \frac{\text{side opposite of } \cos^{-1}(x)}{\text{side adjacent to } \cos^{-1}(x)} = \frac{\sqrt{1-x^2}}{x}.$$


1
$\sqrt{1-x^2}$
$\cos^{-1}(x)$
x

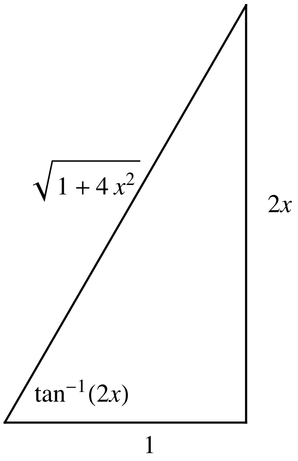
1.4.85

$$\cos(\sec^{-1}(x)) = \frac{\text{side adjacent to } \sec^{-1} x}{\text{hypotenuse}} = \frac{1}{x}.$$


x
$\sqrt{x^2-1}$
$\sec^{-1}(x)$
1

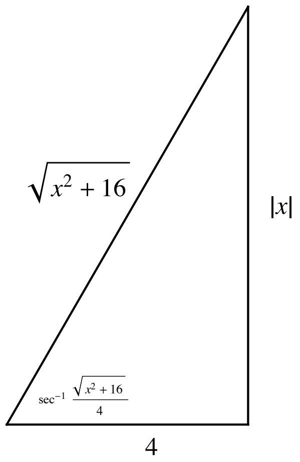
1.4.86

$$\cot(\tan^{-1} 2x) = \frac{\text{side adjacent to } \tan^{-1} 2x}{\text{side opposite of } \tan^{-1} 2x} = \frac{1}{2x}.$$


$\sqrt{1+4x^2}$
$2x$
$\tan^{-1}(2x)$
1

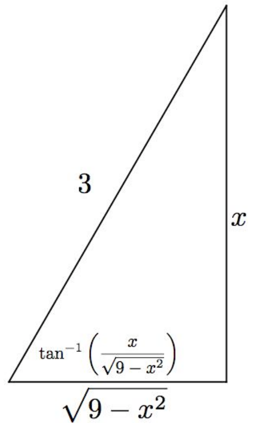
1.4.87

Assume $x > 0$. Then $\sin\left(\sec^{-1}\left(\frac{\sqrt{x^2+16}}{4}\right)\right) =$ $\frac{\text{side opposite of } \sec^{-1}\left(\frac{\sqrt{x^2+16}}{4}\right)}{\text{hypotenuse}} =$ $\frac{x}{\sqrt{x^2+16}}$.



1.4.88

$\cos\left(\tan^{-1}\left(\frac{x}{\sqrt{9-x^2}}\right)\right) =$ $\frac{\text{side adjacent to } \tan^{-1}\left(\frac{x}{\sqrt{9-x^2}}\right)}{\text{hypotenuse}} = \frac{\sqrt{9-x^2}}{3}$.

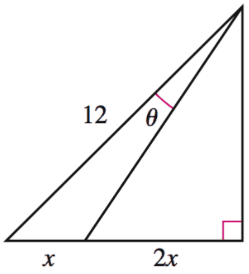


1.4.89 Because $\sin\theta = \frac{x}{6}$, $\theta = \sin^{-1}(x/6)$. Also, $\theta = \tan^{-1}\left(\frac{x}{\sqrt{36-x^2}}\right) = \sec^{-1}\left(\frac{6}{\sqrt{36-x^2}}\right)$.

1.4.90

Note that the vertical side in the adjacent diagram has length $\sqrt{144-9x^2}$ (by the Pythagorean theorem.) Let ψ be the angle in the adjacent diagram which is opposite the side labeled $2x$. First note that $\tan(\psi) = \frac{2x}{\sqrt{144-9x^2}}$, so $\psi = \tan^{-1}\left(\frac{2x}{\sqrt{144-9x^2}}\right)$. Also, $\sin(\theta+\psi) = \frac{3x}{12} = \frac{x}{4}$, so $\theta + \psi = \sin^{-1}(x/4)$. Therefore,

$$\theta = \sin^{-1}(x/4) - \psi = \sin^{-1}(x/4) - \tan^{-1}(2x/\sqrt{144-9x^2}).$$



1.4.91

a. False. For example, $\sin(\pi/2+\pi/2) = \sin(\pi) = 0 \neq \sin(\pi/2) + \sin(\pi/2) = 1 + 1 = 2$.

b. False. That equation has zero solutions, because the range of the cosine function is $[-1, 1]$.

c. False. It has infinitely many solutions of the form $\pi/6 + 2k\pi$, where k is an integer (among others.)

d. False. It has period $\frac{2\pi}{\pi/12} = 24$.

e. True. The others have a range of either $[-1, 1]$ or $(-\infty, -1] \cup [1, \infty)$.

f. False. For example, suppose $x = .5$. Then $\sin^{-1}(x) = \pi/6$ and $\cos^{-1}(x) = \pi/3$, so that $\frac{\sin^{-1}(x)}{\cos^{-1}(x)} = \frac{\pi/6}{\pi/3} = .5$. However, note that $\tan^{-1}(.5) \neq .5$,

g. True. Note that the range of the inverse cosine function is $[0, \pi]$.

h. False. For example, if $x = .5$, we would have $\sin^{-1}(.5) = \pi/6 \neq 1/\sin(.5)$.

1.4.92 If $\sin\theta = -4/5$, then the Pythagorean identity gives $|\cos\theta| = 3/5$. But if $\pi < \theta < 3\pi/2$, then the cosine of θ is negative, so $\cos\theta = -3/5$. Thus $\tan\theta = 4/3$, $\cot\theta = 3/4$, $\sec\theta = -5/3$, and $\csc\theta = -5/4$.

1.4.93 If $\cos\theta = 5/13$, then the Pythagorean identity gives $|\sin\theta| = 12/13$. But if $0 < \theta < \pi/2$, then the sine of θ is positive, so $\sin\theta = 12/13$. Thus $\tan\theta = 12/5$, $\cot\theta = 5/12$, $\sec\theta = 13/5$, and $\csc\theta = 13/12$.

1.4.94 If $\sec\theta = 5/3$, then $\cos\theta = 3/5$, and the Pythagorean identity gives $|\sin\theta| = 4/5$. But if $3\pi/2 < \theta < 2\pi$, then the sine of θ is negative, so $\sin\theta = -4/5$. Thus $\tan\theta = -4/3$, $\cot\theta = -3/4$, and $\csc\theta = -5/4$.

1.4.95 If $\csc\theta = 13/12$, then $\sin\theta = 12/13$, and the Pythagorean identity gives $|\cos\theta| = 5/13$. But if $0 < \theta < \pi/2$, then the cosine of θ is positive, so $\cos\theta = 5/13$. Thus $\tan\theta = 12/5$, $\cot\theta = 5/12$, and $\sec\theta = 13/5$.

1.4.96 The amplitude is 2, and the period is $\frac{2\pi}{2} = \pi$.

1.4.97 The amplitude is 3, and the period is $\frac{2\pi}{1/3} = 6\pi$.

1.4.98 The amplitude is 2.5, and the period is $\frac{2\pi}{1/2} = 4\pi$.

1.4.99 The amplitude is 3.6, and the period is $\frac{2\pi}{\pi/24} = 48$.

1.4.100 Using the given diagram, drop a perpendicular from the point $(b\cos\theta, b\sin\theta)$ to the x axis, and consider the right triangle thus formed whose hypotenuse has length c. By the Pythagorean theorem, $(b\sin\theta)^2 + (a - b\cos\theta)^2 = c^2$. Expanding the binomial gives $b^2\sin^2\theta + a^2 - 2ab\cos\theta + b^2\cos^2\theta = c^2$. Now because $b^2\sin^2\theta + b^2\cos^2\theta = b^2$, this reduces to $a^2 + b^2 - 2ab\cos\theta = c^2$.

1.4.101 Note that $\sin A = \frac{h}{c}$ and $\sin C = \frac{h}{a}$, so $h = c\sin A = a\sin C$. Thus

$$\frac{\sin A}{a} = \frac{\sin C}{c}.$$

Now drop a perpendicular from the vertex A to the line determined by $\overline{BC}$, and let h_2 be the length of this perpendicular. Then $\sin C = \frac{h_2}{b}$ and $\sin B = \frac{h_2}{C}$, so $h_2 = b\sin C = c\sin B$. Thus

$$\frac{\sin C}{c} = \frac{\sin B}{b}.$$

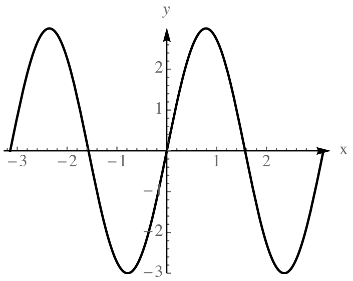
Putting the two displayed equations together gives

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

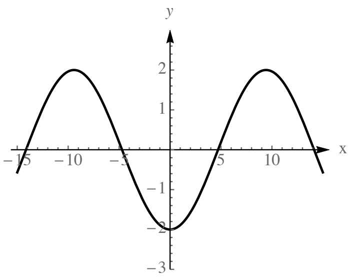
1.4.102 Consider the $\theta = \angle DOA$ where $D(300, 200)$ is the point where the Ditl is anchored, $O(0, 0)$ is the point where the observer is, and $A(300, 0)$ is the point on the x-axis closest to D. Then $\theta = \tan^{-1}(2/3)$. Then consider the angle $\phi = \angle WOB$ where $W(-100, 250)$ is the location of the Windborne and $B(-100, 0)$ is the point on the x-axis closest to W. Then $\phi = \tan^{-1}(250/100) = \tan^{-1}(5/2)$. The angle we are looking for has measure $\pi - \theta - \phi = \pi - \tan^{-1}(5/2) - \tan^{-1}(2/3) \approx 1.3633$ radians.

1.4.103 The area of the entire circle is πr^2. The ratio $\dfrac{\theta}{2\pi}$ represents the proportion of the area swept out by a central angle θ. Thus the area of a sector of a circle is this same proportion of the entire area, so it is $\dfrac{\theta}{2\pi} \cdot \pi r^2 = \dfrac{r^2\theta}{2}$.

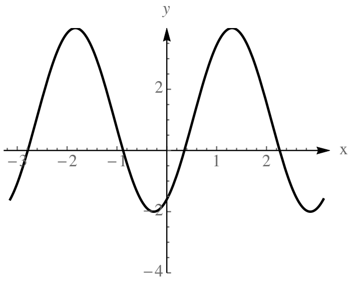
1.4.104



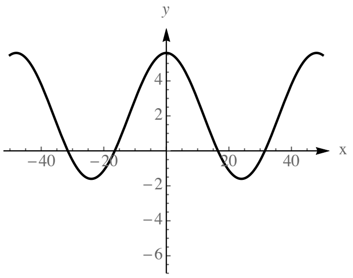
1.4.105



1.4.106



1.4.107



1.4.108 It is helpful to imagine first shifting the function horizontally so that the x intercept is where it should be, then stretching the function horizontally to obtain the correct period, and then stretching the function vertically to obtain the correct amplitude. Because the old x-intercept is at $x = 0$ and the new one should be at $x = 3$ (halfway between where the maximum and the minimum occur), we need to shift the function 3 units to the right. Then to get the right period, we need to multiply (before applying the sine function) by $\pi/6$ so that the new period is $\dfrac{2\pi}{\pi/6} = 12$. Finally, to get the right amplitude and to get the max and min at the right spots, we need to multiply on the outside by 4. Thus, the desired function is:

$$f(x) = 4\sin((\pi/6)(x - 3)) = 4\sin((\pi/6)x - \pi/2).$$

1.4.109 It is helpful to imagine first shifting the function horizontally so that the x intercept is where it should be, then stretching the function horizontally to obtain the correct period, and then stretching the function vertically to obtain the correct amplitude, and then shifting the whole graph up. Because the old x-intercept is at $x = 0$ and the new one should be at $x = 9$ (halfway between where the maximum and the minimum occur), we need to shift the function 9 units to the right. Then to get the right period, we need to multiply (before applying the sine function) by $\pi/12$ so that the new period is $\dfrac{2\pi}{\pi/12} = 24$. Finally, to get the right amplitude and to get the max and min at the right spots, we need to multiply on the outside by 3, and then shift the whole thing up 13 units. Thus, the desired function is:

$$f(x) = 3\sin((\pi/12)(x - 9)) + 13 = 3\sin((\pi/12)x - 3\pi/4) + 13.$$

1.4.110 Let C be the point on the end line so that segment $\overline{AC}$ is perpendicular to the endline. Then the distance $G_1C = 38.\overline{3}$, $G_2C = 15$, and $AC = 69$ and $BC = 84$, where all lengths are in feet. Thus

$$\mathrm{m}(\angle G_1AG_2) = \mathrm{m}(\angle G_1AC) - \mathrm{m}(\angle G_2AC) = \tan^{-1}\left(\frac{38.\overline{3}}{69}\right) - \tan^{-1}\left(\frac{15}{69}\right) \approx 16.79^\circ,$$

while

$$\mathrm{m}(\angle G_1BG_2) = \mathrm{m}(\angle G_1BC) - \mathrm{m}(\angle G_2BC) = \tan^{-1}\left(\frac{38.\overline{3}}{84}\right) - \tan^{-1}\left(\frac{15}{84}\right) \approx 14.4^\circ.$$

The kicking angle was not improved by the penalty.

1.4.111 Let C be the circumference of the earth. Then the first rope has radius $r_1 = \frac{C}{2\pi}$. The circle generated by the longer rope has circumference $C + 38$, so its radius is $r_2 = \frac{C+38}{2\pi} = \frac{C}{2\pi} + \frac{38}{2\pi} \approx r_1 + 6$, so the radius of the bigger circle is about 6 feet more than the smaller circle.

1.4.112

a. The period of this function is $\frac{2\pi}{2\pi/365} = 365$.

b. Because the maximum for the regular sine function is 1, and this function is scaled vertically by a factor of 2.8 and shifted 12 units up, the maximum for this function is $(2.8)(1) + 12 = 14.8$. Similarly, the minimum is $(2.8)(-1) + 12 = 9.2$. Because of the horizontal shift, the point at $t = 81$ is the midpoint between where the max and min occur. Thus the max occurs at $81 + (365/4) \approx 172$ and the min occurs approximately $(365/2)$ days later at about $t = 355$.

c. The solstices occur halfway between these points, at 81 and $81 + (365/2) \approx 264$.

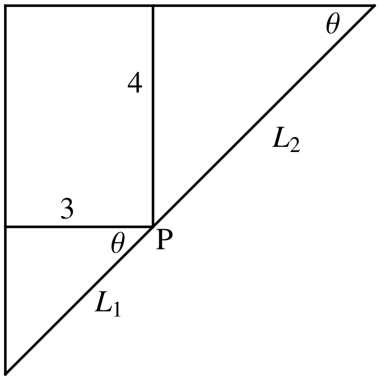
1.4.113 We are seeking a function with amplitude 10 and period 1.5, and value 10 at time 0, so it should have the form $10\cos(kt)$, where $\frac{2\pi}{k} = 1.5$. Solving for k yields $k = \frac{4\pi}{3}$, so the desired function is $d(t) = 10\cos(4\pi t/3)$.

1.4.114 Let θ_1 be the viewing angle to the bottom of the television. Then $\theta_1 = \tan^{-1}\left(\frac{3}{x}\right)$. Now $\tan(\theta + \theta_1) = \frac{10}{x}$, so $\theta + \theta_1 = \tan^{-1}\left(\frac{10}{x}\right)$, so $\theta = \tan^{-1}\left(\frac{10}{x}\right) - \theta_1 = \tan^{-1}\left(\frac{10}{x}\right) - \tan^{-1}\left(\frac{3}{x}\right)$.

1.4.107 The area of the entire circle is πr^2. The ratio $\frac{\theta}{2\pi}$ represents the proportion of the area swept out by a central angle θ. Thus the area of a sector of a circle is this same proportion of the entire area, so it is $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{r^2\theta}{2}$.

1.4.115 Let L be the line segment connecting the tops of the ladders and let M be the horizontal line segment between the walls h feet above the ground. Now note that the triangle formed by the ladders and L is equilateral, because the angle between the ladders is 60 degrees, and the other two angles must be equal and add to 120, so they are 60 degrees as well. Now we can see that the triangle formed by L, M and the right wall is similar to the triangle formed by the left ladder, the left wall, and the ground, because they are both right triangles with one angle of 75 degrees and one of 15 degrees. Thus $M = h$ is the distance between the walls.

1.4.116



Let the corner point P divide the pole into two pieces, L_1 (which spans the 3-ft hallway) and L_2 (which spans the 4-ft hallway.) Then $L = L_1 + L_2$. Now $L_2 = \dfrac{4}{\sin\theta}$, and $\dfrac{3}{L_1} = \cos\theta$ (see diagram.) Thus $L = L_1 + L_2 = \dfrac{3}{\cos\theta} + \dfrac{4}{\sin\theta}$. When $L = 10$, $\theta \approx .9273$.

Chapter One Review

1

a. True. For example, $f(x) = x^2$ is such a function.

b. False. For example, $\cos(\pi/2 + \pi/2) = \cos(\pi) = -1 \neq \cos(\pi/2) + \cos(\pi/2) = 0 + 0 = 0$.

c. False. Consider $f(1+1) = f(2) = 2m + b \neq f(1) + f(1) = (m+b) + (m+b) = 2m + 2b$. (At least these aren't equal when $b \neq 0$.)

d. True. $f(f(x)) = f(1-x) = 1 - (1-x) = x$.

e. False. This set is the union of the disjoint intervals $(-\infty, -7)$ and $(1, \infty)$.

f. False. For example, if $x = y = 10$, then $\log_{10} xy = \log_{10} 100 = 2$, but $\log_{10} 10 \cdot \log_{10} 10 = 1 \cdot 1 = 1$.

g. True. $\sin^{-1}(\sin(2\pi)) = \sin^{-1}(0) = 0$.

2 B represents a function but A doesn't. A does not pass the vertical line test.

3 f is a one-to-one function, but g isn't (it fails the horizontal line test.)

4 Because the quantity under the radical must be non-zero, the domain of f is $[0, \infty)$. The range is also $[0, \infty)$.

5 The denominator must not be zero, so we must have $w \neq 2$. The domain is $\{w : w \neq 2\}$. Note that when $w \neq 2$, the function becomes $\dfrac{(w-2)(2w+1)}{w-2} = 2w + 1$. So the graph of f is a line of slope 2 with the point $(2, 5)$ missing, so the range is $\{y : y \neq 5\}$.

6 It is necessary that $x + 6 > 0$, so $x > -6$. The domain is $(-6, \infty)$ and the range is $(-\infty, \infty)$.

7 Because h can be written $h(z) = \sqrt{(z-3)(z+1)}$, we see that the domain is $(-\infty, -1] \cup [3, \infty)$. The range is $[0, \infty)$. (Note that as z gets large, $h(z)$ gets large as well.)

8 $f(g(2)) = f(-2) = f(2) = 2$, and $g(f(-2) = g(f(2)) = g(2) = -2$.

9 Yes, $\tan(\tan^{-1} x) = x$ because the range of the inverse tangent function is a subset of the domain of the tangent function, and the functions are inverses. However, $\tan^{-1}(\tan x)$ does not always equal x, for example: $\tan^{-1}(\tan \pi) = \tan^{-1} 0 = 0$.

10 $f(g(4)) = f(9) = 11$.

11 $g(f(4)) = g(5) = 8$.

12 $f^{-1}(10) = 8$ (Because $f(8) = 10$.)

13 $g^{-1}(5) = 7$ (Because $g(7) = 5$.)

14 $f^{-1}(g^{-1}(4) = f^{-1}(8) = 6$.

15 $g^{-1}(f(3)) = g^{-1}(4) = 8$.

16 Note that $f(6) = 8$ and recall that f is an odd function. Then $f(-6) = -f(6) = -8$, so $f^{-1}(-8) = -6$.

17 $f^{-1}(1 + f(-3)) = f^{-1}(1 - f(3)) = f^{-1}(1 - 4) = f^{-1}(-3) = -f(3) = -2$.

18 $g(1 - f(f^{-1}(-7))) = g(1 - (-7)) = g(8) = 4$.

19

a. $h(g(\pi/2)) = h(1) = 1$

b. $h(f(x)) = h(x^3) = x^{3/2}$.

c. $f(g(h(x))) = f(g(\sqrt{x})) = f(\sin(\sqrt{x})) = (\sin(\sqrt{x}))^3$.

d The domain of $g(f(x))$ is $\mathbb{R}$, because the domain of both functions is the set of all real numbers.

e. The range of $f(g(x))$ is $[-1, 1]$. This is because the range of g is $[-1, 1]$, and on the restricted domain $[-1, 1]$, the range of f is also $[-1, 1]$.

20

a. If $g(x) = x^2 + 1$ and $f(x) = \sin x$, then $f(g(x)) = f(x^2 + 1) = \sin(x^2 + 1)$.

b. If $g(x) = x^2 - 4$ and $f(x) = x^{-3}$ then $f(g(x)) = f(x^2 - 4) = (x^2 - 4)^{-3}$.

c. If $g(x) = \cos 2x$ and $f(x) = e^x$, then $f(g(x)) = f(\cos 2x) = e^{\cos 2x}$.

21

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - 2(x+h) - (x^2 - 2x)}{h} = \frac{x^2 + 2hx + h^2 - 2x - 2h - x^2 + 2x}{h}$$
$$= \frac{2hx + h^2 - 2h}{h} = 2x + h - 2.$$

$$\frac{f(x) - f(a)}{x - a} = \frac{x^2 - 2x - (a^2 - 2a)}{x - a} = \frac{(x^2 - a^2) - 2(x - a)}{x - a} = \frac{(x - a)(x + a) - 2(x - a)}{x - a} = x + a - 2.$$

22 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{4 - 5(x+h) - (4 - 5x)}{h} = \dfrac{4 - 5x - 5h - 4 + 5x}{h} = -\dfrac{5h}{h} = -5$.

$\dfrac{f(x) - f(a)}{x - a} = \dfrac{4 - 5x - (4 - 5a)}{x - a} = -\dfrac{5(x - a)}{x - a} = -5$.

23

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 + 2 - (x^3 + 2)}{h} = \frac{x^2 + 3x^2h + 3xh^2 + h^3 + 2 - x^3 - 2}{h}$$
$$= \frac{h(3x^2 + 3xh + h^2)}{h} = 3x^2 + 3xh + h^2.$$

$$\frac{f(x) - f(a)}{x - a} = \frac{x^3 + 2 - (a^3 + 2)}{x - a} = \frac{x^3 - a^3}{x - a} = \frac{(x - a)(x^2 + ax + a^2)}{x - a} = x^2 + ax + a^2.$$

24

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{7}{x+h+3}-\frac{7}{x+3}}{h} = \frac{\frac{7x+21-(7x+7h+21)}{(x+3)(x+h+3)}}{h}$$
$$= -\frac{7h}{(h)(x+3)(x+h+3)} = -\frac{7}{(x+3)(x+h+3)}.$$

$$\frac{f(x)-f(a)}{x-a} = \frac{\frac{7}{x+3}-\frac{7}{a+3}}{x-a} = \frac{\frac{7a+21-(7x+21)}{(x+3)(a+3)}}{x-a} = -\frac{7(x-a)}{(x-a)(x+3)(a+3)} = -\frac{7}{(x+3)(a+3)}.$$

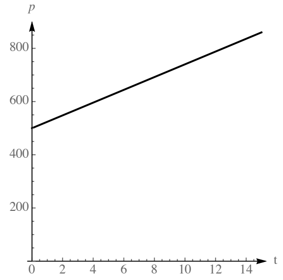
25

a. This line has slope $\frac{2-(-3)}{4-2} = \frac{5}{2}$. Therefore the equation of the line is $y-2 = \frac{5}{2}(x-4)$, so $y = \frac{5}{2}x - 8$.

b. This line has the form $y = \frac{3}{4}x + b$, and because $(-4, 0)$ is on the line, $0 = (3/4)(-4) + b$, so $b = 3$. Thus the equation of the line is given by $y = \frac{3}{4}x + 3$.

c. This line has slope $\frac{0-(-2)}{4-0} = \frac{1}{2}$, and the y-intercept is given to be -2, so the equation of this line is $y = \frac{1}{2}x - 2$.

26 If t is the number of years **after** 2018, then $p(t) = 24t + 500$. Because the year 2033 is 15 years after 2018, the population is predicted by $p(15) = 24 \cdot 15 + 500 = 360 + 500 = 860$.



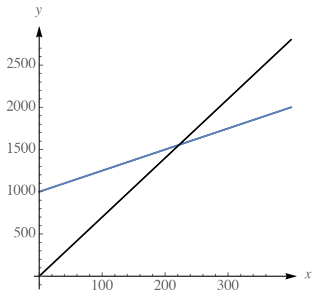
27 We are looking for the line between the points $(0, 212)$ and $(6000, 200)$. The slope is $\frac{212-200}{0-6000} = -\frac{12}{6000} = -\frac{1}{500}$. Because the intercept is given, we deduce that the line is $B = f(a) = -\frac{1}{500}a + 212$.

28

a. The cost of producing x books is $C(x) = 1000 + 2.5x$.

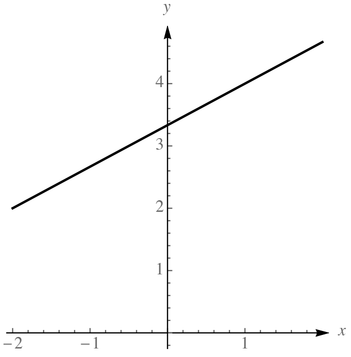
b. The revenue generated by selling x books is $R(x) = 7x$.

c. The break-even point is where $R(x) = C(x)$. This is where $7x = 1000 + 2.5x$, or $4.5x = 1000$. So $x = \frac{1000}{4.5} \approx 222$.

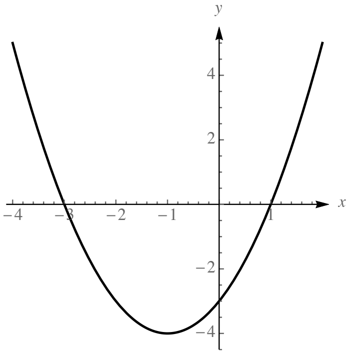


29

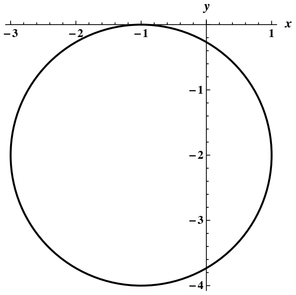
a. This is a straight line with slope 2/3 and y-intercept 10/3.



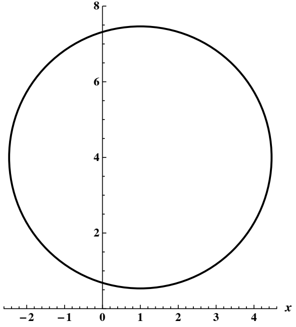
b. Completing the square gives $y = (x^2 + 2x + 1) - 4$, or $y = (x+1)^2 - 4$, so this is the standard parabola shifted one unit to the left and down 4 units.



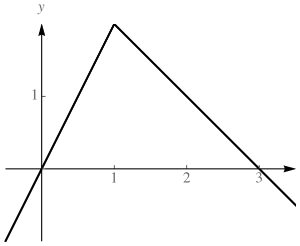
c. Completing the square, we have $x^2 + 2x + 1 + y^2 + 4y + 4 = -1 + 1 + 1 + 4$, so we have $(x+1)^2 + (y+2)^2 = 4$, a circle of radius 2 centered at $(-1, -2)$.



d. Completing the square, we have $x^2 - 2x + 1 + y^2 - 8y + 16 = -5 + 1 + 16$, or $(x-1)^2 + (y-4)^2 = 12$, which is a circle of radius $\sqrt{12}$ centered at $(1, 4)$.



30



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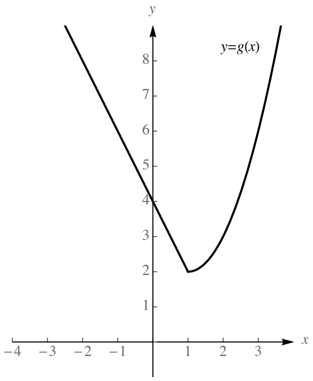

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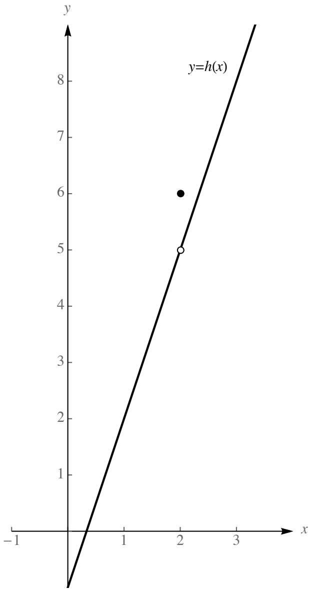
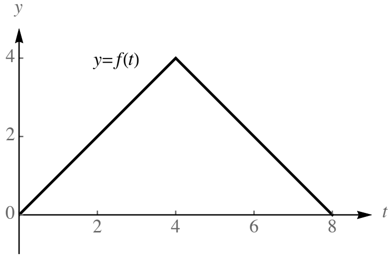


33

a.



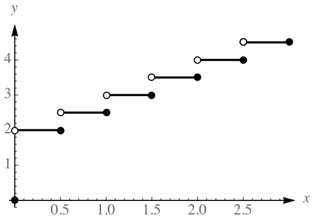
b. $A(2) = \dfrac{1}{2} \cdot 2 \cdot 2 = 2$ and $A(6) = 16 - \dfrac{1}{2} \cdot 2 \cdot 2 = 14$.

c. Note that for $0 \le x \le 4$, the area is that of a triangle with base x and height x. For $4 \le x \le 8$, the area is given by the difference of 16 (the total area under the curve from 0 to 8) minus the area of a triangle with base $8 - x$ and height $8 - x$. So the area for x in that range is $16 - \dfrac{(8-x)^2}{2} = 16 - 32 + 8x - \dfrac{x^2}{2}$. Therefore,

$$A(x) = \begin{cases} \frac{x^2}{2} & \text{if } 0 \le x \le 4 \\ -\frac{x^2}{2} + 8x - 16 & \text{if } 4 \le x \le 8. \end{cases}$$

34

The function is a piecewise step function which jumps up by one every half-hour step.

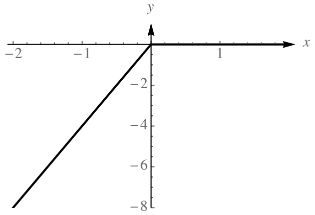


35

Because $|x| = \begin{cases} -x & \text{if } x < 0; \\ x & \text{if } x \geq 0, \end{cases}$

we have

$$2(x - |x|) = \begin{cases} 2(x - (-x)) = 4x & \text{if } x < 0; \\ 2(x - x) = 0 & \text{if } x \geq 0. \end{cases}$$



36 To solve $x^{1/3} = x^{1/4}$ we raise each side to the 12th power, yielding $x^4 = x^3$. This gives $x^4 - x^3 = 0$, or $x^3(x-1) = 0$, so the only solutions are $x = 0$ and $x = 1$ (which can be easily verified as solutions.) Between 0 and 1, $x^{1/4} > x^{1/3}$, but for $x > 1$, $x^{1/3} > x^{1/4}$.

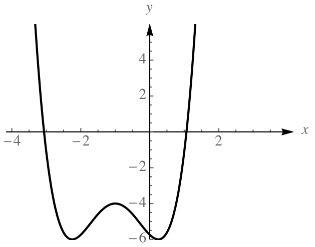
37 The domain of $x^{1/7}$ is the set of all real numbers, as is its range. The domain of $x^{1/4}$ is the set of non-negative real numbers, as is its range.

38 Completing the square in the second equation, we have $x^2 + y^2 - 7y + \frac{49}{4} = -8 + \frac{49}{4}$, which can be written as $x^2 + (y - (7/2))^2 = \frac{17}{4}$. Thus we have a circle of radius $\sqrt{17}/2$ centered at $(0, 7/2)$, along with the standard parabola. These intersect when $y = 7y - y^2 - 8$, which occurs for $y^2 - 6y + 8 = 0$, so for $y = 2$ and $y = 4$, with corresponding x values of ± 2 and $\pm\sqrt{2}$.

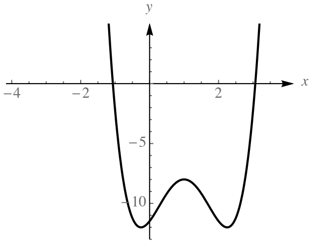
39 Completing the square, we can write $x^2 + 6x - 3 = x^2 + 6x + 9 - 3 - 9 = (x+3)^2 - 12$, so the graph is obtained by shifting $y = x^2$ 3 units right and 12 units down.

40

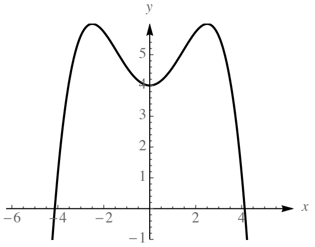
a.



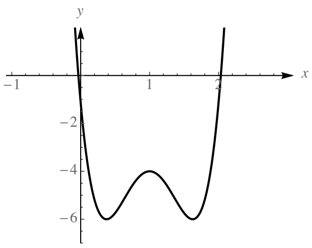
b.



c.



d.



41

a. Because $f(-x) = \cos -3x = \cos 3x = f(x)$, this is an even function, and is symmetric about the y-axis.

b. Because $f(-x) = 3(-x)^4 - 3(-x)^2 + 1 = 3x^4 - 3x^2 + 1 = f(x)$, this is an even function, and is symmetric about the y-axis.

c. Because replacing x by $-x$ and/or replacing y by $-y$ gives the same equation, this represents a curve which is symmetric about the y-axis and about the origin and about the x-axis.

42 We have $8 = e^{4k}$, and so $\ln 8 = 4k$, so $k = \frac{\ln 8}{4}$.

43 If $\log x^2 + 3\log x = \log 32$, then $\log(x^2 \cdot x^3) = \log(32)$, so $x^5 = 32$ and $x = 2$. The answer does not depend on the base of the log.

44 $\ln 3x + \ln(x+2) = \ln(3x(x+2))$. This is zero when $3x(x+2) = 1$, or $3x^2 + 6x - 1 = 0$. By the quadratic formula, we have $\frac{-6\pm\sqrt{36-4\cdot 3\cdot(-1)}}{6} = \frac{-6\pm\sqrt{48}}{6} = -1 \pm \frac{2\sqrt{3}}{3}$. The original equation isn't defined for $-1 - \frac{2\sqrt{3}}{3}$ so the only solution is $-1 + \frac{2\sqrt{3}}{3}$.

45 If $3\ln(5t+4) = 12$, then $\ln(5t+4) = 4$, and $e^4 = 5t+4$. It then follows that $5t = e^4 - 4$ and so $t = \frac{e^4-4}{5}$.

46 If $7^{y-3} = 50$, then $\ln(7^{y-3}) = \ln 50$, so $(y-3)\ln 7 = \ln 50$ and $y - 3 = \frac{\ln 50}{\ln 7}$. Then $y = 3 + \frac{\ln 50}{\ln 7}$.

47 If $1 - 2\sin^2\theta = 0$, then $\sin^2\theta = \frac{1}{2}$, so $|\sin\theta| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. So $\theta = \frac{\pi}{4}, \frac{3\pi}{4} \frac{5\pi}{4} \frac{7\pi}{4}$.

48 First note that if θ is between $-\pi/2$ and $\pi/2$, that 2θ is then between $-\pi$ and π. If $\sin^2 2\theta = \frac{1}{2}$, then $|\sin 2\theta| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. So $2\theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$. Thus $\theta = -\frac{3\pi}{8}, -\frac{\pi}{8}, \frac{\pi}{8}, \frac{3\pi}{8}$

49 First note that if θ is between $-\pi/2$ and $\pi/2$, that 2θ is then between $-\pi$ and π. If $4\cos^2 2\theta = 3$, then $\cos^2 2\theta = \frac{3}{4}$, and $|\cos 2\theta| = \frac{\sqrt{3}}{2}$. Thus $2\theta = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$, and $\theta = \pm\frac{\pi}{12}, \pm\frac{5\pi}{12}$.

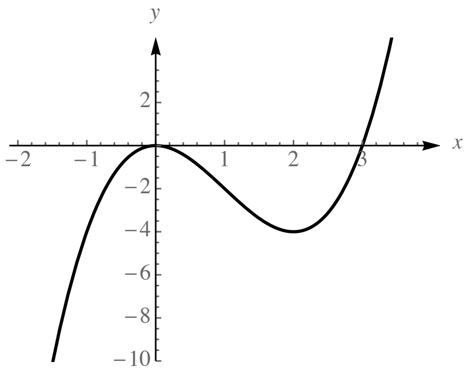
50 First note that if θ is between 0 and π that 3θ is then between 0 and 3π. If $\sqrt{2}\sin 3\theta + 1 = 2$, then $\sin 3\theta = \frac{1}{\sqrt{2}}$. Then $3\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$, so $\theta = \frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}$.

51 In 2010 (when $t = 0$), the population is $P(0) = 100$. So we are seeking t so that $200 = 100e^{t/50}$, or $e^{t/50} = 2$. Taking the natural logarithm of both sides yields $\frac{t}{50} = \ln 2$, or $t = 50\ln 2 \approx 35$ years.

52 Curve A is $y = 3^{-x}$, curve B is $y = 2^x$, and curve C is $y = -\ln x$.

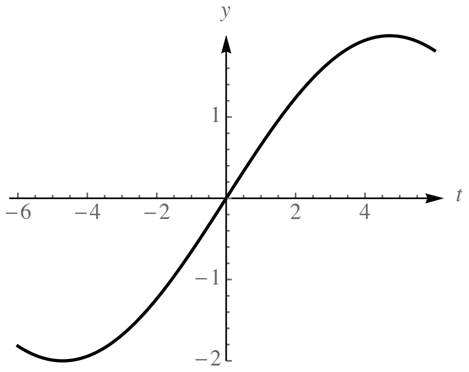
53

By graphing, it is clear that this function is not one-to-one on its whole domain, but it is one-to-one on the interval $(-\infty, 0]$, on the interval $[0, 2]$, and on the interval $[2, \infty)$, so it would have an inverse if we restricted it to any of these particular intervals.



54

This function is a stretched version of the sine function, it is one-to-one on the interval $[-3\pi/2, 3\pi/2]$ (and on other intervals as well ...)



55 Switching x and y gives $x = 6 - 4y$. Then $x - 6 = -4y$, so

$$y = \frac{x-6}{-4} = \frac{6-x}{4} = -\frac{1}{4}x + \frac{3}{2}.$$

56 Switching x and y gives $x = 3y - 4$, so $x + 4 = 3y$ and $y = \frac{x+4}{3}$.

57 Completing the square gives $f(x) = x^2 - 4x + 4 + 1 = (x-2)^2 + 1$. Switching the x and y and solving for y yields $(y-2)^2 = x - 1$, so $|y - 2| = \sqrt{x-1}$, and thus $y = f^{-1}(x) = 2 + \sqrt{x-1}$ (we choose the "+" rather than the "−" because the domain of f is $x \geq 2$, so the range of f^{-1} must also consist of numbers greater than or equal to 2).

58 Switching x and y gives $x = \frac{4y^2}{y^2+10}$. Then $x(y^2+10) = 4y^2$, so $xy^2 - 4y^2 = -10x$. Then $y^2(x-4) = -10x$, so $y^2 = \frac{10x}{4-x}$, and $|y| = \sqrt{\frac{10x}{4-x}}$, so $y = \sqrt{\frac{10x}{4-x}}$ (we choose the "+" rather than the "−" because the domain of f is $x \geq 0$, so the range of f^{-1} must also consist of numbers greater than or equal to 0).

59 Switching x and y gives $x = 3y^2 + 1$, so $3y^2 = x - 1$, and $y^2 = \frac{x-1}{3}$. Then $|y| = \sqrt{\frac{x-1}{3}}$, and $y = -\sqrt{\frac{x-1}{3}}$ (we choose the "−" rather than the "+" because the domain of f is $x \leq 0$, so the range of f^{-1} must also consist of numbers less than or equal to 0).

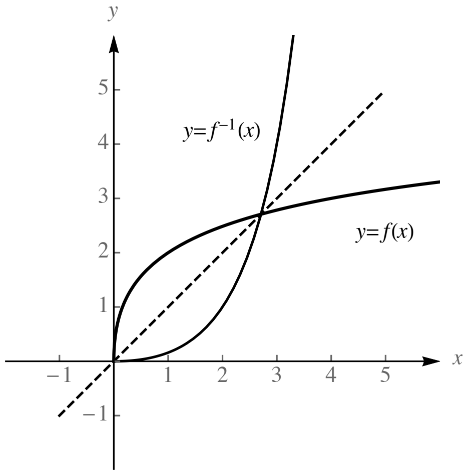
60 If $y = 1/x^2$, then switching x and y gives $x = 1/y^2$, so $y = f^{-1}(x) = 1/\sqrt{x}$.

61 Switching x and y gives $x = e^{y^2+1}$. Then $\ln x = y^2 + 1$, so $y^2 = \ln x - 1$, and $|y| = \sqrt{\ln x - 1}$, so $y = \sqrt{\ln x - 1}$ (we choose the "+" rather than the "−" because the domain of f is $x \geq 0$, so the range of f^{-1} must also consist of numbers greater than or equal to 0).

62 Switching x and y gives $x = \ln(y^2 + 1)$, so $e^x = y^2 + 1$, and $y^2 = e^x - 1$, so $|y| = \sqrt{e^x - 1}$, so $y = \sqrt{e^x - 1}$ (we choose the "+" rather than the "−" because the domain of f is $x \geq 0$, so the range of f^{-1} must also consist of numbers greater than or equal to 0.)

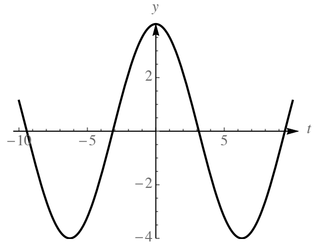
63

Switching x and y gives $x = \frac{6\sqrt{y}}{\sqrt{y}+2}$. Then $x\sqrt{y} + 2x = 6\sqrt{y}$, and so $x\sqrt{y} - 6\sqrt{y} = -2x$. Then $\sqrt{y}(x-6) = -2x$, so $\sqrt{y} = \frac{2x}{6-x}$, and $y = \left(\frac{2x}{6-x}\right)^2 = \frac{4x^2}{(6-x)^2}$. Because the range of f must match the domain of f^{-1}, we must restrict our inverse function to $[0, 6]$.

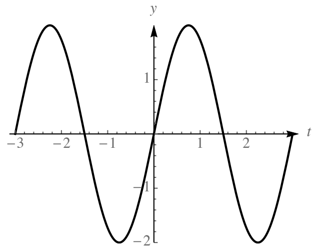


64

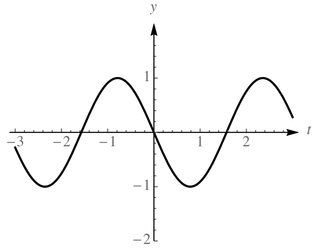
a. This function has period $\frac{2\pi}{1/2} = 4\pi$ and amplitude 4.



b. This function has period $\frac{2\pi}{2\pi/3} = 3$ and amplitude 2.



c. This function has period $\frac{2\pi}{2} = \pi$ and amplitude 1. Compared to the ordinary cosine function it is compressed horizontally, flipped about the x-axis, and shifted $\pi/4$ units to the right.



65

a. We need to scale the ordinary cosine function so that its period is 6, and then shift it 3 units to the right, and multiply it by 2. So the function we seek is $y = 2\cos((\pi/3)(t-3)) = -2\cos(\pi t/3)$.

b. We need to scale the ordinary cosine function so that its period is 24, and then shift it to the right 6 units. We then need to change the amplitude to be half the difference between the maximum and minimum, which would be 5. Then finally we need to shift the whole thing up by 15 units. The function we seek is thus $y = 15 + 5\cos((\pi/12)(t-6)) = 15 + 5\sin(\pi t/12)$.

66 The pictured function has a period of π, an amplitude of 2, and a maximum of 3 and a minimum of -1. It can be described by $y = 1 + 2\cos(2(x - \pi/2))$.

67

a. $-\sin x$ is pictured in F.

b. $\cos 2x$ is pictured in E.

c. $\tan(x/2)$ is pictured in D.

d. $-\sec x$ is pictured in B.

e. $\cot 2x$ is pictured in C.

f. $\sin^2 x$ is pictured in A.

68 If $\sec x = 2$, then $\cos x = \frac{1}{2}$. This occurs for $x = -\pi/3$ and $x = \pi/3$, so the intersection points are $(-\pi/3, 2)$ and $(\pi/3, 2)$.

69 $\sin x = -\frac{1}{2}$ for $x = 7\pi/6$ and for $x = 11\pi/6$, so the intersection points are $(7\pi/6, -1/2)$ and $(11\pi/6, -1/2)$.

70 Note that $\frac{5\pi}{8} = \frac{5\pi/4}{2}$. Using a half-angle identity,

$$\sin\left(\frac{5\pi/4}{2}\right) = \sqrt{\frac{1-\cos 5\pi/4}{2}} = \sqrt{\frac{1+\sqrt{2}/2}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

71 Note that $\frac{7\pi}{8} = \frac{7\pi/4}{2}$. Using the half-angle identity,

$$\cos\left(\frac{7\pi/4}{2}\right) = -\sqrt{\frac{1+\cos 7\pi/4}{2}} = -\sqrt{\frac{1+\sqrt{2}/2}{2}} = -\sqrt{\frac{2+\sqrt{2}}{4}} = -\frac{\sqrt{2+\sqrt{2}}}{2}.$$

72 Because $\sin(\pi/3) = \sqrt{3}/2$, $\sin^{-1}(\sqrt{3}/2) = \pi/3$.

73 Because $\cos(\pi/6) = \sqrt{3}/2$, $\cos^{-1}(\sqrt{3}/2) = \pi/6$.

74 Because $\cos(2\pi/3) = -1/2$, $\cos^{-1}(-1/2) = 2\pi/3$.

75 Because $\sin(-\pi/2) = -1$, $\sin^{-1}(-1) = -\pi/2$.

76 $\cos(\cos^{-1}(-1)) = \cos(\pi) = -1$.

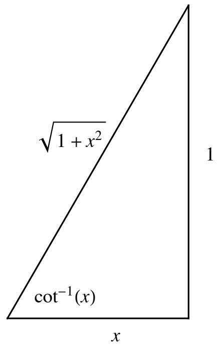
77 $\sin(\sin^{-1}(x)) = x$, for all x in the domain of the inverse sine function.

78 $\cos^{-1}(\sin 3\pi) = \cos^{-1}(0) = \pi/2$.

79 If $\theta = \sin^{-1}(12/13)$, then $0 < \theta < \pi/2$, and $\sin\theta = 12/13$. Then (using the Pythagorean identity) we can deduce that $\cos\theta = 5/13$. It must follow that $\tan\theta = 12/5$, $\cot\theta = 5/12$, $\sec\theta = 13/5$, and $\csc\theta = 13/12$.

80

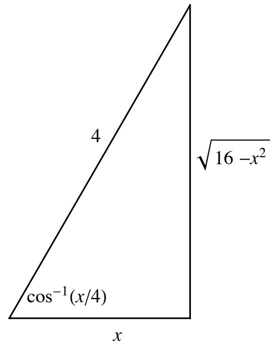
$$\csc(\cot^{-1} x) = \frac{\text{hypotenuse}}{\text{side opposite of } \cot^{-1} x} = \sqrt{1+x^2}$$



$\sqrt{1+x^2}$
1
$\cot^{-1}(x)$
x

81

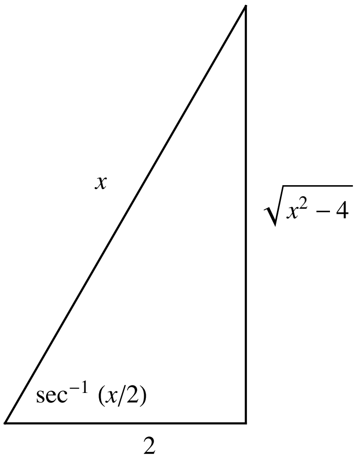
$$\sin(\cos^{-1}(x/4)) = \frac{\text{side opposite of } \cos^{-1}(x/4)}{\text{hypotenuse}} = \frac{\sqrt{16-x^2}}{4}.$$



4
$\sqrt{16-x^2}$
$\cos^{-1}(x/4)$
x

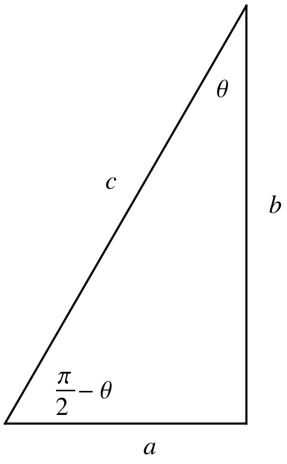
82

$$\tan(\sec^{-1}(x/2)) = \frac{\text{side opposite of } \sec^{-1}(x/2)}{\text{side adjacent to } \sec^{-1}(x/2)} = \frac{\sqrt{x^2-4}}{2}.$$



x
$\sqrt{x^2-4}$
$\sec^{-1}(x/2)$
2

83



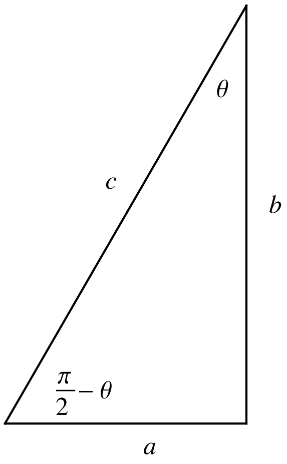
Note that

$$\tan\theta = \frac{a}{b} = \cot(\pi/2-\theta).$$

Thus,

$$\cot^{-1}(\tan\theta) = \cot^{-1}(\cot(\pi/2-\theta)) = \pi/2-\theta.$$

84



Note that

$$\sec\theta = \frac{c}{b} = \csc(\pi/2-\theta).$$

Thus,

$$\csc^{-1}(\sec\theta) = \csc^{-1}(\csc(\pi/2-\theta)) = \pi/2-\theta.$$

85 Let $\theta = \sin^{-1}(x)$. Then $\sin\theta = x$ and note that then $\sin(-\theta) = -\sin\theta = -x$, so $-\theta = \sin^{-1}(-x)$. Then $\sin^{-1}(x)+\sin^{-1}(-x) = \theta + -\theta = 0$.

86 We multiply the quantity $\frac{\sin\theta}{1+\cos\theta}$ by the conjugate of the denominator over itself:

$$\frac{\sin\theta}{1+\cos\theta} = \frac{\sin\theta}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta} = \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} = \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta} = \frac{1-\cos\theta}{\sin\theta}.$$

87 Using the definition of the tangent function in terms of sine and cosine, we have:

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}.$$

If we divide both the numerator and denominator of this last expression by $\cos^2\theta$, we obtain

$$\frac{2\tan\theta}{1-\tan^2\theta}.$$

88 Let N be the north pole, and C the center of the given circle, and consider the angle CNP. This angle measures $\frac{\pi-\varphi}{2}$. (Note that the triangle CNP is isosceles.) Now consider the triangle NOX where O is the origin and X is the point $(x,0)$. Using triangle NOX, we have $\tan\left(\frac{\pi-\varphi}{2}\right) = \frac{x}{2R}$, so $x = 2R\tan\left(\frac{\pi-\varphi}{2}\right)$.

89

a.

n	1	2	3	4	5	6	7	8	9	10
$T(n)$	1	5	14	30	55	91	140	204	285	385

b. The domain of this function consists of the positive integers.

c. Using trial and error and a calculator yields that $T(n) > 1000$ for the first time for $n = 14$.

90

a.

n	1	2	3	4	5	6	7	8	9	10
$S(n)$	1	3	6	10	15	21	28	36	45	55

b. The domain of this function consists of the positive integers. The range is a subset of the set of positive integers.

c. Using trial and error and a calculator yields that $S(n) > 1000$ for the first time for $n = 45$.

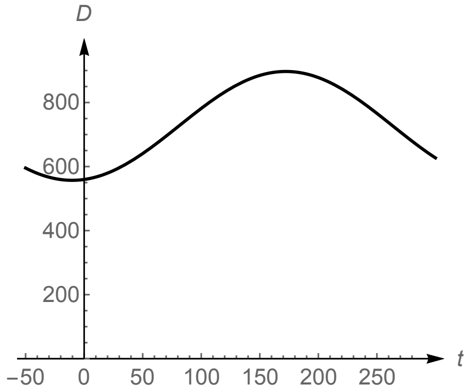
91 To find $s(t)$ note that we are seeking a periodic function with period 365, and with amplitude 87.5 (which is half of the number of minutes between 7:25 and 4:30). We need to shift the function 4 days plus one fourth of 365, which is about 95 days so that the max and min occur at $t = 4$ days and at half a year later. Also, to get the right value for the maximum and minimum, we need to multiply by negative one and add 117.5 (which represents 30 minutes plus half the amplitude, because $s = 0$ corresponds to 4:00 AM.) Thus we have

$$s(t) = 117.5 - 87.5\sin\left(\frac{\pi}{182.5}(t - 95)\right).$$

A similar analysis leads to the formula

$$S(t) = 844.5 + 87.5\sin\left(\frac{\pi}{182.5}(t - 67)\right).$$

The graph pictured shows $D(t) = S(t) - s(t)$, the length of day function, which has its max at the summer solstice which is about the 172nd day of the year, and its min at the winter solstice.



Chapter 2

Limits

2.1 The Idea of Limits

2.1.1 The average velocity of the object between time $t = a$ and $t = b$ is the change in position divided by the elapsed time: $v_{\text{av}} = \dfrac{s(b) - s(a)}{b - a}$.

2.1.2 In order to compute the instantaneous velocity of the object at time $t = a$, we compute the average velocity over smaller and smaller time intervals of the form $[a, t]$, using the formula: $v_{\text{av}} = \dfrac{s(t) - s(a)}{t - a}$. We let t approach a. If the quantity $\dfrac{s(t) - s(a)}{t - a}$ approaches a limit as $t \to a$, then that limit is called the instantaneous velocity of the object at time $t = a$.

2.1.3 The average velocity is $\dfrac{s(3) - s(2)}{3 - 2} = 156 - 136 = 20$.

2.1.4 The average velocity is $\dfrac{s(4) - s(1)}{4 - 1} = \dfrac{144 - 84}{3} = \dfrac{60}{3} = 20$.

2.1.5

a. $\dfrac{s(2) - s(0)}{2 - 0} = \dfrac{72 - 0}{2} = 36$.

b. $\dfrac{s(1.5) - s(0)}{1.5 - 0} = \dfrac{66 - 0}{1.5} = 44$.

c. $\dfrac{s(1) - s(0)}{1 - 0} = \dfrac{52 - 0}{1} = 52$.

d. $\dfrac{s(.5) - s(0)}{.5 - 0} = \dfrac{30 - 0}{.5} = 60$.

2.1.6

a. $\dfrac{s(2.5) - s(.5)}{2.5 - .5} = \dfrac{150 - 46}{2} = 52$.

b. $\dfrac{s(2) - s(.5)}{2 - .5} = \dfrac{136 - 46}{1.5} = 60$.

c. $\dfrac{s(1.5) - s(.5)}{1.5 - .5} = \dfrac{114 - 46}{1} = 68$.

d. $\dfrac{s(1) - s(.5)}{1 - .5} = \dfrac{84 - 46}{.5} = 76$.

2.1.7 $\frac{s(1.01)-s(1)}{.01} = 47.84$, while $\frac{s(1.001)-s(1)}{.001} = 47.984$ and $\frac{s(1.0001)-s(1)}{.0001} = 47.9984$. It appears that the instantaneous velocity at $t = 1$ is approximately 48.

2.1.8 $\frac{s(2.01)-s(2)}{.01} = -4.16$, while $\frac{s(2.001)-s(2)}{.001} = -4.016$ and $\frac{s(2.0001)-s(2)}{.0001} = -4.0016$. It appears that the instantaneous velocity at $t = 2$ is approximately -4.

2.1.9 The slope of the secant line between points $(a, f(a))$ and $(b, f(b))$ is the ratio of the differences $f(b) - f(a)$ and $b - a$. Thus $m_{\text{sec}} = \frac{f(b)-f(a)}{b-a}$.

2.1.10 In order to compute the slope of the tangent line to the graph of $y = f(t)$ at $(a, f(a))$, we compute the slope of the secant line over smaller and smaller time intervals of the form $[a, t]$. Thus we consider $\frac{f(t)-f(a)}{t-a}$ and let $t \to a$. If this quantity approaches a limit, then that limit is the slope of the tangent line to the curve $y = f(t)$ at $t = a$.

2.1.11 Both problems involve the same mathematics, namely finding the limit as $t \to a$ of a quotient of differences of the form $\frac{g(t)-g(a)}{t-a}$ for some function g.

2.1.12 Note that $f(2) = 64$.

a. i. $f(0.5) = 28$. So the slope of the secant line is $\frac{28-64}{0.5-2} = \frac{-36}{-3/2} = 24$.
   ii. $f(1.9) \approx 63.84$. So the slope of the secant line is about $\frac{63.84-64}{1.9-2} = 1.6$.
   iii. $f(1.99) \approx 63.9984$. So the slope of the secant line is about $\frac{63.9984-64}{1.99-2} = 0.16$.
   iv. $f(1.999) \approx 63.999984$. So the slope of the secant line is about $\frac{63.999984-64}{1.999-2} = 0.016$.
   v. $f(1.9999) \approx 63.99999984$. So the slope of the secant line is about $\frac{63.99999984-64}{1.9999-2} = 0.0016$.

b. A good guess is that the limit is 0.

c. The slope of the tangent line is the limit of the slopes of the secant lines, so it is also 0.

2.1.13

a. Over $[1, 4]$, we have $v_{\text{av}} = \frac{s(4)-s(1)}{4-1} = \frac{256-112}{3} = 48$.

b. Over $[1, 3]$, we have $v_{\text{av}} = \frac{s(3)-s(1)}{3-1} = \frac{240-112}{2} = 64$.

c. Over $[1, 2]$, we have $v_{\text{av}} = \frac{s(2)-s(1)}{2-1} = \frac{192-112}{1} = 80$.

d. Over $[1, 1+h]$, we have

$$\begin{aligned} v_{\text{av}} = \frac{s(1+h)-s(1)}{1+h-1} &= \frac{-16(1+h)^2 + 128(1+h) - (112)}{h} = \frac{-16h^2 - 32h + 128h}{h} \\ &= \frac{h(-16h+96)}{h} = 96 - 16h = 16(6-h). \end{aligned}$$

2.1.14

a. Over $[0, 3]$, we have $v_{\text{av}} = \frac{s(3)-s(0)}{3-0} = \frac{65.9-20}{3} = 15.3$.

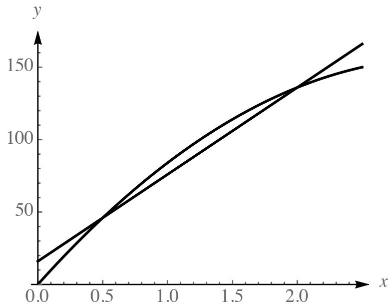
b. Over $[0, 2]$, we have $v_{\text{av}} = \frac{s(2)-s(0)}{2-0} = \frac{60.4-20}{2} = 20.2$.

c. Over $[0,1]$, we have $v_{\text{av}} = \dfrac{s(1)-s(0)}{1-0} = \dfrac{45.1-20}{1} = 25.1$.

d. Over $[0,h]$, we have

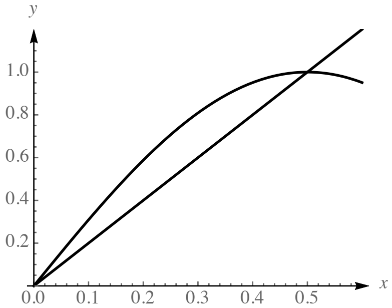
$$\begin{aligned} v_{\text{av}} &= \frac{s(h)-s(0)}{h-0} = \frac{-4.9h^2+30h+20-20}{h} \\ &= \frac{(h)(-4.9h+30)}{h} = -4.9h+30. \end{aligned}$$

2.1.15



The slope of the secant line is given by $\dfrac{s(2)-s(0.5)}{2-0.5} = \dfrac{136-46}{1.5} = 60$. This represents the average velocity of the object over the time interval $[0.5,2]$.

2.1.16



The slope of the secant line is given by $\dfrac{s(0.5)-s(0)}{0.5-0} = \dfrac{1}{0.5} = 2$. This represents the average velocity of the object over the time interval $[0,0.5]$.

2.1.17

Time Interval	$[1,2]$	$[1,1.5]$	$[1,1.1]$	$[1,1.01]$	$[1,1.001]$
Average Velocity	80	88	94.4	95.84	95.984

The instantaneous velocity appears to be 96 ft/s.

2.1.18

Time Interval	$[2,3]$	$[2,2.25]$	$[2,2.1]$	$[2,2.01]$	$[2,2.001]$
Average Velocity	5.5	9.175	9.91	10.351	10.395

The instantaneous velocity appears to be 10.4 m/s.

2.1.19

Time Interval	$[2,3]$	$[2.9,3]$	$[2.99,3]$	$[2.999,3]$	$[2.9999,3]$	$[2.99999,3]$
Average Velocity	20	5.6	4.16	4.016	4.002	4.0002

The instantaneous velocity appears to be 4 ft/s.

2.1.20

Time Interval	$[\pi/2,\pi]$	$[\pi/2,\pi/2+.1]$	$[\pi/2,\pi/2+.01]$	$[\pi/2,\pi/2+.001]$	$[\pi/2,\pi/2+.0001]$
Average Velocity	-1.90986	-0.149875	-0.0149999	-0.0015	-0.00015

The instantaneous velocity appears to be 0 ft/s.

2.1.21

Time Interval	$[3, 3.1]$	$[3, 3.01]$	$[3, 3.001]$	$[3, 3.0001]$
Average Velocity	-17.6	-16.16	-16.016	-16.002

The instantaneous velocity appears to be -16 ft/s.

2.1.22

Time Interval	$[\pi/2, \pi/2 + .1]$	$[\pi/2, \pi/2 + .01]$	$[\pi/2, \pi/2 + .001]$	$[\pi/2, \pi/2 + .0001]$
Average Velocity	-19.9667	-19.9997	-20.0000	-20.0000

The instantaneous velocity appears to be -20 ft/s.

2.1.23

Time Interval	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$	$[0, 0.0001]$
Average Velocity	79.468	79.995	80.000	80.0000

The instantaneous velocity appears to be 80 ft/s.

2.1.24

Time Interval	$[0, 1]$	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$
Average Velocity	-10	-18.1818	-19.802	-19.98

The instantaneous velocity appears to be -20 ft/s.

2.1.25

x **Interval**	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$	$[2, 2.0001]$
Slope of Secant Line	8.2	8.02	8.002	8.0002

The slope of the tangent line appears to be 8.

2.1.26

x **Interval**	$[\pi/2, \pi/2 + .1]$	$[\pi/2, \pi/2 + .01]$	$[\pi/2, \pi/2 + .001]$	$[\pi/2, \pi/2 + .0001]$
Slope of Secant Line	-2.995	-2.99995	-3.0000	-3.0000

The slope of the tangent line appears to be -3.

2.1.27

x **Interval**	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$	$[0, 0.0001]$
Slope of the Secant Line	1.05171	1.00502	1.0005	1.00005

The slope of the tangent line appears to be 1.

2.1.28

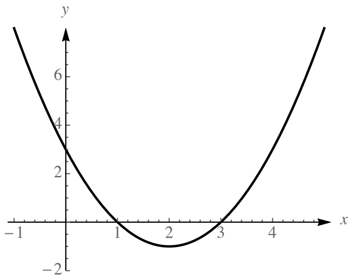
x **Interval**	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$	$[1, 1.0001]$
Slope of the Secant Line	2.31	2.0301	2.003	2.0003

The slope of the tangent line appears to be 2.

2.1.29

a. Note that the graph is a parabola with vertex $(2, -1)$.

b. At $(2, -1)$ the function has tangent line with slope 0.



c.

x **Interval**	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$	$[2, 2.0001]$
Slope of the Secant Line	0.1	0.01	0.001	0.0001

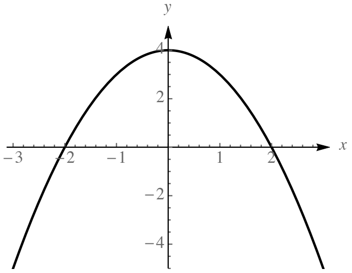
The slope of the tangent line at $(2, -1)$ appears to be 0.

2.1.30

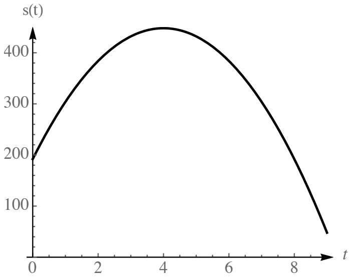
a. Note that the graph is a parabola with vertex $(0, 4)$.

b. At $(0, 4)$ the function has a tangent line with slope 0.

c. This is true for this function – because the function is symmetric about the y-axis and we are taking pairs of points symmetrically about the y axis. Thus $f(0+h) = 4 - (0+h)^2 = 4 - (-h)^2 = f(0-h)$. So the slope of any such secant line is $\dfrac{4 - h^2 - (4 - h^2)}{h - (-h)} = \dfrac{0}{2h} = 0$.



2.1.31



a. Note that the graph is a parabola with vertex $(4, 448)$.

b. At $(4, 448)$ the function has tangent line with slope 0, so $a = 4$.

c.

x **Interval**	$[4, 4.1]$	$[4, 4.01]$	$[4, 4.001]$	$[4, 4.0001]$
Slope of the Secant Line	-1.6	$-.16$	$-.016$	$-.0016$

The slopes of the secant lines appear to be approaching zero.

d. On the interval $[0, 4)$ the instantaneous velocity of the projectile is positive.

e. On the interval $(4, 9]$ the instantaneous velocity of the projectile is negative.

2.1.32

a. The rock strikes the water when $s(t) = 96$. This occurs when $16t^2 = 96$, or $t^2 = 6$, whose only positive solution is $t = \sqrt{6} \approx 2.45$ seconds.

b.

t **Interval**	$[\sqrt{6} - .1, \sqrt{6}]$	$[\sqrt{6} - .01, \sqrt{6}]$	$[\sqrt{6} - .001, \sqrt{6}]$	$[\sqrt{6} - .0001, \sqrt{6}]$
Average Velocity	76.7837	78.2237	78.3677	78.3821

When the rock strikes the water, its instantaneous velocity is about 78.38 ft/s.

2.1.33 For line AD, we have

$$m_{AD} = \frac{y_D - y_A}{x_D - x_A} = \frac{f(\pi) - f(\pi/2)}{\pi - (\pi/2)} = \frac{1}{\pi/2} \approx 0.63662.$$

For line AC, we have

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{f(\pi/2 + .5) - f(\pi/2)}{(\pi/2 + .5) - (\pi/2)} = -\frac{\cos(\pi/2 + .5)}{.5} \approx 0.958851.$$

For line AB, we have

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{f(\pi/2 + .05) - f(\pi/2)}{(\pi/2 + .05) - (\pi/2)} = -\frac{\cos(\pi/2 + .05)}{.05} \approx 0.999583.$$

Computing one more slope of a secant line:

$$m_{\text{sec}} = \frac{f(\pi/2 + .01) - f(\pi/2)}{(\pi/2 + .01) - (\pi/2)} = -\frac{\cos(\pi/2 + .01)}{.01} \approx 0.999983.$$

Conjecture: The slope of the tangent line to the graph of f at $x = \pi/2$ is 1.

2.2 Definition of a Limit

2.2.1 Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to a number L whenever x is sufficiently close to (but not equal to) a, then we write $\lim_{x\to a} f(x) = L$.

2.2.2 False. For example, consider the function $f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 4 & \text{if } x = 0. \end{cases}$

Then $\lim_{x\to 0} f(x) = 0$, but $f(0) = 4$.

2.2.3

a. $h(2) = 5$.

b. $\lim_{x\to 2} h(x) = 3$.

c. $h(4)$ does not exist.

d. $\lim_{x\to 4} f(x) = 1$.

e. $\lim_{x\to 5} h(x) = 2$.

2.2.4

a. $g(0) = 0$.

b. $\lim_{x\to 0} g(x) = 1$.

c. $g(1) = 2$.

d. $\lim_{x\to 1} g(x) = 2$.

2.2.5

a. $f(1) = -1$.

b. $\lim_{x\to 1} f(x) = 1$.

c. $f(0) = 2$.

d. $\lim_{x\to 0} f(x) = 2$.

2.2.6

a. $f(2) = 2$.

b. $\lim_{x\to 2} f(x) = 4$.

c. $\lim_{x\to 4} f(x) = 4$.

d. $\lim_{x\to 5} f(x) = 2$.

2.2.7

a.

x	1.9	1.99	1.999	1.9999	2.1	2.01	2.001	2.0001
$f(x) = \frac{x^2-4}{x-2}$	3.9	3.99	3.999	3.9999	4.1	4.01	4.001	4.0001

b. $\lim_{x\to 2} f(x) = 4$.

2.2.8

a.

x	0.9	0.99	0.999	0.9999	1.1	1.01	1.001	1.0001
$f(x) = \dfrac{x^3-1}{x-1}$	2.71	2.9701	2.997	2.9997	3.31	3.30301	3.003	3.0003

b. $\lim\limits_{x\to1} \dfrac{x^3-1}{x-1} = 3$

2.2.9

a.

t	8.9	8.99	8.999	9.1	9.01	9.001
$g(t) = \dfrac{t-9}{\sqrt{t}-3}$	5.98329	5.99833	5.99983	6.01662	6.00167	6.00017

b. $\lim\limits_{t\to9} \dfrac{t-9}{\sqrt{t}-3} = 6.$

2.2.10

a.

x	0.01	0.001	0.0001	0.00001
$f(x) = (1+x)^{1/x}$	2.70481	2.71692	2.71815	2.71827

x	-0.01	-0.001	-0.0001	-0.00001
$f(x) = (1+x)^{1/x}$	2.732	2.71964	2.71842	2.71830

b. $\lim\limits_{x\to0}(1+x)^{1/x} \approx 2.718.$

c. $\lim\limits_{x\to0}(1+x)^{1/x} = e.$

2.2.11 Suppose the function f is defined for all x near a but greater than a. If $f(x)$ is arbitrarily close to L for x sufficiently close to (but strictly greater than) a, then we write $\lim\limits_{x\to a^+} f(x) = L$.

2.2.12 Suppose the function f is defined for all x near a but less than a. If $f(x)$ is arbitrarily close to L for x sufficiently close to (but strictly less than) a, then we write $\lim\limits_{x\to a^-} f(x) = L$.

2.2.13 It must be true that $L = M$.

2.2.14

a.

x	1.9	1.99	1.999	1.9999
$g(x) = \frac{x^3-4x}{8\lvert x-2\rvert}$	-0.92625	-0.99251	-0.99925	-0.99993

x	2.1	2.01	2.001	2.0001
$g(x) = \frac{x^3-4x}{8\lvert x-2\rvert}$	1.07625	1.00751	1.00075	1.00008

b. $\lim\limits_{x\to2^-} g(x) = -1$. $\lim\limits_{x\to2^+} g(x) = 1$. $\lim\limits_{x\to2} g(x)$ does not exist.

2.2.15

a. $f(1) = 0.$

b. $\lim\limits_{x\to1^-} f(x) = 1.$

c. $\lim\limits_{x\to1^+} f(x) = 0.$

d. $\lim\limits_{x\to1} f(x)$ does not exist, since the two one-sided limits aren't equal.

2.2.16 Because graphing utilities generally just plot a sampling of points and "connect the dots," they can sometimes mislead the user investigating the subtleties of limits.

2.2.17

a. $f(1) = 3.$

b. $\lim_{x\to 1^-} f(x) = 2.$

c. $\lim_{x\to 1^+} f(x) = 2.$

d. $\lim_{x\to 1} f(x) = 2.$

e. $f(3) = 2.$

f. $\lim_{x\to 3^-} f(x) = 4.$

g. $\lim_{x\to 3^+} f(x) = 1.$

h. $\lim_{x\to 3} f(x)$ does not exist.

i. $f(2) = 3.$

j. $\lim_{x\to 2^-} f(x) = 3.$

k. $\lim_{x\to 2^+} f(x) = 3.$

l. $\lim_{x\to 2} f(x) = 3.$

2.2.18

a. $g(2) = 3.$

b. $\lim_{x\to 2^-} g(x) = 2.$

c. $\lim_{x\to 2^+} g(x) = 3.$

d. $\lim_{x\to 2} g(x)$ does not exist.

e. $g(3) = 2.$

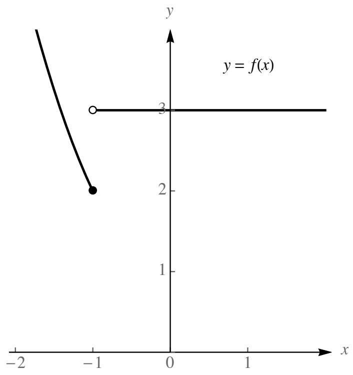
f. $\lim_{x\to 3^-} g(x) = 3.$

g. $\lim_{x\to 3^+} g(x) = 2.$

h. $g(4) = 3.$

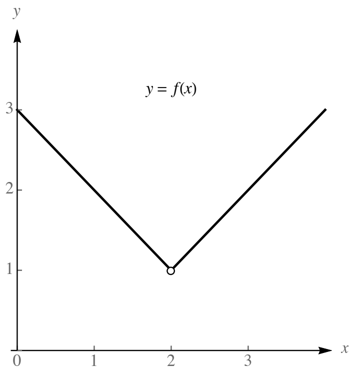
i. $\lim_{x\to 4} g(x) = 3.$

2.2.19



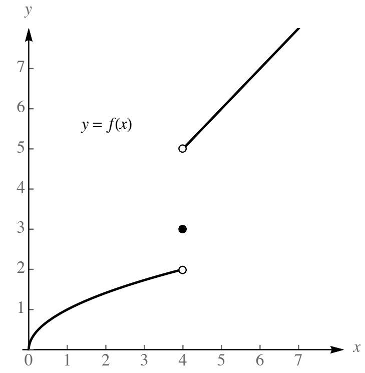
$f(-1) = 2$, $\lim_{x\to -1^-} f(x) = 2$, $\lim_{x\to -1^+} f(x) = 3$, $\lim_{x\to -1} f(x)$ does not exist.

2.2.20



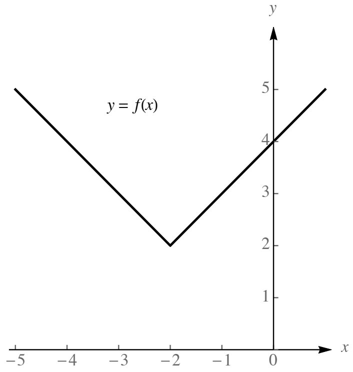
$f(2)$ is undefined. $\lim_{x\to 2^-} f(x) = 1$, $\lim_{x\to 2^+} f(x) = 1$, and $\lim_{x\to 2} f(x) = 1.$

2.2.21



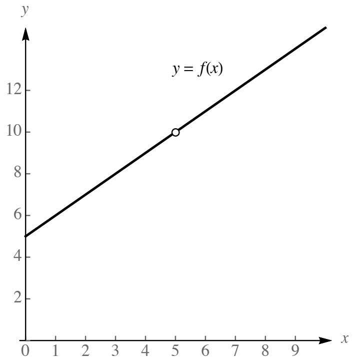
$f(4) = 3$, $\lim_{x\to 4^-} f(x) = 2$, $\lim_{x\to 4^+} f(x) = 5$, $\lim_{x\to 4} f(x)$ does not exist.

2.2.22



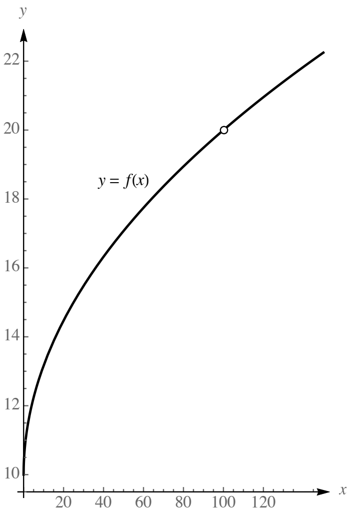
$f(-2) = 2$, $\lim_{x\to -2^-} f(x) = 2$, $\lim_{x\to -2^+} f(x) = 2$, $\lim_{x\to -2} f(x) = 2$.

2.2.23



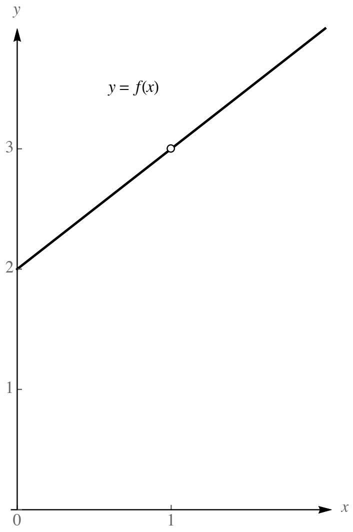
$f(5)$ does not exist. $\lim_{x\to 5^-} f(x) = 10$, $\lim_{x\to 5^+} f(x) = 10$, $\lim_{x\to 5} f(x) = 10$.

2.2.24



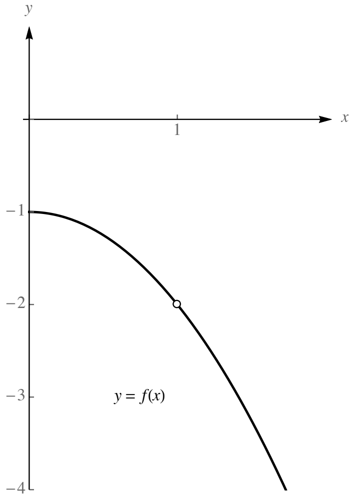
$f(100)$ does not exist. $\lim_{x\to 100^-} f(x) = 20$, $\lim_{x\to 100^+} f(x) = 20$, $\lim_{x\to 100} f(x) = 20$.

2.2.25



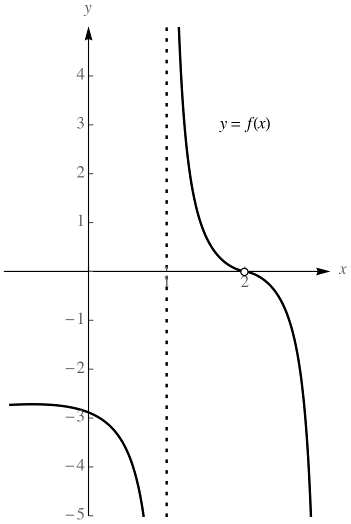
$f(1)$ does not exist. $\lim_{x\to 1^-} f(x) = 3$, $\lim_{x\to 1^+} f(x) = 3$, $\lim_{x\to 1} f(x) = 3$.

2.2.26



$f(1)$ does not exist. $\lim\limits_{x\to 1^-} f(x) = -2$, $\lim\limits_{x\to 1^+} f(x) = -2$, $\lim\limits_{x\to 1} f(x) = -2$.

2.2.27

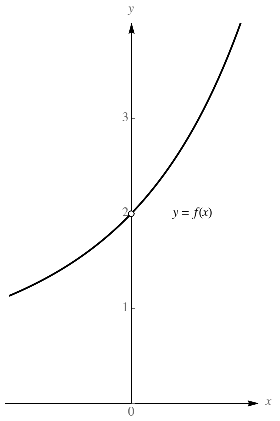


x	1.99	1.999	1.9999
$f(x) = \frac{x-2}{\ln\lvert x-2\rvert}$	0.0021715	0.00014476	0.000010857

x	2.0001	2.001	2.01
$f(x) = \frac{x-2}{\ln\lvert x-2\rvert}$	−0.000010857	−0.00014476	−0.0021715

From the graph and the table, the limit appears to be 0.

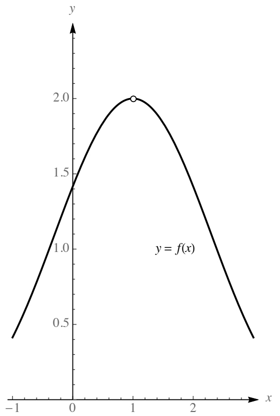
2.2.28



x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	1.8731	1.98673	1.9987	2.0013	2.0134	2.1403

From both the graph and the table, the limit appears to be 2.

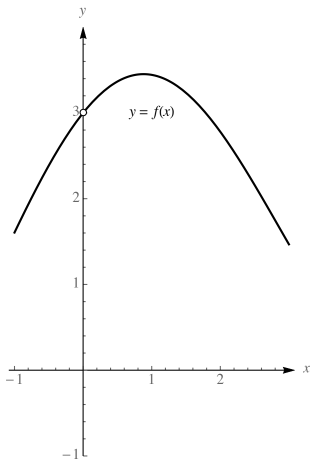
2.2.29



x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	1.993342	1.999933	1.999999	1.999999	1.999933	1.993342

From both the graph and the table, the limit appears to be 2.

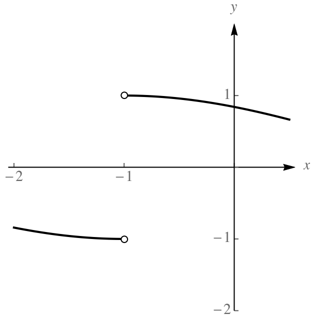
2.2.30



x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.8951	2.99	2.999	3.001	3.0099	3.0949

From both the graph and the table, the limit appears to be 3.

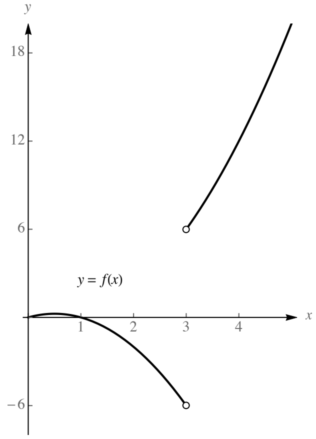
2.2.31



x	-1.1	-1.01	-1.001	-0.999	-0.99	-0.9
$f(x)$	-0.9983342	-0.9999833	-0.9999998	0.9999998	0.9999833	0.9983342

From both the graph and the table, it appears that the limit does not exist.

2.2.32



x	2.9	2.99	2.999	3.001	3.01	3.1
$g(x)$	-5.51	-5.9501	-5.995001	6.005001	6.0501	6.51

From both the graph and the table, it appears that the limit does not exist.

2.2.33

a. False. In fact $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3} = \lim\limits_{x\to 3}(x+3) = 6$.

b. False. For example, if $f(x) = \begin{cases} x^2 & \text{if } x \neq 0; \\ 5 & \text{if } x = 0 \end{cases}$ and if $a = 0$ then $f(a) = 5$ but $\lim\limits_{x\to a} f(x) = 0$.

c. False. For example, the limit in part a of this problem exists, even though the corresponding function is undefined at $a = 3$.

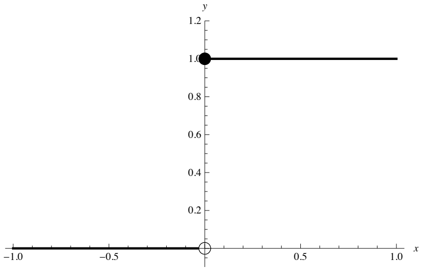
d. False. It is true that the limit of $\sqrt{x}$ as x approaches zero from the right is zero, but because the domain of $\sqrt{x}$ does not include any numbers to the left of zero, the two-sided limit doesn't exist.

e. True. Note that $\lim\limits_{x\to\pi/2}\cos x = 0$ and $\lim\limits_{x\to\pi/2}\sin x = 1$, so $\lim\limits_{x\to\pi/2}\dfrac{\cos x}{\sin x} = \dfrac{0}{1} = 0$.

2.2.34

a. Note that H is piecewise constant.

b. $\lim\limits_{x\to0^-} H(x) = 0$, $\lim\limits_{x\to0^+} H(x) = 1$, and so $\lim\limits_{x\to0} H(x)$ does not exist.

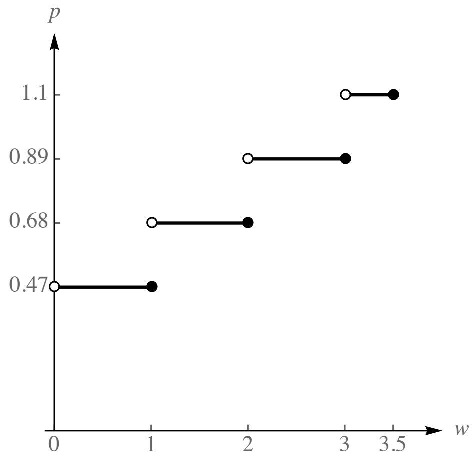


2.2.35

a. Note that the function is piecewise constant.

b. $\lim\limits_{w\to2.3} f(w) = .89$.

c. $\lim\limits_{w\to3^+} f(w) = 1.1$ corresponds to the fact that for any piece of mail that weighs slightly over 3 ounces, the postage will cost \$1.1 cents. $\lim\limits_{w\to3^-} f(w) = \$0.89$ corresponds to the fact that for any piece of mail that weighs slightly less than 3 ounces, the postage will cost 89 cents. Because the two one-sided limits are not equal, $\lim\limits_{w\to3} f(w)$ does not exist.



2.2.36

h	0.01	0.001	0.0001	−0.0001	−0.001	−0.01
$\frac{(1+2h)^{1/h}}{2e^{2+h}}$	0.48535	0.498504	0.49985	0.50015	0.501504	0.515367

$\lim\limits_{h\to0}\dfrac{(1+2h)^{1/h}}{2e^{2+h}} = \dfrac{1}{2}$.

2.2.37

x	1.37	1.47	1.57	1.58	1.68	1.78
$\frac{\cot 3x}{\cos x}$	3.44773	3.10016	3.00001	3.0008	3.11834	3.49316

$\lim\limits_{x\to\pi/2}\dfrac{\cot 3x}{\cos x} = 3$.

2.2.38

x	0.9	0.99	0.999	1.001	1.01	1.1
$\frac{9\sqrt{2x-x^4}-\sqrt[3]{x}}{1-x^{3/4}}$	2.29222	2.02691	2.00267	1.99734	1.97357	1.75541

$\lim\limits_{x\to1}\dfrac{18(\sqrt[3]{x}-1)}{x^3-1} = 2$.

2.2.39

x	0.99	0.999	0.9999	1.0001	1.001	1.01
$\frac{18(\sqrt[3]{x}-1)}{x^3-1}$	15.5803	15.9574	15.9957	16.0043	16.0427	16.4339

$$\lim_{x\to 1} \frac{9\sqrt{2x-x^4}-\sqrt[3]{x}}{1-x^{3/4}} = 16.$$

2.2.40

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$\frac{6^x-3^x}{x\ln 16}$	0.24614	0.249639	0.249964	0.250036	0.250362	0.25364

$$\lim_{x\to 0} \frac{6^x-3^x}{x\ln 16} = \frac{1}{4}.$$

2.2.41

h	0.01	0.001	0.0001	-0.0001	-0.001	-0.01
$\frac{\ln(1+h)}{h}$	0.995033	0.9995	0.99995	1.00005	1.0005	1.00503

$$\lim_{h\to 0} \frac{\ln(1+h)}{h} = 1.$$

2.2.42

h	0.01	0.001	0.0001	-0.0001	-0.001	-0.01
$\frac{4^h-1}{h\ln(h+2)}$	1.99954	1.99994	1.99999	2.00001	2.00006	2.00067

$$\lim_{h\to 0} \frac{4^h-1}{h\ln(h+2)} = 2.$$

2.2.43

a.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$
$f(x)=\sin(1/x)$	1	-1	1	-1	1	-1

If $x_n = \frac{2}{(2n+1)\pi}$, then $f(x_n) = (-1)^n$ where n is a non-negative integer.

b. As $x \to 0$, $1/x \to \infty$. So the values of $f(x)$ oscillate dramatically between -1 and 1.

c. $\lim\limits_{x\to 0} \sin(1/x)$ does not exist.

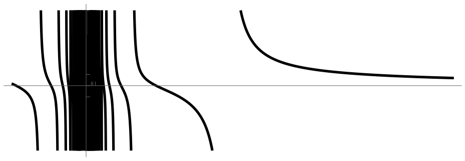
2.2.44

a.

x	$\frac{12}{\pi}$	$\frac{12}{3\pi}$	$\frac{12}{5\pi}$	$\frac{12}{7\pi}$	$\frac{12}{9\pi}$	$\frac{12}{11\pi}$
$f(x)=\tan(3/x)$	1	-1	1	-1	1	-1

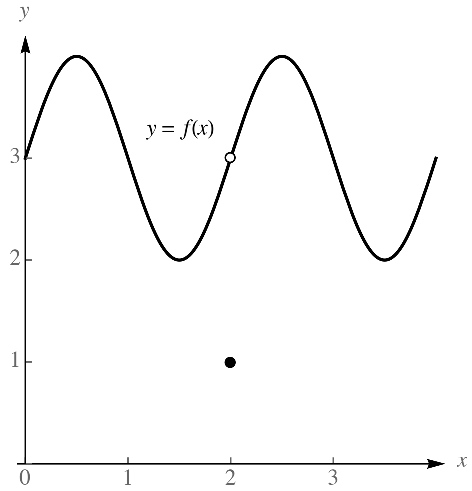
We have alternating 1's and -1's.

b. $\tan 3x$ alternates between 1 and -1 infinitely many times on $(0, h)$ for any $h > 0$.

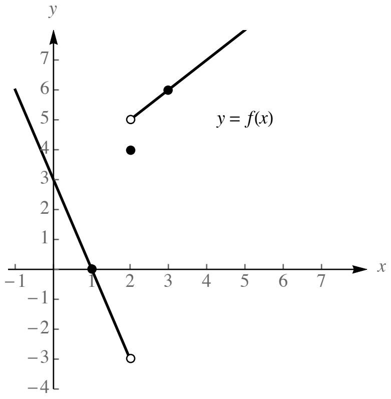


c. $\lim_{x\to 0} \tan(3/x)$ does not exist.

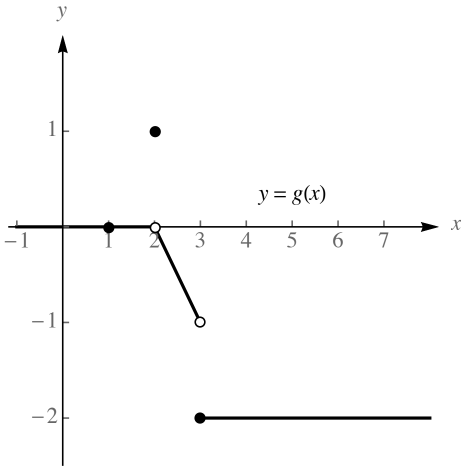
2.2.45



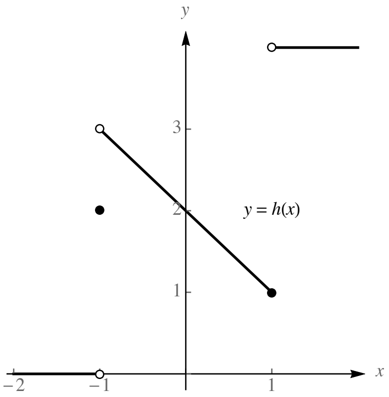
2.2.46



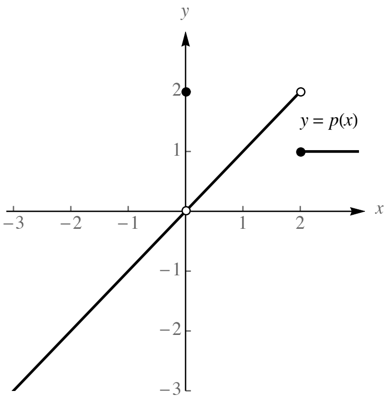
2.2.47



2.2.48



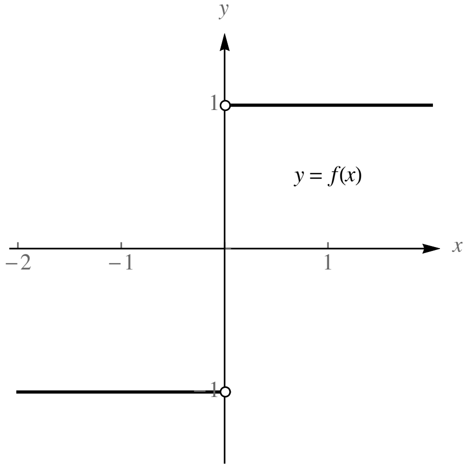
2.2.49



2.2.50

a. Note that $f(x) = \frac{|x|}{x}$ is undefined at 0, and $\lim_{x\to 0^-} f(x) = -1$ and $\lim_{x\to 0^+} f(x) = 1$.

b. $\lim_{x\to 0} f(x)$ does not exist, since the two one-side limits aren't equal.



2.2.51

a. $\lim_{x\to -1^-} \lfloor x\rfloor = -2$, $\lim_{x\to -1^+} \lfloor x\rfloor = -1$, $\lim_{x\to 2^-} \lfloor x\rfloor = 1$, $\lim_{x\to 2^+} \lfloor x\rfloor = 2$.

b. $\lim_{x\to 2.3^-} \lfloor x\rfloor = 2$, $\lim_{x\to 2.3^+} \lfloor x\rfloor = 2$, $\lim_{x\to 2.3} \lfloor x\rfloor = 2$.

c. In general, for an integer a, $\lim_{x\to a^-} \lfloor x\rfloor = a-1$ and $\lim_{x\to a^+} \lfloor x\rfloor = a$.

d. In general, if a is not an integer, $\lim_{x\to a^-} \lfloor x\rfloor = \lim_{x\to a^+} \lfloor x\rfloor = \lfloor a\rfloor$.

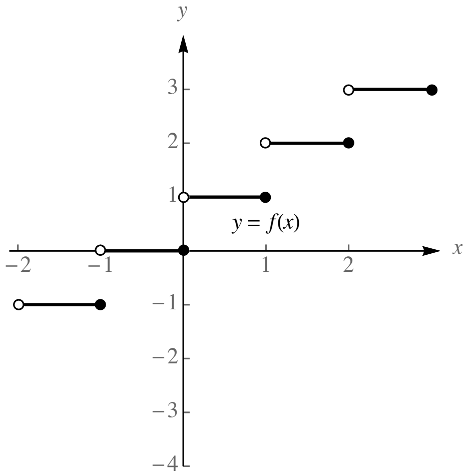
e. $\lim_{x\to a} \lfloor x\rfloor$ exists and is equal to $\lfloor a\rfloor$ for non-integers a.

2.2.52

a. Note that the graph is piecewise constant.

b. $\lim_{x\to 2^-} \lceil x\rceil = 2$, $\lim_{x\to 1^+} \lceil x\rceil = 2$, $\lim_{x\to 1.5} \lceil x\rceil = 2$.

c. $\lim_{x\to a} \lceil x\rceil$ exists and is equal to $\lceil a\rceil$ for non-integers a.



2.2.53

a. Because of the symmetry about the y axis, we must have $\lim_{x\to -2^+} f(x) = 8$.

b. Because of the symmetry about the y axis, we must have $\lim_{x\to -2^-} f(x) = 5$.

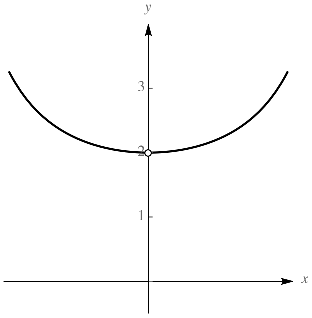
2.2.54

a. Because of the symmetry about the origin, we must have $\lim\limits_{x\to -2^+} g(x) = -8$.

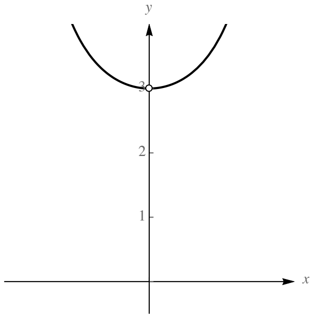
b. Because of the symmetry about the origin, we must have $\lim\limits_{x\to -2^-} g(x) = -5$.

2.2.55

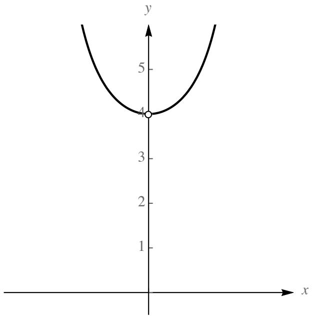
a.



$$\lim_{x\to 0}\frac{\tan 2x}{\sin x} = 2.$$



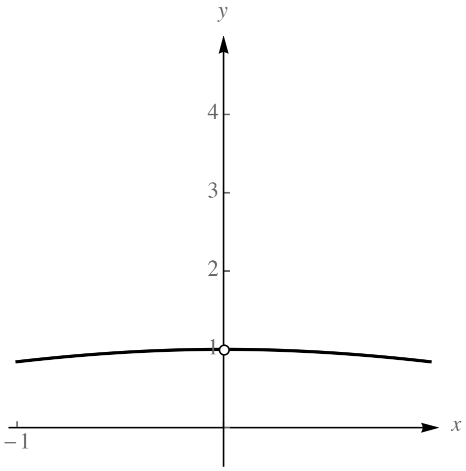
$$\lim_{x\to 0}\frac{\tan 3x}{\sin x} = 3.$$



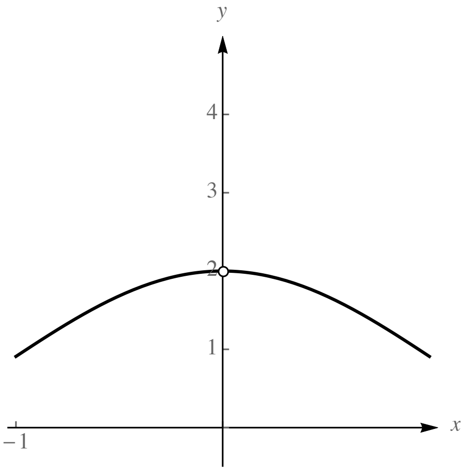
$$\lim_{x\to 0}\frac{\tan 4x}{\sin x} = 4.$$

b. It appears that $\lim\limits_{x\to 0}\dfrac{\tan(px)}{\sin x} = p$.

2.2.56

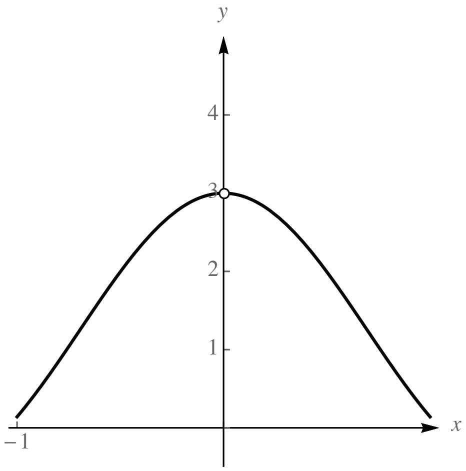


$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

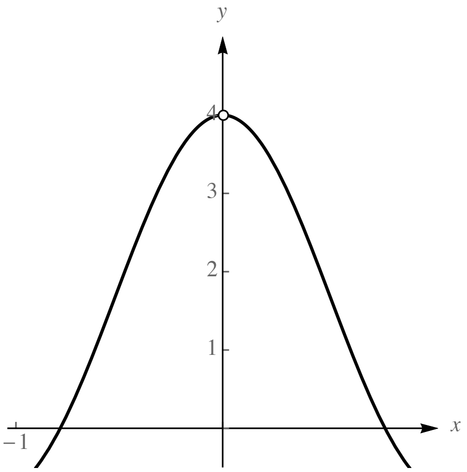


$$\lim_{x\to 0} \frac{\sin 2x}{x} = 2.$$

a.



$$\lim_{x\to 0} \frac{\sin 3x}{x} = 3.$$

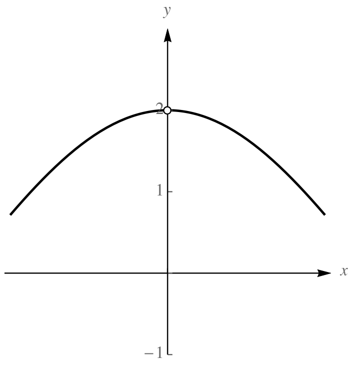


$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4.$$

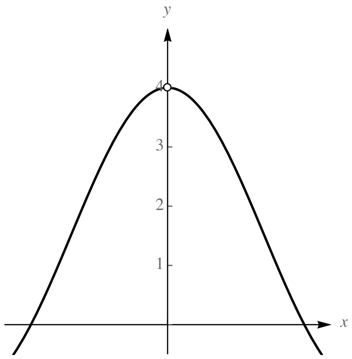
b. It appears that $\lim_{x\to 0} \frac{\sin(px)}{x} = p$.

2.2.57

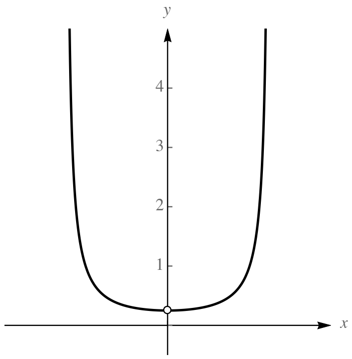
For $p = 8$ and $q = 2$, it appears that the limit is 4.



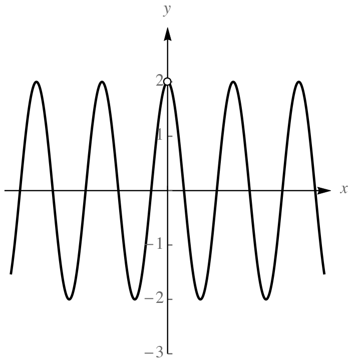
For $p = 12$ and $q = 3$, it appears that the limit is 4.



For $p = 4$ and $q = 16$, it appears that the limit is 1/4.



For $p = 100$ and $q = 50$, it appears that the limit is 2.



Conjecture: $\lim_{x\to 0} \dfrac{\sin px}{\sin qx} = \dfrac{p}{q}$.

2.3 Techniques for Computing Limits

2.3.1 If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $\lim_{x\to a} p(x) = \lim_{x\to a}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$
$= a_n(\lim_{x\to a} x)^n + a_{n-1}(\lim_{x\to a} x)^{n-1} + \cdots + a_1 \lim_{x\to a} x + \lim_{x\to a} a_0$
$= a_n a^n + a_{n-1}a^{n-1} + \cdots + a_1 a + a_0. = p(a)$.

2.3.2 $\lim_{x\to 1}(x^3 + 3x^2 - 3x + 1) = 1 + 3 - 3 + 1 = 2$.

2.3.3 For a rational function $r(x)$, we have $\lim_{x\to a} r(x) = r(a)$ exactly for those numbers a which are in the domain of r. (Which are those for which the denominator isn't zero.)

2.3.4 $\lim_{x\to 4}\left(\dfrac{x^2 - 4x - 1}{3x - 1}\right) = \dfrac{16 - 16 - 1}{12 - 1} = -\dfrac{1}{11}$.

2.3.5 Because $\frac{x^2-7x+12}{x-3} = \frac{(x-3)(x-4)}{x-3} = x - 4$ (for $x \neq 3$), we can see that the graphs of these two functions are the same except that one is undefined at $x = 3$ and the other is a straight line that is defined everywhere. Thus the function $\frac{x^2-7x+12}{x-3}$ is a straight line except that it has a "hole" at $(3, -1)$. The two functions have the same limit as $x \to 3$, namely $\lim_{x\to 3} \dfrac{x^2 - 7x + 12}{x - 3} = \lim_{x\to 3}(x - 4) = -1$.

2.3.6 $\lim_{x\to 5} \dfrac{4x^2 - 100}{x - 5} = \lim_{x\to 5} \dfrac{4(x-5)(x+5)}{x - 5} = \lim_{x\to 5} 4(x + 5) = 40$.

2.3.7 $\lim_{x\to 1} 4f(x) = 4 \lim_{x\to 1} f(x) = 4 \cdot 8 = 32$. This follows from the Constant Multiple Law.

2.3.8 $\lim_{x\to 1} \dfrac{f(x)}{h(x)} = \dfrac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} h(x)} = \dfrac{8}{2} = 4$. This follows from the Quotient Law.

2.3.9 $\lim_{x\to 1}(f(x) - g(x)) = \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 8 - 3 = 5$. This follows from the Difference Law.

2.3.10 $\lim_{x\to 1} f(x)h(x) = \lim_{x\to 1} f(x) \cdot \lim_{x\to 1} h(x) = 8 \cdot 2 = 16$. This follows from the Product Law.

2.3.11 $\lim_{x\to 1} \dfrac{f(x)}{g(x) - h(x)} = \dfrac{\lim_{x\to 1} f(x)}{\lim_{x\to 1}[g(x) - h(x)]} = \dfrac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x) - \lim_{x\to 1} h(x)} = \dfrac{8}{3 - 2} = 8$. This follows from the Quotient and Difference Laws.

2.3.12 $\lim_{x\to1}\sqrt[3]{f(x)g(x)+3} = \sqrt[3]{\lim_{x\to1}(f(x)g(x)+3)} = \sqrt[3]{\lim_{x\to1}f(x)\cdot\lim_{x\to1}g(x)+\lim_{x\to1}3} = \sqrt[3]{8\cdot3+3} = \sqrt[3]{27} = 3$. This follows form the Root, Product, Sum and Constant Laws.

2.3.13 $\lim_{x\to1}f(x)^{2/3} = \left(\lim_{x\to1}f(x)\right)^{2/3} = 8^{2/3} = 2^2 = 4$. This follows from the Root and Power Laws.

2.3.14 If $p(x)$ is a polynomial, then $\lim_{x\to a^-}p(x) = \lim_{x\to a^+}p(x) = p(a)$.

2.3.15 $\lim_{x\to0}g(x) = \lim_{x\to0}(2x+1) = 1$, while $g(0) = 5$.

2.3.16 $\lim_{x\to3^-}f(x) = \lim_{x\to3^-}4 = 4$, and $\lim_{x\to3^+}f(x) = \lim_{x\to3^+}(x+2) = 5$. Because the two one-sided limits differ, the two-sided limit doesn't exist.

2.3.17 If p and q are polynomials then $\lim_{x\to0}\frac{p(x)}{q(x)} = \frac{\lim_{x\to0}p(x)}{\lim_{x\to0}q(x)} = \frac{p(0)}{q(0)}$. Because this quantity is given to be equal to 10, we have $\frac{p(0)}{2} = 10$, so $p(0) = 20$.

2.3.18 By a direct application of the squeeze theorem, $\lim_{x\to2}g(x) = 5$.

2.3.19 $\lim_{x\to4}(3x-7) = 3\lim_{x\to4}x - 7 = 3\cdot4-7 = 5$.

2.3.20 $\lim_{x\to1}(-2x+5) = -2\lim_{x\to1}x+5 = -2\cdot1+5 = 3$.

2.3.21 $\lim_{x\to-9}(5x) = 5\lim_{x\to-9}x = 5\cdot-9 = -45$.

2.3.22 $\lim_{x\to6}4 = 4$.

2.3.23 $\lim_{x\to1}(2x^3-3x^2+4x+5) = \lim_{x\to1}2x^3 - \lim_{x\to1}3x^2 + \lim_{x\to1}4x + \lim_{x\to1}5 = 2(\lim_{x\to1}x)^3 - 3(\lim_{x\to1}x)^2 + 4(\lim_{x\to1}x) + 5 = 2(1)^3 - 3(1)^2 + 4\cdot1 + 5 = 8$.

2.3.24 $\lim_{t\to-2}(t^2+5t+7) = \lim_{t\to-2}t^2 + \lim_{t\to-2}5t + \lim_{t\to-2}7 = \left(\lim_{t\to-2}t\right)^2 + 5\lim_{t\to-2}t + 7 = (-2)^2 + 5\cdot(-2) + 7 = 1$.

2.3.25 $\lim_{x\to1}\frac{5x^2+6x+1}{8x-4} = \frac{\lim_{x\to1}(5x^2+6x+1)}{\lim_{x\to1}(8x-4)} = \frac{5(\lim_{x\to1}x)^2 + 6\lim_{x\to1}x + \lim_{x\to1}1}{8\lim_{x\to1}x - \lim_{x\to1}4} = \frac{5(1)^2+6\cdot1+1}{8\cdot1-4} = 3$.

2.3.26 $\lim_{t\to3}\sqrt[3]{t^2-10} = \sqrt[3]{\lim_{t\to3}(t^2-10)} = \sqrt[3]{\lim_{t\to3}t^2 - \lim_{t\to3}10} = \sqrt[3]{\left(\lim_{t\to3}t\right)^2 - 10} = \sqrt[3]{(3)^2-10} = -1$.

2.3.27 $\lim_{p\to2}\frac{3p}{\sqrt{4p+1}-1} = \frac{\lim_{p\to2}3p}{\lim_{p\to2}(\sqrt{4p+1}-1)} = \frac{3\lim_{p\to2}p}{\lim_{p\to2}\sqrt{4p+1} - \lim_{p\to2}1} = \frac{3\cdot2}{\sqrt{\lim_{p\to2}(4p+1)}-1} = \frac{6}{3-1} = 3$.

2.3.28 $\lim_{x\to2}(x^2-x)^5 = \left(\lim_{x\to2}(x^2-x)\right)^5 = \left(\lim_{x\to2}x^2 - \lim_{x\to2}x\right)^5 = (4-2)^5 = 32$.

2.3.29 $\lim_{x\to3}-\frac{5x}{\sqrt{4x-3}} = \frac{\lim_{x\to3}-5x}{\lim_{x\to3}\sqrt{4x-3}} = \frac{-5\lim_{x\to3}x}{\sqrt{\lim_{x\to3}(4x-3)}} = -\frac{5\cdot3}{\sqrt{4\lim_{x\to3}x - \lim_{x\to3}3}} = -\frac{15}{\sqrt{4\cdot3-3}} = -5$.

2.3.30

$$\lim_{h\to0}\frac{3}{\sqrt{16+3h}+4}=\frac{\lim\limits_{h\to0}3}{\lim\limits_{h\to0}(\sqrt{16+3h}+4)}=\frac{3}{\sqrt{\lim\limits_{h\to0}(16+3h)}+\lim\limits_{h\to0}4}=\frac{3}{\sqrt{\lim\limits_{h\to0}16+\lim\limits_{h\to0}3h}+4}$$
$$=\frac{3}{\sqrt{16+3\cdot0}+4}=\frac{3}{4+4}=\frac{3}{8}.$$

2.3.31 $\lim\limits_{x\to2}(5x-6)^{3/2}=(5\cdot2-6)^{3/2}=4^{3/2}=2^3=8.$

2.3.32 $\lim\limits_{h\to0}\dfrac{100}{(10h-1)^{11}+2}=\dfrac{100}{(-1)^{11}+2}=\dfrac{100}{1}=100.$

2.3.33 $\lim\limits_{x\to1}\dfrac{x^2-1}{x-1}=\lim\limits_{x\to1}\dfrac{(x+1)(x-1)}{x-1}=\lim\limits_{x\to1}(x+1)=2.$

2.3.34 $\lim\limits_{x\to3}\dfrac{x^2-2x-3}{x-3}=\lim\limits_{x\to3}\dfrac{(x-3)(x+1)}{x-3}=\lim\limits_{x\to3}(x+1)=4.$

2.3.35 $\lim\limits_{x\to4}\dfrac{x^2-16}{4-x}=\lim\limits_{x\to4}\dfrac{(x+4)(x-4)}{-(x-4)}=\lim\limits_{x\to4}[-(x+4)]=-8.$

2.3.36 $\lim\limits_{t\to2}\dfrac{3t^2-7t+2}{2-t}=\lim\limits_{t\to2}\dfrac{(t-2)(3t-1)}{-(t-2)}=\lim\limits_{t\to2}[-(3t-1)]=-5.$

2.3.37 $\lim\limits_{x\to b}\dfrac{(x-b)^{50}-x+b}{x-b}=\lim\limits_{x\to b}\dfrac{(x-b)^{50}-(x-b)}{x-b}=\lim\limits_{x\to b}\dfrac{(x-b)((x-b)^{49}-1)}{x-b}=$ $\lim\limits_{x\to b}[(x-b)^{49}-1]=-1.$

2.3.38 $\lim\limits_{x\to-b}\dfrac{(x+b)^7+(x+b)^{10}}{4(x+b)}=\lim\limits_{x\to-b}\dfrac{(x+b)((x+b)^6+(x+b)^9)}{4(x+b)}=\lim\limits_{x\to-b}\dfrac{(x+b)^6+(x+b)^9}{4}=\dfrac{0}{4}=0.$

2.3.39

$$\lim_{x\to-1}\frac{(2x-1)^2-9}{x+1}=\lim_{x\to-1}\frac{(2x-1-3)(2x-1+3)}{x+1}$$
$$=\lim_{x\to-1}\frac{2(x-2)2(x+1)}{x+1}=\lim_{x\to-1}4(x-2)=4\cdot(-3)=-12.$$

2.3.40

$$\lim_{h\to0}\frac{\frac{1}{5+h}-\frac{1}{5}}{h}=\lim_{h\to0}\frac{\left(\frac{1}{5+h}-\frac{1}{5}\right)\cdot5\cdot(5+h)}{h\cdot5\cdot(5+h)}$$
$$=\lim_{h\to0}\frac{5-(5+h)}{5h(5+h)}=\lim_{h\to0}-\frac{h}{5h(5+h)}=\lim_{h\to0}-\frac{1}{5(5+h)}=-\frac{1}{25}.$$

2.3.41 $\lim\limits_{x\to9}\dfrac{\sqrt{x}-3}{x-9}=\lim\limits_{x\to9}\dfrac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}=\lim\limits_{x\to9}\dfrac{x-9}{(x-9)(\sqrt{x}+3)}=\lim\limits_{x\to9}\dfrac{1}{\sqrt{x}+3}=\dfrac{1}{6}.$

2.3.42 $\lim\limits_{w\to1}\left(\dfrac{1}{w^2-w}-\dfrac{1}{w-1}\right)=\lim\limits_{w\to1}\left(\dfrac{1}{w(w-1)}-\dfrac{w}{w(w-1)}\right)=\lim\limits_{w\to1}\dfrac{1-w}{w(w-1)}=-\lim\limits_{w\to1}\dfrac{1}{w}=-1.$

2.3.43

$$\lim_{t\to5}\left(\frac{1}{t^2-4t-5}-\frac{1}{6(t-5)}\right)=\lim_{t\to5}\left(\frac{1}{(t-5)(t+1)}-\frac{1}{6(t-5)}\right)$$
$$=\lim_{t\to5}\left(\frac{6}{6(t-5)(t+1)}-\frac{t+1}{6(t-5)(t+1)}\right)$$
$$=\lim_{t\to5}\frac{5-t}{6(t-5)(t+1)}=-\lim_{t\to5}\frac{1}{6(t+1)}=-\frac{1}{36}.$$

2.3.44 Expanding gives

$$\lim_{t\to 3}\left(\left(4t-\frac{2}{t-3}\right)\left(6+t-t^2\right)\right)=\lim_{t\to 3}\left(4t(6+t-t^2)-\frac{2(6+t-t^2)}{t-3}\right)$$
$$=\lim_{t\to 3}\left(4t(6+t-t^2)-\frac{2(3-t)(2+t)}{t-3}\right).$$

Now because $t-3=-(3-t)$, we have

$$\lim_{t\to 3}\left(4t(6+t-t^2)+2(2+t)\right)=12(6+3-9)+2(2+3)=10.$$

2.3.45 $\displaystyle\lim_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}}=\lim_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}=\lim_{x\to a}\frac{(x-a)(\sqrt{x}+\sqrt{a})}{x-a}=\lim_{x\to a}(\sqrt{x}+\sqrt{a})=2\sqrt{a}.$

2.3.46 $\displaystyle\lim_{x\to a}\frac{x^2-a^2}{\sqrt{x}-\sqrt{a}}=\lim_{x\to a}\frac{x^2-a^2}{\sqrt{x}-\sqrt{a}}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}=\lim_{x\to a}\frac{(x-a)(x+a)(\sqrt{x}+\sqrt{a})}{x-a}=$
$(a+a)(\sqrt{a}+\sqrt{a})=4a^{3/2}$.

2.3.47

$$\lim_{h\to 0}\frac{\sqrt{16+h}-4}{h}=\lim_{h\to 0}\frac{(\sqrt{16+h}-4)(\sqrt{16+h}+4)}{h(\sqrt{16+h}+4)}=\lim_{h\to 0}\frac{(16+h)-16}{h(\sqrt{16+h}+4)}$$
$$=\lim_{h\to 0}\frac{h}{h(\sqrt{16+h}+4)}=\lim_{h\to 0}\frac{1}{(\sqrt{16+h}+4)}=\frac{1}{8}.$$

2.3.48 $\displaystyle\lim_{x\to c}\frac{x^2-2cx+c^2}{x-c}=\lim_{x\to c}\frac{(x-c)^2}{x-c}=\lim_{x\to c}x-c=c-c=0.$

2.3.49 $\displaystyle\lim_{x\to 4}\frac{\frac{1}{x}-\frac{1}{4}}{x-4}=\lim_{x\to 4}\frac{\frac{4-x}{4x}}{x-4}=\lim_{x\to 4}\frac{4-x}{4x(x-4)}=-\lim_{x\to 4}\frac{1}{4x}=-\frac{1}{16}.$

2.3.50

$$\lim_{x\to 3}\frac{\frac{1}{x^2+2x}-\frac{1}{15}}{x-3}=\lim_{x\to 3}\frac{\frac{15-(x^2+2x)}{15(x^2+2x)}}{x-3}=\lim_{x\to 3}\frac{15-(x^2+2x)}{15(x^2+2x)(x-3)}=\lim_{x\to 3}\frac{15-2x-x^2}{15(x^2+2x)(x-3)}$$
$$=\lim_{x\to 3}\frac{(3-x)(5+x)}{15(x^2+2x)(x-3)}=\lim_{x\to 3}-\frac{(5+x)}{15(x^2+2x)}=-\frac{8}{225}.$$

2.3.51

$$\lim_{x\to 1}\frac{\sqrt{10x-9}-1}{x-1}=\lim_{x\to 1}\frac{(\sqrt{10x-9}-1)(\sqrt{10x-9}+1)}{(x-1)(\sqrt{10x-9}+1)}=\lim_{x\to 1}\frac{(10x-9)-1}{(x-1)(\sqrt{10x-9}+1)}$$
$$=\lim_{x\to 1}\frac{10(x-1)}{(x-1)(\sqrt{10x-9}+1)}=\lim_{x\to 1}\frac{10}{(\sqrt{10x-9}+1)}=\frac{10}{2}=5.$$

2.3.52 $\displaystyle\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{2}{x^2-2x}\right)=\lim_{x\to 2}\left(\frac{x}{x(x-2)}-\frac{2}{x(x-2)}\right)=\lim_{x\to 2}\left(\frac{x-2}{x(x-2)}\right)=\lim_{x\to 2}\frac{1}{x}=\frac{1}{2}.$

2.3.53 $\displaystyle\lim_{h\to 0}\frac{(5+h)^2-25}{h}=\lim_{h\to 0}\frac{25+10h+h^2-25}{h}=\lim_{h\to 0}\frac{h(10+h)}{h}=\lim_{h\to 0}(10+h)=10.$

2.3.54 We have

$$\lim_{w\to -k}\frac{w^2+5kw+4k^2}{w^2+kw}=\lim_{w\to -k}\frac{(w+4k)(w+k)}{(w)(w+k)}=\lim_{w\to -k}\frac{w+4k}{w}=\frac{-k+4k}{-k}=-3.$$

If $k=0$, we have $\displaystyle\lim_{w\to -k}\frac{w^2+5kw+4k^2}{w^2+kw}=\lim_{w\to 0}\frac{w^2}{w^2}=1.$

2.3.55 $\lim\limits_{x\to 1}\dfrac{x-1}{\sqrt{x}-1}=\lim\limits_{x\to 1}\dfrac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}=\lim\limits_{x\to 1}\dfrac{(x-1)(\sqrt{x}+1)}{x-1}=\lim\limits_{x\to 1}(\sqrt{x}+1)=2.$

2.3.56

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{\sqrt{4x+5}-3}&=\lim_{x\to 1}\frac{(x-1)(\sqrt{4x+5}+3)}{(\sqrt{4x+5}-3)(\sqrt{4x+5}+3)}=\lim_{x\to 1}\frac{(x-1)(\sqrt{4x+5}+3)}{4x+5-9}\\&=\lim_{x\to 1}\frac{(x-1)(\sqrt{4x+5}+3)}{4(x-1)}=\lim_{x\to 1}\frac{(\sqrt{4x+5}+3)}{4}=\frac{6}{4}=\frac{3}{2}.\end{aligned}$$

2.3.57

$$\begin{aligned}\lim_{x\to 4}\frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}&=\lim_{x\to 4}\frac{3(x-4)(\sqrt{x+5})(3+\sqrt{x+5})}{(3-\sqrt{x+5})(3+\sqrt{x+5})}=\lim_{x\to 4}\frac{3(x-4)(\sqrt{x+5})(3+\sqrt{x+5})}{9-(x+5)}\\&=\lim_{x\to 4}\frac{3(x-4)(\sqrt{x+5})(3+\sqrt{x+5})}{-(x-4)}\\&=\lim_{x\to 4}[-3(\sqrt{x+5})(3+\sqrt{x+5})]=(-3)(3)(3+3)=-54\end{aligned}$$

2.3.58 Assume $c\neq 0$.

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sqrt{cx+1}-1}&=\lim_{x\to 0}\frac{x(\sqrt{cx+1}+1)}{(\sqrt{cx+1}-1)(\sqrt{cx+1}+1)}=\lim_{x\to 0}\frac{x(\sqrt{cx+1}+1)}{(cx+1)-1}\\&=\lim_{x\to 0}\frac{x(\sqrt{cx+1}+1)}{cx}=\lim_{x\to 0}\frac{(\sqrt{cx+1}+1)}{c}=\frac{2}{c}.\end{aligned}$$

2.3.59 $\lim\limits_{x\to 0}x\cos x=0\cdot 1=0.$

2.3.60 $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sin x}=\lim\limits_{x\to 0}\dfrac{2\sin x\cos x}{\sin x}=\lim\limits_{x\to 0}2\cos x=2.$

2.3.61 $\lim\limits_{x\to 0}\dfrac{1-\cos x}{\cos^2 x-3\cos x+2}=\lim\limits_{x\to 0}\dfrac{1-\cos x}{(\cos x-2)(\cos x-1)}=-\lim\limits_{x\to 0}\dfrac{1}{\cos x-2}=-\dfrac{1}{1-2}=1.$

2.3.62 $\lim\limits_{x\to 0}\dfrac{\cos x-1}{\cos^2 x-1}=\lim\limits_{x\to 0}\dfrac{\cos x-1}{(\cos x-1)(\cos x+1)}=\lim\limits_{x\to 0}\dfrac{1}{\cos x+1}=\dfrac{1}{2}.$

2.3.63 $\lim\limits_{x\to 0^-}\dfrac{x^2-x}{|x|}=\lim\limits_{x\to 0^-}\dfrac{x(x-1)}{-x}=-\lim\limits_{x\to 0^-}(x-1)=1.$

2.3.64 $\lim\limits_{w\to 3^-}\dfrac{|w-3|}{w^2-7w+12}=\lim\limits_{w\to 3^-}\dfrac{3-w}{(w-3)(w-4)}=-\lim\limits_{w\to 3^-}\dfrac{1}{w-4}=1.$

2.3.65 $\lim\limits_{t\to 2^+}\dfrac{|2t-4|}{t^2-4}=\lim\limits_{t\to 2^+}\dfrac{2(t-2)}{(t-2)(t+2)}=\dfrac{2}{4}=\dfrac{1}{2}.$

2.3.66 $\lim\limits_{x\to -1^-}g(x)=\lim\limits_{x\to -1^-}\dfrac{x^2-1}{x+1}=\lim\limits_{x\to -1^-}\dfrac{(x+1)(x-1)}{x+1}=\lim\limits_{x\to -1^-}(x-1)=-2$. Also, $\lim\limits_{x\to -1^+}g(x)=\lim\limits_{x\to -1^+}(-2)=-2$. Therefore, $\lim\limits_{x\to -1}g(x)=-2$.

2.3.67 $\lim\limits_{x\to 3^+}\dfrac{x-3}{|x-3|}=\lim\limits_{x\to 3^+}\dfrac{x-3}{x-3}=\lim\limits_{x\to 3^+}1=1$. On the other hand, $\lim\limits_{x\to 3^-}\dfrac{x-3}{|x-3|}=\lim\limits_{x\to 3^-}\dfrac{x-3}{3-x}=\lim\limits_{x\to 3^-}(-1)=-1$. Therefore, $\lim\limits_{x\to 3}\dfrac{x-3}{|x-3|}$ does not exist.

2.3.68 $\lim_{x\to 5^+} \frac{|x-5|}{x^2-25} = \lim_{x\to 5^+} \frac{x-5}{(x-5)(x+5)} = \lim_{x\to 5^+} \frac{1}{x+5} = \frac{1}{10}$. On the other hand, $\lim_{x\to 5^-} \frac{|x-5|}{x^2-25} = \lim_{x\to 5^-} \frac{5-x}{(x-5)(x+5)} = -\lim_{x\to 5^+} \frac{1}{x+5} = -\frac{1}{10}$. Therefore, $\lim_{x\to 5} \frac{|x-5|}{x^2-25}$ does not exist.

2.3.69 Because the domain of $f(x) = \frac{x^3+1}{\sqrt{x-1}}$ is the interval $(1,\infty)$, the limit doesn't exist.

2.3.70 $\lim_{x\to 1^+} \frac{x-1}{\sqrt{x^2-1}} = \lim_{x\to 1^+} \frac{x-1}{(x-1)^{1/2}(x+1)^{1/2}} = \lim_{x\to 1^+} \frac{(x-1)^{1/2}}{(x+1)^{1/2}} = \frac{0}{\sqrt{2}} = 0.$

2.3.71

a. False. For example, if $f(x) = \begin{cases} x & \text{if } x \neq 1; \\ 4 & \text{if } x = 1, \end{cases}$ then $\lim_{x\to 1} f(x) = 1$ but $f(1) = 4$.

b. False. For example, if $f(x) = \begin{cases} x+1 & \text{if } x \leq 1; \\ x-6 & \text{if } x > 1, \end{cases}$ then $\lim_{x\to 1^-} f(x) = 2$ but $\lim_{x\to 1^+} f(x) = -5$.

c. False. For example, if $f(x) = \begin{cases} x & \text{if } x \neq 1; \\ 4 & \text{if } x = 1, \end{cases}$ and $g(x) = 1$, then f and g both have limit 1 as $x \to 1$, but $f(1) = 4 \neq g(1)$.

d. False. For example $\lim_{x\to 2} \frac{x^2-4}{x-2}$ exists and is equal to 4.

e. False. For example, it would be possible for the domain of f to be $[1,\infty)$, so that the one-sided limit exists but the two-sided limit doesn't even make sense. This would be true, for example, if $f(x) = x-1$.

2.3.72

a. $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} (5x-15) = 5.$

b. $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} \sqrt{6x+1} = 5.$

c. Because the two one-sided limits both exist and are equal to 5, $\lim_{x\to 4} g(x) = 5$.

2.3.73

a. $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^2+1) = (-1)^2 + 1 = 2.$

b. $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \sqrt{x+1} = \sqrt{-1+1} = 0.$

c. Because the two one-sided limits differ, $\lim_{x\to -1} f(x)$ does not exist.

2.3.74

a. $\lim_{x\to -5^-} f(x) = \lim_{x\to -5^-} 0 = 0.$

b. $\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+} \sqrt{25-x^2} = \sqrt{25-25} = 0.$

c. $\lim_{x\to -5} f(x) = 0.$

d. $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \sqrt{25-x^2} = \sqrt{25-25} = 0.$

e. $\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} 3x = 15.$

f. $\lim_{x\to 5} f(x)$ does not exist.

2.3.75

a. $\lim\limits_{x\to 2^+} \sqrt{x-2} = \sqrt{2-2} = 0.$

b. The domain of $f(x) = \sqrt{x-2}$ is $[2, \infty)$. Thus, any question about this function that involves numbers less than 2 doesn't make any sense, because those numbers aren't in the domain of f.

2.3.76

a. Note that the domain of $f(x) = \sqrt{\frac{x-3}{2-x}}$ is $(2, 3]$. $\lim\limits_{x\to 3^-} \sqrt{\dfrac{x-3}{2-x}} = 0.$

b. Because the numbers to the right of 3 aren't in the domain of this function, the limit as $x \to 3^+$ of this function doesn't make any sense.

2.3.77 $\lim\limits_{x\to 10} E(x) = \lim\limits_{x\to 10} \dfrac{4.35}{x\sqrt{x^2+0.01}} = \dfrac{4.35}{10\sqrt{100.01}} \approx 0.0435$ N/C.

2.3.78 $\lim\limits_{t\to 200^-} d(t) = \lim\limits_{t\to 200^-} (3 - 0.015t)^2 = (3 - (0.015)(200))^2 = (3-3)^2 = 0$. As time approaches 200 seconds, the depth of the water in the tank is approaching 0.

2.3.79 $\lim\limits_{S\to 0^+} r(S) = \lim\limits_{S\to 0^+} (1/2)\left(\sqrt{100 + \dfrac{2S}{\pi}} - 10\right) = 0.$

The radius of the circular cylinder approaches zero as the surface area approaches zero.

2.3.80

a. $L(c/2) = L_0\sqrt{1 - \frac{(c/2)^2}{c^2}} = L_0\sqrt{1-(1/4)} = \sqrt{3}L_0/2.$

b. $L(3c/4) = L_0\sqrt{1-(1/c^2)(3c/4)^2} = L_0\sqrt{1-(9/16)} = \sqrt{7}L_0/4.$

c. It appears that that the observed length L of the ship decreases as the ship speed increases.

d. $\lim\limits_{x\to c^-} L_0\sqrt{1-(\nu^2/c^2)} = L_0 \cdot 0 = 0$. As the speed of the ship approaches the speed of light, the observed length of the ship shrinks to 0.

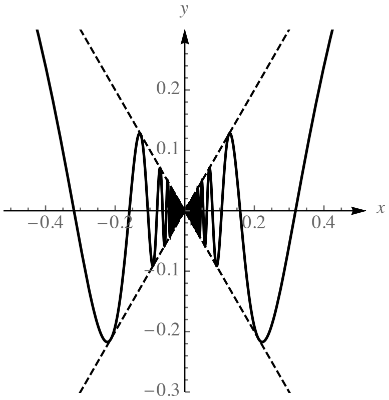
2.3.81

a. The statement we are trying to prove can be stated in cases as follows: For $x > 0$, $-x \le x\sin(1/x) \le x$, and for $x < 0$, $x \le x\sin(1/x) \le -x$.

Now for all $x \ne 0$, note that $-1 \le \sin(1/x) \le 1$ (because the range of the sine function is $[-1, 1]$). We will consider the two cases $x > 0$ and $x < 0$ separately, but in each case, we will multiply this inequality through by x, switching the inequalities for the $x < 0$ case.

For $x > 0$ we have $-x \le x\sin(1/x) \le x$, and for $x < 0$ we have $-x \ge x\sin(1/x) \ge x$, which are exactly the statements we are trying to prove.

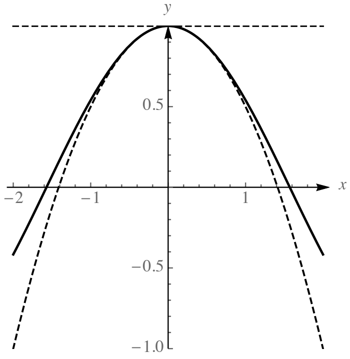
b.



c. Because $\lim\limits_{x\to 0} -|x| = \lim\limits_{x\to 0} |x| = 0$, and because $-|x| \le x\sin(1/x) \le |x|$, the Squeeze Theorem assures us that $\lim\limits_{x\to 0}[x\sin(1/x)] = 0$ as well.

2.3.82

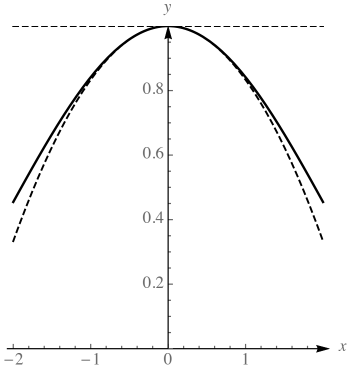
a.



b. Note that $\lim_{x\to 0}\left[1-\frac{x^2}{2}\right] = 1 = \lim_{x\to 0} 1$. So because $1-\frac{x^2}{2} \le \cos x \le 1$, the squeeze theorem assures us that $\lim_{x\to 0}\cos x = 1$ as well.

2.3.83

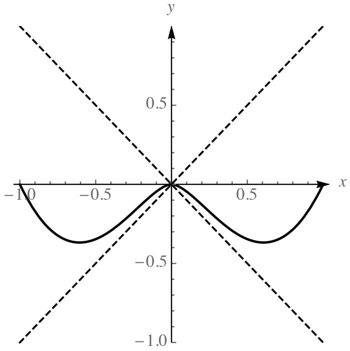
a.



b. Note that $\lim_{x\to 0}\left[1-\frac{x^2}{6}\right] = 1 = \lim_{x\to 0} 1$. So because $1-\frac{x^2}{6} \le \frac{\sin x}{x} \le 1$, the squeeze theorem assures us that $\lim_{x\to 0}\frac{\sin x}{x} = 1$ as well.

2.3.84

a.



b. Note that $\lim_{x\to 0}(-|x|) = 0 = \lim_{x\to 0}|x|$. So because $-|x| \le x^2\ln x^2 \le |x|$, the squeeze theorem assures us that $\lim_{x\to 0}(x^2\ln x^2) = 0$ as well.

2.3.85 Using the definition of $|x|$ given, we have $\lim_{x\to 0^-}|x| = \lim_{x\to 0^-}(-x) = -0 = 0$. Also, $\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} x = 0$. Because the two one-sided limits are both 0, we also have $\lim_{x\to 0}|x| = 0$.

2.3.86

If $a > 0$, then for x near a, $|x| = x$. So in this case, $\lim\limits_{x\to a} |x| = \lim\limits_{x\to a} x = a = |a|$.

If $a < 0$, then for x near a, $|x| = -x$. So in this case, $\lim\limits_{x\to a} |x| = \lim\limits_{x\to a}(-x) = -a = |a|$, (because $a < 0$).

If $a = 0$, we have already seen in a previous problem that $\lim\limits_{x\to 0} |x| = 0 = |0|$.

Thus in all cases, $\lim\limits_{x\to a} |x| = |a|$.

2.3.87 $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \dfrac{x^2-5x+6}{x-3} = \lim\limits_{x\to 3} \dfrac{(x-3)(x-2)}{x-3} = \lim\limits_{x\to 3}(x-2) = 1$. So $a = 1$.

2.3.88 In order for $\lim\limits_{x\to 2} f(x)$ to exist, we need the two one-sided limits to exist and be equal. We have $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (3x+b) = 6+b$, and $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x-2) = 0$. So we need $6+b=0$, so we require that $b = -6$. Then $\lim\limits_{x\to 2} f(x) = 0$.

2.3.89 In order for $\lim\limits_{x\to -1} g(x)$ to exist, we need the two one-sided limits to exist and be equal. We have $\lim\limits_{x\to -1^-} g(x) = \lim\limits_{x\to -1^-} (x^2-5x) = 6$, and $\lim\limits_{x\to -1^+} g(x) = \lim\limits_{x\to -1^+} (ax^3-7) = -a-7$. So we need $-a-7 = 6$, so we require that $a = -13$. Then $\lim\limits_{x\to -1} f(x) = 6$.

2.3.90 $\lim\limits_{x\to 2} \dfrac{x^5-32}{x-2} = \lim\limits_{x\to 2} \dfrac{(x-2)(x^4+2x^3+4x^2+8x+16)}{x-2} = \lim\limits_{x\to 2}(x^4+2x^3+4x^2+8x+16) = 16+16+16+16+16 = 80$.

2.3.91 $\lim\limits_{x\to 1} \dfrac{x^6-1}{x-1} = \lim\limits_{x\to 1} \dfrac{(x-1)(x^5+x^4+x^3+x^2+x+1)}{x-1} = \lim\limits_{x\to 1}(x^5+x^4+x^3+x^2+x+1) = 6$.

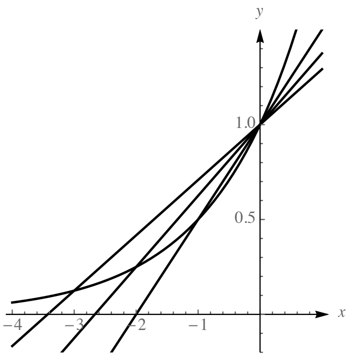
2.3.92 $\lim\limits_{x\to -1} \dfrac{x^7+1}{x+1} = \lim\limits_{x\to -1} \dfrac{(x+1)(x^6-x^5+x^4-x^3+x^2-x+1)}{x+1} = \lim\limits_{x\to -1}(x^6-x^5+x^4-x^3+x^2-x+1) = 7$.

2.3.93 $\lim\limits_{x\to a} \dfrac{x^5-a^5}{x-a} = \lim\limits_{x\to a} \dfrac{(x-a)(x^4+ax^3+a^2x^2+a^3x+a^4)}{x-a} = \lim\limits_{x\to a}(x^4+ax^3+a^2x^2+a^3x+a^4) = 5a^4$.

2.3.94 $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x-a} = \lim\limits_{x\to a} \dfrac{(x-a)(x^{n-1}+ax^{n-2}+\cdots+a^{n-2}x+a^{n-1})}{x-a} = \lim\limits_{x\to a}(x^{n-1}+ax^{n-2}+\cdots+a^{n-2}x+a^{n-1}) = na^{n-1}$.

2.3.95

a.



b. The slope of the secant line between $(0,1)$ and $(x, 2^x)$ is $\frac{2^x-1}{x}$.

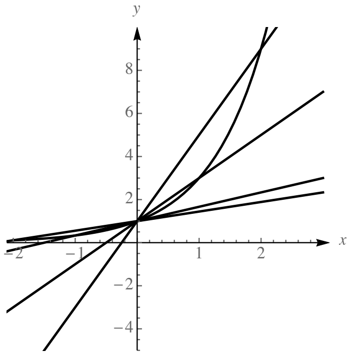
c.

x	-1	-0.1	-0.01	-0.001	-0.0001	-0.00001
$\frac{2^x-1}{x}$	0.5	0.66967	0.69075	0.692907	0.693123	0.693145

It appears that $\lim_{x\to 0^-} \frac{2^x-1}{x} \approx 0.693$.

2.3.96

a.



b. The slope of the secant line between $(0,1)$ and $(x,3^x)$ is $\frac{3^x-1}{x}$.

c.

x	-0.1	-0.01	-0.001	-0.0001	0.0001	0.001	0.01	0.1
$\frac{3^x-1}{x}$	1.04042	1.0926	1.09801	1.09855	1.09867	1.09922	1.10467	1.16123

It appears that $\displaystyle\lim_{x\to 0}\frac{3^x-1}{x}\approx 1.099$.

2.3.97 $\displaystyle\lim_{x\to -1^-} f(x)=\lim_{x\to 1^+} f(x)=6$. $\displaystyle\lim_{x\to -1^+} f(x)=\lim_{x\to 1^-} f(x)=5$.

2.3.98 $\displaystyle\lim_{x\to -1^-} g(x)=-\lim_{x\to 1^+} g(x)=-6$. $\displaystyle\lim_{x\to -1^+} g(x)=-\lim_{x\to 1^-} g(x)=-5$.

2.3.99 $\displaystyle\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}=\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}=\lim_{x\to 1}\frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}=\frac{1}{3}$.

2.3.100 $\displaystyle\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{x-16}=\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{(\sqrt[4]{x}-2)(\sqrt[4]{x^3}+2\sqrt[4]{x^2}+4\sqrt[4]{x}+8)}=\lim_{x\to 16}\frac{1}{\sqrt[4]{x^3}+2\sqrt[4]{x^2}+4\sqrt[4]{x}+8}=\frac{1}{32}$.

2.3.101 Let $f(x)=x-1$ and $g(x)=\frac{5}{x-1}$. Then $\displaystyle\lim_{x\to 1} f(x)=0$, $\displaystyle\lim_{x\to 1} f(x)g(x)=\lim_{x\to 1}\frac{5(x-1)}{x-1}=\lim_{x\to 1}5=5$.

2.3.102 Let $f(x)=x^2-1$. Then $\displaystyle\lim_{x\to 1}\frac{f(x)}{x-1}=\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}(x+1)=2$.

2.3.103 Let $p(x)=x^2+2x-8$. Then $\displaystyle\lim_{x\to 2}\frac{p(x)}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+4)}{x-2}=\lim_{x\to 2}(x+4)=6$.

The constants are unique. We know that 2 must be a root of p (otherwise the given limit couldn't exist), so it must have the form $p(x)=(x-2)q(x)$, and q must be a degree 1 polynomial with leading coefficient 1 (otherwise p wouldn't have leading coefficient 1.) So we have $p(x)=(x-2)(x+d)$, but because $\displaystyle\lim_{x\to 2}\frac{p(x)}{x-2}=\lim_{x\to 2}(x+d)=2+d=6$, we are forced to realize that $d=4$. Therefore, we have deduced that the only possibility for p is $p(x)=(x-2)(x+4)=x^2+2x-8$.

2.3.104 Because $\displaystyle\lim_{x\to 1} f(x)=4$, we know that f is near 4 when x is near 1 (but not equal to 1). It follows that $\displaystyle\lim_{x\to -1} f(x^2)=4$ as well, because when x is near but not equal to -1, x^2 is near 1 but not equal to 1. Thus $f(x^2)$ is near 4 when x is near -1.

2.3.105 As $x\to 0^+$, $(1-x)\to 1^-$. So $\displaystyle\lim_{x\to 0^+} g(x)=\lim_{(1-x)\to 1^-} f(1-x)=\lim_{z\to 1^-} f(z)=6$. (Where $z=1-x$.)

As $x\to 0^-$, $(1-x)\to 1^+$. So $\displaystyle\lim_{x\to 0^-} g(x)=\lim_{(1-x)\to 1^+} f(1-x)=\lim_{z\to 1^+} f(z)=4$. (Where $z=1-x$.)

2.3.106

a. Suppose $0 < \theta < \pi/2$. Note that $\sin\theta > 0$, so $|\sin\theta| = \sin\theta$. Also, $\sin\theta = \frac{|AC|}{1}$, so $|AC| = |\sin\theta|$.

   Now suppose that $-\pi/2 < \theta < 0$. Then $\sin\theta$ is negative, so $|\sin\theta| = -\sin\theta$. We have $\sin\theta = \frac{-|AC|}{1}$, so $|AC| = -\sin\theta = |\sin\theta|$.

b. Suppose $0 < \theta < \pi/2$. Because AB is the hypotenuse of triangle ABC, we know that $|AB| > |AC|$. We have $|\sin\theta| = |AC| < |AB| <$ the length of arc $AB = \theta = |\theta|$.

   If $-\pi/2 < \theta < 0$, we can make a similar argument. We have

$$|\sin\theta| = |AC| < |AB| < \text{the length of arc } AB = -\theta = |\theta|.$$

c. If $0 < \theta < \pi/2$, we have $\sin\theta = |\sin\theta| < |\theta|$, and because $\sin\theta$ is positive, we have $-|\theta| \le 0 < \sin\theta$. Putting these together gives $-|\theta| < \sin\theta < |\theta|$.

   If $-\pi/2 < \theta < 0$, then $|\sin\theta| = -\sin\theta$. From the previous part, we have $|\sin\theta| = -\sin\theta < |\theta|$. Therefore, $-|\theta| < \sin\theta$. Now because $\sin\theta$ is negative on this interval, we have $\sin\theta < 0 \le |\theta|$. Putting these together gives $-|\theta| < \sin\theta < |\theta|$.

d. If $0 < \theta < \pi/2$, we have

$$0 \le 1 - \cos\theta = |OB| - |OC| = |BC| < |AB| < \text{the length of arc } AB = \theta = |\theta|.$$

   For $-\pi/2 < \theta < 0$, we have

$$0 \le 1 - \cos\theta = |OB| - |OC| = |BC| < |AB| < \text{the length of arc } AB = -\theta = |\theta|.$$

e. Using the result of part d, we multiply through by -1 to obtain $-|\theta| \le \cos\theta - 1 \le 0$, and then add 1 to all parts, obtaining $1 - |\theta| \le \cos\theta \le 1$, as desired.

2.3.107

$$\begin{aligned}
\lim_{x\to a} p(x) &= \lim_{x\to a}(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0)\\
&= \lim_{x\to a}(a_nx^n) + \lim_{x\to a}(a_{n-1}x^{n-1}) + \cdots + \lim_{x\to a}(a_1x) + \lim_{x\to a} a_0\\
&= a_n \lim_{x\to a} x^n + a_{n-1}\lim_{x\to a} x^{n-1} + \cdots + a_1 \lim_{x\to a} x + a_0\\
&= a_n(\lim_{x\to a} x)^n + a_{n-1}(\lim_{x\to a} x)^{n-1} + \cdots + a_1(\lim_{x\to a} x) + a_0\\
&= a_na^n + a_{n-1}a^{n-1} + \cdots + a_1a + a_0 = p(a).
\end{aligned}$$

2.4 Infinite Limits

2.4.1 As x approaches a from the right, the values of $f(x)$ are negative and become arbitrarily large in magnitude.

2.4.2 As x approaches a (from either side), the values of $f(x)$ are positive and become arbitrarily large in magnitude.

2.4.3 A vertical asymptote for a function f is a vertical line $x = a$ so that one or more of the following are true: $\lim_{x\to a^-} f(x) = \pm\infty$, $\lim_{x\to a^+} f(x) = \pm\infty$.

2.4.4 No. For example, if $f(x) = x^2 - 4$ and $g(x) = x - 2$ and $a = 2$, we would have $\lim_{x\to 2}\frac{f(x)}{g(x)} = 4$, even though $g(2) = 0$.

2.4.5

x	$\frac{x+1}{(x-1)^2}$	x	$\frac{x+1}{(x-1)^2}$
1.1	210	.9	190
1.01	20,100	.99	19,900
1.001	2,001,000	.999	1,999,000
1.0001	200,010,000	.9999	199,990,000

From the data given, it appears that $\lim_{x\to 1} f(x) = \infty$.

2.4.6 $\lim_{x\to 3} f(x) = \infty$, and $\lim_{x\to -1} f(x) = -\infty$.

2.4.7

a. $\lim_{x\to 1^-} f(x) = \infty$.
b. $\lim_{x\to 1^+} f(x) = \infty$.
c. $\lim_{x\to 1} f(x) = \infty$.
d. $\lim_{x\to 2^-} f(x) = \infty$.
e. $\lim_{x\to 2^+} f(x) = -\infty$.
f. $\lim_{x\to 2} f(x)$ does not exist.

2.4.8

a. $\lim_{x\to 2^-} g(x) = \infty$.
b. $\lim_{x\to 2^+} g(x) = -\infty$.
c. $\lim_{x\to 2} g(x)$ does not exist.
d. $\lim_{x\to 4^-} g(x) = -\infty$.
e. $\lim_{x\to 4^+} g(x) = -\infty$.
f. $\lim_{x\to 4} g(x) = -\infty$.

2.4.9

a. $\lim_{x\to -2^-} h(x) = -\infty$.
b. $\lim_{x\to -2^+} h(x) = -\infty$.
c. $\lim_{x\to -2} h(x) = -\infty$.
d. $\lim_{x\to 3^-} h(x) = \infty$.
e. $\lim_{x\to 3^+} h(x) = -\infty$.
f. $\lim_{x\to 3} h(x)$ does not exist.

2.4.10

a. $\lim_{x\to -2^-} p(x) = -\infty$.
b. $\lim_{x\to -2^+} p(x) = -\infty$.
c. $\lim_{x\to -2} p(x) = -\infty$.
d. $\lim_{x\to 3^-} p(x) = -\infty$.
e. $\lim_{x\to 3^+} p(x) = -\infty$.
f. $\lim_{x\to 3} p(x) = -\infty$.

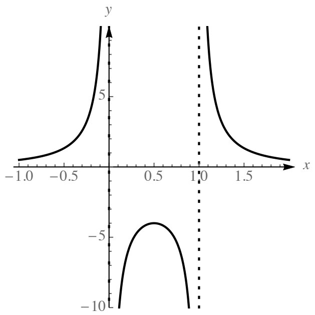
2.4.11

a. $\lim_{x\to 0^-} \dfrac{1}{x^2 - x} = \infty$.

b. $\lim_{x\to 0^+} \dfrac{1}{x^2 - x} = -\infty$.

c. $\lim_{x\to 1^-} \dfrac{1}{x^2 - x} = -\infty$.

d. $\lim_{x\to 1^+} \dfrac{1}{x^2 - x} = \infty$.



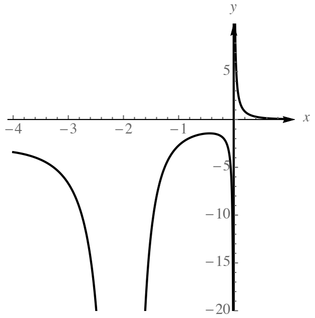
2.4.12

a. $\lim_{x\to-2^+} \frac{e^{-x}}{x(x+2)^2} = -\infty.$

b. $\lim_{x\to-2} \frac{e^{-x}}{x(x+2)^2} = -\infty.$

c. $\lim_{x\to0^-} \frac{e^{-x}}{x(x+2)^2} = -\infty.$

d. $\lim_{x\to0^+} \frac{e^{-x}}{x(x+2)^2} = \infty.$



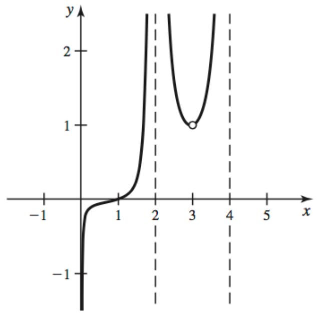
2.4.13 Because the numerator is approaching a non-zero constant while the denominator is approaching zero, the quotient of these numbers is getting big – at least the absolute value of the quotient is getting big. The quotient is actually always negative, because a number near 100 divided by a negative number is always negative. Thus $\lim_{x\to2} \frac{f(x)}{g(x)} = -\infty.$

2.4.14 Using the same sort of reasoning as in the last problem – as $x \to 3$ the numerator is fixed at 1, but the denominator is getting small, so the quotient is getting big. It remains to investigate the sign of the quotient. As $x \to 3^-$, the quantity $x-3$ is negative, so the quotient of the positive number 1 and this small negative number is negative. On the other hand, as $x \to 3^+$, the quantity $x-3$ is positive, so the quotient of 1 and this number is positive. Thus: $\lim_{x\to3^-} \frac{1}{x-3} = -\infty$, and $\lim_{x\to3^+} \frac{1}{x-3} = \infty.$

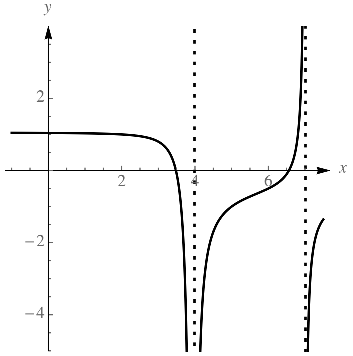
2.4.15 Note that $\lim_{x\to1} \frac{x^2-4x+3}{x^2-3x+2} = \lim_{x\to1} \frac{(x-3)(x-1)}{(x-2)(x-1)} = \lim_{x\to1} \frac{x-3}{x-2} = \frac{-2}{-1} = 2$. So there is *not* a vertical asymptote at $x = 1$. On the other hand, $\lim_{x\to2^+} \frac{x^2-4x+3}{x^2-3x+2} = \lim_{x\to2^+} \frac{(x-3)(x-1)}{(x-2)(x-1)} = \lim_{x\to2^+} \frac{x-3}{x-2} = -\infty$, so there is a vertical asymptote at $x = 2$.

2.4.16 Note the at $x \to 0$ the numerator has limit 1 while the denominator has limit 0, so the quotient is growing without bound. Note also that the denominator is always positive, because for all x, $\cos x \leq 1$ so $1 - \cos x \geq 0$.

2.4.17



2.4.18



2.4.19 Both a and b are true statements.

2.4.20 Both a and c are true statements.

2.4.21

a. $\lim\limits_{x\to 2^+} \dfrac{1}{x-2} = \infty.$

b. $\lim\limits_{x\to 2^-} \dfrac{1}{x-2} = -\infty.$

c. $\lim\limits_{x\to 2} \dfrac{1}{x-2}$ does not exist.

2.4.22

a. $\lim\limits_{x\to 3^+} \dfrac{2}{(x-3)^3} = \infty.$

b. $\lim\limits_{x\to 3^-} \dfrac{2}{(x-3)^3} = -\infty.$

c. $\lim\limits_{x\to 3} \dfrac{2}{(x-3)^3}$ does not exist.

2.4.23

a. $\lim\limits_{x\to 4^+} \dfrac{x-5}{(x-4)^2} = -\infty.$

b. $\lim\limits_{x\to 4^-} \dfrac{x-5}{(x-4)^2} = -\infty.$

c. $\lim\limits_{x\to 4} \dfrac{x-5}{(x-4)^2} = -\infty.$

2.4.24

a. $\lim\limits_{x\to 1^+} \dfrac{x}{|x-1|} = \infty.$

b. $\lim\limits_{x\to 1^-} \dfrac{x}{|x-1|} = \infty.$

c. $\lim\limits_{x\to 1} \dfrac{x}{|x-1|} = \infty.$

2.4.25

a. $\lim\limits_{x\to 3^+} \dfrac{(x-1)(x-2)}{(x-3)} = \infty.$

b. $\lim\limits_{x\to 3^-} \dfrac{(x-1)(x-2)}{(x-3)} = -\infty.$

c. $\lim\limits_{x\to 3} \dfrac{(x-1)(x-2)}{(x-3)}$ does not exist.

2.4.26

a. $\lim\limits_{x\to -2^+} \dfrac{(x-4)}{x(x+2)} = \infty.$

b. $\lim\limits_{x\to -2^-} \dfrac{(x-4)}{x(x+2)} = -\infty.$

c. $\lim\limits_{x\to -2} \dfrac{(x-4)}{x(x+2)}$ does not exist.

2.4.27

a. $\lim_{x\to 2^+} \frac{x^2-4x+3}{(x-2)^2} = -\infty$.

b. $\lim_{x\to 2^-} \frac{x^2-4x+3}{(x-2)^2} = -\infty$.

c. $\lim_{x\to 2} \frac{x^2-4x+3}{(x-2)^2} = -\infty$.

2.4.28

a. $\lim_{x\to -2^+} \frac{x^3-5x^2+6x}{x^4-4x^2} = \lim_{x\to -2^+} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)} = \lim_{x\to -2^+} \frac{x-3}{x(x+2)} = \infty$.

b. $\lim_{x\to -2^-} \frac{x^3-5x^2+6x}{x^4-4x^2} = \lim_{x\to -2^-} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)} = \lim_{x\to -2^-} \frac{x-3}{x(x+2)} = -\infty$.

c. Because the two one-sided limits differ, $\lim_{x\to -2} \frac{x^3-5x^2+6x}{x^4-4x^2}$ does not exist.

d. $\lim_{x\to 2} \frac{x^3-5x^2+6x}{x^4-4x^2} = \lim_{x\to 2} \frac{x-3}{x(x+2)} = \frac{-1}{8}$.

2.4.29

a. $\lim_{x\to 2^+} \frac{1}{\sqrt{x(x-2)}} = \infty$.

b. $\lim_{x\to 2^-} \frac{1}{\sqrt{x(x-2)}}$ does not exist. Note that the domain of the function is $(-\infty,0)\cup(2,\infty)$.

c. $\lim_{x\to 2} \frac{1}{\sqrt{x(x-2)}}$ does not exist.

2.4.30

a. $\lim_{x\to 1^+} \frac{x-3}{\sqrt{x^2-5x+4}}$ does not exist. Note that $x^2-5x+4=(x-4)(x-1)$ so the domain of the function is $(-\infty,1)\cup(4,\infty)$.

b.] $\lim_{x\to 1^-} \frac{x-3}{\sqrt{x^2-5x+4}} = -\infty$.

c. $\lim_{x\to 1} \frac{x-3}{\sqrt{x^2-5x+4}}$ does not exist.

2.4.31

a. $\lim_{x\to 0} \frac{x-3}{x^4-9x^2} = \lim_{x\to 0} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x\to 0} \frac{1}{x^2(x+3)} = \infty$.

b. $\lim_{x\to 3} \frac{x-3}{x^4-9x^2} = \lim_{x\to 3} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x\to 3} \frac{1}{x^2(x+3)} = \frac{1}{54}$.

c. $\lim_{x\to -3} \frac{x-3}{x^4-9x^2} = \lim_{x\to -3} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x\to -3} \frac{1}{x^2(x+3)}$, which does not exist.

2.4.32

a. $\lim_{x\to 0} \frac{x-2}{x^5-4x^3} = \lim_{x\to 0} \frac{x-2}{x^3(x-2)(x+2)} = \lim_{x\to 0} \frac{1}{x^3(x+2)}$, which does not exist.

b. $\lim_{x\to 2}\frac{x-2}{x^5-4x^3}=\lim_{x\to 2}\frac{x-2}{x^3(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x^3(x+2)}=\frac{1}{32}$.

c. $\lim_{x\to -2}\frac{x-2}{x^5-4x^3}=\lim_{x\to -2}\frac{x-2}{x^3(x-2)(x+2)}=\lim_{x\to -2}\frac{1}{x^3(x+2)}$, which does not exist.

2.4.33 $\lim_{x\to 0}\frac{x^3-5x^2}{x^2}=\lim_{x\to 0}\frac{x^2(x-5)}{x^2}=\lim_{x\to 0}(x-5)=-5$.

2.4.34 $\lim_{t\to 5}\frac{4t^2-100}{t-5}=\lim_{t\to 5}\frac{4(t-5)(t+5)}{t-5}=\lim_{t\to 5}[4(t+5)]=40$.

2.4.35 $\lim_{x\to 1^+}\frac{x^2-5x+6}{x-1}=\lim_{x\to 1^+}\frac{(x-2)(x-3)}{x-1}=\infty$. (Note that as $x\to 1^+$, the numerator is near 2, while the denominator is near zero, but is positive. So the quotient is positive and large.)

2.4.36 $\lim_{z\to 4}\frac{z-5}{(z^2-10z+24)^2}=\lim_{z\to 4}\frac{z-5}{(z-4)^2(z-6)^2}=-\infty$. (Note that as $z\to 4$, the numerator is near -1 while the denominator is near zero but is positive. So the quotient is negative with large absolute value.)

2.4.37 $\lim_{x\to 6^+}\frac{x-7}{\sqrt{x-6}}=-\infty$. (Note that as $x\to 6^+$ the numerator is near -1 and the denominator is near zero but is positive. So the quotient is negative with large absolute value.)

2.4.38 $\lim_{x\to 2^-}\frac{x-1}{\sqrt{(x-3)(x-2)}}=\infty$. Note that as $x\to 2^-$ the numerator is near 1 and the denominator is near zero but is positive. So the quotient is positive with large absolute value.)

2.4.39 $\lim_{\theta\to 0^+}\csc\theta=\lim_{\theta\to 0^+}\frac{1}{\sin\theta}=\infty$.

2.4.40 $\lim_{x\to 0^-}\csc x=\lim_{x\to 0^-}\frac{1}{\sin x}=-\infty$.

2.4.41 $\lim_{x\to 0^+}-10\cot x=\lim_{x\to 0^+}\frac{-10\cos x}{\sin x}=-\infty$. (Note that as $x\to 0^+$, the numerator is near -10 and the denominator is near zero, but is positive. Thus the quotient is a negative number whose absolute value is large.)

2.4.42 $\lim_{\theta\to(\pi/2)^+}\frac{1}{3}\tan\theta=\lim_{\theta\to(\pi/2)^+}\frac{\sin\theta}{3\cos\theta}=-\infty$. (Note that as $\theta\to(\pi/2)^+$, the numerator is near 1 and the denominator is near 0, but is negative. Thus the quotient is a negative number whose absolute value is large.)

2.4.43 $\lim_{\theta\to 0}\frac{2+\sin\theta}{1-\cos^2\theta}=\infty$. (Note that as $\theta\to 0$, the numerator is near 2 and the denominator is near 0, but is positive. Thus the quotient is a positive number whose absolute value is large.)

2.4.44 $\lim_{\theta\to 0^-}\frac{\sin\theta}{\cos^2\theta-1}=\lim_{\theta\to 0^-}\frac{\sin\theta}{-\sin^2\theta}=-\lim_{\theta\to 0^-}\frac{1}{\sin\theta}=\infty$.

2.4.45

a. $\lim_{x\to 5}\frac{x-5}{x^2-25}=\lim_{x\to 5}\frac{1}{x+5}=\frac{1}{10}$, so there isn't a vertical asymptote at $x=5$.

b. $\lim_{x\to -5^-}\frac{x-5}{x^2-25}=\lim_{x\to -5^-}\frac{1}{x+5}=-\infty$, so there is a vertical asymptote at $x=-5$.

c. $\lim_{x\to -5^+}\frac{x-5}{x^2-25}=\lim_{x\to -5^+}\frac{1}{x+5}=\infty$. This also implies that $x=-5$ is a vertical asymptote, as we already noted in part b.

2.4.46

a. $\lim_{x\to 7^-} \dfrac{x+7}{x^4-49x^2} = \lim_{x\to 7^-} \dfrac{x+7}{x^2(x+7)(x-7)} = \lim_{x\to 7^-} \dfrac{1}{x^2(x-7)} = -\infty$, so there is a vertical asymptote at $x = 7$.

b. $\lim_{x\to 7^+} \dfrac{x+7}{x^4-49x^2} = \lim_{x\to 7^+} \dfrac{x+7}{x^2(x+7)(x-7)} = \lim_{x\to 7^+} \dfrac{1}{x^2(x-7)} = \infty$. This also implies that there is a vertical asymptote at $x = 7$, as we already noted in part a.

c. $\lim_{x\to -7} \dfrac{x+7}{x^4-49x^2} = \lim_{x\to -7} \dfrac{x+7}{x^2(x+7)(x-7)} = \lim_{x\to -7} \dfrac{1}{x^2(x-7)} = \dfrac{1}{-686}$. So there is not a vertical asymptote at $x = 7$.

d. $\lim_{x\to 0} \dfrac{x+7}{x^4-49x^2} = \lim_{x\to 0} \dfrac{x+7}{x^2(x+7)(x-7)} = \lim_{x\to 0} \dfrac{1}{x^2(x-7)} = -\infty$. So there is a vertical asymptote at $x = 0$.

2.4.47 $f(x) = \dfrac{x^2-9x+14}{x^2-5x+6} = \dfrac{(x-2)(x-7)}{(x-2)(x-3)}$. Note that $x = 3$ is a vertical asymptote, while $x = 2$ appears to be a candidate but isn't one. We have $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \dfrac{x-7}{x-3} = -\infty$ and $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \dfrac{x-7}{x-3} = \infty$, and thus $\lim_{x\to 3} f(x)$ doesn't exist. Note that $\lim_{x\to 2} f(x) = 5$.

2.4.48 $f(x) = \dfrac{\cos x}{x(x+2)}$ has vertical asymptotes at $x = 0$ and at $x = -2$. Note that $\cos x$ is near 1 when x is near 0 , and $\cos x$ is near -0.4 when x is near -2. Thus, $\lim_{x\to 0^+} f(x) = +\infty$, $\lim_{x\to 0^-} f(x) = -\infty$, $\lim_{x\to -2^+} f(x) = \infty$, and $\lim_{x\to -2^-} f(x) = -\infty$.

2.4.49 $f(x) = \dfrac{x+1}{x^3-4x^2+4x} = \dfrac{x+1}{x(x-2)^2}$. There are vertical asymptotes at $x = 0$ and $x = 2$. We have $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{x+1}{x(x-2)^2} = -\infty$, while $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \dfrac{x+1}{x(x-2)^2} = \infty$, and thus $\lim_{x\to 0} f(x)$ doesn't exist.

Also we have $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \dfrac{x+1}{x(x-2)^2} = \infty$, while $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \dfrac{x+1}{x(x-2)^2} = \infty$, and thus $\lim_{x\to 2} f(x) = \infty$ as well.

2.4.50 $g(x) = \dfrac{x^3-10x^2+16x}{x^2-8x} = \dfrac{x(x-2)(x-8)}{x(x-8)}$. This function has no vertical asymptotes.

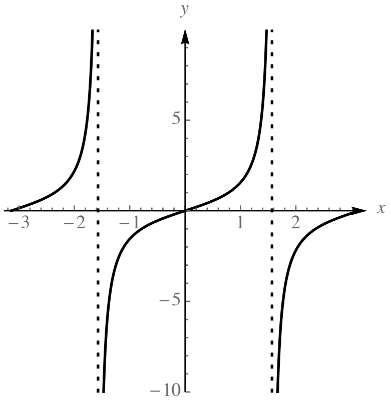
2.4.51

a. $\lim_{x\to(\pi/2)^+} \tan x = -\infty$.

b. $\lim_{x\to(\pi/2)^-} \tan x = \infty$.

c. $\lim_{x\to(-\pi/2)^+} \tan x = -\infty$.

d. $\lim_{x\to(-\pi/2)^-} \tan x = \infty$.



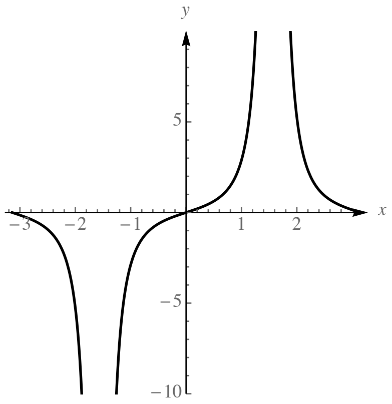
2.4.52

a. $\lim\limits_{x\to(\pi/2)^+} \sec x \tan x = \infty$.

b. $\lim\limits_{x\to(\pi/2)^-} \sec x \tan x = \infty$.

c. $\lim\limits_{x\to(-\pi/2)^+} \sec x \tan x = -\infty$.

d. $\lim\limits_{x\to(-\pi/2)^-} \sec x \tan x = -\infty$.



2.4.53

a. False. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{(x-1)(x-6)}{(x-1)(x+1)} = -\dfrac{5}{2}$.

b. True. For example, $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} \dfrac{(x-1)(x-6)}{(x-1)(x+1)} = -\infty$.

c. False. For example $g(x) = \frac{1}{x-1}$ has $\lim\limits_{x\to 1^+} g(x) = \infty$, but $\lim\limits_{x\to 1^-} g(x) = -\infty$.

2.4.54

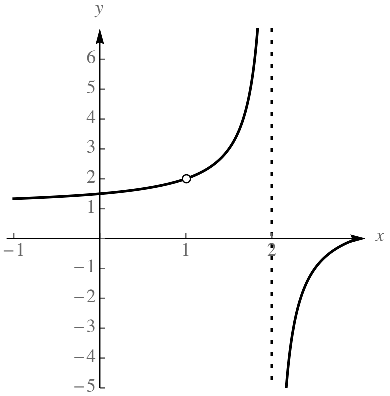
function	a	b	c	d	e	f
graph	D	C	F	B	A	E

2.4.55 We are seeking a function with a factor of $x-1$ in the denominator, but there should be more factors of $x-1$ in the numerator, and there should be a factor of $(x-2)^2$ in the denominator. This will accomplish the desired results. So

$$r(x) = \frac{(x-1)^2}{(x-1)(x-2)^2}.$$

2.4.56

One such function is $f(x) = \frac{x^2-4x+3}{x^2-3x+2} = \frac{(x-1)(x-3)}{(x-1)(x-2)}$.



2.4.57 One example is $f(x) = \frac{1}{x-6}$.

2.4.58 $f(x) = \dfrac{x^2-1}{(x^2-1)(x^2+1)} = \dfrac{1}{x^2+1}$ (for $x \neq \pm 1$). There are no vertical asymptotes, because for all a, $\lim_{x\to a} f(x) = \dfrac{1}{a^2+1}$.

2.4.59 $f(x) = \dfrac{x^2-3x+2}{x^{10}-x^9} = \dfrac{(x-2)(x-1)}{x^9(x-1)}$. f has a vertical asymptote at $x = 0$, because $\lim_{x\to 0^+} f(x) = -\infty$ (and $\lim_{x\to 0^-} f(x) = \infty$.) Note that $\lim_{x\to 1} f(x) = -1$, so there isn't a vertical asymptote at $x = 1$.

2.4.60 $g(x) = 2 - \ln x^2$ has a vertical asymptote at $x = 0$, because $\lim_{x\to 0}(2 - \ln x^2) = \infty$.

2.4.61 $h(x) = \dfrac{e^x}{(x+1)^3}$ has a vertical asymptote at $x = -1$, because

$$\lim_{x\to -1^+} \frac{e^x}{(x+1)^3} = \infty \quad \text{and} \quad \lim_{x\to -1^-} h(x) = -\infty.$$

2.4.62 $p(x) = \sec(\pi x/2) = \dfrac{1}{\cos(\pi x/2)}$ has a vertical asymptote on $(-2, 2)$ at $x = \pm 1$.

2.4.63 $g(\theta) = \tan(\pi\theta/10) = \dfrac{\sin(\pi\theta/10)}{\cos(\pi\theta/10)}$ has a vertical asymptote at each $\theta = 10n + 5$ where n is an integer. This is due to the fact that $\cos(\pi\theta/10) = 0$ when $\pi\theta/10 = \pi/2 + n\pi$ where n is an integer, which is the same as $\{\theta\colon \theta = 10n + 5, n \text{ an integer}\}$. Note that at all of these numbers which make the denominator zero, the numerator isn't zero.

2.4.64 $q(s) = \dfrac{\pi}{s - \sin s}$ has a vertical asymptote at $s = 0$. Note that this is the only number where $\sin s = s$.

2.4.65 $f(x) = \frac{1}{\sqrt{x}\sec x} = \frac{\cos x}{\sqrt{x}}$ has a vertical asymptote at $x = 0$.

2.4.66 $g(x) = e^{1/x}$ has a vertical asymptote at $x = 0$, because $\lim_{x\to 0^+} e^{1/x} = \infty$. (Note that as $x \to 0^+$, $1/x \to \infty$, so $e^{1/x} \to \infty$ as well.)

2.4.67

a. Note that the numerator of the given expression factors as $(x-3)(x-4)$. So if $a = 3$ or if $a = 4$ the limit would be a finite number. In fact, $\lim_{x\to 3} \dfrac{(x-3)(x-4)}{x-3} = -1$ and $\lim_{x\to 4} \dfrac{(x-3)(x-4)}{x-4} = 1$.

b. For any number other than 3 or 4, the limit would be either $\pm\infty$. Because $x - a$ is always positive as $x \to a^+$, the limit would be $+\infty$ exactly when the numerator is positive, which is for a in the set $(-\infty, 3) \cup (4, \infty)$.

c. The limit would be $-\infty$ for a in the set $(3, 4)$.

2.4.68

a. The slope of the secant line is given by $\dfrac{f(h) - f(0)}{h} = \dfrac{h^{1/3}}{h} = h^{-2/3}$.

b. $\lim_{h\to 0} \dfrac{1}{\sqrt[3]{h^2}} = \infty$. This tells us that the slope of the tangent line is infinite – which means that the tangent line at $(0, 0)$ is vertical.

2.4.69

a. The slope of the secant line is $\dfrac{f(h) - f(0)}{h} = \dfrac{h^{2/3}}{h} = h^{-1/3}$.

b. $\lim_{h\to 0^+} \dfrac{1}{h^{1/3}} = \infty$, and $\lim_{h\to 0^-} \dfrac{1}{h^{1/3}} = -\infty$. The tangent line is infinitely steep at the origin (i.e., it is a vertical line.)

2.5 Limits at Infinity

2.5.1 As $x < 0$ becomes arbitrarily large in absolute value, the corresponding values of f approach 10.

2.5.2 $\lim_{x\to\infty} f(x) = -2$ and $\lim_{x\to-\infty} f(x) = 4$.

2.5.3 $\lim_{x\to\infty} x^{12} = \infty$. Note that x^{12} is positive when $x > 0$.

2.5.4 $\lim_{x\to-\infty} 3x^{11} = -\infty$. Note that x^{11} is negative when $x < 0$.

2.5.5 $\lim_{x\to\infty} x^{-6} = \lim_{x\to\infty} \dfrac{1}{x^6} = 0$.

2.5.6 $\lim_{x\to-\infty} x^{-11} = \lim_{x\to-\infty} \dfrac{1}{x^{11}} = 0$.

2.5.7 $\lim_{t\to\infty} (-12t^{-5}) = \lim_{t\to\infty} -\dfrac{12}{t^5} = 0$.

2.5.8 $\lim_{x\to-\infty} 2x^{-8} = \lim_{x\to-\infty} \dfrac{2}{x^8} = 0$.

2.5.9 $\lim_{x\to\infty} (3 + 10/x^2) = 3 + \lim_{x\to\infty} (10/x^2) = 3 + 0 = 3$.

2.5.10 $\lim_{x\to\infty} (5 + 1/x + 10/x^2) = 5 + \lim_{x\to\infty} (1/x) + \lim_{x\to\infty} (10/x^2) = 5 + 0 + 0 = 5$.

2.5.11 If $f(x) \to 100,000$ as $x \to \infty$ and $g(x) \to \infty$ as $x \to \infty$, then the ratio $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$. (Because *eventually* the values of f are small compared to the values of g.)

2.5.12 $\lim_{x\to\infty} \dfrac{3+2x+4x^2}{x^2} = \lim_{x\to\infty} \dfrac{3}{x^2} + \lim_{x\to\infty} \dfrac{2x}{x^2} + \lim_{x\to\infty} \dfrac{4x^2}{x^2} = 0 + \lim_{x\to\infty} \dfrac{2}{x} + \lim_{x\to\infty} 4 = 0 + 0 + 4 = 4$.

2.5.13 $\lim_{t\to\infty} e^t = \infty$, $\lim_{t\to-\infty} e^t = 0$, and $\lim_{t\to\infty} e^{-t} = 0$.

2.5.14 As $x \to \infty$, we note that $e^{-2x} \to 0$, while as $x \to -\infty$, we have $e^{-2x} \to \infty$.

2.5.15 Because $\lim_{x\to\infty} 3 - \dfrac{1}{x^2} = 3$ and $\lim_{x\to\infty} 3 + \dfrac{1}{x^2} = 3$, by the Squeeze Theorem we must have $\lim_{x\to\infty} g(x) = 3$.

Similarly, because $\lim_{x\to-\infty} 3 - \dfrac{1}{x^2} = 3$ and $\lim_{x\to-\infty} 3 + \dfrac{1}{x^2} = 3$, by the Squeeze Theorem we must have $\lim_{x\to-\infty} g(x) = 3$.

2.5.16 $\lim_{x\to-\infty} g(x) = 3$, $\lim_{x\to\infty} g(x) = -1$, $\lim_{x\to-2^-} g(x) = \infty$, $\lim_{x\to 2^+} g(x) = -\infty$.

2.5.17 $\lim_{\theta\to\infty} \dfrac{\cos\theta}{\theta^2} = 0$. Note that $-1 \le \cos\theta \le 1$, so $-\dfrac{1}{\theta^2} \le \dfrac{\cos\theta}{\theta^2} \le \dfrac{1}{\theta^2}$. The result now follows from the Squeeze Theorem.

2.5.18 Note that $\dfrac{5t^2 + t\sin t}{t^2}$ can be written as $5 + \dfrac{\sin t}{t}$. Also, note that because $-1 \le \sin t \le 1$, we have $-\dfrac{1}{t} \le \dfrac{\sin t}{t} \le \dfrac{1}{t}$, so $\dfrac{\sin t}{t} \to 0$ as $t \to \infty$ by the Squeeze Theorem. Therefore,

$$\lim_{t\to\infty} \frac{5t^2 + t\sin t}{t^2} = \lim_{t\to\infty} \left(5 + \frac{\sin t}{t}\right) = 5 + 0 = 5.$$

2.5.19 $\displaystyle\lim_{x\to\infty}\frac{\cos x^5}{\sqrt{x}}=0$. Note that $-1\le\cos x^5\le 1$, so $\frac{-1}{\sqrt{x}}\le\frac{\cos x^5}{\sqrt{x}}\le\frac{1}{\sqrt{x}}$. Because $\displaystyle\lim_{x\to\infty}\frac{1}{\sqrt{x}}=\lim_{x\to\infty}\frac{-1}{\sqrt{x}}=0$, we have $\displaystyle\lim_{x\to\infty}\frac{\cos x^5}{\sqrt{x}}=0$ by the Squeeze Theorem.

2.5.20 $\displaystyle\lim_{x\to-\infty}\left(5+\frac{100}{x}+\frac{\sin^4(x^3)}{x^2}\right)=5+0+0=5$. For this last limit, note that $0\le\sin^4(x^3)\le 1$, so $\displaystyle 0\le\frac{\sin^4(x^3)}{x^2}\le\frac{1}{x^2}$. The result now follows from the Squeeze Theorem.

2.5.21 $\displaystyle\lim_{x\to\infty}(3x^{12}-9x^7)=\infty$.

2.5.22 $\displaystyle\lim_{x\to-\infty}(3x^7+x^2)=-\infty$.

2.5.23 $\displaystyle\lim_{x\to-\infty}(-3x^{16}+2)=-\infty$.

2.5.24 $\displaystyle\lim_{x\to-\infty}(2x^{-8}+4x^3)=0+\lim_{x\to-\infty}4x^3=-\infty$.

2.5.25 $\displaystyle\lim_{x\to\infty}\frac{(14x^3+3x^2-2x)}{(21x^3+x^2+2x+1)}\cdot\frac{1/x^3}{1/x^3}=\lim_{x\to\infty}\frac{14+(3/x)-(2/x^2)}{21+(1/x)+(2/x^2)+(1/x^3)}=\frac{14}{21}=\frac{2}{3}$.

2.5.26 $\displaystyle\lim_{x\to\infty}\frac{(9x^3+x^2-5)}{(3x^4+4x^2)}\cdot\frac{1/x^4}{1/x^4}=\lim_{x\to\infty}\frac{(9/x)+(1/x^2)-(5/x^4)}{3+(4/x^2)}=\frac{0}{3}=0$.

2.5.27 $\displaystyle\lim_{x\to-\infty}\frac{(3x^2+3x)}{(x+1)}\cdot\frac{1/x}{1/x}=\lim_{x\to-\infty}\frac{3x+3}{1+(1/x)}=-\infty$.

2.5.28 $\displaystyle\lim_{x\to\infty}\frac{(x^4+7)}{(x^5+x^2-x)}\cdot\frac{1/x^5}{1/x^5}=\lim_{x\to\infty}\frac{(1/x)+(7/x^5)}{1+(1/x^3)-(1/x^4)}=\frac{0+0}{1+0-0}=0$.

2.5.29 Note that for $w>0$, $w^2=\sqrt{w^4}$. We have

$$\lim_{w\to\infty}\frac{(15w^2+3w+1)}{\sqrt{9w^4+w^3}}\cdot\frac{1/w^2}{1/\sqrt{w^4}}=\lim_{w\to\infty}\frac{15+(3/w)+(1/w^2)}{\sqrt{9+(1/w)}}=\frac{15}{\sqrt{9}}=5.$$

2.5.30 Note that $\sqrt{x^8}=x^4$ (even for $x<0$). We have

$$\lim_{x\to-\infty}\frac{(40x^4+x^2+5x)}{\sqrt{64x^8+x^6}}\cdot\frac{1/x^4}{1/\sqrt{x^8}}=\lim_{x\to-\infty}\frac{40+(1/x^2)+(5/x^3)}{\sqrt{64+(1/x^2)}}=\frac{40}{\sqrt{64}}=\frac{40}{8}=5.$$

2.5.31 Note that for $x<0$, $\sqrt{x^2}=-x$. We have

$$\lim_{x\to-\infty}\frac{\sqrt{16x^2+x}}{x}\cdot\frac{\sqrt{1/x^2}}{-1/x}=\lim_{x\to-\infty}\frac{\sqrt{16+(1/x)}}{-1}=-\sqrt{16}=-4.$$

2.5.32 Note that $x^2=\sqrt{x^4}$ for all x. We have

$$\lim_{x\to\infty}\frac{6x^2}{(4x^2+\sqrt{16x^4+x^2})}\cdot\frac{1/x^2}{1/x^2}=\lim_{x\to\infty}\frac{6}{(4+\sqrt{16+(1/x^2)})}=\frac{6}{4+\sqrt{16}}=\frac{3}{4}.$$

2.5.33 $\displaystyle\lim_{x\to\infty}\frac{(x^2-\sqrt{x^4+3x^2})}{1}\cdot\frac{(x^2+\sqrt{x^4+3x^2})}{(x^2+\sqrt{x^4+3x^2})}=\lim_{x\to\infty}\frac{x^4-(x^4+3x^2)}{x^2+\sqrt{x^4+3x^2}}=\lim_{x\to\infty}\frac{-3x^2}{x^2+\sqrt{x^4+3x^2}}$. Now divide the numerator and denominator by x^2 to give

$$\lim_{x\to\infty}\frac{-3x^2}{x^2+\sqrt{x^4+3x^2}}\cdot\frac{1/x^2}{1/x^2}=\lim_{x\to\infty}\frac{-3}{1+\sqrt{1+(3/x^2)}}=-\frac{3}{2}.$$

2.5.34 $\lim\limits_{x\to-\infty}\dfrac{(x+\sqrt{x^2-5x})}{1}\cdot\dfrac{(x-\sqrt{x^2-5x})}{(x-\sqrt{x^2-5x})}=\lim\limits_{x\to-\infty}\dfrac{x^2-(x^2-5x)}{x-\sqrt{x^2-5x}}=\lim\limits_{x\to-\infty}\dfrac{5x}{x-\sqrt{x^2-5x}}$. Now divide the numerator and denominator by x (and recall that for $x<0$ we have $-\sqrt{x^2}=x$) giving

$$\lim_{x\to-\infty}\frac{5x}{(x-\sqrt{x^2-5x})}\cdot\frac{1/x}{1/x}=\lim_{x\to-\infty}\frac{5}{1+\sqrt{1-(5/x)}}=\frac{5}{2}.$$

2.5.35 Note that because $-1\le\sin x\le 1$, we have $-\dfrac{1}{e^x}\le\dfrac{\sin x}{e^x}\le\dfrac{1}{e^x}$. Then because $\lim\limits_{x\to\infty}\dfrac{\pm 1}{e^x}=0$, the Squeeze Theorem tells us that $\lim\limits_{x\to\infty}\dfrac{\sin x}{e^x}=0$.

2.5.36 Note that because $-1\le\cos x\le 1$, we have $-e^x\le e^x\cos x\le e^x$. Then $3-e^x\le e^x\cos x+3\le e^x+3$. Because $\lim\limits_{x\to-\infty}3-e^x=3$ and $\lim\limits_{x\to-\infty}e^x+3=3$, the Squeeze Theorem tells us that $\lim\limits_{x\to-\infty}e^x\cos x+3=3$.

2.5.37 $\lim\limits_{x\to\infty}\dfrac{4x}{20x+1}=\lim\limits_{x\to\infty}\dfrac{4x}{20x+1}\cdot\dfrac{1/x}{1/x}=\lim\limits_{x\to\infty}\dfrac{4}{20+1/x}=\dfrac{4}{20}=\dfrac{1}{5}$. Thus, the line $y=\frac{1}{5}$ is a horizontal asymptote.

$\lim\limits_{x\to-\infty}\dfrac{4x}{20x+1}=\lim\limits_{x\to-\infty}\dfrac{4x}{20x+1}\cdot\dfrac{1/x}{1/x}=\lim\limits_{x\to-\infty}\dfrac{4}{20+1/x}=\dfrac{4}{20}=\dfrac{1}{5}$. This shows that the curve is also asymptotic to the asymptote in the negative direction.

2.5.38 $\lim\limits_{x\to\infty}\dfrac{3x^2-7}{x^2+5x}=\lim\limits_{x\to\infty}\dfrac{3x^2-7}{x^2+5x}\cdot\dfrac{1/x^2}{1/x^2}=\lim\limits_{x\to\infty}\dfrac{3-(7/x^2)}{1+(5/x)}=\dfrac{3-0}{1+0}=3$. Thus, the line $y=3$ is a horizontal asymptote.

$\lim\limits_{x\to-\infty}\dfrac{3x^2-7}{x^2+5x}=\lim\limits_{x\to-\infty}\dfrac{3x^2-7}{x^2+5x}\cdot\dfrac{1/x^2}{1/x^2}=\lim\limits_{x\to-\infty}\dfrac{3-(7/x^2)}{1+(5/x)}=\dfrac{3-0}{1+0}=3$. Thus, the curve is also asymptotic to the asymptote in the negative direction.

2.5.39 $\lim\limits_{x\to\infty}\dfrac{(6x^2-9x+8)}{(3x^2+2)}\cdot\dfrac{1/x^2}{1/x^2}=\lim\limits_{x\to\infty}\dfrac{6-9/x+8/x^2}{3+2/x^2}=\dfrac{6-0+0}{3+0}=2$. Similarly $\lim\limits_{x\to-\infty}f(x)=2$. The line $y=2$ is a horizontal asymptote.

2.5.40 $\lim\limits_{x\to\infty}\dfrac{(12x^8-3)}{(3x^8-2x^7)}\cdot\dfrac{1/x^8}{1/x^8}=\lim\limits_{x\to\infty}\dfrac{12-3/x^8}{3-2/x}=\dfrac{12-0}{3-0}=4$. Similarly $\lim\limits_{x\to-\infty}f(x)=4$. The line $y=4$ is a horizontal asymptote.

2.5.41 $\lim\limits_{x\to\infty}\dfrac{3x^3-7}{x^4+5x^2}=\lim\limits_{x\to\infty}\dfrac{3x^3-7}{x^4+5x^2}\cdot\dfrac{3/x^4}{1/x^4}=\lim\limits_{x\to\infty}\dfrac{1/x-(7/x^4)}{1+(5/x^2)}=\dfrac{0-0}{1+0}=0$. Thus, the line $y=0$ (the x-axis) is a horizontal asymptote.

$\lim\limits_{x\to-\infty}\dfrac{3x^3-7}{x^4+5x^2}=\lim\limits_{x\to-\infty}\dfrac{3x^3-7}{x^4+5x^2}\cdot\dfrac{3/x^4}{1/x^4}=\lim\limits_{x\to-\infty}\dfrac{1/x-(7/x^4)}{1+(5/x^2)}=\dfrac{0-0}{1+0}=0$. Thus, the curve is asymptotic to the x-axis in the negative direction as well.

2.5.42 $\lim\limits_{x\to\infty}\dfrac{(2x+1)}{(3x^4-2)}\cdot\dfrac{1/x^4}{1/x^4}=\lim\limits_{x\to\infty}\dfrac{2/x^3+1/x^4}{3-2/x^4}=\dfrac{0+0}{3-0}=0$. Similarly $\lim\limits_{x\to-\infty}f(x)=0$. The line $y=0$ is a horizontal asymptote.

2.5.43 $\lim\limits_{x\to\infty}\dfrac{(40x^5+x^2)}{(16x^4-2x)}\cdot\dfrac{1/x^4}{1/x^4}=\lim\limits_{x\to\infty}\dfrac{40x+1/x^2}{16-2/x^3}=\infty$. Similarly $\lim\limits_{x\to-\infty}f(x)=-\infty$. There are no horizontal asymptotes.

2.5.44 Note that for all x, $\sqrt{x^4}=x^2$. Then

$$\lim_{x\to\pm\infty}\frac{(6x^2+1)}{\sqrt{4x^4+3x+1}}\cdot\frac{1/x^2}{\sqrt{1/x^4}}=\lim_{x\to\pm\infty}\frac{6+(1/x^2)}{\sqrt{4+(3/x^3)+(1/x^4)}}=\frac{6}{\sqrt{4}}=3.$$

So $y=3$ is the only horizontal asymptote.

2.5.45 Note that for all x, $\sqrt{x^8} = x^4$. Then $\lim\limits_{x\to\pm\infty} \dfrac{1}{(2x^4 - \sqrt{4x^8 - 9x^4}} \cdot \dfrac{(2x^4 + \sqrt{4x^8 - 9x^4})}{(2x^4 + \sqrt{4x^8 - 9x^4})}$
$= \lim\limits_{x\to\pm\infty} \dfrac{(2x^4 + \sqrt{4x^8 - 9x^4})}{(4x^8 - (4x^8 - 9x^4))} \cdot \dfrac{1/x^4}{1/x^4} = \lim\limits_{x\to\pm\infty} \dfrac{2 + \sqrt{4 - (9/x^4)}}{9} = \dfrac{4}{9}$.
So $y = \dfrac{4}{9}$ is the only horizontal asymptote.

2.5.46 First note that $\sqrt{x^2} = x$ for $x > 0$, while $\sqrt{x^2} = -x$ for $x < 0$. Then $\lim\limits_{x\to\infty} f(x)$ can be written as

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x+1} \cdot \frac{1/\sqrt{x^2}}{1/x} = \lim_{x\to\infty} \frac{\sqrt{1+1/x^2}}{2+1/x} = \frac{1}{2}.$$

However, $\lim\limits_{x\to-\infty} f(x)$ can be written as

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x+1} \cdot \frac{1/\sqrt{x^2}}{-1/x} = \lim_{x\to-\infty} \frac{\sqrt{1+1/x^2}}{-2-1/x} = -\frac{1}{2}.$$

2.5.47 First note that $\sqrt{x^6} = x^3$ if $x > 0$, but $\sqrt{x^6} = -x^3$ if $x < 0$. We have $\lim\limits_{x\to\infty} \dfrac{4x^3+1}{(2x^3 + \sqrt{16x^6+1})} \cdot \dfrac{1/x^3}{1/x^3} =$
$\lim\limits_{x\to\infty} \dfrac{4+1/x^3}{2+\sqrt{16+1/x^6}} = \dfrac{4+0}{2+\sqrt{16+0}} = \dfrac{2}{3}$.

However, $\lim\limits_{x\to-\infty} \dfrac{4x^3+1}{(2x^3+\sqrt{16x^6+1})} \cdot \dfrac{1/x^3}{1/x^3} = \lim\limits_{x\to-\infty} \dfrac{4+1/x^3}{2-\sqrt{16+1/x^6}} = \dfrac{4+0}{2-\sqrt{16+0}} = \dfrac{4}{-2} = -2$.

So $y = \frac{2}{3}$ is a horizontal asymptote (as $x \to \infty$) and $y = -2$ is a horizontal asymptote (as $x \to -\infty$).

2.5.48 First note that for $x > 0$ we have $\sqrt{x^2} = x$, but for $x < 0$, we have $-x = \sqrt{x^2}$. Then we have $\lim\limits_{x\to\infty} x - \sqrt{x^2 - 9x} = \lim\limits_{x\to\infty} \dfrac{(x - \sqrt{x^2-9x})(x + \sqrt{x^2-9x})}{x + \sqrt{x^2-9x}} = \lim\limits_{x\to\infty} \dfrac{9x}{(x+\sqrt{x^2-9x})} \cdot \dfrac{1/x}{1/x} = \lim\limits_{x\to\infty} \dfrac{9}{1+\sqrt{1-9/x}} = \dfrac{9}{2}$.
On the other hand, $\lim\limits_{x\to-\infty} x - \sqrt{x^2-9x} = \lim\limits_{x\to-\infty} \dfrac{(x-\sqrt{x^2-9x})(x+\sqrt{x^2-9x})}{x+\sqrt{x^2-9x}} =$
$\lim\limits_{x\to-\infty} \dfrac{9x}{(x+\sqrt{x^2-9x})} \cdot \dfrac{-1/x}{-1/x} = \lim\limits_{x\to-\infty} \dfrac{-9}{-1+\sqrt{1-9/x}} = -\infty$. The last equal sign follows because $\sqrt{1-9/x} > 1$ but is approaching 1 as $x \to -\infty$. We can therefore conclude that $y = \dfrac{9}{2}$ is the only horizontal asymptote, and is an asymptote as $x \to \infty$.

2.5.49 First note that $\sqrt[3]{x^6} = x^2$ and $\sqrt{x^4} = x^2$ for all x (even when $x < 0$.) We have $\lim\limits_{x\to\infty} \dfrac{\sqrt[3]{x^6+8}}{(4x^2+\sqrt{3x^4+1})} \cdot$
$\dfrac{1/x^2}{1/x^2} = \lim\limits_{x\to\infty} \dfrac{\sqrt[3]{1+8/x^6}}{4+\sqrt{3+1/x^4}} = \dfrac{1}{4+\sqrt{3+0}} = \dfrac{1}{4\sqrt{3}}$.

The calculation as $x \to -\infty$ is similar. So $y = \frac{1}{4\sqrt{3}}$ is a horizontal asymptote.

2.5.50 First note that $\sqrt{x^2} = x$ for $x > 0$ and $\sqrt{x^2} = -x$ for $x < 0$.

We have

$$\begin{aligned}\lim_{x\to\infty} 4x(3x - \sqrt{9x^2+1}) &= \lim_{x\to\infty} \frac{4x(3x-\sqrt{9x^2+1})(3x+\sqrt{9x^2+1})}{3x+\sqrt{9x^2+1}} \\ &= \lim_{x\to\infty} \frac{(4x)(-1)}{(3x+\sqrt{9x^2+1})} \cdot \frac{1/x}{1/x} \\ &= \lim_{x\to\infty} -\frac{4}{3+\sqrt{9+1/x^2}} = -\frac{4}{6} = -\frac{2}{3}.\end{aligned}$$

Moreover, as $x \to -\infty$ we have

$$\begin{aligned}\lim_{x\to-\infty} 4x(3x-\sqrt{9x^2+1}) &= \lim_{x\to-\infty} \frac{4x(3x-\sqrt{9x^2+1})(3x+\sqrt{9x^2+1})}{3x+\sqrt{9x^2+1}}\\ &= \lim_{x\to-\infty} \frac{(4x)(-1)}{(3x+\sqrt{9x^2+1})}\cdot\frac{1/x}{1/x}\\ &= \lim_{x\to-\infty} -\frac{4}{3-\sqrt{9+1/x^2}} = \infty.\end{aligned}$$

Note that this last equality is due to the fact that the numerator is the constant -4 and the denominator is approaching zero (from the left) so the quotient is positive and is getting large.

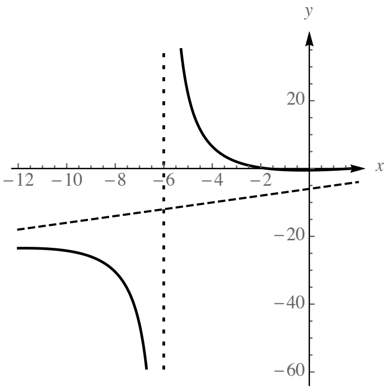
So $y = -\frac{2}{3}$ is the only horizontal asymptote.

2.5.51

a. $f(x) = \dfrac{x^2-3}{x+6} = x - 6 + \dfrac{33}{x+6}$. The oblique asymptote of f is $y = x - 6$.

b. Because $\lim_{x\to-6^+} f(x) = \infty$, there is a vertical asymptote at $x = -6$. Note also that $\lim_{x\to-6^-} f(x) = -\infty$.

c.

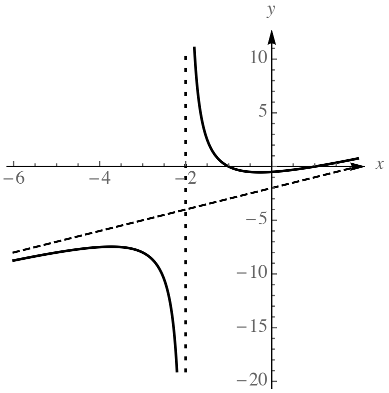


2.5.52

a. $f(x) = \dfrac{x^2-1}{x+2} = x - 2 + \dfrac{3}{x+2}$. The oblique asymptote of f is $y = x - 2$.

b. Because $\lim_{x\to-2^+} f(x) = \infty$, there is a vertical asymptote at $x = -2$. Note also that $\lim_{x\to-2^-} f(x) = -\infty$.

c.

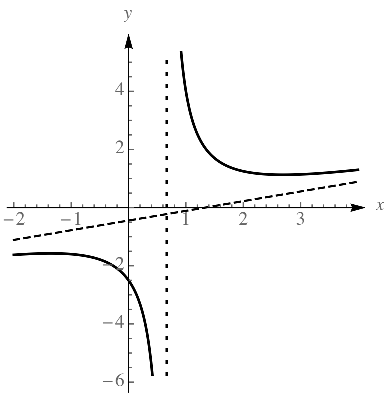


2.5.53

a. $f(x) = \dfrac{x^2-2x+5}{3x-2} = (1/3)x - 4/9 + \dfrac{37}{9(3x-2)}$. The oblique asymptote of f is $y = (1/3)x - 4/9$.

b. Because $\lim\limits_{x\to(2/3)^+} f(x) = \infty$, there is a vertical asymptote at $x = 2/3$. Note also that $\lim\limits_{x\to(2/3)^-} f(x) = -\infty$.

c.

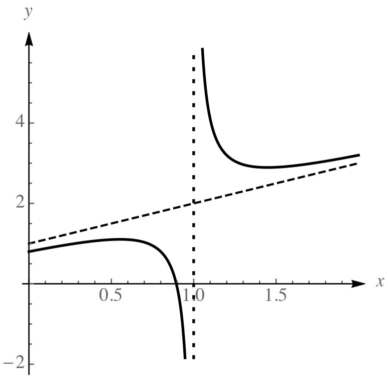


2.5.54

a. $f(x) = \dfrac{5x^2-4}{5x-5} = x + 1 + \dfrac{1}{5x-5}$. The oblique asymptote of f is $y = x + 1$.

b. Because $\lim\limits_{x\to1^+} f(x) = \infty$, there is a vertical asymptote at $x = 1$. Note also that $\lim\limits_{x\to1^-} f(x) = -\infty$.

c.

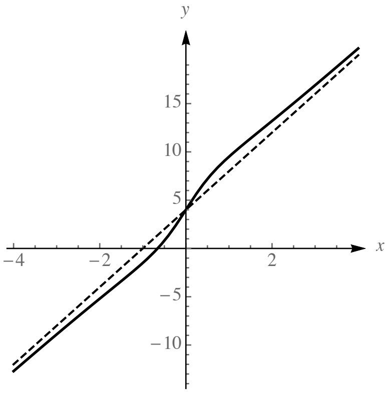


2.5.55

a. $f(x) = \dfrac{4x^3+4x^2+7x+4}{1+x^2} = 4x + 4 + \dfrac{3x}{1+x^2}$. The oblique asymptote of f is $y = 4x + 4$.

b. There are no vertical asymptotes.

c.

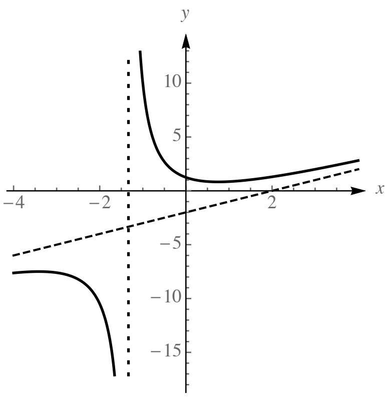


2.5.56

a. $f(x) = \dfrac{3x^2-2x+5}{3x+4} = x - 2 + \dfrac{13}{3x+4}$. The oblique asymptote of f is $y = x - 2$.

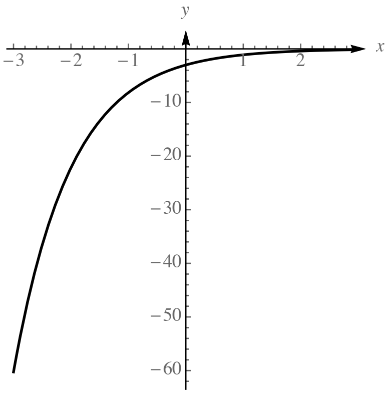
b. Because $\lim_{x\to(-4/3)^+} f(x) = \infty$, there is a vertical asymptote at $x = -4/3$. Note also that $\lim_{x\to(-4/3)^-} f(x) = -\infty$.

c.



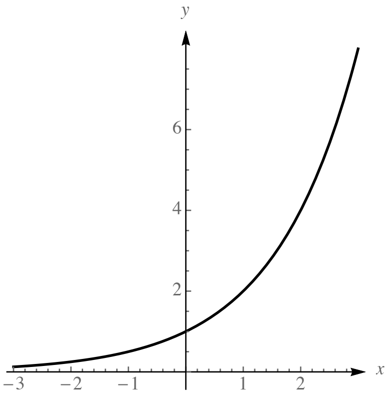
2.5.57

$\lim_{x\to\infty}(-3e^{-x}) = -3\cdot 0 = 0$. $\lim_{x\to-\infty}(-3e^{-x}) = -\infty$.



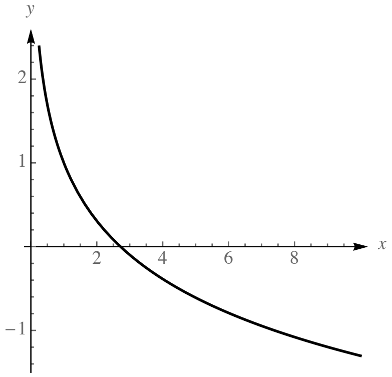
2.5.58

$\lim_{x\to\infty} 2^x = \infty$. $\lim_{x\to-\infty} 2^x = 0$.



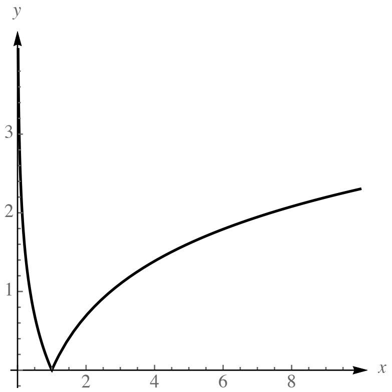
2.5.59

$\lim_{x\to\infty}(1-\ln x) = -\infty$. $\lim_{x\to 0^+}(1-\ln x) = \infty$.



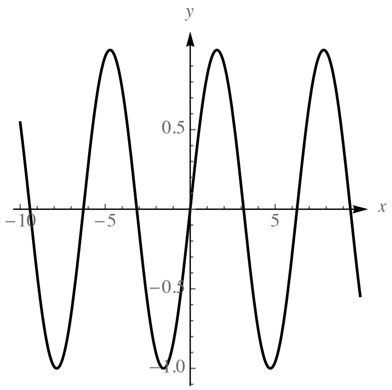
2.5.60

$\lim_{x\to\infty}|\ln x| = \infty$. $\lim_{x\to 0^+}|\ln x| = \infty$.



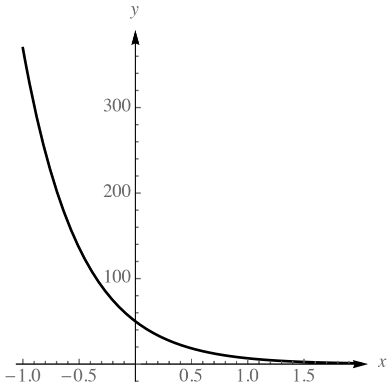
2.5.61

$y = \sin x$ has no asymptotes. $\lim_{x\to\infty}\sin x$ and $\lim_{x\to-\infty}\sin x$ do not exist.



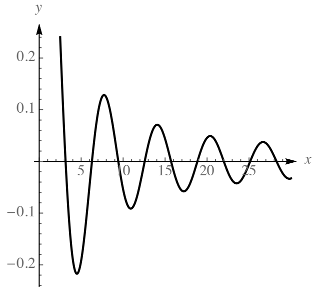
2.5.62

$\lim_{x\to\infty} \frac{50}{e^{2x}} = 0$. $\lim_{x\to-\infty} \frac{50}{e^{2x}} = \infty$.



2.5.63

a. False. For example, the function $y = \frac{\sin x}{x}$ on the domain $[1, \infty)$ has a horizontal asymptote of $y = 0$, and it crosses the x-axis infinitely many times.



b. False. If f is a rational function, and if $\lim_{x\to\infty} f(x) = L \neq 0$, then the degree of the polynomial in the numerator must equal the degree of the polynomial in the denominator. In this case, both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x) = \frac{a_n}{b_n}$ where a_n is the leading coefficient of the polynomial in the numerator and b_n is the leading coefficient of the polynomial in the denominator. In the case where $\lim_{x\to\infty} f(x) = 0$, then the degree of the numerator is strictly less than the degree of the denominator. This case holds for $\lim_{x\to-\infty} f(x) = 0$ as well.

c. True. There are only two directions which might lead to horizontal asymptotes: there could be one as $x \to \infty$ and there could be one as $x \to -\infty$, and those are the only possibilities.

d. False. The limit of the difference of two functions can be written as the difference of the limits only when both limits exist. It is the case that $\lim_{x\to\infty} (x^3 - x) = \infty$.

2.5.64 $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{2500}{t+1} = 0$. The steady state exists. The steady state value is 0.

2.5.65 $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{3500t}{t+1} = 3500$. The steady state exists. The steady state value is 3500.

2.5.66 $\lim_{t\to\infty} m(t) = \lim_{t\to\infty} 200(1 - 2^{-t}) = 200$. The steady state exists. The steady state value is 200.

2.5.67 $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} 1000e^{0.065t} = \infty$. The steady state does not exist.

2.5.68 $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{1500}{3 + 2e^{-.1t}} = \frac{1500}{3} = 500$. The steady state exists. The steady state value is 500.

2.5.69 $\lim_{t\to\infty} a(t) = \lim_{t\to\infty} 2\left(\frac{t+\sin t}{t}\right) = \lim_{t\to\infty} 2\left(1+\frac{\sin t}{t}\right) = 2$. The steady state exists. The steady state value is 2.

2.5.70

a. $\lim_{x\to\infty} \frac{x^2-4x+3}{x-1} = \infty$, and $\lim_{x\to-\infty} \frac{x^2-4x+3}{x-1} = -\infty$. There are no horizontal asymptotes.

b. It appears that $x = 1$ is a candidate to be a vertical asymptote, but note that $f(x) = \frac{x^2-4x+3}{x-1} = \frac{(x-1)(x-3)}{x-1}$. Thus $\lim_{x\to1} f(x) = \lim_{x\to1}(x-3) = -2$. So f has no vertical asymptotes.

2.5.71

a. $\lim_{x\to\infty} \frac{2x^3+10x^2+12x}{x^3+2x^2} \cdot \frac{(1/x^3)}{(1/x^3)} = \lim_{x\to\infty} \frac{2+10/x+12/x^2}{1+2/x} = 2$. Similarly, $\lim_{x\to-\infty} f(x) = 2$. Thus, $y = 2$ is a horizontal asymptote.

b. Note that $f(x) = \frac{2x(x+2)(x+3)}{x^2(x+2)}$. So $\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{2(x+3)}{x} = \infty$, and similarly, $\lim_{x\to0^-} f(x) = -\infty$. There is a vertical asymptote at $x = 0$. Note that there is no asymptote at $x = -2$ because $\lim_{x\to-2} f(x) = -1$.

2.5.72

a. We have $\lim_{x\to\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4} \cdot \frac{(1/x^2)}{(1/x^2)} = \lim_{x\to\infty} \frac{\sqrt{16+64/x^2}+1}{2-4/x^2} = \frac{5}{2}$. Similarly, $\lim_{x\to-\infty} f(x) = \frac{5}{2}$. So $y = \frac{5}{2}$ is a horizontal asymptote.

b. $\lim_{x\to\sqrt{2}^+} f(x) = \lim_{x\to-\sqrt{2}^-} f(x) = \infty$, and $\lim_{x\to\sqrt{2}^-} f(x) = \lim_{x\to-\sqrt{2}^+} f(x) = -\infty$ so there are vertical asymptotes at $x = \pm\sqrt{2}$.

2.5.73

a. We have $\lim_{x\to\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144} \cdot \frac{(1/x^4)}{(1/x^4)} = \lim_{x\to\infty} \frac{3+3/x-36/x^2}{1-25/x^2+144/x^4} = 3$. Similarly, $\lim_{x\to-\infty} f(x) = 3$. So $y = 3$ is a horizontal asymptote.

b. Note that $f(x) = \frac{3x^2(x+4)(x-3)}{(x+4)(x-4)(x+3)(x-3)}$. Thus, $\lim_{x\to-3^+} f(x) = -\infty$ and $\lim_{x\to-3^-} f(x) = \infty$. Also, $\lim_{x\to4^-} f(x) = -\infty$ and $\lim_{x\to4^+} f(x) = \infty$. Thus there are vertical asymptotes at $x = -3$ and $x = 4$.

2.5.74

a. First note that

$$f(x) = x^2(4x^2 - \sqrt{16x^4+1}) \cdot \frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}} = -\frac{x^2}{4x^2+\sqrt{16x^4+1}}.$$

We have $\lim_{x\to\infty} -\frac{x^2}{4x^2+\sqrt{16x^4+1}} \cdot \frac{(1/x^2)}{(1/x^2)} = \lim_{x\to\infty} -\frac{1}{4+\sqrt{16+1/x^4}} = -\frac{1}{8}$. Similarly, the limit as $x \to -\infty$ of $f(x)$ is $-\frac{1}{8}$ as well. So $y = -\frac{1}{8}$ is a horizontal asymptote.

b. f has no vertical asymptotes.

2.5.75

a. $\lim\limits_{x\to\infty} \dfrac{x^2-9}{x^2-3x}\cdot\dfrac{(1/x^2)}{(1/x^2)} = \lim\limits_{x\to\infty}\dfrac{1-9/x^2}{1-3/x} = 1$. A similar result holds as $x \to -\infty$. So $y = 1$ is a horizontal asymptote.

b. Because $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\dfrac{x+3}{x} = \infty$ and $\lim\limits_{x\to 0^-} f(x) = -\infty$, there is a vertical asymptote at $x = 0$.

2.5.76

a. $\lim\limits_{x\to\pm\infty}\dfrac{x^4-1}{x^2-1} = \lim\limits_{x\to\pm\infty}\dfrac{(x^2-1)(x^2+1)}{x^2-1} = \lim\limits_{x\to\pm\infty} x^2+1 = \infty$. There are no horizontal or slant asymptotes.

b. It appears that $x = \pm 1$ may be candidates for vertical asymptotes, but because

$$\frac{x^4-1}{x^2-1} = \frac{(x^2-1)(x^2+1)}{x-1} = x^2+1$$

for $x \neq \pm 1$ there are no vertical asymptotes either.

2.5.77

a. First note that $f(x)$ can be written s

$$\frac{\sqrt{x^2+2x+6}-3}{x-1}\cdot\frac{\sqrt{x^2+2x+6}+3}{\sqrt{x^2+2x+6}+3} = \frac{x^2+2x+6-9}{(x-1)(\sqrt{x^2+2x+6}+3)} = \frac{(x-1)(x+3)}{(x-1)(\sqrt{x^2+2x+6}+3)}.$$

Thus

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{x+3}{\sqrt{x^2+2x+6}+3}\cdot\frac{1/x}{1/x} = \lim_{x\to\infty}\frac{1+3/x}{\sqrt{1+2/x+6/x^2}+3/x} = 1.$$

Using the fact that $\sqrt{x^2} = -x$ for $x < 0$, we have $\lim\limits_{x\to-\infty} f(x) = -1$. Thus the lines $y = 1$ and $y = -1$ are horizontal asymptotes.

b. f has no vertical asymptotes.

2.5.78

a. Note that when x is large $|1-x^2| = x^2-1$. We have $\lim\limits_{x\to\infty}\dfrac{|1-x^2|}{x^2+x} = \lim\limits_{x\to\infty}\dfrac{x^2-1}{x^2+x} = 1$. Likewise $\lim\limits_{x\to-\infty}\dfrac{|1-x^2|}{x^2+x} = \lim\limits_{x\to-\infty}\dfrac{x^2-1}{x^2+x} = 1$. So there is a horizontal asymptote at $y = 1$.

b. Note that when x is near 0, we have $|1-x^2| = 1-x^2 = (1-x)(1+x)$. So $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\dfrac{1-x}{x} = \infty$. Similarly, $\lim\limits_{x\to 0^-} f(x) = -\infty$. There is a vertical asymptote at $x = 0$.

2.5.79

a. Note that when $x > 1$, we have $|x| = x$ and $|x-1| = x-1$. Thus

$$f(x) = (\sqrt{x}-\sqrt{x-1})\cdot\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}} = \frac{1}{\sqrt{x}+\sqrt{x-1}}.$$

Thus $\lim\limits_{x\to\infty} f(x) = 0$.

When $x < 0$, we have $|x| = -x$ and $|x-1| = 1-x$. Thus

$$f(x) = (\sqrt{-x}-\sqrt{1-x})\cdot\frac{\sqrt{-x}+\sqrt{1-x}}{\sqrt{-x}+\sqrt{1-x}} = -\frac{1}{\sqrt{-x}+\sqrt{1-x}}.$$

Thus, $\lim\limits_{x\to-\infty} f(x) = 0$. There is a horizontal asymptote at $y = 0$.

b. f has no vertical asymptotes.

2.5.80

a. $\lim_{x\to\infty} \frac{(3e^x+10)}{e^x} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to\infty} \frac{3+(10/e^x)}{1} = 3$. On the other hand, $\lim_{x\to-\infty} \frac{3e^x+10}{e^x} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to-\infty} \frac{3+(10e^{-x})}{1} = \infty$. $y=3$ is a horizontal asymptote as $x\to\infty$.

b. f has no vertical asymptotes.

2.5.81

a. $\lim_{x\to\infty} \frac{\cos x + 2\sqrt{x}}{\sqrt{x}} = \lim_{x\to\infty} \left(2 + \frac{\cos x}{\sqrt{x}}\right) = 2$. $y=2$ is a horizontal asymptote.

b. $\lim_{x\to0^+} \frac{\cos x + 2\sqrt{x}}{\sqrt{x}} = \infty$. and $\lim_{x\to0^-} \frac{\cos x + 2\sqrt{x}}{\sqrt{x}}$ does not exist. $x=0$ is a vertical asymptote.

2.5.82

a. $\lim_{x\to\infty} \cot^{-1} x = 0$.

b. $\lim_{x\to-\infty} \cot^{-1} x = \pi$.

2.5.83

a. $\lim_{x\to\infty} \sec^{-1} x = \pi/2$.

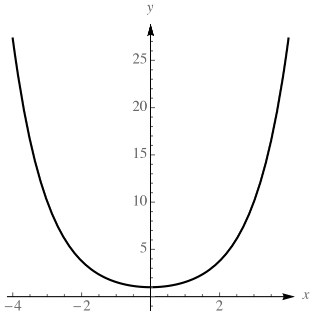
b. $\lim_{x\to-\infty} \sec^{-1} x = \pi/2$.

2.5.84

a. $\lim_{x\to\infty} \frac{e^x+e^{-x}}{2} = \infty$.

$\lim_{x\to-\infty} \frac{e^x+e^{-x}}{2} = \infty$.

b. $\cosh(0) = \frac{e^0+e^0}{2} = \frac{1+1}{2} = 1$.

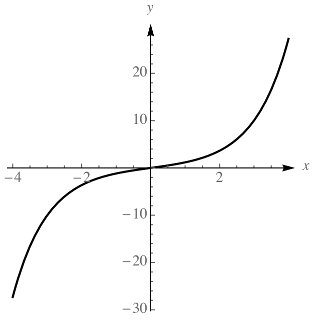


2.5.85

a. $\lim_{x\to\infty} \frac{e^x-e^{-x}}{2} = \infty$.

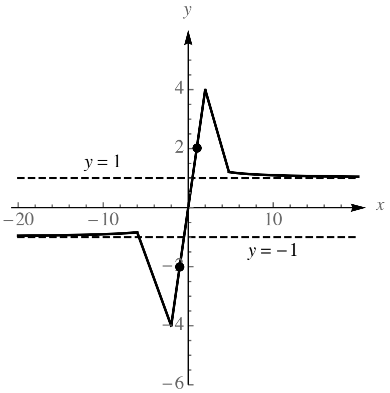
$\lim_{x\to-\infty} \frac{e^x-e^{-x}}{2} = -\infty$.

b. $\sinh(0) = \frac{e^0 - e^0}{2} = \frac{1-1}{2} = 0.$



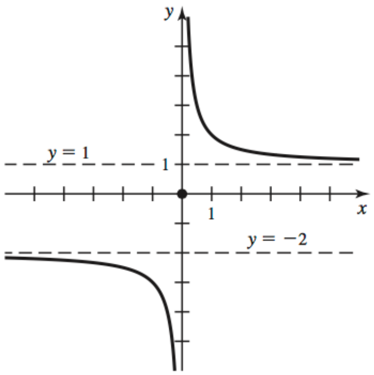
2.5.86

One possible such graph is:



2.5.87

One possible such graph is:



2.5.88 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \dfrac{4}{n} = 0.$

2.5.89 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \dfrac{n-1}{n} = \lim_{n\to\infty} [1 - (1/n)] = 1.$

2.5.90 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \dfrac{n^2}{n+1} = \lim_{n\to\infty} \dfrac{n}{1+1/n} = \infty$, so the limit does not exist.

2.5.91 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} [1/n + 1/n^2] = 0.$

2.5.92

a. Suppose $m = n$.

$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{p(x)}{q(x)} \cdot \frac{1/x^n}{1/x^n} \\ &= \lim_{x\to\pm\infty} \frac{a_n + a_{n-1}/x + \cdots + a_1/x^{n-1} + a_0/x^n}{b_n + b_{n-1}/x + \cdots + b_1/x^{n-1} + b_0/x^n} \\ &= \frac{a_n}{b_n}.\end{aligned}$$

b. Suppose $m < n$.

$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{p(x)}{q(x)} \cdot \frac{1/x^n}{1/x^n} \\ &= \lim_{x\to\pm\infty} \frac{a_n/x^{n-m} + a_{n-1}/x^{n-m+1} + \cdots + a_1/x^{n-1} + a_0/x^n}{b_n + b_{n-1}/x + \cdots + b_1/x^{n-1} + b_0/x^n} \\ &= \frac{0}{b_n} = 0.\end{aligned}$$

2.5.93

a. No. If $m = n$, there will be a horizontal asymptote, and if $m = n + 1$, there will be a slant asymptote.

b. Yes. For example, $f(x) = \frac{x^4}{\sqrt{x^6+1}}$ has slant asymptote $y = x$ as $x \to \infty$ and slant asymptote $y = -x$ as $x \to -\infty$.

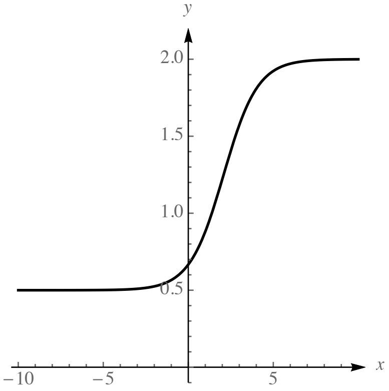
2.5.94

a. $\lim_{x\to\infty} \frac{4e^x + 2e^{2x}}{8e^x + e^{2x}} = \lim_{x\to\infty} \frac{(4e^x + 2e^{2x})}{(8e^x + e^{2x})} \cdot \frac{1/e^{2x}}{1/e^{2x}} = \lim_{x\to\infty} \frac{(4/e^x) + 2}{(8/e^x) + 1} = 2.$

b. $\lim_{x\to-\infty} \frac{4e^x + 2e^{2x}}{8e^x + e^{2x}} = \lim_{x\to-\infty} \frac{(4e^x + 2e^{2x})}{(8e^x + e^{2x})} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to-\infty} \frac{4 + 2e^x}{8 + e^x} = \frac{1}{2}.$

c. The lines $y = 2$ and $y = \frac{1}{2}$ are horizontal asymptotes.

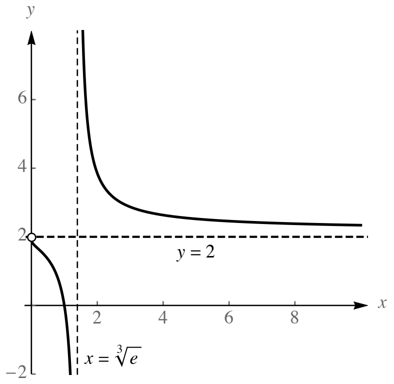
d.



2.5.95 $\lim_{x\to\infty}\frac{2e^x+3}{e^x+1} = \lim_{x\to\infty}\frac{(2e^x+3)}{(e^x+1)}\cdot\frac{1/e^x}{1/e^x} = \lim_{x\to\infty}\frac{2+3/e^x}{1+1/e^x} = \frac{2+0}{1+0} = 2$. Thus the line $y = 2$ is a horizontal asymptote. Also $\lim_{x\to-\infty}\frac{2e^x+3}{e^x+1} = \frac{0+3}{0+1} = 3$, so $y = 3$ is a horizontal asymptote.

2.5.96 $\lim_{x\to\infty}\frac{3e^{5x}+7e^{6x}}{9e^{5x}+14e^{6x}} = \lim_{x\to\infty}\frac{(3e^{5x}+7e^{6x})}{(9e^{5x}+14e^{6x})}\cdot\frac{1/e^{6x}}{1/e^{6x}} = \lim_{x\to\infty}\frac{3e^{-x}+7}{9e^{-x}+14} = \frac{7}{14} = \frac{1}{2}$. So $y = \frac{1}{2}$ is a horizontal asymptote. Also, $\lim_{x\to-\infty}\frac{3e^{5x}+7e^{6x}}{9e^{5x}+14e^{6x}} = \lim_{x\to-\infty}\frac{(3e^{5x}+7e^{6x})}{(9e^{5x}+14e^{6x})}\cdot\frac{1/e^{5x}}{1/e^{5x}} = \lim_{x\to-\infty}\frac{3+7e^x}{9+14e^x} = \frac{3}{9} = \frac{1}{3}$. So $y = \frac{1}{3}$ is a horizontal asymptote.

2.5.97 Using the rules of logarithms, $f(x) = \frac{6\ln x}{3\ln x - 1}$. The domain of f is $(0, \sqrt[3]{e}) \cup (\sqrt[3]{e}, \infty)$. We first examine the end behavior of the function. Observe that $\lim_{x\to\infty}\frac{6\ln x}{3\ln x-1} = \lim_{x\to\infty}\frac{6}{3-(1/\ln x)} = \frac{6}{3} = 2$ and $\lim_{x\to 0^+}\frac{6\ln x}{3\ln x-1} = \lim_{x\to 0^+}\frac{6}{3-(1/\ln x)} = \frac{6}{3} = 2$. So the function has a horizontal asymptote of $y = 2$ and it is undefined at $x = 0$ but has limit 2 as x approaches 0 from the right. Notice also that as $x \to \sqrt[3]{e}^+$, $6\ln x \to 2$ and $3\ln x - 1$ is positive and approaches 0. Therefore, $\lim_{x\to\sqrt[3]{e}^+}\frac{6\ln x}{3\ln x-1} = \infty$ and by a similar argument, $\lim_{x\to\sqrt[3]{e}^-}\frac{6\ln x}{3\ln x-1} = -\infty$



2.6 Continuity

2.6.1

a. $a(t)$ is a continuous function during the time period from when she jumps from the plane and when she touches down on the ground, because her position is changing continuously with time.

b. $n(t)$ is not a continuous function of time. The function "jumps" at the times when a quarter must be added.

c. $T(t)$ is a continuous function, because temperature varies continuously with time.

d. $p(t)$ is not continuous – it jumps by whole numbers when a player scores a point.

2.6.2 In order for f to be continuous at $x = a$, the following conditions must hold:

- f must be defined at a (i.e. a must be in the domain of f),
- $\lim_{x\to a} f(x)$ must exist, and

- $\lim_{x\to a} f(x)$ must equal $f(a)$.

2.6.3 A function f is continuous on an interval I if it is continuous at all points in the interior of I, and it must be continuous from the right at the left endpoint (if the left endpoint is included in I) and it must be continuous from the left at the right endpoint (if the right endpoint is included in I.)

2.6.4 The words "hole" and "break" are not mathematically precise, so a strict mathematical definition can not be based on them.

2.6.5 f is discontinuous at $x = 1$, at $x = 2$, and at $x = 3$. At $x = 1$, $f(1)$ is not defined (so the first condition is violated). At $x = 2$, $f(2)$ is defined and $\lim_{x\to 2} f(x)$ exists, but $\lim_{x\to 2} f(x) \neq f(2)$ (so condition 3 is violated). At $x = 3$, $\lim_{x\to 3} f(x)$ does not exist (so condition 2 is violated).

2.6.6 f is discontinuous at $x = 1$, at $x = 2$, and at $x = 3$. At $x = 1$, $\lim_{x\to 1} f(x) \neq f(1)$ (so condition 3 is violated). At $x = 2$, $\lim_{x\to 2} f(x)$ does not exist (so condition 2 is violated). At $x = 3$, $f(3)$ is not defined (so condition 1 is violated).

2.6.7 f is discontinuous at $x = 1$, at $x = 2$, and at $x = 3$. At $x = 1$, $\lim_{x\to 1} f(x)$ does not exist, and $f(1)$ is not defined (so conditions 1 and 2 are violated). At $x = 2$, $\lim_{x\to 2} f(x)$ does not exist (so condition 2 is violated). At $x = 3$, $f(3)$ is not defined (so condition 1 is violated).

2.6.8 f is discontinuous at $x = 2$, at $x = 3$, and at $x = 4$. At $x = 2$, $\lim_{x\to 2} f(x)$ does not exist (so condition 2 is violated). At $x = 3$, $f(3)$ is not defined and $\lim_{x\to 3} f(x)$ does not exist (so conditions 1 and 2 are violated). At $x = 4$, $\lim_{x\to 4} f(x) \neq f(4)$ (so condition 3 is violated).

2.6.9

a. A function f is continuous from the left at $x = a$ if a is in the domain of f, and $\lim_{x\to a^-} f(x) = f(a)$.

b. A function f is continuous from the right at $x = a$ if a is in the domain of f, and $\lim_{x\to a^+} f(x) = f(a)$.

2.6.10 If f is right-continuous at $x = 3$, then $f(3) = \lim_{x\to 3^+} f(x) = 6$, so $f(3) = 6$.

2.6.11 f is continuous on $(0, 1)$, on $(1, 2)$, on $(2, 3]$, and on $(3, 4)$. It is continuous from the left at 3.

2.6.12 f is continuous on $(0, 1)$, on $(1, 2]$, on $(2, 3)$, and on $(3, 4)$. It is continuous from the left at 2.

2.6.13 f is continuous on $[0, 1)$, on $(1, 2)$, on $[2, 3)$, and on $(3, 5)$. It is continuous from the right at 2.

2.6.14 f is continuous on $(0, 2]$, on $(2, 3)$, on $(3, 4)$, and on $(4, 5)$. It is continuous from the left at 2.

2.6.15 The domain of $f(x) = \dfrac{e^x}{x}$ is $(-\infty, 0) \cup (0, \infty)$, and f is continuous everywhere on this domain.

2.6.16 The function is continuous on $(0, 15]$, on $(15, 30]$, on $(30, 45]$, and on $(45, 60]$.

2.6.17 The number -5 is not in the domain of f, because the denominator is equal to 0 when $x = -5$. Thus, the function is not continuous at -5.

2.6.18 The function is defined at 5, in fact $f(5) = \frac{50+15+1}{25+25} = \frac{66}{50} = \frac{33}{25}$. Also, $\lim_{x\to 5} f(x) = \lim_{x\to 5} \dfrac{2x^2 + 3x + 1}{x^2 + 5x} = \dfrac{33}{25} = f(5)$. The function is continuous at $a = 5$.

2.6.19 f is discontinuous at 1, because 1 is not in the domain of f; $f(1)$ is not defined.

2.6.20 g is discontinuous at 3 because 3 is not in the domain of g; $g(3)$ is not defined.

2.6.21 f is discontinuous at 1, because $\lim_{x\to 1} f(x) \neq f(1)$. In fact, $f(1) = 3$, but $\lim_{x\to 1} f(x) = 2$.

2.6.22 f is continuous at 3, because $\lim_{x\to 3} f(x) = f(3)$. In fact, $f(3) = 2$ and $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{(x-3)(x-1)}{x-3} = \lim_{x\to 3}(x-1) = 2$.

2.6.23 f is discontinuous at 4, because 4 is not in the domain of f; $f(4)$ is not defined.

2.6.24 f is discontinuous at -1 because $\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{x(x+1)}{x+1} = \lim_{x\to -1} x = -1 \neq f(-1) = 2$.

2.6.25 Because p is a polynomial, it is continuous on all of $\mathbb{R} = (-\infty, \infty)$.

2.6.26 Because g is a rational function, it is continuous on its domain, which is all of $\mathbb{R} = (-\infty, \infty)$. (Because $x^2 + x + 1$ has no real roots.)

2.6.27 Because f is a rational function, it is continuous on its domain. Its domain is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

2.6.28 Because s is a rational function, it is continuous on its domain. Its domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

2.6.29 Because f is a rational function, it is continuous on its domain. Its domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

2.6.30 Because f is a rational function, it is continuous on its domain. Its domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

2.6.31 Because $f(x) = \left(x^8 - 3x^6 - 1\right)^{40}$ is a polynomial, it is continuous everywhere, including at 0. Thus $\lim_{x\to 0} f(x) = f(0) = (-1)^{40} = 1$.

2.6.32 Because $f(x) = \left(\frac{3}{2x^5 - 4x^2 - 50}\right)^4$ is a rational function, it is continuous at all points in its domain, including at $x = 2$. So $\lim_{x\to 2} f(x) = f(2) = \frac{81}{16}$.

2.6.33 Because $x^3 - 2x^2 - 8x = x(x^2 - 2x - 8) = x(x-4)(x+2)$, we have (as long as $x \neq 4$)

$$\sqrt{\frac{x^3 - 2x^2 - 8x}{x-4}} = \sqrt{x(x+2)}.$$

Thus, $\lim_{x\to 4} \sqrt{\frac{x^3 - 2x^2 - 8x}{x-4}} = \lim_{x\to 4} \sqrt{x(x+2)} = \sqrt{24}$, using Theorem 2.12 and the fact that the square root is a continuous function.

2.6.34 Note that $t - 4 = (\sqrt{t} - 2)(\sqrt{t} + 2)$, so for $t \neq 4$, we have

$$\frac{t-4}{\sqrt{t}-2} = \sqrt{t} + 2.$$

Thus, $\lim_{t\to 4} \frac{t-4}{\sqrt{t}-2} = \lim_{t\to 4}(\sqrt{t} + 2) = 4$. Then using Theorem 2.12 and the fact that the tangent function is continuous at 4, we have $\lim_{t\to 4} \tan\left(\frac{t-4}{\sqrt{t}-2}\right) = \tan\left(\lim_{t\to 4} \frac{t-4}{\sqrt{t}-2}\right) = \tan 4$.

2.6.35 Because $f(x) = \left(\frac{x+5}{x+2}\right)^4$ is a rational function, it is continuous at all points in its domain, including at $x = 1$. Thus $\lim\limits_{x\to 1} f(x) = f(1) = 16$.

2.6.36 $\lim\limits_{x\to\infty}\left(\dfrac{2x+1}{x}\right)^3 = \lim\limits_{x\to\infty}(2+(1/x))^3 = 2^3 = 8$.

2.6.37 Note that

$$\begin{aligned}\lim_{x\to 5}\frac{6(\sqrt{x^2-16}-3)}{5(x-5)} &= \lim_{x\to 5}\frac{6(\sqrt{x^2-16}-3)}{5(x-5)}\cdot\frac{(\sqrt{x^2-16}+3)}{(\sqrt{x^2-16}+3)} = \lim_{x\to 5}\frac{6(x^2-25)}{5(x-5)(\sqrt{x^2-16}+3)}\\ &= \lim_{x\to 5}\frac{6(x+5)}{5(\sqrt{x^2-16}+3)} = \frac{60}{30} = 2.\end{aligned}$$

2.6.38 First note that

$$\lim_{x\to 0}\frac{x}{\sqrt{16x+1}-1} = \lim_{x\to 0}\frac{x}{(\sqrt{16x+1}-1)}\cdot\frac{(\sqrt{16x+1}+1)}{(\sqrt{16x+1}+1)} = \lim_{x\to 0}\frac{x(\sqrt{16x+1}+1)}{16x} = \frac{2}{16} = \frac{1}{8}.$$

Then because $f(x) = x^{1/3}$ is continuous at $1/8$, we have $\lim\limits_{x\to 0}\left(\dfrac{x}{\sqrt{16x+1}-1}\right)^{1/3} = \left(\dfrac{1}{8}\right)^{1/3} = \dfrac{1}{2}$, by Theorem 2.12.

2.6.39

a. f is defined at 1. We have $f(1) = 1^2+(3)(1) = 4$. To see whether or not $\lim\limits_{x\to 1} f(x)$ exists, we investigate the two one-sided limits. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 2x = 2$, and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(x^2+3x) = 4$, so $\lim\limits_{x\to 1} f(x)$ does not exist. Thus f is discontinuous at $x = 1$.

b. f is continuous from the right, because $\lim\limits_{x\to 1^+} f(x) = 4 = f(1)$.

c. f is continuous on $(-\infty, 1)$ and on $[1, \infty)$.

2.6.40

a. f is defined at 0, in fact $f(0) = 1$. However, $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}(x^3+4x+1) = 1$, while $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 2x^3 = 0$. So $\lim\limits_{x\to 0} f(x)$ does not exist.

b. f is continuous from the left at 0, because $\lim\limits_{x\to 0^-} f(x) = f(0) = 1$.

c. f is continuous on $(-\infty, 0]$ and on $(0, \infty)$.

2.6.41 f is defined and is continuous on $(-\infty, 5]$. It is continuous from the left at 5.

2.6.42 f is defined and is continuous on $[-5, 5]$. It is continuous from the right at -5 and is continuous from the left at 5.

2.6.43 f is continuous on $(-\infty, -\sqrt{8}]$ and on $[\sqrt{8}, \infty)$. It is continuous from the left at $-\sqrt{8}$ and from the right at $\sqrt{8}$.

2.6.44 $g(x) = \sqrt{x^2-3x+2} = \sqrt{(x-1)(x-2)}$ is defined and is continuous on $(-\infty, 1]$ and on $[2, \infty)$. It is continuous from the left at 1 and from the right at 2.

2.6.45 Because f is the composition of two functions which are continuous on $(-\infty, \infty)$, it is continuous on $(-\infty, \infty)$.

2.6.46 f is continuous on $(-\infty,-1]$ and on $[1,\infty)$. It is continuous from the left at -1 and from the right at 1.

2.6.47 Because f is the composition of two functions which are continuous on $(-\infty,\infty)$, it is continuous on $(-\infty,\infty)$.

2.6.48 f is continuous on $[1,\infty)$. It is continuous from the right at 1.

2.6.49 $\lim\limits_{x\to 2}\sqrt{\dfrac{4x+10}{2x-2}} = \sqrt{\dfrac{18}{2}} = 3.$

2.6.50 $\lim\limits_{x\to -1}\left(x^2-4+\sqrt[3]{x^2-9}\right) = (-1)^2-4+\sqrt[3]{(-1)^2-9} = -3+\sqrt[3]{-8} = -3+-2 = -5.$

2.6.51 $\lim\limits_{x\to\pi}\dfrac{\cos^2 x+3\cos x+2}{\cos x+1} = \lim\limits_{x\to\pi}\dfrac{(\cos x+1)(\cos x+2)}{\cos x+1} = \lim\limits_{x\to\pi}(\cos x+2) = 1.$

2.6.52 $\lim\limits_{x\to 3\pi/2}\dfrac{\sin^2 x+6\sin x+5}{\sin^2 x-1} = \lim\limits_{x\to 3\pi/2}\dfrac{(\sin x+5)(\sin x+1)}{(\sin x-1)(\sin x+1)} = \lim\limits_{x\to 3\pi/2}\dfrac{\sin x+5}{\sin x-1} = \dfrac{4}{-2} = -2.$

2.6.53 $\lim\limits_{x\to 3}\sqrt{x^2+7} = \sqrt{9+7} = 4.$

2.6.54 $\lim\limits_{t\to 2}\dfrac{t^2+5}{1+\sqrt{t^2+5}} = \dfrac{9}{1+\sqrt{9}} = \dfrac{9}{4}.$

2.6.55 $\lim\limits_{x\to\pi/2}\dfrac{\sin x-1}{\sqrt{\sin x}-1} = \lim\limits_{x\to\pi/2}(\sqrt{\sin x}+1) = 2.$

2.6.56 $\lim\limits_{\theta\to 0}\dfrac{\frac{1}{2+\sin\theta}-\frac{1}{2}}{\sin\theta}\cdot\dfrac{(2)(2+\sin\theta)}{(2)(2+\sin\theta)} = \lim\limits_{\theta\to 0}\dfrac{2-(2+\sin\theta)}{(\sin\theta)(2)(2+\sin\theta)} = \lim\limits_{\theta\to 0}-\dfrac{1}{2(2+\sin\theta)} = -\dfrac{1}{4}.$

2.6.57 $\lim\limits_{x\to 0}\dfrac{\cos x-1}{\sin^2 x} = \lim\limits_{x\to 0}\dfrac{\cos x-1}{1-\cos^2 x} = \lim\limits_{x\to 0}\dfrac{\cos x-1}{(1-\cos x)(1+\cos x)} = \lim\limits_{x\to 0}-\dfrac{1}{1+\cos x} = -\dfrac{1}{2}.$

2.6.58 $\lim\limits_{x\to 0^+}\dfrac{1-\cos^2 x}{\sin x} = \lim\limits_{x\to 0^+}\dfrac{\sin^2 x}{\sin x} = \lim\limits_{x\to 0^+}\sin x = 0.$

2.6.59 $\lim\limits_{x\to 0}\dfrac{e^{4x}-1}{e^x-1} = \lim\limits_{x\to 0}\dfrac{(e^{2x}+1)(e^{2x}-1)}{e^x-1} = \lim\limits_{x\to 0}\dfrac{(e^{2x}+1)(e^x-1)(e^x+1)}{e^x-1} =$
$\lim\limits_{x\to 0}(e^{2x}+1)(e^x+1) = 2\cdot 2 = 4.$

2.6.60 $\lim\limits_{x\to e^2}\dfrac{\ln^2 x-5\ln x+6}{\ln x-2} = \lim\limits_{x\to e^2}\dfrac{(\ln x-2)(\ln x-3)}{\ln x-2} = \lim\limits_{x\to e^2}(\ln x-3) = -1.$

2.6.61 $f(x)=\csc x$ isn't defined at $x=k\pi$ where k is an integer, so it isn't continuous at those points. So it is continuous on intervals of the form $(k\pi,(k+1)\pi)$ where k is an integer. $\lim\limits_{x\to\pi/4}\csc x = \sqrt{2}$. $\lim\limits_{x\to 2\pi^-}\csc x = -\infty$.

2.6.62 f is defined on $[0,\infty)$, and it is continuous there, because it is the composition of continuous functions defined on that interval. $\lim\limits_{x\to 4} f(x) = e^2$. $\lim\limits_{x\to 0} f(x)$ does not exist—but $\lim\limits_{x\to 0^+} f(x) = e^0 = 1$, because f is continuous from the right.

2.6.63 f isn't defined for any number of the form $\pi/2+k\pi$ where k is an integer, so it isn't continuous there. It is continuous on intervals of the form $(\pi/2+k\pi,\pi/2+(k+1)\pi)$, where k is an integer.

$\lim\limits_{x\to\pi/2^-} f(x) = \infty$. $\lim\limits_{x\to 4\pi/3} f(x) = \dfrac{1-\sqrt{3}/2}{-1/2} = \sqrt{3}-2.$

2.6.64 The domain of f is $(0, 1]$, and f is continuous on this interval because it is the quotient of two continuous functions and the function in the denominator isn't zero on that interval.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{\ln x}{\sin^{-1}(x)} = \frac{\ln 1}{\sin^{-1}(1)} = \frac{0}{\pi/2} = 0.$$

2.6.65 This function is continuous on its domain, which is $(-\infty, 0) \cup (0, \infty)$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^x}{1-e^x} = \infty, \text{ while } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^x}{1-e^x} = -\infty.$$

2.6.66 This function is continuous on its domain, which is $(-\infty, 0) \cup (0, \infty)$.

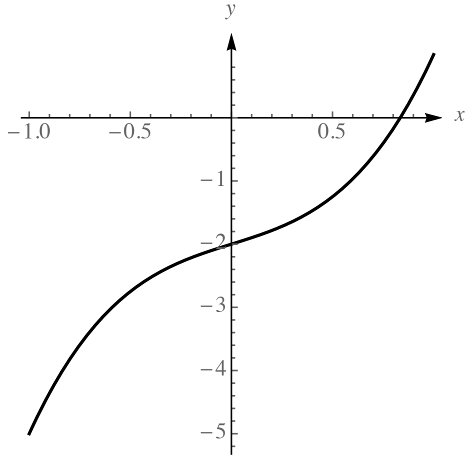
$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{2x}-1}{e^x-1} = \lim_{x\to 0} \frac{(e^x+1)(e^x-1)}{e^x-1} = \lim_{x\to 0}(e^x+1) = 2.$$

2.6.67

a. Note that $f(x) = 2x^3 + x - 2$ is continuous everywhere, so in particular it is continuous on $[-1, 1]$. Note that $f(-1) = -5 < 0$ and $f(1) = 1 > 0$. Because 0 is an intermediate value between $f(-1)$ and $f(1)$, the Intermediate Value Theorem guarantees a number c between -1 and 1 where $f(c) = 0$.

b. Using a graphing calculator and a computer algebra system, we see that the root of f is about 0.835.

c.

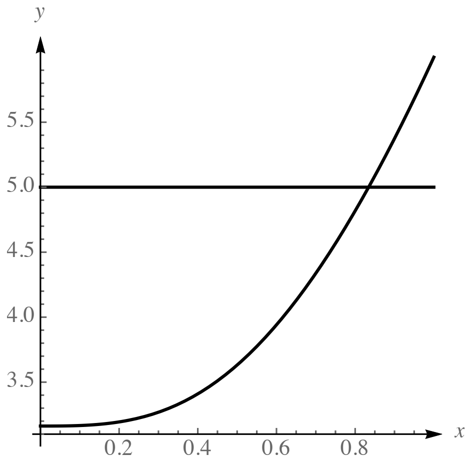


2.6.68

a. Note that $f(x) = \sqrt{x^4 + 25x^3 + 10} - 5$ is continuous on its domain, so in particular it is continuous on $[0, 1]$. Note that $f(0) = \sqrt{10} - 5 < 0$ and $f(1) = 6 - 5 = 1 > 0$. Because 0 is an intermediate value between $f(0)$ and $f(1)$, the Intermediate Value Theorem guarantees a number c between 0 and 1 where $f(c) = 0$.

b. Using a graphing calculator and a computer algebra system, we see the root of $f(x)$ is at about .834.

c.

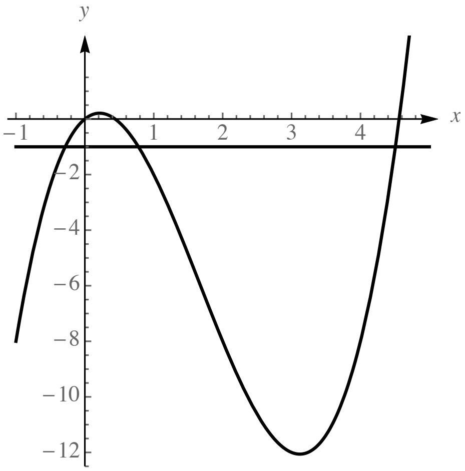


2.6.69

a. Note that $f(x) = x^3 - 5x^2 + 2x$ is continuous everywhere, so in particular it is continuous on $[-1, 5]$. Note that $f(-1) = -8 < -1$ and $f(5) = 10 > -1$. Because -1 is an intermediate value between $f(-1)$ and $f(5)$, the Intermediate Value Theorem guarantees a number c between -1 and 5 where $f(c) = -1$.

b. Using a graphing calculator and a computer algebra system, we see that there are actually three different values of c between -1 and 5 for which $f(c) = -1$. They are $c \approx -0.285$, $c \approx 0.778$, and $c \approx 4.507$.

c.

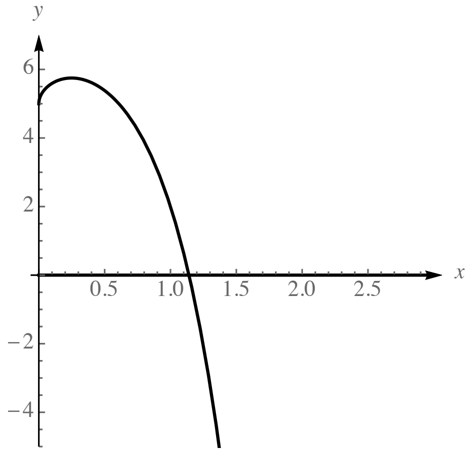


2.6.70

a. Note that $f(x) = -x^5 - 4x^2 + 2\sqrt{x} + 5$ is continuous on its domain, so in particular it is continuous on $[0, 3]$. Note that $f(0) = 5 > 0$ and $f(3) \approx -270.5 < 0$. Because 0 is an intermediate value between $f(0)$ and $f(3)$, the Intermediate Value Theorem guarantees a number c between 0 and 3 where $f(c) = 0$.

b. Using a graphing calculator and a computer algebra system, we see that the value of c guaranteed by the theorem is about 1.141.

c.

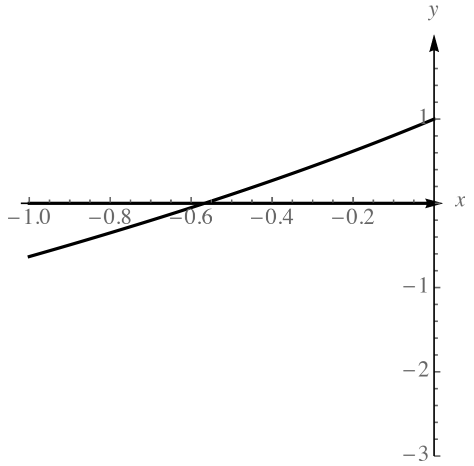


2.6.71

a. Note that $f(x) = e^x + x$ is continuous on its domain, so in particular it is continuous on $[-1, 0]$. Note that $f(-1) = \frac{1}{e} - 1 < 0$ and $f(0) = 1 > 0$. Because 0 is an intermediate value between $f(-1)$ and $f(0)$, the Intermediate Value Theorem guarantees a number c between -1 and 0 where $f(c) = 0$.

b. Using a graphing calculator and a computer algebra system, we see that the value of c guaranteed by the theorem is about -0.567.

c.

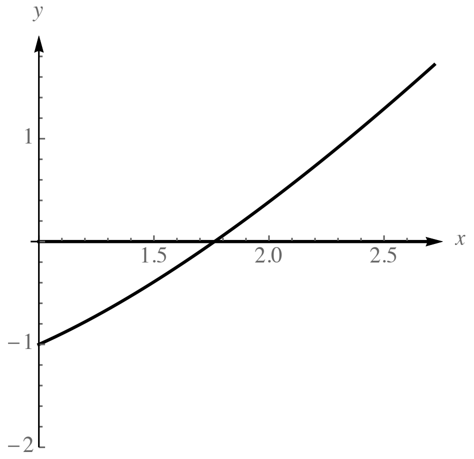


2.6.72

a. Note that $f(x) = x\ln x - 1$ is continuous on its domain, so in particular it is continuous on $[1, e]$. Note that $f(1) = \ln 1 - 1 = -1 < 0$ and $f(e) = e - 1 > 0$. Because 0 is an intermediate value between $f(1)$ and $f(e)$, the Intermediate Value Theorem guarantees a number c between 1 and e where $f(c) = 0$.

b. Using a graphing calculator and a computer algebra system, we see that the value of c guaranteed by the theorem is about 1.76322.

c.



2.6.73

a. True. If f is right continuous at a, then $f(a)$ exists and the limit from the right at a exists and is equal to $f(a)$. Because it is left continuous, the limit from the left exists — so we now know that the limit as $x \to a$ of $f(x)$ exists, because the two one-sided limits are both equal to $f(a)$.

b. True. If $\lim\limits_{x\to a} f(x) = f(a)$, then $\lim\limits_{x\to a^+} f(x) = f(a)$ and $\lim\limits_{x\to a^-} f(x) = f(a)$.

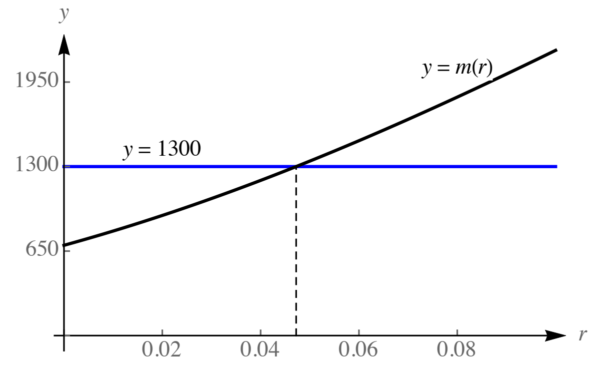
c. False. The statement would be true if f were continuous. However, if f isn't continuous, then the statement doesn't hold. For example, suppose that $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1; \\ 1 & \text{if } 1 \le x \le 2, \end{cases}$ Note that $f(0) = 0$ and $f(2) = 1$, but there is no number c between 0 and 2 where $f(c) = 1/2$.

d. False. Consider $f(x) = x^2$ and $a = -1$ and $b = 1$. Then f is continuous on $[a, b]$, but $\frac{f(1)+f(-1)}{2} = 1$, and there is no c on (a, b) with $f(c) = 1$.

2.6.74

a. Because m is a continuous function of r on $[0.04, 0.05]$, and because $m(0.04) \approx 1193.54$ and $m(0.05) \approx 1342.05$, (and 1300 is an intermediate value between these two numbers) the Intermediate Value Theorem guarantees a value of r between 0.04 and 0.05 where $m(r) = 1300$.

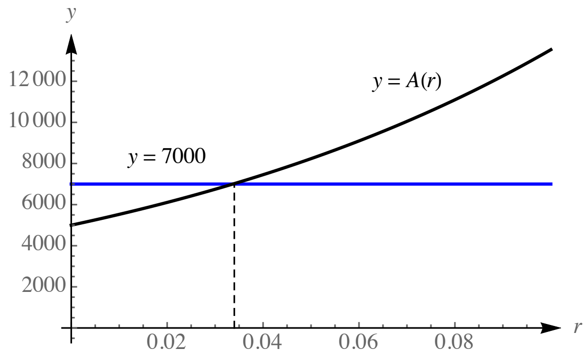
b. Using a computer algebra system, we see that the required interest rate is about 0.047.



2.6.75

a. Because A is a continuous function of r on $[0, 0.08]$, and because $A(0) = 5000$ and $A(0.08) \approx 11098.2$, (and 7000 is an intermediate value between these two numbers) the Intermediate Value Theorem guarantees a value of r between 0 and 0.08 where $A(r) = 7000$.

b. Solving $5000(1 + (r/12))^{120} = 7000$ for r, we see that $(1 + (r/12))^{120} = 7/5$, so $1 + r/12 = \sqrt[120]{7/5}$, so $r = 12(\sqrt[120]{7/5} - 1) \approx 0.034$.



2.6.76

a. Note that $A(0.01) \approx 2615.55$ and $A(0.1) \approx 3984.36$. By the Intermediate Value Theorem, there must be a number r_0 between 0.01 and 0.1 so that $A(r_0) = 3500$.

b. The desired value is $r_0 \approx 0.0728$ or 7.28%.

2.6.77 Consider the function $f(x) = \cos x - 2x$ on the interval $[0, 1]$. Note that $f(0) = 1$ and $f(\pi/2) = -\pi < 0$. So by the Intermediate Value Theorem, there must be a root of f on the interval $[0, \pi/2]$. Using a computer algebra system, we find a root of approximately 0.45.

2.6.78 Let $f(x) = |x|$.

For values of a other than 0, it is clear that $\lim_{x\to a} |x| = |a|$ because f is defined to be either the polynomial x (for values greater than 0) or the polynomial $-x$ (for values less than 0.) For the value of $a = 0$, we have $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0 = f(0)$. Also, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x) = -0 = 0$. Thus $\lim_{x\to 0} f(x) = f(0)$, so f is continuous at 0.

2.6.79 Because $f(x) = x^3 + 3x - 18$ is a polynomial, it is continuous on $(-\infty, \infty)$, and because the absolute value function is continuous everywhere, $|f(x)|$ is continuous everywhere.

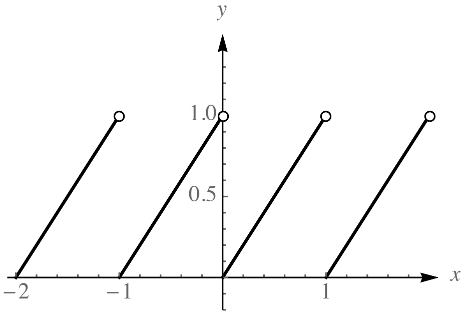
2.6.80 Let $f(x) = \dfrac{x+4}{x^2-4}$. Then f is continuous on $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. So $g(x) = |f(x)|$ is also continuous on this set.

2.6.81 Let $f(x) = \dfrac{1}{\sqrt{x}-4}$. Then f is continuous on $[0, 16) \cup (16, \infty)$. So $h(x) = |f(x)|$ is continuous on this set as well.

2.6.82 Because $x^2 + 2x + 5$ is a polynomial, it is continuous everywhere, as is $|x^2 + 2x + 5|$. So $h(x) = |x^2 + 2x + 5| + \sqrt{x}$ is continuous on its domain, namely $[0, \infty)$.

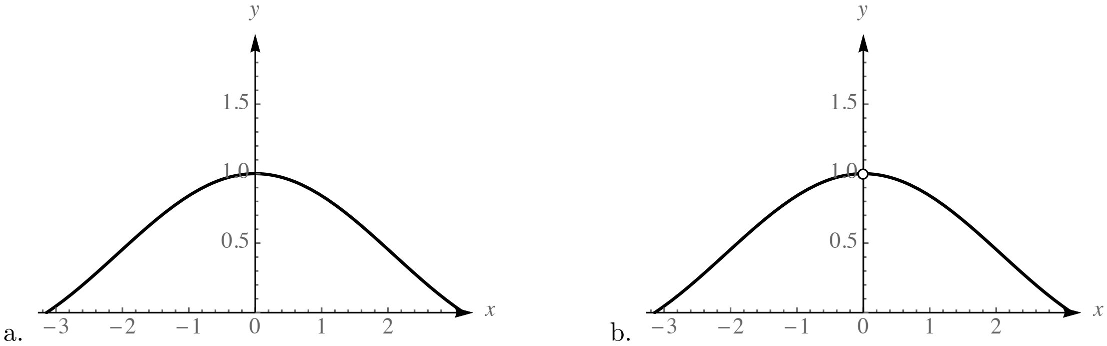
2.6.83

The graph shown isn't drawn correctly at the integers. At an integer a, the value of the function is 0, whereas the graph shown appears to take on all the values from 0 to 1.
Note that in the correct graph, $\lim_{x \to a^-} f(x) = 1$ and $\lim_{x \to a^+} f(x) = 0$ for every integer a.



2.6.84

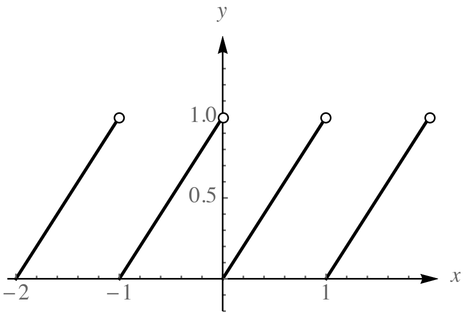
The graph as drawn on most graphing calculators appears to be continuous at $x = 0$, but it isn't, of course (because the function isn't defined at $x = 0$). A better drawing would show the "hole" in the graph at (0, 1).



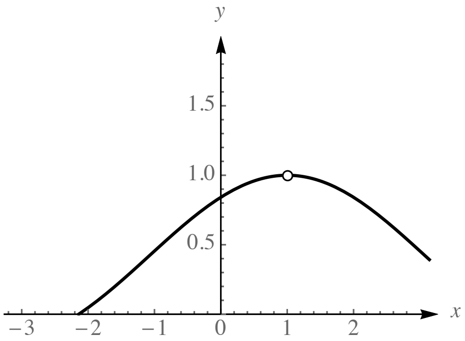
c. It appears that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

2.6.85 With slight modifications, we can use the examples from the previous two problems.

a. The function $y = x - \lfloor x \rfloor$ is defined at $x = 1$ but isn't continuous there.



b. The function $y = \frac{\sin(x-1)}{x-1}$ has a limit at $x = 1$, but isn't defined there, so isn't continuous there.



2.6.86 In order for this function to be continuous at $x = -1$, we require $\lim_{x\to -1} f(x) = f(-1) = a$. So the value of a must be equal to the value of $\lim_{x\to -1} \dfrac{x^2+3x+2}{x+1} = \lim_{x\to -1} \dfrac{(x+2)(x+1)}{x+1} = \lim_{x\to -1}(x+2) = 1$. Thus we must have $a = 1$.

2.6.87

a. In order for g to be continuous from the left at $x = 1$, we must have $\lim_{x\to 1^-} g(x) = g(1) = a$. We have $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-}(x^2+x) = 2$. So we must have $a = 2$.

b. In order for g to be continuous from the right at $x = 1$, we must have $\lim_{x\to 1^+} g(x) = g(1) = a$. We have $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+}(3x+5) = 8$. So we must have $a = 8$.

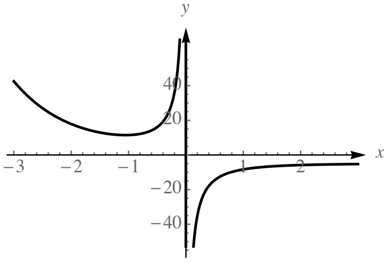
c. Because the limit from the left and the limit from the right at $x = 1$ don't agree, there is no value of a which will make the function continuous at $x = 1$.

2.6.88 $\lim_{x\to 0^-} \dfrac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to 0^-} \dfrac{2e^x+5e^{3x}}{e^{2x}(1-e^x)} = \infty$.

$\lim_{x\to 0^+} \dfrac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to 0^+} \dfrac{2e^x+5e^{3x}}{e^{2x}(1-e^x)} = -\infty$.

$\lim_{x\to -\infty} \dfrac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to -\infty} \dfrac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} \cdot \dfrac{e^{-2x}}{e^{-2x}} = \lim_{x\to -\infty} \dfrac{2e^{-x}+5e^{x}}{1-e^x} = \infty$.

$\lim_{x\to \infty} \dfrac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to \infty} \dfrac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} \cdot \dfrac{e^{-3x}}{e^{-3x}} = \lim_{x\to \infty} \dfrac{2e^{-2x}+5}{e^{-x}-1} = -5$.

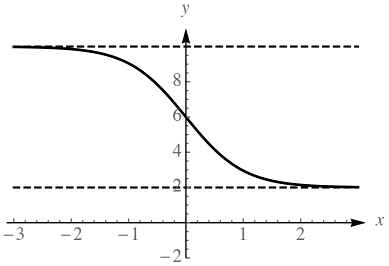


There is a vertical asymptote at $x = 0$, and the line $y = -5$ is a horizontal asymptote.

2.6.89 $\lim_{x\to 0} \dfrac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \dfrac{12}{2} = 6.$

$$\lim_{x\to -\infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \lim_{x\to -\infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \lim_{x\to -\infty} \frac{2e^{2x} + 10}{e^{2x} + 1} = \frac{10}{1} = 10.$$

$$\lim_{x\to \infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \lim_{x\to \infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to \infty} \frac{2 + 10e^{-2x}}{1 + e^{-2x}} = \frac{2}{1} = 2.$$



There are no vertical asymptotes. The lines $y = 2$ and $y = 10$ are horizontal asymptotes.

2.6.90 Let $f(x) = x^3 + 10x^2 - 100x + 50$. Note that $f(-20) < 0$, $f(-5) > 0$, $f(5) < 0$, and $f(10) > 0$. Because the given polynomial is continuous everywhere, the Intermediate Value Theorem guarantees us a root on $(-20, -5)$, at least one on $(-5, 5)$, and at least one on $(5, 10)$. Because there can be at most 3 roots and there are at least 3 roots, there must be exactly 3 roots. The roots are $x_1 \approx -16.32$, $x_2 \approx 0.53$ and $x_3 \approx 5.79$.

2.6.91 Let $f(x) = 70x^3 - 87x^2 + 32x - 3$. Note that $f(0) < 0$, $f(0.2) > 0$, $f(0.55) < 0$, and $f(1) > 0$. Because the given polynomial is continuous everywhere, the Intermediate Value Theorem guarantees us a root on $(0, 0.2)$, at least one on $(0.2, 0.55)$, and at least one on $(0.55, 1)$. Because there can be at most 3 roots and there are at least 3 roots, there must be exactly 3 roots. The roots are $x_1 = 1/7$, $x_2 = 1/2$ and $x_3 = 3/5$.

2.6.92

a. We have $f(0) = 0$, $f(2) = 3$, $g(0) = 3$ and $g(2) = 0$.

b. $h(t) = f(t) - g(t)$, $h(0) = -3$ and $h(2) = 3$.

c. By the Intermediate Value Theorem, because h is a continuous function and 0 is an intermediate value between -3 and 3, there must be a time c between 0 and 2 where $h(c) = 0$. At this point $f(c) = g(c)$, and at that time, the distance from the car is the same on both days, so the hiker is passing over the exact same point at that time.

2.6.93 We can argue essentially like the previous problem, or we can imagine an identical twin to the original monk, who takes an identical version of the original monk's journey up the winding path while the monk is taking the return journey down. Because they must pass somewhere on the path, that point is the one we are looking for.

2.6.94

a. Because $|-1| = 1$, $|g(x)| = 1$, for all x.

b. The function g isn't continuous at $x = 0$, because $\lim_{x\to 0^+} g(x) = 1 \neq -1 = \lim_{x\to 0^-} g(x)$.

c. This constant function is continuous everywhere, in particular at $x = 0$.

d. This example shows that in general, the continuity of $|g|$ does not imply the continuity of g.

2.6.95 The discontinuity is not removable, because $\lim_{x\to a} f(x)$ does not exist. The discontinuity pictured is a jump discontinuity.

2.6.96 The discontinuity is not removable, because $\lim_{x\to a} f(x)$ does not exist. The discontinuity pictured is an infinite discontinuity.

2.6.97 Note that $\lim_{x\to 2} \dfrac{x^2 - 7x + 10}{x - 2} = \lim_{x\to 2} \dfrac{(x-2)(x-5)}{x-2} = \lim_{x\to 2}(x - 5) = -3$. Because this limit exists, the discontinuity is removable.

2.6.98 Note that $\lim_{x\to 1} \dfrac{x^2 - 1}{1 - x} = \lim_{x\to 1} \dfrac{(x-1)(x+1)}{1-x} = \lim_{x\to 1}[-(x+1)] = -2$. Because this limit exists, the discontinuity is removable.

2.6.99 Note that $h(x) = \dfrac{x^3 - 4x^2 + 4x}{x(x-1)} = \dfrac{x(x-2)^2}{x(x-1)}$. Thus $\lim_{x\to 0} h(x) = -4$, and the discontinuity at $x = 0$ is removable. However, $\lim_{x\to 1} h(x)$ does not exist, and the discontinuity at $x = 1$ is not removable (it is infinite.)

2.6.100 This is a jump discontinuity, because $\lim_{x\to 2^+} f(x) = 1$ and $\lim_{x\to 2^-} f(x) = -1$.

2.6.101

a. Note that $-1 \le \sin(1/x) \le 1$ for all $x \neq 0$, so $-x \le x\sin(1/x) \le x$ (for $x > 0$. For $x < 0$ we would have $x \le x\sin(1/x) \le -x$.) Because both $x \to 0$ and $-x \to 0$ as $x \to 0$, the Squeeze Theorem tells us that $\lim_{x\to 0} x\sin(1/x) = 0$ as well. Because this limit exists, the discontinuity is removable.

b. Note that as $x \to 0^+$, $1/x \to \infty$, and thus $\lim_{x\to 0^+} \sin(1/x)$ does not exist. So the discontinuity is not removable.

2.6.102 Because g is continuous at a, as $x \to a$, $g(x) \to g(a)$. Because f is continuous at $g(a)$, as $z \to g(a)$, $f(z) \to f(g(a))$. Let $z = g(x)$, and suppose $x \to a$. Then $g(x) = z \to g(a)$, so $f(z) = f(g(x)) \to f(g(a))$, as desired.

2.6.103

a. Consider $g(x) = x + 1$ and $f(x) = \frac{|x-1|}{x-1}$. Note that both g and f are continuous at $x = 0$. However $f(g(x)) = f(x+1) = \frac{|x|}{x}$ is not continuous at 0.

b. The previous theorem says that the composition of f and g is continuous at a if g is continuous at a and f is continuous at $g(a)$. It does not say that if g and f are both continuous at a that the composition is continuous at a.

2.6.104 The Intermediate Value Theorem requires that our function be continuous on the given interval. In this example, the function f is not continuous on $[-2, 2]$ because it isn't continuous at 0.

2.6.105

a. Using the hint, we have

$$\sin x = \sin(a + (x - a)) = \sin a \cos(x - a) + \sin(x - a) \cos a.$$

Note that as $x \to a$, we have that $\cos(x - a) \to 1$ and $\sin(x - a) \to 0$.

So,

$$\lim_{x\to a} \sin x = \lim_{x\to a} \sin(a+(x-a)) = \lim_{x\to a} (\sin a \cos(x - a) + \sin(x - a) \cos a) = (\sin a)\cdot 1+0\cdot \cos a = \sin a.$$

b. Using the hint, we have

$$\cos x = \cos(a + (x - a)) = \cos a \cos(x - a) - \sin a \sin(x - a).$$

So,

$$\begin{aligned}\lim_{x\to a} \cos x = \lim_{x\to a} \cos(a + (x - a)) &= \lim_{x\to a} ((\cos a) \cos(x - a) - (\sin a) \sin(x - a)) \\ &= (\cos a) \cdot 1 - (\sin a) \cdot 0 = \cos a.\end{aligned}$$

2.7 Precise Definitions of Limits

2.7.1 Note that all the numbers in the interval $(1, 3)$ are within 1 unit of the number 2. So $|x - 2| < 1$ is true for all numbers in that interval. In fact, $\{x\colon 0 < |x - 2| < 1\}$ is exactly the set $(1, 3)$ with $x \neq 2$.

2.7.2 Note that all the numbers in the interval $(2, 6)$ are within 2 units of the number 4. So $|f(x) - 4| < \varepsilon$ for $\varepsilon = 2$ (or any number greater than 2).

2.7.3

a. This is symmetric about $x = 5$, because $\frac{1+9}{2} = 5$.

b. This is symmetric about $x = 5$, because $\frac{4+6}{2} = 5$.

c. This is not symmetric about $x = 5$, because $\frac{3+8}{2} \neq 5$.

d. This is symmetric about $x = 5$, because $\frac{4.5+5.5}{2} = 5$.

2.7.4 The set $\{x : |x - a| < \delta\}$ is the interval $(a - \delta, a + \delta)$ and $\{x : 0 < |x - a| < \delta\}$ is the set of all points in the interval $(a - \delta, a + \delta)$ excluding the point a.

2.7.5 $\lim_{x\to a} f(x) = L$ if for any arbitrarily small positive number ε, there exists a number δ, so that $f(x)$ is within ε units of L for any number x within δ units of a (but not including a itself).

2.7.6 The set of all x for which $|f(x) - L| < \varepsilon$ is the set of numbers so that the value of the function f at those numbers is within ε units of L.

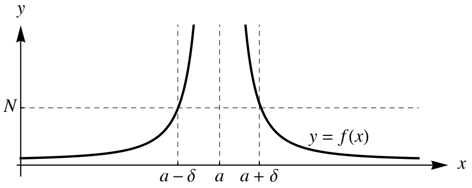
2.7.7 We are given that $|f(x) - 5| < 0.1$ for values of x in the interval $(0, 5)$, so we need to ensure that the set of x values we are allowing fall in this interval.

Note that the number 0 is two units away from the number 2 and the number 5 is three units away from the number 2. In order to be sure that we are talking about numbers in the interval $(0, 5)$ when we write $|x - 2| < \delta$, we would need to have $\delta = 2$ (or a number less than 2). In fact, the set of numbers for which $|x - 2| < 2$ is the interval $(0, 4)$ which is a subset of $(0, 5)$.

If we were to allow δ to be any number greater than 2, then the set of all x so that $|x - 2| < \delta$ would include numbers less than 0, and those numbers aren't on the interval $(0, 5)$.

2.7.8

$\lim_{x \to a} f(x) = \infty$, if for any $N > 0$, there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $f(x) > N$.



2.7.9

a. In order for f to be within 2 units of 5, it appears that we need x to be within 1 unit of 2. So $\delta = 1$.

b. In order for f to be within 1 unit of 5, it appears that we would need x to be within 1/2 unit of 2. So $\delta = 0.5$.

2.7.10

a. In order for f to be within 1 unit of 4, it appears that we would need x to be within 1 unit of 2. So $\delta = 1$.

b. In order for f to be within 1/2 unit of 4, it appears that we would need x to be within 1/2 unit of 2. So $\delta = 1/2$.

2.7.11

a. In order for f to be within 3 units of 6, it appears that we would need x to be within 2 units of 3. So $\delta = 2$.

b. In order for f to be within 1 unit of 6, it appears that we would need x to be within 1/2 unit of 3. So $\delta = 1/2$.

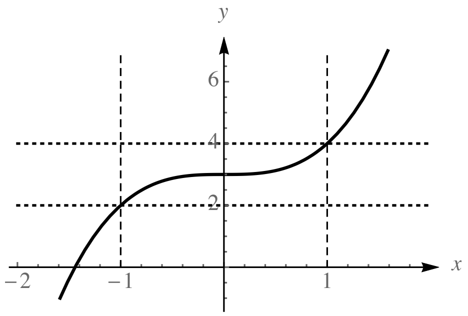
2.7.12

a. In order for f to be within 1 unit of 5, it appears that we would need x to be within 3 units of 4. So $\delta = 3$.

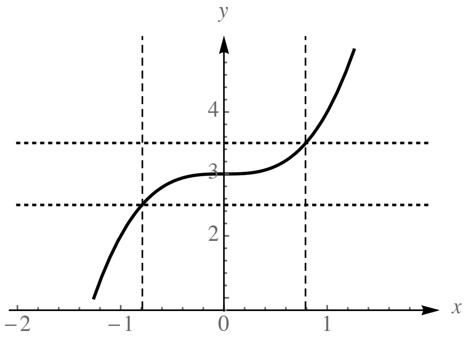
b. In order for f to be within 1/2 unit of 5, it appears that we would need x to be within 2 units of 4. So $\delta = 2$.

2.7.13

a. If $\varepsilon = 1$, we need $|x^3 + 3 - 3| < 1$. So we need $|x| < \sqrt[3]{1} = 1$ in order for this to happen. Thus $\delta = 1$ will suffice.

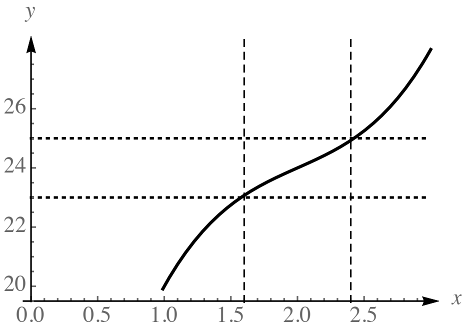


b. If $\varepsilon = 0.5$, we need $|x^3 + 3 - 3| < 0.5$. So we need $|x| < \sqrt[3]{0.5}$ in order for this to happen. Thus $\delta = \sqrt[3]{0.5} \approx 0.79$ will suffice.

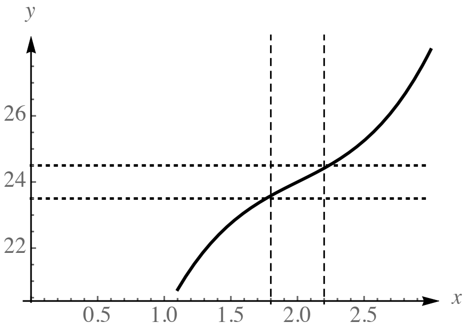


2.7.14

a. By looking at the graph, it appears that for $\varepsilon = 1$, we would need δ to be about 0.4 or less.



b. By looking at the graph, it appears that for $\varepsilon = 0.5$, we would need δ to be about 0.2 or less.



2.7.15

a. For $\varepsilon = 1$, the required value of δ would also be 1. A larger value of δ would work to the right of 2, but this is the largest one that would work to the left of 2.

b. For $\varepsilon = 1/2$, the required value of δ would also be $1/2$.

c. It appears that for a given value of ε, it would be wise to take $\delta = \min(\varepsilon, 2)$. This assures that the desired inequality is met on both sides of 2.

2.7.16

a. For $\varepsilon = 2$, the required value of δ would be 1 (or smaller). This is the largest value of δ that works on either side.

b. For $\varepsilon = 1$, the required value of δ would be $1/2$ (or smaller). This is the largest value of δ that works on the right of 4.

c. It appears that for a given value of ε, the corresponding value of $\delta = \min(5/2, \varepsilon/2)$.

2.7.17

a. For $\varepsilon = 2$, it appears that a value of $\delta = 1$ (or smaller) would work.

b. For $\varepsilon = 1$, it appears that a value of $\delta = 1/2$ (or smaller) would work.

c. For an arbitrary ε, a value of $\delta = \varepsilon/2$ or smaller appears to suffice.

2.7.18

a. For $\varepsilon = 1/2$, it appears that a value of $\delta = 1$ (or smaller) would work.

b. For $\varepsilon = 1/4$, it appears that a value of $\delta = 1/2$ (or smaller) would work.

c. For an arbitrary ε, a value of 2ε or smaller appears to suffice.

2.7.19 For any $\varepsilon > 0$, let $\delta = \varepsilon/8$. Then if $0 < |x-1| < \delta$, we would have $|x-1| < \varepsilon/8$. Then $|8x-8| < \varepsilon$, so $|(8x+5)-13| < \varepsilon$. This last inequality has the form $|f(x)-L| < \varepsilon$, which is what we were attempting to show. Thus, $\lim\limits_{x\to 1}(8x+5) = 13$.

2.7.20 For any $\varepsilon > 0$, let $\delta = \varepsilon/2$. Then if $0 < |x-3| < \delta$, we would have $|x-3| < \varepsilon/2$. Then $|2x-6| < \varepsilon$, so $|-2x+6| < \varepsilon$, so $|(-2x+8)-2| < \varepsilon$. This last inequality has the form $|f(x)-L| < \varepsilon$, which is what we were attempting to show. Thus, $\lim\limits_{x\to 3}(-2x+8) = 2$.

2.7.21 First note that if $x \neq 4$, $f(x) = \frac{x^2-16}{x-4} = x+4$.

Now if $\varepsilon > 0$ is given, let $\delta = \varepsilon$. Now suppose $0 < |x-4| < \delta$. Then $x \neq 4$, so the function $f(x)$ can be described by $x+4$. Also, because $|x-4| < \delta$, we have $|x-4| < \varepsilon$. Thus $|(x+4)-8| < \varepsilon$. This last inequality has the form $|f(x)-L| < \varepsilon$, which is what we were attempting to show. Thus, $\lim\limits_{x\to 4}\dfrac{x^2-16}{x-4} = 8$.

2.7.22 First note that if $x \neq 3$, $f(x) = \frac{x^2-7x+12}{x-3} = \frac{(x-4)(x-3)}{x-3} = x-4$.

Now if $\varepsilon > 0$ is given, let $\delta = \varepsilon$. Now suppose $0 < |x-3| < \delta$. Then $x \neq 3$, so the function $f(x)$ can be described by $x-4$. Also, because $|x-3| < \delta$, we have $|x-3| < \varepsilon$. Thus $|(x-4)-(-1)| < \varepsilon$. This last inequality has the form $|f(x)-L| < \varepsilon$, which is what we were attempting to show. Thus, $\lim\limits_{x\to 3} f(x) = -1$.

2.7.23 Let $\varepsilon > 0$ be given and assume that $0 < |x-0| < \delta$ where $\delta = \varepsilon$. It follows that $||x|-0| = |x| = |x-0| < \delta = \varepsilon$. We have shown that for any $\varepsilon > 0$, $||x|-0| < \varepsilon$ whenever $0 < |x-0| < \delta$, provided $0 < \delta \leq \varepsilon$.

2.7.24 Let $\varepsilon > 0$ be given and assume that $0 < |x-0| < \delta$ where $\delta = \frac{\varepsilon}{5}$. It follows that $||5x|-0| = 5|x-0| < 5\delta = 5\left(\frac{\varepsilon}{5}\right) = \varepsilon$. We have shown that for any $\varepsilon > 0$, $||5x|-0| < \varepsilon$ whenever $0 < |x-0| < \delta$, provided $0 < \delta \leq \frac{\varepsilon}{5}$.

2.7.25 Let $\varepsilon > 0$ be given and assume that $0 < |x-7| < \delta$ where $\delta = \varepsilon/3$. If $x < 7$, $|f(x)-9| = |3x-12-9| = 3|x-7| < 3\delta = 3(\varepsilon/3) = \varepsilon$; if $x > 7$, then $|f(x)-9| = |x+2-9| = |x-7| < \delta = \varepsilon/3 < \varepsilon$. We've shown that for any $\varepsilon > 0$, $|f(x)-9| < \varepsilon$ whenever $0 < |x-7| < \delta$, provided $0 < \delta \leq \varepsilon/3$.

2.7.26 Let $\varepsilon > 0$ be given and assume that $0 < |x-5| < \delta$ where $\delta = \varepsilon/4$. If $x < 5$, $|f(x)-4| = |2x-6-4| = 2|x-5| < 2\delta = 2(\varepsilon/4) = \varepsilon/2 < \varepsilon$; if $x > 5$, then $|f(x)-4| = |-4x+24-4| = |-4(x-5)| = 4|x-5| < 4\delta = 4(\varepsilon/4) = \varepsilon$. We've shown that for any $\varepsilon > 0$, $|f(x)-4| < \varepsilon$ whenever $0 < |x-5| < \delta$, provided $0 < \delta \leq \varepsilon/4$.

2.7.27 Let $\varepsilon > 0$ be given. Let $\delta = \sqrt{\varepsilon}$. Then if $0 < |x-0| < \delta$, we would have $|x| < \sqrt{\varepsilon}$. But then $|x^2| < \varepsilon$, which has the form $|f(x)-L| < \varepsilon$. Thus, $\lim\limits_{x\to 0} f(x) = 0$.

2.7.28 Let $\varepsilon > 0$ be given. Let $\delta = \sqrt{\varepsilon}$. Then if $0 < |x-3| < \delta$, we would have $|x-3| < \sqrt{\varepsilon}$. But then $|(x-3)^2| < \varepsilon$, which has the form $|f(x) - L| < \varepsilon$. Thus, $\lim\limits_{x\to 3} f(x) = 0$.

2.7.29 Let $\varepsilon > 0$ be given and assume that $0 < |x-2| < \delta$ where $\delta = \min\{1, \varepsilon/8\}$. By factoring $x^2+3x-10$, we find that $|x^2+3x-10| = |x-2||x+5|$. Because $|x-2| < \delta$ and $\delta \le 1$, we have $|x-2| < 1$, which implies that $-1 < x-2 < 1$, or $1 < x < 3$. It follows that $|x+5| = x+5 \le 8$. We also know that $|x-2| < \varepsilon/8$ because $0 < |x-2| < \delta$ and $\delta \le \varepsilon/8$. Therefore $|x^2+3x-10| = |x-2||x+5| < (\varepsilon/8)\cdot 8 = \varepsilon$. We have shown that for any $\varepsilon > 0$, $|x^2+3x-10| < \varepsilon$ whenever $0 < |x-2| < \delta$, provided $0 < \delta \le \min\{1, \varepsilon/8\}$.

2.7.30 Let $\varepsilon > 0$ be given and assume that $0 < |x-4| < \delta$ where $\delta = \min\{1, \varepsilon/14\}$. Observe that $|2x^2-4x+1-17| = |2x^2-4x-16| = 2|x-4||x+2|$. Because $|x-4| < \delta$ and $\delta \le 1$, we have $|x-4| < 1$ which implies that $-1 < x-4 < 1$ or $3 < x < 5$. It follows that $|x+2| = x+2 \le 7$. We also know that $|x-4| < \varepsilon/14$ because $0 < |x-4| < \delta$ and $\delta \le \varepsilon/14$. Therefore $|2x^2-4x+1| = 2|x-4||x+2| < 2(\varepsilon/14)\cdot 7 = \varepsilon$. We have shown that for any $\varepsilon > 0$, $|2x^2-4x+1-17| < \varepsilon$ provided $0 < \delta \le \min\{1, \varepsilon/14\}$.

2.7.31 Let $\varepsilon > 0$ be given and assume that $0 < |x-(-3)| < \delta$ where $\delta = \varepsilon/2$. Using the inequality $||a|-|b|| \le |a-b|$ with $a = 2x$ and $b = -6$, it follows that $||2x|-6| = ||2x|-|-6|| \le |2x-(-6)| = 2|x-(-3)| < 2\delta = 2(\varepsilon/2) = \varepsilon$ and therefore $||2x|-6| < \varepsilon$. We have shown that for any $\varepsilon > 0$, $||2x|-6| < \varepsilon$ whenever $0 < |x-(-3)| < \delta$, provided $0 < \delta \le \varepsilon/2$.

2.7.32 Let $\varepsilon > 0$ be given and assume that $0 < |x-25| < \delta$ where $\delta = \min\{25, 5\varepsilon\}$. Because $|x-25| < \delta$ and $\delta \le 25$, we have $|x-25| < 25$, which implies that $-25 < x-25 < 25$ or $0 < x < 50$. Because $x > 0$, we have $\sqrt{x}+5 > 5$ and it follows that $\frac{1}{\sqrt{x}+5} < \frac{1}{5}$. Therefore

$$|\sqrt{x}-5| = \frac{|x-25|}{\sqrt{x}+5} \le \frac{|x-25|}{5} < \frac{\delta}{5} \le \frac{5\varepsilon}{5} = \varepsilon.$$

We have shown that for any $\varepsilon > 0$, $|\sqrt{x}-5| < \varepsilon$, provided $0 < \delta \le \min\{25, 5\varepsilon\}$.

2.7.33 Assume $|x-3| < 1$, as indicated in the hint. Then $2 < x < 4$, so $\frac{1}{4} < \frac{1}{x} < \frac{1}{2}$, and thus $\left|\frac{1}{x}\right| < \frac{1}{2}$.

Also note that the expression $\left|\frac{1}{x} - \frac{1}{3}\right|$ can be written as $\left|\frac{x-3}{3x}\right|$.

Now let $\varepsilon > 0$ be given. Let $\delta = \min(6\varepsilon, 1)$. Now assume that $0 < |x-3| < \delta$. Then

$$|f(x) - L| = \left|\frac{x-3}{3x}\right| < \left|\frac{x-3}{6}\right| < \frac{6\varepsilon}{6} = \varepsilon.$$

Thus we have established that $\left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon$ whenever $0 < |x-3| < \delta$.

2.7.34 Note that for $x \ne 4$, the expression $\frac{x-4}{\sqrt{x}-2} = \frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \sqrt{x}+2$. Also note that if $|x-4| < 1$, then x is between 3 and 5, so $\sqrt{x} > 0$. Then it follows that $\sqrt{x}+2 > 2$, and therefore $\frac{1}{\sqrt{x}+2} < \frac{1}{2}$. We will use this fact below.

Let $\varepsilon > 0$ be given. Let $\delta = \min(2\varepsilon, 1)$. Suppose that $0 < |x-4| < \delta$, so $|x-4| < 2\varepsilon$. We have

$$\begin{aligned}|f(x) - L| = |\sqrt{x}+2-4| = |\sqrt{x}-2| &= \left|\frac{x-4}{\sqrt{x}+2}\right| \\ &< \frac{|x-4|}{2} < \frac{2\varepsilon}{2} = \varepsilon.\end{aligned}$$

2.7.35 Assume $|x-(1/10)| < (1/20)$, as indicated in the hint. Then $1/20 < x < 3/20$, so $\frac{20}{3} < \frac{1}{x} < \frac{20}{1}$, and thus $\left|\frac{1}{x}\right| < 20$.

Also note that the expression $\left|\frac{1}{x} - 10\right|$ can be written as $\left|\frac{10x-1}{x}\right|$.

Let $\varepsilon > 0$ be given. Let $\delta = \min(\varepsilon/200, 1/20)$. Now assume that $0 < |x-(1/10)| < \delta$. Then

$$\begin{aligned}|f(x) - L| = \left|\frac{10x-1}{x}\right| &< |(10x-1)\cdot 20| \\ &\le |x-(1/10)|\cdot 200 < \frac{\varepsilon}{200}\cdot 200 = \varepsilon.\end{aligned}$$

Thus we have established that $\left|\frac{1}{x} - 10\right| < \varepsilon$ whenever $0 < |x-(1/10)| < \delta$.

2.7.36 Multiplying both sides of the inequality $|\sin\frac{1}{x}| \le 1$ by $|x|$, we have $|x\sin\frac{1}{x}| \le |x|$. Let $\varepsilon > 0$ be given and assume that $0 < |x-0| < \delta$ where $\delta = \varepsilon$. We have $|x\sin\frac{1}{x} - 0| = |x\sin\frac{1}{x}| \le |x| \le |x-0| < \delta = \varepsilon$. Therefore it as been shown that for any $\varepsilon > 0$, $|x\sin\frac{1}{x} - 0| < \varepsilon$ whenever $0 < |x-0| < \delta$, provided $0 < \delta \le \varepsilon$.

2.7.37 Let $\varepsilon > 0$ be given and assume that $0 < |x-0| < \delta$ where $\delta = \min\{1, \sqrt{\varepsilon}/\sqrt{2}\}$. Because $|x-0| < \delta$, we have $|x| < 1$ and $|x| < \sqrt{\varepsilon}/\sqrt{2}$, which implies that $x^2 < 1$ and $x^2 < \varepsilon/2$. It follows that $|x^2+x^4-0| = x^2+x^4 = x^2(1+x^2) \le \frac{\varepsilon}{2}\cdot 2 = \varepsilon$. We have shown that for any $\varepsilon > 0$, $|x^2+x^4-0| < \varepsilon$ whenever $0 < |x-0| < \delta$, provided $0 < \delta \le \min\{1, \sqrt{\varepsilon}/\sqrt{2}\}$.

2.7.38 Let $f(x) = b$. Let $\varepsilon > 0$ be given and assume that $0 < |x-a| < \delta$ where $\delta = 1$ (or any other positive number). Then $|f(x) - b| = |b-b| = 0 < \varepsilon$. We have shown that for any $\varepsilon > 0$, $|b-b| < \varepsilon$ whenever $0 < |x-a| < \delta$, provided δ equals any positive number.

2.7.39 Let $m = 0$, then the proof is as follows: Let $\varepsilon > 0$ be given and assume that $0 < |x-a| < \delta$ where $\delta = 1$ (or any other positive number). Then $|f(x) - b| = |b-b| = 0 < \varepsilon$. We have shown that for any $\varepsilon > 0$, $|b-b| < \varepsilon$ whenever $0 < |x-a| < \delta$, provided δ equals any positive number.

Now assume that $m \ne 0$. Let $\varepsilon > 0$ be given and assume that $0 < |x-a| < \delta$ where $\delta = \varepsilon/|m|$. Then

$$|(mx+b) - (ma+b)| = |m||x-a| < |m|\delta = |m|(\varepsilon/|m|) = \varepsilon.$$

Therefore it has been shown that for any $\varepsilon > 0$, $|(mx+b)-(ma+b)| < \varepsilon$ whenever $0 < |x-a| < \delta$, provided $\delta = \varepsilon/|m|$.

2.7.40 Let $\varepsilon > 0$ be given and assume that $0 < |x-3| < \delta$ where $\delta = \min\{1, \varepsilon/37\}$. By factoring $x^3 - 27$, we find that $|x^3-27| = |x-3||x^2+3x+9|$. Because $|x-3| < \delta$ and $\delta \le 1$, we have $|x-3| \le 1$, which implies that $-1 < x-3 < 1$ or $2 < x < 4$. It follows that $|x^2+3x+9| = x^2+3x+9 \le 4^2+3(4)+9 = 37$. We also know that $|x-3| < \varepsilon/37$ because $0 < |x-3| < \delta$ and $\delta \le \varepsilon/37$. Therefore $|x^3-27| = |x-3||x^2+3x+9| \le (\varepsilon/37)\cdot 37 = \varepsilon$. We have shown that for any $\varepsilon > 0$, $|x^3-27| < \varepsilon$ whenever $0 < |x-3| < \delta$, provided $0 < \delta \le \min\{1, \varepsilon/37\}$.

2.7.41 Let $\varepsilon > 0$ be given and assume that $0 < |x-1| < \delta$ where $\delta = \min\left\{\dfrac{1}{2}, \dfrac{8\varepsilon}{65}\right\}$. Observe that $|x^4-1| = |(x^2-1)(x^2+1)| = |x-1||x+1||x^2+1|$. Because $|x-1| < \delta$ and $\delta \le \dfrac{1}{2}$, we have $|x-1| < \dfrac{1}{2}$, which implies that $-\dfrac{1}{2} < x-1 < \dfrac{1}{2}$, or $\dfrac{1}{2} < x < \dfrac{3}{2}$. It follows that $|x+1| = x+1 \le \dfrac{5}{2}$. Also $x^2 < \dfrac{9}{4}$, so $|x^2+1| = x^2+1 \le \dfrac{13}{4}$. We also know that $|x-1| < \dfrac{8\varepsilon}{65}$ because $|x-1| < \delta$ and $\delta \le \dfrac{8\varepsilon}{65}$. Therefore

$$|x^4-1| = |x-1||x+1||x^2+1| \le \frac{8\varepsilon}{65}\cdot\frac{5}{2}\cdot\frac{13}{4} = \varepsilon.$$

We have shown that for any $\varepsilon > 0$, $|x^4-1| < \varepsilon$ whenever $0 < |x-1| < \delta$, provided $0 < \delta = \min\left\{\dfrac{1}{2}, \dfrac{8\varepsilon}{65}\right\}$.

2.7.42 Note that if $|x-5| < 1$, then $4 < x < 6$, so that $9 < x+5 < 11$, so $|x+5| < 11$. Note also that $16 < x^2 < 36$, so $\frac{1}{x^2} < \frac{1}{16}$.

Let $\varepsilon > 0$ be given. Let $\delta = \min(1, \frac{400}{11}\varepsilon)$. Assume that $0 < |x-5| < \delta$. Then

$$\begin{aligned}|f(x) - L| &= \left|\frac{1}{x^2} - \frac{1}{25}\right| = \frac{|x+5||x-5|}{25x^2} \\ &< \frac{11|x-5|}{25x^2} < \frac{11}{25\cdot 16}|x-5| < \frac{11}{400}\frac{400\varepsilon}{11} = \varepsilon.\end{aligned}$$

2.7.43 Let $\varepsilon > 0$ be given.

Because $\lim_{x\to a} f(x) = L$, we know that there exists a $\delta_1 > 0$ so that $|f(x)-L| < \varepsilon/2$ when $0 < |x-a| < \delta_1$. Also, because $\lim_{x\to a} g(x) = M$, there exists a $\delta_2 > 0$ so that $|g(x) - M| < \varepsilon/2$ when $0 < |x-a| < \delta_2$.

Now let $\delta = \min(\delta_1, \delta_2)$.

Then if $0 < |x-a| < \delta$, we would have $|f(x) - g(x) - (L-M)| = |(f(x) - L) + (M - g(x))| \le |f(x) - L| + |M - g(x)| = |f(x) - L| + |g(x) - M| \le \varepsilon/2 + \varepsilon/2 = \varepsilon$. Note that the key inequality in this sentence follows from the triangle inequality.

2.7.44 First note that the theorem is trivially true if $c = 0$. So assume $c \neq 0$.

Let $\varepsilon > 0$ be given. Because $\lim_{x \to a} f(x) = L$, there exists a $\delta > 0$ so that if $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon/|c|$. But then $|c||f(x) - L| = |cf(x) - cL| < \varepsilon$, as desired. Thus, $\lim_{x \to a} cf(x) = cL$.

2.7.45 Let $N > 0$ be given. Let $\delta = 1/\sqrt{N}$. Then if $0 < |x - 4| < \delta$, we have $|x - 4| < 1/\sqrt{N}$. Taking the reciprocal of both sides, we have $\dfrac{1}{|x-4|} > \sqrt{N}$, and squaring both sides of this inequality yields $\dfrac{1}{(x-4)^2} > N$. Thus $\lim_{x \to 4} f(x) = \infty$.

2.7.46 Let $N > 0$ be given. Let $\delta = 1/\sqrt[4]{N}$. Then if $0 < |x - (-1)| < \delta$, we have $|x+1| < 1/\sqrt[4]{N}$. Taking the reciprocal of both sides, we have $\dfrac{1}{|x+1|} > \sqrt[4]{N}$, and raising both sides to the 4th power yields $\dfrac{1}{(x+1)^4} > N$. Thus $\lim_{x \to -1} f(x) = \infty$.

2.7.47 Let $N > 1$ be given. Let $\delta = 1/\sqrt{N-1}$. Suppose that $0 < |x - 0| < \delta$. Then $|x| < 1/\sqrt{N-1}$, and taking the reciprocal of both sides, we see that $1/|x| > \sqrt{N-1}$. Then squaring both sides yields $dsfrac1x^2 > N - 1$, so $\dfrac{1}{x^2} + 1 > N$. Thus $\lim_{x \to 0} f(x) = \infty$.

2.7.48 Let $N > 0$ be given. Let $\delta = 1/\sqrt[4]{N+1}$. Then if $0 < |x - 0| < \delta$, we would have $|x| < 1/\sqrt[4]{N+1}$. Taking the reciprocal of both sides yields $\dfrac{1}{|x|} > \sqrt[4]{N+1}$, and then raising both sides to the 4th power gives $\dfrac{1}{x^4} > N + 1$, so $\dfrac{1}{x^4} - 1 > N$. Now because $-1 \le \sin x \le 1$, we can surmise that $\frac{1}{x^4} - \sin x > N$ as well, because $\dfrac{1}{x^4} - \sin x \ge \dfrac{1}{x^4} - 1$. Hence $\lim_{x \to 0} \left(\dfrac{1}{x^4} - \sin x \right) = \infty$.

2.7.49

a. False. In fact, if the statement is true for a specific value of δ_1, then it would be true for any value of $\delta < \delta_1$. This is because if $0 < |x - a| < \delta$, it would automatically follow that $0 < |x - a| < \delta_1$.

b. False. This statement is not equivalent to the definition – note that it says "for an arbitrary δ there exists an ε" rather than "for an arbitrary ε there exists a δ."

c. True. This is the definition of $\lim_{x \to a} f(x) = L$.

d. True. Both inequalities describe the set of x's which are within δ units of a.

2.7.50

a. We want it to be true that $|f(x) - 2| < 0.25$. So we need $|x^2 - 2x + 3 - 2| = |x^2 - 2x + 1| = (x-1)^2 < 0.25$. Therefore we need $|x - 1| < \sqrt{0.25} = 0.5$. Thus we should let $\delta = 0.5$.

b. We want it to be true that $|f(x) - 2| < \varepsilon$. So we need $|x^2 - 2x + 3 - 2| = |x^2 - 2x + 1| = (x-1)^2 < \varepsilon$. Therefore we need $|x - 1| < \sqrt{\varepsilon}$. Thus we should let $\delta = \sqrt{\varepsilon}$.

2.7.51 Because we are approaching a from the right, we are only considering values of x which are close to, but a little larger than a. The numbers x to the right of a which are within δ units of a satisfy $0 < x - a < \delta$.

2.7.52 Because we are approaching a from the left, we are only considering values of x which are close to, but a little smaller than a. The numbers x to the left of a which are within δ units of a satisfy $0 < a - x < \delta$.

2.7.53

a. Let $\varepsilon > 0$ be given. let $\delta = \varepsilon/2$. Suppose that $0 < x < \delta$. Then $0 < x < \varepsilon/2$ and

$$\begin{aligned}|f(x) - L| &= |2x - 4 - (-4)| = |2x| = 2|x| \\ &= 2x < \varepsilon.\end{aligned}$$

b. Let $\varepsilon > 0$ be given. let $\delta = \varepsilon/3$. Suppose that $0 < 0 - x < \delta$. Then $-\delta < x < 0$ and $-\varepsilon/3 < x < 0$, so $\varepsilon > -3x$. We have

$$\begin{aligned}|f(x) - L| &= |3x - 4 - (-4)| = |3x| = 3|x| \\ &= -3x < \varepsilon.\end{aligned}$$

c. Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon/3$. Because $\varepsilon/3 < \varepsilon/2$, we can argue that $|f(x) - L| < \varepsilon$ whenever $0 < |x| < \delta$ exactly as in the previous two parts of this problem.

2.7.54

a. This statement holds for $\delta = 2$ (or any number less than 2).

b. This statement holds for $\delta = 2$ (or any number less than 2).

c. This statement holds for $\delta = 1$ (or any number less than 1).

d. This statement holds for $\delta = .5$ (or any number less than 0.5).

2.7.55 Let $\varepsilon > 0$ be given, and let $\delta = \varepsilon^2$. Suppose that $0 < x < \delta$, which means that $x < \varepsilon^2$, so that $\sqrt{x} < \varepsilon$. Then we have

$$|f(x) - L| = |\sqrt{x} - 0| = \sqrt{x} < \varepsilon.$$

as desired.

2.7.56

a. Suppose that $\lim\limits_{x\to a^-} f(x) = L$ and $\lim\limits_{x\to a^+} f(x) = L$. Let $\varepsilon > 0$ be given. There exists a number δ_1 so that $|f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta_1$, and there exists a number δ_2 so that $|f(x) - L| < \varepsilon$ whenever $0 < a - x < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. It immediately follows that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$, as desired.

b. Suppose $\lim\limits_{x\to a} f(x) = L$, and let $\varepsilon > 0$ be given. We know that a δ exists so that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$. In particular, it must be the case that $|f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta$ and also that $|f(x) - L| < \varepsilon$ whenever $0 < a - x < \delta$. Thus $\lim\limits_{x\to a^+} f(x) = L$ and $\lim\limits_{x\to a^-} f(x) = L$.

2.7.57

a. We say that $\lim\limits_{x\to a^+} f(x) = \infty$ if for each positive number N, there exists $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad a < x < a + \delta.$$

b. We say that $\lim\limits_{x\to a^-} f(x) = -\infty$ if for each negative number N, there exists $\delta > 0$ such that

$$f(x) < N \quad \text{whenever} \quad a - \delta < x < a.$$

c. We say that $\lim\limits_{x\to a^-} f(x) = \infty$ if for each positive number N, there exists $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad a - \delta < x < a.$$

2.7.58 Let $N < 0$ be given. Let $\delta = -1/N$, and suppose that $1 < x < 1 + \delta$. Then $1 < x < \dfrac{N-1}{N}$, so $\dfrac{1-N}{N} < -x < -1$, and therefore $1 + \dfrac{1-N}{N} < 1 - x < 0$, which can be written as $\dfrac{1}{N} < 1 - x < 0$. Taking reciprocals yields the inequality $N > \dfrac{1}{1-x}$, as desired.

2.7.59 Let $N > 0$ be given. Let $\delta = 1/N$, and suppose that $1 - \delta < x < 1$. Then $\dfrac{N-1}{N} < x < 1$, so $\dfrac{1-N}{N} > -x > -1$, and therefore $1 + \dfrac{1-N}{N} > 1 - x > 0$, which can be written as $\dfrac{1}{N} > 1 - x > 0$. Taking reciprocals yields the inequality $N < \dfrac{1}{1-x}$, as desired.

2.7.60 Let $M < 0$ be given. Let $\delta = \sqrt{-2/M}$. Suppose that $0 < |x-1| < \delta$. Then $(x-1)^2 < -2/M$, so $\dfrac{1}{(x-1)^2} > \dfrac{M}{-2}$, and $\dfrac{-2}{(x-1)^2} < M$, as desired.

2.7.61 Let $M < 0$ be given. Let $\delta = \sqrt[4]{-10/M}$. Suppose that $0 < |x+2| < \delta$. Then $(x+2)^4 < -10/M$, so $\dfrac{1}{(x+2)^4} > \dfrac{M}{-10}$, and $-\dfrac{10}{(x+2)^4} < M$, as desired.

2.7.62 Let $N > 0$ be given and let $N_1 = \max\{1, N - c\}$. Because $\lim\limits_{x\to a} f(x) = \infty$ there exists $\delta > 0$ such that $f(x) > N_1$ whenever $0 < |x-a| < \delta$. It follows that $f(x) + c > N_1 + c \geq N - c + c = N$. So for any $N > 0$, there exists $\delta > 0$ such that $f(x) + c > N$ whenever $0 < |x-a| < \delta$.

2.7.63 Let $N > 0$ be given. Because $\lim\limits_{x\to a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $f(x) > \frac{N}{2}$ whenever $0 < |x-a| < \delta_1$. Similarly, because $\lim\limits_{x\to a} g(x) = \infty$, there exists $\delta_2 > 0$ such that $g(x) > \frac{N}{2}$ whenever $0 < |x-a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$ and assume that $0 < |x-a| < \delta$. Because $\delta = \min\{\delta_1, \delta_2\}$, $\delta \leq \delta_1$ and $\delta \leq \delta_2$. It follows that $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$ and therefore $f(x) + g(x) > \frac{N}{2} + \frac{N}{2} = N$. So for any $N > 0$, there exists $\delta > 0$ such that $f(x) + g(x) > N$ whenever $0 < |x-a| < \delta$.

2.7.64 Let $\varepsilon > 0$ be given. Let $N = \dfrac{10}{\varepsilon}$. Suppose that $x > N$. Then $x > \dfrac{10}{\varepsilon}$ so $0 < \dfrac{10}{x} < \varepsilon$. Thus, $|\dfrac{10}{x} - 0| < \varepsilon$, as desired.

2.7.65 Let $\varepsilon > 0$ be given. Let $N = 1/\varepsilon$. Suppose that $x > N$. Then $\dfrac{1}{x} < \varepsilon$, and so

$$|f(x) - L| = |2 + \frac{1}{x} - 2| < \varepsilon.$$

2.7.66 Let $M > 0$ be given. Let $N = 100M$. Suppose that $x > N$. Then $x > 100M$, so $\dfrac{x}{100} > M$, as desired.

2.7.67 Let $M > 0$ be given. Let $N = M - 1$. Suppose that $x > N$. Then $x > M - 1$, so $x + 1 > M$, and thus $\dfrac{x^2 + x}{x} > M$, as desired.

2.7.68 Let $\varepsilon > 0$ be given. Because $\lim\limits_{x\to a} f(x) = L$, there exists a number δ_1 so that $|f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta_1$. And because $\lim\limits_{x\to a} h(x) = L$, there exists a number δ_2 so that $|h(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$, and suppose that $0 < |x-a| < \delta$. Because $f(x) \leq g(x) \leq h(x)$ for x near a, we also have that $f(x) - L \leq g(x) - L \leq h(x) - L$. Now whenever x is within δ units of a (but $x \neq a$), we also note that $-\varepsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \varepsilon$. Therefore $|g(x) - L| < \varepsilon$, as desired.

2.7.69 Let $\varepsilon > 0$ be given. Let $N = \lfloor (1/\varepsilon) \rfloor + 1$. By assumption, there exists an integer $M > 0$ so that $|f(x) - L| < 1/N$ whenever $|x - a| < 1/M$. Let $\delta = 1/M$.

Now assume $0 < |x - a| < \delta$. Then $|x - a| < 1/M$, and thus $|f(x) - L| < 1/N$. But then

$$|f(x) - L| < \frac{1}{\lfloor (1/\varepsilon) \rfloor + 1} < \varepsilon,$$

as desired.

2.7.70 Suppose that $\varepsilon = 1$. Then no matter what δ is, there are numbers in the set $0 < |x - 2| < \delta$ so that $|f(x) - 2| > \varepsilon$. For example, when x is only slightly greater than 2, the value of $|f(x) - 2|$ will be 2 or more.

2.7.71 Let $f(x) = \dfrac{|x|}{x}$ and suppose $\lim\limits_{x\to 0} f(x)$ does exist and is equal to L. Let $\varepsilon = 1/2$. There must be a value of δ so that when $0 < |x| < \delta$, $|f(x) - L| < 1/2$. Now consider the numbers $\delta/3$ and $-\delta/3$, both of which are within δ of 0. We have $f(\delta/3) = 1$ and $f(-\delta/3) = -1$. However, it is impossible for both $|1 - L| < 1/2$ and $|-1 - L| < 1/2$, because the former implies that $1/2 < L < 3/2$ and the latter implies that $-3/2 < L < -1/2$. Thus $\lim\limits_{x\to 0} f(x)$ does not exist.

2.7.72 Suppose that $\lim\limits_{x\to a} f(x)$ exists and is equal to L. Let $\varepsilon = 1/2$. By the definition of limit, there must be a number δ so that $|f(x) - L| < \frac{1}{2}$ whenever $0 < |x - a| < \delta$. Now in every set of the form $(a, a + \delta)$ there are both rational and irrational numbers, so there will be value of f equal to both 0 and 1. Thus we have $|0 - L| < 1/2$, which means that L lies in the interval $(-1/2, 1/2)$, and we have $|1 - L| < 1/2$, which means that L lies in the interval $(1/2, 3/2)$. Because these both can't be true, we have a contradiction.

2.7.73 Because f is continuous at a, we know that $\lim\limits_{x\to a} f(x)$ exists and is equal to $f(a) > 0$. Let $\varepsilon = f(a)/3$. Then there is a number $\delta > 0$ so that $|f(x) - f(a)| < f(a)/3$ whenever $|x - a| < \delta$. Then whenever x lies in the interval $(a - \delta, a + \delta)$ we have $-f(a)/3 \le f(x) - f(a) \le f(a)/3$, so $2f(a)/3 \le f(x) \le 4f(a)/3$, so f is positive in this interval.

2.7.74 Using the triangle inequality, we have $|a| = |(a-b)+b| \le |a-b|+|b|$. This implies that $|a| \le |a-b|+|b|$ or $|a| - |b| \le |a - b|$. A similar argument shows that $|b| - |a| \le |a - b|$. Because the expression $||a| - |b||$ is equal to either $|a| - |b|$ or $|b| - |a|$, it follows that $||a| - |b|| \le |a - b|$.

Chapter Two Review

1

a. False. Because $\lim\limits_{x\to 1} \dfrac{x - 1}{x^2 - 1} = \lim\limits_{x\to 1} \dfrac{1}{x + 1} = \dfrac{1}{2}$, f doesn't have a vertical asymptote at $x = 1$.

b. False. In general, these methods are too imprecise to produce accurate results.

c. False. For example, the function $f(x) = \begin{cases} 2x & \text{if } x < 0; \\ 1 & \text{if } x = 0; \\ 4x & \text{if } x > 0 \end{cases}$ has a limit of 0 as $x \to 0$, but $f(0) = 1$.

d. True. When we say that a limit exists, we are saying that there is a real number L that the function is approaching. If the limit of the function is ∞, it is still the case that there is no real number that the function is approaching. (There is no real number called "infinity.")

e. False. It could be the case that $\lim\limits_{x\to a^-} f(x) = 1$ and $\lim\limits_{x\to a^+} f(x) = 2$.

f. False.

g. False. For example, the function $f(x) = \begin{cases} 2 & \text{if } 0 < x < 1; \\ 3 & \text{if } 1 \le x < 2, \end{cases}$ is continuous on $(0,1)$, and on $[1,2)$, but isn't continuous on $(0,2)$.

h. True. $\lim_{x\to a} f(x) = f(a)$ if and only if f is continuous at a.

2 $s(1) = 48$ and $s(1.5) = 60$, so the average velocity over the time period is

$$\frac{s(1.5) - s(1)}{1.5 - 1} = \frac{60 - 48}{0.5} = 24\,\text{ft/s}.$$

3 For various values of b, we calculate $v_{\text{avg}} = \dfrac{s(b) - s(1.5)}{b - 1.5}$.

b	1.6	1.51	1.501	1.5001	1.50001
v_{avg}	10.4	11.84	11.984	11.9984	11.9998

We estimate that the instantaneous velocity is 12.

4

a. $f(-1) = 1$

b. $\lim_{x\to -1^-} f(x) = 3.$

c. $\lim_{x\to -1^+} f(x) = 1.$

d. $\lim_{x\to -1} f(x)$ does not exist.

e. $f(1) = 5.$

f. $\lim_{x\to 1} f(x) = 5.$

g. $\lim_{x\to 2} f(x) = 4.$

h. $\lim_{x\to 3^-} f(x) = 3.$

i. $\lim_{x\to 3^+} f(x) = 5.$

j. $\lim_{x\to 3} f(x)$ does not exist.

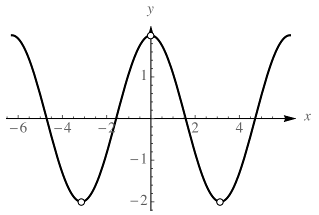
5 This function is discontinuous at $x = -1$, at $x = 1$, and at $x = 3$. At $x = -1$ it is discontinuous because $\lim_{x\to -1} f(x)$ does not exist. At $x = 1$, it is discontinuous because $\lim_{x\to 1} f(x) \ne f(1)$. At $x = 3$, it is discontinuous because $f(3)$ does not exist, and because $\lim_{x\to 3} f(x)$ does not exist.

6

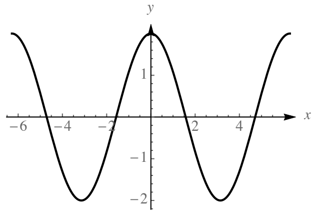
a. The graph drawn by most graphing calculators and computer algebra systems doesn't show the discontinuities where $\sin\theta = 0$.

b. It appears to be equal to 2

c. Using a trigonometric identity, $\lim_{\theta\to 0} \dfrac{\sin 2\theta}{\sin\theta} = \lim_{\theta\to 0} \dfrac{2\sin\theta\cos\theta}{\sin\theta}$. This can then be seen to be $\lim_{\theta\to 0} 2\cos\theta = 2.$



True graph, showing discontinuities where $\sin\theta = 0$.



Graph shown without discontinuities.

7

a.

x	$0.9\pi/4$	$0.99\pi/4$	$0.999\pi/4$	$0.9999\pi/4$
$f(x)$	1.4098	1.4142	1.4142	1.4142

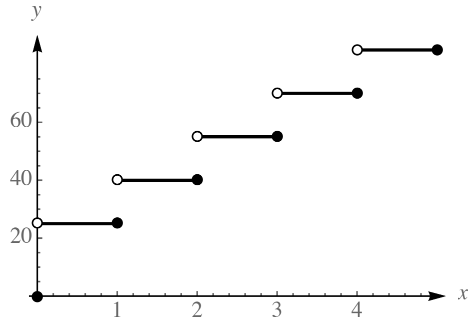
x	$1.1\pi/4$	$1.01\pi/4$	$1.001\pi/4$	$1.0001\pi/4$
$f(x)$	1.4098	1.4142	1.4142	1.4142

The limit appears to be approximately 1.4142.

b. $\displaystyle\lim_{x\to\pi/4} \frac{\cos 2x}{\cos x - \sin x} = \lim_{x\to\pi/4} \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} = \lim_{x\to\pi/4} (\cos x + \sin x) = \sqrt{2}.$

8

a.



b. $\displaystyle\lim_{t\to 2.9} f(t) = 55.$

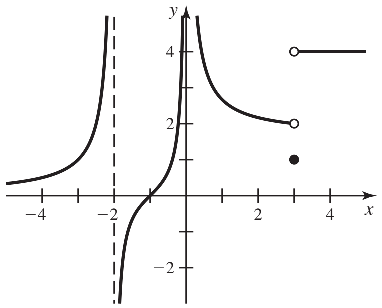
c. $\displaystyle\lim_{t\to 3^-} f(t) = 55$ and $\displaystyle\lim_{t\to 3^+} f(t) = 70.$

d. The cost of the rental jumps by \$15 exactly at $t = 3$. A rental lasting slightly less than 3 days cost \$55 and rentals lasting slightly more than 3 days cost \$70.

e. The function f is continuous everywhere except at the integers. The cost of the rental jumps by \$15 at each integer.

9

There are infinitely many different correct functions which you could draw. One of them is:



10 $\displaystyle\lim_{x\to 1000} 18\pi^2 = 18\pi^2.$

11 $\displaystyle\lim_{x\to 1} \sqrt{5x+6} = \sqrt{11}.$

12

$$\lim_{h\to 0} \frac{\sqrt{5x+5h} - \sqrt{5x}}{h} \cdot \frac{\sqrt{5x+5h} + \sqrt{5x}}{\sqrt{5x+5h} + \sqrt{5x}} = \lim_{h\to 0} \frac{(5x+5h) - 5x}{h(\sqrt{5x+5h} + \sqrt{5x})} = \lim_{h\to 0} \frac{5}{\sqrt{5x+5h} + \sqrt{5x}} = \frac{5}{2\sqrt{5x}}.$$

13

$$\lim_{h\to 0} \frac{(h+6)^2 + (h+6) - 42}{h} = \lim_{h\to 0} \frac{h^2 + 12h + 36 + h + 6 - 42}{h} = \lim_{h\to 0} \frac{h^2 + 13h}{h} = \lim_{h\to 0} (h + 13) = 13.$$

14 Factoring the numerator as the difference of squares, we have

$$\begin{aligned}\lim_{x\to a}\frac{(3x+1)^2-(3a+1)^2}{x-a} &= \lim_{x\to a}\frac{((3x+1)-(3a+1))((3x+1)+(3a+1))}{x-a}\\ &= \lim_{x\to a}\frac{(3x-3a)(3x+3a+2)}{x-a}\\ &= 3\lim_{x\to a}(3x+3a+2)\\ &= 3(3a+3a+2) = 18a+6.\end{aligned}$$

15 $\displaystyle\lim_{x\to 1}\frac{x^3-7x^2+12x}{4-x} = \frac{1-7+12}{4-1} = \frac{6}{3} = 2.$

16 $\displaystyle\lim_{x\to 4}\frac{x^3-7x^2+12x}{4-x} = \lim_{x\to 4}\frac{x(x-3)(x-4)}{4-x} = \lim_{x\to 4} x(3-x) = -4.$

17 $\displaystyle\lim_{x\to 1}\frac{1-x^2}{x^2-8x+7} = \lim_{x\to 1}\frac{(1-x)(1+x)}{(x-7)(x-1)} = \lim_{x\to 1}\frac{-(x+1)}{x-7} = \frac{1}{3}.$

18 $\displaystyle\lim_{x\to 3}\frac{\sqrt{3x+16}-5}{x-3}\cdot\frac{\sqrt{3x+16}+5}{\sqrt{3x+16}+5} = \lim_{x\to 3}\frac{3(x-3)}{(x-3)(\sqrt{3x+16}+5)} = \lim_{x\to 3}\frac{3}{\sqrt{3x+16}+5} = \frac{3}{10}.$

19

$$\begin{aligned}\lim_{x\to 3}\frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{2}\right) &= \lim_{x\to 3}\frac{2-\sqrt{x+1}}{2(x-3)\sqrt{x+1}}\cdot\frac{(2+\sqrt{x+1})}{(2+\sqrt{x+1})}\\ &= \lim_{x\to 3}\frac{4-(x+1)}{2(x-3)(\sqrt{x+1})(2+\sqrt{x+1})}\\ &= \lim_{x\to 3}\frac{-(x-3)}{2(x-3)(\sqrt{x+1})(2+\sqrt{x+1})}\\ &= \lim_{x\to 3}-\frac{1}{2\sqrt{x+1}(2+\sqrt{x+1})} = -\frac{1}{16}.\end{aligned}$$

20 $\displaystyle\lim_{t\to 1/3}\frac{t-\frac{1}{3}}{(3t-1)^2} = \lim_{t\to 1/3}\frac{3t-1}{3(3t-1)^2} = \lim_{t\to 1/3}\frac{1}{3(3t-1)}$, which does not exist.

21 $\displaystyle\lim_{x\to 3}\frac{x^4-81}{x-3} = \lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{x-3} = \lim_{x\to 3}(x+3)(x^2+9) = 108.$

22 Note that $\displaystyle\frac{p^5-1}{p-1} = p^4+p^3+p^2+p+1$. (Use long division.)
$\displaystyle\lim_{p\to 1}\frac{p^5-1}{p-1} = \lim_{p\to 1}(p^4+p^3+p^2+p+1) = 5.$

23 $\displaystyle\lim_{x\to 81}\frac{\sqrt[4]{x}-3}{x-81} = \lim_{x\to 81}\frac{\sqrt[4]{x}-3}{(\sqrt{x}+9)(\sqrt[4]{x}+3)(\sqrt[4]{x}-3)} = \lim_{x\to 81}\frac{1}{(\sqrt{x}+9)(\sqrt[4]{x}+3)} = \frac{1}{108}.$

24 $\displaystyle\lim_{\theta\to\pi/2}\frac{\sin^2\theta-5\sin\theta+4}{\sin^2\theta-1} = \lim_{\theta\to\pi/2}\frac{(\sin\theta-4)(\sin\theta-1)}{(\sin\theta-1)(\sin\theta+1)} = \lim_{\theta\to\pi/2}\frac{\sin\theta-4}{\sin\theta+1} = \frac{1-4}{1+1} = -\frac{3}{2}.$

25 $\displaystyle\lim_{x\to\pi/2}\frac{\frac{1}{\sqrt{\sin x}}-1}{x+\pi/2} = \frac{0}{\pi} = 0.$

26 The domain of $f(x) = \sqrt{\dfrac{x-1}{x-3}}$ is $(-\infty, 1]$ and $(3,\infty)$, so $\displaystyle\lim_{x\to 1^+} f(x)$ doesn't exist.
However, we have $\displaystyle\lim_{x\to 1^-} f(x) = 0.$

27 $\lim\limits_{x\to 5} \dfrac{x-7}{x(x-5)^2} = -\infty.$

28 $\lim\limits_{x\to -5^+} \dfrac{x-5}{x+5} = -\infty.$

29 $\lim\limits_{x\to 3^-} \dfrac{x-4}{x^2-3x} = \lim\limits_{x\to 3^-} \dfrac{x-4}{x(x-3)} = \infty.$

30 $\lim\limits_{x\to 0^+} \dfrac{u-1}{\sin u} = -\infty.$

31 $\lim\limits_{x\to 1^+} \dfrac{4x^3-4x^2}{|x-1|} = \lim\limits_{x\to 1^+} \dfrac{4x^2(x-1)}{x-1} = \lim\limits_{x\to 1^+} 4x^2 = 4.$

32 The expression $2x-4 = 2(x-2)$ is negative for $x < 2$, so $|2x-4| = -2(x-2)$. Therefore, $\lim\limits_{x\to 2^-} \dfrac{|2x-4|}{x^2-5x+6} = \lim\limits_{x\to 2^-} \dfrac{-2(x-2)}{(x-2)(x-3)} = \lim\limits_{x\to 2^-} \dfrac{-2}{x-3} = 2.$

33 $\lim\limits_{x\to 0^-} \dfrac{2}{\tan x} = -\infty.$

34 First note that for all x, $\sqrt{x^4} = x^2$. Then we have

$$\lim_{x\to\infty} \frac{(4x^2+3x+1)}{\sqrt{8x^4+2}} \cdot \frac{1/x^2}{1/\sqrt{x^4}} = \lim_{x\to\infty} \frac{4+(3/x)+(1/x^2)}{\sqrt{8+(2/x^4)}} = \frac{4}{\sqrt{8}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

35 $\lim\limits_{x\to\infty} \dfrac{2x-3}{4x+10} = \lim\limits_{x\to\infty} \dfrac{2-(3/x)}{4+(10/x)} = \dfrac{2}{4} = \dfrac{1}{2}.$

36 $\lim\limits_{x\to\infty} \dfrac{x^4-1}{x^5+2} = \lim\limits_{x\to\infty} \dfrac{(1/x)-(1/x^5)}{1+(2/x^5)} = \dfrac{0-0}{1+0} = 0.$

37 Note that for $x < 0$, $x = -\sqrt{x^2}$. Then we have

$$\lim_{x\to-\infty} \frac{(3x+1)}{\sqrt{ax^2+2}} \cdot \frac{1/x}{-1/\sqrt{x^2}} = -\lim_{x\to-\infty} \frac{3+(1/x)}{\sqrt{a+(2/x^2)}} = -\frac{3}{\sqrt{a}}.$$

38 We have

$$\begin{aligned}\lim_{x\to\infty} \sqrt{x^2+ax} - \sqrt{x^2-b} &= \lim_{x\to\infty} \frac{(\sqrt{x^2+ax}-\sqrt{x^2-b})}{1} \cdot \frac{(\sqrt{x^2+ax}+\sqrt{x^2-b})}{(\sqrt{x^2+ax}+\sqrt{x^2-b})} \\ &= \lim_{x\to\infty} \frac{ax+b}{\sqrt{x^2+ax}+\sqrt{x^2-b}}.\end{aligned}$$

Now noting that $x = \sqrt{x^2}$ for $x > 0$ we have

$$\lim_{x\to\infty} \frac{(ax+b)}{(\sqrt{x^2+ax}+\sqrt{x^2-b})} \cdot \frac{1/x}{1/\sqrt{x^2}} = \lim_{x\to\infty} \frac{a+b/x}{\sqrt{1+a/x}+\sqrt{1-b/x}} = \frac{a}{2}.$$

39 We multiply the numerator and denominator by the conjugate of the denominator (i.e., the expression $\sqrt{x^2-ax}+\sqrt{x^2-x}$). This gives

$$\lim_{x\to\infty} \frac{\sqrt{x^2-ax}+\sqrt{x^2-x}}{(x^2-ax)-(x^2-x)} = \lim_{x\to\infty} \frac{\sqrt{x^2-ax}+\sqrt{x^2-x}}{-ax+x}.$$

We now multiply by $\dfrac{1/\sqrt{x^2}}{1/x}$ to obtain

$$\lim_{x\to\infty} \frac{\sqrt{1-(a/x)}+\sqrt{1-(1/x)}}{-a+1} = \frac{2}{1-a}.$$

40 $\lim_{z\to\infty}\left(e^{-2z}+\frac{2}{z}\right)=0+0=0.$

41 $\lim_{x\to\infty}(3\tan^{-1}x+2)=\frac{3\pi}{2}+2.$

42 $\lim_{x\to-\infty}(-3x^3+5)=\infty.$

43 For $x<1$, $|x-1|+x=-(x-1)+x=1$. Therefore $\lim_{x\to-\infty}(|x-1|+x)=\lim_{x\to-\infty}1=1.$
For $x>1$, $|x-1|+x=x-1+x=2x-1$. Therefore $\lim_{x\to\infty}(|x-1|+x)=\lim_{x\to\infty}2x-1=\infty.$

44 For $x<2$, $|x-2|+x=-(x-2)+x=2$. Therefore $\lim_{x\to-\infty}\frac{|x-2|+x}{x}=\lim_{x\to-\infty}\frac{2}{x}=0.$
For $x>2$, $|x-2|+x=x-2+x=2x-2$. Therefore $\lim_{x\to\infty}\frac{|x-2|+x}{x}=\lim_{x\to\infty}\frac{2x-2}{x}=2.$

45 $\lim_{w\to\infty}\frac{\ln w^2}{\ln w^3+1}=\lim_{w\to\infty}\frac{2\ln w}{(3\ln w+1)}\cdot\frac{1/\ln w}{1/\ln w}=\lim_{w\to\infty}\frac{2}{3+(1/\ln w)}=\frac{2}{3}.$

46 $\lim_{r\to-\infty}\frac{1}{2+e^r}=\frac{1}{2+0}=\frac{1}{2}.$
$\lim_{r\to\infty}\frac{1}{2+e^r}=0.$

47 $\lim_{r\to\infty}\frac{(2e^{4r}+3e^{5r})}{(7e^{4r}-9e^{5r})}\cdot\frac{1/e^{5r}}{1/e^{5r}}=\lim_{r\to\infty}\frac{(2/e^r)+3}{(7/e^r)-9}=\frac{3}{-9}=-\frac{1}{3}.$
$\lim_{r\to-\infty}\frac{(2e^{4r}+3e^{5r})}{(7e^{4r}-9e^{5r})}\cdot\frac{1/e^{4r}}{1/e^{4r}}=\lim_{r\to-\infty}\frac{2+3e^r}{7-9e^r}=\frac{2}{7}.$

48 Because $-1\le\sin x\le1$, $-e^x\le e^x\sin x\le e^x$. Because $\lim_{x\to-\infty}-e^x=\lim_{x\to-\infty}e^x=0$, we can conclude that $\lim_{x\to-\infty}e^x\sin x=0$ by the Squeeze Theorem.

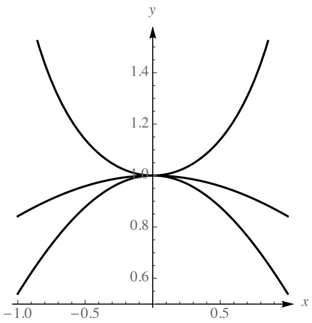
49 We know that $0\le\cos^4x\le1$. Dividing each part of this inequality by x^2+x+1 and then adding 5, we have $5\le5+\frac{\cos^4x}{x^2+x+1}\le5+\frac{1}{x^2+x+1}$. Note that $\lim_{x\to\infty}5=5$ and $\lim_{x\to\infty}\left(5+\frac{1}{x^2+x+1}\right)=5$, so by the Squeeze Theorem we can conclude that $\lim_{x\to\infty}\left(5+\frac{\cos^4x}{x^2+x+1}\right)=5.$

50 Recall that $-1\le\cos t\le1$, and that $e^{3t}>0$ for all t. Thus $-\frac{1}{e^{3t}}\le\frac{\cos t}{e^{3t}}\le\frac{1}{e^{3t}}$. Because $\lim_{t\to\infty}\frac{1}{e^{3t}}=\lim_{t\to\infty}-\frac{1}{e^{3t}}=0$, we can conclude $\lim_{t\to\infty}\frac{\cos t}{e^{3t}}=0$ by the Squeeze Theorem.

51 $\lim_{x\to1^-}\frac{x}{\ln x}=-\infty.$

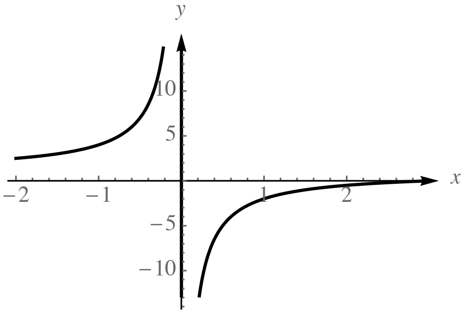
52 Note that $\lim_{x\to0}(\sin^2x+1)=1$. Thus if $1\le g(x)\le\sin^2x+1$, the Squeeze Theorem assures us that $\lim_{x\to0}g(x)=1$ as well.

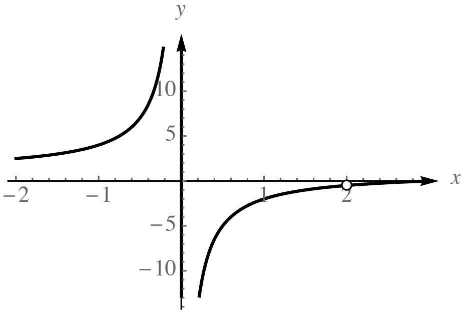
53

a. 

b. Because $\lim_{x\to 0}\cos x = \lim_{x\to 0}\dfrac{1}{\cos x} = 1$, the Squeeze Theorem assures us that $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$ as well.

54





First note that $f(x) = \frac{x^2-5x+6}{x^2-2x} = \frac{(x-3)(x-2)}{x(x-2)}$.

a. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\dfrac{(x-3)(x-2)}{x(x-2)} = \infty.$
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\dfrac{(x-3)(x-2)}{x(x-2)} = -\infty.$
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\dfrac{x-3}{x} = -\dfrac{1}{2}.$
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\dfrac{x-3}{x} = -\dfrac{1}{2}.$

b. By the above calculations and the definition of vertical asymptote, f has a vertical asymptote at $x = 0$.

c. Note that the actual graph has a "hole" at the point $(2, -1/2)$, because $x = 2$ isn't in the domain, but $\lim_{x\to 2} f(x) = -1/2$.

55 $\lim_{x\to\infty}\dfrac{4x^3+1}{1-x^3} = \lim_{x\to\infty}\dfrac{4+(1/x^3)}{(1/x^3)-1} = \dfrac{4+0}{0-1} = -4$. A similar result holds as $x \to -\infty$. Thus, $y = -4$ is a horizontal asymptote as $x \to \infty$ and as $x \to -\infty$.

56 Note that $\sqrt{x^{12}} = x^6$ for all x. We have $\lim_{x\to\pm\infty}\dfrac{(x^6+1)}{\sqrt{16x^{14}+1}}\cdot\dfrac{1/x^6}{1/\sqrt{x^{12}}} = \lim_{x\to\pm\infty}\dfrac{1+(1/x^6)}{\sqrt{16x^2+(1/x^{12})}} = 0.$

57 $\lim_{x\to\infty}(1-e^{-2x}) = 1$, while $\lim_{x\to-\infty}(1-e^{-2x}) = -\infty$.
$y = 1$ is a horizontal asymptote as $x \to \infty$.

58 $\lim_{x\to\infty}\dfrac{1}{\ln x^2} = 0$, and $\lim_{x\to-\infty}\dfrac{1}{\ln x^2} = 0$, so $y = 0$ is a horizontal asymptote as $x \to \infty$ and as $x \to -\infty$.

59 $\lim_{x\to\infty}\dfrac{(6e^x+20)}{(3e^x+4)}\cdot\dfrac{1/e^x}{1/e^x} = \lim_{x\to\infty}\dfrac{6+(20/e^x)}{3+(4/e^x)} = \dfrac{6}{3} = 2.$
$\lim_{x\to-\infty}\dfrac{6e^x+20}{3e^x+4} = \dfrac{0+20}{0+4} = 5.$

60 First note that $\sqrt{\dfrac{1}{x^2}} = \left|\dfrac{1}{x}\right| = \begin{cases} \frac{1}{x} & \text{if } x > 0; \\ -\frac{1}{x} & \text{if } x < 0. \end{cases}$.

$$\lim_{x\to\infty} \frac{x+1}{\sqrt{9x^2+x}} = \lim_{x\to\infty} \frac{1+(1/x)}{\sqrt{9+\frac{1}{x}}} = \frac{1}{3}.$$

On the other hand, $\displaystyle\lim_{x\to-\infty} \frac{x+1}{\sqrt{9x^2+x}} = \lim_{x\to\infty} \frac{1+(1/x)}{-\sqrt{9+\frac{1}{x}}} = -\frac{1}{3}$.

So $y = \frac{1}{3}$ is a horizontal asymptote as $x \to \infty$, and $y = -\frac{1}{3}$ is a horizontal asymptote as $x \to -\infty$.

61

a. $\displaystyle\lim_{x\to\infty} \frac{(3x^2+5x+7)}{(x+1)} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{3x+5+(7/x)}{1+(1/x)} = \infty.$

$\displaystyle\lim_{x\to-\infty} \frac{(3x^2+5x+7)}{(x+1)} \cdot \frac{1/x}{1/x} = \lim_{x\to-\infty} \frac{3x+5+(7/x)}{1+(1/x)} = -\infty.$

b. By long division, we see that $\frac{3x^2+5x+7}{x+1} = 3x+2+\frac{5}{x+1}$, so the line $y = 3x+2$ is a slant asymptote.

62

a. $\displaystyle\lim_{x\to\infty} \frac{9x^2+4}{(2x-1)^2} = \lim_{x\to\infty} \frac{9x^2+4}{4x^2-4x+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{9+4/x^2}{4-4/x+1/x^2} = \frac{9}{4}.$

$\displaystyle\lim_{x\to-\infty} \frac{9x^2+4}{(2x-1)^2} = \lim_{x\to-\infty} \frac{9x^2+4}{4x^2-4x+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{9+4/x^2}{4-4/x+1/x^2} = \frac{9}{4}.$

b. Because there is a horizontal asymptote, there is not a slant asymptote.

63

a. $\displaystyle\lim_{x\to\infty} \frac{1+x-2x^2-x^3}{x^2+1} = \lim_{x\to\infty} \frac{1+x-2x^2-x^3}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{1/x^2+1/x-2-x}{1+1/x^2} = -\infty.$

$\displaystyle\lim_{x\to-\infty} \frac{1+x-2x^2-x^3}{x^2+1} = \lim_{x\to-\infty} \frac{1+x-2x^2-x^3}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{1/x^2+1/x-2-x}{1+1/x^2} = \infty.$

b. By long division, we can write $f(x)$ as $f(x) = -x-2+\frac{2x+3}{x^2+1}$, so the line $y = -x-2$ is the slant asymptote.

64

a. $\displaystyle\lim_{x\to\infty} \frac{x(x+2)^3}{3x^2-4x} = \lim_{x\to\infty} \frac{x^4+6x^3+12x^2+8x}{3x^2-4x} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{x^2+6x+12+8/x}{3-4/x} = \infty.$

$\displaystyle\lim_{x\to-\infty} \frac{x(x+2)^3}{3x^2-4x} = \lim_{x\to-\infty} \frac{x^4+6x^3+12x^2+8x}{3x^2-4x} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{x^2+6x+12+8/x}{3-4/x} = \infty.$

b. Because the degree of the numerator of this rational function is two more than the degree of the denominator, there is no slant asymptote.

65

a. $\displaystyle\lim_{x\to\infty} \frac{(4x^3+x^2+7)}{(x^2-x+1)} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{4x+1+(7/x)}{1-(1/x)+(1/x^2)} = \infty.$

$\displaystyle\lim_{x\to-\infty} \frac{(4x^3+x^2+7)}{(x^2-x+1)} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{4x+1+(7/x)}{1-(1/x)+(1/x^2)} = -\infty.$

b. By long division, we can write $\dfrac{4x^3+x^2+7}{x^2-x+1} = 4x+5+\dfrac{x+2}{x^2-x+1}$. Therefore $y = 4x+5$ is a slant asymptote.

66 Note that $f(x) = \dfrac{2x^2+6}{2x^2+3x-2} = \dfrac{2(x^2+3)}{(2x-1)(x+2)}$.

We have $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{2+6/x^2}{2+3/x-2/x^2} = 1$. A similar result holds as $x \to -\infty$.
$\lim\limits_{x\to 1/2^-} f(x) = -\infty$. $\lim\limits_{x\to 1/2^+} f(x) = \infty$.
$\lim\limits_{x\to -2^-} f(x) = \infty$. $\lim\limits_{x\to -2^+} f(x) = -\infty$.
Thus, $y = 1$ is a horizontal asymptote as $x \to \infty$ and as $x \to -\infty$. Also, $x = \frac{1}{2}$ and $x = -2$ are vertical asymptotes.

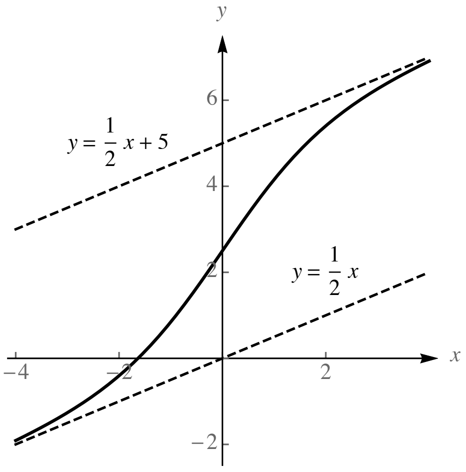
67 Recall that $\tan^{-1} x = 0$ only for $x = 0$. The only vertical asymptote is $x = 0$.
$\lim\limits_{x\to\infty} \dfrac{1}{\tan^{-1} x} = \dfrac{1}{\pi/2} = \dfrac{2}{\pi}$.
$\lim\limits_{x\to-\infty} \dfrac{1}{\tan^{-1} x} = \dfrac{1}{-\pi/2} = -\dfrac{2}{\pi}$. So $y = \frac{2}{\pi}$ is a horizontal asymptote as $x \to \infty$ and $y = -\frac{2}{\pi}$ is a horizontal asymptote as $x \to -\infty$.

68 By long division, we can write $\dfrac{2x^2-7}{x-2} = 2x+4+\dfrac{1}{x-2}$, so $y = 2x+4$ is a slant asymptote. Also, $\lim\limits_{x\to 2^+} \dfrac{2x^2-7}{x-2} = \infty$, so $x = 2$ is a vertical asymptote.

69 Observe that

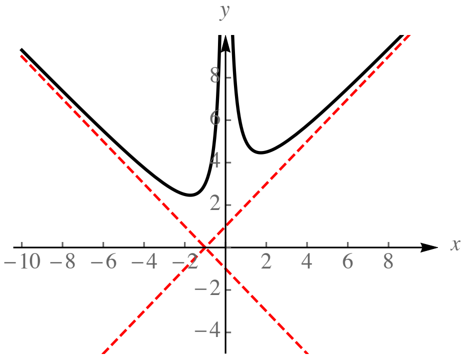
$$f(x) = \frac{x+xe^x+10e^x}{2(e^x+1)} = \frac{x(1+e^x)+10e^x}{2(e^x+1)} = \frac{1}{2}x + \frac{5e^x}{e^x+1}.$$

Because $\lim\limits_{x\to\infty} \dfrac{5e^x}{e^x+1} = \lim\limits_{x\to\infty} \dfrac{5}{1+(1/e^x)} = 5$, the graph of f and the line $y = \frac{1}{2}x+5$ approach each other as $x \to \infty$. Similarly, $\lim\limits_{x\to-\infty} \dfrac{5e^x}{e^x+1} = \dfrac{0}{0+1} = 0$ and therefore the graph of f and the line $y = \frac{1}{2}x$ approach each other as $x \to -\infty$.



70 Observe that $\lim\limits_{x\to 0^+} \dfrac{x^2+x+3}{|x|} = \lim\limits_{x\to 0^+} \dfrac{x^2+x+3}{x} = \lim\limits_{x\to 0^+} x+1+(3/x) = \infty$ and $\lim\limits_{x\to 0^-} \dfrac{x^2+x+3}{|x|} = \lim\limits_{x\to 0^-} \dfrac{x^2+x+3}{-x} = -\lim\limits_{x\to 0^-} (x+1+(3/x)) = -\infty$. For $x > 0$, we have $f(x) = \dfrac{x^2+x+3}{x} = x+1+\dfrac{3}{x}$, so $y = x+1$ is a slant asymptote as $x \to \infty$. For $x < 0$, we have $f(x) = \dfrac{x^2+x+3}{-x} = -x-1-\dfrac{3}{x}$, so

$y = -x - 1$ is a slant asymptote as $x \to -\infty$. So the function has one vertical asymptote $x = 0$ and two slant asymptotes, $y = x + 1$ and $y = -x - 1$.



71 The function f is not continuous at 5 because $f(5)$ is not defined.

72 g is discontinuous at 4 because $\lim_{x\to 4} g(x) = \lim_{x\to 4} \frac{(x+4)(x-4)}{x-4} = 8 \neq g(4)$.

73 Observe that $h(5) = -2(5) + 14 = 4$. Because $\lim_{x\to 5^-} h(x) = \lim_{x\to 5^-} (-2x + 14) = 4$ and $\lim_{x\to 5^+} h(x) = \lim_{x\to 5^+} \sqrt{x^2 - 9} = \sqrt{25 - 9} = 4$, we have $\lim_{x\to 5} h(x) = 4$. Thus f is continuous at $x = 5$.

74 Observe that $g(2) = -2$ and $\lim_{x\to 2} g(x) = \lim_{x\to 2} \frac{x^3 - 5x^2 + 6x}{x - 2} = \lim_{x\to 2} \frac{x(x-2)(x-3)}{x-2} = \lim_{x\to 2} x(x-3) = -2$. Therefore g is continuous at $x = 2$.

75 The domain of f is $(-\infty, -\sqrt{5}]$ and $[\sqrt{5}, \infty)$, and f is continuous on that domain. It is left continuous at $-\sqrt{5}$ and right continuous at $\sqrt{5}$.

76 The domain of g is $[2, \infty)$, and it is continuous on that domain. It is continuous from the right at $x = 2$.

77 The domain of h is $(-\infty, -5)$, $(-5, 0)$, $(0, 5)$, $(5, \infty)$, and like all rational functions, it is continuous on its domain.

78 g is the composition of two functions which are defined and continuous on $(-\infty, \infty)$, so g is continuous on that interval as well.

79 In order for g to be left continuous at 1, it is necessary that $\lim_{x\to 1^-} g(x) = g(1)$, which means that $a = 3$. In order for g to be right continuous at 1, it is necessary that $\lim_{x\to 1^+} g(x) = g(1)$, which means that $a + b = 3 + b = 3$, so $b = 0$.

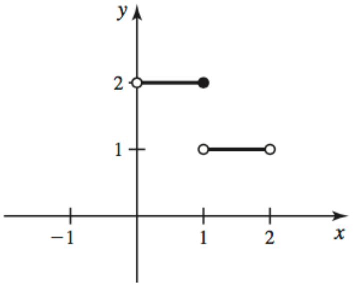
80

a. Because the domain of h is $(-\infty, -3]$ and $[3, \infty)$, there is no way that h can be left continuous at 3.

b. h is right continuous at 3, because $\lim_{x\to 3^+} h(x) = 0 = h(3)$.

81

One such possible graph is pictured to the right.



82

a. Consider the function $f(x) = x^5 + 7x + 5$. f is continuous everywhere, and $f(-1) = -3 < 0$ while $f(0) = 5 > 0$. Therefore, 0 is an intermediate value between $f(-1)$ and $f(0)$. By the Intermediate Value Theorem, there must a number c between 0 and 1 so that $f(c) = 0$.

b. Using a computer algebra system, one can find that $c \approx -0.691671$ is a root.

83

a. Rewrite The equation as $x - \cos x = 0$ and let $f(x) = x - \cos x$. Because x and $\cos x$ are continuous on the given interval, so is f. Because $f(0) = -1 < 0$ and $f(\pi/2) = \pi/2 > 0$, it follows from the Intermediate Value Theorem that the equation has a solution on $(0, \pi/2)$.

b. Using a computer algebra system, one can find that $c \approx 0.739085$ is a root.

84 Temperature changes gradually, so it is reasonable to assume that T is a continuous function and therefore f is also continuous. Because$f(0) = -33 < 0$ and $f(12) = 33 > 0$, it follows from the Intermediate Value Theorem that there is a value t_0 in $(0, 12)$ satisfying $f(t_0) = 0$. . Therefore, $T(t_0) - T(t_0 + 12) = 0$, or $T(t_0) = T(t_0 + 12)$.

85

a. Note that $m(0) = 0$ and $m(5) \approx 38.34$ and $m(15) \approx 21.2$. Thus, 30 is an intermediate value between both $m(0)$ and $m(5)$, and $m(5)$ and $m(15)$. Note also that m is a continuous function. By the IVT, there must be a number c_1 between 0 and 5 with $m(c_1) = 30$, and a number c_2 between 5 and 15 with $m(c_2) = 30$.

b. A little trial and error leads $c_1 \approx 2.4$ and $c_2 \approx 10.8$.

c. No. The graph of the function on a graphing calculator suggests that it peaks at about 38.5

86 Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon/5$. Now suppose that $0 < |x - 1| < \delta$.
Then

$$\begin{aligned} |f(x) - L| &= |(5x - 2) - 3| = |5x - 5| \\ &= 5|x - 1| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon. \end{aligned}$$

87 Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon$. Now suppose that $0 < |x - 5| < \delta$.
Then

$$\begin{aligned} |f(x) - L| &= \left| \frac{x^2 - 25}{x - 5} - 10 \right| = \left| \frac{(x - 5)(x + 5)}{x - 5} - 10 \right| = |x + 5 - 10| \\ &= |x - 5| < \varepsilon. \end{aligned}$$

88 Let $\varepsilon > 0$ be given. Let $\delta = \frac{\varepsilon}{4}$ and assume that $0 < |x-3| < \delta$. For $x < 3$,

$$|f(x)-5| = |3x-4-5| = 3|x-3| < 3\delta = 3\cdot\frac{\varepsilon}{4} < \varepsilon.$$

For $x > 3$,

$$|f(x)-5| = |-4x+17-5| = 4|x-3| < 4\delta = 4\cdot\frac{\varepsilon}{4} = \varepsilon.$$

We have shown that for any $\varepsilon > 0$, $|f(x)-5| < \varepsilon$ whenever $0 < |x-2| < \delta$, provided $0 < \delta \le \frac{\varepsilon}{4}$.

89 Let $\varepsilon > 0$ be given. Let $\delta = \min\{1, \varepsilon/15\}$ and assume that $0 < |x-2| < \delta$. Then $|3x^2-4-8| = 3|x^2-4| = 3|x-2||x+2|$. Because $0 < |x-2| < \delta$ and $\delta \le 1$, $-1 < x-2 < 1$ and so $1 < x < 3$. It follows that $x+2 < 5$. Therefore $|3x^2-4-8| = 3|x-2||x+2| < 3\cdot\frac{\varepsilon}{15}\cdot 5 = \varepsilon$. So we've shown that for any $\varepsilon > 0$, $|3x^2-4-8| < \varepsilon$ whenever $0 < |x-2| < \delta$, provided $0 < \delta \le \min\{1, \varepsilon/15\}$.

90 Let $\varepsilon > 0$ be given. Let $\delta = \frac{\varepsilon^2}{4}$ and assume that $0 < x-2 < \delta$. Then $|\sqrt{4x-8}-0| = 2\sqrt{x-2} < 2\sqrt{\delta} = 2\sqrt{\frac{\varepsilon^2}{4}} = \varepsilon$. So we've shown that for any $\varepsilon > 0$, $|\sqrt{4x-8}-0| < \varepsilon$ whenever $0 < x-2 < \delta$, provided $0 < \delta \le \frac{\varepsilon^2}{4}$.

91 Let $N > 0$ be given. Let $\delta = 1/\sqrt[4]{N}$. Suppose that $0 < |x-2| < \delta$. Then $|x-2| < \dfrac{1}{\sqrt[4]{N}}$, so $\dfrac{1}{|x-2|} > \sqrt[4]{N}$, and $\dfrac{1}{(x-2)^4} > N$, as desired.

92

a. Assume $L > 0$. (If $L = 0$, the result follows immediately because that would imply that the function f is the constant function 0, and then $f(x)g(x)$ is also the constant function 0.) Assume that δ_1 is a number so that $|f(x)| \le L$ for $|x-a| < \delta_1$.

   Let $\varepsilon > 0$ be given. Because $\lim_{x\to a} g(x) = 0$, we know that there exists a number $\delta_2 > 0$ so that $|g(x)| < \varepsilon/L$ whenever $0 < |x-a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$.

   Then
   $$|f(x)g(x)-0| = |f(x)||g(x)| < L\cdot\frac{\varepsilon}{L} = \varepsilon,$$
   whenever $0 < |x-a| < \delta$.

b. Let $f(x) = \dfrac{x^2}{x-2}$. Then
   $$\lim_{x\to 2} f(x)(x-2) = \lim_{x\to 2}\frac{x^2(x-2)}{x-2} = \lim_{x\to 2} x^2 = 4 \ne 0.$$
   This doesn't violate the previous result because the given function f is not bounded near $x = 2$.

c. Because $|H(x)| \le 1$ for all x, the result follows directly from part a) of this problem (using $L = 1$, $a = 0$, $f(x) = H(x)$, and $g(x) = x$).

Chapter 3

Derivatives

3.1 Introducing the Derivative

3.1.1 The secant line through the points $(a, f(a))$ and $(x, f(x))$ for x near a, of the graph of f, is given by $m_{\text{sec}} = \dfrac{f(x)-f(a)}{x-a}$. As x approaches a, we obtain the limit $m_{\tan} = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a} m_{\text{sec}}$.

3.1.2 The slope of the secant line through the points $(a, f(a))$ and $(x, f(x))$ for x near a, of the graph of f, is given by $m_{\text{sec}} = \dfrac{f(x)-f(a)}{x-a}$. So the slope is the change of f divided by the length of the interval $[a, x]$ over which the change occurs, that is, the average rate of change of f over $[a, x]$.

3.1.3 The average rate of change of f over $[a, x]$ is the slope of the secant line $m_{\text{sec}} = \dfrac{f(x)-f(a)}{x-a}$. As x approaches a, the length of the interval $x - a$ goes to zero, and in the limit we obtain the instantaneous rate of change of f at a given by $m_{\tan} = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$.

3.1.4 The slope of the tangent line, the instantaneous rate of change, and the value of the derivative of a function at a given point are all the same.

3.1.5 $f'(a)$ is the value of the derivative of f at a. Also, $f'(a)$ is the slope of the tangent line to the graph of f at $(a, f(a))$. Furthermore, $f'(a)$ is the instantaneous rate of change of f at a.

3.1.6 $f(6)$ is the y-value of the point on the graph associated with $x = 6$, so $f(6) = 5$. $f'(6)$ is the slope of the tangent line at the point $(6, 5)$, so is $f'(6) = -2$.

3.1.7 $f(2) = 4(2) - 1 = 7$. $f'(2)$ is the slope of the tangent line, so $f'(2) = 4$.

3.1.8 $g(3) = 5(3) + 4 = 19$. $g'(3)$ is the slope of the tangent line, so $g'(3) = 5$.

3.1.9 Using the point-slope form of the equation of a line, we have $y - 2 = 3(x - 1)$, or $y = 3x - 1$.

3.1.10 Using the point-slope form of the equation of a line, we have $y - 4 = 7(x - (-2))$, or $y = 7x + 18$.

3.1.11 $m_{\tan} = \lim\limits_{x\to 1} \dfrac{-5x+1+4}{x-1} = \lim\limits_{x\to 1} \dfrac{-5x+5}{x-1} = \lim\limits_{x\to 1} \left(-5\left(\dfrac{x-1}{x-1}\right)\right) = -5$.

3.1.12 $m_{\tan} = \lim\limits_{x\to 1} \dfrac{5-5}{x-1} = 0$.

3.1.13

$$\begin{aligned} s'(1) &= \lim_{t\to 1} \frac{-16t^2+100t-84}{t-1} = -4\lim_{t\to 1} \frac{4t^2-25t+21}{t-1} = -4\lim_{t\to 1} \frac{(4t-21)(t-1)}{t-1} \\ &= -4\lim_{t\to 1}(4t-21) = 68\,\text{ft/s}. \end{aligned}$$

3.1.14

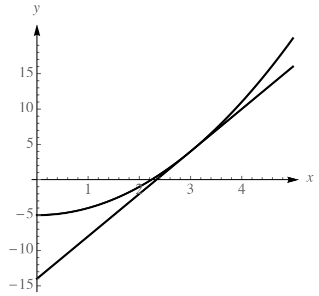
$$s'(2) = \lim_{t\to 2} \frac{-16t^2+128t+192-384}{t-2} = -16\lim_{t\to 2}\frac{t^2-8t+12}{t-2} = -16\lim_{t\to 2}\frac{(t-2)(t-6)}{t-2} = -16\lim_{t\to 2}(t-6) = 64\,\text{ft/s}.$$

3.1.15

a. $m_{\tan} = \lim_{x\to 3}\frac{x^2-5-4}{x-3} = \lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}\frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 6.$

b. Using the point-slope form of the equation of a line, we obtain $y-4=6(x-3)$, or $y=6x-14$.

c.

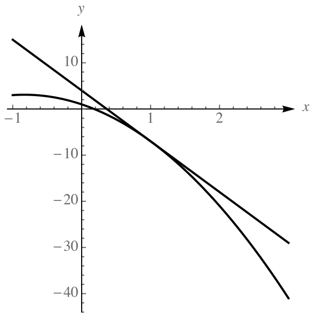


3.1.16

a. $m_{\tan} = \lim_{x\to 1}\frac{-3x^2-5x+1-(-7)}{x-1} = \lim_{x\to 1}\frac{-3x^2-5x+8}{x-1} = \lim_{x\to 1}\frac{-(3x+8)(x-1)}{x-1} = \lim_{x\to 1}(-3x-8) = -11.$

b. Using the point-slope form of the equation of a line, we get $y+7=-11(x-1)$, or $y=-11x+4$.

c.

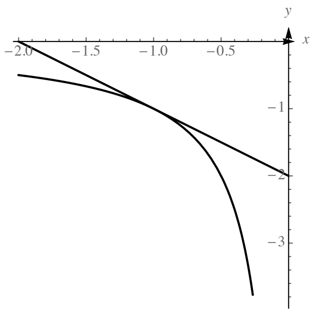


3.1.17

a. $m_{\tan} = \lim_{x\to -1}\frac{\frac{1}{x}+1}{x+1} = \lim_{x\to -1}\frac{\frac{1+x}{x}}{x+1} = \lim_{x\to -1}\frac{1}{x} = -1.$

b. $y-(-1)=-1(x+1)$, or $y=-x-2$.

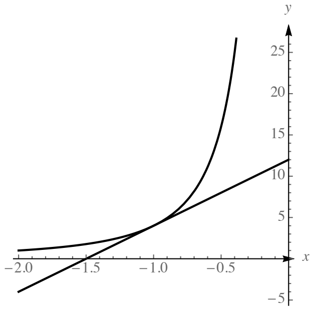
c.



3.1.18

a. $m_{\tan} = \lim\limits_{x\to -1} \dfrac{\frac{4}{x^2}-4}{x+1} = \lim\limits_{x\to -1} \dfrac{4(1-x)(x+1)}{x^2(x+1)} = \lim\limits_{x\to -1} \dfrac{4(1-x)}{x^2} = 8.$

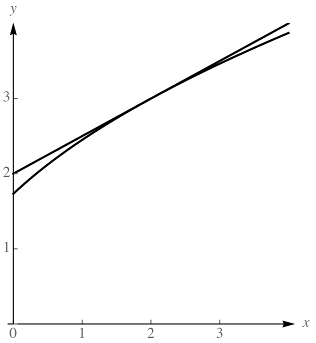
b. $y-4 = 8(x+1)$, or $y = 8x+12$.

c. 

3.1.19

a. $m_{\tan} = \lim\limits_{x\to 2} \dfrac{\sqrt{3x+3}-3}{(x-2)} \cdot \dfrac{\sqrt{3x+3}+3}{\sqrt{3x+3}+3} = \lim\limits_{x\to 2} \dfrac{3(x-2)}{(x-2)(\sqrt{3x+3}+3)} = \lim\limits_{x\to 2} \dfrac{3}{\sqrt{3x+3}+3} = \dfrac{1}{2}.$

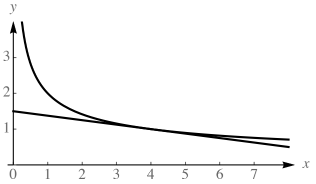
b. $y-3 = \frac{1}{2}(x-2)$, or $y = \frac{1}{2}x+2$.

c. 

3.1.20

a. $m_{\tan} = \lim\limits_{x\to 4} \dfrac{\frac{2}{\sqrt{x}}-1}{(x-4)} = \lim\limits_{x\to 4} \dfrac{(2-\sqrt{x})}{\sqrt{x}(x-4)} \cdot \dfrac{(2+\sqrt{x})}{(2+\sqrt{x})} = \lim\limits_{x\to 4} \dfrac{4-x}{\sqrt{x}(x-4)(2+\sqrt{x})} = -\lim\limits_{x\to 4} \dfrac{1}{\sqrt{x}(2+\sqrt{x})} = -\dfrac{1}{2(2+2)} = -\dfrac{1}{8}.$

b. $y-1 = -\frac{1}{8}(x-4)$ or $y = -\frac{1}{8}x+\frac{3}{2}$.

c. 

3.1.21

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{2(0+h)+1-1}{h} = \lim\limits_{h\to 0} \dfrac{2h}{h} = 2.$

b. $y-1 = 2x$, or $y = 2x+1$.

3.1.22

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{-7(-1+h)-7}{h} = \lim\limits_{h\to 0} -\dfrac{7h}{h} = \lim\limits_{h\to 0} -7 = -7.$

b. $y-7 = -7(x+1)$ or $y = -7x$.

3.1.23

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{3(1+h)^2-4(1+h)+1}{h} = \lim\limits_{h\to 0} \dfrac{3+6h+3h^2-4-4h+1}{h} = \lim\limits_{h\to 0} \dfrac{3h^2+2h}{h} = \lim\limits_{h\to 0}(3h+2) = 2.$

b. $y+1 = 2(x-1)$, or $y = 2x-3$.

3.1.24

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{8-2h^2-8}{h} = \lim\limits_{h\to 0} -\dfrac{2h^2}{h} = \lim\limits_{h\to 0} -2h = 0.$

b. $y - 8 = 0(x-0)$ or $y = 8$.

3.1.25

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{(2+h)^2-4-0}{h} = \lim\limits_{h\to 0} \dfrac{4+4h+h^2-4}{h} = \lim\limits_{h\to 0}(4+h) = 4.$

b. $y - 0 = 4(x-2)$, or $y = 4x - 8$.

3.1.26

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{\frac{1}{1+h}-1}{h} = \lim\limits_{h\to 0} \dfrac{1-(1+h)}{(1+h)h} = \lim\limits_{h\to 0} \dfrac{-1}{1+h} = -1.$

b. $y - 1 = -1(x-1)$, or $y = -x + 2$.

3.1.27

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{(1+h)^3-1}{h} = \lim\limits_{h\to 0} \dfrac{1+3h+3h^2+h^3-1}{h} = \lim\limits_{h\to 0} \dfrac{h(3+3h+h^2)}{h} = \lim\limits_{h\to 0}(3+3h+h^2) = 3.$

b. $y - 1 = 3(x-1)$, or $y = 3x - 2$.

3.1.28

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{\frac{1}{2h+1}-1}{h} = \lim\limits_{h\to 0} \dfrac{\frac{1-2h-1}{2h+1}}{h} = \lim\limits_{h\to 0} -\dfrac{2h}{h(2h+1)} = \lim\limits_{h\to 0} -\dfrac{2}{2h+1} = -2\ .$

b. $y - 1 = -2x$, or $y = -2x + 1$.

3.1.29

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{\frac{1}{3-2(h-1)}-\frac{1}{5}}{h} = \lim\limits_{h\to 0} \dfrac{\frac{5-(3-2h+2)}{15-10(h-1)}}{h} = \lim\limits_{h\to 0} \dfrac{2}{15-10(h-1)} = \dfrac{2}{25}.$

b. $y - \frac{1}{5} = \frac{2}{25}(x+1)$, or $y = \dfrac{2}{25}x + \dfrac{7}{25}$.

3.1.30

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{\sqrt{h+2-1}-1}{h} = \lim\limits_{h\to 0} \dfrac{(\sqrt{h+1}-1)(\sqrt{h+1}+1)}{h(\sqrt{h+1}+1)} = \lim\limits_{h\to 0} \dfrac{h+1-1}{h(\sqrt{h+1}+1)} =$
$\lim\limits_{h\to 0} \dfrac{1}{\sqrt{h+1}+1} = \dfrac{1}{2}.$

b. $y - 1 = \frac{1}{2}(x-2)$, or $y = \frac{1}{2}x$.

3.1.31

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{\sqrt{1+h+3}-2}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{4+h}-2}{h} \cdot \dfrac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \lim\limits_{h\to 0} \dfrac{4+h-4}{h(\sqrt{4+h}+2)} =$
$\lim\limits_{h\to 0} \dfrac{1}{\sqrt{4+h}+2} = \dfrac{1}{4}.$

b. $y - 2 = \frac{1}{4}\,(x-1)$, or $y = \dfrac{1}{4}x + \dfrac{7}{4}$.

3.1.32

a. $m_{\tan} = \lim_{h\to 0} \frac{\frac{-2+h}{-2+h+1} - 2}{h} = \lim_{h\to 0} \frac{-2+h-2(-2+h+1)}{(h)(-2+h+1)} = \lim_{h\to 0} -\frac{1}{-2+h+1} = 1.$

b. $y - 2 = (x - (-2))$ or $y = x + 4$.

3.1.33

a. $f'(-3) = \lim_{h\to 0} \frac{8(-3+h)+24}{h} = \lim_{h\to 0} \frac{8h}{h} = 8.$

b. $y - (-24) = 8(x+3)$, or $y = 8x$.

3.1.34

a. $f'(3) = \lim_{h\to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h\to 0} \frac{(9+6h+h^2)-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = 6.$

b. $y - 9 = 6(x-3)$, or $y = 6x - 9$.

3.1.35

a. $f'(-2) = \lim_{h\to 0} \frac{4(-2+h)^2 + 2(-2+h) - 12}{h} = \lim_{h\to 0} \frac{16-16h+4h^2-4+2h-12}{h} =$
$\lim_{h\to 0} \frac{-14h+4h^2}{h} = -14.$

b. $y - 12 = -14(x+2)$, or $y = -14x - 16$.

3.1.36

a. $f'(10) = \lim_{h\to 0} \frac{2(10+h)^3 - 2000}{h} = \lim_{h\to 0} \frac{2(1000+300h+30h^2+h^3)-2000}{h} =$
$\lim_{h\to 0}(600+60h+2h^2) = 600.$

b. $y - 2000 = 600(x-10)$, or $y = 600x - 4000$.

3.1.37

a. $f'\left(\frac{1}{4}\right) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{\frac{1}{4}+h}} - 2}{h} = \lim_{h\to 0} \frac{1-2\sqrt{\frac{1}{4}+h}}{h\sqrt{\frac{1}{4}+h}} = \lim_{h\to 0} \frac{\left(1-2\sqrt{\frac{1}{4}+h}\right)\left(1+2\sqrt{\frac{1}{4}+h}\right)}{h\sqrt{\frac{1}{4}+h}\left(1+2\sqrt{\frac{1}{4}+h}\right)} =$
$\lim_{h\to 0} \frac{1-4\left(\frac{1}{4}+h\right)}{h\sqrt{\frac{1}{4}+h}\left(1+2\sqrt{\frac{1}{4}+h}\right)} = \lim_{h\to 0} -\frac{4}{\sqrt{\frac{1}{4}+h}\left(1+2\sqrt{\frac{1}{4}+h}\right)} = -4.$

b. $y - 2 = -4\left(x - \frac{1}{4}\right)$, or $y = -4x + 3$.

3.1.38

a. $f'(1) = \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - 1}{h} = \lim_{h\to 0} \frac{1-(1+h)^2}{h(1+h)^2} = \lim_{h\to 0} \frac{1-1-2h-h^2}{h(1+h)^2} = \lim_{h\to 0} \frac{-2-h}{(1+h)^2} = -2.$

b. $y - 1 = -2(x-1)$, or $y = -2x + 3$.

3.1.39

a. $f'(4) = \lim_{h\to 0} \frac{\sqrt{2(4+h)+1} - 3}{h} = \lim_{h\to 0} \frac{\sqrt{9+2h}-3}{h} \cdot \frac{\sqrt{9+2h}+3}{\sqrt{9+2h}+3} = \lim_{h\to 0} \frac{9+2h-9}{h(\sqrt{9+2h}+3)} =$
$\lim_{h\to 0} \frac{2}{\sqrt{9+2h}+3} = \frac{1}{3}.$

b. $y-3=\frac{1}{3}(x-4)$, or $y=\frac{1}{3}x+\frac{5}{3}$.

3.1.40

a. $f'(12)=\lim\limits_{h\to 0}\dfrac{\sqrt{3(12+h)}-6}{h}=\lim\limits_{h\to 0}\dfrac{\sqrt{36+3h}-6}{h}\cdot\dfrac{\sqrt{36+3h}+6}{\sqrt{36+3h}+6}=\lim\limits_{h\to 0}\dfrac{36+3h-36}{h(\sqrt{36+3h}+6)}=$ $\lim\limits_{h\to 0}\dfrac{3}{\sqrt{36+3h}+6}=\dfrac{1}{4}$.

b. $y-6=\frac{1}{4}(x-12)$, or $y=\frac{1}{4}x+3$.

3.1.41

a. $f'(5)=\lim\limits_{h\to 0}\dfrac{\frac{1}{5+h+5}-\frac{1}{10}}{h}=\lim\limits_{h\to 0}\dfrac{10-(10+h)}{10h(10+h)}=\lim\limits_{h\to 0}\dfrac{-1}{10(10+h)}=-\dfrac{1}{100}$.

b. $y-\frac{1}{10}=-\frac{1}{100}(x-5)$, or $y=-\frac{1}{100}x+\frac{3}{20}$.

3.1.42

a. $f'(2)=\lim\limits_{h\to 0}\dfrac{\frac{1}{3(2+h)-1}-\frac{1}{5}}{h}=\lim\limits_{h\to 0}\dfrac{5-(3(2+h)-1)}{h(3(2+h)-1)5}=\lim\limits_{h\to 0}-\dfrac{3h}{h(5+3h)5}=\lim\limits_{h\to 0}-\dfrac{3}{(5+3h)5}=-\dfrac{3}{25}$.

b. $y-\frac{1}{5}=-\frac{3}{25}(x-2)$, or $y=-\frac{3}{25}x+\frac{11}{25}$.

3.1.43 $m_{\tan}=\lim\limits_{h\to 0}\dfrac{\frac{1}{1+h+1}-\frac{1}{2}}{h}=\lim\limits_{h\to 0}\dfrac{2-(2+h)}{h(2+h)2}=\lim\limits_{h\to 0}\dfrac{-1}{(2+h)2}=-\dfrac{1}{4}$.

3.1.44 $m_{\tan}=\lim\limits_{h\to 0}\dfrac{2+h-(2+h)^2-(2-4)}{h}=\lim\limits_{h\to 0}\dfrac{2+h-4-4h-h^2+2}{h}=\lim\limits_{h\to 0}\dfrac{-3h-h^2}{h}=$ $\lim\limits_{h\to 0}(-3-h)=-3$.

3.1.45 $m_{\tan}=\lim\limits_{h\to 0}\dfrac{2\sqrt{25+h}-1-(2\sqrt{25}-1)}{h}=\lim\limits_{h\to 0}\dfrac{2(\sqrt{25+h}-\sqrt{25})}{h}=$ $\lim\limits_{h\to 0}\dfrac{2(\sqrt{25+h}-\sqrt{25})(\sqrt{25+h}+\sqrt{25})}{h(\sqrt{25+h}+\sqrt{25})}=\lim\limits_{h\to 0}\dfrac{2(25+h-25)}{h(\sqrt{25+h}+\sqrt{25})}=\lim\limits_{h\to 0}\dfrac{2}{\sqrt{25+h}+\sqrt{25}}=\dfrac{1}{5}$.

3.1.46 $m_{\tan}=\lim\limits_{h\to 0}\dfrac{\pi(3+h)^2-9\pi}{h}=\lim\limits_{h\to 0}\dfrac{\pi(9+6h+h^2)-9\pi}{h}=\lim\limits_{h\to 0}\dfrac{6\pi h+\pi h^2}{h}=\lim\limits_{h\to 0}(6\pi+\pi h)=6\pi$.

3.1.47

a. True. Because the graph is a line, any secant line has the same graph as the function and thus the same slope.

b. False. For example, take $f(x)=x^2$, $P=(0,0)$ and $Q=(1,1)$. Then the secant line has slope $m_{\sec}=\frac{1-0}{1-0}=1$, but the the graph has a horizontal tangent at P so $m_{\tan}=0$ and $m_{\sec}>m_{\tan}$.

c. True. $m_{\sec}=\dfrac{(x+h)^2-x^2}{h}=\dfrac{2xh+h^2}{h}=2x+h$, while $m_{\tan}=\lim\limits_{h\to 0}(2x+h)=2x$. Because we assume that $h>0$, we have $m_{\sec}=2x+h>2x=m_{\tan}$.

3.1.48 $V'(12)=\lim\limits_{t\to 12}\dfrac{3t-36}{t-12}=3\lim\limits_{t\to 12}\dfrac{t-12}{t-12}=3\,\text{gal/min}$. The instantaneous rate of change of the volume of water in the tub is 3 gal/min 12 minutes after the faucet was turned on.

3.1.49 $d'(4)=\lim\limits_{t\to 4}\dfrac{16t^2-256}{t-4}=16\lim\limits_{t\to 4}\dfrac{(t-4)(t+4)}{t-4}=16\lim\limits_{t\to 4}(t+4)=16\cdot 8=128\,\text{ft/s}$. The object is falling with an instantaneous speed of 128 ft/s 4 seconds after being dropped.

3.1.50 $F'(10) = \lim\limits_{x\to 10} \dfrac{\frac{k}{x^2} - \frac{k}{100}}{x-10} = k \lim\limits_{x\to 10} \dfrac{100 - x^2}{100x^2(x-10)} = -k \lim\limits_{x\to 10} \dfrac{(x-10)(x+10)}{100x^2(x-10)} = -k \lim\limits_{x\to 10} \dfrac{x+10}{100x^2} = -0.002k\,\text{N/m}$. The instantaneous rate of change in the force between the two masses is $-0.002k$ N/m at a distance of separation of 10 m.

3.1.51 $v'(3) = \lim\limits_{t\to 3} \dfrac{(t-5)^2 - 4}{t-3} = \lim\limits_{t\to 3} \dfrac{(t-5-2)(t-5+2)}{t-3} = \lim\limits_{t\to 3}(t-7) = -4\,\text{m/s}$ per second. The instantaneous rate of change in the car's speed is $-4\,\text{m/s}^2$ at time 3 seconds.

3.1.52

a. The average rate of growth is the slope of the secant line between $t = 20$ and $t = 30$ so $m_{\sec} = \dfrac{528,000 - 304,744}{30-20} \approx 22,326$, which means that the average population growth is a bit over 22,000 people per year (Census data only provide estimates; there is no use in calculating more accurately).

b. Drawing the secant line from part a) and the approximate tangent line for 1975 (corresponding to $t = 25$), we see that they have about the same slope.

c. The average rate of growth is given by $m_{\sec} = \dfrac{1,563,282 - 304,744}{50-20} \approx 41,951$, which means that the average rate of growth of Las Vegas from 1970 to 2000 was about 41,951 people/year. This is an underestimate of the rate of growth in 2000 because the slope of the line tangent to the curve at $t = 50$ is steeper. The approximate slope of the tangent line at $t = 50$ is 60,000, which means the population was growing at about 60,000 people/year in 2000.

3.1.53

a. $L'(15) \approx \dfrac{6.75 - 3.5}{2 - 1.25} \approx 4.3\,\text{mm/week}$. At 1.5 weeks, the talon is growing at a rate of about 4.3 mm/week.

b. $L'(a) \approx 0$ for $a \geq 4$. At 4 weeks the talons have stopped growing.

3.1.54 $A'(7) \approx -5\,\text{mg/hour}$. $A'(15) \approx -2\,\text{mg/hour}$. The caffeine levels are dropping at about 5 mg/hour after 7 hours and at 2 mg/hr after 15 hours.

3.1.55 $D'(60) \approx 0.05\,\text{hr/day}$. 60 days after January 1, the daylight hours are increasing at about 0.05 hours per day (3 minutes per day). $D'(170) \approx 0$ hr/day. The number of daylight hours is neither increasing nor decreasing 170 days after January 1.

3.1.56 Consider $a = 1$ and $f(x) = 3x^2 + 4x$.

Then $f'(1) = \lim\limits_{x\to 1} \dfrac{3x^2 + 4x - 7}{x-1}$, as desired.

We have $f'(1) = \lim\limits_{x\to 1} \dfrac{3x^2 + 4x - 7}{x-1} = \lim\limits_{x\to 1} \dfrac{(x-1)(3x+7)}{x-1} = \lim\limits_{x\to 1}(3x+7) = 10.$

3.1.57 Consider $a = 2$ and $f(x) = 5x^2$.

Then $f'(2) = \lim\limits_{x\to 2} \dfrac{5x^2 - 20}{x-2}$ as desired.

We have $f'(2) = \lim\limits_{x\to 2} \dfrac{5x^2 - 20}{x-2} = \lim\limits_{x\to 2} \dfrac{5(x-2)(x+2)}{x-2} = \lim\limits_{x\to 2} 5(x+2) = 20.$

3.1.58 Consider $a = 2$ and $f(x) = \frac{1}{x+1}$.

Then $f'(2) = \lim\limits_{x\to 2} \dfrac{\frac{1}{x+1} - \frac{1}{3}}{x-2}$ as desired.

We have $f'(2) = \lim\limits_{x\to 2} \dfrac{\frac{1}{x+1} - \frac{1}{3}}{x-2} = \lim\limits_{x\to 2} \dfrac{3 - (x+1)}{(x-2)3(x+1)} = \lim\limits_{x\to 2} \dfrac{-(x-2)}{(x-2)3(x+1)} = \lim\limits_{x\to 2} \dfrac{-1}{3(x+1)} = -\dfrac{1}{9}.$

3.1.59 Consider $a = 2$ and $f(x) = x^4$.

Then $f'(2) = \lim_{h\to 0} \frac{(2+h)^4 - 16}{h}$ as desired.

We have

$$f'(2) = \lim_{h\to 0} \frac{(2+h)^4 - 16}{h} = \lim_{h\to 0} \frac{16 + 32h + 24h^2 + 8h^3 + h^4 - 16}{h}$$
$$= \lim_{h\to 0} \frac{h\left(32 + 24h + 8h^2 + h^3\right)}{h} = \lim_{h\to 0} \left(32 + 24h + 8h^2 + h^3\right) = 32.$$

3.1.60 Consider $a = 2$ and $f(x) = \sqrt{x}$.

Then $f'(2) = \lim_{h\to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h}$, as desired.

We have

$$f'(2) = \lim_{h\to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{2+h} - \sqrt{2}\right)\left(\sqrt{2+h} + \sqrt{2}\right)}{h\left(\sqrt{2+h} + \sqrt{2}\right)}$$
$$= \lim_{h\to 0} \frac{2 + h - 2}{h\left(\sqrt{2+h} + \sqrt{2}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

3.1.61 Consider $a = -1$ and $f(x) = |x|$.

Then $f'(-1) = \lim_{h\to 0} \frac{|-1+h| - 1}{h}$ as desired.

We have $f'(-1) = \lim_{h\to 0} \frac{|-1+h| - 1}{h} = \lim_{h\to 0} \frac{1 - h - 1}{h} = \lim_{h\to 0} -1 = -1.$

3.1.62

a. $f'(4) = \lim_{x\to 4} \frac{\sqrt{x} - \sqrt{4}}{x - 4} = \lim_{x\to 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x\to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4}.$

b. For $|h|$ near zero we have $f'(4) \approx \frac{f(4+h) - f(4)}{h} = \frac{\sqrt{4+h} - 2}{h}$.

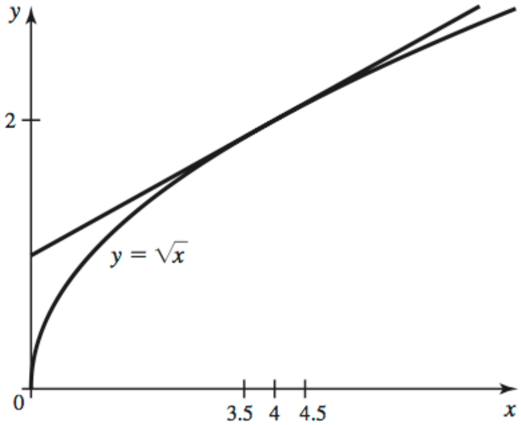
c.

h	$\frac{\sqrt{4+h}-2}{h}$	Error	h	$\frac{\sqrt{4+h}-2}{h}$	Error
0.1	0.248457	-0.001543	-0.1	0.251582	.001582
0.01	0.249844	-0.000156	-0.01	0.250156	.000156
0.001	0.249984	-0.000016	-0.001	0.250016	.000016
0.0001	0.249998	-0.000002	-0.0001	0.250002	.000002

d. The error approaches zero as h approaches zero.

3.1.63

a. Note that the slope generated by the centered difference quotient is $\frac{f(4.5) - f(3.5)}{2(0.5)} = \sqrt{4.5} - \sqrt{3.5} \approx 0.250492$. The centered difference quotient line is very close to the tangent line, which closely approximates the function near the point of tangency.



b.

h	Approximation	Error
0.1	0.25002	2.0×10^{-5}
0.01	≈ 0.25000	2.0×10^{-7}
0.001	≈ 0.25000	2.0×10^{-9}

c. The centered difference quotient is symmetric about zero, so using the corresponding negative values yields the same results.

d. The centered difference quotient appears to be more accurate than the approximation in the previous problem.

3.1.64

a. $\dfrac{f(2.5)-f(2)}{2.5-2} = \dfrac{81-55}{0.5} = 52.$

b. $\dfrac{f(2.5)-f(1.5)}{2(0.5)} = 81 - 33 = 48.$

3.1.65

a. Forward:

$$\frac{\operatorname{erf}(1.05)-\operatorname{erf}(1)}{0.05} = \frac{0.862436-0.842701}{0.05} = 0.3947.$$

Centered:

$$\frac{\operatorname{erf}(1.05)-\operatorname{erf}(0.95)}{2(0.05)} = \frac{0.862436-0.820891}{0.1} = 0.41545.$$

b. Forward:

$$\left|0.3947 - \frac{2}{e\sqrt{\pi}}\right| \approx 0.02.$$

Centered:

$$\left|0.41545 - \frac{2}{e\sqrt{\pi}}\right| \approx 0.003$$

3.2 The Derivative as a Function

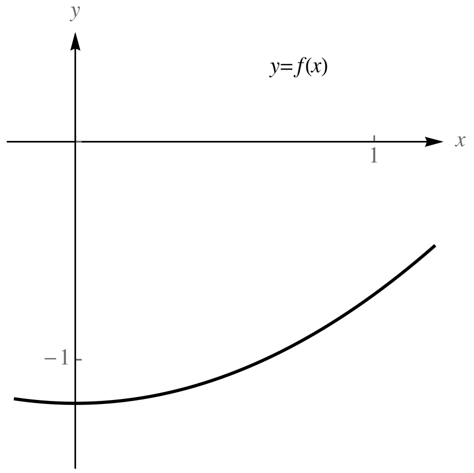
3.2.1 It represents the slope function of f.

3.2.2 $f'(1) = 5$, $f'(2) = 8$, $f'(3) = 11$.

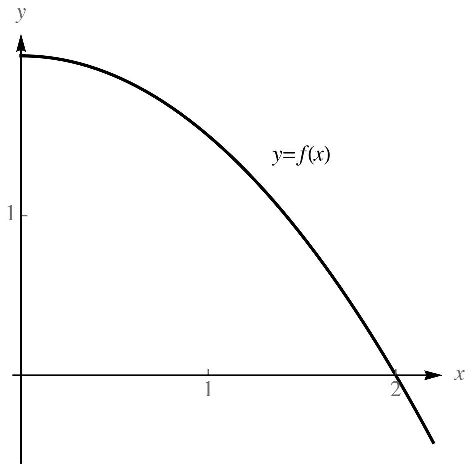
3.2.3 $\dfrac{dy}{dx}$ is the limit of $\dfrac{\Delta y}{\Delta x}$ and is the rate of change of y with respect to x.

3.2.4 The derivative of f with respect to x can be written as $f'(x)$ or $\dfrac{df}{dx}$ or $D_x(f)$ or $\frac{d}{dx}(f)$.

3.2.5



3.2.6



3.2.7 Yes, differentiable functions are continuous by Theorem 3.1.

3.2.8 No, there are continuous functions which are not differentiable. For example $f(x) = |x|$ is continuous everywhere but the graph of f has a corner at 0, and thus f is not differentiable at 0.

3.2.9 The graph is a line with y-intercept of 1 and a slope of 3.

3.2.10 $f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$

3.2.11 $f'(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{7(x+h) - 7x}{h} = \lim_{h\to 0} \frac{7h}{h} = \lim_{h\to 0} 7 = 7.$

3.2.12 $f'(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{-3(x+h) - (-3x)}{h} = \lim_{h\to 0} \frac{-3h}{h} = -\lim_{h\to 0} 3 = -3.$

3.2.13 $\dfrac{dy}{dx} = \displaystyle\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} 2x + h = 2x$. Therefore, $\left.\dfrac{dy}{dx}\right|_{x=3} = 6$ and $\left.\dfrac{dy}{dx}\right|_{x=-2} = -4$.

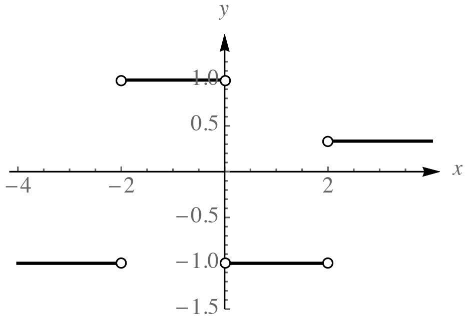
3.2.14 $w'(x) = \displaystyle\lim_{h\to 0} \frac{1.5(x+h) - 35.8 - (1.5x - 35.8)}{h} = \lim_{h\to 0} \frac{1.5h}{h} = \lim_{h\to 0} 1.5 = 1.5$. The weight of an Atlantic salmon increases at a rate of 1.5 lb/in when its length is between 33 and 48 inches long.

3.2.15 (c) is the only line with negative slope, so it corresponds to derivative (A). Since (d) contains the points $(2, 0)$ and $(0, 1)$, it has slope $\frac{1}{2}$, so it corresponds to derivative (B). Finally, lines (a) and (b) are parallel; since (b) contains the points $(0, 1)$ and $(-1, 0)$, it has slope 1, so that (a) has slope 1 as well. They both correspond to derivative (C).

3.2.16 Note that (A) and (C) have positive slope, while (B) and (D) have negative slope. Since (a) is the largest positive derivative, it corresponds to (A), which has larger slope than (B). Thus (b), the other positive derivative, corresponds to (C). Since (d) is the negative derivative of largest magnitude, it corresponds to (D), since the slope of (D) has larger magnitude than that of (B). So (c), the other negative derivative, corresponds to (B).

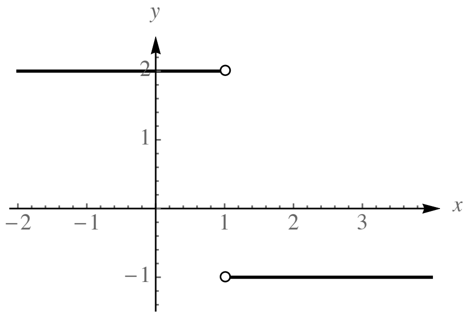
3.2.17

The function f is not differentiable at $x = -2, 0, 2$, so f' is not defined at those points. Elsewhere, the slope is constant.



3.2.18

The function f is not differentiable at $x = 1$ so f' is not defined there. Elsewhere, the slope is constant.



3.2.19

a. f is not continuous at $x = 1$.

b. f is not differentiable at $x = 1$ and at $x = 0$.

3.2.20

a. g is not continuous at $x = -1$ and $x = 0$.

b. g is not differentiable at $x = -1$ and at $x = 0$ and at $x = 1$.

3.2.21

a. $f'(x) = \lim_{h\to 0} \dfrac{5(x+h)+2-(5x+2)}{h} = \lim_{h\to 0} \dfrac{5h}{h} = \lim_{h\to 0} 5 = 5.$

b. $f'(1) = 5$ and $f'(2) = 5$.

3.2.22

a. $f'(x) = \lim_{h\to 0} \dfrac{7-7}{h} = \lim_{h\to 0} 0 = 0.$

b. $f'(-1) = 0$ and $f'(2) = 0$.

3.2.23

a. $f'(x) = \lim_{h\to 0} \dfrac{4(x+h)^2+1-(4x^2+1)}{h} = \lim_{h\to 0} \dfrac{4x^2+8hx+4h^2+1-4x^2-1}{h} =$
$\lim_{h\to 0} \dfrac{h(8x+4h)}{h} = \lim_{h\to 0} 8x+4h = 8x.$

b. $f'(2) = 16$ and $f'(4) = 32$.

3.2.24

a. $f'(x) = \lim_{h\to 0} \dfrac{(x+h)^2+3(x+h)-(x^2+3x)}{h} = \lim_{h\to 0} \dfrac{x^2+2hx+h^2+3x+3h-x^2-3x}{h} =$
$\lim_{h\to 0} \dfrac{h(2x+h+3)}{h} = \lim_{h\to 0}(2x+h+3) = 2x+3.$

b. $f'(-1) = 1$ and $f'(4) = 11$.

3.2.25

a. $f'(x) = \lim_{h\to 0} \dfrac{\frac{1}{x+h+1}-\frac{1}{x+1}}{h} = \lim_{h\to 0} \dfrac{(x+1)-(x+h+1)}{h(x+1)(x+h+1)} = \lim_{h\to 0} \dfrac{-h}{h(x+1)(x+h+1)} =$
$-\lim_{h\to 0} \dfrac{1}{(x+1)(x+h+1)} = -\dfrac{1}{(x+1)^2}.$

b. $f'(-1/2) = -4$ and $f'(5) = -\dfrac{1}{36}$.

3.2.26

a. $f'(x) = \lim_{h\to 0} \dfrac{\frac{x+h}{x+h+2}-\frac{x}{x+2}}{h} = \lim_{h\to 0} \dfrac{(x+2)(x+h)-(x+h+2)(x)}{h(x+h+2)(x+2)} =$
$\lim_{h\to 0} \dfrac{x^2+hx+2x+2h-x^2-hx-2x}{h(x+h+2)(x+2)} = \lim_{h\to 0} \dfrac{2}{(x+h+2)(x+2)} = \dfrac{2}{(x+2)^2}.$

b. $f'(-1) = 2$ and $f'(0) = \frac{1}{2}$.

3.2.27

a. $f'(t) = \lim_{h\to 0} \dfrac{\frac{1}{\sqrt{t+h}}-\frac{1}{\sqrt{t}}}{h} = \lim_{h\to 0} \dfrac{\sqrt{t}-\sqrt{t+h}}{h\sqrt{t}\sqrt{t+h}} = \lim_{h\to 0} \dfrac{(\sqrt{t}-\sqrt{t+h})}{h\sqrt{t}\sqrt{t+h}} \cdot \dfrac{(\sqrt{t}+\sqrt{t+h})}{(\sqrt{t}+\sqrt{t+h})} =$
$\lim_{h\to 0} \dfrac{t-(t+h)}{h\sqrt{t}\sqrt{t+h}(\sqrt{t}+\sqrt{t+h})} = -\lim_{h\to 0} \dfrac{1}{\sqrt{t}\sqrt{t+h}(\sqrt{t}+\sqrt{t+h})} = -\dfrac{1}{2t^{3/2}}.$

b. $f'(9) = -\frac{1}{54}$ and $f'(1/4) = -4$.

3.2.28

a. $f'(w) = \lim_{h\to 0} \dfrac{(\sqrt{4(w+h)-3}-\sqrt{4w-3})}{h} \cdot \dfrac{(\sqrt{4(w+h)-3}+\sqrt{4w-3})}{(\sqrt{4(w+h)-3}+\sqrt{4w-3})} =$
$\lim_{h\to 0} \dfrac{4(w+h)-3-(4w-3)}{h(\sqrt{4(w+h)-3}+\sqrt{4w-3})} = \lim_{h\to 0} \dfrac{4}{\sqrt{4(w+h)-3}+\sqrt{4w-3}} = \dfrac{2}{\sqrt{4w-3}}.$

b. $f'(1) = 2$ and $f'(3) = \frac{2}{3}$.

3.2.29

a.

$$\begin{aligned} f'(s) &= \lim_{h\to 0} \frac{4(s+h)^3 + 3(s+h) - (4s^3+3s)}{h} \\ &= \lim_{h\to 0} \frac{4s^3 + 12s^2h + 12sh^2 + 4h^3 + 3s + 3h - 4s^3 - 3s}{h} \\ &= \lim_{h\to 0} \frac{h(12s^2 + 12sh + 4h^2 + 3)}{h} = \lim_{h\to 0}(12s^2 + 12sh + 4h^2 + 3) = 12s^2 + 3. \end{aligned}$$

b. $f'(-3) = 111$ and $f'(-1) = 15$.

3.2.30

a.

$$\begin{aligned} f'(t) &= \lim_{h\to 0} \frac{3(t+h)^4 - 3t^4}{h} \\ &= 3\lim_{h\to 0} \frac{t^4 + 4t^3h + 6t^2h^2 + 4th^3 + h^4 - t^4}{h} \\ &= 3\lim_{h\to 0}(4t^3 + 6t^2h + 4th^2 + h^3) = 12t^3 \end{aligned}$$

b. $f'(-2) = -96$ and $f'(2) = 96$.

3.2.31

a. $v(t) = s'(t) = \lim\limits_{h\to 0} \dfrac{-16(t+h)^2 + 100(t+h) - (-16t^2 + 100t)}{h} =$
$\lim\limits_{h\to 0} \dfrac{-16t^2 - 32th - 16h^2 + 100t + 100h + 16t^2 - 100t}{h} = \lim\limits_{h\to 0} \dfrac{h(-32t - 16h + 100)}{h} = -32t + 100.$

b. $v(1) = 68$ ft/s and $v(2) = 36$ ft/s.

3.2.32

a. $v(t) = s'(t) = \lim\limits_{h\to 0} \dfrac{-16(t+h)^2 + 128(t+h) + 192 - (-16t^2 + 128t + 192)}{h} =$
$\lim\limits_{h\to 0} \dfrac{-16t^2 - 32th - 16h^2 + 128t + 128h + 192 + 16t^2 - 128t - 192}{h} =$
$\lim\limits_{h\to 0} \dfrac{h(-32t - 16h + 128)}{h} = -32t + 128.$

b. $v(1) = 96$ ft/s and $v(2) = 64$ ft/s.

3.2.33

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{\frac{x+h+1}{x+h+2} - \frac{x+1}{x+2}}{h} \\ &= \lim_{h\to 0} \frac{(x+2)(x+h+1) - (x+1)(x+h+2)}{h(x+h+2)(x+2)} \\ &= \lim_{h\to 0} \frac{x^2 + xh + x + 2x + 2h + 2 - (x^2 + xh + 2x + x + h + 2)}{h(x+h+2)(x+2)} \\ &= \lim_{h\to 0} \frac{h}{h(x+h+2)(x+2)} = \frac{1}{(x+2)^2} \end{aligned}$$

Therefore $\left.\dfrac{dy}{dx}\right|_2 = \dfrac{1}{16}$.

3.2.34

$$\begin{aligned}\frac{ds}{dt}&=\lim_{h\to 0}\frac{11(t+h)^3+t+h+1-(11t^3+t+1)}{h}\\&=\lim_{h\to 0}\frac{11t^3+33t^2h+33th^2+11h^3+t+h+1-11t^3-t-1}{h}\\&=\lim_{h\to 0}\frac{h(33t^2+33th+11h^2+1)}{h}\\&=\lim_{h\to 0}(33t^2+33th+11h^2+1)=33t^2+1.\end{aligned}$$

Therefore $\left.\dfrac{ds}{dt}\right|_{-1}=34.$

3.2.35

a.

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\frac{3(x+h)^2+2(x+h)-10-(3x^2+2x-10)}{h}\\&=\lim_{h\to 0}\frac{3x^2+6xh+3h^2+2x+2h-10-3x^2-2x+10}{h}\\&=\lim_{h\to 0}\frac{6xh+2h+3h^2}{h}=\lim_{h\to 0}(6x+2+3h)=6x+2.\end{aligned}$$

b. We have $f'(1)=8$, and the tangent line is given by $y+5=8(x-1)$, or $y=8x-13$.

3.2.36

a. $f'(x)=\lim_{h\to 0}\dfrac{5(x+h)^2-6(x+h)+1-(5x^2-6x+1)}{h}=$
$\lim_{h\to 0}\dfrac{5x^2+10xh+5h^2-6x-6h-5x^2+6x}{h}=\lim_{h\to 0}\dfrac{10xh+5h^2-6h}{h}=\lim_{h\to 0}(10x+5h-6)=10x-6.$

b. We have $f'(2)=14$, so the tangent line is given by $y-9=14(x-2)$, or $y=14x-19$.

3.2.37

a.

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}\\&=\lim_{h\to 0}\frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}\cdot\frac{\sqrt{3x+3h+1}+\sqrt{3x+1}}{\sqrt{3x+3h+1}+\sqrt{3x+1}}\\&=\lim_{h\to 0}\frac{3x+3h+1-3x-1}{h(\sqrt{3x+3h+1}+\sqrt{3x+1})}=\lim_{h\to 0}\frac{3}{\sqrt{3x+3h+1}+\sqrt{3x+1}}=\frac{3}{2\sqrt{3x+1}}.\end{aligned}$$

b. We have $f'(8)=\frac{3}{10}$. Using the point-slope form, we get that the tangent line has equation $y-5=\frac{3}{10}(x-8)$, which can be written as $y=\frac{3}{10}x+\frac{13}{5}$.

3.2.38

a.

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\frac{\sqrt{x+h+2}-\sqrt{x+2}}{h}\\&=\lim_{h\to 0}\frac{\sqrt{x+h+2}-\sqrt{x+2}}{h}\cdot\frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}}\\&=\lim_{h\to 0}\frac{x+h+2-x-2}{h(\sqrt{x+h+2}+\sqrt{x+2})}=\lim_{h\to 0}\frac{1}{\sqrt{x+h+2}+\sqrt{x+2}}=\frac{1}{2\sqrt{x+2}}.\end{aligned}$$

b. We have $f'(7) = \frac{1}{6}$. Using the point-slope form, we get that the tangent line has equation $y - 3 = \frac{1}{6}(x-7)$, or $y = \frac{1}{6}x + \frac{11}{6}$.

3.2.39

a. $f'(x) = \lim\limits_{h\to 0} \dfrac{\frac{2}{3(x+h)+1} - \frac{2}{3x+1}}{h} = \lim\limits_{h\to 0} \dfrac{6x+2-(6x+6h+2)}{h(3x+1)(3x+3h+1)} = \lim\limits_{h\to 0} \dfrac{-6h}{h(3x+1)(3x+3h+1)} = -\dfrac{6}{(3x+1)^2}$.

b. We have $f'(-1) = -\frac{3}{2}$. Using the point-slope form, we get that the tangent line has equation $y + 1 = -\frac{3}{2}(x+1)$, which can be written as $y = -\frac{3}{2}x - \frac{5}{2}$.

3.2.40

a. $f'(x) = \lim\limits_{h\to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim\limits_{h\to 0} \dfrac{x-x-h}{h(x+h)x} = \lim\limits_{h\to 0} \dfrac{-1}{x^2+xh} = -\dfrac{1}{x^2}$.

b. We have $f'(-5) = -\frac{1}{25}$. Using the point-slope form, we get that the tangent line has equation $y + \frac{1}{5} = -\frac{1}{25}(x+5)$, which can be written as $y = \frac{-1}{25}x - \frac{2}{5}$.

3.2.41

a. From the graph we approximate the derivative by the slope of a secant line: For example we see that $E(6) = 250\,\text{kWh}$ and $E(18) = 350$ kWh, so the power after 10 hours is approximately the slope of the secant line through these points, so $P(10) \approx m_{sec} = \dfrac{E(18) - E(6)}{18-6} = \dfrac{350\,\text{kWh} - 250\,\text{kWh}}{12\text{h}} \approx 8.3\,\text{kW}$. Similarly, after 20 hours, using 18 hours and 25 hours, that $P(20) \approx m_{\text{sec}} = \dfrac{E(22) - E(18)}{22-18} = \dfrac{325\,\text{kWh} - 350\,\text{kWh}}{4\text{h}} \approx -6.25\text{kW}$.

b. The power is zero where the graph of $E(t)$ has a horizontal tangent line, which happens approximately at $t = 6$ hours and $t = 18$ hours.

c. The power has a maximum where the graph of $E(t)$ has the steepest increase, which is approximately at $t = 12$ hours.

3.2.42 $m_{\tan} = \lim\limits_{h\to 0} \dfrac{m(x+h)+b-(mx+b)}{h} = \lim\limits_{h\to 0} \dfrac{mh}{h} = m$. Thus the derivative has the same value as the slope of the line and the graph and formula of the tangent line are the same as those of the function, namely $mx + b$.

3.2.43

a. $\frac{d}{dx}\left(ax^2+bx+c\right)=$
$\lim_{h\to 0}\frac{a(x+h)^2+b(x+h)+c-\left(ax^2+bx+c\right)}{h}=\lim_{h\to 0}\frac{ax^2+2axh+ah^2+bx+bh+c-ax^2-bx-c}{h}$
$=\lim_{h\to 0}\frac{2axh+ah^2+bh}{h}=\lim_{h\to 0}(2ax+ah+b)=2ax+b.$

b. With $a=4, b=-3, c=10$ we have $\frac{d}{dx}(4x^2-3x+10)=2\cdot 4\cdot x+(-3)=8x-3.$

c. From part (b), $f'(1)=8\cdot 1-3=5.$

3.2.44

a. $\frac{d}{dx}\sqrt{ax+b}=\lim_{h\to 0}\frac{\sqrt{a(x+h)+b}-\sqrt{ax+b}}{h}=$
$\lim_{h\to 0}\frac{\left(\sqrt{a(x+h)+b}-\sqrt{ax+b}\right)\left(\sqrt{a(x+h)+b}+\sqrt{ax+b}\right)}{h\left(\sqrt{a(x+h)+b}+\sqrt{ax+b}\right)}=$
$\lim_{h\to 0}\frac{a(x+h)+b-(ax+b)}{h\left(\sqrt{a(x+h)+b}+\sqrt{ax+b}\right)}=\lim_{h\to 0}\frac{ax+ah+b-ax-b}{h\left(\sqrt{a(x+h)+b}+\sqrt{ax+b}\right)}=$
$\lim_{h\to 0}\frac{a}{\sqrt{a(x+h)+b}+\sqrt{ax+b}}=\frac{a}{2\sqrt{ax+b}}$, provided $ax+b>0$.

b. With $a=5, b=9$, we have $\frac{d}{dx}\sqrt{5x+9}=\frac{5}{2\sqrt{5x+9}}$.

c. From part (b), $f'(-1)=\frac{5}{2\sqrt{-5+9}}=\frac{5}{2\sqrt{4}}=\frac{5}{4}$.

3.2.45

a. At C and D, the slope of the tangent line (and thus of the curve) is negative.

b. At A, B, and E, the slope of the curve is positive.

c. The graph is in its steepest ascent at A followed by B. At E it barely increases, at D it slightly decreases and at C it is decreasing the most, so the points in decreasing order of slope are A, B, E, D, C.

3.2.46

a. The graph of the function has negative slope to the right of the vertical axis so the slope is negative at D and E.

b. The graph of the function has positive slope to the left of the vertical axis so the slope is positive at A, B, C.

c. The slope at D and E is negative, with a slightly larger absolute value at D. The slope at A and C is about equal and positive, and the slope is steepest at B. So the order is B, A, C, E, D, where A and C could be switched.

3.2.47

a. The function has non-negative slope everywhere, and as there is a horizontal tangent at $x=0$, so the derivative has to be zero at zero. The graph of the derivative has to be above the x-axis and touching it at $x=0$, so (D) is the graph of the derivative.

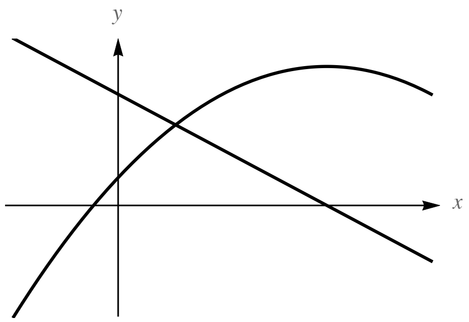
b. The graph of this function has three horizontal tangent lines, at $x=-1,0,1$, and the matching graph of the derivative with three zeros is (C).

c. The function has negative slope on $(-1, 0)$, and positive slope on $(0, 1)$ and has a horizontal tangent at $x = 0$, so the derivative has to be negative on $(-1, 0)$, positive on $(0, 1)$ and zero at $x = 0$; the graph is (B).

d. The function has negative slope everywhere so the graph of the derivative has to be negative everywhere, which is graph (A).

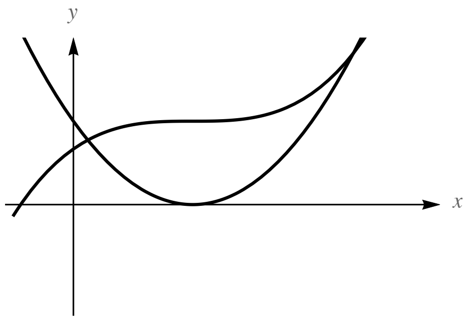
3.2.48

The function has a positive slope for $x < 1$ and a negative slope for $x > 1$, and a horizontal tangent line at $x = 1$.



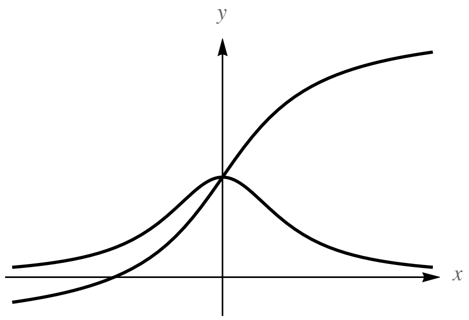
3.2.49

The function always has non-negative slope, so the derivative is never below the x axis.
However, it does have slope zero at about $x = 2$.

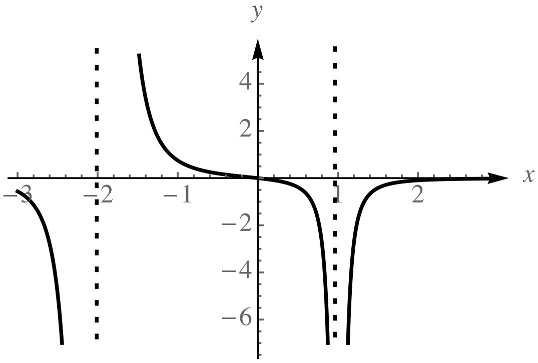


3.2.50

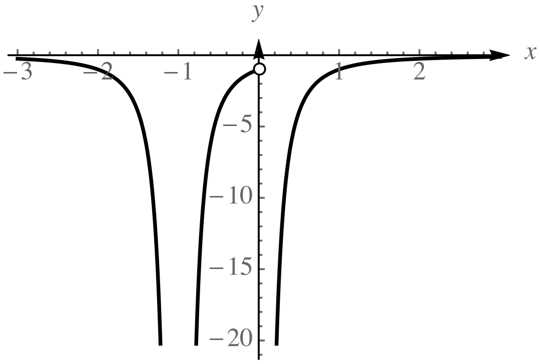
The slope increases until the function crosses the y axis at $x = 0$, and then the slope is still positive, but decreases.



3.2.51 Note that f is undefined at $x = -2$ and $x = 1$, but is differentiable elsewhere. It is decreasing, and increasingly rapidly, as x increases towards $x = -2$. It decreases, but increasingly slowly, and towards a zero slope, as x increases from 1. Finally, between $x = -2$ and $x = 1$, the function increases, but increasingly slowly, until x =0 and then decreases, but increasingly rapidly, as x approaches 1. A graph of the derivative is



3.2.52 Note that f is undefined at $x = -1$ and $x = 0$, but is differentiable elsewhere. It is decreasing everywhere it is defined. On $(-\infty, -1)$, it decreases increasingly rapidly as $x \to -1$, while on $(-1, 0)$ it decreases more and more slowly from a very large negative slope just to the right of -1 until it is almost flat near $x = 0$. Finally, on $(0, \infty)$ it decreases more and more slowly from a very large negative slope and approaches a zero slope as $x \to \infty$. A graph of the derivative is

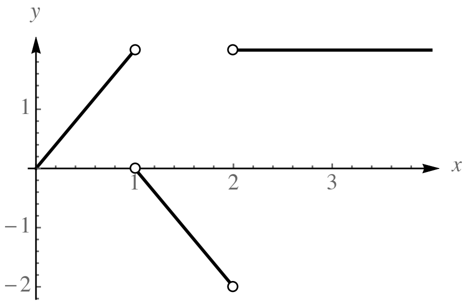


3.2.53

a. The function f is not continuous at $x = 1$, because the graph has a jump there.

b. The function f is not differentiable at $x = 1$ because it is not continuous at that point (Theorem 3.1 Alternate Version), and it is also not differentiable at $x = 2$ because the graph has a corner there.

c.

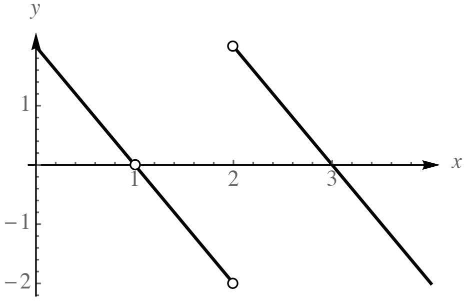


3.2.54

a. The function g is not defined at $x = 1$, because the graph has a hole there.

b. The function g is not differentiable at $x = 1$ because it is not defined at that point, and it is also not differentiable at $x = 2$ because the graph has a cusp there.

c.



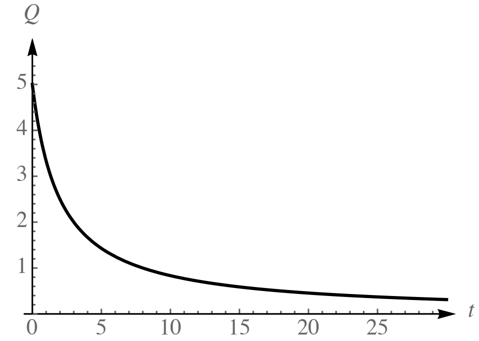
3.2.55

a. The tangent line for Q looks to be the steepest at $t = 0$.

b. All tangent lines for Q have positive slope, so Q' is positive for $t \geq 0$.

c. The tangent lines appear to be getting less steep as t increases, so Q' is decreasing.

d.



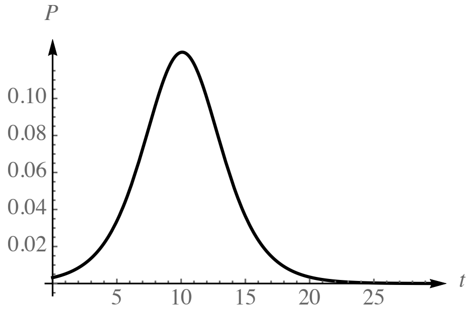
3.2.56

a. The tangent line appears to be the steepest at $t = 10$.

b. The tangent lines to P all have positive slope, so $P' > 0$ for $t \geq 0$.

c. P' appears to increase from $t = 0$ to $t = 10$, and then decrease after that.

d.



3.2.57

a. True. Differentiability implies continuity, by Theorem 3.1.

b. True. Because the absolute value function is continuous, and $y = x + 1$ is continuous, and the composition of continuous functions is continuous, we know that this function is continuous. Note that it is not differentiable at $x = -1$ because the absolute value function is not differentiable at $x = 0$.

c. False. In order for f to be differentiable on $[a, b]$, f would need to be defined at a and at b. Because the domain of f doesn't include these endpoints, this situation is not possible.

3.2.58

a. $f'(x) = 2x$

b. $f'(x) = 3x^2$

c. $f'(x) = 4x^3$

d. $f'(x) = nx^{n-1}$.

3.2.59 In order for f to be differentiable at $x = 1$, it would need to be continuous there. Thus, $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 2x^2 = 2 = \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} ax - 2 = a - 2$, so the only possible value for a is 4. Now checking the differentiability at 1, we have (from the left)

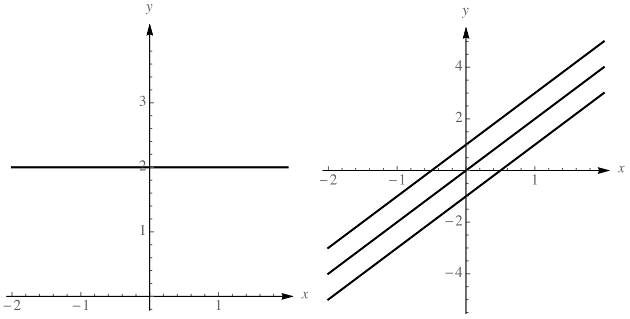
$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{2x^2 - 2}{x - 1} = \lim_{x\to 1^-} 2(x + 1) = 4.$$

Also, from the right we have

$$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{4x - 2 - 2}{x - 1} = \lim_{x\to 1^+} 4 = 4,$$

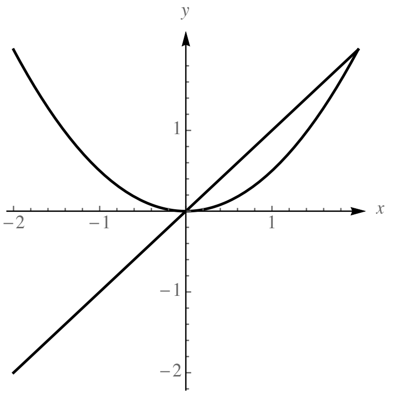
so f is differentiable at 1 for $a = 4$.

3.2.60 The graph of $f'(x) = 2$ is a horizontal line. The possible graphs of f are all lines with slope 2 (and are thus parallel).



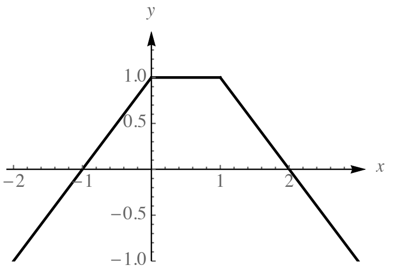
3.2.61

Because $f'(x) = x$ is negative for $x < 0$ and positive for $x > 0$, we have that the graph of f has to have negative slope on $(-\infty, 0)$ and positive slope on $(0, \infty)$ and has to have a horizontal tangent at $x = 0$. Because f' only gives us the slope of the tangent line and not the actual value of f, there are infinitely many graphs possible, they all have the same shape, but are shifted along the y−axis.



3.2.62

Because the derivative is constant on $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$, the graph of f has to consist of pieces of straight lines on these intervals. There are infinitely many possible functions f that have f' for its derivative. Because f is assumed to be continuous, each possible f is a shift, up or down, of another possible f.



3.2.63 With $f(x) = 3x - 4$, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3(x+h) - 4 - (3x-4)}{h} = \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3.$$

The slope of the tangent line at $(1, -1)$ is 3, so the slope of the normal line is $-\frac{1}{3}$. The equation of the normal line is thus $y - (-1) = -\frac{1}{3}(x-1)$, or $y = -\frac{1}{3}x - \frac{2}{3}$.

3.2.64 With $f(x) = \sqrt{x}$, we have $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$. Thus, the slope of the tangent line at $(4, 2)$ is $\frac{1}{4}$, so the slope of the normal line is -4. The equation of the normal line is $y - 2 = -4(x-4)$, or $y = -4x + 18$.

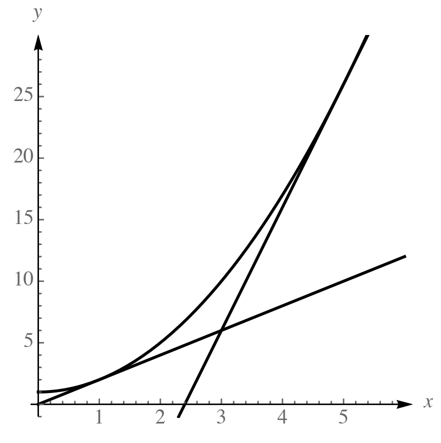
3.2.65 With $f(x) = \frac{2}{x}$, we have $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} = \lim_{h\to 0} \frac{\frac{2x-(2(x+h))}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-2h}{hx(x+h)} = \lim_{h\to 0} \frac{-2}{x(x+h)} = -\frac{2}{x^2}$. At the point $(1, 2)$ the slope of the tangent line is -2, so the slope of the normal line is $\frac{1}{2}$. The equation of the normal line is $y - 2 = \frac{1}{2}(x-1)$ or $y = \frac{x}{2} + \frac{3}{2}$.

3.2.66 With $f(x) = x^2 - 3x$, we have $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h} = \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - 3x - 3h - x^2 + 3x}{h} = \lim_{h\to 0} \frac{2hx + h^2 - 3h}{h} = \lim_{h\to 0} \frac{h(2x+h-3)}{h} == 2x - 3$. At the point $(3, 0)$ the slope of the tangent line is 3, so the slope of the normal line is $-\frac{1}{3}$. The equation of the normal line is $y - 0 = -\frac{1}{3}(x-3)$ or $y = -\frac{x}{3} + 1$.

3.2.67 With $f(x) = x^2 + 1$, we have $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + 1 - (x^2+1)}{h} = \lim_{h\to 0} \frac{x^2 + 2hx + h^2 + 1 - x^2 - 1}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = 2x$. We are looking for points (x, x^2+1) where the slope of the line between this point and $Q(3, 6)$ is equal to $2x$. So we seek solutions to the equation

$$\frac{6 - (x^2+1)}{3 - x} = 2x,$$

which can be written as $5 - x^2 = 2x(3-x)$, or $x^2 - 6x + 5 = 0$. Factoring, we obtain $(x-5)(x-1) = 0$, so the solutions are $x = 5$ and $x = 1$. Note that at the point $(5, 26)$ on the curve the tangent line is $y - 26 = 10(x-5)$ which does contain the point $Q(3, 6)$ and at the point $(1, 2)$ the equation of the tangent line is $y - 2 = 2(x-1)$, which also contains the point $Q(3, 6)$.



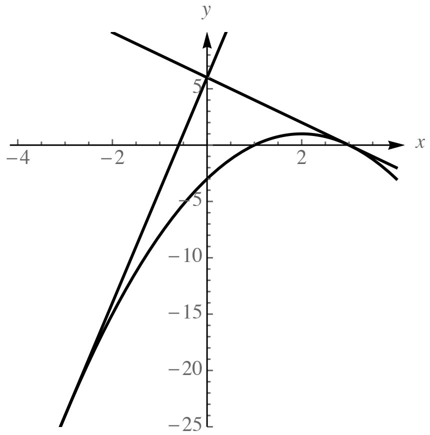
3.2.68 With $f(x) = -x^2 + 4x - 3$, we have

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{-(x+h)^2 + 4(x+h) - 3 - (-x^2+4x-3)}{h} \\ &= \lim_{h\to 0} \frac{-x^2 - 2hx - h^2 + 4x + 4h - 3 + x^2 - 4x + 3}{h} \\ &= \lim_{h\to 0} \frac{-2hx - h^2 + 4h}{h} = \lim_{h\to 0} \frac{h(-2x-h+4)}{h} \\ &= \lim_{h\to 0}(-2x - h + 4) = -2x + 4. \end{aligned}$$

We are looking for points $(x, -x^2+4x-3)$ where the slope of the line between this point and $Q(0,6)$ is equal to $-2x+4$. So we seek solutions to the equation

$$\frac{-x^2+4x-3-6}{x-0} = -2x+4,$$

which can be written as $-x^2+4x-9 = -2x^2+4x$, or $x^2-9=0$. This factors as $(x-3)(x+3)=0$, so the solutions are $x = \pm 3$. Note that at the point $(3,0)$ the equation of the tangent line is $y = -2(x-3)$ which does contain the point $Q(0,6)$, and at the point $(-3,-24)$ the equation of the tangent line is $y+24 = 10(x+3)$, which also contains the point $Q(0,6)$.



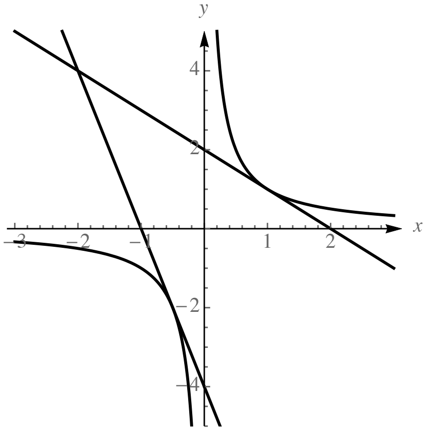
3.2.69 With $f(x) = \frac{1}{x}$, we have

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}. \end{aligned}$$

We are looking for points $(x, 1/x)$ where the slope of the line between this point and $Q(-2,4)$ is equal to $\frac{-1}{x^2}$. So we seek solutions to the equation

$$\frac{1/x - 4}{x+2} = -\frac{1}{x^2},$$

which can be written as $x - 4x^2 = -x - 2$, or $4x^2 - 2x - 2 = 0$, or $2x^2 - x - 1 = 0$. This factors as $(2x+1)(x-1) = 0$, so the solutions are $x = 1$ and $x = -1/2$. Note that at $x = 1$ the equation of the tangent line is $y - 1 = -1(x-1)$ which does contain the point $Q(-2,4)$, and at $x = -1/2$ the equation of the tangent line is $y + 2 = -4\left(x + \frac{1}{2}\right)$ or $y = -4x - 4$ which also contains the point $Q(-2,4)$.



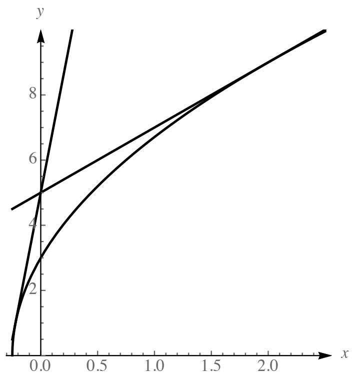
3.2.70 With $f(x) = 3\sqrt{4x+1}$, we have

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{3\sqrt{4(x+h)+1} - 3\sqrt{4x+1}}{h} \\ &= 3\lim_{h\to 0} \frac{(\sqrt{4(x+h)+1} - \sqrt{4x+1})}{h} \cdot \frac{(\sqrt{4(x+h)+1} + \sqrt{4x+1})}{(\sqrt{4(x+h)+1} + \sqrt{4x+1})} \\ &= 3\lim_{h\to 0} \frac{4(x+h)+1-(4x+1)}{h(\sqrt{4(x+h)+1} + \sqrt{4x+1})} \\ &= 12\lim_{h\to 0} \frac{1}{\sqrt{4(x+h)+1} + \sqrt{4x+1}} = \frac{6}{\sqrt{4x+1}} \end{aligned}$$

We are looking for points $(x, 3\sqrt{4x+1})$ where the slope of the line between this point and $Q(0,5)$ is equal to $\frac{6}{\sqrt{4x+1}}$. So we seek solutions to the equation

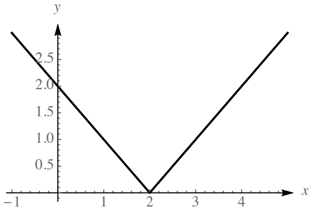
$$\frac{3\sqrt{4x+1}-5}{x} = \frac{6}{\sqrt{4x+1}}.$$

Cross multiplying gives $6x = 3(4x+1) - 5\sqrt{4x+1}$. This can be written as $5\sqrt{4x+1} = 6x+3$. Squaring both sides and gathering like terms gives $36x^2 - 64x - 16 = 0$ which can be written as $4(9x+2)(x-2) = 0$, so the solutions are $x = 2$ and $x = -\frac{2}{9}$. At $x = 2$ the tangent line is $y = 2(x-2)+9$ or $y = 2x+5$ and at $x = -\frac{2}{9}$ the tangent line is $y = 18x+5$.



3.2.71

a.

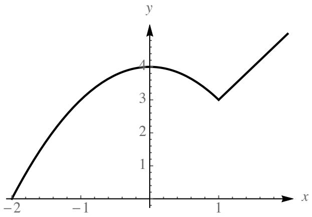


b. $f'_+(2) = \lim_{h\to 0^+} \frac{|2+h-2|-0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$, because for $h > 0$, we have $|h| = h$. Similarly, $f'_-(2) = \lim_{h\to 0^-} \frac{|2+h-2|-0}{h} = \lim_{h\to 0^-} -\frac{h}{h} = -1$, because for $h < 0$, we have $|h| = -h$.

c. Because f is defined at $a = 2$ and the graph of f does not jump, f is continuous at $a = 2$. Because the left-hand and right-hand derivatives are not equal, f is not differentiable at $a = 2$.

3.2.72

a.

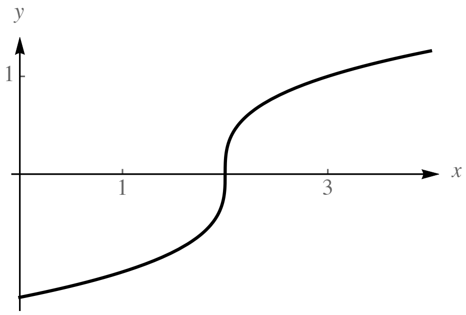


b. Assuming the point $a = 1$, we get $f'_+(1) = \lim_{h\to 0^+} \frac{2(1+h)+1-3}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2$, because for $h > 0$, we have $1 + h > 1$. Similarly, $f'_-(1) = \lim_{h\to 0^-} \frac{4-(1+h)^2-3}{h} = \lim_{h\to 0^-} \frac{-2h-h^2}{h} = -2$, because for $h < 0$, we have $1 + h < 1$.

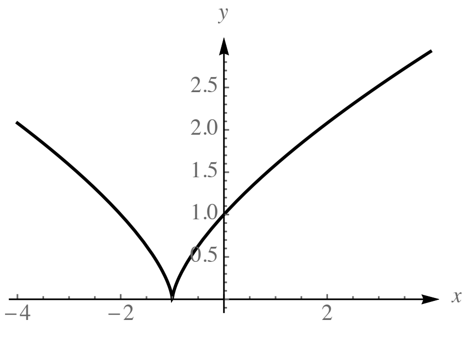
c. Because f is defined at $a = 1$ and the graph of f does not jump, f is continuous at $a = 1$. Because the left-hand and right-hand derivatives are not equal, f is not differentiable at $a = 1$.

3.2.73

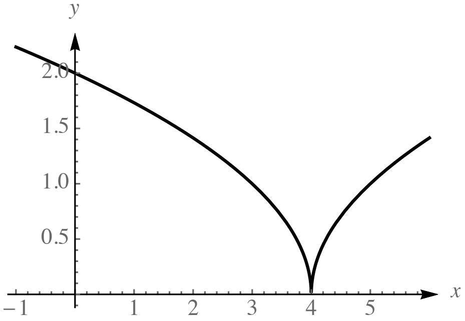
a. The graph has a vertical tangent at $x = 2$.



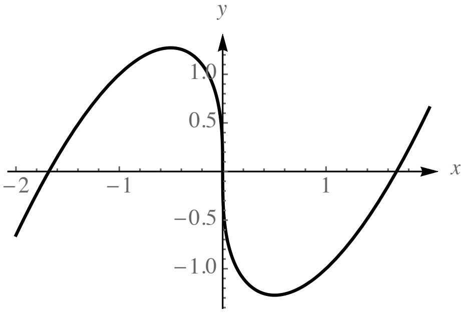
b. The graph has a vertical tangent at $x = -1$.



c. The graph has a vertical tangent at $x = 4$.

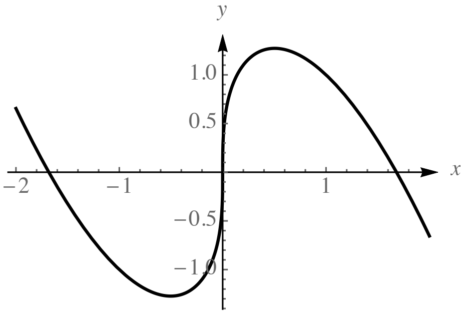


d. The graph has a vertical tangent at $x = 0$.

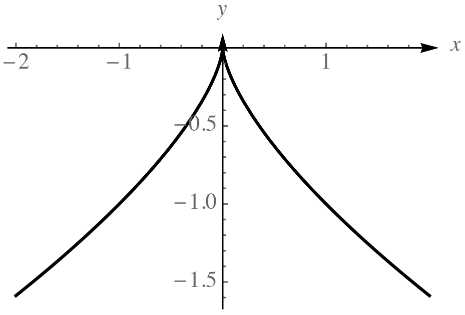


3.2.74

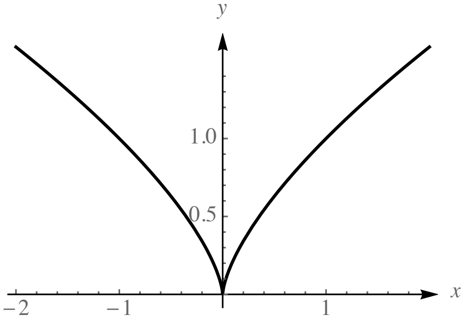
a. This graph has $\lim_{x\to a^-} f'(x) = \lim_{x\to a^+} f'(x) = +\infty$, where $a = 0$.



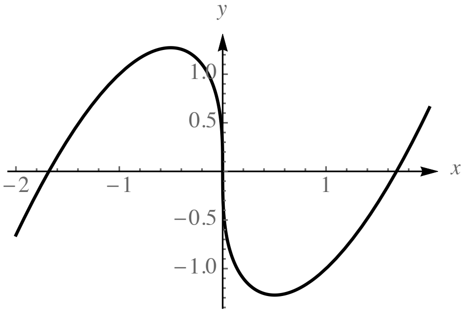
b. This graph has $\lim_{x\to a^-} f'(x) = +\infty$ and $\lim_{x\to a^+} f'(x) = -\infty$, where $a = 0$.



c. This graph has $\lim_{x\to a^-} f'(x) = -\infty$ and $\lim_{x\to a^+} f'(x) = +\infty$, where $a = 0$.



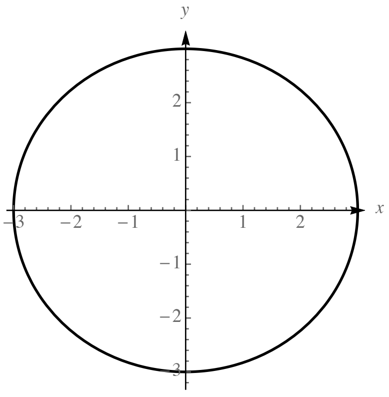
d. This graph has $\lim_{x\to a^-} f'(x) = -\infty$ and $\lim_{x\to a^+} f'(x) = -\infty$, where $a = 0$.



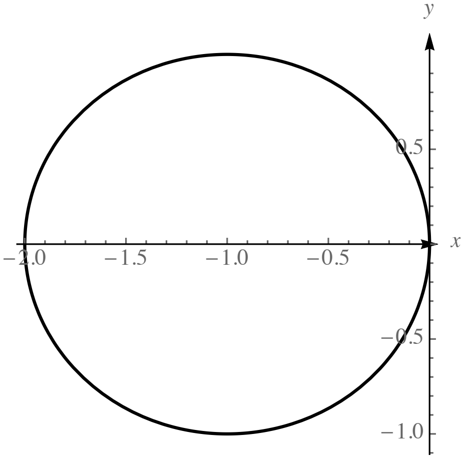
3.2.75 $f'(0) = \lim_{h\to 0} \frac{h^{1/3}}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = +\infty$. Thus the graph of f has a vertical tangent at $x = 0$.

3.2.76

a. This circle has vertical tangents at $x = 3$ and $x = -3$.

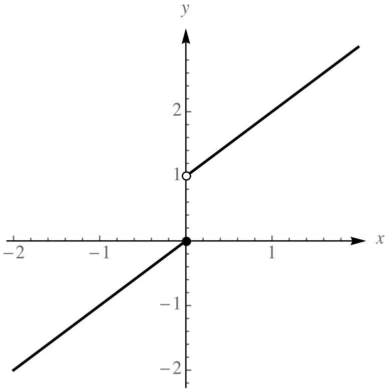


b. This circle has vertical tangents at $x = -2$ and $x = 0$.



3.2.77

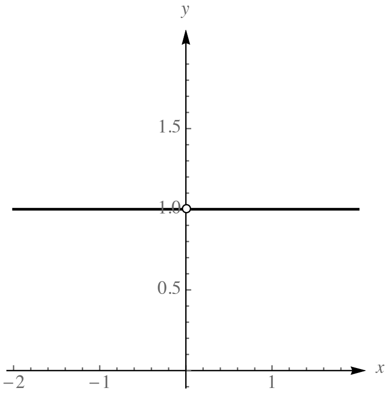
a.



b. For $x < 0$, we have $f'(x) = \lim_{h\to 0} \dfrac{x+h-x}{h} = \lim_{h\to 0} \dfrac{h}{h} = \lim_{h\to 0} 1 = 1$.

c. For $x > 0$, we have $f'(x) = \lim_{h\to 0} \dfrac{x+h+1-(x+1)}{h} = \lim_{h\to 0} \dfrac{h}{h} = \lim_{h\to 0} 1 = 1$.

d.



e. f is not differentiable at $x = 0$ as it is not continuous there. Also, if we were to compute the derivative of f from the right at 0 we would have

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h+1-0}{h} = \lim_{h\to 0^+} \frac{h+1}{h},$$

which does not exist.

3.2.78 It is not differentiable at $x = 2$. The denominator of f is zero when $x = 2$, so f is not defined at $x = 2$, and is therefore not differentiable there.

3.3 Rules of Differentiation

3.3.1 Often the limit definition of f' is difficult to compute, especially for functions which are reasonably complicated. The rules for differentiation allow us to easily compute the derivatives of complex functions.

3.3.2 It is shown to be valid for all positive integers n. In future sections we will see that it holds for all real numbers.

3.3.3 The function $f(x) = e^x$ is an example of a function with this property.

3.3.4 The sum rule tells us that the derivative of $f + g$ is $f' + g'$. That is, the derivative of the sum of two functions is the sum of the derivatives of those functions.

3.3.5 By the constant multiple rule, the derivative of the function cf where c is a constant and f is a function is cf'. That is, the derivative of a constant times a function is that same constant times the derivative of the function.

3.3.6 The 5th derivative of a function is found by differentiating the 4th derivative of the function. The 4th derivative is found by differentiating the 3rd derivative of the function, and so on. Thus, one would need to compute f' and then four more derivatives to arrive at the 5th derivative of f.

3.3.7 $(f + g)'(3) = f'(3) + g'(3) = 6 + (-2) = 4$.

3.3.8 $g'(x) = f'(x) + e^x$, so $g'(0) = f'(0) + e^0 = 6 + 1 = 7$.

3.3.9 $F'(2) = f'(2) + g'(2) = -1 + \frac{1}{2} = -\frac{1}{2}$.

3.3.10 $G'(6) = f'(6) - g'(6) = -1 - \frac{1}{2} = -\frac{3}{2}$.

3.3.11 $H'(2) = 3f'(2) + 2g'(2) = 3(-1) + 2(\frac{1}{2}) = -2$.

3.3.12 $\frac{d}{dx}\left[f(x) + g(x)\right]_{x=1} = f'(1) + g'(1) = 3 + 2 = 5$.

3.3.13 $\frac{d}{dx}\left[1.5f(x)\right]_{x=2} = 1.5f'(2) = 1.5 \cdot 5 = 7.5$.

3.3.14 $\frac{d}{dx}\left[2x - 3g(x)\right]_{x=4} = 2 - 3g'(4) = 2 - (3 \cdot 1) = -1$.

3.3.15 $f'(t) = 10t^9$, $f''(t) = 90t^8$, and $f'''(t) = 720t^7$.

3.3.16 $y(0) = e^0 = 1$ and $y'(0) = e^0 = 1$. The equation of the tangent line is $y - 1 = 1(x - 0)$, or $y = x + 1$.

3.3.17 $\frac{d}{dx}\left.4f(x))\right|_{x=5} = 4f'(5) = \frac{4}{10} = \frac{2}{5}$.

3.3.18 $(f + g)(3) = f(3) + g(3) = 10 - 14 = -4$ and $(f + g)'(3) = f'(3) + g'(3) = 4 - 5 = -1$.

3.3.19 By the power rule, $y' = 5x^{5-1} = 5x^4$.

3.3.20 By the power rule $f'(t) = 1t^{1-1} = t^0 = 1$.

3.3.21 By the constant rule, $f'(x) = 0$.

3.3.22 By the constant rule, $g'(x) = 0$. (Note that e^3 is a constant, its value does not depend on x.)

3.3.23 By the constant multiple rule and the power rule, $f'(x) = 5 \cdot \frac{d}{dx}x^3 = 5 \cdot 3x^2 = 15x^2$.

3.3.24 By the constant multiple and power rules, $g'(w) = \frac{5}{6} \cdot \frac{d}{dw}w^{12} = \frac{5}{6} \cdot 12w^{11} = 10w^{11}$.

3.3.25 By the power rule, the sum rule, the constant multiplier rule, and the constant rule, $h'(x) = \frac{2t}{2} + 0 = t$.

3.3.26 By the power rule, the constant rule, and the sum rule, $f'(v) = 100v^{99} + e^v$.

3.3.27 By the constant multiple and power rules, $p'(x) = 8 \cdot \frac{d}{dx}x = 8 \cdot 1 = 8$.

3.3.28 By the constant multiple rule and the power rule, $g'(t) = 6 \cdot \frac{d}{dt}\sqrt{t} = 6 \cdot \frac{1}{2\sqrt{t}} = \frac{3}{\sqrt{t}}$.

3.3.29 By the constant multiple and power rules, $g'(t) = 100\frac{d}{dt}t^2 = 100 \cdot 2t = 200t$.

3.3.30 By the constant multiple rule and the power rule, $f'(s) = \frac{1}{4} \cdot \frac{d}{ds}\sqrt{s} = \frac{1}{4} \cdot \frac{1}{2\sqrt{s}} = \frac{1}{8\sqrt{s}}$.

3.3.31 $f'(x) = \frac{d}{dx}(3x^4 + 7x) = \frac{d}{dx}(3x^4) + \frac{d}{dx}(7x) = 12x^3 + 7$.

3.3.32 $g'(x) = \frac{d}{dx}(6x^5 - \frac{5}{2}x^2 + x + 5) = \frac{d}{dx}(6x^5) - \frac{d}{dx}\left(\frac{5}{2}x^2\right) + \frac{d}{dx}(x) + \frac{d}{dx}(5) = 30x^4 - 5x + 1$.

3.3.33 $f'(x) = \frac{d}{dx}(10x^4 - 32x + e^2) = \frac{d}{dx}(10x^4) - \frac{d}{dx}(32x) + \frac{d}{dx}(e^2) = 40x^3 - 32 - 0 = 40x^3 - 32$.

3.3.34 $f'(t) = \frac{d}{dt}(6\sqrt{t} - 4t^3 + 9) = \frac{d}{dt}(6\sqrt{t}) - \frac{d}{dt}(4t^3) + \frac{d}{dt}(9) = \frac{6}{2\sqrt{t}} - 12t^2 + 0 = \frac{3}{\sqrt{t}} - 12t^2$.

3.3.35 $g'(w) = \frac{d}{dw}(2w^3 + 3w^2 + 10w) = 2\frac{d}{dw}(w^3) + 3\frac{d}{dw}(w^2) + 10\frac{d}{dw}(w) = 2(3w^2) + 3(2w) + 10(1) = 6w^2 + 6w + 10$.

3.3.36 $s'(t) = \frac{d}{dt}(4\sqrt{t} - \frac{1}{4}t^4 + t + 1) = \frac{d}{dt}(4\sqrt{t}) - \frac{d}{dt}\left(\frac{1}{4}t^4\right) + \frac{d}{dt}(t) + \frac{d}{dt}(1) = \frac{4}{2\sqrt{t}} - t^3 + 1 + 0 = \frac{2}{\sqrt{t}} - t^3 + 1$.

3.3.37 $f'(x) = \frac{d}{dx}(3e^x + 5x + 5) = 3\frac{d}{dx}(e^x) + 5\frac{d}{dx}(x) + \frac{d}{dx}(5) = 3e^x + 5$.

3.3.38 $g'(w) = \frac{d}{dw}(e^w - e^2 + 8) = \frac{d}{dw}(e^w) - \frac{d}{dw}(e^2) + \frac{d}{dw}(8) = e^w - 0 + 0 = e^w$.

3.3.39 $f'(x) = \begin{cases} 2x & \text{if } x < 0 \\ 4x + 1 & \text{if } x > 0. \end{cases}$

3.3.40 $g'(w) = \begin{cases} 1 + 5e^w & \text{if } w < 1 \\ 6w^2 + 4 & \text{if } w > 1. \end{cases}$

3.3.41

a. $d'(t) = 32t$ is the velocity of the stone after t seconds, measured in feet per second.

b. The stone travels $d(6) = 16 \cdot 6^2 = 576$ feet and strikes the ground with a velocity of $32 \cdot 6 = 192$ feet per second. Converting to miles per hour, we have $192 \cdot \frac{3600}{5280} \approx 130.9$ miles per hour.

3.3.42

a. The instantaneous velocity is given by $v(t) = \frac{d}{dt}s(t) = -10t + 40$, $0 \le t \le 10$.

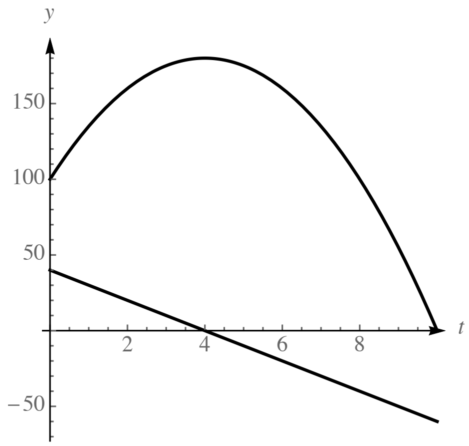
b. $v(t) = 0$ when $-10t + 40 = 0$, which occurs at $t = 4$.

c. The magnitude of the velocity is $|v(t)|$. Note that $v(t) \ge 0$ for $0 \le t \le 4$, and $v(t) < 0$ for $4 < t \le 10$. Thus

$$|v(t)| = \begin{cases} -10t + 40 & \text{for } 0 \le t \le 4, \\ 10t - 40 & \text{for } 4 < t \le 10. \end{cases}$$

Note that $|v(0)| = 40$ and $|v(10)| = 60$. The greatest magnitude over this time interval is 60 meters per second.

d.



3.3.43

a. $A'(t) = -\frac{1}{25}t + 2$ square miles per year.

b. First we must find when $A(t) = 38$. This occurs when $-\frac{t^2}{50} + 2t + 28 = 38$, which can be written as $t^2 - 100t + 900 = 0$. This factors as $(t - 90)(t - 10) = 0$, so the only solution on the given domain is $t = 10$. At this time, we have $A'(10) = -0.4 + 2 = 1.6$ square miles per year.

c. Note that $A'(20) = -0.8 + 2 = 1.2$ square miles per year. In order to maintain a density of 1000 people per square mile, we must multiply the density of people per square mile times the number of square miles per year in order to obtain the rate of people per year required to maintain that density. Thus the population growth rate must be $1000 \cdot 1.2 = 1200$ people per year.

3.3.44

a. Because $p(t) = 1200e^t$, we get $p'(t) = 1200 \cdot e^t$ cells per hour.

b. Because $p'(t)$ is also exponential, it is smallest when $t = 0$, and largest when $t = 4$.

3.3.45 $w'(x) = \begin{cases} 0.4 & \text{if } 19 < x < 21 \\ 0.8 & \text{if } 21 < x < 32 \\ 1.5 & \text{if } x > 32. \end{cases}$ The derivative measures the rate of change (in lb/inch) of Atlantic salmon. Longer salmon put on more weight per inch than shorter salmon.

3.3.46 Expanding the product gives $f(x) = x - 1$, so $f'(x) = 1$.

3.3.47 Expanding the product yields $f(x) = 6x^3 + 3x^2 + 4x + 2$. So

$$\begin{aligned} f'(x) &= \frac{d}{dx}(6x^3 + 3x^2 + 4x + 2) = \frac{d}{dx}(6x^3) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(4x) + \frac{d}{dx}(2) \\ &= 18x^2 + 6x + 4. \end{aligned}$$

3.3.48 Expanding the product yields $g(r) = 5r^5 + 18r^3 + r^2 + 9r + 3$. So

$$\begin{aligned} g'(r) &= \frac{d}{dr}(5r^5 + 18r^3 + r^2 + 9r + 3) = \frac{d}{dr}(5r^5) + \frac{d}{dr}(18r^3) + \frac{d}{dr}(r^2) + \frac{d}{dr}(9r) + \frac{d}{dr}(3) \\ &= 25r^4 + 54r^2 + 2r + 9. \end{aligned}$$

3.3.49 f simplifies as $f(w) = w^2 - 1$, so $f'(w) = 2w$ for $w \neq 0$.

3.3.50 y simplifies as $y = \dfrac{(4s)(3s^2 - 2s + 3)}{4s} = 3s^2 - 2s + 3$. Thus $y' = 6s - 2$, for $s \neq 0$.

3.3.51 Expanding the product yields $h(x) = x^4 + 2x^2 + 1$. So

$$\begin{aligned} h'(x) &= \frac{d}{dx}(x^4 + 2x^2 + 1) = \frac{d}{dx}(x^4) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(1) \\ &= 4x^3 + 4x. \end{aligned}$$

3.3.52 Expanding the product gives $h(x) = x - x^2$, so $h'(x) = 1 - 2x$.

3.3.53 g simplifies as $g(x) = \dfrac{(x-1)(x+1)}{x-1} = x + 1$. Thus $g'(x) = 1$ for $x \neq 1$.

3.3.54 h simplifies as $h(x) = \dfrac{(x)(x-2)(x-4)}{(x)(x-2)} = x - 4$. Thus, $h'(x) = 1$ for $x \neq 0, 2$.

3.3.55 y simplifies as $y = \dfrac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{\sqrt{x} - \sqrt{a}} = \sqrt{x} + \sqrt{a}$. Thus $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$ for $x \neq a$.

3.3.56 y simplifies as $y = \dfrac{(x-a)^2}{x-a} = x - a$. Thus $\dfrac{dy}{dx} = 1$ for $x \neq a$.

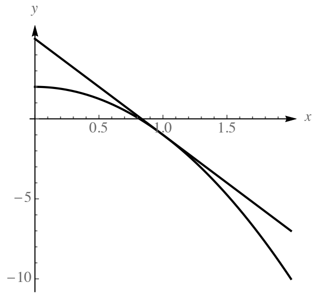
3.3.57 g simplifies as $g(w) = \dfrac{e^{2w}}{e^w} + \dfrac{e^w}{e^w} = e^w + 1$. So $g'(w) = e^w$.

3.3.58 r simplifies as $r(t) = \dfrac{(e^t + 2)(e^t + 1)}{e^t + 2} = e^t + 1$, so $r'(t) = e^t$.

3.3.59

a. $y' = -6x$, so the slope of the tangent line at $a = 1$ is -6. Thus, the tangent line at the point $(1, -1)$ is $y + 1 = -6(x - 1)$, or $y = -6x + 5$.

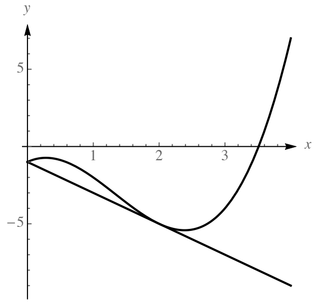
b.



3.3.60

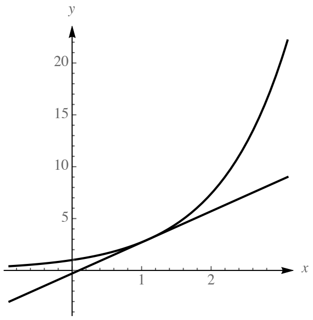
a. $y' = 3x^2 - 8x + 2$, so the slope of the tangent line at $a = 2$ is -2. Thus, the tangent line at the point $(2, -5)$ is $y + 5 = -2(x - 2)$, or $y = -2x - 1$.

b.



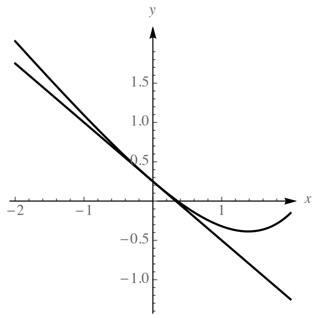
3.3.61

a. $y' = e^x$, so the slope of the tangent line at $a = \ln 3$ is 3. Thus, the tangent line at the point $(\ln 3, 3)$ is $y - 3 = 3(x - \ln 3)$, or $y = 3x + 3 - 3\ln 3$.

b. 

3.3.62

a. $y' = \frac{e^x}{4} - 1$, so the slope of the tangent line at $a = 0$ is $-\frac{3}{4}$. Thus, the tangent line at the point $(0, 1/4)$ is $y - 1/4 = -\frac{3}{4} \cdot (x - 0)$, or $y = -\frac{3x}{4} + \frac{1}{4}$.

b. 

3.3.63

a. $f'(x) = 2x - 6$, so the slope is zero when $2x - 6 = 0$, which is at $x = 3$.

b. The slope is 2 when $2x - 6 = 2$ which is at $x = 4$.

3.3.64

a. $f'(t) = 3t^2 - 27$, so the slope of the tangent line is zero when $3t^2 - 27 = 0$, which is at $t = 3$ and $t = -3$.

b. The slope is 21 where $f'(t) = 3t^2 - 27 = 21$, or when $t^2 = 16$, so at $t = 4$ and $t = -4$.

3.3.65

a. The slope of the tangent line is given by $f'(x) = 6x^2 - 6x - 12$, and this quantity is zero when $x^2 - x - 2 = 0$, or $(x - 2)(x + 1) = 0$. The two solutions are thus $x = -1$ and $x = 2$, so the points on the graph are $(-1, 11)$ and $(2, -16)$.

b. The slope of the tangent line is 60 when $6x^2 - 6x - 12 = 60$, which is when $6x^2 - 6x - 72 = 0$. Simplifying this quadratic expression yields the equation $x^2 - x - 12 = 0$, which has solutions $x = -3$ and $x = 4$, so the points on the graph are $(-3, -41)$, and $(4, 36)$.

3.3.66

a. The slope of the tangent line is given by $f'(x) = 2e^x - 6$. This is equal to zero when $2e^x = 6$, which occurs for $x = \ln 3$. The point on the graph is therefore $(\ln 3, 6 - 6\ln 3)$.

b. The slope of the tangent line is 12 when $2e^x - 6 = 12$, or $e^x = 9$. This occurs for $x = \ln 9$. The point on the graph is therefore $(\ln 9, 18 - 6\ln 9)$.

3.3.67

a. The slope of the tangent line is given by $\dfrac{2}{\sqrt{x}} - 1$. This is equal to zero when $\sqrt{x} = 2$, or $x = 4$. The point on the graph is $(4, 4)$.

b. The slope of the tangent line is $-\frac{1}{2}$ when $\dfrac{2}{\sqrt{x}} - 1 = -\dfrac{1}{2}$. Solving for x gives $x = 16$. The point on the graph is $(16, 0)$.

3.3.68 $f'(x) = 9x^2 + 10x + 6$, $f''(x) = 18x + 10$, and $f^{(3)}(x) = 18$.

3.3.69 $f'(x) = 20x^3 + 30x^2 + 3$, $f''(x) = 60x^2 + 60x$, and $f^{(3)}(x) = 120x + 60$.

3.3.70 $f'(x) = 6x + 5e^x$, $f''(x) = 6 + 5e^x$, and $f^{(3)}(x) = 5e^x$.

3.3.71 f simplifies as $f(x) = \frac{(x-8)(x+1)}{x+1} = x - 8$. So for $x \neq -1$, $f'(x) = 1$, $f''(x) = 0$, and $f^{(3)}(x) = 0$.

3.3.72 $f'(x) = f''(x) = f^{(3)}(x) = 10e^x$.

3.3.73

a. False. 10^5 is a constant, so the constant rule assures us that $\frac{d}{dx}(10^5) = 0$.

b. True. This follows because the slope is given by $f'(x) = e^x > 0$ for all x.

c. False. $\frac{d}{dx}(e^3) = 0$.

d. False. $\frac{d}{dx}(e^x) = e^x$, not xe^{x-1}.

e. False. We have $\frac{d}{dx}(5x^3 + 2x + 5) = 15x^2 + 2$. Thus we have $\frac{d^2}{dx^2}(5x^3 + 2x + 5) = 30x$, and $\frac{d^3}{dx^3}(5x^3 + 2x + 5) = 30$. It is true that $\frac{d^n}{dx^n}(5x^3 + 2x + 5) = 0$ for $n \geq 4$.

3.3.74

a. The slope of the tangent line to g at x is given by $g'(x) = 2x + f'(x)$, so $g'(3) = 6 + f'(3) = 10$. The point on the curve $y = g(x)$ at $x = 3$ is $(3, 9 + f(3)) = (3, 9 + 1) = (3, 10)$. Thus the equation of the tangent line at this point is $y - 10 = 10(x - 3)$, or $y = 10x - 20$.

b. The slope of the tangent line to h at x is given by $h'(x) = 3f'(x)$, so $h'(3) = 3f'(3) = 3 \cdot 4 = 12$. The point on the curve $y = h(x)$ at $x = 3$ is $(3, 3 \cdot f(3)) = (3, 3)$. Thus the equation of the tangent line at this point is $y - 3 = 12(x - 3)$, or $y = 12x - 33$.

3.3.75 First note that because the slope of $4x + 1$ is 4, it must be the case that $f'(2) = 4$. Also, at $x = 2$, we have $y = 4 \cdot 2 + 1 = 9$, so $f(2) = 9$. Because the line tangent to the graph of g at 2 has slope 3, we know that $g'(2) = 3$. The tangent line to g at $x = 2$ must be $y - (-2) = 3(x - 0)$, so the value of the tangent line at 2 (which must also be the value of $g(2)$) is 4. So $g(2) = 4$.

a. $y'(2) = f'(2) + g'(2) = 4 + 3 = 7$. The line contains the point $(2, f(2) + g(2)) = (2, 13)$. Thus, the equation of the tangent line is $y - 13 = 7(x - 2)$, or $y = 7x - 1$.

b. $y'(2) = f'(2) - 2g'(2) = 4 - 2 \cdot 3 = -2$. The line contains the point $(2, f(2) - 2g(2)) = (2, 1)$. Thus, the equation of the tangent line is $y - 1 = -2(x - 2)$, or $y = -2x + 5$.

c. $y'(2) = 4f'(2) = 4 \cdot 4 = 16$. The line contains the point $(2, 4f(2)) = (2, 36)$. Thus, the equation of the tangent line is $y - 36 = 16(x - 2)$, or $y = 16x + 4$.

3.3.76 For $y = x + \sqrt{x}$ we have $y' = 1 + \frac{1}{2}\left(x^{-1/2}\right) = 1 + \frac{1}{2\sqrt{x}}$. Setting this equal to 2 yields $1 + \frac{1}{2\sqrt{x}} = 2$, or $\sqrt{x} = \frac{1}{2}$. Thus the tangent line has slope 2 for $x = 1/4$.

3.3.77 For $f(x) = x^2 + bx + c$ we have $f'(x) = 2x + b$, so $f'(1) = 2 + b$. Because the slope of $4x + 2$ is 4, we require $2 + b = 4$, so $b = 2$. Also, because the value of $4x + 2$ at $x = 1$ is 6, we must have $f(1) = 1 + 2 + c = 6$, so $c = 3$. Thus the curve $f(x) = x^2 + 2x + 3$ has $y = 4x + 2$ as its tangent line at $x = 1$.

3.3.78 $F'(2) = f'(2) + g'(2) = -3 + 1 = -2$.

3.3.79 $G'(2) = 3f'(2) - g'(2) = 3(-3) - 1 = -10$.

3.3.80 $F'(5) = f'(5) + g'(5) = 1 - 1 = 0$.

3.3.81 $G'(5) = 3f'(5) - g'(5) = 3 \cdot 1 - (-1) = 4$.

3.3.82

a. Let $f(x) = e^x$ and $a = 0$. Then $\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x} = \lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = f'(0)$.

b. $f'(x) = e^x$, so $f'(0) = e^0 = 1$, so $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$.

3.3.83

a. Let $f(x) = x + e^x$ and $a = 0$. Then $\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x} = \lim\limits_{x\to 0} \dfrac{x + e^x - 1}{x} = f'(0)$.

b. $f'(x) = 1 + e^x$, so $f'(0) = 1 + e^0 = 2$, so $\lim\limits_{x\to 0} \dfrac{x + e^x - 1}{x} = 2$.

3.3.84

a. Let $f(x) = x^{100}$ and $a = 1$. Then $\lim\limits_{x\to 1} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x\to 1} \dfrac{x^{100} - 1}{x - 1} = f'(1)$.

b. Because $f'(x) = 100x^{99}$, we have $f'(1) = 100$, so this is the value of the original limit.

3.3.85

a. Let $f(x) = \sqrt{x}$ and $a = 9$. Then $\lim\limits_{h\to 0} \dfrac{f(a + h) - f(a)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{9 + h} - \sqrt{9}}{h} = f'(9)$.

b. Because $f'(x) = \frac{1}{2\sqrt{x}}$, we have $f'(9) = \frac{1}{6}$, so this is the value of the original limit.

3.3.86

a. Let $f(x) = x^8 + x^3$, and $a = 1$. Then $\lim\limits_{h\to 0} \dfrac{f(a + h) - f(a)}{h} = \lim\limits_{h\to 0} \dfrac{(1 + h)^8 + (1 + h)^3 - 2}{h} = f'(1)$.

b. Because $f'(x) = 8x^7 + 3x^2$, we have $f'(1) = 8 + 3 = 11$, so this is the value of the original limit.

3.3.87

a. Let $f(x) = e^x$, and $a = 3$. Then $\lim\limits_{h\to 0} \dfrac{f(a + h) - f(a)}{h} = \lim\limits_{h\to 0} \dfrac{e^{3+h} - e^3}{h} = f'(3)$.

b. Because $f'(x) = e^x$, we have $f'(3) = e^3$, so this is the value of the original limit.

3.3.88

h	$\frac{2^h-1}{h}$	$\frac{3^h-1}{h}$
-1.0	0.5	0.666667
-0.1	.66967	1.040415
-0.01	.69075	1.0926
-0.001	.692907	1.098009
-0.0001	.693123	1.098552
$-.00001$	.693145	1.098606

It appears that $\lim\limits_{h\to 0^-} \dfrac{2^h - 1}{h} \approx .6931$ and $\lim\limits_{h\to 0^-} \dfrac{3^h - 1}{h} \approx 1.0986$.

3.3.89 $\lim_{x\to 0}\dfrac{e^{3x}-1}{x} = 3.$

3.3.90 $\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n \approx 2.7183.$

3.3.91 $\lim_{x\to 0^+} x^x = 1.$

3.3.92 $f'(x) = \lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\dfrac{c-c}{h} = \lim_{h\to 0}\dfrac{0}{h} = \lim_{h\to 0} 0 = 0.$

3.3.93

$$\begin{aligned}\frac{d}{dx}x^n &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\left(\frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n - x^n}{h}\right)\\ &= \lim_{h\to 0}\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right) = nx^{n-1} + 0 + 0 + \cdots + 0 = nx^{n-1}\end{aligned}$$

3.3.94

a. Let $m = -n$ so that $m > 0$.

$$\begin{aligned}f'(a) &= \lim_{x\to a}\frac{x^n - a^n}{x-a} = \lim_{x\to a}\frac{x^{-m}-a^{-m}}{x-a} = \lim_{x\to a}\frac{1}{a^m x^m}\cdot\frac{a^m - x^m}{x-a}\\ &= \left(-\lim_{x\to a}\frac{1}{a^m x^m}\right)\left(\lim_{x\to a}\frac{x^m - a^m}{x-a}\right)\\ &= \left(-\lim_{x\to a}\frac{1}{a^m x^m}\right)\left(\lim_{x\to a}\frac{(x-a)(x^{m-1}+ax^{m-2}+a^2x^{m-3}+\cdots+a^{m-1})}{x-a}\right)\\ &= \left(-\lim_{x\to a}\frac{1}{a^m x^m}\right)\left(\lim_{x\to a}\left(x^{m-1}+ax^{m-2}+a^2x^{m-3}+\cdots+a^{m-1}\right)\right)\\ &= -\frac{1}{a^{2m}}\cdot ma^{m-1} = -ma^{-m-1} = na^{n-1}.\end{aligned}$$

b. $\dfrac{d}{dx}(x^{-7}) = -7x^{-8}$. Also, $\dfrac{d}{dx}(x^{-10}) = -10x^{-11} = -\dfrac{10}{x^{11}}$.

3.3.95

a. $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{d}{dx}x^{1/2} = \dfrac{1}{2}\cdot x^{-1/2} = \dfrac{1}{2\sqrt{x}}.$

b.

$$\begin{aligned}\frac{d}{dx}x^{3/2} &= \lim_{h\to 0}\frac{(x+h)^{3/2}-x^{3/2}}{h} = \lim_{h\to 0}\frac{((x+h)^{3/2}-x^{3/2})((x+h)^{3/2}+x^{3/2})}{h((x+h)^{3/2}+x^{3/2})}\\ &= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h((x+h)^{3/2}+x^{3/2})} = \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h((x+h)^{3/2}+x^{3/2})}\\ &= \lim_{h\to 0}\frac{3x^2+3xh+h^2}{((x+h)^{3/2}+x^{3/2})} = \frac{3x^2+0+0}{x^{3/2}+x^{3/2}} = \frac{3x^2}{2x^{3/2}} = \frac{3}{2}x^{1/2}.\end{aligned}$$

c.

$$\begin{aligned}\frac{d}{dx}x^{5/2} &= \lim_{h\to 0}\frac{(x+h)^{5/2}-x^{5/2}}{h} = \lim_{h\to 0}\frac{((x+h)^{5/2}-x^{5/2})((x+h)^{5/2}+x^{5/2})}{h((x+h)^{5/2}+x^{5/2})}\\ &= \lim_{h\to 0}\frac{(x+h)^5-x^5}{h((x+h)^{5/2}+x^{5/2})} = \lim_{h\to 0}\frac{x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5-x^5}{h((x+h)^{5/2}+x^{5/2})}\\ &= \lim_{h\to 0}\frac{5x^4+10x^3h+10x^2h^2+5xh^3+h^4}{((x+h)^{5/2}+x^{5/2})}\\ &= \frac{5x^4+0+0+0+0}{x^{5/2}+x^{5/2}} = \frac{5x^4}{2x^{5/2}} = \frac{5}{2}x^{3/2}.\end{aligned}$$

d. It appears that $\frac{d}{dx}x^{n/2} = \frac{n}{2}\cdot x^{(n/2)-1}$.

3.3.96

a. $\frac{d}{dx}(e^{-x}) = \lim_{h\to 0}\frac{e^{-(x+h)}-e^{-x}}{h} = \lim_{h\to 0}\frac{e^{-x}(e^{-h}-1)}{h} = e^{-x}\lim_{h\to 0}\frac{e^{-h}-1}{h}$.

b. Let $w=-h$. Then $\lim_{h\to 0}\frac{e^{-h}-1}{h} = \lim_{w\to 0}\frac{e^{w}-1}{-w} = -\lim_{w\to 0}\frac{e^{w}-1}{w} = -1$.

c. $\frac{d}{dx}(e^{-x}) = e^{-x}\lim_{h\to 0}\frac{e^{-h}-1}{h} = -e^{-x}$.

3.3.97

a. $\frac{d}{dx}(e^{2x}) = \lim_{h\to 0}\frac{e^{2(x+h)}-e^{2x}}{h} = \lim_{h\to 0}\frac{e^{2x}(e^{2h}-1)}{h} = e^{2x}\lim_{h\to 0}\frac{e^{2h}-1}{h}$.

b. Let $z=2h$. Then $\lim_{h\to 0}\frac{e^{2h}-1}{h} = \lim_{z\to 0}\frac{e^{z}-1}{z/2} = 2\lim_{z\to 0}\frac{e^{z}-1}{z} = 2$.

c. $\frac{d}{dx}(e^{2x}) = e^{2x}\lim_{h\to 0}\frac{e^{2h}-1}{h} = 2e^{2x}$.

3.3.98

a.

$$\begin{aligned}\frac{d}{dx}(x^2e^x) &= \lim_{h\to 0}\frac{(x+h)^2e^{x+h}-x^2e^x}{h}\\ &= \lim_{h\to 0} e^x\cdot\frac{(x^2+2xh+h^2)e^h-x^2}{h} = e^x\cdot\lim_{h\to 0}\frac{(x^2+2xh+h^2)e^h-x^2}{h}.\end{aligned}$$

b.

$$\begin{aligned}&e^x\cdot\lim_{h\to 0}\frac{(x^2+2xh+h^2)e^h-x^2}{h} = e^x\left(\lim_{h\to 0}\frac{x^2e^h+2xhe^h+h^2e^h-x^2}{h}\right)\\ &= e^x\left(\lim_{h\to 0}\frac{x^2(e^h-1)+2xhe^h+h^2e^h}{h}\right) = e^x\left(x^2\lim_{h\to 0}\frac{e^h-1}{h}+2x\lim_{h\to 0}e^h+\lim_{h\to 0}he^h\right)\\ &= e^x\left(x^2+2x\right).\end{aligned}$$

3.4 The Product and Quotient Rules

3.4.1 The derivative of the product fg with respect to x is given by $f'(x)g(x)+f(x)g'(x)$.

3.4.2 The derivative of the quotient $\frac{f}{g}$ with respect to x is given by $\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$.

3.4.3 $\frac{d}{dx}((x+1)(3x+2)) = \left(\frac{d}{dx}(x+1)\right)(3x+2)+(x+1)\frac{d}{dx}(3x+2) = 1(3x+2)+(x+1)3 = 6x+5$.

3.4.4 $f'(x) = 4x^3e^x+x^4e^x = e^x(4x^3+x^4) = x^3e^x(x+4)$. Thus $f'(1)=5e$.

3.4.5 $\frac{d}{dx}\left(\frac{x-1}{3x+2}\right) = \frac{(3x+2)\frac{d}{dx}(x-1)-(x-1)\frac{d}{dx}(3x+2)}{(3x+2)^2} = \frac{3x+2-3(x-1)}{(3x+2)^2} = \frac{5}{(3x+2)^2}$.

3.4.6 $g'(x) = \frac{(x+1)(2x)-x^2(1)}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$, so $g'(1)=\frac{3}{4}$.

3.4.7

a. Using the product rule: $f'(x) = 1(x-1) + x(1) = 2x - 1$.

b. By expanding first: $f(x) = x^2 - x$ so $f'(x) = 2x - 1$.

3.4.8

a. By the product rule: $g'(t) = 1(t^2 - t + 1) + (t+1)(2t-1) = t^2 - t + 1 + 2t^2 - t + 2t - 1 = 3t^2$.

b. By expanding first: $g(t) = t^3 - t^2 + t + t^2 - t + 1 = t^3 + 1$, so $g'(t) = 3t^2$.

3.4.9

a. By the product rule: $y' = (2t+7)(3t-4) + (t^2 + 7t) \cdot 3 = 9t^2 + 34t - 28$.

b. By expanding first: $y' = \frac{d}{dt}(3t^3 + 17t^2 - 28t) = 9t^2 + 34t - 28$.

3.4.10

a. By the product rule: $h'(z) = (3z^2 + 8z + 1)(z-1) + (z^3 + 4z^2 + z)(1) = 3z^3 + 8z^2 + z - 3z^2 - 8z - 1 + z^3 + 4z^2 + z = 4z^3 + 9z^2 - 6z - 1$.

b. By expanding first: $h'(z) = \dfrac{d}{dz}(z^4 + 4z^3 + z^2 - z^3 - 4z^2 - z) = \dfrac{d}{dz}(z^4 + 3z^3 - 3z^2 - z) = 4z^3 + 9z^2 - 6z - 1$.

3.4.11

a. By the quotient rule:

$f'(w) = \dfrac{w(3w^2-1) - (w^3 - w) \cdot 1}{w^2} = \dfrac{2w^3}{w^2} = 2w$ for $w \neq 0$.

b. For $w \neq 0$ this simplifies as $w^2 - 1$. $f'(w) = \dfrac{d}{dw}(w^2 - 1) = 2w$.

3.4.12

a. By the quotient rule:

$$\begin{aligned} g'(s) &= \frac{4s(12s^2 - 16s + 4) - (4s^3 - 8s^2 + 4s) \cdot 4}{(4s)^2} = \frac{48s^3 - 64s^2 + 16s - 16s^3 + 32s^2 - 16s}{16s^2} \\ &= \frac{32s^3 - 32s^2}{16s^2} = 2s - 2 \end{aligned}$$

for $s \neq 0$.

b. For $s \neq 0$, the function simplifies to $g(s) = s^2 - 2s + 1$. Then $g'(s) = \dfrac{d}{ds}(s^2 - 2s + 1) = 2s - 2$.

3.4.13

a. By the quotient rule: $y' = \dfrac{(x-a)(2x) - (x^2 - a^2)}{(x-a)^2} = \dfrac{2x^2 - 2ax - x^2 + a^2}{(x-a)^2} = \dfrac{x^2 - 2ax + a^2}{(x-a)^2} = \dfrac{(x-a)^2}{(x-a)^2} = 1$ for $x \neq a$.

b. For $x \neq a$, this simplifies as $y = \dfrac{(x+a)(x-a)}{x-a} = x + a$. $y' = \dfrac{d}{dx}(x+a) = 1$.

3.4.14

a. By the quotient rule: $y' = \dfrac{(x-a)(2x-2a) - (x^2 - 2ax + a^2) \cdot 1}{(x-a)^2} = \dfrac{x^2 - 2ax + a^2}{(x-a)^2} = \dfrac{(x-a)^2}{(x-a)^2} = 1$ for $x \neq a$.

b. For $x \neq a$, this simplifies as $y = \dfrac{(x-a)^2}{(x-a)} = x - a$. $y' = \dfrac{d}{dx}(x-a) = 1$.

3.4.15 $\dfrac{d}{dx}(f(x)g(x))\Big|_{x=1} = f'(1)g(1) + f(1)g'(1) = 4 \cdot 2 + 5 \cdot 3 = 23$.
$\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)\Bigg|_{x=1} = \dfrac{g(1)f'(1) - f(1)g'(1)}{g(1)^2} = \dfrac{2 \cdot 4 - 5 \cdot 3}{2^2} = -\dfrac{7}{4}$.

3.4.16 By the extended power rule, $\dfrac{d}{dx}\dfrac{1}{x^{10}} = \dfrac{d}{dx}x^{-10} = -10x^{-11} = -\dfrac{10}{x^{11}}$.
By the quotient rule,

$$\frac{d}{dx}\frac{1}{x^{10}} = \frac{x^{10} \cdot 0 - 1 \cdot 10x^9}{(x^{10})^2} = -\frac{10x^9}{x^{20}} = -\frac{10}{x^{11}}.$$

3.4.17 By the quotient rule, $f'(x) = \dfrac{(x+6) \cdot 1 - x \cdot 1}{(x+6)^2} = \dfrac{6}{(x+6)^2}$, so $f'(3) = \dfrac{6}{81} = \dfrac{2}{27}$ and $f'(-2) = \dfrac{3}{8}$.

3.4.18 We have $f'(x) = 0 + 1 \cdot e^x + xe^x = e^x(1+x)$, so $f'(0) = 1$.

3.4.19 $f'(x) = 12x^3(2x^2-1) + 3x^4 \cdot 4x = 24x^5 - 12x^3 + 12x^5 = 36x^5 - 12x^3$.

3.4.20 $g'(x) = 6 - (2e^x + 2xe^x)$.

3.4.21 $f'(x) = \dfrac{(x+1) \cdot 1 - x \cdot 1}{(x+1)^2} = \dfrac{1}{(x+1)^2}$.

3.4.22 $f'(x) = \dfrac{(x-2)(3x^2-8x+1) - (x^3-4x^2+x)(1)}{(x-2)^2} = \dfrac{2x^3 - 10x^2 + 16x - 2}{(x-2)^2}$.

3.4.23 $f'(t) = \dfrac{5}{3}t^{2/3}e^t + t^{5/3}e^t = e^t t^{2/3}(\dfrac{5}{3} + t)$.

3.4.24 $g'(w) = (10w+3)e^w + (5w^2+3w+1)e^w = e^w(5w^2+13w+4)$.

3.4.25 $f'(x) = \dfrac{(e^x+1)e^x - e^x(e^x)}{(e^x+1)^2} = \dfrac{e^x}{(e^x+1)^2}$.

3.4.26 $f'(x) = \dfrac{(2e^x+1)(2e^x) - (2e^x-1)2e^x}{(2e^x+1)^2} = \dfrac{2e^x(2e^x+1-(2e^x-1))}{(2e^x+1)^2} = \dfrac{4e^x}{(2e^x+1)^2}$.

3.4.27 $f'(x) = (1)e^{-x} + x(-e^{-x}) = e^{-x}(1-x)$.

3.4.28 $f'(x) = e^x \cdot \sqrt[3]{x} + e^x \cdot \dfrac{1}{3}x^{-2/3} = e^x\left(\dfrac{3x+1}{3x^{2/3}}\right)$.

3.4.29 $y' = \dfrac{d}{dt}\left(\dfrac{3t-1}{2t-2}\right) = \dfrac{(2t-2) \cdot 3 - (3t-1) \cdot 2}{(2t-2)^2} = -\dfrac{4}{(2t-2)^2} = -\dfrac{1}{(t-1)^2}$.

3.4.30 $h'(w) = \dfrac{(w^2+1)(2w) - (w^2-1)(2w)}{(w^2+1)^2} = \dfrac{2w^3 + 2w - 2w^3 + 2w}{(w^2+1)^2} = \dfrac{4w}{(w^2+1)^2}$.

3.4.31 $h'(x) = (1)(x^3+x^2+x+1) + (x-1)(3x^2+2x+1) = x^3+x^2+x+1+3x^3+2x^2+x-3x^2-2x-1 = 4x^3$.

3.4.32 $f'(x) = -\dfrac{2}{x^3} \cdot (x^2+1) + \left(1 + \dfrac{1}{x^2}\right)(2x) = -\dfrac{2}{x} - \dfrac{2}{x^3} + 2x + \dfrac{2}{x} = 2x - \dfrac{2}{x^3}$.

3.4.33 $g'(w) = e^w(w^3-1) + e^w \cdot 3w^2 = e^w(w^3+3w^2-1)$.

3.4.34 $s'(t) = \dfrac{e^t \cdot \frac{4}{3}t^{1/3} - t^{4/3}e^t}{e^{2t}} = \dfrac{t^{1/3}e^t(\frac{4}{3} - t)}{e^{2t}} = t^{1/3}\left(\dfrac{4-3t}{3e^t}\right)$.

3.4.35 $f'(t) = e^t(t^2 - 2t + 2) + e^t(2t - 2) = e^t(t^2 - 2t + 2 + 2t - 2) = t^2e^t$.

3.4.36 $f'(x) = e^x(x^3 - 3x^2 + 6x - 6) + e^x(3x^2 - 6x + 6) = e^x(x^3 - 3x^2 + 6x - 6 + 3x^2 - 6x + 6) = x^3e^x$.

3.4.37 $g'(x) = \dfrac{(x^2-1)\cdot e^x - e^x \cdot 2x}{(x^2-1)^2} = \dfrac{e^x(x^2 - 2x - 1)}{(x^2-1)^2}$.

3.4.38

$$\begin{aligned} y' &= \frac{d}{dx}\left(\frac{2\sqrt{x}-1}{4x+1}\right) = \frac{(4x+1)\left(\frac{1}{\sqrt{x}}\right) - (2\sqrt{x}-1)4}{(4x+1)^2} \\ &= \frac{4\sqrt{x} + \frac{1}{\sqrt{x}} - 8\sqrt{x} + 4}{(4x+1)^2} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{-4x + 1 + 4\sqrt{x}}{\sqrt{x}(4x+1)^2}. \end{aligned}$$

3.4.39 $f'(x) = (-9)\cdot 3 \cdot x^{-9-1} = -27x^{-10}$.

3.4.40 $y' = \dfrac{d}{dp}(4p^{-3}) = -12p^{-4}$.

3.4.41 $g'(t) = \dfrac{d}{dt}(3t^2 + 6t^{-7}) = 6t - 42t^{-8}$.

3.4.42 $y' = \dfrac{d}{dw}(w^2 + 5 + w^{-1}) = 2w - w^{-2}$.

3.4.43 $g'(t) = \dfrac{d}{dt}(1 + 3t^{-1} + t^{-2}) = -3t^{-2} - 2t^{-3}$.

3.4.44 $p'(x) = \dfrac{d}{dx}(2x^{-2} + \frac{3}{2}x^{-4} + \frac{1}{2}x^{-5}) = -4x^{-3} - 6x^{-5} - \frac{5}{2}x^{-6}$.

3.4.45 $g'(x) = \dfrac{(x-2)((x+1)e^x + e^x) - (x+1)e^x}{(x-2)^2} = \dfrac{e^x}{(x-2)^2} \cdot \dfrac{(x-2)(x+2) - (x+1)}{1} = \dfrac{e^x}{(x-2)^2} \cdot \big(x^2 - x - 5\big)$.

3.4.46 First note that $x^3 - 1 = (x-1)(x^2+x+1)$. So we can simplify $h(x)$ as $h(x) = \dfrac{2x^2-1}{x^2+x+1}$. Thus

$h'(x) = \dfrac{(x^2+x+1)(4x) - (2x^2-1)(2x+1)}{(x^2+x+1)^2} = \dfrac{(4x^3+4x^2+4x) - (4x^3+2x^2-2x-1)}{(x^2+x+1)^2} = \dfrac{2x^2+6x+1}{(x^2+x+1)^2}$.

3.4.47 $h'(x) = \dfrac{(x+1)e^x - xe^x}{(x+1)^2} = \dfrac{e^x}{(x+1)^2}$.

3.4.48 $h'(x) = \dfrac{x^2e^x \cdot 1 - (x+1)(2xe^x + x^2e^x)}{x^4e^{2x}} = \dfrac{x^2e^x - 2x^2e^x - x^3e^x - 2xe^x - x^2e^x}{x^4e^{2x}} = -\dfrac{x^2+2x+2}{x^3e^x}$.

3.4.49

$$\begin{aligned} g'(w) &= \frac{(\sqrt{w}-w)(\frac{1}{2\sqrt{w}}+1) - (\sqrt{w}+w)(\frac{1}{2\sqrt{w}}-1)}{(\sqrt{w}-w)^2} = \frac{\frac{1}{2} + \sqrt{w} - \frac{1}{2}\sqrt{w} - w - (\frac{1}{2} - \sqrt{w} + \frac{1}{2}\sqrt{w} - w)}{(\sqrt{w}-w)^2} \\ &= \frac{2\sqrt{w} - \sqrt{w}}{(\sqrt{w}-w)^2} = \frac{\sqrt{w}}{(\sqrt{w}-w)^2}. \end{aligned}$$

3.4.50 $f'(x) = \frac{d}{dx}\left(\frac{(2-x)(2+x)}{x-2}\right) = \frac{d}{dx}(-(2+x)) = \frac{d}{dx}(-2-x) = -1.$

3.4.51 $h'(w) = \frac{(w^{5/3}+1)\frac{5}{3}w^{2/3} - w^{5/3}(\frac{5}{3}w^{2/3})}{(w^{5/3}+1)^2} = \frac{\frac{5}{3}w^{2/3}}{(w^{5/3}+1)^2} = \frac{5w^{2/3}}{3(w^{5/3}+1)^2}.$

3.4.52 $g'(x) = \frac{(x^{4/3}+1)\frac{4}{3}x^{1/3} - (x^{4/3}-1)\frac{4}{3}x^{1/3}}{(x^{4/3}+1)^2} = \frac{\frac{8}{3}x^{1/3}}{(x^{4/3}+1)^2} = \frac{8x^{1/3}}{3(x^{4/3}+1)^2}.$

3.4.53 $f'(x) = \frac{d}{dx}\left(4x^2 - \frac{2x}{5x+1}\right) = 8x - \frac{(5x+1)2-(2x)(5)}{(5x+1)^2} = 8x - \frac{2}{(5x+1)^2}.$

3.4.54

$$\begin{aligned} f'(z) &= \left(\frac{z(2z)-(z^2+1)}{z^2}\right)e^z + \left(\frac{z^2+1}{z}\right)e^z = e^z\left(\left(\frac{z^2-1}{z^2}\right) + \left(\frac{z^2+1}{z}\right)\right) \\ &= e^z\left(\frac{z^3+z^2+z-1}{z^2}\right). \end{aligned}$$

3.4.55

$$\begin{aligned} h'(r) &= \frac{(r+1)(-1-\frac{1}{2\sqrt{r}}) - (2-r-\sqrt{r})\cdot 1}{(r+1)^2} = \frac{-r-\frac{\sqrt{r}}{2}-1-\frac{1}{2\sqrt{r}}-2+r+\sqrt{r}}{(r+1)^2} \\ &= \frac{\frac{\sqrt{r}}{2}-\frac{1}{2\sqrt{r}}-3}{(r+1)^2}\cdot\frac{2\sqrt{r}}{2\sqrt{r}} = \frac{r-1-6\sqrt{r}}{2\sqrt{r}(r+1)^2}. \end{aligned}$$

3.4.56

$$\begin{aligned} y' &= \frac{(\sqrt{x}-\sqrt{a})\cdot 1-(x-a)\frac{1}{2\sqrt{x}}}{(\sqrt{x}-\sqrt{a})^2} = \left(\frac{(\sqrt{x}-\sqrt{a})\cdot 1-(x-a)\frac{1}{2\sqrt{x}}}{(\sqrt{x}-\sqrt{a})^2}\right)\cdot\frac{2\sqrt{x}}{2\sqrt{x}} \\ &= \frac{2x-2\sqrt{ax}-x+a}{2\sqrt{x}(\sqrt{x}-\sqrt{a})^2} = \frac{x-2\sqrt{ax}+a}{2\sqrt{x}(\sqrt{x}-\sqrt{a})^2} = \frac{(\sqrt{x}-\sqrt{a})^2}{2\sqrt{x}(\sqrt{x}-\sqrt{a})^2} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

3.4.57 $h'(x) = (35x^6+5)(6x^3+3x^2+3) + (5x^7+5x)(18x^2+6x) = 15((7x^6+1)(2x^3+x^2+1) + (x^7+x)(6x^2+2x)) = 300x^9+135x^8+105x^6+120x^3+45x^2+15.$

3.4.58 Consider $s(t)$ as the product of $(t+1)(t+2)$ and $(t+3)$. Then by the product rule, $s'(t) = ((1)(t+2)+(t+1)(1))(t+3)+(t+1)(t+2)(1) = (t+2)(t+3)+(t+1)(t+3)+(t+1)(t+2) = t^2+5t+6+t^2+4t+3+t^2+3t+2 = 3t^2+12t+11$. Alternatively, one could multiply out $s(t)$ from the outset giving $s(t) = t^3+6t^2+11t+6$ and so $s'(t) = 3t^2+12t+11$.

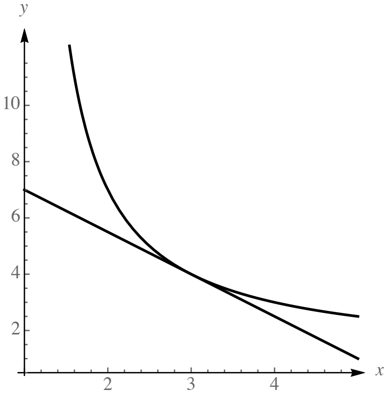
3.4.59 $f(x) = \sqrt{(e^x+4x^2)^2} = |e^x+4x^2| = e^x+4x^2$. Therefore, $f'(x) = e^x+8x$.

3.4.60 $g(x) = \frac{(e^x-1)(e^x+1)}{e^x-1} = e^x+1$. Therefore, $g'(x) = e^x$.

3.4.61

a. $y' = \dfrac{(x-1)-(x+5)}{(x-1)^2} = -\dfrac{6}{(x-1)^2}$.
At $a = 3$ we have $y' = -\dfrac{6}{4} = -\dfrac{3}{2}$ and $y = 4$, so the equation of the tangent line is $y - 4 = -\dfrac{3}{2}\cdot(x-3)$, or $y = -\dfrac{3}{2}x + \dfrac{17}{2}$.

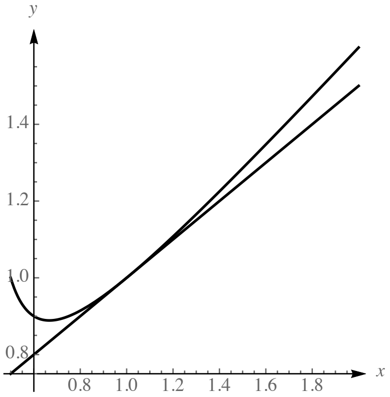
b.



3.4.62

a. $y' = \dfrac{(3x-1)4x-(2x^2)3}{(3x-1)^2} = \dfrac{6x^2-4x}{(3x-1)^2}$.
At $a = 1$ we have $y' = \dfrac{1}{2}$ and $y = 1$, so the equation of the tangent line is $y - 1 = \dfrac{1}{2}\cdot(x-1)$, or $y = \dfrac{1}{2}x + \dfrac{1}{2}$.

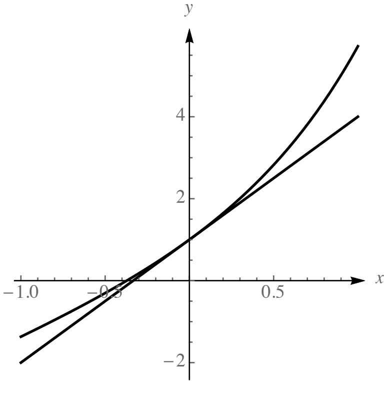
b.



3.4.63

a. $y' = 2 + (1)e^x + xe^x$.
At $a = 0$ we have $y' = 2+1+0 = 3$ and $y = 1$. So the equation of the tangent line is $y - 1 = 3(x-0)$, or $y = 3x + 1$.

b.

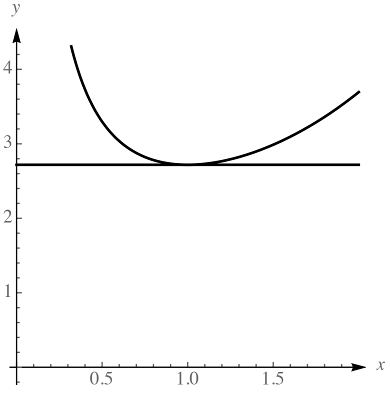


3.4.64

a. $y' = \dfrac{xe^x - e^x}{x^2}$.

At $a = 1$ we have $y' = \dfrac{e-e}{1} = 0$, and $y = e$. Thus, the equation of the tangent line is $y - e = 0$, or $y = e$.

b.



3.4.65

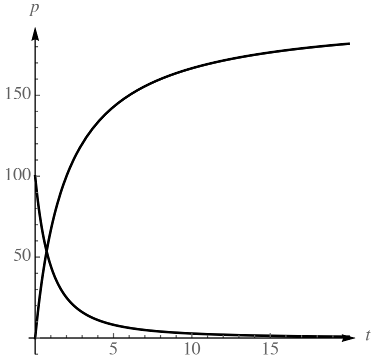
a. $p'(t) = \dfrac{(t+2)200 - 200t}{(t+2)^2} = \dfrac{400}{(t+2)^2}$.

b. $p'(5) = \dfrac{400}{49} \approx 8.16$.

c. The value of p' is as large as possible when its denominator is as small as possible, which is when $t = 0$. The value of $p'(0)$ is 100.

d. $\lim\limits_{t\to\infty} p'(t) = \lim\limits_{t\to\infty} \dfrac{400}{(t+2)^2} = 0$. This means that the population eventually has a growth rate of 0, which means that the population approaches a steady state.

e.



3.4.66

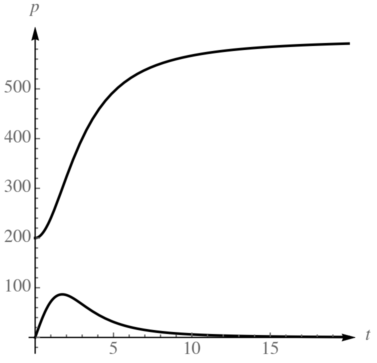
a. The instantaneous growth rate is simply the derivative of the function, so $p'(t) = 600\dfrac{(t^2+9)(2t) - 2t(t^2+3)}{(t^2+9)^2} = 600\dfrac{12t}{(t^2+9)^2}$.

b. $p'(5) = 600\left(\dfrac{60}{34^2}\right) = \dfrac{9000}{289} \approx 31.14$.

c. Consider the graph of $y = p'(t)$, shown in part (e). It appears that $p'(t)$ is at a maximum somewhere between $t = 1$ and $t = 2$; in fact, the value is $t = \sqrt{3} \approx 1.732$.

d. We have $\lim\limits_{t\to\infty} p(t) = 600 \lim\limits_{t\to\infty} \dfrac{1 + 3/t^2}{1 + 9/t^2} = 600$. This means that the population eventually approaches a steady state of 600.

e.



3.4.67

a. The instantaneous rate of change is $\dfrac{d}{dx}F(x) = -\dfrac{2kQq}{x^3}$ N/m $= -\dfrac{1.8 \times 10^{10}Qq}{x^3}$ N/m.

b. $\left[\frac{d}{dx}F(x)\right]\Big|_{x=0.001} = -\frac{2(9\times 10^9)}{(0.001)^3} = -\frac{18\times 10^9}{10^{-9}} = -18\times 10^{18} = -1.8\times 10^{19}$ Newtons per meter.

c. Because the distance x appears in the denominator of $F'(x)$, the absolute value of the instantaneous rate of change decreases with the separation.

3.4.68

a. The instantaneous rate of change if $\frac{d}{dx}F(x) = \frac{2GMm}{x^3}$ Newtons per meter.

b. $\left[\frac{d}{dx}F(x)\right]\Big|_{x=0.01} = \frac{13.4\times 10^{-11}\cdot(0.1)^2}{(0.01)^3} = 1.34\times 10^{-6}$ Newtons per meter.

c. Because the distance x appears in the denominator of $F'(x)$, the instantaneous rate of change decreases with the separation.

3.4.69

a. False. In fact, because e^5 is a constant, its derivative is zero.

b. False. It is certainly a reasonable way to proceed, but one could also write the given quantity as $x+3+2x^{-1}$, and then proceed using the sum rule and the power rule and the extended power rule.

c. False. $\frac{d}{dx}\left(\frac{1}{x^5}\right) = \frac{d}{dx}\left(x^{-5}\right) = -5x^{-6} = -\frac{5}{x^6}$.

d. False. The derivative is $3x^2e^x + x^3e^x = x^2e^x(3+x)$.

3.4.70 $f'(x) = -x^{-2} = -\frac{1}{x^2}$. $f''(x) = 2x^{-3} = \frac{2}{x^3}$. $f'''(x) = -6x^{-4} = -\frac{6}{x^4}$.

3.4.71 $f'(x) = 2x(2+x^{-3}) + x^2(-3x^{-4}) = 4x + 2x^{-2} - 3x^{-2} = 4x - x^{-2} = 4x - \frac{1}{x^2}$.

$f''(x) = 4 + 2x^{-3} = 2(2 + \frac{1}{x^3})$.

$f'''(x) = -6x^{-4} = -\frac{6}{x^4}$.

3.4.72

$$f'(x) = \frac{d}{dx}\left(\frac{x}{x+2}\right) = \frac{(x+2)-x}{(x+2)^2} = \frac{2}{(x+2)^2} = \frac{2}{x^2+4x+4}.$$

$$\begin{aligned} f''(x) = \frac{d}{dx}\left(\frac{2}{x^2+4x+4}\right) &= \frac{(x^2+4x+4)\cdot 0 - 2(2x+4)}{(x+2)^4} \\ &= \frac{-4(x+2)}{(x+2)^4} = -\frac{4}{(x+2)^3} = -\frac{4}{x^3+6x^2+12x+8}. \end{aligned}$$

3.4.73

$$f'(x) = \frac{d}{dx}\left(\frac{x^2-7x}{x+1}\right) = \frac{(x+1)(2x-7)-(x^2-7x)\cdot 1}{(x+1)^2} = \frac{x^2+2x-7}{x^2+2x+1} = \frac{x^2+2x-7}{(x+1)^2}.$$

$$\begin{aligned} f''(x) = \frac{d}{dx}\left(\frac{x^2+2x-7}{x^2+2x+1}\right) &= \frac{(x^2+2x+1)(2x+2)-(x^2+2x-7)(2x+2)}{(x+1)^4} \\ &= \frac{(2x^2+4x+2)-(2x^2+4x-14)}{(x+1)^3} = \frac{16}{(x+1)^3}. \end{aligned}$$

3.4.74

a. $g'(x) = 2xf(x) + x^2f'(x)$, so $g'(2) = 2 \cdot 2 \cdot f(2) + 4 \cdot f'(2) = 8 + 12 = 20$. Thus, the tangent line is given by $y - 8 = 20(x - 2)$, or $y = 20x - 32$.

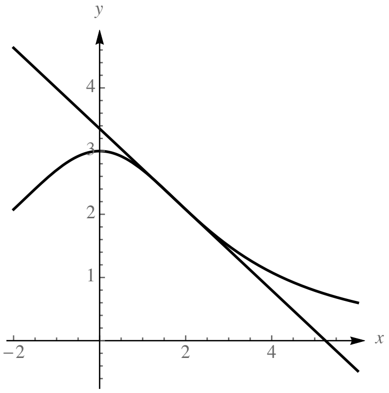
b. $h'(x) = \dfrac{(x-3)f'(x) - f(x)}{(x-3)^2}$, so $h'(2) = \dfrac{(-1) \cdot 3 - 2}{(-1)^2} = -5$. Also, $h(2) = -2$. Thus, the tangent line is given by $y + 2 = (-5)(x - 2)$, or $y = -5x + 8$.

3.4.75

a. $y' = -\dfrac{54x}{(x^2+9)^2}$. At $x = 2$, $y' = -\dfrac{108}{169}$ and $y = \dfrac{27}{13}$. Thus the tangent line is given by

$$y - \frac{27}{13} = -\frac{108}{169}(x - 2),$$

or $y = -\dfrac{108}{169}x + \dfrac{567}{169}$.

b. 

3.4.76 $\dfrac{d}{dx}\left[f(x)g(x)\right]\Big|_{x=1} = f'(1)g(1) + f(1)g'(1) = 3 \cdot 4 + 5 \cdot 2 = 22.$

3.4.77 $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]\Bigg|_{x=2} = \dfrac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2} = \dfrac{2 \cdot 5 - 4 \cdot 4}{4} = -\dfrac{3}{2}.$

3.4.78 $\dfrac{d}{dx}\left[xf(x)\right]\Big|_{x=3} = f(3) + 3 \cdot f'(3) = 3 + 3 \cdot 2 = 9.$

3.4.79 $\dfrac{d}{dx}\left[\dfrac{f(x)}{x+2}\right]\Bigg|_{x=4} = \dfrac{(4+2)f'(4) - f(4)}{36} = \dfrac{6-2}{36} = \dfrac{1}{9}.$

3.4.80 $\dfrac{d}{dx}\left[\dfrac{xf(x)}{g(x)}\right]\Bigg|_{x=4} = \dfrac{g(4)(f(4) + 4f'(4)) - 4f(4)g'(4)}{(g(4))^2} = \dfrac{3(2 + 4 \cdot 1) - 4 \cdot 2 \cdot 1}{9} = \dfrac{10}{9}.$

3.4.81 $\dfrac{d}{dx}\left[\dfrac{f(x)g(x)}{x}\right]\Bigg|_{x=4} = \dfrac{4(f'(4)g(4) + f(4)g'(4)) - f(4)g(4)}{16} = \dfrac{4(1 \cdot 3 + 2 \cdot 1) - (2 \cdot 3)}{16} = \dfrac{14}{16} = \dfrac{7}{8}.$

3.4.82

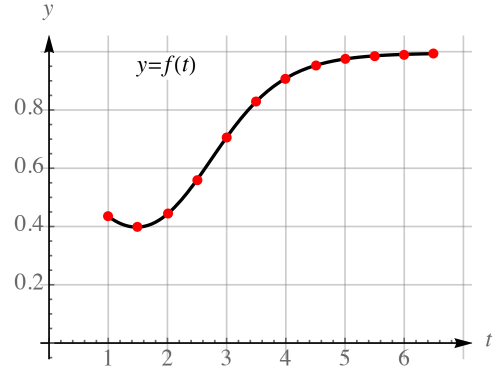
a. The derivative $m'(t)$ has units of g/week and measures the rate at which the owlets are increasing in mass.

b. The derivative $w'(t)$ has units of mm/week and measures the rate at which the wing chord is increasing.

c. The derivative $f'(t)$ has units of mm/g per week, which measures the rate at which the ratio of wing chord length to mass is changing per week.

3.4.83

a.



b. For $t \approx 3$.

c. $f'(t) = \dfrac{115.48(41.38) - 12.48(81.55)}{115.48^2} \approx 0.28\ mm/g$ per week. At a young age, the bird's wings are growing quickly relative to its age.

d. It appears that $f'(t) \approx 0$ for $t \geq 6.5$. $f'(6.5) = \dfrac{121.45(0.38) - 0.01(120.61)}{121.45^2} \approx 0.003\ mm/g$ per week. As the bird matures the bird's growth rate slows down, so the ratio of wing chord length to mass doesn't change much, which implies that the rate of change of wing chord length to mass, f', is almost 0.

3.4.84 $p'(x) = f'(x)g(x) + f(x)g'(x)$, so $p'(3) = f'(3)g(3) + f(3)g'(3) = 5(2) + 2(-10) = -10$.

3.4.85 $q'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$, so $q'(3) = \dfrac{f'(3)g(3) - f(3)g'(3)}{g(3)^2} = \dfrac{5(2) - 2(-10)}{4} = \dfrac{30}{4} = \dfrac{15}{2}$.

3.4.86 $y' = 1 \cdot f(x) + xf'(x)$, so at $x = 1$ we have $y'(1) = f(1) + f'(1) = 2 + 2 = 4$.

3.4.87 $\dfrac{d}{dx}(f(x)g(x))\Big|_{x=4} = f'(4)g(4) + f(4)g'(4) = \dfrac{1}{2}(1) + 3(-1) = -\dfrac{5}{2}$.

3.4.88 $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)\Bigg|_{x=4} = \dfrac{g(4)f'(4) - f(4)g'(4)}{g(4)^2} = \dfrac{1\left(\dfrac{1}{2}\right) - 3(-1)}{1^2} = \dfrac{7}{2}$.

3.4.89 $\dfrac{d}{dx}(xg(x))\Big|_{x=2} = g(2) + 2g'(2) = 3 + 2(-1) = 1$.

3.4.90 $\dfrac{d}{dx}\left(\dfrac{x^2}{f(x)}\right)\Bigg|_{x=2} = \dfrac{f(2)(4) - 4f'(2)}{f(2)^2} = \dfrac{2(4) - 4(1/2)}{4} = \dfrac{3}{2}$.

3.4.91

a. $y' = (2x - 3)h(x) + (x^2 - 3x)h'(x)$ so $y'(4) = 5(2) + 4(-3) = -2$. Also, $y(4) = 4(2) = 8$. So the equation is $y - 8 = -2(x - 4)$ or $y = -2x + 16$.

b. $y' = \dfrac{(x+2)h'(x) - h(x)}{(x+2)^2}$, so $y'(4) = \dfrac{6(-3) - 2}{36} = -\dfrac{20}{36} = -\dfrac{5}{9}$. Also, $y(4) = \dfrac{1}{3}$. So the equation of the tangent line is $y - \dfrac{1}{3} = -\dfrac{5}{9}(x - 4)$ or $y = -\dfrac{5}{9}x + \dfrac{23}{9}$.

3.4.92

a. Because the slope of f at 2 is $f'(2) = 4$ and the slope of g at 2 is $g'(2) = 3$ and $f(2) = 4 \cdot 2 + 1 = 9$, $g(2) = 3 \cdot 2 - 2 = 4$ we have $y'(2) = f'(2)g(2) + f(2)g'(2) = 4 \cdot 4 + 9 \cdot 3 = 43$. Thus, the tangent line at this point is $y - 36 = 43(x - 2)$, or $y = 43x - 50$.

b. $y'(2) = \dfrac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2} = \dfrac{4 \cdot 4 - 9 \cdot 3}{16} = -\dfrac{11}{16}$. So the equation of the tangent line at this point is $y - \dfrac{9}{4} = -\dfrac{11}{16}(x - 2)$, or $y = -\dfrac{11}{16}x + \dfrac{29}{8}$.

3.4.93 Following the hint, we note that $f(0) = 100$ and we have $f'(x) = 40x^7 + 30x^4 + 20x^3 + 6x + 20$, so $f'(0) = 20$. Also note that $g(0) = 2$ and $g'(x) = 100x^9 + 72x^8 + 30x^4 + 12x + 4$, so $g'(0) = 4$. Then

$$q'(0) = \frac{g(0)f'(0) - f(0)g'(0)}{g(0)^2} = \frac{2(20) - 100(4)}{4} = -90.$$

3.4.94 Following the hint in the previous problem, let $p(x) = f(x)g(x)$. Then $f'(x) = 5e^x + 50x^4 + 60x^2 + 200x + 5$ and $g'(x) = 50x^4 + 120x^2 + 40x + 4$. We have $f(0) = 25$ and $f'(0) = 10$ and $g(0) = 10$ and $g'(0) = 4$. Then

$$p'(0) = f(0)g'(0) + f'(0)g(0) = 25(4) + 10(10) = 200.$$

3.4.95

a. The tangent line at $x = a$ is $y - a^2 = 2a(x - a)$ and at $x = b$ is $y - b^2 = 2b(x - b)$.

These intersect when $a^2 + 2ax - 2a^2 = b^2 + 2bx - 2b^2$, or $(2a - 2b)x = a^2 - b^2$, which is met when $x = \dfrac{a+b}{2}$. So $c = \dfrac{a+b}{2}$.

b. The tangent line at $x = a$ is $y - \sqrt{a} = \dfrac{1}{2\sqrt{a}}(x - a)$ and at $x = b$ is $y - \sqrt{b} = \dfrac{1}{2\sqrt{b}}(x - b)$.

These intersect when $\sqrt{a} + \dfrac{1}{2\sqrt{a}}(x - a) = \sqrt{b} + \dfrac{1}{2\sqrt{b}}(x - b)$, or $\left(\dfrac{1}{2\sqrt{a}} - \dfrac{1}{2\sqrt{b}}\right) x = \dfrac{\sqrt{b} - \sqrt{a}}{2}$, which is met when $x = \sqrt{ab}$. So $c = \sqrt{ab}$.

c. The tangent line at $x = a$ is $y - \dfrac{1}{a} = -\dfrac{1}{a^2}(x - a)$ and at $x = b$ is $y - \dfrac{1}{b} = -\dfrac{1}{b^2}(x - b)$.

These intersect when $\dfrac{1}{a} - \dfrac{1}{a^2}(x - a) = \dfrac{1}{b} - \dfrac{1}{b^2}(x - b)$, or $\left(\dfrac{2}{a} - \dfrac{x}{a^2}\right) = \left(\dfrac{2}{b} - \dfrac{x}{b^2}\right)$, which is met when $x\left(\dfrac{1}{b^2} - \dfrac{1}{a^2}\right) = \dfrac{2}{b} - \dfrac{2}{a}$, or $x \cdot \left(\dfrac{a^2 - b^2}{a^2b^2}\right) = \dfrac{2(a-b)}{ab}$. Thus we arrive at $x = \dfrac{2ab}{a+b}$. So $c = \dfrac{2ab}{a+b}$.

d. The tangent line at $x = a$ is $y - f(a) = f'(a)(x - a)$ and at $x = b$ is $y - f(b) = f'(b)(x - b)$.

These intersect when $f(a) + f'(a)(x - a) = f(b) + f'(b)(x - b)$, or $(f'(a) - f'(b))x = f(b) - f(a) - f'(b)b + f'(a)a$. Solving for x yields $x = \dfrac{f(b) - f(a) - f'(b)b + f'(a)a}{f'(a) - f'(b)}$ provided $f'(a) \neq f'(b)$ (which occurs when the tangent lines are parallel and don't intersect.)

3.4.96

a.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \lim_{h \to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h \cdot g(x+h)g(x)}.$$

b.

$$\begin{aligned}&\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{h\cdot g(x+h)g(x)}\\&=\lim_{h\to 0}\frac{g(x)\left(\frac{f(x+h)-f(x)}{h}\right)-f(x)\left(\frac{g(x+h)-g(x)}{h}\right)}{g(x+h)g(x)}=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}.\end{aligned}$$

c. F' exists provided that f and g are differentiable, and $g(x)\neq 0$. Note that we used the fact that $\lim_{h\to 0} g(x+h)=g(x)$, which is true because g is continuous (because it is differentiable).

3.4.97

$$\begin{aligned}\frac{d^2}{dx^2}(f(x)g(x))&=\frac{d}{dx}(f'(x)g(x)+f(x)g'(x))=f''(x)g(x)+f'(x)g'(x)+f'(x)g'(x)+f(x)g''(x)\\&=f''(x)g(x)+2f'(x)g'(x)+f(x)g''(x).\end{aligned}$$

3.4.98

a. $(fg)^{(2)}=(f'g+fg')'=f''g+f'g'+f'g'+fg''=f''g+2f'g'+fg''$.

b. We proceed by induction. For $n=1$, we have that

$$(fg)'=\sum_{k=0}^{1}f^{(k)}g^{(1-k)}=f'g+fg'.$$

Now suppose that the rule holds for $n=m$. We will show that the rule holds for $n=m+1$.

$$\begin{aligned}(fg)^{(m+1)}&=\left((fg)^{(m)}\right)'=\sum_{k=0}^{n}\binom{n}{k}(f^{(k)}g^{(n-k)})'=\sum_{k=0}^{n}\binom{n}{k}\left(f^{(k+1)}g^{(n-k)}+f^{(k)}g^{(n+1-k)}\right)\\&=\sum_{k=0}^{n}\binom{n}{k}\left(f^{(k+1)}g^{(n+1-(k+1))}+f^{(k)}g^{(n+1-k)}\right)\\&=\sum_{k=1}^{n+1}\binom{n}{k-1}f^{(k)}g^{(n+1-k)}+\sum_{k=0}^{n}\binom{n}{k}f^{(k)}g^{(n+1-k)}=\sum_{k=0}^{n+1}\binom{n+1}{k}f^{(k)}g^{(n+1-k)},\end{aligned}$$

because $\binom{n}{0}=\binom{n}{n+1}=1$ and $\binom{n}{k-1}+\binom{n}{k}=\binom{n+1}{k}$.

c. $(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}$ follows a similar pattern.

3.4.99

a.

$$\begin{aligned}\frac{d}{dx}[(f(x)g(x))h(x)]&=\frac{d}{dx}[f(x)g(x)]\cdot h(x)+f(x)g(x)\cdot\frac{d}{dx}h(x)\\&=[f'(x)g(x)+f(x)g'(x)]h(x)+f(x)g(x)h'(x)\\&=f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x).\end{aligned}$$

b. $\frac{d}{dx}[e^x(x-1)(x+3)]=e^x(x-1)(x+3)+e^x(x+3)+e^x(x-1)=e^x(x^2+2x-3+x+3+x-1)=e^x(x^2+4x-1)$.

3.5 Derivatives of Trigonometric Functions

3.5.1 A direct substitution would yield the quotient of zero with itself, which isn't defined

3.5.2 It is an important ingredient in the derivation of the formula $\frac{d}{dx}\sin x = \cos x$.

3.5.3 Because $\tan x = \frac{\sin x}{\cos x}$, and $\cot x = \frac{\cos x}{\sin x}$, we can use the quotient rule to compute these derivatives, because we know the derivatives of $\sin x$ and of $\cos x$.

3.5.4 Remember the rule that the derivative of a "co" function can be obtained from the derivative of a function by changing all of the functions in the formula to their cofunctions, and introducing a factor of negative one. Thus, for example, because $\frac{d}{dx}\tan x = \sec^2 x$, we would have $\frac{d}{dx}\cot x = -\csc^2 x$.

3.5.5 $f'(x) = \cos x$ and $f'(\pi) = \cos\pi = -1$.

3.5.6 $\left.\frac{d}{dx}(\tan x)\right|_{x=\pi/3} = \sec^2(\pi/3) = 4$.

3.5.7 $y' = \cos x$, so $y'(0) = \cos 0 = 1$. So the equation of the tangent line is $y - 0 = 1(x - 0)$ or $y = x$.

3.5.8 Because $\frac{d}{dx}\sin x = \cos x$, the graph of $\sin x$ will have a horizontal tangent line where the cosine function is 0. This happens at all real numbers of the form $x = \frac{2n+1}{2}\cdot\pi$ where n is an integer.

3.5.9 $\frac{d}{dx}(\sin x + \cos x) = \cos x - \sin x$, so $\frac{d^2}{dx^2}(\sin x + \cos x) = \frac{d}{dx}(\cos x - \sin x) = -\sin x - \cos x = -(\sin x + \cos x)$.

3.5.10 $\frac{d^2}{dx^2}(\sec x) = \frac{d}{dx}(\sec x\tan x) = \sec x\tan x\tan x + \sec x\sec^2 x = \sec x(\tan^2 x + \sec^2 x)$.

3.5.11 $\lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{3\sin 3x}{3x} = 3\lim_{x\to 0}\frac{\sin 3x}{3x} = 3\cdot 1 = 3$.

3.5.12 $\lim_{x\to 0}\frac{\sin 5x}{3x} = \frac{1}{3}\lim_{x\to 0}\frac{5\sin 5x}{5x} = \frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x} = \frac{5}{3}\cdot 1 = \frac{5}{3}$.

3.5.13 $\lim_{x\to 0}\frac{\sin 7x}{\sin 3x} = \lim_{x\to 0}\frac{\frac{7\sin 7x}{7x}}{\frac{3\sin 3x}{3x}} = \frac{7}{3}\cdot\lim_{x\to 0}\frac{\frac{\sin 7x}{7x}}{\frac{\sin 3x}{3x}} = \frac{7}{3}\cdot\frac{1}{1} = \frac{7}{3}$.

3.5.14 $\lim_{x\to 0}\frac{\sin 3x}{\tan 4x} = \lim_{x\to 0}\frac{\sin 3x\cos 4x}{\sin 4x} = \lim_{x\to 0}\frac{\frac{3\sin 3x}{3x}\cdot\cos 4x}{\frac{4\sin 4x}{4x}} = \frac{3\cdot 1\cdot 1}{4\cdot 1} = \frac{3}{4}$.

3.5.15 $\lim_{x\to 0}\frac{\tan 5x}{x} = \lim_{x\to 0}\frac{5\sin 5x}{5x\cos 5x} = 5\lim_{x\to 0}\frac{\sin 5x}{5x}\cdot\lim_{x\to 0}\frac{1}{\cos 5x} = 5\cdot 1\cdot 1 = 5$.

3.5.16 $\lim_{\theta\to 0}\frac{\cos^2\theta - 1}{\theta} = \left(\lim_{\theta\to 0}(\cos\theta + 1)\right)\cdot\left(\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta}\right) = 2\cdot 0 = 0$.

3.5.17 $\lim_{x\to 0}\frac{\tan 7x}{\sin x} = \lim_{x\to 0}\frac{\sin 7x}{\cos 7x\cdot\sin x} = \lim_{x\to 0}\left(\frac{1}{\cos 7x}\cdot\frac{x}{\sin x}\cdot\frac{7\sin 7x}{7x}\right) =$
$7\cdot\lim_{x\to 0}\frac{1}{\cos 7x}\cdot\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}\frac{\sin 7x}{7x} = 7\cdot 1\cdot 1\cdot 1 = 7$.

3.5.18 $\lim\limits_{\theta\to 0}\dfrac{\sec\theta-1}{\theta}=\lim\limits_{\theta\to 0}\dfrac{\frac{1}{\cos\theta}-1}{\theta}=\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta\cos\theta}=\lim\limits_{\theta\to 0}\dfrac{1}{\cos\theta}\cdot\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta}=1\cdot 0=0.$

3.5.19 $\lim\limits_{x\to 2}\dfrac{\sin(x-2)}{x^2-4}=\lim\limits_{x\to 2}\left(\dfrac{1}{x+2}\cdot\dfrac{\sin(x-2)}{x-2}\right)=\lim\limits_{x\to 2}\dfrac{1}{x+2}\cdot\lim\limits_{x\to 2}\dfrac{\sin(x-2)}{x-2}=\dfrac{1}{4}\cdot 1=\dfrac{1}{4}.$

3.5.20 $\lim\limits_{x\to -3}\dfrac{\sin(x+3)}{x^2+8x+15}=\lim\limits_{x\to -3}\dfrac{\sin(x+3)}{(x+5)(x+3)}=\lim\limits_{x\to -3}\dfrac{1}{x+5}\cdot\lim\limits_{x\to -3}\dfrac{\sin(x+3)}{(x+3)}=\dfrac{1}{2}\cdot 1=\dfrac{1}{2}.$

3.5.21 $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}=\lim\limits_{x\to 0}\dfrac{a\sin ax}{ax}\cdot\dfrac{bx}{b\sin bx}=\dfrac{a}{b}\lim\limits_{x\to 0}\dfrac{\sin ax}{ax}\cdot\lim\limits_{x\to 0}\dfrac{bx}{\sin bx}=\dfrac{a}{b}\cdot 1\cdot 1=\dfrac{a}{b}.$

3.5.22 $\lim\limits_{x\to 0}\dfrac{\sin ax}{bx}=\dfrac{a}{b}\lim\limits_{x\to 0}\dfrac{\sin ax}{ax}=\dfrac{a}{b}\cdot 1=\dfrac{a}{b}.$

3.5.23 $\dfrac{dy}{dx}=\cos x-\sin x.$

3.5.24 $\dfrac{dy}{dx}=10x-\sin x.$

3.5.25 $\dfrac{dy}{dx}=-e^{-x}\sin x+e^{-x}\cos x=e^{-x}(\cos x-\sin x).$

3.5.26 $\dfrac{dy}{dx}=\cos x+4e^x.$

3.5.27 $\dfrac{dy}{dx}=\sin x+x\cos x.$

3.5.28 $\dfrac{dy}{dx}=e^x(-\sin x+\cos x)+e^x(\cos x+\sin x)=2e^x\cos x.$

3.5.29

$$\begin{aligned}\frac{dy}{dx}&=\frac{(\sin x+1)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}=\frac{-1(\sin^2 x+\cos^2 x)-\sin x}{(1+\sin x)^2}\\&=\frac{-1(1+\sin x)}{(1+\sin x)^2}=-\frac{1}{1+\sin x}.\end{aligned}$$

3.5.30 $\dfrac{dy}{dx}=\dfrac{(1+\sin x)(-\cos x)-(1-\sin x)(\cos x)}{(1+\sin x)^2}=\dfrac{-2\cos x}{(1+\sin x)^2}.$

3.5.31 $\dfrac{dy}{dx}=\cos x\cos x+\sin x\cdot(-\sin x)=\cos^2 x-\sin^2 x=\cos(2x).$

3.5.32

$$\begin{aligned}&\frac{d}{dx}\left(\frac{a\sin x+b\cos x}{a\sin x-b\cos x}\right)=\frac{(a\sin x-b\cos x)(a\cos x-b\sin x)-(a\sin x+b\cos x)(a\cos x+b\sin x)}{(a\sin x-b\cos x)^2}\\&=\frac{a^2\sin x\cos x-ab\sin^2 x-ab\cos^2 x+b^2\cos x\sin x-a^2\sin x\cos x-ab\sin^2 x-ab\cos^2 x-b^2\sin x\cos x}{(a\sin x-b\cos x)^2}\\&=\frac{-2ab(\sin^2 x+\cos^2 x)}{(a\sin x-b\cos x)^2}=-\frac{2ab}{(a\sin x-b\cos x)^2}.\end{aligned}$$

3.5.33 $\dfrac{dy}{dx}=-\sin x\cos x+\cos x(-\sin x)=-2\sin x\cos x=-\sin(2x).$

3.5.34

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+\cos x)(\sin x + x\cos x) - x\sin x(-\sin x)}{(1+\cos x)^2} = \frac{\sin x + x\cos x + \sin x\cos x + x\cos^2 x + x\sin^2 x}{(1+\cos x)^2} \\ &= \frac{\sin x + x\cos x + \sin x\cos x + x}{(1+\cos x)^2} = \frac{\sin x(1+\cos x) + x(1+\cos x)}{(1+\cos x)^2} = \frac{\sin x + x}{1+\cos x}.\end{aligned}$$

3.5.35 $\frac{dy}{dw} = 2w\sin w + w^2\cos w + 2\cos w - 2w\sin w - 2\cos w = w^2\cos w.$

3.5.36 $\frac{dy}{dx} = -3x^2\cos x + x^3\sin x + 6x\sin x + 3x^2\cos x + 6\cos x - 6x\sin x - 6\cos x = x^3\sin x.$

3.5.37 $\frac{dy}{dx} = \cos x\sin x + x(-\sin x)\sin x + x\cos x\cos x = \sin x\cos x - x\sin^2 x + x\cos^2 x = \frac{1}{2}\sin 2x + x\cos 2x.$

3.5.38 $\frac{dy}{dx} = \frac{0 - 1\cdot\cos x}{(2+\sin x)^2} = -\frac{\cos x}{(2+\sin x)^2}.$

3.5.39 $\frac{dy}{dx} = \frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2} = \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x}.$

3.5.40 $\frac{dy}{dx} = \frac{e^x(\cos x - \sin x) - e^x(\sin x + \cos x)}{e^{2x}} = \frac{-2\sin x}{e^x} = -2e^{-x}\sin x.$

3.5.41 $\frac{dy}{dx} = \frac{(1+\cos x)\sin x - (1-\cos x)(-\sin x)}{(1+\cos x)^2} = \frac{2\sin x}{(1+\cos x)^2}.$

3.5.42 $\frac{dy}{dx} = \sec^2 x - \csc^2 x.$

3.5.43 $\frac{dy}{dx} = \sec x\tan x - \csc x\cot x.$

3.5.44 $\frac{dy}{dx} = \sec x\tan x\tan x + \sec x\sec^2 x = \sec x(\tan^2 x + \sec^2 x).$

3.5.45 $\frac{dy}{dx} = e^x\csc x + e^x(-\csc x\cot x) = e^x\csc x(1-\cot x).$

3.5.46 $\frac{dy}{dw} = \frac{(1+\tan w)\sec^2 w - \tan w\sec^2 w}{(1+\tan w)^2} = \frac{\sec^2 w}{(1+\tan w)^2}.$

3.5.47

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+\csc x)(-\csc^2 x) - \cot x(-\csc x\cot x)}{(1+\csc x)^2} = \frac{-\csc^2 x - \csc^3 x + \csc x(\csc^2 x - 1)}{(1+\csc x)^2} \\ &= \frac{-\csc x(1+\csc x)}{(1+\csc x)^2} = -\frac{\csc x}{1+\csc x}\end{aligned}$$

3.5.48

$$\begin{aligned}\frac{dy}{dt} &= \frac{(1+\sec t)\sec^2 t - \tan t(\sec t\tan t)}{(1+\sec t)^2} = \frac{\sec^2 t + \sec^3 t - \sec t(\sec^2 t - 1)}{(1+\sec t)^2} \\ &= \frac{\sec t(1+\sec t)}{(1+\sec t)^2} = \frac{\sec t}{1+\sec t}.\end{aligned}$$

3.5.49

$$\begin{aligned}\frac{dy}{dz} &= \frac{0-(\sec z\tan z\csc z-\sec z\csc z\cot z)}{\sec^2 z\csc^2 z} = \frac{\sec z\csc z(\cot z-\tan z)}{\sec^2 z\csc^2 z}\\ &= \frac{\cot z-\tan z}{\sec z\csc z} = \cos^2 z-\sin^2 z = \cos(2z).\end{aligned}$$

3.5.50 Because $\csc^2\theta-1=\cot^2\theta$, we have $\dfrac{dy}{d\theta}=\dfrac{d}{d\theta}\cot^2\theta=-\csc^2\theta\cot\theta+\cot\theta(-\csc^2\theta)=-2\csc^2\theta\cot\theta$.

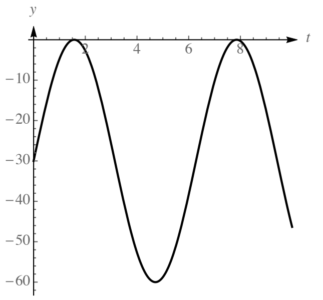
3.5.51 $\dfrac{dy}{dx}=1-(-\sin x\sin x+\cos x\cos x)=1+\sin^2 x-\cos^2 x=1+\sin^2 x-(1-\sin^2 x)=2\sin^2 x$.

3.5.52 $\dfrac{d}{dx}(\cot x)=\dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right)=\dfrac{\sin x(-\sin x)-\cos x(\cos x)}{\sin^2 x}=\dfrac{-(\sin^2 x+\cos^2 x)}{\sin^2 x}=-\dfrac{1}{\sin^2 x}=-\csc^2 x$.

3.5.53 $\dfrac{d}{dx}(\sec x)=\dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right)=\dfrac{0-(-\sin x)}{\cos^2 x}=\dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x}=\sec x\tan x$.

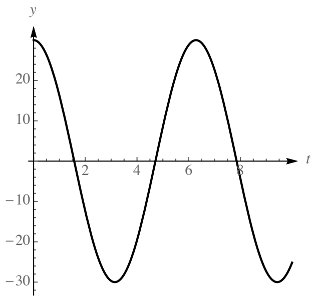
3.5.54 $\dfrac{d}{dx}(\csc x)=\dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right)=\dfrac{0-\cos x}{\sin^2 x}=-\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x}=-\csc x\cot x$.

3.5.55



a

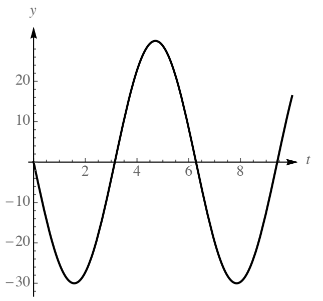
b. $v(t)=y'(t)=30\cos t$ cm per second.



c.

d. $v(t)=30\cos t=0$ when $t=\dfrac{2n+1}{2}\cdot\pi$ where n is a non-negative integer. At those times, the position is given by $\begin{cases}0 & \text{if } n \text{ is even}\\ -60 & \text{if } n \text{ is odd.}\end{cases}$

e. The maximum velocity is 30 cm per second because $|\cos t|\le 1$ for all t. We have $\cos t=1$ for $t=2n\pi$ for a positive integer n. At those times, $y(2n\pi)=-30$.

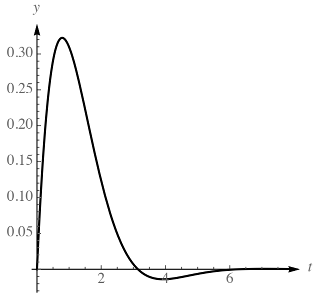


f.

$a(t)=v'(t)=-30\sin t$.

3.5.56

a.



The graph of $f(t) = e^{-k} \sin t$ oscillates between $-e^{-t}$ and e^{-t} because $-1 \leq \sin t \leq 1$.

b. We have $f'(t) = -e^{-t} \sin t + e^{-t} \cos t$, which is zero when $\cos t - \sin t = 0$, which occurs for $t = \frac{\pi}{4} + n\pi$ where n is any integer.

c. Because $-1 \leq \sin t \leq 1$ and $e^{-t} > 0$ for all t, we have $-e^{-t} \leq e^{-t} \sin t \leq e^{-t}$. And because $\lim_{t\to\infty} e^{-t} = 0$, we have that $\lim_{t\to\infty} e^{-t} \sin t = 0$ by the Squeeze Theorem. This means that the vibrations approach zero in the long run.

3.5.57 $y' = \sin x + x \cos x$, so $y'' = \cos x + \cos x + -x \sin x = 2\cos x - x \sin x$.

3.5.58 $y' = 2x \cos x + x^2(-\sin x) = x(2\cos x - x \sin x)$, so $y'' = (2\cos x - x\sin x) + x(-2\sin x - (\sin x + x\cos x)) = 2\cos x - x \sin x - 2x \sin x - x \sin x - x^2 \cos x = 2\cos x - 4x \sin x - x^2 \cos x$.

3.5.59 $y' = e^x \sin x + e^x \cos x$, so $y'' = e^x \sin x + e^x \cos x + e^x \cos x + e^x(-\sin x) = 2e^x \cos x$.

3.5.60 $y' = \dfrac{1}{2}e^x \cos x + \dfrac{1}{2}e^x(-\sin x)$, so $y'' = \dfrac{1}{2}e^x \cos x + \dfrac{1}{2}e^x(-\sin x) + \dfrac{1}{2}e^x(-\sin x) + \dfrac{1}{2}e^x(-\cos x) = -e^x \sin x$.

3.5.61 $y' = -\csc^2 x$ and $y'' = -((-\csc x \cot x)\csc x + \csc x(-\csc x \cot x)) = 2\cot x \csc^2 x$.

3.5.62 $y' = \sec^2 x$ and $y'' = \sec x \tan x(\sec x) + \sec x(\sec x \tan x) = 2\tan x \sec^2 x$.

3.5.63

$$y' = \sec x \tan x \csc x - \sec x \csc x \cot x = \sec x \csc x(\tan x - \cot x) = \sec^2 x - \csc^2 x.$$

$$\begin{aligned} y'' &= \sec x(\sec x \tan x) + (\sec x \tan x)\sec x - ((-\csc x \cot x)\csc x + \csc x(-\csc x \cot x)) \\ &= 2\sec^2 x \tan x + 2\csc^2 x \cot x. \end{aligned}$$

3.5.64 $y' = (-\sin x)\sin x + \cos x \cos x = \cos^2 x - \sin^2 x = \cos(2x)$.
$y'' = 2\cos x(-\sin x) - 2\sin x \cos x = -2(2\sin x \cos x) = -2\sin(2x)$.

3.5.65

a. False. $\dfrac{d}{dx}\sin^2 x = \sin x \cos x + \cos x \sin x = 2\sin x \cos x \neq \cos^2 x$.

b. False. $\dfrac{d^2}{dx^2}\sin x = \dfrac{d}{dx}\cos x = -\sin x \neq \sin x$.

c. True. $\dfrac{d^4}{dx^4}\cos x = \dfrac{d^3}{dx^3}(-\sin x) = \dfrac{d^2}{dx^2}(-\cos x) = \dfrac{d}{dx}\sin x = \cos x$.

d. True. In fact, $\pi/2$ isn't even in the domain of $\sec x$.

3.5.66 $\displaystyle\lim_{x\to\pi/2}\frac{\cos x}{x-\pi/2} == \lim_{x\to\pi/2}\frac{\cos x-\cos\pi/2}{x-\pi/2} = \frac{d}{dx}\cos x\Big|_{x=\pi/2} = -\sin\pi/2 = -1.$

3.5.67 $\displaystyle\lim_{x\to\pi/4}\frac{\tan x-1}{x-\pi/4} = \lim_{x\to\pi/4}\frac{\tan x-\tan\pi/4}{x-\pi/4} = \frac{d}{dx}\tan x\Big|_{x=\pi/4} = \sec^2\pi/4 = 2.$

3.5.68 $\displaystyle\lim_{h\to 0}\frac{\sin(\pi/6+h)-(1/2)}{h} = \lim_{h\to 0}\frac{\sin(\pi/6+h)-\sin\pi/6}{h} = \frac{d}{dx}\sin x\Big|_{x=\pi/6} = \cos(\pi/6) = \sqrt{3}/2.$

3.5.69 $\displaystyle\lim_{h\to 0}\frac{\cos(\pi/6+h)-(\sqrt{3}/2)}{h} = \lim_{h\to 0}\frac{\cos(\pi/6+h)-(\cos\pi/6)}{h} = \frac{d}{dx}\cos x\Big|_{x=\pi/6} = -\sin(\pi/6) = -1/2.$

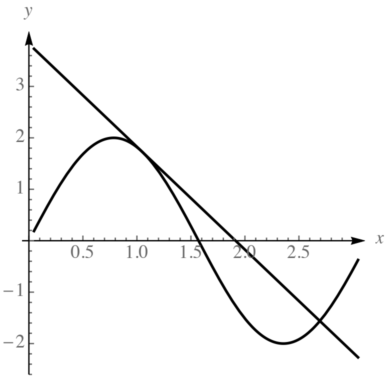
3.5.70 $\displaystyle\lim_{x\to\pi/4}\frac{\cot(x)-1}{x-\pi/4} = \lim_{x\to\pi/4}\frac{\cot(x)-\cot\pi/4}{x-\pi/4} = \frac{d}{dx}\cot x\Big|_{x=\pi/4} = -\csc^2(\pi/4) = -2.$

3.5.71 $\displaystyle\lim_{h\to 0}\frac{\tan(5\pi/6+h)+(1/\sqrt{3})}{h} = \lim_{h\to 0}\frac{\tan(5\pi/6+h)-(\tan 5\pi/6)}{h} = \frac{d}{dx}\tan x\Big|_{x=\pi/6} =$ $\sec^2(5\pi/6) = 4/3.$

3.5.72

a. $y' = 4\cos^2 x - 4\sin^2 x$, so $y'(\pi/3) = 4\left(\frac{1}{4}-\frac{3}{4}\right) = -2$. $y(\pi/3) = 4\cdot(\sqrt{3}/2)\cdot(1/2) = \sqrt{3}$. The tangent line is thus given by $y-\sqrt{3} = -2(x-\pi/3)$, or $y = -2x+\sqrt{3}+\frac{2\pi}{3}$.

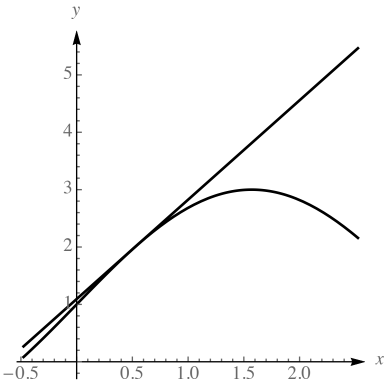
b.



3.5.73

a. $y' = 2\cos x$, so $y'(\pi/6) = \sqrt{3}$. $y(\pi/6) = 2$. The tangent line is thus given by $y-2 = \sqrt{3}(x-\pi/6)$, or $y = \sqrt{3}x+2-\frac{\pi\sqrt{3}}{6}$.

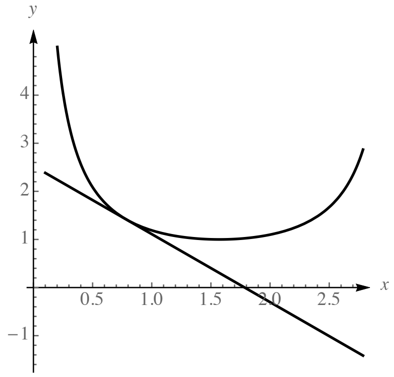
b.



3.5.74

a. $y' = -\csc x \cot x$, so $y'(\pi/4) = -\sqrt{2}$. $y(\pi/4) = \sqrt{2}$. The tangent line is thus given by $y - \sqrt{2} = -\sqrt{2}(x - \pi/4)$, or $y = -\sqrt{2}x + \sqrt{2} + \frac{\sqrt{2}\pi}{4}$.

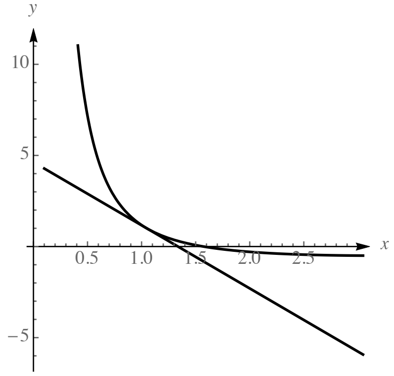
b.



3.5.75

a. $y' = \dfrac{(1-\cos x)(-\sin x) - \cos x \sin x}{(1-\cos x)^2} = -\dfrac{\sin x}{(1-\cos x)^2}$, so $y'(\pi/3) = -2\sqrt{3}$. $y(\pi/3) = 1$. The tangent line is thus given by $y - 1 = -2\sqrt{3}(x - \pi/3)$, or $y = -2\sqrt{3}x + \frac{2\sqrt{3}\pi}{3} + 1$.

b.



3.5.76

a. A horizontal tangent line occurs when $g'(x) = 1 - \cos x = 0$, which is when $\cos x = 1$. This occurs when $x = 2n\pi$, where n is any integer.

b. A slope of 1 occurs when $g'(x) = 1 - \cos x = 1$, which is when $\cos x = 0$. This occurs when $x = \dfrac{2n+1}{2} \cdot \pi$, where n is any integer.

3.5.77 For a horizontal tangent line we need $f'(x) = 1 + 2\sin x = 0$, or $\sin x = -\dfrac{1}{2}$. This occurs for $x = \frac{7\pi}{6} + 2n\pi$ where n is any integer, or for $x = \frac{11\pi}{6} + 2n\pi$ where n is any integer.

3.5.78

a. The derivative of graph (a) is graph (D), because graph (a) has a positive slope everywhere, its derivative must be positive everywhere, and graph (D) is the only one with this property.

b. The derivative of graph (b) is graph (B), because graph (b) has negative slope everywhere, its derivative must be negative everywhere, and graph (B) is the only one with this property.

c. The derivative of graph (c) is graph (A), because graph (c) has horizontal tangents at 0 and ±π, its derivative needs to be 0 at these points, and only graph (A) has this property.

d. The derivative of graph (d) is graph (C), because graph (d) has horizontal tangents at $\pm\pi/2$ and $\pm 3\pi/2$, its derivative needs to be 0 at these points, and only graph (C) has this property.

3.5.79

a. $y'(t) = A\cos t$, $y''(t) = -A\sin t$, so $y''(t) + y(t) = -A\sin t + A\sin t = 0$ for all A and all t.

b. $y'(t) = -B\sin t$, $y''(t) = -B\cos t$, so $y''(t) + y(t) = -B\cos t + B\cos t = 0$ for all B and all t.

c. $y' = A\cos t - B\sin t$, $y'' = -A\sin t - B\cos t$, so $y''(t) + y(t) = -A\sin t - B\cos t + A\sin t + B\cos t = 0$ for all A, B, t.

3.5.80 $\frac{d}{dx}(\sin 2x) = \frac{d}{dx}(2\sin x\cos x) = (2\cos x)\cos x + (2\sin x)(-\sin x) = 2(\cos^2 x - \sin^2 x) = 2\cos 2x$.

3.5.81

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = \lim_{x\to 0}\frac{(\cos x - 1)(\cos x + 1)}{x(\cos x + 1)} = \lim_{x\to 0}\frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x\to 0} -\frac{\sin^2 x}{x(\cos x + 1)}$$
$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0} -\frac{\sin x}{(\cos x + 1)} = 1\cdot\frac{0}{2} = 0.$$

3.5.82 We will use this version of the half-angle formula: $\frac{1-\cos x}{2} = \sin^2(x/2)$.

We have $\lim_{x\to 0}\frac{\cos x - 1}{x} = \lim_{x\to 0} -\frac{1-\cos x}{2x/2} = \lim_{x\to 0} -\frac{\sin^2(x/2)}{x/2} = -\lim_{x\to 0}\sin(x/2)\cdot\lim_{x\to 0}\frac{\sin x/2}{x/2} = -1\cdot 0\cdot 1 = 0.$

3.5.83

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$
$$= \cos x\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) - \sin x\left(\lim_{h\to 0}\frac{\sin h}{h}\right) = \cos x\cdot 0 - \sin x\cdot 1 = -\sin x.$$

3.5.84 f is continuous at 0 if and only if $\lim_{x\to 0} f(x) = f(0)$. Because $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{3\sin x}{x} = 3$, we require $a = 3$ in order for f to be continuous.

3.5.85 g is continuous at 0 if and only if $\lim_{x\to 0} g(x) = g(0)$. Because $\lim_{x\to 0} g(x) = \lim_{x\to 0}\frac{1-\cos x}{2x} = \frac{1}{2}\cdot 0 = 0$, we require $a = 0$ in order for g to be continuous.

3.5.86

a. The unit circle consists of 360 degrees and 2π radians, so each degree corresponds to $\frac{2\pi}{360} = \frac{\pi}{180}$ radians.

b. $\lim_{x\to 0}\frac{s(x)}{x} = \lim_{x\to 0}\frac{\sin(\pi x/180)}{x} = \frac{\pi}{180}\lim_{x\to 0}\frac{\sin(\pi x/180)}{\pi x/180} = \frac{\pi}{180}\cdot 1 = \frac{\pi}{180}$.

3.5.87

a. $\frac{d}{dx}\sin^2 x = \sin x\cos x + \cos x\sin x = 2\sin x\cos x$.

b. $\frac{d}{dx}\sin^3 x = \frac{d}{dx}(\sin^2 x)(\sin x) = (2\sin x\cos x)\sin x + \sin^2 x\cdot\cos x = 3\sin^2 x\cos x$.

c. $\frac{d}{dx}\sin^4 x = \frac{d}{dx}(\sin^3 x)(\sin x) = (3\sin^2 x\cos x)(\sin x) + (\sin^3 x)(\cos x) = 4\sin^3 x\cos x$.

d. We guess that $\frac{d}{dx}\sin^n x = n\sin^{n-1}x\cos x$.

We have already seen that the claim is valid for $n = 2$. Suppose our guess is valid for a given positive integer n. Then

$$\frac{d}{dx}\sin^{n+1}x = \frac{d}{dx}(\sin^n x)(\sin x) = (n\sin^{n-1}x\cos x)(\sin x) + \sin^n x\cos x = (n+1)\sin^n x\cos x.$$

Thus by induction, the result holds for all n.

3.5.88 Consider the statement $\frac{d^{2n}}{dx^{2n}}(\sin x) = (-1)^n\sin x$. This statement is valid for $n = 1$ because

$$\frac{d^2}{dx^2}\sin x = \frac{d}{dx}\cos x = -\sin x.$$

Now suppose the statement is valid for some positive integer n. Then

$$\frac{d^{2n+2}}{dx^{2n+2}}\sin x = \frac{d^2}{dx^2}\left(\frac{d^{2n}}{dx^{2n}}\sin x\right) = \frac{d^2}{dx^2}\left((-1)^n\sin x\right) = (-1)^n\cdot(-1)\sin x = (-1)^{n+1}\sin x,$$

which completes the proof.

Similarly, consider the statement $\frac{d^{2n}}{dx^{2n}}(\cos x) = (-1)^n\cos x$ This statement is valid for $n = 1$ because

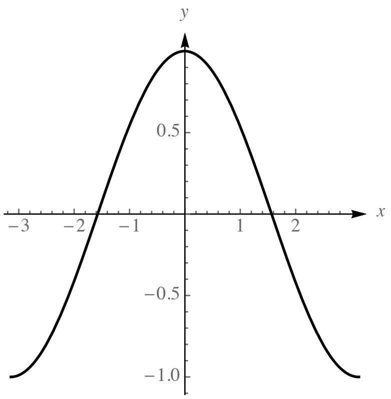
$$\frac{d^2}{dx^2}\cos x = \frac{d}{dx}(-\sin x) = -\cos x.$$

Now suppose the statement is valid for some positive integer n. Then

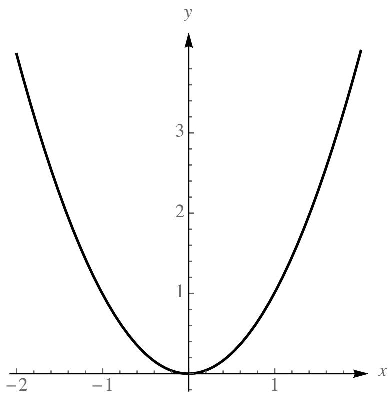
$$\frac{d^{2n+2}}{dx^{2n+2}}\cos x = \frac{d^2}{dx^2}\left(\frac{d^{2n}}{dx^{2n}}\cos x\right) = \frac{d^2}{dx^2}\left((-1)^n\cos x\right) = (-1)^n\cdot(-1)\cos x = (-1)^{n+1}\cos x,$$

which completes the proof.

3.5.89 Because D is a difference quotient, and because $h = 0.01$ is small, D is a good approximation to f'. Therefore, the graph of D is nearly indistinguishable from the graph of $f'(x) = \cos x$.



3.5.90 Because D is a difference quotient, and because $h = 0.01$ is small, D is a good approximation to f'. Therefore, the graph of D is nearly indistinguishable from the graph of $f'(x) = x^2$.



3.6 Derivatives as Rates of Change

3.6.1 The average rate of change is $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$, whereas the instantaneous rate of change is the limit of this quotient as $\Delta x \to 0$.

3.6.2 If $\dfrac{dy}{dx}$ is large, then small changes in x will result in relatively large changes in the value of y.

3.6.3 If $\dfrac{dy}{dx}$ is small, then small changes in x will result in relatively small changes in the value of y.

3.6.4

a. It is negative – it is equal to the force of gravity which is -32 ft/sec^2.

b. At its highest point it is not moving – so its velocity is 0.

3.6.5 At 15 weeks, the puppy grows at a rate of 1.75 lb/week.

3.6.6 The speed of an object is the absolute value of its velocity. Thus, velocity encompasses the direction that the object is moving, while speed does not (it is always positive).

3.6.7 Acceleration is the instantaneous rate of change of the velocity; that is, if $s(t)$ is the position of an object at time t, then $s''(t) = \dfrac{d}{dt}(v(t)) = a(t)$ is the acceleration of the object at time t.

3.6.8 If the object is moving in the positive direction, the velocity will decrease. If it is moving in the negative direction, the velocity will increase.

3.6.9 $V'(T) = 0.6$. The speed of sound increases by 0.6 m/s for each increase of 1° in temperature.

3.6.10 $\dfrac{dV}{dt}$ represents the rate in ft^3/hr at which the pool is being filled.

3.6.11

a. $v_{\text{avg}} = \dfrac{f(0.75)-f(0)}{0.75} = \dfrac{30-0}{0.75} = 40$ mph.

b. $v_{\text{avg}} = \dfrac{f(0.75)-f(0.25)}{0.75-0.25} = \dfrac{30-10}{0.5} = 40$ mph.

This is a pretty good estimate, since the graph is nearly linear over that time interval.

c. $v_{\text{avg}} = \dfrac{f(2.25) - f(1.75)}{2.25 - 1.75} = \dfrac{-14 - 16}{0.5} = -60$ mph.

At 11 a.m. the velocity is $v(2) \approx -60$ mph. The car is moving south with a speed of approximately 60 mph.

d. From 9 a.m. until about 10:08 a.m., the car moves north, away from the station. Then it moves south, passing the station at approximately 11:02 a.m., and continues south until about 11:40 a.m. Then the car drives north until 12:00 noon stopping south of the station.

3.6.12

a. $v_{\text{avg}} = \dfrac{s(1.5) - s(0)}{1.5 - 0} = \dfrac{600 - 0}{1.5} = 400$ mph.

b. $v_{\text{avg}} = \dfrac{s(8.5) - s(7.5)}{8.5 - 7.5} = \dfrac{0 - 300}{1} = -300$ mph.

c. The velocity is zero from about 9 a.m. until 11:10 a.m. when the plane is at the gate in Minneapolis.

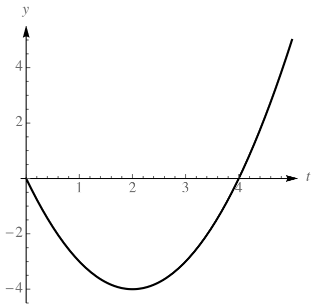
d. $v(6) \approx \dfrac{800 - 1400}{1} = -600$ mph. The velocity is negative as the plane returns to Seattle.

3.6.13 Each of the first 200 stoves cost on average \$70 to produce, while the 201st stove costs \$65 to produce.

3.6.14 If $D(p)$ is decreasing, then $\dfrac{dD}{dp}$ is negative. Both p and D are positive, so $E(p) = \dfrac{dD}{dp}\dfrac{p}{D}$ is negative.

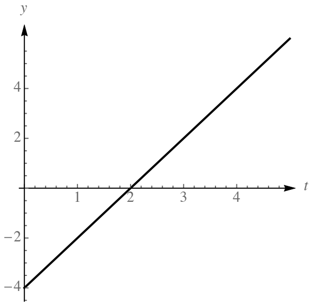
3.6.15

a.



$f(t) = t^2 - 4t$

b.



$f'(t) = 2t - 4$

b. $f'(t) = 0$ when $t = 2$ – that is when the object is stationary. For $0 \le t < 2$ we have $f'(t) < 0$ so the object is moving to the left. For $2 < t \le 5$ we have $f'(t) > 0$ so the object is moving to the right.

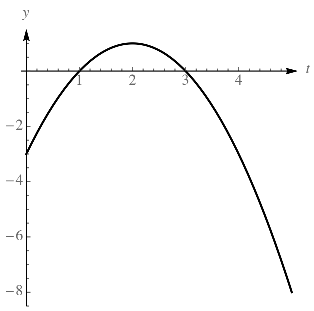
c. $f'(1) = -2$ ft/sec and $f''(t) = 2\,\text{ft/sec}^2$, so in particular, $f''(1) = 2\,\text{ft/sec}^2$.

d. $f'(t) = 0$ when $t = 2$ and $f''(2) = 2\,\text{ft/sec}^2$.

e. On the interval $(2, 5]$ the velocity and acceleration are both positive, so the object's speed is increasing.

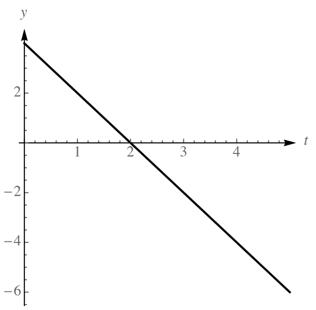
3.6.16

a.



$f(t) = -t^2 + 4t - 3$

b.



$f'(t) = -2t + 4$

b. $f'(t) = 0$ when $t = 2$ – that is when the object is stationary. For $0 \leq t < 2$ we have $f'(t) > 0$ so the object is moving to the right. For $2 < t \leq 5$ we have $f'(t) < 0$ so the object is moving to the left.

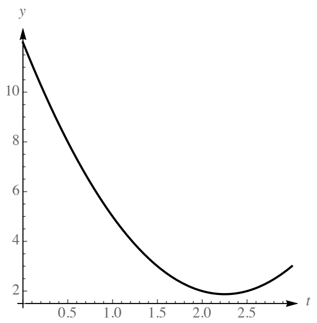
c. $f'(1) = 2$ ft/sec and $f''(t) = -2\,\text{ft/sec}^2$, so in particular, $f''(1) = -2\,\text{ft/sec}^2$.

d. $f'(t) = 0$ when $t = 2$ and $f''(2) = -2\,\text{ft/sec}^2$.

e. On the interval $(2, 5]$ the velocity and the acceleration are both negative, so the object's speed is increasing.

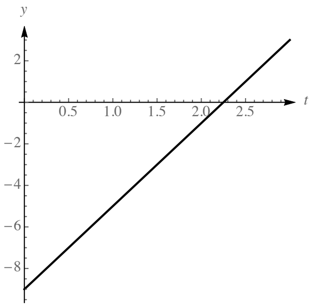
3.6.17

a.



$f(t) = 2t^2 - 9t + 12$

b.



$f'(t) = 4t - 9$

b. $f'(t) = 0$ when $t = 9/4$ – that is when the object is stationary. For $0 \leq t < 9/4$ we have $f'(t) < 0$ so the object is moving to the left. For $9/4 < t \leq 3$ we have $f'(t) > 0$ so the object is moving to the right.

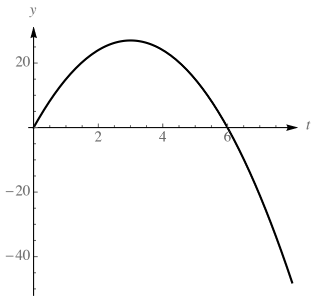
c. $f'(1) = -5$ ft/sec and $f''(t) = 4\,\text{ft/sec}^2$, so in particular, $f''(1) = 4\,\text{ft/sec}^2$.

d. $f'(t) = 0$ when $t = 9/4$ and $f''(9/4) = 4\,\text{ft/sec}^2$.

e. On the interval $(9/4, 3]$ both the velocity and acceleration are positive, so the object's speed is increasing.

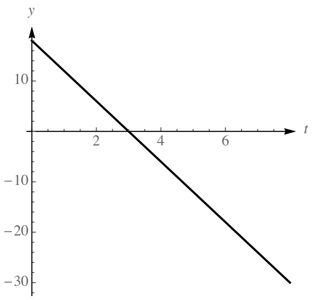
3.6.18

a.



$f(t) = 18t - 3t^2$

b.



$f'(t) = 18 - 6t$

b. $f'(t) = 0$ when $t = 3$ – that is when the object is stationary. For $0 \le t < 3$ we have $f'(t) > 0$ so the object is moving to the right. For $3 < t \le 8$ we have $f'(t) < 0$ so the object is moving to the left.

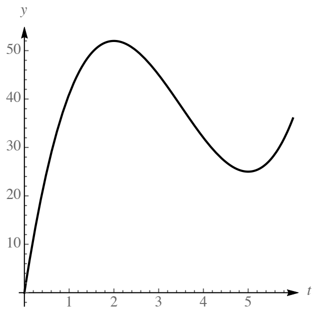
c. $f'(1) = 12$ ft/sec and $f''(t) = -6\,\text{ft/sec}^2$, so in particular, $f''(1) = -6\,\text{ft/sec}^2$.

d. $f'(t) = 0$ when $t = 3$ and $f''(3) = -6\,\text{ft/sec}^2$.

e. On the interval $(3, 8]$ both the velocity and acceleration are negative, so the object's speed is increasing on that interval.

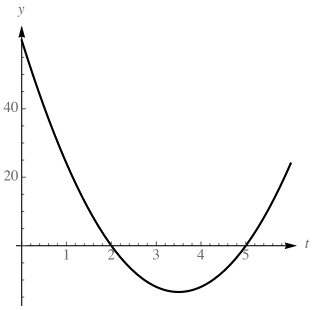
3.6.19

a.



$f(t) = 2t^3 - 21t^2 + 60t$

b.



$f'(t) = 6t^2 - 42t + 60$

b. $f'(t) = 0$ when $6(t-2)(t-5) = 0$, which is at $t = 2$ and $t = 5$ – that is when the object is stationary. For $0 \le t < 2$ we have $f'(t) > 0$ so the object is moving to the right. For $2 < t < 5$ we have $f'(t) < 0$ so the object is moving to the left. For $5 < t \le 8$ we have $f'(t) > 0$, so the object is moving to the right again.

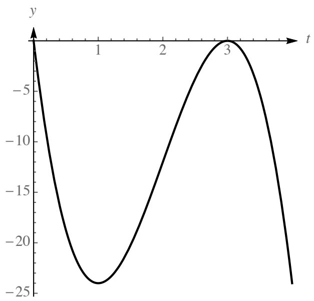
c. $f'(1) = 24$ ft/sec and $f''(t) = 12t - 42$, so $f''(1) = -30\,\text{ft/sec}^2$.

d. $f'(t) = 0$ when $t = 2$ and $t = 5$. We have $f''(2) = -18\,\text{ft/sec}^2$ and $f''(5) = 18\,\text{ft/sec}^2$.

e. $f''(t) = 12t - 42$ is positive for $t > \frac{42}{12} = \frac{7}{2}$ and negative for $t < \frac{7}{2}$. So f' and f'' are both positive on $(5, 6]$ and are both negative on $(2, 3.5)$, so that is where the object is speeding up.

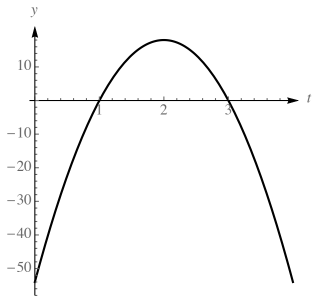
3.6.20

a.



$f(t) = -6t^3 + 36t^2 - 54t$

b.



$f'(t) = -18t^2 + 72t - 54$

b. $f'(t) = 0$ when $-18(t-3)(t-1) = 0$, which is at $t = 1$ and $t = 3$ – that is when the object is stationary. For $0 \le t < 1$ we have $f'(t) < 0$ so the object is moving to the left. For $1 < t < 3$ we have $f'(t) > 0$ so the object is moving to the right. For $3 < t \le 4$ we have $f'(t) < 0$, so the object is moving to the left again.

c. $f'(1) = 0$ ft/sec and $f''(t) = -36t + 72$, so $f''(1) = 36\,\text{ft/sec}^2$.

d. $f'(t) = 0$ when $t = 1$ and $t = 3$. We have $f''(1) = 36\,\text{ft/sec}^2$ and $f''(3) = -36\,\text{ft/sec}^2$.

e. $f''(t) = -36t + 72$ is positive on $[0, 2)$, so the object's velocity and acceleration are both negative on $(1, 2)$ and they are both negative on $(3, 4]$, so that is where the object is speeding up.

3.6.21 To find out when it hits the water, we set $s(t) = -16t^2 + 64$ equal to 0. This gives $-16t^2 = -64$, or $t^2 = 4$, so $t = 2$ is the time the stone hits the water. The velocity at time t is $v(t) = s'(t) = -32t$, so $v(2) = -64$, so the speed is $|-64| = 64$ ft/sec.

3.6.22 To determine when it hits the ground, we set $s(t) = -6t^2 + 54$ equal to 0. This gives $-6t^2 = -54$ or $t^2 = 9$, so $t = 3$ is the time the stone hits the ground. The velocity at time t is $v(t) = s'(t) = -12t$, so $v(3) = -36$, so the speed is $|-36| = 36$ ft/sec.

3.6.23

a. $v(t) = s'(t) = -32t + 32$.

b. $v(t) = 0$ when $-32t + 32 = 0$, so at $t = 1$ s.

c. $s(1) = -16 + 32 + 48 = 64$ ft.

d. $s(t) = 0$ when $-16t^2 + 32t + 48 = 0$, or $-16(t + 1)(t - 3) = 0$, so at $t = 3$ s.

e. $v(3) = -96 + 32 = -64$ ft/s.

f. Note that the acceleration is always negative, so the stone will be speeding up when its velocity is also negative, which occurs on the interval $(1, 3)$. This corresponds to the downward portion of the stone's trip.

3.6.24

a. $v(t) = s'(t) = -9.8t + 19.6$.

b. $v(t) = 0$ when $-9.8t + 19.6 = 0$, so at $t = 2$ s.

c. $s(2) = -4.9 \cdot 4 + 19.6(2) + 24.5 = 44.1$ m.

d. $s(t) = 0$ when $-4.9t^2 + 19.6t + 24.5 = 0$, or $t^2 - 4t - 5 = (t + 1)(t - 5) = 0$, so at $t = 5$ s.

e. $v(5) = -9.8(5) + 19.6 = -29.4$ ft/s.

f. Note that the acceleration is always negative, so the stone will be speeding up when its velocity is also negative, which occurs on the interval $(2, 5)$. This corresponds to the downward portion of the stone's trip.

3.6.25

a. $v(t) = s'(t) = -32t + 64$ ft/sec.

b. $v(t) = 0$ when $-32t + 64 = 0$, which is at $t = 2$.

c. $s(2) = -16 \cdot 4 + 64 \cdot 2 + 32 = 96$ feet.

d. $s(t) = 0$ when $-16t^2 + 64t + 32 = 0$. Using the quadratic formula we see that this occurs when $t = 2 + \sqrt{6} \approx 4.45$ seconds.

e. The velocity when the stone hits the ground is $v(2+\sqrt{6}) = -32(2+\sqrt{6}) + 64 = -32\sqrt{6} \approx -78.38$ feet per second.

f. The acceleration due to gravity is always negative, so the object is speeding up when its velocity is negative; that is, on its downward journey during the interval $(2, 2+\sqrt{6})$.

3.6.26

a. The ball is at its maximum height when $v(t) = -32t + v_0 = 0$, so $t = \dfrac{v_0}{32}$. Its height at this time is $s(v_0/32) = -16\left(\dfrac{v_0}{32}\right)^2 + v_0\left(\dfrac{v_0}{32}\right) = \dfrac{v_0^2}{64}$.

b. The ball hits the ground when $s(t) = -16t^2 + v_0 t = 0$, so at $t = \dfrac{v_0}{16}$. Its velocity at this time is $v\left(\dfrac{v_0}{16}\right) = -32\left(\dfrac{v_0}{16}\right) + v_0 = -v_0$.

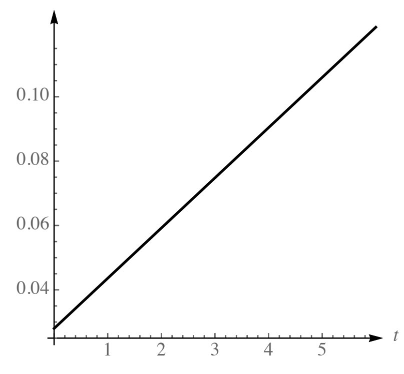
3.6.27 Because the maximum height is $\dfrac{v_0^2}{64}$ (see the previous problem), we need $\dfrac{v_0^2}{64} = 128$, so $v_0 = \sqrt{128 \cdot 64} \approx 90.5$ ft/s.

3.6.28

a. $\dfrac{p(6) - p(0)}{6 - 0} = 0.0748$ million people/year or 74,800 people/year.

b. $p'(1) = 0.0436$ million people/year or 43,60 people/year. $p'(5) = 0.106$ million people/year or 106,000 people/year.

c. The rate of growth in Washington is increasing over this time period.



3.6.29

a. The average cost function is given by $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{1000}{x} + .1$. The marginal cost function is given by $M(x) = C'(x) = .1$.

b. At $a = 2000$ we have $\overline{C}(2000) = \dfrac{1000}{2000} + .1 = .6$, and $M(2000) = .1$.

c. The average cost per item when producing 2000 items is \$0.60. The cost of producing the next item is \$0.10.

3.6.30

a. The average cost function is given by $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{500}{x} + .02$. The marginal cost function is given by $M(x) = C'(x) = .02$.

b. At $a = 1000$ we have $\overline{C}(1000) = \dfrac{500}{1000} + .02 = .52$, and $M(1000) = .02$.

c. The average cost per item when producing 1000 items is \$0.52. The cost of producing the next item is \$0.02.

3.6.31

a. The average cost function is given by $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{100}{x} + 40 - 0.01x$. The marginal cost function is given by $M(x) = C'(x) = 40 - 0.02x$.

b. At $a = 1000$ we have $\overline{C}(1000) = \dfrac{100}{1000} + 40 - (.01)(1000) = 30.1$, and $M(1000) = 20$.

c. The average cost per item when producing 1000 items is \$30.10. The cost of producing the next item is \$20.00.

3.6.32

a. The average cost function is given by $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{800}{x} + 100 - 0.04x$. The marginal cost function is given by $M(x) = C'(x) = 100 - 0.08x$.

b. At $a = 500$ we have $\overline{C}(500) = \dfrac{800}{500} + 100 - (.04)(500) = 81.6$, and $M(500) = 60$.

c. The average cost per item when producing 500 items is \$81.60. The cost of producing the next item is \$60.00.

3.6.33

a. $D(10) = 40 - 20 = 20$ DVDs per day.

b. Demand is zero when $D(p) = 40 - 2p = 0$, which occurs for $p = 20$ dollars.

c. The elasticity is $E(p) = \dfrac{dD}{dp}\dfrac{p}{D} = -2\left(\dfrac{p}{40-2p}\right) = \dfrac{p}{p-20}$.

d. This quantity satisfies $-1 < E(p) < 0$ when $-1 < \frac{p}{p-20} < 0$ which occurs when $p < 20 - p$, or $p < 10$. So for prices in the interval $(0, 10)$ the demand is inelastic, while for prices in the interval $(10, 20)$ the demand is elastic.

e. If the price goes up from 10 to 10.25, that is a $\frac{.25}{10} = .025 = 2.5\%$ increase in price.

f. If the price goes up from 10 to 10.25, the demand goes from $D(10) = 40 - 20 = 20$ to $D(10.25) = 40 - 20.5 = 19.5$, which is a $\frac{.5}{20} = 2.5\%$ decrease.

3.6.34

a. The domain of the demand function is $(40, \infty)$.

b. $D(60) = \dfrac{1800}{60-40} = \dfrac{1800}{20} = 90$.

c. $E(p) = \dfrac{dD}{dp}\dfrac{p}{D} = \dfrac{-1800}{(p-40)^2} \cdot \dfrac{p(p-40)}{1800} = \dfrac{p}{40-p}$.

d. Note that $E(p) = \dfrac{p}{40-p}$ is always less than -1 on the interval $(40, \infty)$. (Note that the equation $\dfrac{p}{40-p} = -1$ has no solutions). So the demand is always elastic.

e. $E(60) = \dfrac{60}{40-60} = \dfrac{60}{-20} = -3$. So a $\frac{2}{60} = 3.\overline{33}\%$ change in price will decrease the demand by about $3 \cdot 3.\overline{33}\% = 10\%$.

3.6.35

a. False. For example, when a ball is thrown up in the air near the surface of the earth, its acceleration is constant (due to gravity) but its velocity changes during its trip.

b. True. If the rate of change of velocity is zero, then velocity must be constant.

c. False. If the velocity is constant over an interval, then the average velocity is equal to the instantaneous velocity over the interval.

d. True. For example, a ball dropped from a tower has negative acceleration and increasing speed as it falls toward the earth.

3.6.36 The velocity is $v(t) = s'(t) = -1.6t$. The feather strikes the surface of the moon when $s(t) = 40 - 0.8t^2 = 0$. This occurs when $t = \sqrt{50} \approx 7.07$ seconds. The velocity at this time is $v(\sqrt{50}) = -1.6\sqrt{50} \approx -11.31$ meters per second, and $a(\sqrt{50}) = -1.6$ meters per second^2.

3.6.37 In each case, the stone reaches its maximum height when its velocity is zero.

On Mars, this occurs when $v(t) = s'(t) = 96 - 12t = 0$, or when $t = 8$ seconds. So the maximum height on Mars is $s(8) = 384$ feet.

On Earth, this occurs when $v(t) = s'(t) = 96 - 32t = 0$, or when $t = 3$ seconds. So the maximum height on Earth is $s(3) = 144$ feet.

The stone will travel $384 - 144 = 240$ feet higher on Mars.

3.6.38

a. Both stones reach their highest points when the derivative of their position functions are 0. Note, however that $f'(t) = -32t + 48 = g'(t)$. Thus both stones reach their maximum height at $t = \dfrac{48}{32} = \dfrac{3}{2}$.

b. The height of the stone thrown from the bridge at $t = 1.5$ seconds is $f(1.5) = 68$ feet, while the other stone reaches $g(1.5) = 36$ feet, so the one thrown from the bridge goes 32 feet higher.

c. The stone from ground level hits the ground when $g(t) = 0$, which occurs when $t = 3$. The velocity at this time is $g'(3) = -48$ feet per second. The stone thrown from the bridge hits the ground when $f(t) = 0$, which occurs when $-16t^2 + 48t + 32 = 0$, or $t^2 - 3t - 2 = 0$, or $t = \dfrac{3 + \sqrt{9 - (4)(-2)}}{2} = \dfrac{3 + \sqrt{17}}{2} \approx 3.56$. At that time, the velocity is approximately -65.97 feet per second.

3.6.39 The first stone reaches its maximum height when $f'(t) = -32t + 32 = 0$, so after 1 second, and its maximum height is therefore $f(1) = -16 + 32 + 48 = 64$ feet.

The second stone reaches its maximum height when $g'(t) = -32t + v_0 = 0$, so when $t = \frac{v_0}{32}$. Its height at that time is $g(v_0/32) = -16(v_0/32)^2 + (v_0^2/32) = \frac{v_0^2}{64}$. This is equal to 64 when $v_0 = 64$ feet per second.

3.6.40

a. The slope of the curve (which is the velocity) increases until about 5:30 p.m., so the car is speeding up over that time interval. From 5:30 p.m. until about 6:20 p.m. the velocity is decreasing. After that it is speeding up until 7:00 p.m.

b. The slope is the largest at about 5:30 p.m. and smallest at about 6:20 p.m.

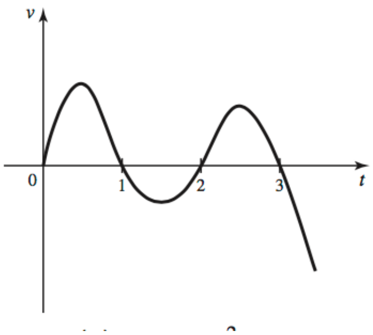
c. The maximum velocity is approximately 40 mph and the minimum is about 5 mph. These are estimates based on visually computing slopes of tangent lines. Your mileage may vary.

3.6.41

a. The velocity is zero at $t = 1, 2, \text{and}\, 3$.

b. The object is moving in the positive direction when the slope of s is positive, so from $t = 0$ to $t = 1$, and from $t = 2$ to $t = 3$. It is moving in the negative direction from $t = 1$ to $t = 2$, and for $t > 3$.

c.



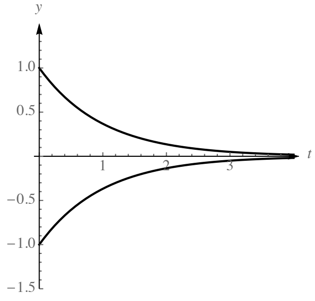
d. The speed is increasing on $(0, 1/2)$ as the velocity is positive and the acceleration is positive there. On $(1/2, 1)$ the speed is decreasing as the velocity is positive but the acceleration is negative. On $(1, 3/2)$ the speed is increasing as the velocity is negative and the acceleration is negative, but on $(3/2, 2)$ the speed is decreasing as the velocity is negative but the acceleration is positive. On approximately $(2, 2.6)$ the speed is increasing as the velocity is positive and the acceleration is positive, but on about $(2.6, 3)$ the velocity is positive but the acceleration is negative, so the speed is decreasing. On $(3, \infty)$ both the velocity and the acceleration are negative, so the object is speeding up.

3.6.42

a. $\frac{dL}{dt}$ represents the rate of change of the length of the species. The derivative is decreasing over time.

b. Over time, the species is getting bigger, but the rate of change is approaching zero.

c.



3.6.43

a. $P(x) = xp(x) - C(x) = 100x + 0.02x^2 - 50x - 100 = 0.02x^2 + 50x - 100$.

b. The average profit is $\overline{P}(x) = \dfrac{P(x)}{x} = 0.02x + 50 - \dfrac{100}{x}$. The marginal profit is $P'(x) = .04x + 50$.

c. $\overline{P}(500) = 59.8$. $P'(500) = 70$.

d. The average profit for the first 500 items sold is \$59.80, while the profit on the 501st item is \$70.00.

3.6.44

a. $P(x) = xp(x) - C(x) = 100x - 0.1x^2 - (-0.02x^2 + 50x + 100) = 50x - 0.08x^2 - 100$.

b. The average profit is $\overline{P}(x) = \dfrac{P(x)}{x} = -0.08x + 50 - \dfrac{100}{x}$. The marginal profit is $P'(x) = -0.16x + 50$.

c. $\overline{P}(500) = 9.8$. $P'(500) = -30$.

d. The average profit for the first 500 items sold is \$9.80, while the profit on the 501st item is −\$30.00.

3.6.45

a. $P(x) = xp(x) - C(x) = 100x + 0.04x^2 - 800$.

b. The average profit is $\overline{P}(x) = \dfrac{P(x)}{x} = .04x + 100 - \dfrac{800}{x}$. The marginal profit is $P'(x) = .08x + 100$.

c. $\overline{P}(1000) = 139.2$. $P'(1000) = 180$.

d. The average profit for the first 1000 items sold is \$139.20, while the profit on the 1001st item is \$180.00.

3.6.46

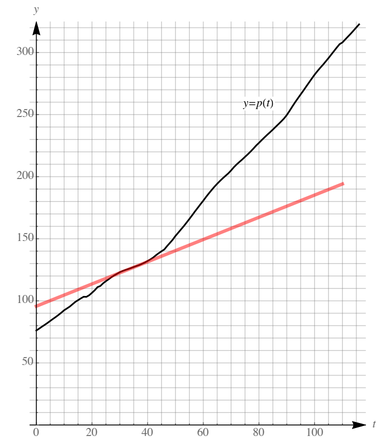
a. $P(x) = xp(x) - C(x) = 100x - 0.06x^2 - 800$.

b. The average profit is $\overline{P}(x) = \dfrac{P(x)}{x} = -0.06x + 100 - \dfrac{800}{x}$. The marginal profit is $P'(x) = -0.12x + 100$.

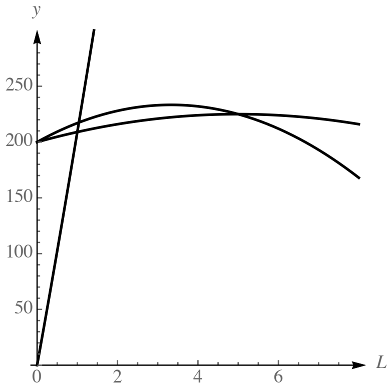
c. $\overline{P}(1000) = 39.2$. $P'(1000) = -20$.

d. The average profit for the first 1000 items sold is \$39.20, while the profit on the 1001st item is −\$20.00.

3.6.47 Since 1925, the population appears to have grown most slowly in about 1935. Using a straight edge and drawing the (approximate) tangent line, we obtain the following figure. The tangent line appears to pass through the points $(5, 100)$ and $(50, 140)$. Therefore the slope of the tangent line is approximately $\dfrac{(140 - 100)}{(50 - 5)} = 0.89$. So the population was growing at about 890,000 people per year in 1935 (answers will vary — between about 700,000 and 1,000,000 people per year are acceptable).



3.6.48

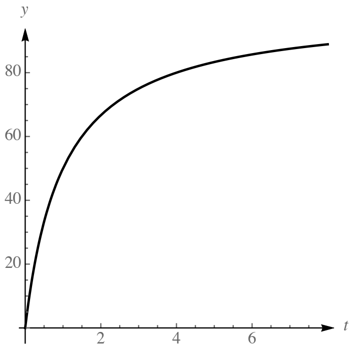


a.

b. The peak of $A(L) = \dfrac{P(L)}{L} = -L^2 + 10L + 200$ occurs when the slope is zero. Note that $P'(L) = -3L^2 + 20L + 200$.

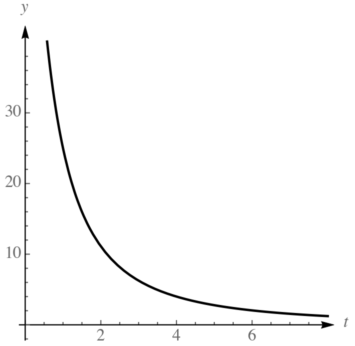
We seek L_0 so that $\dfrac{dA}{dL}(L_0) = 0$, which occurs when $\dfrac{L_0 \cdot P'(L_0) - P(L_0)}{L_0^2} = 0$, or when $P'(L_0) = \dfrac{P(L_0)}{L_0} = A(L_0)$. Thus if the peak of A occurs at L_0, we have $M(L_0) = A(L_0)$.

3.6.49



a.

b. $v(t) = s'(t) = \dfrac{(t+1)100 - 100t \cdot 1}{(t+1)^2} = \dfrac{100}{(t+1)^2}$.



c.

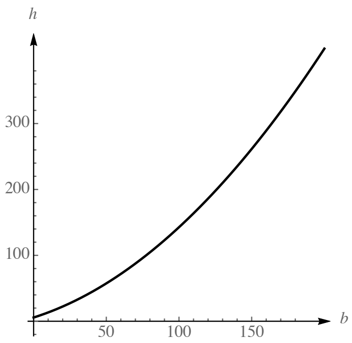
The velocity of the marble is decreasing.

d. $s(t) = 80$ when $\dfrac{100t}{t+1} = 80$, or $100t = 80t + 80$, which occurs when $t = 4$ seconds.

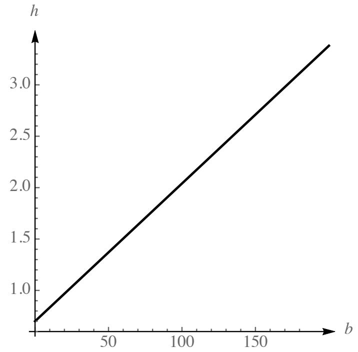
e. $v(t) = 50$ when $\dfrac{100}{(t+1)^2} = 50$, or $(t+1)^2 = 2$. This occurs for $t = \sqrt{2} - 1 \approx 0.414$ seconds.

3.6.50

a.



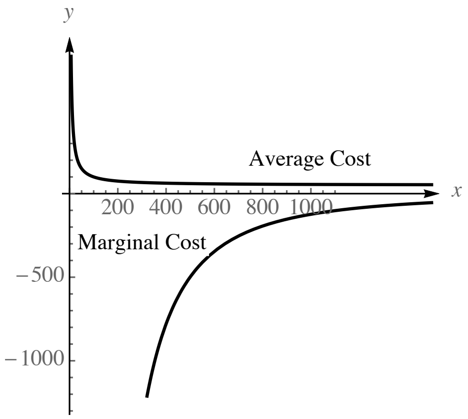
b.



The function $\frac{dh}{db}$ shows the rate of increase in height (in meters) per cm increase in the base diameter of the tree.

3.6.51

a. The average cost function is $\overline{C}(x) = \frac{C(x)}{25000} = 50 + \frac{5000}{x} + 0.00006x$. The marginal cost function is $C'(x) = -\frac{125000000}{x^2} + 1.5$.
The average cost decreases to about 50 per unit as the batch size increases, while the marginal cost is negative but increases.

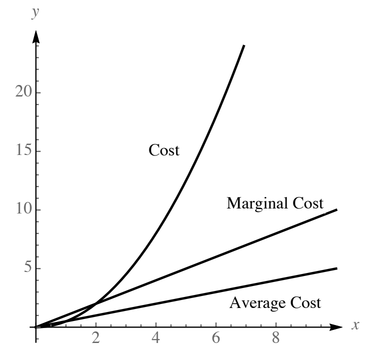


b. $\overline{C}(5000) = 51.3$, $C'(5000) = -3.5$.

c. If the batch size is 5000, then the average cost of producing 25000 items is $51.30 per item. If the batch size is increased from 5000 to 5001, then the cost of producing 25000 items would decrease by about $3.50.

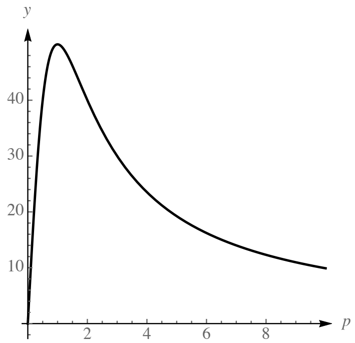
3.6.52

All these functions are increasing, so the cost per item increases as more items are produced. And we will have less revenue unless we charge more.



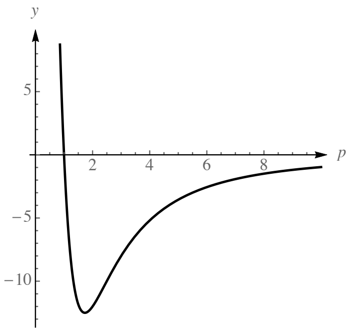
3.6.53

a.



$$R(p) = \frac{100p}{p^2+1}$$

b.

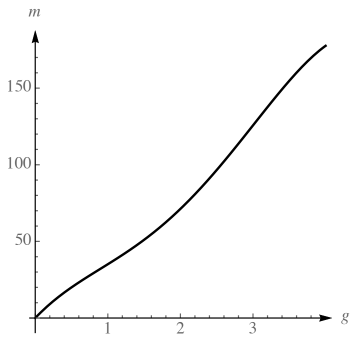


$$R'(p) = \frac{100(1-p^2)}{(p^2+1)^2}$$

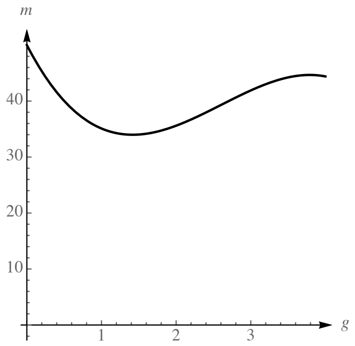
c. $R'(p)$ is zero at $p = 1$, and the maximum of $R(p)$ occurs at this same value of p, so that is the price to charge in order to maximize revenue. The revenue at this price is \$50.00.

3.6.54

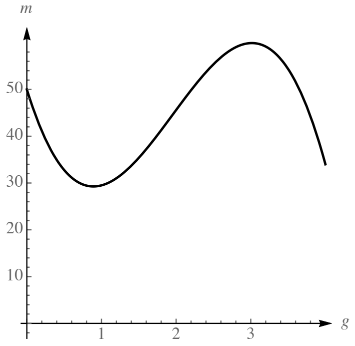
a. The number of miles increases with the number of gallons of gasoline.



b. The gas mileage is $m(g)/g$. The number of miles per gallon decreases during the first 1.5 gallons or so, then increases until it peaks again just short of 4 gallons.

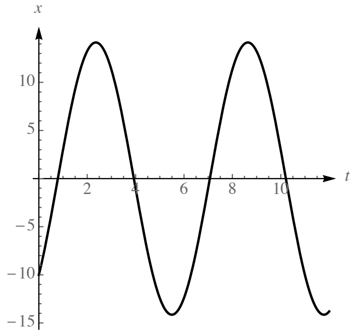


c. $\frac{dm}{dg}$ represents the instantaneous rate of change of the number of miles driven per unit of gasoline consumed.



3.6.55

a. The mass oscillates about the equilibrium point.



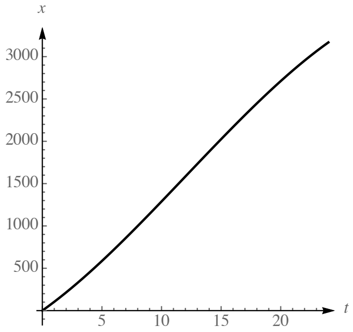
b. $\frac{dx}{dt} = 10\cos t + 10\sin t$ is the velocity of the mass at time t.

c. $\frac{dx}{dt} = 0$ when $\sin t = -\cos t$, which occurs when $t = \frac{4n+3}{4} \cdot \pi$ where n is any positive integer.

d. The model is unrealistic as it ignores the effects of friction and gravity. In reality, the amplitude would decrease as the mass oscillates.

3.6.56

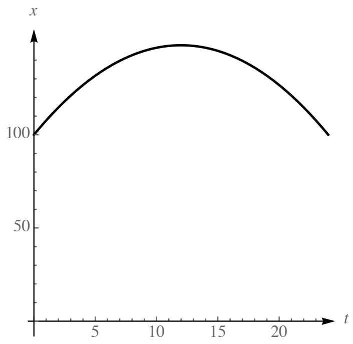
a.



The energy function.

b. $P(t) = E'(t) = 100 + 8t - \frac{t^2}{3}$, $0 \le t \le 24$.

c. The power increases from midnight to noon, then decreases again until midnight. The units are kilowatts.



The power function.

3.6.57

a. Juan starts out faster, but slows toward the end, while Jean starts slower but increases her speed toward the end.

b. Because both start and finish at the same time, they finish with the same average angular velocity.

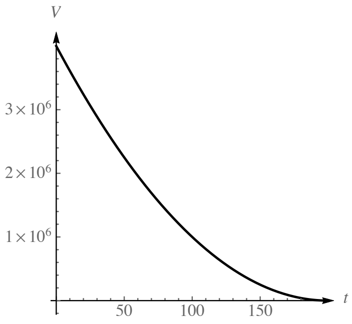
c. It is a tie.

d. Jean's velocity is given by $\theta'(t) = \frac{\pi t}{4}$. At $t = 2$, $\theta'(2) = \frac{\pi}{2}$ radians per minute. Her velocity is greatest at $t = 4$.

e. Juan's velocity is given by $\phi'(t) = \pi - \frac{\pi t}{4}$. At $t = 2$, $\phi'(2) = \frac{\pi}{2}$ radians per minute as well. His velocity is greatest at $t = 0$.

3.6.58

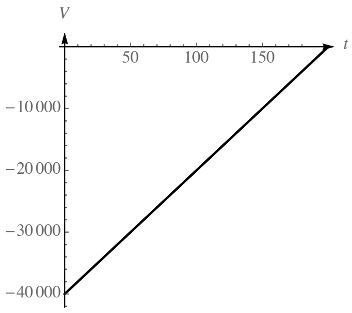
a.



At the beginning the volume is 4,000,000 cubic meters.

b. The tank is empty when $V(t) = 100(200 - t)^2 = 0$, which occurs when $t = 200$.

c.



Because $V(t)$ can be written as $V(t) = 4,000,000 - 40,000t + 100t^2$, the flow rate is $V'(t) = -40,000 + 200t$ cubic meters per minute.

d. The flow rate is largest (in absolute value) when $t = 0$ and smallest when $t = 200$.

3.6.59

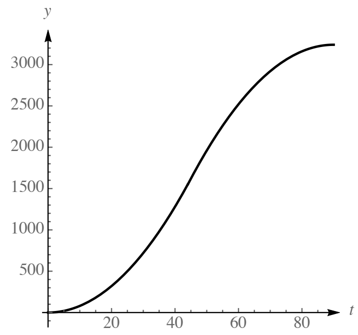
a. $v(t) = y'(t) = -15e^{-t}\cos t - 15e^{-t}\sin t$, so $v(1) \approx -7.625$ meters per second, and $v(3) \approx .63$ meters per second.

b. She is moving down for approximately 2.4 seconds, and then up until about 5.5 seconds, and then down again until about 8.6 seconds, and then up again.

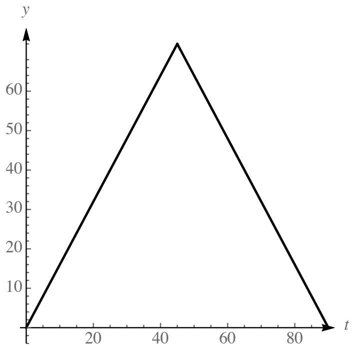
c. The maximum velocity going up appears to be about about 0.65 meters per second.

3.6.60

a.



b.



$$V'(t) = \begin{cases} \frac{8}{5}t & \text{for } 0 \le t \le 45, \\ -\frac{8}{5}t + 144 & \text{for } 45 \le t \le 90. \end{cases} \quad \text{This is in cubic feet per day.}$$

c. The flow increases for the first 45 days, then decreases. The flow rate is at a maximum at 45 days.

3.6.61

a. $T'(t) = 160 - 80x$, so $T'(1) = 80$, so the heat flux at 1 is -80. At $x = 3$ we have $T'(3) = -80$, so the heat flux at 3 is 80.

b. The heat flux $-T'(x)$ is negative for $0 \le x < 2$ and positive for $2 < x \le 4$.

c. At any point other than the midpoint of the rod, heat flows toward the closest end of the rod, and "out the end."

3.7 The Chain Rule

3.7.1 If $y = f(x)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$. Alternatively, we have $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$.

3.7.2 The inner function is $x^2 + 10$ and the outer function is u^{-5}, so with $y = f(u)$ and $u = g(x)$, we have $f(u) = u^{-5}$ and $g(x) = x^2 + 10$. Then $y = (x^2 + 10)^{-5}$.

3.7.3 The inner function is $u = x^3 + x + 1$ and the outer function is $y = u^4$. So

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u^3(3x^2 + 1) = 4(x^3 + x + 1)^3(3x^2 + 1).$$

3.7.4 The inner function is $u = x^3 + 2x$ and the outer function is $y = e^u$. So

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u(3x^2 + 2) = e^{x^3+2x}(3x^2 + 2).$$

3.7.5

a. $u = g(x) = \cos x$, $y = f(u) = u^3$. So $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = 3\cos^2 x \cdot (-\sin x) = -3\cos^2 x \sin x$.

b. $u = g(x) = x^3$, $y = f(u) = \cos u$. So $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = -\sin x^3 \cdot 3x^2 = -3x^2 \sin x^3$.

3.7.6 We would need to know $f'(3)$. This is because $h'(1) = f'(g(1))g'(1) = f'(3) \cdot 5$, but we can't finish this calculation unless we know $f'(3)$.

3.7.7 The derivative of $f(g(x))$ equals f' evaluated at $g(x)$ multiplied by g' evaluated at x.

3.7.8 Let $g(x) = x^2 + 2x + 1$ and $f(x) = x^2$. Then $f(g(x)) = (x^2 + 2x + 1)^2$. We have $\dfrac{d}{dx} f(g(x)) = f'(g(x))g'(x) = 2(g(x))(2x + 2) = 2(x^2 + 2x + 1)(2x + 2) = 4(x^2 + 2x + 1)(x + 1) = 4(x + 1)^3$.

3.7.9 Let $g(x) = 4x + 1$ and $f(x) = \sqrt{x}$ so that $f(g(x)) = \sqrt{4x + 1}$. Then $\dfrac{d}{dx} f(g(x)) = f'(g(x))g'(x) = \dfrac{1}{2\sqrt{g(x)}} \cdot 4 = \dfrac{2}{\sqrt{4x + 1}}$.

3.7.10 Let $h(x) = x^2 + 1$, $g(u) = \cos u$, and $f(v) = v^4$. Then $f(g(h(x))) = f(g(x^2 + 1)) = f(\cos(x^2 + 1)) = (\cos(x^2 + 1))^4 = Q(x)$.

3.7.11 $h'(3) = f'(g(3))g'(3) = f'(4)g'(3) = 10 \cdot 5 = 50$. Note that the value of $f(4)$ isn't needed to calculate this.

3.7.12 $h'(4) = f'(g(4))g'(4) = f'(5)g'(4) = -\dfrac{1}{2} \cdot \dfrac{1}{4} = -\dfrac{1}{8}$.

3.7.13 $y' = e^{kx} \cdot k = ke^{kx}$.

3.7.14 $f'(x) = 15e^{3x} \cdot 3 = 45e^{3x}$.

3.7.15 With $u = 3x + 7$ and $y = u^{10}$ we have $\dfrac{dy}{dx} = \dfrac{dy}{dx} \cdot \dfrac{du}{dx} = 10u^9 \cdot 3 = 30(3x + 7)^9$.

3.7.16 With $u = 5x^2 + 11x$ and $y = u^{4/3}$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \dfrac{4}{3} u^{1/3} \cdot (10x + 11) = \dfrac{4}{3} \sqrt[3]{5x^2 + 11x}(10x + 11)$.

3.7.17 With $u = \sin x$ and $y = u^5$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = 5u^4 \cdot \cos x = 5\sin^4 x \cos x$.

3.7.18 With $u = x^5$ and $y = \sin u$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \cos u \cdot 5x^4 = 5x^4 \cos x^5$.

3.7.19 With $u = x^2 + 1$ and $y = \sqrt{u}$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \dfrac{1}{2\sqrt{u}} \cdot (2x) = \dfrac{x}{\sqrt{x^2 + 1}}$.

3.7.20 With $u = 7x - 1$ and $y = \sqrt{u}$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \dfrac{du}{dx} = \dfrac{1}{2\sqrt{u}} \cdot 7 = \dfrac{7}{2\sqrt{7x - 1}}$.

3.7.21 With $u = 4x^2 + 1$ and $y = e^u$, we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \dfrac{du}{dx} = e^u \cdot 8x = 8xe^{4x^2+1}$.

3.7.22 With $u = \sqrt{x}$ and $y = e^u$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = e^u \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$.

3.7.23 With $u = 5x^2$ and $y = \tan u$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \sec^2 u \cdot (10x) = 10x \sec^2 5x^2$.

3.7.24 With $u = x/4$ and $y = \sin u$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \cos u \cdot (1/4) = \dfrac{1}{4} \cdot \cos(x/4)$.

3.7.25

a. $h'(3) = f'(g(3))g'(3) = f'(1) \cdot 20 = 5 \cdot 20 = 100$.

b. $h'(2) = f'(g(2))g'(2) = f'(5) \cdot 10 = -10 \cdot 10 = -100$.

c. $p'(4) = g'(f(4))f'(4) = g'(1) \cdot (-8) = 2 \cdot (-8) = -16$.

d. $p'(2) = g'(f(2))f'(2) = g'(3) \cdot 2 = 20 \cdot 2 = 40$.

e. $h'(5) = f'(g(5))g'(5) = f'(2) \cdot 20 = 2 \cdot 20 = 40$.

3.7.26

a. $h'(1) = f'(g(1))g'(1) = f'(4) \cdot 9 = 7 \cdot 9 = 63$.

b. $h'(2) = f'(g(2))g'(2) = f'(1) \cdot 7 = (-6) \cdot 7 = -42$.

c. $h'(3) = f'(g(3))g'(3) = f'(5) \cdot 3 = 2 \cdot 3 = 6$.

d. $k'(3) = g'(g(3))g'(3) = g'(5) \cdot 3 = (-5) \cdot 3 = -15$.

e. $k'(1) = g'(g(1))g'(1) = g'(4) \cdot 9 = (-1) \cdot 9 = -9$.

f. $k'(5) = g'(g(5))g'(5) = g'(3) \cdot (-5) = 3 \cdot (-5) = -15$.

3.7.27 With $g(x) = 3x^2 + 7x$ and $f(u) = u^{10}$ we have $\dfrac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = 10(3x^2 + 7x)^9(6x + 7)$.

3.7.28 With $g(x) = x^2 + 2x + 7$ and $f(u) = u^8$, we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = 8(x^2 + 2x + 7)^7(2x + 2) = 16(x^2 + 2x + 7)^7(x + 1).$$

3.7.29 With $g(x) = 10x + 1$ and $f(u) = \sqrt{u}$, we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \frac{1}{2\sqrt{10x + 1}} \cdot 10 = \frac{5}{\sqrt{10x + 1}}.$$

3.7.30 With $g(x) = x^2 + 9$ and $f(u) = \sqrt[3]{u}$ we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \frac{1}{3u^{2/3}} \cdot (2x) = \frac{2x}{3(x^2 + 9)^{2/3}}.$$

3.7.31 With $g(x) = 7x^3 + 1$ and $f(u) = 5u^{-3}$ we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = -15(7x^3 + 1)^{-4}(21x^2) = -315(7x^3 + 1)^{-4} \cdot x^2.$$

3.7.32 With $g(t) = 5t$ and $f(u) = \cos u$ we have $\dfrac{d}{dt}[f(g(t))] = f'(g(t))g'(t) = -\sin 5t \cdot 5 = -5\sin 5t$.

3.7.33 With $g(x) = 3x + 1$ and $f(u) = \sec u$, we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \sec(3x + 1)\tan(3x + 1) \cdot 3 = 3\sec(3x + 1)\tan(3x + 1).$$

3.7.34 With $g(x) = e^x$ and $f(u) = \csc u$, we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = -(\csc e^x \cot e^x)e^x = -e^x \csc e^x \cot e^x.$$

3.7.35 With $g(x) = e^x$ and $f(u) = \tan u$ we have $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \sec^2 u \cdot e^x = e^x \sec^2 e^x$.

3.7.36 With $g(t) = \tan t$ and $f(u) = e^u$ we have $\frac{d}{dt}[f(g(t))] = f'(g(t))g'(t) = e^{\tan t} \cdot \sec^2 t$.

3.7.37 With $g(x) = 4x^3 + 3x + 1$ and $f(u) = \sin u$ we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \cos u \cdot (12x^2 + 3) = (12x^2 + 3) \cdot \cos(4x^3 + 3x + 1).$$

3.7.38 With $g(t) = t^2 + t$ and $f(u) = \csc u$ we have

$$\frac{d}{dt}[f(g(t))] = f'(g(t))g'(t) = -(\csc u)(\cot u) \cdot (2t + 1) = -(2t + 1)\csc(t^2 + t)\cot(t^2 + t).$$

3.7.39 With $g(x) = 5x + 1$ and $f(u) = x^{2/3}$ we have

$$\frac{d}{dx}(f(g(x)) = f'(g(x)g'(x) = \frac{2}{3}(5x + 1)^{-\frac{1}{3}} \cdot 5 = \frac{10}{3(5x + 1)^{\frac{1}{3}}}.$$

3.7.40 We use the product rule and the chain rule when calculating the derivative of $y = (x + 1)^{1/3}$.

$$y' = 1 \cdot (x + 1)^{1/3} + x \cdot (1/3)(x + 1)^{-2/3} = \frac{3(x + 1)}{3(x + 1)^{2/3}} + \frac{x}{3(x + 1)^{2/3}} = \frac{4x + 3}{3(x + 1)^{2/3}}.$$

3.7.41 With $u = \frac{2x}{4x - 3}$ and $f(u) = u^{1/4}$ we have

$$\frac{dy}{dx} = \frac{1}{4}\left(\frac{2x}{4x - 3}\right)^{-\frac{3}{4}} \cdot \left(\frac{2(4x - 3) - 2x \cdot 4}{(4x - 3)^2}\right) = -\frac{3}{2}\left(\frac{4x - 3}{2x}\right)^{\frac{3}{4}} \cdot \frac{1}{(4x - 3)^2} = -\frac{3}{2^{7/4}x^{3/4}(4x - 3)^{5/4}}.$$

3.7.42 First note that $\frac{dy}{d\theta} = \frac{d}{d\theta}\left(\cos^4 \theta\right) + \frac{d}{d\theta}\left(\sin^4 \theta\right)$. To compute the first term, let $g_1(\theta) = \cos \theta$ and $f(u) = u^4$. Then $\frac{d}{d\theta}\cos^4 \theta = 4\cos^3(\theta)(-\sin \theta) = -4\sin \theta \cos^3 \theta$.

Similarly, to compute the second term, let $g_2(\theta) = \sin \theta$ and $f(u) = u^4$. Then $\frac{d}{d\theta}\sin^4 \theta = 4\sin^3 \theta \cos \theta = 4\cos \theta \sin^3 \theta$. Thus, $\frac{dy}{d\theta} = -4\sin \theta \cos^3 \theta + 4\cos \theta \sin^3 \theta$. This can be further simplified to $4\cos \theta \sin \theta(\sin^2 \theta - \cos^2 \theta) = 2\sin 2\theta(-\cos 2\theta) = -\sin 4\theta$.

3.7.43 With $g(x) = \sec x + \tan x$ and $f(u) = u^5$ we have $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = 5u^4 \cdot (\sec x \tan x + \sec^2 x) = 5(\sec x + \tan x)^4(\sec x \tan x + \sec^2 x) = 5\sec x(\sec x + \tan x)^5$.

3.7.44 With $g(z) = 4\cos z$ and $f(u) = \sin u$ we have

$$\frac{dy}{dz} = f'(g(z))g'(z) = \cos(4\cos z) \cdot (-4\sin z) = -4\sin z \cos(4\cos z).$$

3.7.45 Take $g(x) = 2x^6 - 3x^3 + 3$, and $n = 25$. Then $y' = n(g(x))^{n-1}g'(x) = 25(2x^6 - 3x^3 + 3)^{24}(12x^5 - 9x^2)$.

3.7.46 Take $g(x) = \cos x + 2\sin x$, and $n = 8$. Then $y' = n(g(x))^{n-1}g'(x) = 8(\cos x + 2\sin x)^7(2\cos x - \sin x)$.

3.7.47 Take $g(x) = 1 + 2\tan u$, and $n = 4.5$. Then

$$y' = n(g(x))^{n-1}g'(x) = 4.5(1 + 2\tan u)^{3.5}(2\sec^2 u) = 9(1 + 2\tan u)^{3.5}\sec^2 u.$$

3.7.48 Take $g(x) = 1 - e^x$, and $n = 4$. Then $y' = n(g(x))^{n-1}g'(x) = 4(1 - e^x)^3(-e^x) = -4e^x(1 - e^x)^3$.

3.7.49

$$\begin{aligned}\frac{d}{dx}\sqrt{1+\cot^2 x} &= \frac{1}{2\sqrt{1+\cot^2 x}}\cdot\frac{d}{dx}(1+\cot^2 x) = \frac{1}{2\sqrt{1+\cot^2 x}}\cdot 2\cot x\cdot\frac{d}{dx}\cot x\\ &= \frac{1}{2\sqrt{1+\cot^2 x}}\cdot 2\cot x\cdot(-\csc^2 x) = -\frac{\cot x\csc^2 x}{\sqrt{1+\cot^2 x}}.\end{aligned}$$

3.7.50 $g'(x) = \dfrac{e^{3x} - x3e^{3x}}{e^{6x}} = \dfrac{e^{3x}(1-3x)}{e^{6x}} = \dfrac{1-3x}{e^{3x}}$.

3.7.51 Note that $\dfrac{d}{dx}e^{-x} = -e^{-x}$. Then we have $y' = \dfrac{2e^x - 3e^{-x}}{3}$.

3.7.52 Using the product rule, we have $f'(x) = e^{7x} + xe^{7x}\cdot 7 = e^{7x}(1+7x)$.

3.7.53

$$\begin{aligned}\frac{d}{dx}\sin(\sin(e^x)) &= \cos(\sin(e^x))\frac{d}{dx}\sin(e^x)\\ &= \cos(\sin(e^x))\cdot\cos(e^x)\cdot e^x\end{aligned}$$

3.7.54

$$\begin{aligned}\frac{d}{dx}\sin^2(e^{3x+1}) &= 2\sin(e^{3x+1})\frac{d}{dx}\sin(e^{3x+1})\\ &= 2\sin(e^{3x+1})\cos(e^{3x+1})\frac{d}{dx}e^{3x+1}\\ &= 2\sin(e^{3x+1})\cos(e^{3x+1})e^{3x+1}\cdot 3\\ &= 3e^{3x+1}\sin(2e^{3x+1})\end{aligned}$$

3.7.55

$$\begin{aligned}\frac{d}{dx}\sin^5(\cos 3x) &= 5\sin^4(\cos 3x)\cdot\frac{d}{dx}(\sin(\cos 3x))\\ &= 5\sin^4(\cos 3x)\cdot\cos(\cos 3x)\cdot\frac{d}{dx}\cos 3x\\ &= 5\sin^4(\cos 3x)\cdot\cos(\cos 3x)\cdot(-\sin 3x)\cdot 3\\ &= -15\sin^4(\cos 3x)\cos(\cos 3x)\sin 3x.\end{aligned}$$

3.7.56

$$\begin{aligned}\frac{d}{dx}\cos^{7/4}(4x^3) &= \frac{7}{4}\cos^{3/4}(4x^3)\cdot\frac{d}{dx}\cos(4x^3) = \frac{7}{4}\cos^{3/4}(4x^3)(-\sin(4x^3))\cdot\frac{d}{dx}(4x^3)\\ &= \frac{7}{4}\cos^{3/4}(4x^3)(-\sin(4x^3))\cdot 12x^2 = -21x^2\cos^{3/4}(4x^3)\sin(4x^3).\end{aligned}$$

3.7.57

$$\begin{aligned}\frac{d}{dt}\left(\frac{e^{2t}}{1+e^{2t}}\right) &= \frac{(1+e^{2t})2e^{2t} - e^{2t}\cdot 2e^{2t}}{(1+e^{2t})^2}\\ &= \frac{2e^{2t}(1+e^{2t}-e^{2t})}{(1+e^{2t})^2}\\ &= \frac{2e^{2t}}{(1+e^{2t})^2}.\end{aligned}$$

3.7.58

$$\begin{aligned}\frac{d}{dx}(1-e^{-0.05x})^{-1} &= -\frac{1}{(1-e^{-0.05x})^2}\cdot\frac{d}{dx}(1-e^{-0.05x})\\ &= -\frac{1}{(1-e^{-0.05x})^2}\cdot(0.05e^{-0.05x}) = -\frac{0.05e^{-0.05x}}{(1-e^{-0.05x})^2}.\end{aligned}$$

3.7.59 $\frac{d}{dx}\sqrt{x+\sqrt{x}} = \frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \frac{d}{dx}(x+\sqrt{x}) = \frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \left(1+\frac{1}{2\sqrt{x}}\right)$.

3.7.60

$$\begin{aligned}\frac{d}{dx}\sqrt{x+\sqrt{x+\sqrt{x}}} &= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \frac{d}{dx}(x+\sqrt{x+\sqrt{x}})\\ &= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \left(1+\frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \left(1+\frac{1}{2\sqrt{x}}\right)\right).\end{aligned}$$

Note that on the last step, we used the result of the previous problem.

3.7.61 $\frac{d}{dx}f(g(x^2)) = f'(g(x^2)) \cdot \frac{d}{dx}(g(x^2)) = f'(g(x^2)) \cdot g'(x^2) \cdot 2x$.

3.7.62 $\frac{d}{dx}[f(g(x^m))]^n = n[f(g(x^m))]^{n-1}f'(g(x^m))g'(x^m)(mx^{m-1})$.

3.7.63 $y' = 5\left(\frac{x}{x+1}\right)^4 \cdot \frac{(x+1)(1)-x(1)}{(x+1)^2} = \frac{5x^4}{(x+1)^6}$.

3.7.64 $y' = 8\left(\frac{e^x}{x+1}\right)^7 \cdot \frac{(x+1)e^x-e^x}{(x+1)^2} = \frac{8xe^{8x}}{(x+1)^9}$.

3.7.65 $y' = e^{x^2+1}(2x)\sin x^3 + e^{x^2+1}(\cos x^3)3x^2 = xe^{x^2+1}(2\sin x^3 + 3x\cos x^3)$.

3.7.66 $y' = \sec^2(xe^x)((1)e^x + xe^x) = e^x(1+x)\sec^2(xe^x)$.

3.7.67 $\frac{dy}{d\theta} = 2\theta\sec 5\theta + \theta^2(5\sec 5\theta\tan 5\theta) = \theta\sec 5\theta(2+5\theta\tan 5\theta)$.

3.7.68 $5\left(\frac{3x}{4x+2}\right)^4 \cdot \frac{(4x+2)3-(3x)4}{(4x+2)^2} = 5\left(\frac{3x}{4x+2}\right)^4 \cdot \frac{6}{(4x+2)^2} = \frac{5(3x)^4}{(4x+2)^4} \cdot \frac{6}{(4x+2)^2} = \frac{2430x^4}{(4x+2)^6}$.

3.7.69 $y' = 4((x+2)(x^2+1))^3 \cdot ((1)(x^2+1) + (x+2)(2x)) = 4((x+2)(x^2+1))^3(3x^2+4x+1) =$
$4(x+2)^2(x^2+1)^3(3x+1)(x+1)$.

3.7.70

$$\begin{aligned}y' &= 2e^{2x}(2x-7)^5 + e^{2x}(5(2x-7)^4 2) = e^{2x}(2x-7)^4(2(2x-7)+10)\\ &= e^{2x}(2x-7)^4(4x-4) = 4e^{2x}(2x-7)^4(x-1).\end{aligned}$$

3.7.71 $y' = \frac{1}{5}(x^4+\cos 2x)^{-4/5}(4x^3-2\sin 2x) = \frac{4x^3-2\sin 2x}{5(x^4+\cos 2x)^{4/5}}$.

3.7.72 $y' = \frac{(t+1)(1\cdot e^t + te^t) - te^t\cdot 1}{(t+1)^2} = \frac{te^t + t^2e^t + e^t + te^t - te^t}{(t+1)^2} = \frac{e^t(t^2+t+1)}{(t+1)^2}$.

3.7.73

$$\begin{aligned}y' &= 2(p+3)^1\sin p^2 + (p+3)^2(\cos p^2)(2p) = (p+3)(2\sin p^2 + 2p^2\cos p^2 + 6p\cos p^2)\\ &= 2(p+3)(\sin p^2 + p^2\cos p^2 + 3p\cos p^2).\end{aligned}$$

3.7.74 $y' = 1.75(2z+5)^{0.75}(2)\tan z + (2z+5)^{1.75}\sec^2 z = (2z+5)^{0.75}(3.5\tan z + (2z+5)\sec^2 z)$.

3.7.75 $\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}} \cdot f'(x)$.

3.7.76 $\frac{d}{dx}\sqrt[5]{f(x)g(x)} = \frac{1}{5(f(x)g(x))^{4/5}} \cdot \frac{d}{dx}(f(x)g(x)) = \frac{f'(x)g(x) + f(x)g'(x)}{5(f(x)g(x))^{4/5}}$.

3.7.77

a. True. The product rule alone will suffice.

b. True. This function is the composition of e^x with $\sqrt{x+1}$.

c. True. The derivative of the composition $f(g(x))$ is the product of $f'(g(x))$ with $g'(x)$, so it is the product of two derivatives.

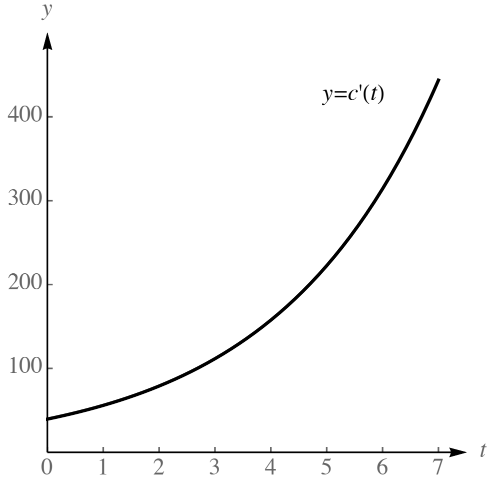
d. False. In fact, $\frac{d}{dx}P(Q(x)) = P'(Q(x))Q'(x)$.

3.7.78

a. The average growth rate between 2007 and 2009 was $\frac{c(2) - c(0)}{2 - 0} \approx 57.1$ million smartphones sold per year. Between 2012 and 2014 it was $\frac{c(7) - c(5)}{7 - 5} \approx 320.4$ million smartphones sold per year. Therefore the average growth rate was much larger between 2012 and 2014.

b. The growth rate in 2008 was $c'(1) \approx 56$ million smartphones sold per year and the growth rate in 2013 was $c'(6) \approx 314.1$ million smartphones sold per year. The growth rate was much larger in 2013.

c.



The growth rate is positive and increasing from 2007 to 2014.

3.7.79 Note that $a(70) = 13330$.

$$\frac{d}{dt}\, p(a(t))\big|_{t=70} = p'(a(70))a'(70) \approx \frac{738 - 765}{14330 - 13330} \cdot \frac{13440 - 13330}{80 - 70} = -0.297\,\text{hPa per minute.}$$

3.7.80

a. $\frac{dT}{dt} = \frac{dT}{dA}\frac{dA}{dt} \approx 6.5 \cdot \frac{2.5 - 2.1}{2 - 1.5} = 6.5 \cdot .8 = 5.2$ degrees per hour. The temperature is dropping at about 5.2 degrees per hour.

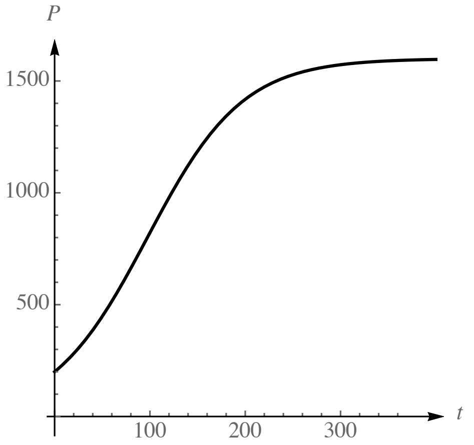
b. An increase in lapse rate would increase the answer to part (a).

c. No. The calculation depends on the lapse rate and on the rate at which the balloon is ascending, but not on the actual temperature.

3.7.81 $m'(t) = 64e^{0.004t} \cdot (0.004) = 0.256e^{0.004t}$. $m'(65) = 0.256e^{0.26} \approx 0.33$. 65 days after the diet switch, the mass of the tortoise is increasing at about 1/3 of a gram per day.

3.7.82

a.

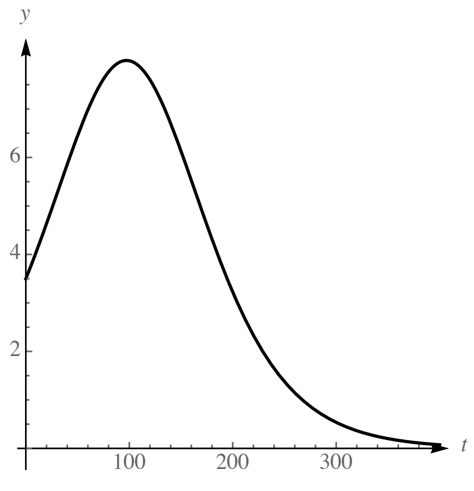


b. The average growth rate for the first ten days is $\dfrac{P(10)-P(0)}{10} \approx \dfrac{237.7-200}{10} = 3.77$ cells per day.

c. The maximum growth rate is where the curve $P(t)$ is the steepest, which appears to be at just shy of 100 days.

d. $P'(t) = \dfrac{0 - 1600 \cdot (-.14e^{-.02t})}{(1+7e^{-.02t})^2} = \dfrac{224e^{-.02t}}{(1+7e^{-.02t})^2}$.

e. At 100 days the populations is a little larger than 800. By doing a little bit of zooming, we can see that the maximum occurs at about $t = 97.3$ days with a population of 800.



3.7.83

a. After 10 years we will have $A(10) = 200e^{0.0398} \approx \$297.77$.

b. The growth rate is $A'(t) = 200 \cdot (0.0398e^{0.0398t}) = 7.96e^{0.0398t}$. After 10 years, the growth rate is $A'(10) = 7.96e^{0.0398} \approx 11.85$ dollars/year.

c. The tangent line is given by $y - 297.77 = 11.85(t-10)$, or $y = 11.85t + 179.27$.

3.7.84

a. $p(10) = 1000e^{-1} \approx 368$ mb, so the pressure on Mt. Everest is about 632 mb less than at sea level.

b. The average pressure change is $\dfrac{p(5)-p(0)}{5} = \dfrac{1000(e^{-0.5}-1)}{5} \approx -78.7$ mb per km.

c. The rate of change in pressure is $p'(5) = -100e^{-5/10} \approx -60.7$ mb per km.

d. Because $p'(z) = -100e^{-z/10}$, it increases as z increases.

e. $\lim\limits_{z\to\infty} p(z) = 0$ means that if we go high enough, there is essentially no atmospheric pressure.

3.7.85

a. The slope is $f'(x) = e^{2x} + 2xe^{2x}$. This is zero when $e^{2x}(1+2x) = 0$, which occurs when $x = -\frac{1}{2}$.

b. The graph of f has a horizontal tangent line at $x = -1/2$.

3.7.86 $\dfrac{d^2}{dx^2}(x\cos(x^2)) = \dfrac{d}{dx}(\cos(x^2) - 2x^2\sin(x^2)) = -2x\sin(x^2) - 4x\sin(x^2) - 4x^3\cos(x^2) = -6x\sin(x^2) - 4x^3\cos(x^2)$.

3.7.87

$$\begin{aligned}\frac{d^2}{dx^2}\sin x^2 &= \frac{d}{dx}(2x\cos x^2) = 2(\cos x^2 - 2x^2\sin x^2)\\ &= 2\cos x^2 - 4x^2\sin x^2.\end{aligned}$$

Note that in the middle of this calculation we used a result from the middle of the previous problem – namely the derivative of $x\cos x^2$.

3.7.88

$$\begin{aligned}\frac{d^2}{dx^2}\sqrt{x^2+2} &= \frac{d}{dx}\left(\frac{1}{2}(x^2+2)^{-1/2}2x\right)\\ &= \frac{d}{dx}\left(\frac{x}{\sqrt{x^2+2}}\right)\\ &= \frac{\sqrt{x^2+2} - x\cdot\left(\frac{x}{\sqrt{x^2+2}}\right)}{x^2+2}\\ &= \frac{\frac{x^2+2-x^2}{\sqrt{x^2+2}}}{x^2+2}\\ &= \frac{2}{(x^2+2)^{3/2}}.\end{aligned}$$

3.7.89 $\dfrac{d^2}{dx^2}e^{-2x^2} = \dfrac{d}{dx}\left(-4xe^{-2x^2}\right) = -4e^{-2x^2} + 16x^2e^{-2x^2} = 4e^{-2x^2}(4x^2-1)$.

3.7.90

a. $\dfrac{d}{dx}(x^2+x)^2 = 2(x^2+x)\cdot\dfrac{d}{dx}(x^2+x) = 2(x^2+x)(2x+1) = 4x^3+6x^2+2x$.

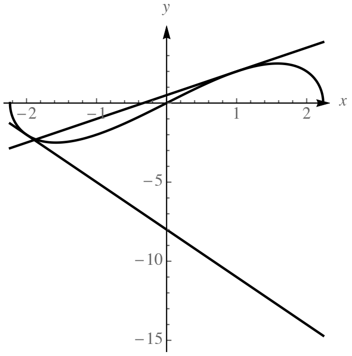
b. $\dfrac{d}{dx}(x^2+x)^2 = \dfrac{d}{dx}(x^4+2x^3+x^2) = 4x^3+6x^2+2x$.

3.7.91 $y' = \dfrac{(x^3-6x-1)(2)(x^2-1)(2x) - (x^2-1)^2(3x^2-6)}{(x^3-6x-1)^2}$,

so $y'(0) = \dfrac{(-1)(2)(-1)(0) - (-1)^2(-6)}{(-1)^2} = 6$. The equation of the tangent line is thus $y+1 = 6(x-0)$, or $y = 6x-1$.

3.7.92

$y' = \sqrt{5-x^2} - \frac{x^2}{\sqrt{5-x^2}}$. Thus we have $y'(1) = 2-(1/2) = 3/2$, and $y'(-2) = 1-(4/1) = -3$. The tangents line we are seeking are $y-2 = (3/2)(x-1)$ and $y+2 = -3(x+2)$, or $y = \frac{3}{2}x + \frac{1}{2}$ and $y = -3x-8$.



3.7.93

a. $g'(4) = 3$, $g(4) = 3 \cdot 4 - 5 = 7$. $f'(7) = -2$, $f(7) = -2 \cdot 7 + 23 = 9$. Thus, $h(4) = f(g(4)) = f(7) = 9$, and $h'(4) = f'(g(4))g'(4) = f'(7) \cdot 3 = -2 \cdot 3 = -6$.

b. The tangent line to h at $(4, 9)$ is given by $y - 9 = -6(x-4)$, or $y = -6x + 33$.

3.7.94

a. $g(1) = f(1^2) = f(1) = 4$.

b. $g'(x) = f'(x^2) \cdot 2x$.

c. Using the previous result, $g'(1) = f'(1) \cdot 2 = 3 \cdot 2 = 6$.

d. The tangent line is given by $y - 4 = 6(x-1)$, or $y = 6x - 2$.

3.7.95 $y'(x) = 2e^{2x}$, so $y'\left(\frac{\ln 3}{2}\right) = 2e^{\ln 3} = 6$. Also, $y\left(\frac{\ln 3}{2}\right) = e^{\ln 3} = 3$. The tangent line is therefore given by $y - 3 = 6\left(x - \frac{\ln 3}{2}\right)$, or $y = 6x + 3 - 3\ln 3$.

3.7.96 First, note that $g'(x) = f'(\sin x) \cdot \cos x$.

a. $g'(0) = f'(0) \cdot \cos 0 = 3 \cdot 1 = 3$.

b. $g'(\pi/2) = f'(1) \cdot \cos(\pi/2) = 5 \cdot 0 = 0$.

c. $g'(\pi) = f'(0) \cdot \cos\pi = 3 \cdot (-1) = -3$.

3.7.97 First, note that $g'(x) = \cos(\pi f(x)) \cdot \pi f'(x)$.

a. $g'(0) = \cos(\pi \cdot f(0)) \cdot \pi f'(0) = \cos(-3\pi) \cdot 3\pi = -3\pi$.

b. $g'(1) = \cos(\pi \cdot f(1)) \cdot \pi f'(1) = \cos(3\pi) \cdot 5\pi = -5\pi$.

3.7.98

a. $\dfrac{dy}{dt} = -y_0\sqrt{\dfrac{k}{m}}\sin\left(t\sqrt{\dfrac{k}{m}}\right)$.

b. The amplitude of the velocity (which is $y_0\sqrt{\frac{k}{m}}$) would decrease by a factor of 2, and the period would increase by a factor of 2.

c. The amplitude of the velocity would increase by a factor of 2, and the period would decrease by a factor of 2.

d. The units for $-y_0\sqrt{\frac{k}{m}}$ would be meters $\cdot \sqrt{\frac{\text{kg/sec}^2}{\text{kg}}} = \frac{\text{meters}}{\text{sec}}$. Inside the sine function the units for $t \cdot \sqrt{\frac{k}{m}}$ are sec $\cdot \frac{1}{\text{sec}} = 1$, so the factor involving the sine function is unit-less (as it should be).

3.7.99

a. $\dfrac{d^2y}{dt^2} = \dfrac{d}{dt}\left(-y_0\sqrt{\dfrac{k}{m}}\sin\left(t\sqrt{\dfrac{k}{m}}\right)\right) = -y_0 \cdot \dfrac{k}{m} \cdot \cos\left(t\sqrt{\dfrac{k}{m}}\right)$.

b. $-\frac{k}{m}y = -\frac{k}{m}\left(y_0\cos\left(t\sqrt{\frac{k}{m}}\right)\right) = \frac{d^2y}{dt^2}$.

3.7.100

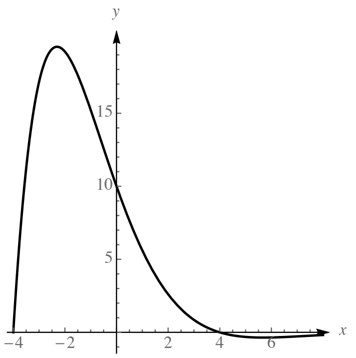
a. The period of $\cos x$ is 2π. The period of a function of the form $y = a\cos bx$ is $\dfrac{2\pi}{b}$. Thus, the period of y is $\dfrac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\dfrac{m}{k}}$.

b. $\dfrac{dT}{dm} = \dfrac{d}{dm}\left(2\pi\sqrt{\dfrac{m}{k}}\right) = \dfrac{2\pi}{\sqrt{k}} \cdot \dfrac{1}{2\sqrt{m}} = \dfrac{\pi}{\sqrt{mk}}$.

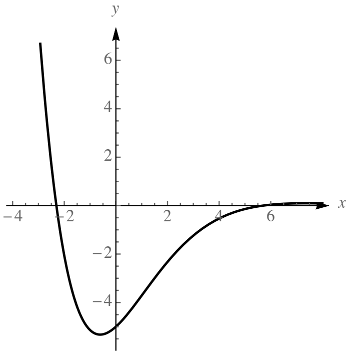
c. Because k and m are greater than 0, and π is greater than 0, this quotient is greater than 0. Physically this means that the period is increasing as mass increases: the oscillations get slower.

3.7.101

a.



b.



$$\frac{dy}{dt} = -5e^{-t/2}\cos\left(\frac{\pi t}{8}\right) - \frac{5\pi}{4}e^{-t/2}\sin\left(\frac{\pi t}{8}\right).$$

c. The velocity is zero at about -2.3 and at about 5.7, and the displacement has a maximum and a minimum at these points.

3.7.102 $\dfrac{dy}{dt} = -e^{-t}(\sin 2t - 2\cos 2t) + e^{-t}(2\cos 2t + 4\sin 2t) = e^{-t}(3\sin 3t + 4\cos 2t)$.

$\dfrac{d^2y}{dt^2} = -e^{-t}(3\sin 2t + 4\cos 2t) + e^{-t}(6\cos 2t - 8\sin 2t) = e^{-t}(-11\sin 2t + 2\cos 2t)$.

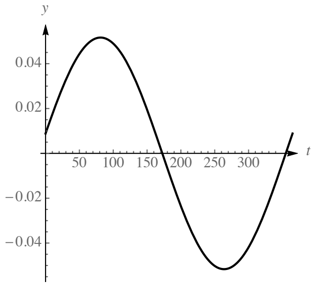
Then $y''(t) + 2y'(t) + 5y(t) = e^{-t}(-11\sin 2t + 2\cos 2t) + e^{-t}(6\sin 2t + 8\cos 2t) + e^{-t}(5\sin 2t - 10\cos 2t) = e^{-t}((-11 + 6 + 5)\sin 2t + (2 + 8 - 10)\cos 2t) = e^{-t}(0 + 0) = 0$, as desired.

3.7.103

a. Assuming a non leap year, March 1st corresponds to $t = 59$. We have $D(59) = 12 - 3\cos\left(\frac{2\pi(69)}{365}\right) \approx$ 10.88 hours.

b. $\dfrac{d}{dt}D(t) = 3 \cdot \dfrac{2\pi}{365} \sin\left(\dfrac{2\pi(t+10)}{365}\right)$ hours per day.

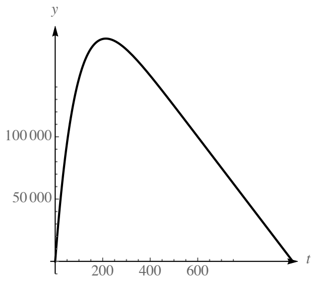
c. $D'(59) \approx 0.048$ hours per day ≈ 2 minutes and 52 seconds per day. This means that on March 1st, the days are getting longer by just shy of 3 minutes per day.



d.

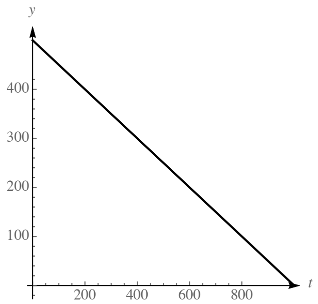
e. The largest increase in the length of the days appears to be at about $t = 81$, and the largest decrease at about $t = 265$. These correspond to March 22nd and to September 22nd. The least rapid changes occur at about $t = 172$ and $t = 355$. These correspond to June 21st and December 21st.

3.7.104



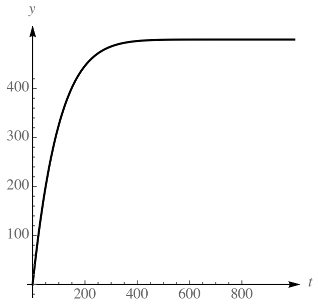
a.

$M(0) = 250(1000)(1 - (10)^{-30} \cdot 10^{30}) = 250,000 \cdot (1-1) = 0$ grams.



b.

$V(1000) = 500 - (.5)(1000) = 500 - 500 = 0.$



c.

$C(0) = \frac{M(0)}{V(0)} = \frac{0}{500} = 0$. $C(1000)$ isn't defined because $V(1000) = 0$, but it appears that $\lim_{t\to1000} C(t) = 500$. The concentration of the salt in the tank increases with time, although it levels off as it nears 500 grams per liter.

d. $M'(t) = 250(1000-t)(10^{-30}(10(1000-t)^9)) - 250(1 - 10^{-30}(1000-t)^{10}) = \frac{250(1-t)^{10}}{10^{29}} - 250\left(1 - \frac{(1000-t)^{10}}{10^{30}}\right)$.

e. It is convenient to rewrite $C(t)$ first. We can rewrite

$$C(t) = \frac{M(t)}{V(t)} = \frac{250(1000-t)}{\frac{1000-t}{2}} \cdot \left(1 - \frac{(1000-t)^{10}}{1000^{10}}\right) = 500 \cdot \left(1 - \left(1 - \frac{t}{1000}\right)^{10}\right).$$

Then $C'(t) = 500 \cdot \left(0 - 10\left(1 - \frac{t}{1000}\right)^9 \cdot -\frac{1}{1000}\right) = 5\left(1 - \frac{t}{1000}\right)^9$.

f. The derivative is positive for $0 \le t \le 1000$, so the concentration is increasing on this interval.

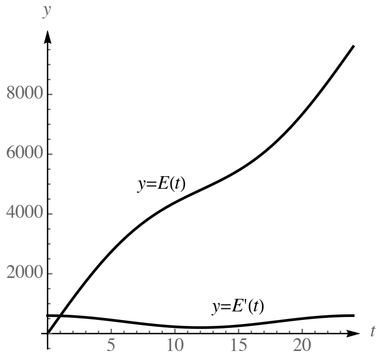
3.7.105

a. $E'(t) = 400 + 200\cos\left(\frac{\pi t}{12}\right)$ MW.

b. Because the maximum value of $\cos\theta$ is 1, the maximum value of $E'(t)$ will be 600 MW, where $\cos\left(\frac{\pi t}{12}\right) = 1$, which is where $t = 0$, which corresponds to noon.

c. Because the minimum value of $\cos\theta$ is -1, the minimum value of $E'(t)$ will be 200 MW, where $\cos\left(\frac{\pi t}{12}\right) = -1$, which is where $\frac{\pi t}{12} = \pi$, or $t = 12$, which corresponds to midnight.

d.



3.7.106

a. $\frac{d}{dt}\cos 2t = -2\sin 2t$, and $\frac{d}{dt}(\cos^2 t - \sin^2 t) = -2\sin t\cos t - 2\sin t\cos t = -4\sin t\cos t$. Thus, $-2\sin 2t = -4\sin t\cos t$, so $\sin 2t = 2\sin t\cos t$.

b. $\frac{d}{dt}(2\cos^2 t - 1) = -4\cos t\sin t$, so again, $\sin 2t = 2\sin t\cos t$.

c. $\frac{d}{dt}(\sin 2t) = 2\cos 2t$, and $\frac{d}{dt}(2\sin t\cos t) = 2\cos t\cos t + 2\sin t(-\sin t) = 2\cos^2 t - 2\sin^2 t$, so $\cos 2t = \cos^2 t - \sin^2 t$.

3.7.107

$$\begin{aligned}\frac{d}{dx}\left(f(x)(g(x))^{-1}\right) &= f'(x)(g(x))^{-1} + f(x)(-(g(x))^{-2}g'(x)) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2}\\ &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.\end{aligned}$$

3.7.108

a.

$$\begin{aligned}\frac{d^2}{dx^2}[f(g(x))] &= \frac{d}{dx}[f'(g(x))g'(x)]\\ &= f''(g(x))g'(x)g'(x) + f'(g(x))g''(x)\\ &= f''(g(x))(g'(x))^2 + f'(g(x))g''(x).\end{aligned}$$

b. Let $g(x) = 3x^4 + 5x^2 + 2$. Then $g'(x) = 12x^3 + 10x$ and $g''(x) = 36x^2 + 10$. Let $f(u) = \sin u$. Then $f'(u) = \cos u$ and $f''(u) = -\sin u$.

We have $\frac{d^2}{dx^2}\sin(3x^4+5x^2+2) = -\sin(3x^4+5x^2+2)\cdot(12x^3+10x)^2+\cos(3x^4+5x^2+2)\cdot(36x^2+10)$.

3.7.109

a. $h(x) = (x^2-3)^5$, $a = 2$.

b. $h'(x) = 5(x^2-3)^4(2x) = 10x(x^2-3)^4$, so the value of this limit is $h'(2) = 20$.

3.7.110

a. $h(x) = \sqrt{4+\sin x}$, $a = 0$.

b. $h'(x) = \frac{1}{2}\cdot(4+\sin x)^{-1/2}\cdot\cos x = \frac{\cos x}{2\sqrt{4+\sin x}}$, so the value of this limit is $h'(0) = \frac{1}{4}$.

3.7.111

a. $h(x) = \sin x^2$, $a = \frac{\pi}{2}$.

b. $h'(x) = (\cos x^2)(2x)$, so the value of this limit is $h'\left(\frac{\pi}{2}\right) = \pi\cdot\cos\left(\frac{\pi^2}{4}\right) \approx -2.45$.

3.7.112

a. $h(x) = \frac{1}{3(x^5+7)^{10}}$, $a = 1$.

b. $h'(x) = -\frac{50x^4}{3(x^5+7)^{11}}$, so the value of this limit is $h'(1) = -\frac{50}{3\cdot 8^{11}} \approx -1.94\times 10^{-9}$.

3.7.113 $\lim_{x\to 5}\frac{f(x^2)-f(25)}{x-5} = \frac{d}{dx}\left[f(x^2)\right]_{x=5} = 2\cdot 5\cdot f'(25) = 10f'(25)$.

3.7.114

a. First note that $\frac{d}{dx}f(-x) = -f'(-x)$.

If f is even then $f(-x) = f(x)$. Because the derivatives of both sides of this equation must be equal, we have $-f'(-x) = f'(x)$ or $f'(-x) = -f'(x)$, so f' must be an odd function.

b. If f is odd then $f(-x) = -f(x)$. Because the derivatives of both sides of this equation must be equal, we have $-f'(-x) = -f'(x)$ or $f'(-x) = f'(x)$, so f' must be an even function.

3.7.115

a. $\lim_{v\to u} H(v) = \lim_{v\to u}\left(\frac{f(v)-f(u)}{v-u} - f'(u)\right) = \lim_{v\to u}\left(\frac{f(v)-f(u)}{v-u}\right) - f'(u) = f'(u) - f'(u) = 0$.

b. Suppose $u = v$. Then clearly both sides of the given expression are 0, so they are equal. Suppose $u \neq v$. Then $H(v) = \frac{f(v)-f(u)}{v-u} - f'(u)$, so $H(v) + f'(u) = \frac{f(v)-f(u)}{v-u}$, so the result holds by multiplying both sides of this equation by $v - u$.

c.

$$\begin{aligned} h'(a) &= \lim_{x\to a}\frac{f(g(x))-f(g(a))}{x-a} = \lim_{x\to a}\frac{H(g(x))+f'(g(a))}{x-a}\cdot(g(x)-g(a)) \\ &= \lim_{x\to a}\left[(H(g(x))+f'(g(a)))\cdot\frac{g(x)-g(a)}{x-a}\right]. \end{aligned}$$

d. $h'(a) = \lim_{x\to a}\left[(H(g(x))+f'(g(a)))\cdot\frac{g(x)-g(a)}{x-a}\right] = (0+f'(g(a)))\cdot g'(a) = f'(g(a))g'(a)$.

3.8 Implicit Differentiation

3.8.1 Implicit differentiation gives a single unified derivative, whereas solving for y explicitly yields two different functions.

3.8.2 In implicit differentiation, the independent and dependent variables may both appear on the same side of an equation, so one must keep track of which is which.

3.8.3 The result of implicit differentiation is often an expression involving both the dependent and independent variables, so one would need to know both in order to calculate the value of the derivative.

3.8.4 The derivative of y^2 with respect to x is not $2y$ as stated in the argument. Instead, $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$. Remember to treat y as a function of x when differentiating terms involving y. With this correction, we have $2y\frac{dy}{dx} + 2\frac{dy}{dx} = 6x^2$, which implies that $\frac{dy}{dx} = \frac{3x^2}{y+1}$.

3.8.5 Differentiating both sides with respect to x gives $1 = 2y\frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{1}{2y}$.

3.8.6 Differentiating both sides with respect to x gives $3 + 12y^2\frac{dy}{dx} = 0$.Thus, $\frac{dy}{dx} = -\frac{3}{12y^2} = -\frac{1}{4y^2}$.

3.8.7 Differentiating both sides with respect to x gives $\cos y \cdot \frac{dy}{dx} + 0 = 1$, so $\frac{dy}{dx} = \frac{1}{\cos y}$.

3.8.8 Differentiating both sides with respect to x gives $e^y\frac{dy}{dx} - e^x = 0$, so $\frac{dy}{dx} = \frac{e^x}{e^y} = e^{x-y}$.

3.8.9

a. When $x = 0$ we have $-y + y^3 = 0$, so $y(y^2 - 1) = 0$, so y can be either 0 or ± 1. So the y-intercepts are $(0, 0)$, $(0, 1)$, and $(0, -1)$.

b. Differentiating both sides with respect to x gives $2 - \frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$. Thus $2 = \frac{dy}{dx} - 3y^2\frac{dy}{dx}$, so $2 = \frac{dy}{dx}\left(1 - 3y^2\right)$ and $\frac{dy}{dx} = \frac{2}{1-3y^2}$.

c. At $(0, 0)$ we have $\frac{dy}{dx} = 2$, at $(0, 1)$ and at $(0, -1)$ we have $\frac{dy}{dx} = \frac{2}{1-3} = -1$.

3.8.10 Differentiating both sides with respect to x gives $2x + 3y^2\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{2x}{3y^2}$. The slope at $(-1, 1)$ is thus $\frac{2}{3}$ and at $(1, 1)$ is $-\frac{2}{3}$.

3.8.11 Differentiating both sides with respect to x gives $1 = 3y^2\frac{dy}{dx}$. Thus $\frac{dy}{dx} = \frac{1}{3y^2} = \frac{1}{3}y^{-2}$. Differentiating both sides again gives $\frac{d^2y}{dx^2} = -\frac{2}{3}y^{-3}\frac{dy}{dx} = -\frac{2}{3}y^{-3} \cdot \frac{1}{3}y^{-2} = -\frac{2}{9y^5}$.

3.8.12 Differentiating both sides with respect to x gives $1 = e^y\frac{dy}{dx}$. Thus $\frac{dy}{dx} = \frac{1}{e^y} = e^{-y}$. Differentiating both sides again gives $\frac{d^2y}{dx^2} = -e^{-y}\frac{dy}{dx} = -e^{-y} \cdot e^{-y} = -e^{-2y}$.

3.8.13

a. $4x^3 + 4y^3\frac{dy}{dx} = 0$. Thus $4y^3\frac{dy}{dx} = -4x^3$, so $\frac{dy}{dx} = -\frac{x^3}{y^3}$.

b. When $x = 1$ and $y = -1$, we have $\frac{dy}{dx} = \frac{-1}{-1} = 1$.

3.8.14

a. $1 = e^y \frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$.

b. When $x = 2$, $\frac{dy}{dx} = \frac{1}{x} = \frac{1}{2}$.

3.8.15

a. $2y\frac{dy}{dx} = 4$, so $\frac{dy}{dx} = \frac{2}{y}$.

b. $\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{2}{2} = 1$.

3.8.16

a. $2y\frac{dy}{dx} + 3 = 0$, so $\frac{dy}{dx} = -\frac{3}{2y}$.

b. $\left.\frac{dy}{dx}\right|_{(1,\sqrt{5})} = -\frac{3}{2\sqrt{5}} = -\frac{3\sqrt{5}}{10}$.

3.8.17

a. $\frac{dy}{dx}\cos y = 20x^3$, so $\frac{dy}{dx} = \frac{20x^3}{\cos y}$.

b. $\left.\frac{dy}{dx}\right|_{(1,\pi)} = \frac{20}{\cos \pi} = -20$.

3.8.18

a. $\frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{y}}\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = \frac{\sqrt{y}}{2\sqrt{x}}$.

b. When $x = 4$ and $y = 1$ we have $\frac{dy}{dx} = \frac{\sqrt{1}}{2\sqrt{4}} = \frac{1}{4}$.

3.8.19

a. $-\frac{dy}{dx}\sin y = 1$, so $\frac{dy}{dx} = -\frac{1}{\sin y} = -\csc y$.

b. $\left.\frac{dy}{dx}\right|_{(0,\pi/2)} = -\csc(\pi/2) = -1$.

3.8.20

a. $(y+x\frac{dy}{dx})\sec^2(xy) = 1+\frac{dy}{dx}$, so $x\frac{dy}{dx}\sec^2(xy) - \frac{dy}{dx} = 1 - y\sec^2(xy)$. Factoring out $\frac{dy}{dx}$ on the left-hand side gives
$\frac{dy}{dx}(x\sec^2(xy) - 1) = 1 - y\sec^2(xy)$, so $\frac{dy}{dx} = \frac{1 - y\sec^2(xy)}{x\sec^2(xy) - 1}$.

b. $\left.\frac{dy}{dx}\right|_{(0,0)} = \frac{1-0}{0-1} = -1$.

3.8.21

a. $1 \cdot y + x\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{y}{x}$

b. $\left.\frac{dy}{dx}\right|_{(1,7)} = -\frac{7}{1} = -7$.

3.8.22

a. We can rewrite the equation as $x = y^2 + 1$. Then $1 = 2y\frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{1}{2y}$.

b. $\left.\dfrac{dy}{dx}\right|_{(10,3)} = \dfrac{1}{6}$.

3.8.23

a. $\dfrac{1}{3}x^{-\frac{2}{3}} + \dfrac{4}{3}y^{\frac{1}{3}}\dfrac{dy}{dx} = 0$, so $\dfrac{4}{3}y^{1/3}\dfrac{dy}{dx} = -\dfrac{1}{3x^{2/3}}$, so $\dfrac{dy}{dx} = -\dfrac{1}{4x^{2/3}y^{1/3}}$.

b. $\left.\dfrac{dy}{dx}\right|_{(1,1)} = -\dfrac{1}{4}$.

3.8.24

a. $\dfrac{2}{3}x^{-\frac{1}{3}} + \dfrac{2}{3}y^{-\frac{1}{3}}\dfrac{dy}{dx} = 0$, so $y^{-1/3}\dfrac{dy}{dx} = -\dfrac{1}{x^{1/3}}$, so $\dfrac{dy}{dx} = -\dfrac{y^{1/3}}{x^{1/3}}$.

b. $\left.\dfrac{dy}{dx}\right|_{(1,1)} = -1$.

3.8.25

a. $y^{1/3} + \dfrac{1}{3}xy^{-2/3}\dfrac{dy}{dx} + \dfrac{dy}{dx} = 0$, so $\dfrac{1}{3}xy^{-2/3}\dfrac{dy}{dx} + \dfrac{dy}{dx} = -y^{1/3}$ and therefore $\dfrac{dy}{dx}\left(\dfrac{1}{3}xy^{-2/3}+1\right) = -y^{1/3}$, so $\dfrac{dy}{dx} = -\dfrac{y^{1/3}}{\frac{1}{3}xy^{-2/3}+1} = -\dfrac{3y}{x+3y^{2/3}}$.

b. $\left.\dfrac{dy}{dx}\right|_{(1,8)} = -\dfrac{24}{13}$.

3.8.26

a. $\dfrac{2}{3}(x+y)^{-\frac{1}{3}}\left(1+\dfrac{dy}{dx}\right) = \dfrac{dy}{dx}$, so $1+\dfrac{dy}{dx} = \dfrac{3}{2}(x+y)^{1/3}\dfrac{dy}{dx}$, and therefore $1 = \dfrac{3}{2}(x+y)^{1/3}\dfrac{dy}{dx} - \dfrac{dy}{dx}$. Then $1 = \left(\dfrac{3}{2}(x+y)^{1/3}-1\right)\dfrac{dy}{dx}$, and so $\dfrac{dy}{dx} = \dfrac{1}{\left(\frac{3}{2}(x+y)^{1/3}-1\right)} = \dfrac{2}{3(x+y)^{1/3}-2}$.

b. $\left.\dfrac{dy}{dx}\right|_{(4,4)} = \dfrac{2}{4} = \dfrac{1}{2}$.

3.8.27 $\cos x + \cos y\dfrac{dy}{dx} = \dfrac{dy}{dx}$, so $\cos x = \dfrac{dy}{dx} - \cos y\dfrac{dy}{dx}$, and therefore $\cos x = \dfrac{dy}{dx}(1-\cos y)$, and thus $\dfrac{dy}{dx} = \dfrac{\cos x}{1-\cos y}$.

3.8.28 $\dfrac{dy}{dx} = 1\cdot e^y + xe^y\dfrac{dy}{dx}$, so $\dfrac{dy}{dx} - xe^y\dfrac{dy}{dx} = e^y$, so $\dfrac{dy}{dx}(1-xe^y) = e^y$, and therefore $\dfrac{dy}{dx} = \dfrac{e^y}{1-xe^y}$. Now by the original equation, $y = xe^y$, so this answer can be written $\dfrac{dy}{dx} = \dfrac{e^y}{1-y}$.

3.8.29 $1+\dfrac{dy}{dx} = -\sin y\cdot\dfrac{dy}{dx}$, so $\dfrac{dy}{dx} + (\sin y)\dfrac{dy}{dx} = -1$, and $\dfrac{dy}{dx} = -\dfrac{1}{1+\sin y}$.

3.8.30 $1+2\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx}$, so $1 = \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} - 2\dfrac{dy}{dx}$, and thus $1 = \dfrac{dy}{dx}\left(\dfrac{1}{2\sqrt{y}}-2\right)$. Because the right-hand side of this equation can be written as $\dfrac{dy}{dx}\left(\dfrac{1-4\sqrt{y}}{2\sqrt{y}}\right)$, we have $\dfrac{dy}{dx} = \dfrac{2\sqrt{y}}{1-4\sqrt{y}}$.

3.8.31 $(y + x\frac{dy}{dx})\cos(xy) = 1 + \frac{dy}{dx}$, so $y\cos(xy) + x\frac{dy}{dx}\cos(xy) = 1 + \frac{dy}{dx}$. If we rearrange terms in order to have the terms with a factor of $\frac{dy}{dx}$ all on the same side, we obtain $y\cos(xy) - 1 = \frac{dy}{dx} - x\frac{dy}{dx}\cos(xy)$. Factoring out the $\frac{dy}{dx}$ factor gives $y\cos(xy) - 1 = \frac{dy}{dx}(1 - x\cos(xy))$, so $\frac{dy}{dx} = \frac{y\cos(xy) - 1}{1 - x\cos(xy)}$.

3.8.32 $(y + x\frac{dy}{dx})e^{xy} = 2\frac{dy}{dx}$, so $ye^{xy} + x\frac{dy}{dx}e^{xy} = 2\frac{dy}{dx}$. We can write this as $ye^{xy} = 2\frac{dy}{dx} - x\frac{dy}{dx}e^{xy}$, and factoring out the factor of $\frac{dy}{dx}$ on the right yields $ye^{xy} = \frac{dy}{dx}(2 - xe^{xy})$. Finally, we can divide to obtain $\frac{dy}{dx} = \frac{ye^{xy}}{2 - xe^{xy}}$.

3.8.33 $-2y\frac{dy}{dx}\sin y^2 + 1 = \frac{dy}{dx}e^y$, which we can write as $1 = \frac{dy}{dx}e^y + 2y\frac{dy}{dx}\sin y^2$, or $1 = \frac{dy}{dx}(e^y + 2y\sin y^2)$. Thus, $\frac{dy}{dx} = \frac{1}{e^y + 2y\sin y^2}$.

3.8.34 $\frac{dy}{dx} = \frac{(y-1) - (x+1)\frac{dy}{dx}}{(y-1)^2}$, which we can write as $(y-1)^2 \cdot \frac{dy}{dx} = y - 1 - (x+1)\frac{dy}{dx}$. If we rearrange terms in order to have terms with a factor of $\frac{dy}{dx}$ on the same side, we obtain $\frac{dy}{dx}(y-1)^2 + \frac{dy}{dx}(x+1) = y - 1$. Factoring out the common factor of $\frac{dy}{dx}$ yields $\frac{dy}{dx}((y-1)^2 + (x+1)) = y - 1$, so $\frac{dy}{dx} = \frac{y-1}{(y-1)^2 + x + 1}$.

3.8.35

$$\begin{aligned}
3x^2 &= \frac{(x-y)(1+\frac{dy}{dx}) - (x+y)(1-\frac{dy}{dx})}{(x-y)^2} \\
3x^2(x-y)^2 &= x + x\frac{dy}{dx} - y - y\frac{dy}{dx} - x + x\frac{dy}{dx} - y + y\frac{dy}{dx} \\
3x^2(x-y)^2 + 2y &= 2x\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{3x^2(x-y)^2 + 2y}{2x}
\end{aligned}$$

3.8.36

$$\begin{aligned}
3(y + x\frac{dy}{dx})(xy+1)^2 &= 1 - 2y\frac{dy}{dx} \\
3x\frac{dy}{dx}(xy+1)^2 + 2y\frac{dy}{dx} &= 1 - 3y(xy+1)^2 \\
\frac{dy}{dx}(3x(xy+1)^2 + 2y) &= 1 - 3y(xy+1)^2 \\
\frac{dy}{dx} &= \frac{1 - 3y(xy+1)^2}{3x(xy+1)^2 + 2y}.
\end{aligned}$$

3.8.37

$$\begin{aligned}
18x^2 + 21\frac{dy}{dx}y^2 &= 13(y + x\frac{dy}{dx}) \\
21\frac{dy}{dx}y^2 - 13x\frac{dy}{dx} &= 13y - 18x^2 \\
\frac{dy}{dx} &= \frac{13y - 18x^2}{21y^2 - 13x}.
\end{aligned}$$

3.8.38

$$\begin{aligned}\cos x\cos y+\sin x(-\sin y)\frac{dy}{dx}&=\cos x+(-\sin y)\frac{dy}{dx}\\ \cos x\cos y-\cos x&=\sin x\sin y\frac{dy}{dx}-\sin y\frac{dy}{dx}\\ \cos x\cos y-\cos x&=(\sin x\sin y-\sin y)\frac{dy}{dx}\\ \frac{dy}{dx}&=\frac{\cos x\cos y-\cos x}{\sin x\sin y-\sin y}.\end{aligned}$$

3.8.39

$$\begin{aligned}\frac{4x^3+2y\frac{dy}{dx}}{2\sqrt{x^4+y^2}}&=5+6y^2\frac{dy}{dx}\\ y\frac{dy}{dx}-6\frac{dy}{dx}y^2\sqrt{x^4+y^2}&=5\sqrt{x^4+y^2}-2x^3\\ \frac{dy}{dx}&=\frac{5\sqrt{x^4+y^2}-2x^3}{y-6y^2\sqrt{x^4+y^2}}.\end{aligned}$$

3.8.40

$$\begin{aligned}\frac{1}{2}(x+y^2)^{-1/2}\left(1+2y\frac{dy}{dx}\right)&=(\cos y)\frac{dy}{dx}\\ \frac{1}{2\sqrt{x+y^2}}&=\cos y\frac{dy}{dx}-\frac{y\cdot dy/dx}{\sqrt{x+y^2}}\\ \frac{1}{2\sqrt{x+y^2}}&=\left(\cos y-\frac{y}{\sqrt{x+y^2}}\right)\frac{dy}{dx}\\ \frac{1}{2\sqrt{x+y^2}}&=\left(\frac{\cos y\sqrt{x+y^2}-y}{\sqrt{x+y^2}}\right)\frac{dy}{dx}\\ \frac{dy}{dx}&=\frac{1}{2(\cos y\sqrt{x+y^2}-y)}\end{aligned}$$

3.8.41

a. $1280=40L^{1/3}K^{2/3}$, so $0=\frac{40}{3}L^{-2/3}K^{2/3}+\frac{80}{3}L^{1/3}K^{-1/3}\cdot\frac{dK}{dL}$. Multiplying both sides by $\frac{3}{40}L^{2/3}K^{1/3}$ yields
$0=K+2L\frac{dK}{dL}$, so $\frac{dK}{dL}=-\frac{1}{2}\frac{K}{L}$.

b. With $L=8$ and $K=64$, $\frac{dK}{dL}=-\frac{64}{16}=-4$.

3.8.42

a. $A=\pi r\sqrt{r^2+h^2}=1500\pi$. So $\pi r'\sqrt{r^2+h^2}+\pi r\frac{rr'+h}{\sqrt{r^2+h^2}}=0$. So $r'(r^2+h^2)+r^2r'+rh=0$, so $r'=-\frac{rh}{2r^2+h^2}$.

b. At $r=30$ and $h=40$, we have $r'=-\frac{1200}{1800+1600}=-\frac{6}{17}$.

3.8.43

a. $V = \dfrac{\pi h^2(3r-h)}{3} = \dfrac{5\pi}{3}$. So

$$\frac{1}{3}[2\pi h(3r-h) + \pi h^2(3r'-1)] = 0,$$

$$6rh - 2h^2 + 3h^2r' - h^2 = 0,$$

so $r' = 1 - \dfrac{2r}{h}$.

b. At $r = 2$ and $h = 1$, we have $r' = 1 - 4 = -3$.

3.8.44

a. When $V = \dfrac{\pi^2(b+a)(b-a)^2}{4} = 64\pi^2$, then we have

$$(b'+1)(b-a)^2 + (b+a)\cdot 2\cdot(b-a)(b'-1) = 0,$$

$$b'(b-a)^2 + 2(b^2-a^2)b' = 2(b^2-a^2) - (b-a)^2,$$

$$b'(b-a)(b-a+2b+2a) = (b-a)(2b+2a-(b-a)),$$

$$\frac{db}{da} = \frac{(b-a)(b+3a)}{(b-a)(3b+a)} = \frac{b+3a}{3b+a}.$$

b. When $a = 6$ and $b = 10$ we have $\dfrac{db}{da} = \dfrac{28}{36} = \dfrac{7}{9}$.

3.8.45

a. $\sin 0 + 5\cdot 0 = 0^2$, so the point $(0,0)$ does lie on the curve.

b. $y'\cos y + 5 = 2yy'$, so $5 = y'(2y - \cos y)$, so $y' = \dfrac{5}{2y-\cos y}$. At the given point we have $y' = -5$. The equation of the tangent line is therefore $y - 0 = -5(x-0)$, or $y = -5x$.

3.8.46

a. $1^3 + 1^3 = 2\cdot 1\cdot 1$, so the point $(1,1)$ does lie on the curve.

b. $3x^2 + 3y^2y' = 2(y + xy')$, which can be written $(3y^2 - 2x)y' = 2y - 3x^2$, so $y' = \dfrac{2y-3x^2}{3y^2-2x}$. At the given point we have $y' = \dfrac{2-3}{3-2} = -1$. The equation of the tangent line is therefore $y - 1 = -1(x-1)$, or $y = -x + 2$.

3.8.47

a. $2^2 + 2\cdot 1 + 1^2 = 7$, so the point $(2,1)$ does lie on the curve.

b. $2x + y + xy' + 2yy' = 0$, which can be written $(x+2y)y' = -2x - y$. Solving for y' yields $y' = \dfrac{-2x-y}{x+2y}$. Thus, at the point $(2,1)$ we have $y' = -\dfrac{5}{4}$. The equation of the tangent line is therefore $y - 1 = -\dfrac{5}{4}(x-2)$, or $y = -\dfrac{5}{4}x + \dfrac{7}{2}$.

3.8.48

a. $(-1)^4 - (-1)^2\cdot 1 + 1^4 = 1$, so the point $(-1,1)$ does lie on the curve.

b. $4x^3 - 2xy - x^2y' + 4y^3y' = 0$, which can be written $y'(4y^3 - x^2) = 2xy - 4x^3$. Thus, $y' = \dfrac{2xy - 4x^3}{4y^3 - x^2}$. Thus, at the point $(-1, 1)$ we have $y' = \dfrac{2}{3}$. The equation of the tangent line is therefore $y - 1 = \dfrac{2}{3}(x+1)$, or $y = \dfrac{2}{3}x + \dfrac{5}{3}$.

3.8.49

a. $\cos(\pi/2 - \pi/4) + \sin(\pi/4) = (\sqrt{2}/2) + (\sqrt{2}/2) = \sqrt{2}$, so the point $(\pi/2, \pi/4)$ does lie on the curve.

b. $(1 - y')(-\sin(x - y)) + y'\cos y = 0$, which can be written as $y'(\cos y + \sin(x - y)) = \sin(x - y)$, so $y' = \dfrac{\sin(x - y)}{\cos y + \sin(x - y)}$. At the given point we have $y' = 1/2$. The equation of the tangent line is therefore $y - (\pi/4) = (1/2)(x - \pi/2)$, or $y = \dfrac{1}{2}x$.

3.8.50

a. $(1 + 2^2)^2 = 25 = \frac{25}{4} \cdot 1 \cdot 2^2$, so the point $(1, 2)$ does lie on the curve.

b. $2(x^2 + y^2)(2x + 2yy') = \frac{25}{4} \cdot (y^2 + 2xyy')$, which can be written as $y'[4y(x^2 + y^2) - (25/2)xy] = (25/4)y^2 - 4x(x^2 + y^2)$, so $y' = \dfrac{25y^2/4 - 4x(x^2 + y^2)}{4y(x^2 + y^2) - (25xy/2)}$. At the given point we have $y' = \dfrac{25 - 20}{40 - 25} = \dfrac{1}{3}$. The equation of the tangent line is therefore $y - 2 = \dfrac{1}{3}(x - 1)$, or $y = \dfrac{1}{3}x + \dfrac{5}{3}$.

3.8.51 $1 + 2yy' = 0$, so $y' = -\dfrac{1}{2y}$. Differentiating again, we obtain

$$y'' = -\frac{1}{2} \cdot \frac{-y'}{y^2} = \frac{y'}{2y^2} = \left(-\frac{1}{2y}\right) \cdot \frac{1}{2y^2} = -\frac{1}{4y^3}.$$

3.8.52 $4x + 2yy' = 0$, so $y' = -\dfrac{2x}{y}$. Differentiating again, we obtain

$$y'' = \frac{-2y + 2xy'}{y^2} = \frac{-2y + \frac{-4x^2}{y}}{y^2} = \frac{-2y^2 - 4x^2}{y^3}.$$

3.8.53 $1 + \dfrac{dy}{dx} = (\cos y)\dfrac{dy}{dx}$, so $1 = \dfrac{dy}{dx}(\cos y - 1)$, and thus $\dfrac{dy}{dx} = \dfrac{1}{\cos y - 1}$. Thus

$$\frac{d^2y}{dx^2} = -\frac{1}{(\cos y - 1)^2} \cdot \left(-\sin y \frac{dy}{dx}\right) = \frac{\sin y}{(\cos y - 1)^2} \cdot \frac{1}{(\cos y - 1)} = \frac{\sin y}{(\cos y - 1)^3}.$$

3.8.54 $4x^3 + 4y'y^3 = 0$, so $y' = -\dfrac{x^3}{y^3}$. Differentiating again, we obtain

$$y'' = -\frac{3x^2y^3 - x^3 \cdot 3y^2y'}{y^6} = \frac{3x^3y' - 3x^2y}{y^4} = \frac{3x^3 \cdot \left(-\frac{x^3}{y^3}\right) - 3x^2y}{y^4} = \frac{-3x^6 - 3x^2y^4}{y^7}.$$

3.8.55 $2y'e^{2y} + 1 = y'$, so $y' = \dfrac{1}{1 - 2e^{2y}}$. Differentiating again, we obtain

$$y'' = -\left(1 - 2e^{2y}\right)^{-2}(-4e^{2y}y') = \frac{4e^{2y}}{(1 - 2e^{2y})^3}.$$

3.8.56 $\cos x + 2xy + x^2y' = 0$, so $y' = -\dfrac{2xy+\cos x}{x^2}$. Differentiating again, we obtain

$$\begin{aligned} y'' &= -\frac{x^2(2y+2xy'-\sin x) - 2x(2xy+\cos x)}{x^4} = \frac{x^2\sin x + 2x\cos x + 2x^2y - 2x^3y'}{x^4} \\ &= \frac{x\sin x + 2\cos x + 2xy + 2(2xy+\cos x)}{x^3} = \frac{x\sin x + 4\cos x + 6xy}{x^3}. \end{aligned}$$

3.8.57

a. False. For example, the equation $y\cos(xy) = x$, cannot be solved explicitly for y in terms of x.

b. True. We have $2x + 2yy' = 0$, and the result follows by solving for y'.

c. False. The equation $x = 1$ doesn't represent any sort of function – it is either just a number, or perhaps a vertical line, but it doesn't represent a differentiable function.

d. False. $y + xy' = 0$, so $y' = -\frac{y}{x}$, $x \neq 0$.

3.8.58

a. $y^{5/2} + \dfrac{5}{2}xy^{3/2}y' + \dfrac{3}{2}x^{1/2}y + x^{3/2}y' = 0$. Then we have

$$\begin{aligned} \frac{5}{2}xy^{3/2}y' + x^{3/2}y' &= -y^{5/2} - \frac{3}{2}x^{1/2}y \\ y'\left(\frac{5}{2}xy^{3/2} + x^{3/2}\right) &= -y^{5/2} - \frac{3}{2}x^{1/2}y \\ y' &= \frac{-y^{5/2} - \frac{3}{2}x^{1/2}y}{\frac{5}{2}xy^{3/2} + x^{3/2}} = -\frac{y(3\sqrt{x} + 2y^{3/2})}{x(2\sqrt{x} + 5y^{3/2})}. \end{aligned}$$

b. At $(4,1)$ we have $y' = -\dfrac{6+2}{4(4+5)} = -\dfrac{2}{9}$.

3.8.59

a. $y + xy' + \dfrac{3}{2}x^{1/2}y^{-1/2} - \dfrac{1}{2}x^{3/2}y^{-3/2}y' = 0$. Multiplying through by $2y^{3/2}$ gives

$$2y^{5/2} + 2xy^{3/2}y' + 3x^{1/2}y - x^{3/2}y' = 0.$$

Then

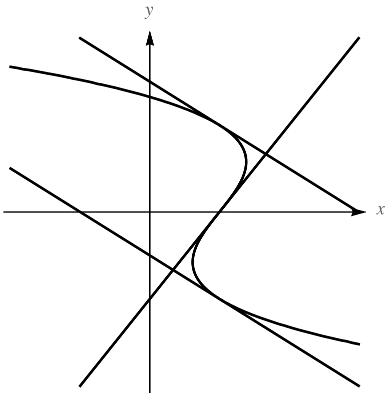
$$\begin{aligned} 2xy^{3/2}y' - x^{3/2}y' &= -2y^{5/2} - 3x^{1/2}y \\ y'(2xy^{3/2} - x^{3/2}) &= -(2y^{5/2} + 3x^{1/2}y) \\ y' &= -\frac{2y^{5/2} + 3x^{1/2}y}{2xy^{3/2} - x^{3/2}} \\ &= \frac{y(2y^{3/2} + 3\sqrt{x})}{x(\sqrt{x} - 2y^{3/2})}. \end{aligned}$$

b. At $(1,1)$, $y' = -5$.

3.8.60

a. There are three points on the curve associated with $x = 1$. When $x = 1$, we have $1 + y^3 - y = 1$, so $y(y^2 - 1) = 0$. The three points are thus $(1, 0)$, $(1, 1)$ and $(1, -1)$. Differentiating yields $1 + 3y^2y' - y' = 0$, so $y' = \dfrac{1}{1 - 3y^2}$.
At $(1, 0)$, we have $y' = 1$, so the tangent line is given by $y = x - 1$.
At $(1, 1)$, we have $y' = -\dfrac{1}{2}$, so the tangent line is given by $y - 1 = -\dfrac{1}{2}(x - 1)$, or $y = -\frac{1}{2}x + \frac{3}{2}$.
At $(1, -1)$, we have $y' = -\dfrac{1}{2}$, so the tangent line is given by $y + 1 = -\frac{1}{2}(x - 1)$, or $y = -\dfrac{1}{2}x - \dfrac{1}{2}$.

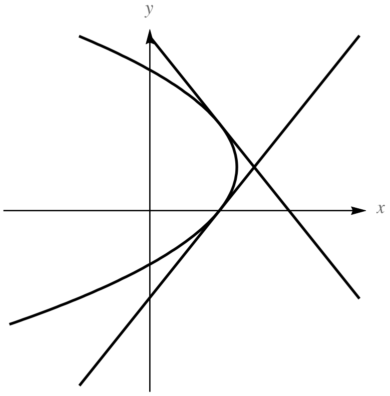
b.



3.8.61

a. There are two points on the curve associated with $x = 1$. When $x = 1$, we have $1 + y^2 - y = 1$, so $y(y - 1) = 0$. The two points are thus $(1, 0)$ and $(1, 1)$. Differentiating yields $1 + 2yy' - y' = 0$, so $y' = \frac{1}{1-2y}$.
At $(1, 0)$, we have $y' = 1$, so the tangent line is given by $y = x - 1$.
At $(1, 1)$, we have $y' = -1$, so the tangent line is given by $y - 1 = -1(x - 1)$, or $y = -x + 2$.

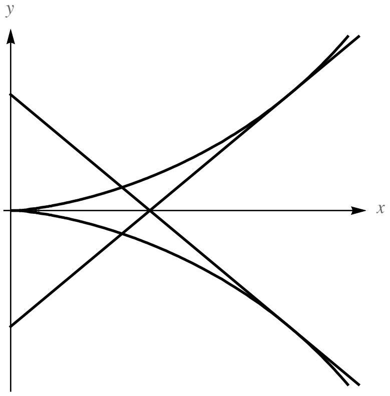
b.



3.8.62

a. There are two points on the curve associated with $x = 2$. When $x = 2$, we have $32 = 2y^2$, so $y^2 = 16$, so $y = \pm 4$. The two points are thus $(2, 4)$ and $(2, -4)$. Differentiating yields $12x^2 = 2yy'(4 - x) + -y^2$.
At $(2, -4)$, we have $48 = -8y'(2) - 16$, so $y' = -4$. Thus the tangent line is given by $y + 4 = -4(x - 2)$, or $y = -4x + 4$.
At $(2, 4)$, we have $y' = 4$, so the tangent line is given by $y - 4 = 4(x - 2)$, or $y = 4x - 4$.

b.



3.8.63

a. $y(2x) + (x^2 + 4)y' = 0$, so $y' = -\dfrac{2xy}{x^2 + 4}$.

b. At $y = 1$ we have $x^2 + 4 = 8$, so $x = \pm 2$. At the point $(2, 1)$ we have $y' = -\frac{4}{8} = -\frac{1}{2}$. At the point $(-2, 1)$ we have $y' = \frac{4}{8} = \frac{1}{2}$. Thus, the equations of the tangent lines are given by $y - 1 = -\frac{1}{2}(x - 2)$ and $y - 1 = \frac{1}{2}(x + 2)$, or $y = -\frac{1}{2}x + 2$ and $y = \frac{1}{2}x + 2$.

c. $y = \frac{8}{x^2 + 4}$, so $y' = \frac{0 - 8 \cdot 2x}{(x^2 + 4)^2} = -\frac{16x}{(x^2 + 4)^2}$.

d. $y' = -\frac{16x}{(x^2 + 4)^2} = -\frac{2x}{x^2 + 4} \cdot \frac{8}{x^2 + 4} = -\frac{2x}{x^2 + 4} \cdot y = -\frac{2xy}{x^2 + 4}$.

3.8.64

a. From exercise 60 we have that $y' = \frac{1}{1 - 3y^2}$. A vertical tangent would occur at a point whose y value would make $1 - 3y^2$ equal to zero. So we are looking for where $3y^2 = 1$ or $y = \pm\frac{1}{\sqrt{3}}$.

If $y = \frac{1}{\sqrt{3}}$, then $x + \left(\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{\sqrt{3}} = 1$, so $x = 1 + \frac{2\sqrt{3}}{9}$, and there is a vertical tangent at $\left(\frac{1}{\sqrt{3}}, 1 + \frac{2\sqrt{3}}{9}\right)$.

If $y = -\frac{1}{\sqrt{3}}$, then $x + \left(-\frac{1}{\sqrt{3}}\right)^3 - -\frac{1}{\sqrt{3}} = 1$, so $x = 1 - \frac{2\sqrt{3}}{9}$, and there is a vertical tangent at $\left(-\frac{1}{\sqrt{3}}, 1 - \frac{2\sqrt{3}}{9}\right)$.

b. Because y' is never zero, there are no horizontal tangent lines.

3.8.65

a. From exercise 61, we have that $y' = \frac{1}{1 - 2y}$. A vertical tangent would occur at a point whose y value would make $1 - 2y$ equal to zero. So we are looking for where $2y = 1$ or $y = \frac{1}{2}$.

If $y = \frac{1}{2}$, then $x + \frac{1}{4} - \frac{1}{2} = 1$, so $x = \frac{5}{4}$, and there is a vertical tangent at $\left(\frac{5}{4}, \frac{1}{2}\right)$.

b. Because y' is never zero, there are no horizontal tangent lines.

3.8.66 Differentiating with respect to x gives $2x + 8y\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0$, so

$$\begin{aligned} 8y\frac{dy}{dx} + 2x\frac{dy}{dx} &= -2x - 2y \\ \frac{dy}{dx}(8y + 2x) &= -2x - 2y \\ \frac{dy}{dx} &= \frac{-x - y}{4y + x}. \end{aligned}$$

This quantity is zero when $y = -x$ (and not $x = y = 0$). Using the original equation, this means that $x^2 + 4x^2 - 2x^2 = 12$, or $3x^2 = 12$, so $x^2 = 4$ and we have $x = \pm 2$. Thus the points on the curve where the tangent line is horizontal are $(2, -2)$ and $(-2, 2)$.

The curve has vertical tangent lines where $x = -4y$ (and not $x = y = 0$). Using the original equation, this means that $16y^2 + 4y^2 - 8y^2 = 12$, or $12y^2 = 12$, so $y = \pm 1$. Thus the points on the curve where the tangent line is vertical are $(-4, 1)$ and $(4, -1)$.

3.8.67 Differentiating with respect to x gives $18x + 2y\dfrac{dy}{dx} - 36 + 6\dfrac{dy}{dx} = 0$, so

$$\begin{aligned} 2y\frac{dy}{dx} + 6\frac{dy}{dx} &= -18x + 36 \\ (2y+6)\frac{dy}{dx} &= -18x + 36 \\ \frac{dy}{dx} &= \frac{-9x+18}{y+3}. \end{aligned}$$

This is zero when $x = 2$. Using the original equation, we have $36 + y^2 - 72 + 6y + 36 = 0$, or $y^2 + 6y = 0$, or $y = 0$ and $y = -6$. Thus there are horizontal tangent lines at $(2, 0)$ and $(2, -6)$. So $y = 0$ and $y = -6$ are horizontal tangent lines.

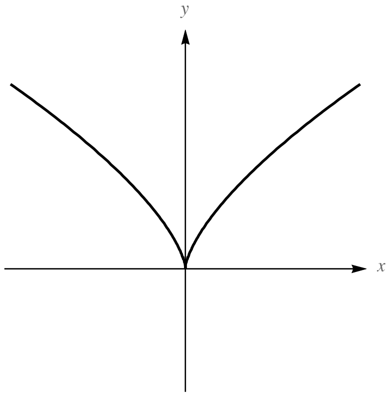
This curve has vertical tangent lines when $y = -3$. Using the original equation, we have $9x^2 + 9 - 36x - 18 + 36 = 0$ or $9x^2 - 36x + 27 = 0$, or $x^2 - 4x + 3 = 0$. This factors as $(x-3)(x-1) = 0$, so the corresponding values of x are 3 and 1. Thus the vertical tangent lines occur at $(3, -3)$ and $(1, -3)$.

3.8.68

a. $3y^2y' = 2ax$, so $y' = \frac{2ax}{3y^2}$.

b. $y = \sqrt[3]{ax^2}$.

c.



3.8.69

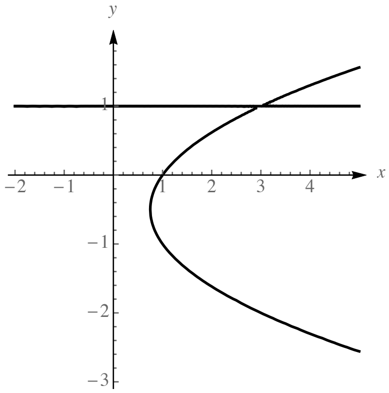
a. If we write $y^3 - 1 = xy - x$, we have $(y-1)(y^2 + y + 1) = x(y-1)$, so $y^2 + y + 1 = x$. Differentiating gives $2y\dfrac{dy}{dx} + \dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = \dfrac{1}{1+2y}$. Note also that $y = 1$ satisfies the equation; $y' = 0$ on this branch.

b.

$$\begin{aligned} y^3 - 1 &= x(y-1) \\ (y-1)(y^2+y+1) &= x(y-1) \\ y^2 + y + 1 &= x \\ y^2 + y + (1-x) &= 0, \end{aligned}$$

so by the quadratic formula we have $y = \dfrac{-1 \pm \sqrt{4x-3}}{2}$. Note that this means that $\pm\sqrt{4x-3} = 2y+1$.

c.

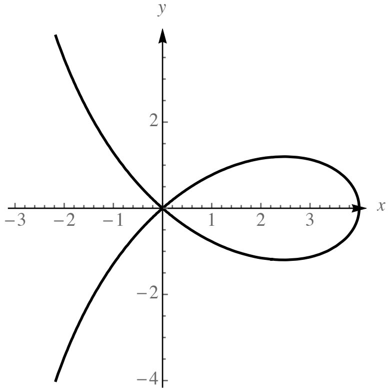


3.8.70

a. $2yy' = \dfrac{(2x(4-x)-x^2)(4+x)-x^2(4-x)}{(4+x)^2}$, so $2yy' = \dfrac{(8x-3x^2)(4+x)-4x^2+x^3}{(4+x)^2}$, and $2yy' = \dfrac{32x-8x^2-2x^3}{(4+x)^2}$. Thus $y' = \dfrac{16x-4x^2-x^3}{y(4+x)^2}$.

b. $y = \pm\sqrt{\dfrac{x^2(4-x)}{4+x}}$.

c.

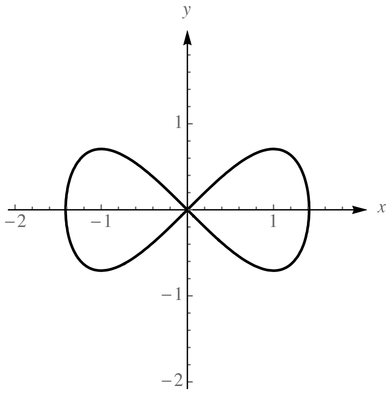


3.8.71

a. $4x^3 = 4x - 4yy'$, so $y' = \dfrac{x-x^3}{y}$.

b. $y = \pm\sqrt{x^2 - \dfrac{x^4}{2}}$.

c.

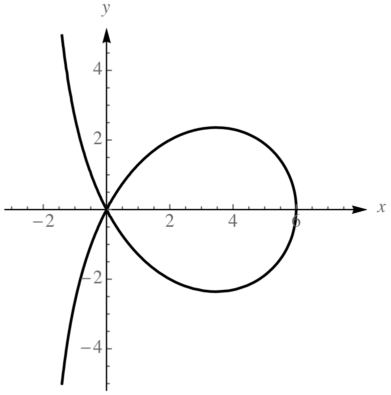


3.8.72

a. $2yy'(x+2)+y^2=12x-3x^2$. So $y'=\frac{12x-3x^2-y^2}{2y(x+2)}$.

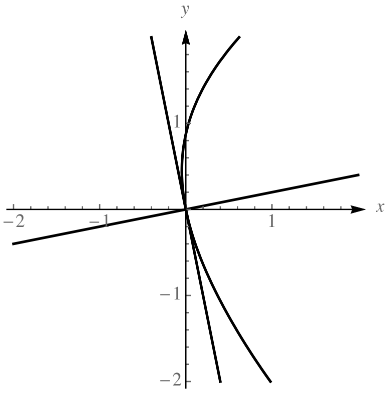
b. $y^2=\dfrac{x^2(6-x)}{x+2}$, so $y=\pm\sqrt{\dfrac{x^2(6-x)}{x+2}}$.

c.



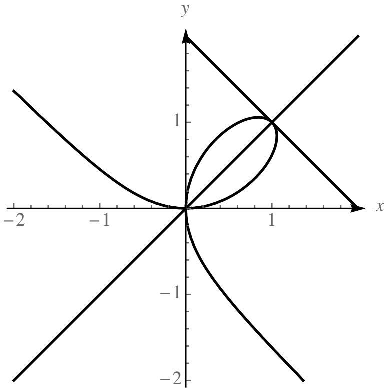
3.8.73

The slope of the normal line is the negative reciprocal of the slope of the tangent line. From 45: $y'=-5$, so the slope of the normal line is $\dfrac{1}{5}$. At the point (0,0), we have the normal line $y=\dfrac{1}{5}x$.



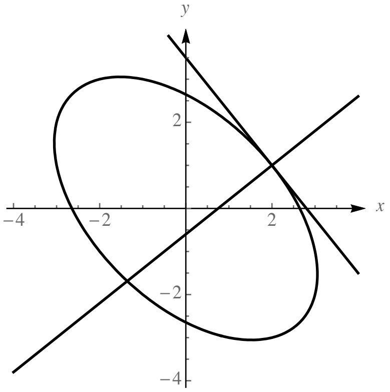
3.8.74

The slope of the normal line is the negative reciprocal of the slope of the tangent line. From 46: $y' = -1$, so the slope of the normal line is 1. At the point $(1, 1)$ we have the line $y - 1 = x - 1$, or $y = x$.



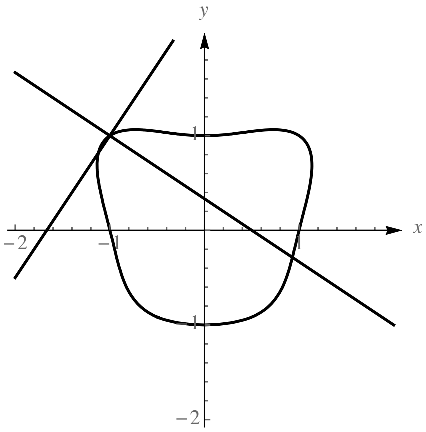
3.8.75

The slope of the normal line is the negative reciprocal of the slope of the tangent line. From 47: $y' = -\frac{5}{4}$, so the slope of the normal line is $\frac{4}{5}$. At the point $(2, 1)$ we have the line $y - 1 = \frac{4}{5}(x - 2)$, or $y = \frac{4}{5}x - \frac{3}{5}$.



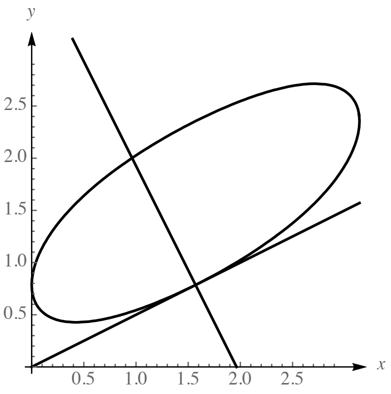
3.8.76

The slope of the normal line is the negative reciprocal of the slope of the tangent line. From 48: $y' = \frac{2}{3}$, so the slope of the normal line is $-\frac{3}{2}$. At the point $(-1, 1)$ we have the line $y - 1 = -\frac{3}{2}(x + 1)$, or $y = -\frac{3}{2}x - \frac{1}{2}$.



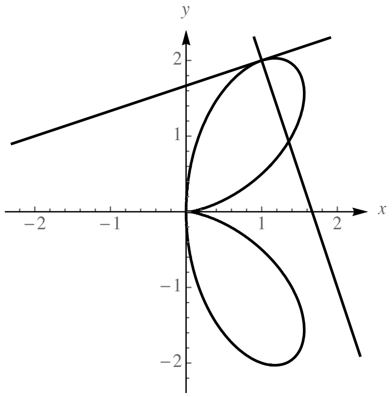
3.8.77

The slope of the normal line is the negative reciprocal of the slope of the tangent line. From 49: $y' = \frac{1}{2}$, so the slope of the normal line is -2. At the point $(\frac{\pi}{2}, \frac{\pi}{4})$ we have the line $y - \frac{\pi}{4} = -2(x - \frac{\pi}{2})$, or $y = -2x + \frac{5\pi}{4}$.



3.8.78

The slope of the normal line is the negative reciprocal of the slope of the tangent line. From 50: $y' = \frac{1}{3}$, so the slope of the normal line is -3. At the point $(1, 2)$ we have the line $y-2 = -3(x-1)$, or $y = -3x + 5$.

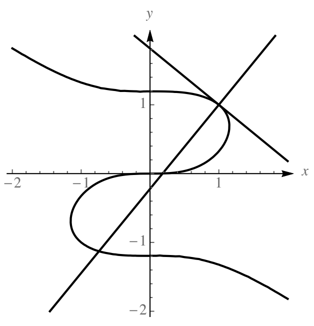


3.8.79

a. We have $9x^2+21y^2y' = 10y'$, so at the point $(1, 1)$ we have $9 + 21y' = 10y'$, so $y' = -\frac{9}{11}$.

Thus, the tangent line is given by $y - 1 = -\frac{9}{11}(x-1)$, or $y = \frac{-9}{11}x+\frac{20}{11}$. The normal line is given by $y-1 = \frac{11}{9}(x-1)$, or $y = \frac{11}{9}x-\frac{2}{9}$.

b.

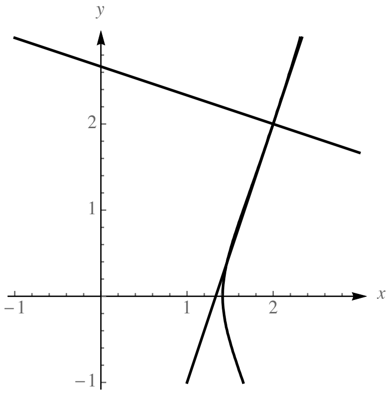


3.8.80

a. We have $4x^3 = 4x + 4yy'$, so at the point $(2,2)$ we have $32 = 8 + 8y'$, so $y' = 3$.

Thus, the tangent line is given by $y - 2 = 3(x - 2)$, or $y = 3x - 4$. The normal line is given by $y - 2 = -\frac{1}{3}(x - 2)$, or $y = -\frac{1}{3}x + \frac{8}{3}$.

b.

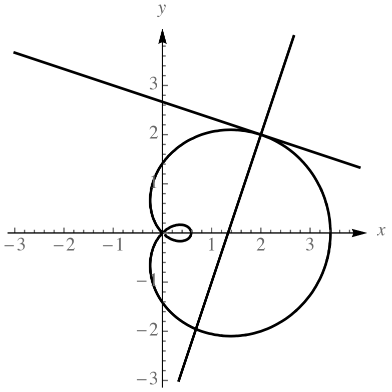


3.8.81

a. We have $2(x^2+y^2-2x)(2x+2yy'-2) = 4x+4yy'$, so at the point $(2,2)$ we have $2(4+4-4)(4+4y'-2) = 8+8y'$, so $16+32y' = 8+8y'$, so $y' = -\frac{1}{3}$.

Thus, the tangent line is given by $y - 2 = -\frac{1}{3}(x - 2)$, or $y = \frac{-1}{3}x + \frac{8}{3}$. The normal line is given by $y - 2 = 3(x - 2)$, or $y = 3x - 4$.

b.

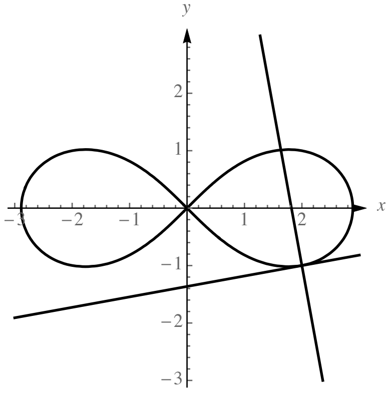


3.8.82

a. We have $2(x^2 + y^2)(2x + 2yy') = \frac{25}{3}(2x - 2yy')$, so at the point $(2, -1)$ we have $2\cdot5\cdot(4-2y') = \frac{25}{3}(4+2y')$, so $40 - 20y' = \frac{100}{3} + \frac{50}{3}y'$, so $120 - 60y' = 100 + 50y'$, and thus $y' = \frac{2}{11}$.

Thus, the tangent line is given by $y + 1 = \frac{2}{11}(x-2)$, or $y = \frac{2}{11}x - \frac{15}{11}$. The normal line is given by $y+1 = -\frac{11}{2}(x-2)$, or $y = \frac{-11}{2}x+10$.

b.



3.8.83 Note for $y = mx$, $\dfrac{dy}{dx} = m = \dfrac{y}{x}$, and for $x^2 + y^2 = a^2$, $\dfrac{dy}{dx} = -\dfrac{x}{y}$. So for any point (x, y), we have $\dfrac{y}{x}$ and $-\dfrac{x}{y}$ are negative reciprocals.

3.8.84 For $y = cx^2$ we have $y' = 2cx$ and for $x^2 + 2y^2 = k$, we have $y' = -\dfrac{x}{2y}$. Let (a, b) be a point on both curves. Then $b = ca^2$, so the point has the form (a, ca^2). A normal line to the ellipse $x^2 + 2y^2 = k$ would have slope $\frac{2y}{x} = \frac{2ca^2}{a} = 2ca$, which is the slope of the tangent line to the parabola $y = cx^2$ at the point in question. Thus the two curves are orthogonal at any points of intersection.

3.8.85 For $xy = a$ we have $xy' + y = 0$, so $y' = -\frac{y}{x}$. For $x^2 - y^2 = b$, we have $2x - 2yy' = 0$, so $y' = \frac{x}{y}$. Let (c, d) be a point on both curves. Then the slope of the normal line to the first curve is $\frac{c}{d}$, but that is the slope of the tangent line to the second curve. Thus the two curves are orthogonal at any points of intersection.

3.8.86
$\frac{5}{2\sqrt{x}} - \frac{5y'}{\sqrt{y}} = \cos x$, so $\frac{5y'}{\sqrt{y}} = \frac{5}{2\sqrt{x}} - \cos x$, and thus $y' = \frac{\sqrt{y}}{5} \cdot \left(\frac{5}{2\sqrt{x}} - \cos x\right) = \frac{\sqrt{y}}{5} \cdot \left(\frac{5 - 2\sqrt{x}\cos x}{2\sqrt{x}}\right)$

At the point $(4\pi, \pi)$ we have $y'(4\pi, \pi) = \frac{\sqrt{\pi}}{5} \cdot \left(\frac{5 - 2\sqrt{4\pi}(1)}{2\sqrt{4\pi}}\right) = \frac{5 - 4\sqrt{\pi}}{20}$.

3.8.87

$$\begin{aligned}
(2x + 2yy')(x^2 + y^2 + x) + (x^2 + y^2)(2x + 2yy' + 1) &= 8y^2 + 16xyy' \\
2yy'(x^2 + y^2 + x) + (x^2 + y^2)2yy' - 16xyy' &= 8y^2 - 2x(x^2 + y^2 + x) - (x^2 + y^2)(2x + 1) \\
y' &= \frac{8y^2 - 2x(x^2 + y^2 + x) - (x^2 + y^2)(2x + 1)}{2y(x^2 + y^2 + x) + 2y(x^2 + y^2) - 16xy} \\
&= \frac{8y^2 - 2x^3 - 2xy^2 - 2x^2 - 2x^3 - x^2 - 2xy^2 - y^2}{2y(x^2 + y^2 + x + x^2 + y^2 - 8x)} \\
&= \frac{7y^2 - 3x^2 - 4xy^2 - 4x^3}{2y(2x^2 + 2y^2 - 7x)}.
\end{aligned}$$

3.8.88

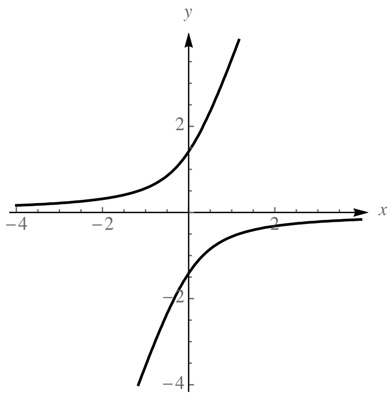
$$\begin{aligned}
\frac{21x^6 + 2yy'}{2\sqrt{3x^7 + y^2}} &= 2y' \sin y \cos y + 100(y + xy') \\
21x^6 + 2yy' &= 4y' \sin y \cos y \sqrt{3x^7 + y^2} + 200y\sqrt{3x^7 + y^2} + 200xy'\sqrt{3x^7 + y^2} \\
200y\sqrt{3x^7 + y^2} - 21x^6 &= 2yy' - 4y' \sin y \cos y \sqrt{3x^7 + y^2} - 200xy'\sqrt{3x^7 + y^2} \\
y' &= \frac{200y\sqrt{3x^7 + y^2} - 21x^6}{2y - 4\sin y \cos y \sqrt{3x^7 + y^2} - 200x\sqrt{3x^7 + y^2}}.
\end{aligned}$$

3.8.89 $\frac{y'}{2\sqrt{y}} + y + xy' = 0$, so $y' + 2x\sqrt{y}y' = -2y\sqrt{y}$, so $y' = \frac{-2y\sqrt{y}}{2x\sqrt{y} + 1} = -\frac{2y^{3/2}}{2x\sqrt{y} + 1}$.
Differentiating again we obtain

$$\begin{aligned}
y'' &= -\frac{(2x\sqrt{y} + 1)(3\sqrt{y}y') - 2y^{3/2}\left(2\sqrt{y} + \frac{xy'}{\sqrt{y}}\right)}{(2x\sqrt{y} + 1)^2} = \frac{-(2x\sqrt{y} + 1)(3\sqrt{y}y') + 4y^2 + 2xyy'}{(2x\sqrt{y} + 1)^2} \\
&= \left(\frac{-(2x\sqrt{y} + 1)(3\sqrt{y})\left(\frac{-2y\sqrt{y}}{2x\sqrt{y}+1}\right) + 4y^2 + 2xy\left(\frac{-2y\sqrt{y}}{1+2x\sqrt{y}}\right)}{(2x\sqrt{y} + 1)^2}\right) \cdot \frac{2x\sqrt{y} + 1}{2x\sqrt{y} + 1} \\
&= \frac{(2x\sqrt{y} + 1)(6y^2) + 4y^2(1 + 2x\sqrt{y}) - 4xy^{5/2}}{(2x\sqrt{y} + 1)^3} = \frac{10y^2 + 16xy^2\sqrt{y}}{(2x\sqrt{y} + 1)^3}.
\end{aligned}$$

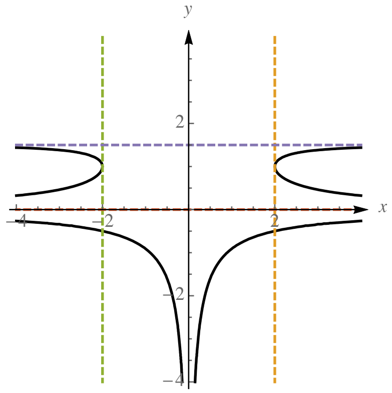
3.8.90 Differentiating with respect to x yields $2yy' - 3y - 3xy' = 0$; simplifying gives $y' = \frac{3y}{2y-3x}$. There could be a horizontal tangent line when $y = 0$, but using the original equation we see that there are no points on the curve where this occurs. (Letting $y = 0$ in the original equation gives the untrue equation $0 = 2$).

Considering where the tangent line might be vertical we consider points where $2y = 3x$. Using the original equation and replacing $3x$ by $2y$, we obtain $y^2 - 2y^2 = 2$, or $-y^2 = 2$, which is again impossible. So the tangent is never vertical nor horizontal.

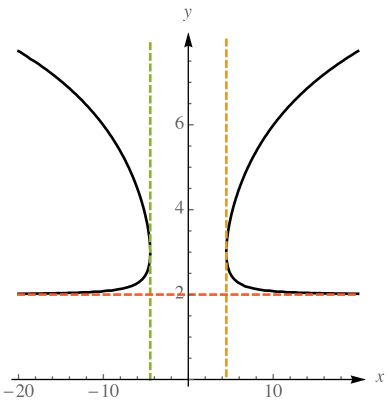


3.8.91 Differentiating with respect to x yields $2x(3y^2 - 2y^3) + x^2(6y - 6y^2)\frac{dy}{dx} = 0$. Solving for $\frac{dy}{dx}$ yields $\frac{dy}{dx} = \frac{-2x(3y^2 - 2y^3)}{6x^2(y - y^2)} = \frac{3y^2 - 2y^3}{3x(y^2 - y)} = \frac{3y - 2y^2}{3x(y - 1)}$. The numerator is zero when $y = 0$ or when $y = \frac{3}{2}$. Using the original equation, there are no points where $y = 0$, and there are also no points where $y = \frac{3}{2}$ because we obtain the equation $x^2(\frac{27}{4} - \frac{27}{4}) = 4$, which has no solutions. So there are no horizontal tangent lines.

There could be vertical tangent lines where $x = 0$ or $y = 1$; in fact, when $y = 1$ the original equation becomes $x^2(3 - 2) = 4$, so $x = \pm 2$. There are vertical tangents at $(2, 1)$ and $(-2, 1)$. Letting $x = 0$ doesn't yield any points in the original equation.

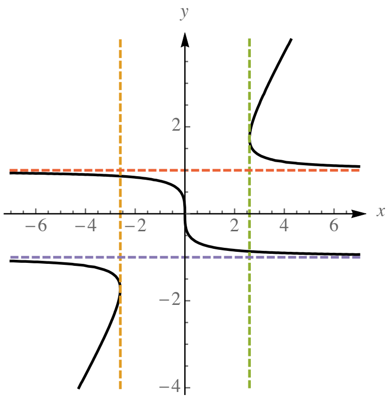


3.8.92 Differentiating with respect to x yields $2x(y-2)+x^2y'-e^yy' = 0$, and solving for y' yields $y' = \frac{2x(2-y)}{x^2-e^y}$. This quantity is zero for $x = 0$ or $y = 2$. Using the original equation we find no solutions for $x = 0$ or $y = 2$, so there are no horizontal tangent lines. There could be vertical tangent lines where $x^2 = e^y$. Using the original equation (and disallowing $x^2 = 0$) we have $x^2(y - 2) - x^2 = 0$, so $x^2(y - 2 - 1) = 0$, so $y = 3$. Using the original equation with $y = 3$ we have $x^2 - e^3 = 0$, so $x = \pm\sqrt{e^3}$. There are vertical tangent lines at $(\sqrt{e^3}, 3)$ and $-\sqrt{e^3}, 3)$.



3.8.93 Differentiating with respect to x yields $(1-y^2)+x(-2yy')+3y^2y'=0$. Solving for y' yields $y'=\dfrac{y^2-1}{3y^2-2xy}=\dfrac{y^2-1}{y(3y-2x)}$. There could be a horizontal tangent line where $y=\pm1$, but the original equation with $y=\pm1$ yields $\pm1=0$, so there are no horizontal tangent lines.

There could be a vertical tangent line for $y=0$ or for $y=\dfrac{2}{3}x$. Using the original equation, letting $y=0$ yields $x=0$, and letting $y=\dfrac{2}{3}x$ yields $x\left(1-\dfrac{4x^2}{9}\right)+\dfrac{8x^3}{27}=0$. For $x\neq0$, this yields $27-12x^2+8x^2=0$, or $x^2=\frac{27}{4}$, so $x=\pm\dfrac{\sqrt{27}}{2}=\pm\dfrac{3\sqrt{3}}{2}$. The corresponding y values are $y=\pm\dfrac{2}{3}x=\pm\sqrt{3}$. So there are vertical tangent lines at $(0,0)$ and at $(3\sqrt{3}/2,\sqrt{3})$ and $(-3\sqrt{3}/2,-\sqrt{3})$.

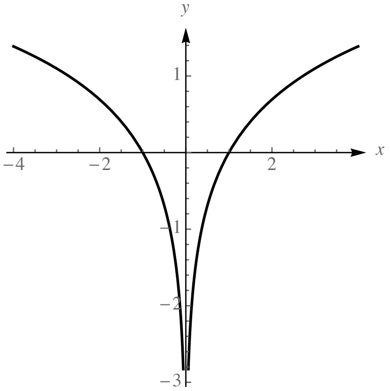


3.9 Derivatives of Logarithmic and Exponential Functions

3.9.1 $y=\ln x$ if and only if $x=e^y$. Differentiating implicitly yields $1=e^y\cdot y'$, so $y'=\frac{1}{e^y}=\frac{1}{e^{\ln x}}=\frac{1}{x}$ for $x>0$.

3.9.2

We have already established that if $y = \ln x$ for $x > 0$ then $y' = \frac{1}{x}$. By the symmetry about the y-axis, we know that for $x < 0$, the derivative of $y = \ln|x|$ should have the same absolute value but the opposite sign of the derivative for the corresponding positive x value. But this is the property that $\frac{1}{x}$ has for $x < 0$ – it is negative and has the right absolute value. So we see that for both $x > 0$ and $x < 0$, $\frac{d}{dx}\ln|x| = \frac{1}{x}$.



3.9.3 $\frac{d}{dx}\ln(kx) = \frac{1}{kx}\cdot k = \frac{1}{x}$. This is valid for $x > 0$ if $k > 0$ and $x < 0$ if $k < 0$. Also, we can write $\ln(kx) = \ln(k) + \ln(x)$, so its derivative is $0 + \frac{1}{x} = \frac{1}{x}$.

3.9.4 $\frac{d}{dx}b^x = b^x \ln b$, for $b > 0$ and all x. In the case $b = e$, the rule states that $\frac{d}{dx}e^x = e^x \ln e = e^x$ because $\ln e = 1$.

3.9.5 $\frac{d}{dx}\log_b x = \frac{1}{x\ln b}$ for $b > 0$, $b \neq 1$ and $x > 0$. If $b = e$, we have $\frac{d}{dx}\log_e x = \frac{1}{x\ln e} = \frac{1}{x}$.

3.9.6 We use inverse property 3 from the text: $b^x = (e^{\ln b})^x = e^{(\ln b)\cdot x} = e^{x\ln b}$.

3.9.7 $e^{x\ln(x^2+1)} = e^{\ln(x^2+1)^x} = (x^2+1)^x$.

3.9.8 $\frac{d}{dx}\left(\ln\left(\frac{x}{x^2+1}\right)\right) = \frac{d}{dx}\left(\ln x - \ln(x^2+1)\right) = \frac{1}{x} - \frac{2x}{x^2+1} = \frac{x^2+1}{x(x^2+1)} - \frac{2x^2}{x(x^2+1)} = \frac{1-x^2}{x(x^2+1)}$.

3.9.9 $\frac{d}{dx}\left(\left(\ln\sqrt{x^2+1}\right)\right) = \frac{d}{dx}\left(\ln(x^2+1)^{1/2}\right) = \frac{d}{dx}\left(\frac{1}{2}\ln(x^2+1)\right) = \frac{1}{2}\frac{2x}{(x^2+1)} = \frac{x}{x^2+1}$.

3.9.10 $\frac{d}{dx}(x^e + e^x) = ex^{e-1} + e^x$.

3.9.11 $f(x) = e^{\ln(g(x)^{h(x)})} = e^{h(x)\cdot\ln(g(x))}$.

3.9.12 $\frac{d}{dx}\left(\ln\sqrt{f(x)}\right) = \frac{d}{dx}(\ln(f(x)^{1/2})) = \frac{d}{dx}\left(\frac{1}{2}\ln(f(x))\right) = \frac{1}{2}\cdot\frac{d}{dx}(\ln(f(x))) = \frac{1}{2}\cdot\frac{f'(x)}{f(x)}$.

3.9.13 $\frac{d}{dx}(\ln(xe^x)) = \frac{d}{dx}(\ln x + \ln e^x) = \frac{d}{dx}(\ln x + x) = \frac{1}{x} + 1 = \frac{1+x}{x}$.

3.9.14 $\frac{d}{dx}(\ln x^{101}) = \frac{d}{dx}(101\ln x) = \frac{101}{x}$.

3.9.15 $\frac{d}{dx}(\ln(7x)) = \frac{d}{dx}(\ln 7 + \ln x) = 0 + \frac{1}{x} = \frac{1}{x}$.

3.9.16 $\frac{d}{dx}(x^2\ln x) = 2x\ln x + x^2\cdot\frac{1}{x} = 2x\ln x + x = x(2\ln x + 1)$.

3.9.17 $\frac{d}{dx}(\ln(x^2)) = \frac{d}{dx}(2\ln x) = \frac{2}{x}.$

3.9.18 $\frac{d}{dx}(\ln(2x^8)) = \frac{d}{dx}(8(\ln 2 + \ln x)) = 8(0 + \frac{1}{x}) = \frac{8}{x}.$

3.9.19 $\frac{d}{dx}(\ln|\sin x|) = \frac{1}{\sin x} \cdot (\cos x) = \cot x.$

3.9.20

$$\begin{aligned} \frac{d}{dx}\left(\frac{1+\ln x^3}{4x^3}\right) &= \frac{d}{dx}\left(\frac{1+3\ln x}{4x^3}\right) = \frac{4x^3\left(\frac{3}{x}\right) - (1+3\ln x)12x^2}{16x^4} \\ &= \frac{12x^2 - 12x^2 - 36x^2\ln x}{16x^4} = -\frac{9\ln x}{4x^4}. \end{aligned}$$

3.9.21 $\frac{d}{dx}\ln(x^4+1) = \frac{1}{x^4+1} \cdot 4x^3 = \frac{4x^3}{x^4+1}.$

3.9.22 $\frac{d}{dx}\left(\ln\sqrt{x^4+x^2}\right) = \frac{d}{dx}\left(\frac{1}{2}\ln(x^4+x^2)\right) = \frac{4x^3+2x}{2(x^4+x^2)} = \frac{2x^3+x}{x^4+x^2} = \frac{2x^2+1}{x^3+x}.$

3.9.23 $\frac{d}{dx}\left(\ln\left(\frac{x+1}{x-1}\right)\right) = \frac{(x-1)}{(x+1)}\left(\frac{(x-1)-(x+1)}{(x-1)^2}\right) = -\frac{2}{(x+1)(x-1)} = \frac{2}{1-x^2}.$

3.9.24 $\frac{d}{dx}(x\ln x - x) = 1 \cdot \ln x + \frac{x}{x} - 1 = \ln x.$

3.9.25 $\frac{d}{dx}((x^2+1)\ln x) = 2x\ln x + \frac{x^2+1}{x}.$

3.9.26 $\frac{d}{dx}(\ln|x^2-1|) = \frac{1}{x^2-1} \cdot 2x = \frac{2x}{x^2-1}.$

3.9.27 $\frac{d}{dx}(x^2(1-\ln x^2)) = 2x(1-\ln x^2) + x^2\left(\frac{-1}{x^2}\right)2x = 2x - 2x\ln x^2 - 2x = -2x\ln x^2.$

3.9.28 $\frac{d}{dx}(3x^3\ln x - x^3) = 9x^2\ln x + 3x^3\left(\frac{1}{x}\right) - 3x^2 = 9x^2\ln x.$

3.9.29 $\frac{d}{dx}\ln(\ln x) = \frac{1}{\ln x} \cdot \frac{1}{x}.$

3.9.30 $\frac{d}{dx}(\ln(\cos^2 x)) = \frac{1}{\cos^2 x} \cdot (-2\sin x\cos x) = -2\tan x.$

3.9.31 $\frac{d}{dx}\left(\frac{\ln x}{\ln x + 1}\right) = \frac{(\ln x + 1)(1/x) - (\ln x)(1/x)}{(\ln x + 1)^2} = \frac{1}{x(\ln x + 1)^2}.$

3.9.32 $\frac{d}{dx}(\ln(e^x + e^{-x})) = \frac{1}{e^x + e^{-x}}(e^x - e^{-x}) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$

3.9.33 $y' = ex^{e-1}.$

3.9.34 $y' = e^x x^e + e^{x+1}x^{e-1} = e^x x^{e-1}(x+e).$

3.9.35 $y' = \pi(2^x+1)^{\pi-1}2^x\ln 2.$

3.9.36 $y' = \frac{d}{dx}(\pi \cdot \ln(x^3+1)) = \pi \cdot \frac{3x^2}{x^3+1} = \frac{3\pi x^2}{x^3+1}.$

3.9.37 $y' = 8^x \ln 8$.

3.9.38 $y' = 5^{3t} \cdot \ln 5 \cdot 3 = 3(\ln 5)(5^{3t})$.

3.9.39 $y' = 5 \cdot \dfrac{d}{dx} 4^x = 5 \cdot \ln 4 \cdot 4^x$.

3.9.40 $y' = -\ln 4 \cdot 4^{-x} \cdot \sin x + 4^{-x} \cos x$.

3.9.41 $y' = 2^{3+\sin x}(\ln 2)\cos x$

3.9.42 $y' = 10^{\ln 2x}(\ln 10)\dfrac{1}{x}$.

3.9.43 $y' = 3x^2 3^x + x^3 3^x \ln 3 = 3^x x^2(3 + x\ln 3)$.

3.9.44 $\dfrac{dP}{dt} = \dfrac{0 - 40(-\ln 2 \cdot 2^{-t})}{(1+2^{-t})^2} = \dfrac{40 \ln 2 \cdot 2^{-t}}{(1+2^{-t})^2}$.

3.9.45 $\dfrac{dA}{dt} = 250(1.045)^{4t} \cdot \ln(1.045) \cdot 4 = 1000 \cdot \ln 1.045 \cdot (1.045^{4t})$.

3.9.46 $\dfrac{d}{dx} 10^x(\ln 10^x - 1) = \dfrac{d}{dx}(10^x(x \ln 10 - 1)) = (10^x \ln 10)(x\ln 10 - 1) + 10^x \ln 10 = x10^x(\ln 10)^2$.

3.9.47 $f'(x) = \dfrac{(2^x+1)2^x \ln 2 - 2^x(2^x \ln 2)}{(2^x+1)^2} = \dfrac{2^x \ln 2}{(2^x+1)^2}$.

3.9.48 $s'(t) = -\sin(2^t) \cdot 2^t \ln 2$.

3.9.49 Let $y = x^{\cos x}$. Then $\ln y = \cos x \ln x$. Differentiating both sides gives

$$\frac{1}{y} \cdot y' = (-\sin x)\ln x + \cos x \cdot \frac{1}{x}.$$

Therefore,

$$y' = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right).$$

At $\pi/2$ we have $y'(\pi/2) = (\pi/2)^0(0/(\pi/2) - \ln(\pi/2)) = -\ln(\pi/2)$.

3.9.50 Let $y = x^{\ln x}$. Then $\ln y = \ln x \ln x = (\ln x)^2$. Differentiating both sides gives

$$\frac{1}{y} \cdot y' = 2\ln x \cdot \frac{1}{x}.$$

Therefore,

$$y' = x^{\ln x - 1} \cdot 2\ln x.$$

At e we have $y'(e) = e^{1-1} \cdot 2\ln e = 2$.

3.9.51 Let $y = x^{\sqrt{x}}$. Then $\ln y = \sqrt{x} \cdot \ln x$. Differentiating both sides gives

$$\frac{1}{y} \cdot y' = \frac{1}{2\sqrt{x}} \ln x + \frac{\sqrt{x}}{x}.$$

Therefore,

$$y' = x^{\sqrt{x}}\left(\frac{\ln x + 2}{2\sqrt{x}}\right).$$

At 4 we have $y'(4) = 4^2\left(\frac{\ln 4 + 2}{4}\right) = 4\ln 4 + 8$.

3.9.52 Because $f(x) = (x^2+1)^x$, we have $\ln f(x) = x\ln(x^2+1)$. Thus,

$$\frac{1}{f(x)}f'(x) = (1)\ln(x^2+1) + x\cdot\frac{1}{x^2+1}\cdot 2x.$$

Therefore,

$$f'(x) = (x^2+1)^x\left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right).$$

We have $f'(1) = (1+1)^1(\ln(1+1) + \frac{2}{1+1}) = 2(\ln 2 + 1) = 2\ln 2 + 2$.

3.9.53 Because $f(x) = (\sin x)^{\ln x}$, we have $\ln f(x) = \ln x \ln\sin x$. Differentiating both sides gives

$$\frac{1}{f(x)}f'(x) = \frac{1}{x}\cdot\ln\sin x + \ln x\cdot\frac{1}{\sin x}\cos x.$$

Therefore,

$$f'(x) = (\sin x)^{\ln x}\left(\frac{\ln\sin x + x\ln x\cot x}{x}\right).$$

We have $f'(\pi/2) = 0$ because $\cot\pi/2 = 0$ and $\ln\sin(\pi/2) = \ln 1 = 0$.

3.9.54 Because $f(x) = \tan^{x-1}x$, we have $\ln f(x) = (x-1)\ln\tan x$. Differentiating both sides gives

$$\frac{1}{f(x)}f'(x) = (1)\ln\tan x + (x-1)\frac{1}{\tan x}\sec^2 x.$$

Therefore,

$$f'(x) = (\tan^{x-1}x)(\ln\tan x + (x-1)\csc x\sec x).$$

We have $f'(\pi/4) = (1)^{\pi/4-1}(\ln 1 + (\pi/4-1)(\sqrt{2})(\sqrt{2})) = \pi/2 - 2$.

3.9.55 Because $f(x) = (4\sin x + 2)^{\cos x}$, we have $\ln f(x) = \cos x\ln(4\sin x + 2)$. Differentiating both sides gives

$$\frac{1}{f(x)}f'(x) = -\sin x\cdot\ln(4\sin x + 2) + \cos x\cdot\frac{4\cos x}{(4\sin x + 2)}.$$

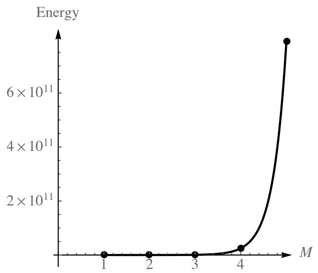
Therefore,

$$f'(x) = (4\sin x + 2)^{\cos x}\left(-\sin x\ln(4\sin x + 2) + \frac{2\cos^2 x}{(2\sin x + 1)}\right).$$

We have $f'(\pi) = 2^{-1}\left(0 + \frac{2}{1}\right) = \frac{1}{2}\cdot\frac{2}{1} = 1$.

3.9.56

a.



b. $\frac{dE}{dM} = 25000\cdot 1.5\cdot\ln 10\cdot 10^{1.5M}$. At $M = 3$ we have $\frac{dE}{dM} = 25000\cdot 1.5\cdot\ln 10\cdot 10^{9/2} \approx 2,730,530,025$ Joules per unit change in M. As the magnitude goes from 3 to 4, the energy goes up by this amount.

3.9.57

a. $T = 10\cdot 2^{-0.274\cdot 16}$ minutes ≈ 28.7 seconds.

b. $\dfrac{\Delta T}{\Delta a} = \dfrac{10 \cdot 2^{-0.274 \cdot 8} - 10 \cdot 2^{-0.274 \cdot 2}}{8-2} \approx -0.78$ minutes per 1000 feet, which is about -46.512 seconds per 1000 feet.

c. $\dfrac{dT}{da} = -2.74 \cdot 2^{-0.274 \cdot a} \cdot \ln 2$. At $a = 8$ we have $\dfrac{dT}{da} = -2.74 \cdot 2^{-0.274 \cdot 8} \cdot \ln 2 \approx -0.42$ minutes per 1000 feet. Every 1000 feet the airplane climbs, leaves about .42 minutes less time of consciousness, which corresponds to about 24.94 seconds.

3.9.58

a. At $Q = 10\mu$Ci we have $10 = 350 \cdot \left(\frac{1}{2}\right)^{t/13.1}$, so $\ln(1/35) = \frac{t}{13.1}\ln(1/2)$, so $t = 13.1 \cdot \frac{\ln 35}{\ln 2} \approx 67.19$ hours.

b. $\dfrac{dQ}{dt} = \dfrac{350}{13.1} \cdot \ln\left(\dfrac{1}{2}\right) \cdot \left(\dfrac{1}{2}\right)^{t/13.1}$. We have $Q'(12) \approx -9.81453$, $Q'(24) \approx -5.20136$, and $Q'(48) \approx -1.46087$. The rate at which the iodine decreases is decreasing in absolute value as time increases.

3.9.59 Let $y = x^{\sin x}$. Then $\ln y = \sin x \ln x$, so $\dfrac{1}{y} \cdot y' = \cos x \ln x + \dfrac{\sin x}{x}$. At the point $(1,1)$ we have $y' = \sin 1$, so the tangent line is given by $y - 1 = (\sin 1)(x-1)$, or $y = (\sin 1)x + 1 - \sin 1$.

3.9.60 Let $y = x^{\sqrt{x}}$. Then we have $\ln y = \sqrt{x} \cdot \ln x$, so $\dfrac{1}{y} \cdot y' = \dfrac{\ln x}{2\sqrt{x}} + \dfrac{\sqrt{x}}{x}$, so $y' = x^{\sqrt{x}}\left(\dfrac{\ln x}{2\sqrt{x}} + \dfrac{\sqrt{x}}{x}\right) = x^{\sqrt{x}}\left(\dfrac{\sqrt{x}\ln x + 2\sqrt{x}}{2x}\right)$. This expression is zero only when $\ln x + 2 = 0$, or $x = e^{-2}$.

3.9.61 Let $y = (x^2)^x = x^{2x}$. Then $\ln y = x \ln x^2$ and $\dfrac{1}{y} \cdot y' = \ln x^2 + 2$, so $y' = x^{2x}(\ln x^2 + 2)$. This quantity is zero when $\ln x^2 = -2$, or $x^2 = e^{-2}$. Thus there are horizontal tangents at $|x| = e^{-1}$, so for $x = \pm\frac{1}{e}$. The two tangent lines are given by $y = \dfrac{1}{e^{2/e}}$ (at $\left(\dfrac{1}{e}, \dfrac{1}{e^{2/e}}\right)$) and $y = e^{2/e}$ (at $\left(-\dfrac{1}{e}, e^{2/e}\right)$.)

3.9.62 Let $y = x^{\ln x}$. Then $\ln y = (\ln x)^2$. Thus $\dfrac{1}{y} \cdot y' = 2\ln x \cdot \dfrac{1}{x}$, so $y' = x^{\ln x}\left(\dfrac{2\ln x}{x}\right)$. This quantity is zero when $\ln x = 0$, which is at $x = 1$. The equation of the tangent line at $(1,1)$ is therefore $y = 1$.

3.9.63 $y' = 4 \cdot \dfrac{2x}{(x^2-1) \cdot \ln 3} = \dfrac{8x}{(x^2-1) \cdot \ln 3}$.

3.9.64 $y' = \dfrac{1}{x \ln 10}$.

3.9.65 $y' = -\sin x(\ln(\cos^2 x)) + \cos x \cdot \left(\dfrac{2\cos x(-\sin x)}{\cos^2 x}\right) = (-\sin x)(\ln(\cos^2 x) + 2)$.

3.9.66 $y' = \dfrac{1}{\ln 8 \tan x} \cdot \sec^2 x$.

3.9.67 $y' = \dfrac{d}{dx}(\log_4 x)^{-1} = -(\log_4 x)^{-2} \cdot \dfrac{1}{x \ln 4} = -\dfrac{1}{x(\ln 4)(\log_4 x)^2} = -\dfrac{\ln 4}{x \ln^2 x}$.

3.9.68 $y' = \dfrac{1}{(\ln 2)(\log_2 x)} \cdot \dfrac{1}{x \ln 2} = \dfrac{1}{(\ln 2)^2 \cdot x \cdot \log_2 x} = \dfrac{1}{x(\ln 2)\ln x}$.

3.9.69 $f'(x) = \dfrac{d}{dx}(4\ln(3x+1)) = \dfrac{4}{3x+1} \cdot 3 = \dfrac{12}{3x+1}$.

3.9.70 $f'(x) = \dfrac{d}{dx}(\ln 2x - 3\ln(x^2+1)) = \dfrac{1}{x} - \dfrac{6x}{x^2+1}$.

3.9.71 $f'(x) = \dfrac{d}{dx}\left(\dfrac{1}{2}\ln 10x\right) = \dfrac{d}{dx}\dfrac{1}{2}[\ln 10 + \ln x] = \dfrac{1}{2x}$.

3.9.72 $f'(x) = \dfrac{d}{dx}\left(\log_2 2^3 - \dfrac{1}{2}\log_2(x+1)\right) = 0 - \dfrac{1}{2}\cdot\dfrac{1}{(x+1)\ln 2} = -\dfrac{1}{(\ln 4)(x+1)}$.

3.9.73 $f'(x) = \dfrac{d}{dx}(\ln(2x-1) + 3\ln(x+2) - 2\ln(1-4x)) = \dfrac{2}{2x-1} + \dfrac{3}{x+2} + \dfrac{8}{1-4x}$.

3.9.74 $f'(x) = \dfrac{d}{dx}(4\ln(\sec x) + 2\ln(\tan x)) = \dfrac{4\sec x\tan x}{\sec x} + \dfrac{2\sec^2 x}{\tan x} = 4\tan x + 2\sec x\csc x$.

3.9.75 Let $y = x^{10x}$. Then $\ln y = 10x\ln x$, so $\dfrac{1}{y}\cdot y' = 10\ln x + 10$, and $y' = x^{10x}(10)(\ln x + 1)$.

3.9.76 Let $y = (2x)^{2x}$. Then $\ln y = 2x\ln(2x)$, and $\dfrac{1}{y}\cdot y' = 2\ln(2x) + 2$, so $y' = (2x)^{2x}(2)(\ln(2x)+1)$.

3.9.77 Let $y = \dfrac{(x+1)^{10}}{(2x-4)^8}$, so $\ln y = \ln\left(\dfrac{(x+1)^{10}}{(2x-4)^8}\right) = 10\ln(x+1) - 8\ln(2x-4)$. Then

$$\frac{1}{y}\cdot y' = \frac{10}{x+1} - \frac{8}{2x-4}\cdot 2,$$

$$y' = \frac{(x+1)^{10}}{(2x-4)^8}\cdot\left(\frac{10}{x+1} - \frac{8}{x-2}\right).$$

3.9.78 Let $y = x^2\cos x$. Then $\ln y = \ln(x^2\cos x) = 2\ln x + \ln(\cos x)$. So $\dfrac{1}{y}\cdot y' = \dfrac{2}{x} + \dfrac{1}{\cos x}\cdot(-\sin x)$, so

$$y' = x^2\cos x\cdot\left(\frac{2}{x} + \frac{1}{\cos x}\cdot(-\sin x)\right) = 2x\cos x - x^2\sin x.$$

3.9.79 Let $y = x^{\ln x}$. Then $\ln y = (\ln x)^2$. Thus $\dfrac{1}{y}\cdot y' = 2\ln x\cdot\dfrac{1}{x}$, so $y' = x^{\ln x}\left(\dfrac{2\ln x}{x}\right)$.

3.9.80 Let $y = \dfrac{\tan^{10} x}{(5x+3)^6}$. Then $\ln y = \ln\left(\dfrac{\tan^{10} x}{(5x+3)^6}\right) = 10\ln(\tan x) - 6\ln(5x+3)$. Then

$$\frac{1}{y}\cdot y' = \frac{10}{\tan x}\sec^2 x - \frac{6}{5x+3}\cdot 5,$$

$$y' = \frac{\tan^{10} x}{(5x+3)^6}\left(\frac{10\sec^2 x}{\tan x} - \frac{30}{5x+3}\right).$$

3.9.81 Let $y = \dfrac{(x+1)^{3/2}(x-4)^{5/2}}{(5x+3)^{2/3}}$. Then $\ln y = \ln\left(\dfrac{(x+1)^{3/2}(x-4)^{5/2}}{(5x+3)^{2/3}}\right) = \dfrac{3}{2}\ln(x+1) + \dfrac{5}{2}\ln(x-4) - \dfrac{2}{3}\ln(5x+3)$. Then

$$\frac{1}{y}\cdot y' = \frac{3}{2(x+1)} + \frac{5}{2(x-4)} - \frac{10}{3(5x+3)},$$

$$y' = \frac{(x+1)^{3/2}(x-4)^{5/2}}{(5x+3)^{2/3}}\cdot\left(\frac{3}{2(x+1)} + \frac{5}{2(x-4)} - \frac{10}{3(5x+3)}\right).$$

3.9.82 Let $y = \dfrac{x^8\cos^3 x}{\sqrt{x-1}}$. Then $\ln y = \ln\left(\dfrac{x^8\cos^3 x}{\sqrt{x-1}}\right) = 8\ln x + 3\ln\cos x - \dfrac{1}{2}\ln(x-1)$. Then

$$\frac{1}{y}\cdot y' = \frac{8}{x} - \frac{3\sin x}{\cos x} - \frac{1}{2x-2},$$

$$y' = \frac{x^8\cos^3 x}{\sqrt{x-1}}\left(\frac{8}{x} - 3\tan x - \frac{1}{2x-2}\right).$$

3.9.83 Let $y = (\sin x)^{\tan x}$, and assume $0 < x < \pi$, $x \neq \frac{\pi}{2}$. Then $\ln y = (\tan x)\ln(\sin x)$. Then

$$\frac{1}{y}\cdot y' = (\sec^2 x)\ln(\sin x) + \frac{\tan x\cos x}{\sin x},$$

$$y' = (\sin x)^{\tan x}\left((\sec^2 x)\ln(\sin x) + 1\right).$$

3.9.84 Let $y = (1+x^2)^{\sin x}$. Then $\ln y = \sin x\cdot\ln(1+x^2)$, so $\dfrac{1}{y}\cdot y' = \cos x\cdot\ln(1+x^2) + \sin x\cdot\dfrac{2x}{1+x^2}$. Therefore we have $y' = (1+x^2)^{\sin x}\left(\cos x\cdot\ln(1+x^2) + \dfrac{2x\sin x}{1+x^2}\right)$.

3.9.85 Let $y = \left(1+\dfrac{1}{x}\right)^x$. Then $\ln y = x\ln\left(1+\dfrac{1}{x}\right)$, so $\dfrac{1}{y}\cdot y' = \ln\left(1+\dfrac{1}{x}\right) + x\left(\dfrac{-1/x^2}{1+1/x}\right) = \ln\left(1+\dfrac{1}{x}\right) - \dfrac{1}{x+1}$. Therefore, $y' = \left(1+\dfrac{1}{x}\right)^x\left(\ln\left(1+\dfrac{1}{x}\right) - \dfrac{1}{x+1}\right)$.

3.9.86 Let $y = x^{x^{10}}$. Then $\ln y = x^{10}\ln x$, so $\dfrac{1}{y}\cdot y' = 10x^9\ln x + \dfrac{x^{10}}{x} = x^9(10\ln x + 1)$. Thus $y' = x^{x^{10}}\cdot x^9(10\ln x + 1)$.

3.9.87

a. False. $\log_2 9$ is a constant, so its derivative is 0.

b. False. If $x < -1$, then the right-hand side is defined while the left-hand side isn't.

c. False. The correct way to write that function would be $e^{(x+1)\ln 2}$.

d. False. $\dfrac{d}{dx}(\sqrt{2})^x = (\sqrt{2})^x\ln(\sqrt{2})$.

e. True. This follows from the generalized power rule.

f. True. Note that $\ln((4x+1)^{\ln x}) = (\ln x)\ln(4x+1) = \ln(4x+1)(\ln x) = \ln(x^{\ln(4x+1)})$. Then we have $\ln((4x+1)^{\ln x}) = \ln(x^{\ln(4x+1)})$, and exponentiating both sides gives the desired result.

3.9.88 $\dfrac{d^2}{dx^2}\left(\ln(x^2+1)\right) = \dfrac{d}{dx}\left(\dfrac{2x}{x^2+1}\right) = \dfrac{(x^2+1)\cdot 2 - 2x(2x)}{(x^2+1)^2} = \dfrac{2-2x^2}{(x^2+1)^2} = \dfrac{2(1-x^2)}{(x^2+1)^2}$.

3.9.89 $\dfrac{d^2}{dx^2}(\log x) = \dfrac{d}{dx}\left(\dfrac{1}{x\ln 10}\right) = -\dfrac{1}{x^2\ln 10}$.

3.9.90 $\dfrac{d}{dx}(2^x) = (2^x)\ln 2$. $\dfrac{d^2}{dx^2}(2^x) = \dfrac{d}{dx}(2^x)\ln 2 = (2^x)(\ln 2)^2$. Clearly, each new derivative is the same as the old multiplied by a factor of $\ln 2$. So after n derivatives, the result is $\dfrac{d^n}{dx^n}(2^x) = (2^x)\cdot(\ln 2)^n$.

3.9.91 $\dfrac{d^3}{dx^3}(x^2\ln x) = \dfrac{d^2}{dx^2}(2x\ln x + x) = \dfrac{d}{dx}(2\ln x + 2 + 1) = \dfrac{2}{x}$.

3.9.92

i. $y' = \dfrac{d}{dx}e^{x\ln(x^2+1)} = e^{x\ln(x^2+1)}\left(\ln(x^2+1)+\dfrac{2x^2}{x^2+1}\right) = (x^2+1)^x\left(\ln(x^2+1)+\dfrac{2x^2}{x^2+1}\right).$

ii. Let $y = (x^2+1)^x$. Then $\ln y = x\ln(x^2+1)$, so $\dfrac{1}{y}\cdot y' = \ln(x^2+1)+\dfrac{2x^2}{x^2+1}$, and thus $y' = (x^2+1)^x\left(\ln(x^2+1)+\dfrac{2x^2}{x^2+1}\right)$.

3.9.93

i. $y' = \dfrac{d}{dx}\left(e^{x\ln 3}\right) = \left(e^{x\ln 3}\right)\cdot\ln 3 = 3^x\ln 3.$

ii. Let $y = 3^x$. Then $\ln y = x\ln 3$. So $\dfrac{1}{y}\cdot y' = \ln 3$, and $y' = 3^x\ln 3$.

3.9.94

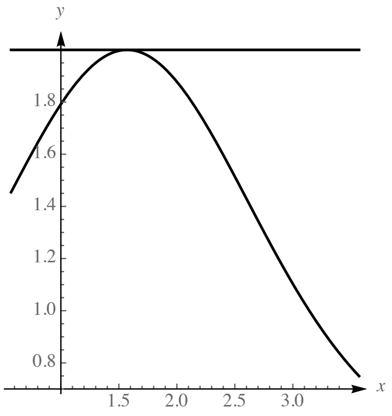
a.

$$\begin{aligned} y' &= \frac{d}{dx}\left(e^{\ln x\ln(4x+1)}\right) = \left(e^{\ln x\ln(4x+1)}\right)\cdot\left(\frac{\ln(4x+1)}{x}+\frac{4\ln x}{4x+1}\right)\\ &= (4x+1)^{\ln x}\left(\frac{(4x+1)\ln(4x+1)+4x\ln x}{x(4x+1)}\right). \end{aligned}$$

b. Let $y = \ln(4x+1)^{\ln x}$ so that $\ln y = (\ln x)\ln(4x+1)$. Then $\dfrac{1}{y}\cdot y' = \dfrac{\ln(4x+1)}{x}+\dfrac{4\ln x}{4x+1}$, so $y' = (4x+1)^{\ln x}\left(\dfrac{(4x+1)\ln(4x+1)+4x\ln x}{x(4x+1)}\right)$.

3.9.95

$y' = \dfrac{d}{dx}e^{\sin x\ln 2} = (\cos x)(\ln 2)2^{\sin x}$. At $x = \pi/2$ we have $y' = 0$, so the tangent line is given by $y = 2$.



3.9.96 We have

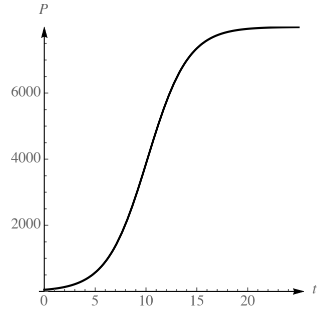
$$y' = -\sin x(\ln(\cos^2 x)) + \cos x\left(\frac{2(\cos x)(-\sin x)}{\cos^2 x}\right) = -\sin x(2+\ln(\cos^2 x)).$$

This quantity is zero when $\sin x = 0$ or $2+\ln(\cos^2 x) = 0$, and the latter occurs when $\cos^2 x = e^{-2}$, or $\cos x = \pm e^{-1}$.

$\sin x = 0$ for $x = 0, \pi, 2\pi$. $\cos x = e^{-1}$ for $x \approx 1.194$ and $x \approx 5.089$. Finally, $\cos x = -e^{-1}$ for $x \approx 1.948$ and $x \approx 4.336$. These seven numbers represent the locations of the horizontal tangent lines on $[0, 2\pi]$.

3.9.97

a. We used a graphing rectangle of $[0, 25] \times [0, 8000]$.



b. To find when $P(t)$ hits 5000, we solve $5000 = \frac{400000}{50+7950e^{-0.5t}}$, or $50 + 7950e^{-0.5t} = 80$. This leads to $7950e^{-0.5t} = 30$, or $-0.5t = \ln\left(\frac{30}{7950}\right)$. So we have $t = 2\ln(265) \approx 11.16$ years.

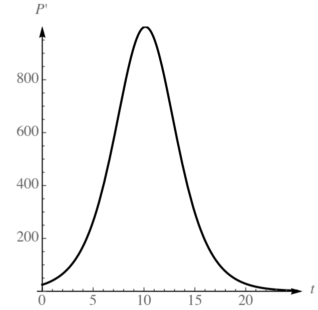
The carrying capacity is $\lim_{t\to\infty} P(t) = \dfrac{400,000}{50} = 8000$. Ninety percent of 8000 is 7200, so we seek the time when $P(t) = 7200$. We have $7200 = \frac{400000}{50+7950e^{-0.5t}}$, or $50 + 7950e^{-0.5t} = \frac{500}{9}$. This leads to $7950e^{-0.5t} = \frac{50}{9}$, or $-0.5t = \ln\left(\frac{50}{71550}\right)$. So we have $t = 2\ln\left(\frac{71550}{50}\right) \approx 14.53$ years.

c. $\dfrac{dP}{dt} = -\dfrac{400000}{(50 + 7950e^{-0.5t})^2} \cdot (7950)(-0.5)e^{-0.5t}$.

At $t = 0$ we have $P'(0) = \frac{400,000 \cdot 7950 \cdot .5}{8000^2} = \frac{1,590,000,000}{8000^2} \approx 25$ fish per year.

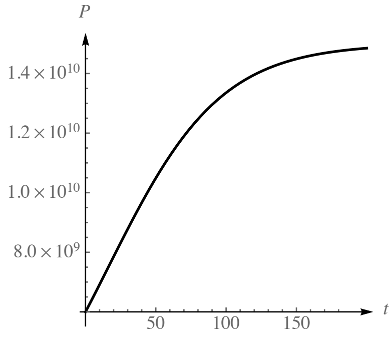
At $t = 5$ we have $P'(5) = \frac{1,590,000,000e^{-5/2}}{(50+7950e^{-5/2})^2} \approx 264$ fish per year.

d. The maximum is at about $t = 10$ years.



3.9.98

a. $P(t) = \frac{6\times10^9 \cdot 15\times10^9}{6\times10^9 + 9\times10^9 \cdot e^{-0.025t}} = \frac{3\times10^{10}}{2+3\cdot e^{-0.025t}}$.



b. $P(21) = \frac{3\times 10^{10}}{2+3\cdot e^{-0.525}} \approx 7.95 \times 10^9$.

$P(t) = 12,000,000,000$ when $2 + 3e^{-0.025t} = \frac{5}{2}$, which is when $e^{-0.025t} = \frac{1}{6}$. This occurs for $t = 40 \ln 6 \approx 71.67$ years.

3.9.99

a. $\ln(P(t)) = \ln(3\cdot 10^{10}) - \ln(2+3e^{-0.025t})$. $\frac{d}{dt}\ln(P(t)) = \frac{P'(t)}{P(t)} = r(t) = \frac{0.075\cdot e^{-0.025t}}{2+3e^{-0.025t}}$. $r(0) = \frac{0.075}{5} = 0.015$, so the population is growing at 1.5% per year in 1999.

b. $r(11) = \frac{0.075e^{-0.275}}{2+3e^{-0.275}} \approx 0.0133$.

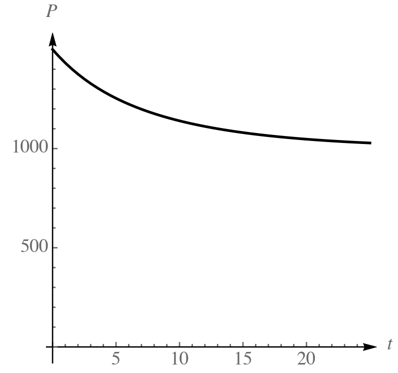
$r(21) = \frac{0.075e^{-0.525}}{2+3e^{-0.525}} \approx 0.0118$.

The relative growth rate decreases over time.

c. $\lim_{t\to\infty} r(t) = \lim_{t\to\infty} \frac{0.075}{3+2e^{0.025t}} = 0$, because the denominator increases without bound. The relative growth rate becomes smaller and smaller as the population nears the carrying capacity.

3.9.100

a. $P(t) = \frac{1500\cdot 1000}{1500-500e^{-0.1t}}$. As $t \to \infty$, the population decreases and gets closer to the carrying capacity of 1000.



b. $P'(t) = -\frac{7.5\times 10^7 e^{-0.1t}}{(1500-500e^{-0.1t})^2}$. $P'(0) = -\frac{7.5\times 10^7}{(1000)^2} = -75$ deer per year.

c. The population reaches 1200 deer when $1200 = \frac{1500\cdot 1000}{1500-500e^{-0.1t}}$. This occurs when $-500e^{-0.1t} = \frac{1.5\times 10^6}{1200} - 1500$, or when $-0.1t = \ln(0.5)$, or when $t = -10\ln(0.5) \approx 6.93$ years. It will take almost 7 years until the deer population reaches 1200.

3.9.101

a.

t	$A(t)$
5	\$17,442.50
15	\$72,704.68
25	\$173,248.49
35	\$356,177.57

Average growth on $[5, 15]$ is $\frac{A(15)-A(5)}{10} \approx \$5526$ per year.

Average growth on $[15, 25]$ is $\frac{A(25)-A(15)}{10} \approx \$10,054$ per year.

Average growth on $[25,35]$ is $\dfrac{A(35)-A(25)}{10} \approx \$18,293$ per year.

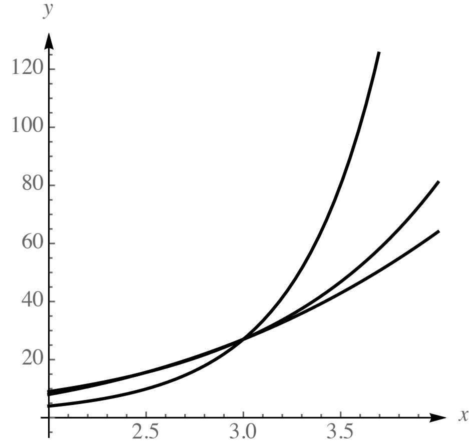
b. $A(40) \approx \$497,872.68$.

c. $A'(t) = 50,000 \cdot 12 \cdot (1.005)^{12t} \cdot \ln(1.005) \approx 2993 \cdot (1.005)^{12t}$. The rate of growth of the investment increases over time, so the earlier you start saving, the higher the rate of increase will be when you retire.

3.9.102 We search for a solution to $x^p = e^x$. If the two curves will have only one point of intersection, then they should be tangent at the point of intersection. So we need $px^{p-1} = e^x$, so we require $px^{p-1} = x^p$, so $x = p$. So $p^p = e^p$, and therefore we must have $p = e$. So we have $x^e = e^x$ intersecting at the point (e, e^e), and that is the only point of intersection.

3.9.103 We search for a solution to $x = p^x$. If the two curves will have only one point of intersection, then they should be tangent at the point of intersection. So we need $1 = p^x \ln p$, or $\dfrac{1}{\ln p} = p^x = x$. So $\ln p = \dfrac{1}{x}$ and $p = e^{1/x}$. Then we have $x = p^x = (e^{1/x})^x = e$. So the point of intersection is (e,e) and the value of p is $e^{1/e} \approx 1.44467$.

3.9.104 By inspection, we see that the point $(3,27)$ is on all three curves



3.9.105 Let $f(x) = \ln x$ and $a = e$. Then $f'(e) = \lim\limits_{x\to e} \dfrac{\ln x - 1}{x - e} = \dfrac{1}{e}$.

3.9.106 Let $f(x) = \ln x$ and $a = e^8$. Then $f'(e^8) = \lim\limits_{h\to 0} \dfrac{\ln(e^8 + h) - 8}{h} = \dfrac{1}{e^8}$.

3.9.107 Let $f(x) = x^x$ and $a = 3$. Then $f'(3) = \lim\limits_{h\to 0} \dfrac{(3+h)^{3+h} - 27}{h} = 3^3 \cdot (\ln 3 + 1) = 27(1 + \ln 3)$.

3.9.108 Let $f(x) = 5^x$ and $a = 2$. Then $f'(2) = \lim\limits_{x\to 2} \dfrac{5^x - 25}{x - 2} = 25 \ln 5$.

3.9.109 Let $y = u(x)^{v(x)}$. Then $\ln y = v(x) \ln u(x)$, so $\dfrac{1}{y} \cdot y' = v'(x) \ln u(x) + v(x) \cdot \dfrac{u'(x)}{u(x)}$. Thus we have $y' = u(x)^{v(x)} \cdot \left(v'(x) \ln u(x) + v(x) \cdot \dfrac{u'(x)}{u(x)} \right) = u(x)^{v(x)} \cdot \left(\dfrac{v(x)}{u(x)} \dfrac{du}{dx} + \ln u(x) \dfrac{dv}{dx} \right)$.

3.9.110 The slope of the tangent line to $y = b^x$ is $y' = b^x \ln b$. We are seeking the y-coordinate of the point (x, b^x) where $b^x \ln b = \dfrac{b^x}{x}$. we must have $x = \dfrac{1}{\ln b}$, in which case the y-coordinate is $b^{\frac{1}{\ln b}} = e$. To see that $b^{\frac{1}{\ln b}} = e$, let $z = b^{\frac{1}{\ln b}}$. Then $\ln z = \ln b^{\frac{1}{\ln b}} = \dfrac{\ln b}{\ln b} = 1$, so $z = e$.

3.10 Derivatives of Inverse Trigonometric Functions

3.10.1
$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$.
$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$, $-\infty < x < \infty$.
$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$.

3.10.2 $y' = \frac{1}{\sqrt{1-x^2}}$. At $x = 0$ we have $y'(0) = 1$.

3.10.3 $y' = \frac{1}{1+x^2}$. At $x = -2$ we have $y'(-2) = \frac{1}{1+4} = \frac{1}{5}$.

3.10.4 $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} = - -\frac{1}{\sqrt{1-x^2}} = -\frac{d}{dx}\cos^{-1} x$.

3.10.5 $(f^{-1})'(8) = \frac{1}{f'(2)} = \frac{1}{4}$.

3.10.6 $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$ where $f(x_0) = y_0$.

3.10.7

a. Note that $f(0) = 4$, so $f^{-1}(4) = 0$. $(f^{-1})'(4) = \frac{1}{f'(0)} = \frac{1}{2}$.

b. Note that $f(1) = 6$, so $f^{-1}(6) = 1$. $(f^{-1})'(6) = \frac{1}{f'(1)} = \frac{1}{3/2} = \frac{2}{3}$.

c. Note that there is no given x so that $f(x) = 1$, so the desired derivative cannot be determined.

d. From the table directly, $f'(1) = \frac{3}{2}$.

3.10.8

a. $f'(f(0)) = f'(2) = 2$.

b. Note that $f(-4) = 0$, so $f^{-1}(0) = -4$. $(f^{-1})'(0) = \frac{1}{f'(-4)} = \frac{1}{5}$.

c. Note that $f(-2) = 1$, so $f^{-1}(1) = -2$. $(f^{-1})'(1) = \frac{1}{f'(-2)} = \frac{1}{4}$.

d. $(f^{-1})'(f(4)) = \frac{1}{f'(4)} = \frac{1}{1} = 1$.

3.10.9 Because $(f^{-1})'(8) = \frac{1}{f'(3)} = \frac{1}{7}$, the equation of the tangent line at $x = 8$ is $y - 3 = \frac{1}{7}(x-8)$ or $y = \frac{1}{7}x + \frac{13}{7}$.

3.10.10 $f(3) = 16$ and $f'(3) = 5$, so $f^{-1}(16) = 3$ and $(f^{-1})'(16) = \frac{1}{5}$.

3.10.11 Let $f(x) = \sin x$. Then $f'(\pi/6) = \cos(\pi/6) = \frac{\sqrt{3}}{2}$, which implies that the slope of the curve $y = \sin^{-1} x$ at $(1/2, \pi/6)$ is $\frac{2}{\sqrt{3}}$.

3.10.12 Let $f(x) = \tan x$. Then $f'(\pi/4) = \sec^2(\pi/4) = 2$. So the slope of the curve $y = \tan^{-1} x$ at $(1, \pi/4)$ is $\frac{1}{2}$.

3.10.13 $\frac{d}{dx}\sin^{-1}(2x) = \frac{2}{\sqrt{1-4x^2}}$.

3.10.14 $\frac{d}{dx}(x\sin^{-1} x) = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$.

3.10.15 $\frac{d}{dw}\cos(\sin^{-1}(2w)) = (-\sin(\sin^{-1}(2w)))\cdot\frac{d}{dw}\left(\sin^{-1}(2w)\right) = -2w\cdot\frac{2}{\sqrt{1-4w^2}} = -\frac{4w}{\sqrt{1-4w^2}}$.

3.10.16 $\frac{d}{dx}\sin^{-1}(\ln x) = \frac{1}{\sqrt{1-(\ln x)^2}}\cdot\frac{d}{dx}\ln x = \frac{1}{x\sqrt{1-(\ln x)^2}}$.

3.10.17 $\frac{d}{dx}\sin^{-1}(e^{-2x}) = \frac{1}{\sqrt{1-e^{-4x}}}\cdot\frac{d}{dx}e^{-2x} = -\frac{2e^{-2x}}{\sqrt{1-e^{-4x}}}$.

3.10.18 $\frac{d}{dx}\sin^{-1}(e^{\sin x}) = \frac{1}{\sqrt{1-e^{2\sin x}}}\cdot\frac{d}{dx}e^{\sin x} = \frac{(\cos x)\cdot(e^{\sin x})}{\sqrt{1-e^{2\sin x}}}$.

3.10.19 $f'(x) = \frac{1}{1+100x^2}\cdot 10 = \frac{10}{100x^2+1}$.

3.10.20 $f'(x) = 2\tan^{-1} x + \frac{2x}{1+x^2} - \frac{2x}{1+x^2} = 2\tan^{-1} x$.

3.10.21 $\frac{d}{dy}\tan^{-1}(2y^2-4) = \frac{1}{1+(2y^2-4)^2}\cdot\frac{d}{dy}(2y^2-4) = \frac{4y}{1+(2y^2-4)^2}$.

3.10.22 $\frac{d}{dz}\tan^{-1}(1/z) = \frac{1}{1+(1/z)^2}\cdot\frac{d}{dz}\frac{1}{z} = -\frac{1}{z^2(1+(1/z)^2)} = -\frac{1}{z^2+1}$.

3.10.23 $\frac{d}{dz}\cot^{-1}\sqrt{z} = -\frac{1}{1+\sqrt{z}^2}\cdot\frac{d}{dz}\sqrt{z} = -\frac{1}{1+z}\cdot\frac{1}{2\sqrt{z}} = -\frac{1}{2\sqrt{z}(1+z)}$.

3.10.24 $\frac{d}{dx}\sec^{-1}\sqrt{x} = \frac{1}{\sqrt{x}\sqrt{x-1}}\cdot\frac{d}{dx}\sqrt{x} = \frac{1}{2x\sqrt{x-1}}$ for $x > 1$.

3.10.25

$$\begin{aligned}f'(x) &= 2x + 6x^2\cot^{-1} x - \frac{2x^3}{1+x^2} - \frac{2x}{1+x^2} = 2x + 6x^2\cot^{-1} x - \frac{2x^3+2x}{1+x^2}\\ &= 2x + 6x^2\cot^{-1} x - \frac{2x(x^2+1)}{1+x^2} = 6x^2\cot^{-1} x.\end{aligned}$$

3.10.26 $f'(x) = \cos^{-1} x - \frac{x}{\sqrt{1-x^2}} - \frac{-2x}{2\sqrt{1-x^2}} = \cos^{-1} x$.

3.10.27 $f'(2) = 2w - \frac{2w}{1+w^4} = \frac{2w+2w^5}{1+w^4} - \frac{2w}{1+w^4} = \frac{2w^5}{1+w^4}$.

3.10.28 $f'(t) = \frac{1}{\sin^{-1} t^2}\cdot\frac{1}{\sqrt{1-(t^2)^2}}\cdot 2t = \frac{2t}{(\sin^{-1} t^2)\sqrt{1-t^4}}$.

3.10.29 $\frac{d}{dx}\cos^{-1}\frac{1}{x} = -\frac{1}{\sqrt{1-\frac{1}{x^2}}}\cdot\left(-\frac{1}{x^2}\right) = \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}} = \frac{|x|}{x^2\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$.

3.10.30 $\frac{d}{dt}(\cos^{-1}t)^2 = 2\cos^{-1}t\cdot\left(-\frac{1}{\sqrt{1-t^2}}\right) = -\frac{2\cos^{-1}t}{\sqrt{1-t^2}}.$

3.10.31 $\frac{d}{du}\csc^{-1}(2u+1) = -\frac{1}{|2u+1|\sqrt{(2u+1)^2-1}}\cdot 2 = -\frac{2}{|2u+1|\sqrt{(2u+1)^2-1}} = -\frac{1}{|2u+1|\sqrt{u^2+u}}.$

3.10.32 $\frac{d}{dt}\ln(\tan^{-1}t) = \frac{1}{\tan^{-1}t}\cdot\frac{1}{1+t^2}.$

3.10.33 $\frac{d}{dy}\cot^{-1}\left(\frac{1}{1+y^2}\right) = \left(-\frac{1}{1+\left(\frac{1}{1+y^2}\right)^2}\right)\cdot\left(-\frac{2y}{(1+y^2)^2}\right) = \frac{2y}{(1+y^2)^2+1}.$

3.10.34 $\frac{d}{dw}\sin[\sec^{-1}2w] = \cos[\sec^{-1}2w]\cdot\frac{2}{|2w|\sqrt{4w^2-1}} = \frac{1/w}{2|w|\sqrt{4w^2-1}} = \frac{1}{2w|w|\sqrt{4w^2-1}}.$

3.10.35 $\frac{d}{dx}\sec^{-1}(\ln x) = \frac{1}{|\ln x|\sqrt{(\ln x)^2-1}}\cdot\frac{1}{x} = \frac{1}{x|\ln x|\sqrt{(\ln x)^2-1}}.$

3.10.36 $\frac{d}{dx}\tan^{-1}(e^{4x}) = \frac{1}{1+e^{8x}}\cdot 4e^{4x} = \frac{4e^{4x}}{1+e^{8x}}.$

3.10.37 $\frac{d}{dx}\csc^{-1}(\tan e^x) = -\frac{1}{|\tan e^x|\sqrt{(\tan e^x)^2-1}}\cdot(\sec^2 e^x)\cdot e^x.$

3.10.38 $\frac{d}{dx}\sin(\tan^{-1}(\ln x)) = \cos(\tan^{-1}(\ln x))\cdot\frac{1}{1+(\ln x)^2}\cdot\frac{1}{x} = \frac{\cos(\tan^{-1}(\ln x))}{x(1+(\ln x)^2)}.$

3.10.39 $\frac{d}{ds}\cot^{-1}(e^s) = -\frac{1}{1+e^{2s}}\cdot e^s = -\frac{e^s}{1+e^{2s}}.$

3.10.40

$$\begin{aligned}\frac{d}{dx}\frac{1}{\tan^{-1}(x^2+4)} &= \frac{d}{dx}\left(\tan^{-1}(x^2+4)\right)^{-1} = -\left(\tan^{-1}(x^2+4)\right)^{-2}\cdot\frac{1}{1+(x^2+4)^2}\cdot 2x\\ &= -\frac{2x}{(1+(x^2+4)^2)\cdot(\tan^{-1}(x^2+4))^2}.\end{aligned}$$

3.10.41 $f'(x) = \frac{1}{1+4x^2}\cdot 2$, so $f'(1/2) = \frac{1}{1+1}\cdot 2 = 1$. Thus the equation of the tangent line is $y-\pi/4 = 1(x-1/2)$, or $y = x+\frac{\pi}{4}-\frac{1}{2}$.

3.10.42 $f'(x) = \frac{1}{\sqrt{1-x^2/16}}\cdot\frac{1}{4} = \frac{1}{\sqrt{16-x^2}}$, so $f'(2) = \frac{1}{\sqrt{12}}$. Thus the equation of the tangent line is $y-\pi/6 = \frac{1}{\sqrt{12}}(x-2)$, or $y = \frac{1}{2\sqrt{3}}x+\frac{\pi}{6}-\frac{1}{\sqrt{3}}$.

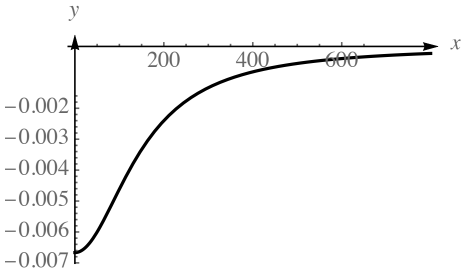
3.10.43 $f'(x) = -\frac{1}{\sqrt{1-x^4}}\cdot 2x = -\frac{2x}{\sqrt{1-x^4}}$, so $f'(1/\sqrt{2}) = -\frac{\sqrt{2}}{\sqrt{1-(1/4)}} = -\frac{2\sqrt{2}}{\sqrt{3}}$. Thus the equation of the tangent line is $y-\pi/3 = -\frac{2\sqrt{2}}{\sqrt{3}}\left(x-1/\sqrt{2}\right)$, or $y = -\frac{2\sqrt{2}}{\sqrt{3}}x+\frac{\pi}{3}+\frac{2}{\sqrt{3}}$.

3.10.44 $f'(x) = \frac{1}{e^x\sqrt{e^{2x}-1}}\cdot e^x$, so $f'(\ln 2) = \frac{2}{2\sqrt{4-1}} = \frac{1}{\sqrt{3}}$. Thus the equation of the tangent line is $y-\pi/3 = \frac{1}{\sqrt{3}}(x-\ln 2)$, or $y = \frac{1}{\sqrt{3}}x+\frac{\pi}{3}-\frac{\ln 2}{\sqrt{3}}$.

3.10.45

a. $\frac{x}{150} = \cot\theta$, so $\theta = \cot^{-1}\left(\frac{x}{150}\right)$. Then $\frac{d\theta}{dx} = -\frac{1}{1+\left(\frac{x}{150}\right)^2} \cdot \frac{1}{150} = -\frac{150}{(150)^2 + x^2}$. When $x = 500$, we have $\frac{d\theta}{dx} = -\frac{150}{150^2 + 500^2} \approx -0.00055$ radians per meter.

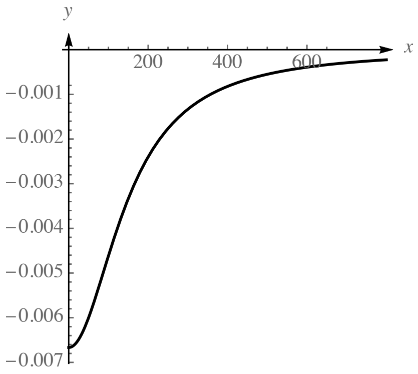
b. The most rapid change is at $x = 0$ where $\frac{d\theta}{dx} = -\frac{1}{150} \approx -0.0067$ radians per meter.



3.10.46

a. $\frac{x}{400} = \cot\theta$, so $\theta = \cot^{-1}\left(\frac{x}{400}\right)$. Then $\frac{d\theta}{dx} = -\frac{1}{1+\left(\frac{x}{400}\right)^2} \cdot \frac{1}{400} = -\frac{400}{(400)^2 + x^2}$. When $x = 500$, we have $\frac{d\theta}{dx} = -\frac{400}{400^2 + 500^2} \approx -0.000976$ radians per meter.

b. The most rapid change is at $x = 0$, where the plane is directly over head.



3.10.47 $f(4) = 16$ so $(f^{-1})'(16) = \frac{1}{f'(4)} = \frac{1}{3}$.

3.10.48 $f(4) = 10$ so $(f^{-1})'(10) = \frac{1}{f'(4)} = \frac{1}{1/2} = 2$.

3.10.49 $f(0) = 1$ so $(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{5/e} = \frac{e}{5}$.

3.10.50 $f(2) = 5$ so $(f^{-1})'(5) = \dfrac{1}{f'(2)} = \dfrac{1}{4}$.

3.10.51 $f\left(\dfrac{\pi}{4}\right) = 1$ so $(f^{-1})'(1) = \dfrac{1}{f'\left(\dfrac{\pi}{4}\right)} = \dfrac{1}{\sec^2\left(\dfrac{\pi}{4}\right)} = \dfrac{1}{2}$.

3.10.52 $f(-3) = 12$ so $(f^{-1})'(12) = \dfrac{1}{f'(-3)} = \dfrac{1}{-8} = -\dfrac{1}{8}$.

3.10.53 $f(4) = 2$ so $(f^{-1})'(2) = \dfrac{1}{f'(4)} = \dfrac{1}{(1/2\sqrt{4})} = 4$.

3.10.54 $f(0) = 4$ so $(f^{-1})'(4) = \dfrac{1}{f'(0)} = \dfrac{1}{40}$.

3.10.55 $f(4) = 36$ and $(f^{-1})'(36) = \dfrac{1}{f'(4)} = \dfrac{1}{2(4+2)} = \dfrac{1}{12}$.

3.10.56 $f(1/3) = 0$ so $(f^{-1})'(0) = \dfrac{1}{f'(1/3)} = \dfrac{1}{3/\ln 10} = \dfrac{\ln 10}{3}$.

3.10.57 Note that $f(1) = 3$. So $(f^{-1})'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{4}$.

3.10.58 $(f^{-1})'(4) = \dfrac{1}{f'(7)} = \dfrac{1}{2/3} = \dfrac{3}{2}$.

3.10.59 $(f^{-1})'(4) = \dfrac{1}{f'(7)} = \dfrac{4}{5}$, so $f'(7) = \dfrac{5}{4}$.

3.10.60 $(f^{-1})'(7) = \dfrac{1}{f'(4)} = \dfrac{1}{1/5} = 5$.

3.10.61

a. True, because $\dfrac{d}{dx}\sin^{-1}x = -\dfrac{d}{dx}\cos^{-1}x$.

b. False. $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$ for all x, and this doesn't equal $\sec^2 x$ anywhere except at the origin (one curve is always less than or equal to one, and the other is always greater than or equal to one).

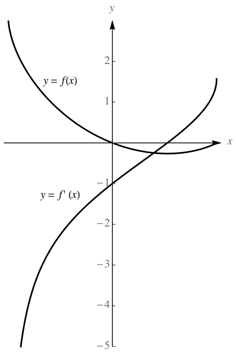
c. True. $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$, and this is minimal when its denominator is as big as possible, which occurs when $x = 0$. So the smallest possible slope of a tangent line for this function on $(-1, 1)$ is $\dfrac{1}{\sqrt{1-0^2}} = 1$.

d. True. $\dfrac{d}{dx}\sin x = \cos x$ and $\cos x = 1$ for $x = 0$ and $-1 \le \cos x \le 1$ for all x. Thus 1 is the largest possible slope for a tangent line to the sine function.

e. True. This follows because the function $\dfrac{1}{x}$ is its own inverse. (Note that $f(f(x)) = \dfrac{1}{f(x)} = \dfrac{1}{1/x} = x$.) Thus, the derivative of the inverse of f is the derivative of f, which is $-\dfrac{1}{x^2}$.

3.10.62

a.

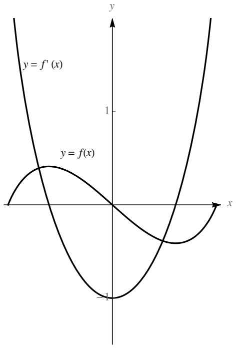


b. $f'(x) = \sin^{-1}(x) + \dfrac{x-1}{\sqrt{1-x^2}}$.

c. Note that f' is zero and f has a horizontal tangent line at about $x = 0.53$.

3.10.63

a.

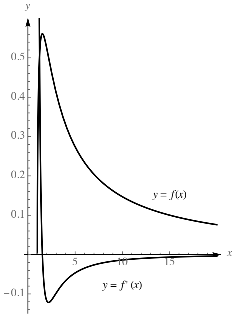


b. $f'(x) = 2x\sin^{-1}(x) + \dfrac{x^2-1}{\sqrt{1-x^2}}$.

c. Note that f' is zero and f has a horizontal tangent line at about $x = -0.61$ and at about $x = 0.61$.

3.10.64

a.

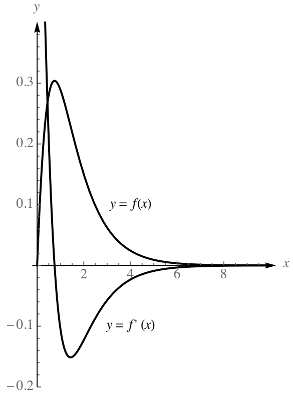


b. $$f'(x) = \frac{x \cdot \dfrac{1}{|x|\sqrt{x^2-1}} - \sec^{-1} x}{x^2} = \frac{1}{x|x|\sqrt{x^2-1}} - \frac{\sec^{-1} x}{x^2}.$$

c. Note that f' is zero and f has a horizontal tangent line at about $x = 1.53$.

3.10.65

a. ,

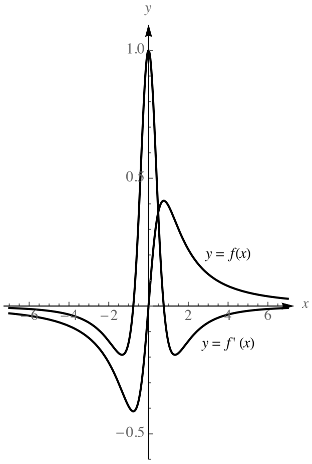


b. $f'(x) = -e^{-x}\tan^{-1}x + e^{-x}\dfrac{1}{1+x^2}$.

c. Note that f' is zero and f has a horizontal tangent line at about $x = 0.75$.

3.10.66

a.



b. $f'(x) = \dfrac{(x^2+1)\dfrac{1}{x^2+1} - \tan^{-1}(x)(2x)}{(x^2+1)^2} = \dfrac{1-2x\tan^{-1}(x)}{(x^2+1)^2}$.

c. Note that f' is zero and f has a horizontal tangent line at about $x = 0.765$ and at about $x = -0.765$.

3.10.67 Let $f(y) = 3y - 4$. Then $f'(y) = 3$ for all y in the domain of f. Let $y = f^{-1}(x)$. $(f^{-1})'(x) = \dfrac{1}{f'(y)} = \dfrac{1}{3}$.

3.10.68 Let $x = y^3 + 3$. Then $1 = 3y^2y'$, so $y' = \dfrac{1}{3y^2} = \dfrac{1}{3(x-3)^{2/3}}$, where $x \neq 3$.

3.10.69 Let $x = f(y) = y^2 - 4$ for $y > 0$. Note that this means that $y = \sqrt{x+4}$. Then $f'(y) = 2y$. So $(f^{-1})'(x) = \dfrac{1}{f'(y)} = \dfrac{1}{2y} = \dfrac{1}{2\sqrt{x+4}}$.

3.10.70 Let $x = \dfrac{y}{y+5}$. Then $x(y+5) = y$, so $y - xy = 5x$. Thus, $y = f^{-1}(x) = \dfrac{5x}{1-x}$. Therefore $(f^{-1})'(x) = \dfrac{(1-x)5 - 5x(-1)}{(1-x)^2} = \dfrac{5}{(1-x)^2}$.

3.10.71 $y = e^{3x+1}$, so $\ln y = 3x + 1$, and thus $x = \dfrac{\ln y - 1}{3}$. Then $f^{-1}(x) = \dfrac{\ln x - 1}{3}$, so $(f^{-1})'(x) = \dfrac{1}{3x}$.

3.10.72 $y = \ln(5x+4)$, so $e^y = 5x+4$, so $x = \dfrac{e^y - 4}{5}$. Then $f^{-1}(x) = \dfrac{e^x - 4}{5}$ and $(f^{-1})'(x) = \dfrac{e^x}{5}$.

3.10.73 $y = 10^{12x-6}$ so $\log_{10} y = 12x - 6$, so $x = \dfrac{\log_{10} y + 6}{12}$. Then $f^{-1}(x) = \dfrac{\log_{10} x + 6}{12}$ and $(f^{-1})'(x) = \dfrac{1}{12x \ln 10}$.

3.10.74 $y = \log_{10} 2x+6$ so $10^y = 2x+6$, so $x = \dfrac{10^y - 6}{2}$. Then $f^{-1}(x) = \dfrac{10^x - 6}{2}$ and $(f^{-1})'(x) = \dfrac{10^x \ln 10}{2}$.

3.10.75 For $y \geq -2$, let $x = \sqrt{y+2}$. Note that it then follows that $x \geq 0$. Then $1 = \dfrac{y'}{2\sqrt{y+2}}$, and $x^2 = y+2$, so $y = x^2 - 2$. Thus we have $(f^{-1})'(x) = y' = 2\sqrt{x^2-2+2} = 2|x| = 2x$, because $x \geq 0$.

3.10.76 For $y > 0$, let $x = y^{2/3}$. Then $1 = \dfrac{2}{3} y^{-1/3} y'$. So $y' = \dfrac{3}{2} y^{1/3} = \dfrac{3}{2}(x^{3/2})^{1/3} = \dfrac{3}{2} x^{1/2}$ where $x > 0$.

3.10.77 For $y > 0$, let $x = y^{-1/2}$. Then $1 = -\dfrac{1}{2} y^{-3/2} y'$, so $y' = -2y^{3/2} = -2(x^{-2})^{3/2} = -2x^{-3}$ where $x > 0$.

3.10.78 Let $f(y) = |y+2|$ for $y \leq -2$. Then $f(y) = -(y+2)$, and $f'(y) = -1$. Thus $(f^{-1})'(x) = \dfrac{1}{f'(y)} = \dfrac{1}{-1} = -1$.

3.10.79

a. Because $\dfrac{l}{10} = \csc(\theta)$, $\theta = \csc^{-1}\left(\dfrac{l}{10}\right)$, and $\dfrac{d\theta}{dl} = -\dfrac{1}{(l/10)\sqrt{(l/10)^2 - 1}} \cdot \dfrac{1}{10} = -\dfrac{10}{l\sqrt{l^2 - 100}}$.

b. $\left.\dfrac{d\theta}{dl}\right|_{l=50} = -\dfrac{10}{50\sqrt{2500-100}} \approx -0.0041$ radians per foot.

$\left.\dfrac{d\theta}{dl}\right|_{l=20} = -\dfrac{10}{20\sqrt{400-100}} \approx -0.029$ radians per foot.

$\left.\dfrac{d\theta}{dl}\right|_{l=11} = -\dfrac{10}{11\sqrt{121-100}} \approx -0.198$ radians per foot.

c. $\displaystyle\lim_{l \to 10^+} -\frac{10}{l\sqrt{l^2-100}} = -\infty$. The angle changes very quickly as we approach the dock.

d. $\dfrac{d\theta}{dl}$ is negative because this measures the change in θ as l increases – but when the boat is approaching the dock, l is decreasing.

3.10.80

a. Because the triangle from the top of the cliff to the falcon is isosceles and has a base of $80-h$, we get that the falcon is also $80-h$ feet from the cliff. So $\tan\theta = \dfrac{h}{80-h}$, or $\theta = \tan^{-1}\left(\dfrac{h}{80-h}\right)$.

b. $\dfrac{d\theta}{dh} = \dfrac{d}{dh}\tan^{-1}\left(\dfrac{h}{80-h}\right) = \dfrac{1}{1+\left(\dfrac{h}{80-h}\right)^2} \cdot \dfrac{80-h+h}{(80-h)^2} = \dfrac{80}{(80-h)^2+h^2}$.

$\left.\dfrac{d\theta}{dh}\right|_{h=60} = \dfrac{80}{20^2+60^2} = \dfrac{1}{50}$ radians per foot.

3.10.81

a. $\sin\theta = \dfrac{c}{D}$, so $\theta = \sin^{-1}\left(\dfrac{c}{D}\right)$. Thus $\dfrac{d\theta}{dc} = \dfrac{1/D}{\sqrt{1-\left(\dfrac{c}{D}\right)^2}} = \dfrac{1}{\sqrt{D^2-c^2}}$.

b. $\left.\dfrac{d\theta}{dc}\right|_{c=0} = \dfrac{1}{\sqrt{D^2}} = \dfrac{1}{D}$.

3.10.82

a. $\cos\theta = \dfrac{c}{D}$, so $\theta = \cos^{-1}\left(\dfrac{c}{D}\right)$. Thus $\dfrac{d\theta}{dc} = -\dfrac{1/D}{\sqrt{1-\left(\dfrac{c}{D}\right)^2}} = -\dfrac{1}{\sqrt{D^2-c^2}}$.

b. $\left.\dfrac{d\theta}{dc}\right|_{c=0} = -\dfrac{1}{D}$. This is the opposite result of exercise 69, as θ now increases with decreasing c.

3.10.83 $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$ where $y_0 = f(x_0)$.
$\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\cos(\sin^{-1}x)} = \dfrac{1}{\sqrt{1-\sin^2(\sin^{-1}x)}} = \dfrac{1}{\sqrt{1-x^2}}$.

3.10.84

a. $\dfrac{d}{dx}\cos^{-1}x = \dfrac{1}{-\sin(\cos^{-1}x)} = -\dfrac{1}{\sqrt{1-\cos^2(\cos^{-1}x)}} = -\dfrac{1}{\sqrt{1-x^2}}$.

b. $\dfrac{d}{dx}(\sin^{-1}x + \cos^{-1}x) = \dfrac{d}{dx}\dfrac{\pi}{2} = 0$, so $\dfrac{d}{dx}\sin^{-1}x = -\dfrac{d}{dx}\cos^{-1}x$. But $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$, so $\dfrac{d}{dx}\cos^{-1}x = -\dfrac{1}{\sqrt{1-x^2}}$.

3.10.85 Using the identity $\cot^{-1}x + \tan^{-1}x = \dfrac{\pi}{2}$, we have the $\dfrac{d}{dx}\cot^{-1}x + \dfrac{d}{dx}\tan^{-1}x = 0$, so $\dfrac{d}{dx}\cot^{-1}x = -\dfrac{d}{dx}\tan^{-1}x$. Likewise, because $\csc^{-1}x + \sec^{-1}x = \dfrac{\pi}{2}$, we have $\dfrac{d}{dx}\csc^{-1}x + \dfrac{d}{dx}\sec^{-1}x = 0$, so $\dfrac{d}{dx}\csc^{-1}x = -\dfrac{d}{dx}\sec^{-1}x$.

3.10.86

a. $y_0 = f(x_0)$, so $y_0 = ax_0 + b$ and $b = y_0 - ax_0$.

b. $x_0 = f^{-1}(y_0)$, so $x_0 = cy_0 + d$ and $c = \dfrac{x_0 - d}{y_0}$. Also, because $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$and $f'(x_0) = a$, $(f^{-1})'(y_0) = c$, we have that $c = \dfrac{1}{a}$. So $d = x_0 - \dfrac{y_0}{a}$.

c. We show that $L(M(x)) = x$.
$L(M(x)) = a(cx+d)+b = acx+ad+b = x+ad+b = x+a(x_0-cy_0)+(y_0-ax_0) = x+ax_0-acy_0+y_0-ax_0 = x-y_0+y_0 = x$.

3.10.87 $\cos(\sin^{-1}(x)) = \sqrt{1-\sin^2(\sin^{-1}(x))} = \sqrt{1-x^2}$ for $-1 \le x \le 1$.

3.10.88 $\cos(2\sin^{-1}(x)) = \cos^2(\sin^{-1}(x)) - \sin^2(\sin^{-1}(x)) = 1 - \sin^2(\sin^{-1}(x)) - \sin^2(\sin^{-1}(x)) = 1 - 2x^2$ for $-1 \le x \le 1$.

3.10.89 $\tan(2\tan^{-1}(x)) = \dfrac{2\tan(\tan^{-1}(x))}{1-\tan^2(\tan^{-1}(x))} = \dfrac{2x}{1-x^2}$ for $-1 < x < 1$.

3.10.90 $\sin(2\sin^{-1}(x)) = 2\sin(\sin^{-1}(x))\cos(\sin^{-1}(x)) = 2x\sqrt{1-x^2}$ for $-1 \le x \le 1$.

3.11 Related Rates

3.11.1 The area of a circle of radius r is $A(r) = \pi r^2$. If the radius $r = r(t)$ changes with time, then the area of the circle is a function of r and r is a function of t, so ultimately A is a function of t. If the radius changes at rate $\frac{dr}{dt}$, then the area changes at rate $2\pi r \frac{dr}{dt}$.

3.11.2 $\frac{dV}{dt} = k\frac{dT}{dt}$.

3.11.3 Because area is width times height, if one increases, the other must decrease in order for the area to remain constant.

3.11.4

a $\frac{dF}{dt} = \frac{9}{5}\frac{dC}{dt}$.

b. $\frac{dF}{dt} = \frac{9}{5}(10) = \frac{90}{5} = 18$ degrees F per minute.

3.11.5

a $V = 200h$, so $\frac{dV}{dt} = 200\frac{dh}{dt}$.

b. $\frac{dV}{dt} = 200 \cdot \frac{1}{4} = 50\,\text{ft}^3/\text{min}$.

c. $\frac{dh}{dt} = \frac{dV/dt}{200} = \frac{10}{200} = \frac{1}{20}$ ft/min.

3.11.6

a $A = lw = 2w \cdot w = 2w^2$.

b. $\frac{dA}{dt} = 4w\frac{dw}{dt}$.

3.11.7

a. $V = \frac{4}{3}\pi r^3$ so $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$.

b. $\frac{dV}{dt} = 4\pi(4)^2(2) = 128\,\text{in}^3/\text{min}$.

c. $\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} = \frac{10}{4\pi 5^2} = \frac{1}{10\pi}$ in/min.

3.11.8 $A = \frac{3}{2}x^2$, so $\frac{dA}{dt} = 3x\frac{dx}{dt}$.

3.11.9 $\frac{dz}{dt} = \frac{dx}{dt} + 3y^2\frac{dy}{dt}$, so at the aforementioned time, $\frac{dz}{dt} = -1 + 3(2)^2(5) = 59$.

3.11.10 $\frac{dw}{dt} = \left(2x\frac{dx}{dt}\right)y^4 + x^2\left(4y^3\frac{dy}{dt}\right)$, so when $x = 3$, $\frac{dx}{dt} = 2$, $y = 1$, and $\frac{dy}{dt} = 4$, we have $\frac{dw}{dt} = 2(3)(2)(1)^4 + (3)^2(4(1)^3(4)) = 156$.

3.11.11

$A(x) = x^2$, $\frac{dx}{dt} = 2$ meters per second.

a. $\frac{dA}{dt} = 2x\frac{dx}{dt}$, so at $x = 10$ we have $\frac{dA}{dt} = 2 \cdot 10\text{m} \cdot 2\text{m/s} = 40\text{m}^2/\text{s}$.

b. At $x = 20\text{m}$ we have $\dfrac{dA}{dt} = 2 \cdot 20\text{m} \cdot 2\text{m/s} = 80\text{m}^2/\text{s}$.

3.11.12

a. Let x be the length of a side of the square. Then $\dfrac{dx}{dt} = -1$ meters per second. Because $A = x^2$, we have $\dfrac{dA}{dt} = \dfrac{dA}{dx}\dfrac{dx}{dt} = 2x \cdot (-1) = -2x$ square meters per second. Thus $A'(5) = -10$, and so the area of the square is decreasing at 10 square meters per second when $x = 5$.

b. If l is the length of a diagonal of a square with side length x, then $x^2 + x^2 = l^2$ by the Pythagorean Theorem, so $l(x) = \sqrt{2}x$. Thus $\dfrac{dl}{dt} = \dfrac{dl}{dx}\dfrac{dx}{dt} = \sqrt{2} \cdot (-1) = -\sqrt{2}$. The diagonals are decreasing at a rate of $\sqrt{2}$ meters per second.

3.11.13

a. Let x be the length of a leg of an isosceles right triangle. Then $\dfrac{dx}{dt} = 2$ meters per second. The area is given by $A(x) = \frac{1}{2}x^2$. Thus, $\dfrac{dA}{dt} = \dfrac{dA}{dx}\dfrac{dx}{dt} = x \cdot 2 = 2x$ square meters per second. When $x = 2$, we have $\dfrac{dA}{dx} = 4$, so the area is increasing at 4 square meters per second.

b. When the hypotenuse is 1 meter long, the legs are $1/\sqrt{2}$ meters long. So $A'(1/\sqrt{2}) = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$, so the area is increasing at $\sqrt{2}$ square meters per second.

c. If h is the length of the hypotenuse, then $x^2 + x^2 = h^2$, so $h = \sqrt{2}x$. So $\dfrac{dh}{dt} = \dfrac{dh}{dx}\dfrac{dx}{dt} = \sqrt{2}\dfrac{dx}{dt} = \sqrt{2} \cdot 2 = 2\sqrt{2}$ meters per second.

3.11.14

a. Let x be the length of a leg, and h the length of the hypotenuse. Then $x^2 + x^2 = h^2$, so $h = \sqrt{2}x$. Thus $\dfrac{dh}{dt} = \sqrt{2}\dfrac{dx}{dt}$, and because we are given that $\dfrac{dh}{dt} = -4$, we must have $\dfrac{dx}{dt} = -\dfrac{4}{\sqrt{2}} = -2\sqrt{2}$ meters per second.

Therefore, $\dfrac{dA}{dt} = \dfrac{dA}{dx}\dfrac{dx}{dt} = 2x \cdot -2\sqrt{2} = -4\sqrt{2}x$. When $x = 5$, we have $\dfrac{dA}{dt} = -20\sqrt{2}$ square meters per second. The area is decreasing at a rate of $20\sqrt{2}$ square meters per second.

b. As mentioned above, $\dfrac{dx}{dt} = -2\sqrt{2}$, so the legs are decreasing at a rate of $2\sqrt{2}$ meters per second.

c. When the triangle has area 4 square meters, the legs have length $x = 2\sqrt{2}$ meters. At that time, $\dfrac{dA}{dt} = \dfrac{dA}{dx}\dfrac{dx}{dt} = 2x(-2\sqrt{2}) = -4\sqrt{2}x = -4\sqrt{2}(2\sqrt{2}) = -16$. The area is decreasing at 16 square meters per second.

3.11.15

a. Let r be the radius of the circle and A the area, and note that we are given $\dfrac{dA}{dt} = 1$ square cm per second. Because $A = \pi r^2$, we have $\dfrac{dA}{dt} = \dfrac{dA}{dr}\dfrac{dr}{dt}$, so

$$1 = 2\pi r\frac{dr}{dt},$$

and thus

$$\frac{dr}{dt} = \frac{1}{2\pi r}.$$

When $r = 2$, we have $\dfrac{dr}{dt} = \dfrac{1}{4\pi}$ cm per second.

b. When $c = 2\pi r = 2$, we have $r = 1/\pi$. At this time, $\dfrac{dr}{dt} = \dfrac{1}{2\pi r} = \dfrac{1}{2\pi(1/\pi)} = \dfrac{1}{2}$ cm per second.

3.11.16 $V(x) = x^3$, so $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$. At $x = 50\,\text{cm}$ and $\dfrac{dx}{dt} = 2\,\text{cm/s}$ we have $\dfrac{dV}{dt} = 3\cdot(50)^2\cdot 2 = 15000\,\text{cm}^3/\text{s}$.

3.11.17 $A(x) = \pi x^2$, so $\dfrac{dA}{dt} = 2\pi x\dfrac{dx}{dt}$. At $x = 10\,\text{ft}$ and $\dfrac{dx}{dt} = -2\,\text{ft/min}$ we have $\dfrac{dA}{dt} = 2\pi\cdot 10\cdot(-2) = -40\pi\,\text{ft}^2/\text{min}$.

3.11.18 $V(x) = x^3$, so $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt} = -0.5\,\text{ft}^3/\text{min}$. When $x = 12\,\text{ft}$ we have $3(144)\text{ft}^2\frac{dx}{dt} = -0.5\,\text{ft}^3/\text{min}$, so $\dfrac{dx}{dt} = -\dfrac{1}{864}\,\text{ft/min} \approx -0.0012\,\text{ft/min}$.

3.11.19 $V(r) = \frac{4}{3}\pi r^3$, so $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt} = 15\,\text{in}^3/\text{min}$. At $r = 10$ inches we have $4\pi(10\,\text{in})^2\frac{dr}{dt} = 15\,\text{in}^3/\text{min}$. Thus, $\dfrac{dr}{dt} = \dfrac{3}{80\pi}\,\text{in/min} \approx 0.012\,\text{in/min}$.

3.11.20 Let x be the excess of the short side of the rectangle over the original 2 cm, so that the at time t the rectangle has dimensions $2+x$ by $4+x$. Then the area $A(x)$ is given by $A(x) = (2+x)(4+x) = 8+6x+x^2$. So $\dfrac{dA}{dt} = (6+2x)\dfrac{dx}{dt}$. With $\dfrac{dx}{dt} = 1\,\text{cm/s}$, and at $t = 20\,\text{s}$ we have $x = 20\,\text{cm}$, so $\dfrac{dA}{dt} = (6+2\cdot 20)\,\text{cm}\cdot 1\,\text{cm/s} = 46\,\text{cm}^2/\text{s}$.

3.11.21 $V(r) = \dfrac{4}{3}\pi r^3$, and $S(r) = 4\pi r^2$. $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt} = k\cdot 4\pi r^2$, so $\dfrac{dr}{dt} = k$, the constant of proportionality.

3.11.22

a. Let $x(t)$ be the distance that the westbound boat has traveled at time t and $y(t)$ the distance the southbound boat has traveled at time t. Then $x(0.5) = 10$ mi and $y(0.5) = 7.5$ mi.

b. The distance s between them is given by $s = \sqrt{x^2+y^2}$. Also $\dfrac{dx}{dt} = 20$ and $\dfrac{dy}{dt} = 15$. We have

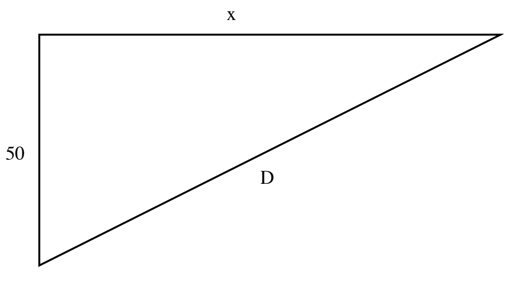
$$\frac{ds}{dt} = \frac{1}{2\sqrt{x^2+y^2}}\cdot\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right) = \frac{20x+15y}{\sqrt{x^2+y^2}}.$$

After 30 minutes $\sqrt{x^2+y^2} = \sqrt{10^2+(7.5)^2} = 12.5$. So $\dfrac{ds}{dt} = \dfrac{200+112.5}{12.5} = 25$ miles per hour.

3.11.23 Let x be the distance the westbound airliner has traveled between noon and t hours after 1:00, and let y be the distance the northbound airliner has traveled t hours after 1:00, and let D be the distance between the planes. We have $D^2 = x^2+y^2$, so $2D\dfrac{dD}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$. We are given that $\dfrac{dx}{dt} = 500$ mph and $\dfrac{dy}{dt} = 550$ mph. At 2:30, we have that $x = 500 + 500\cdot 1.5 = 1250$, and $y = 550\cdot 1.5 = 825$ miles. $D = \sqrt{2243125} \approx 1497.7$ miles. Thus $\dfrac{dD}{dt} \approx \dfrac{1250\cdot 500 + 825\cdot 550}{1497.7} \approx 720.27$ miles per hour.

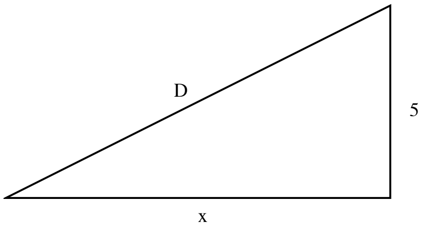
3.11.24

Let x be the horizontal distance of the kite, and let D be the length of the string. Then $D^2 = x^2 + 2500$, so $2D\dfrac{dD}{dt} = 2x\dfrac{dx}{dt}$, so $\dfrac{dD}{dt} = \dfrac{x}{D}\dfrac{dx}{dt}$. When $D = 120$ feet, then $x = \sqrt{11900} \approx 109$ feet. Therefore, $\dfrac{dD}{dt} \approx \dfrac{109}{120}\cdot 5 \approx 4.55$ feet per second.



3.11.25

Let D be the length of the rope from the boat to the capstan, and let x be the horizontal distance from the boat to the dock. By the Pythagorean Theorem, $x^2 + 25 = D^2$, so $2x\dfrac{dx}{dt} = 2D\dfrac{dD}{dt}$, so $\dfrac{dx}{dt} = \dfrac{D}{x}\dfrac{dD}{dt}$. We are given that $\dfrac{dD}{dt} = -3$ feet per second, so when $x = 10$, we have $\dfrac{dx}{dt} = \dfrac{\sqrt{125}}{10} \cdot (-3) = -\dfrac{3\sqrt{5}}{2}$ feet per second. The boat is approaching the dock at $\dfrac{3\sqrt{5}}{2}$ feet per second.



3.11.26

a. Let s be the distance from the origin to the bug's position $P(x, x^2)$ on the parabola. By the Pythagorean Theorem, $s^2 = x^2 + x^4$. Differentiating with respect to t, we have

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 4x^3\frac{dx}{dt} = 2x(1+2x^2)\frac{dx}{dt}.$$

Dividing through by 2 and solving for $\dfrac{dx}{dt}$ gives

$$\frac{dx}{dt} = \frac{s(ds/dt)}{x(1+2x^2)}.$$

When the bug is at $(2, 4)$ we have $s = \sqrt{20} = 2\sqrt{5}$, so

$$\frac{dx}{dt} = \frac{2\sqrt{5}(1)}{2(1+2(4))} = \frac{\sqrt{5}}{9} \text{ cm/min.}$$

b. Differentiating both sides of $y = x^2$ with respect to t gives

$$\frac{dy}{dt} = 2x\frac{dx}{dt}.$$

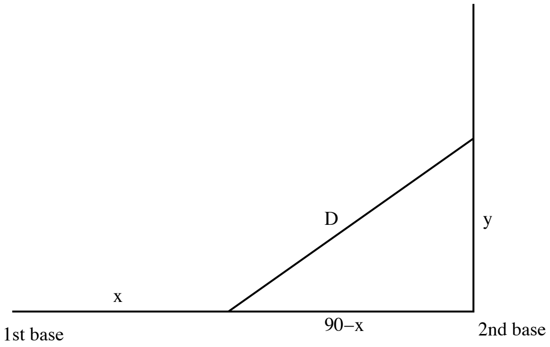
c. Using the previous parts of this problem, we have

$$\frac{dy}{dt} = 2(2)\left(\frac{\sqrt{5}}{9}\right) = \frac{4\sqrt{5}}{9} \text{ cm/min.}$$

3.11.27 Let x be the distance the motorcycle has traveled since the instant it went under the balloon, and let y be the height of the balloon above the ground t seconds after the motorcycle went under it. We have $x^2 + y^2 = D^2$ where D is the distance between the motorcycle and the balloon. Thus, $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2D\dfrac{dD}{dt}$, and we are given that $\dfrac{dy}{dt} = 10$ feet per second, and $\dfrac{dx}{dt} = 40\,\text{mph} = \dfrac{176}{3}$ ft/s. After 10 seconds have passed, we have that $y = 150 + 100 = 250\,\text{ft}$, $x = \dfrac{1760}{3}$ ft and $D = \sqrt{250^2 + \left(\frac{1760}{3}\right)^2} \approx 638\,\text{ft}$. Thus, $\dfrac{dD}{dt} \approx \dfrac{1}{638}\left(\dfrac{1760}{3} \cdot \dfrac{176}{3} + 2500\right) \approx 57.86$ feet per second.

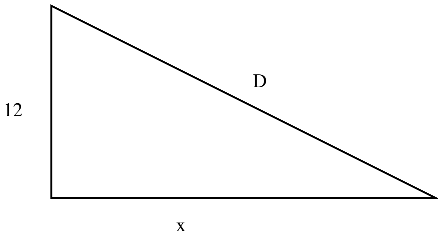
3.11.28

Let D, x and y be as pictured. By the Pythagorean theorem, we know that $D^2 = (90-x)^2 + y^2$. We are given that $\frac{dx}{dt} = 18$ feet per second, and $\frac{dy}{dt} = 20$ feet per second. Differentiating, we obtain $2D\frac{dD}{dt} = -2(90-x)\frac{dx}{dt} + 2y\frac{dy}{dt}$. After 1 second, we have that $x = 18$ and $y = 20$, and $D = 4\sqrt{349}$ feet. So $\frac{dD}{dt} = \frac{1}{4\sqrt{349}}(-72\cdot 18 + 20\cdot 20) \approx -11.99$ feet per second. So the distance between the runners is decreasing at a rate of about 11.99 feet per second.



3.11.29

Let x be the distance between the fish and the fisherman's feet, and let D be the distance between the fish and the tip of the pole. Then $D^2 = x^2 + 144$, so $2D(dD/dt) = 2x(dx/dt)$. Note that $dD/dt = -1/3\,\text{ft/sec}$, so when $x = 20\,\text{ft}$, we have $dx/dt = \sqrt{400+144}/20\cdot(-1/3) \approx -0.3887\,\text{ft/sec} \approx -4.66\,\text{in/sec}$. The fish is moving toward the fisherman at about 4.66 in/sec.



3.11.30 $y = 50x - x^2$, so $\frac{dy}{dt} = 50\frac{dx}{dt} - 2x\frac{dx}{dt}$. We are given that $\frac{dx}{dt} = 30$ feet per second. For $x = 10$, we have $\frac{dy}{dt} = 1500 - 600 = 900$ feet per second. For $x = 40$, $\frac{dy}{dt} = 1500 - 2400 = -900$ feet per second.

3.11.31 Let $h(t)$ be the height of the water in the tank at time t. Then the volume of the water in the tank at time t is given by $V = \pi r^2 h = \pi h$. We are seeking $\frac{dV}{dt}$ when $\frac{dh}{dt} = -1/2$ foot per minute. Because $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = -\frac{1}{2}\pi$, the volume of the water in the tank is decreasing at $\pi/2$ cubic feet per minute, so the water is draining out at $\pi/2$ cubic feet per minute.

3.11.32 We have that $V = \pi r^2 h$, and r is a constant 2 inches, so $V = 4\pi h$, and $\frac{dV}{dt} = 4\pi\frac{dh}{dt}$. Because we are given that $\frac{dh}{dt} = -0.25$ inches per second, we have that $\frac{dV}{dt} = 4\pi(-0.25) = -\pi\,\text{in}^3/\text{s}$. Thus, the soda is being sucked out at a rate of π cubic inches per second.

3.11.33 Let x be the distance from the bottom of the cylinder to the position of the piston. Let $V(x)$ be the volume of the cylinder when the piston is at position x. $V(x) = 25\pi x$, so $\frac{dV}{dt} = 25\pi\frac{dx}{dt}$. Because $\frac{dx}{dt} = -3\,\text{cm/s}$ we have $\frac{dV}{dt} = 25\pi(-3)\,\text{cm}^3/\text{s} = -75\pi\,\text{cm}^3/\text{s}$.

3.11.34 For the small pool, $V_s = 25\pi h_s$, so $\frac{dV_s}{dt} = 25\pi\frac{dh_s}{dt}$, and we are given that $\frac{dh_s}{dt} = .5$ meters per minutes, so $\frac{dV_s}{dt} = 12.5\pi\,\text{m}^3/\text{min}$. Because the pools are being filled at the same rate, this number is also $\frac{dV_L}{dt}$ for the large pool. We have $V_L = 64\pi h_L$, so $\frac{dV_L}{dt} = 12.5\pi = 64\pi\frac{dh_L}{dt}$, so $\frac{dh_L}{dt} = \frac{25}{128}$ meters per minute.

3.11.35 $V = \frac{1}{3}\pi r^2 h$ where $r = 3h$, so $V = 3\pi h^3$. We have that $\dfrac{dV}{dt} = 9\pi h^2 \dfrac{dh}{dt}$, and we given that $\dfrac{dh}{dt} = 2$ at the moment when $h = 12$, so at that time, $\dfrac{dV}{dt} = 9\pi \cdot 144\,\text{cm}^2 \cdot 2\,\text{cm/sec} = 2592\pi\,\text{cm}^3/\text{s}$. This is the rate at which the volume of the sandpile is increasing, so it must also be the rate at which the sand is leaving the bin, because there is no other sand involved.

3.11.36 Let h be the depth of the water in the tank at time t, and let r be the radius of the cone-shaped water at time t. By similar triangles, we have that $\dfrac{h}{r} = \dfrac{12}{6}$, so $h = 2r$. The volume of the water in the tank is given by $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{h^2}{4} \cdot h = \frac{\pi h^3}{12}$. Thus, $\dfrac{dV}{dt} = \dfrac{\pi h^2}{4}\dfrac{dh}{dt}$, and so when $h = 3$ we have $-2\,\text{ft}^3/\text{s} = \dfrac{9\pi\,\text{ft}^2}{4}\dfrac{dh}{dt}$, so $\dfrac{dh}{dt} = -\dfrac{8}{9\pi}$ ft/s. So the depth of the water is decreasing at a rate of $8/(9\pi)$ feet per second.

3.11.37 Let h be the depth of the water in the tank at time t, and let r be the radius of the cone-shaped water at time t. By similar triangles, we have that $\dfrac{h}{r} = \dfrac{12}{6}$, so $h = 2r$. The volume of the water in the tank is given by $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\dfrac{h^2}{4} \cdot h = \dfrac{\pi h^3}{12}$. Thus, $\dfrac{dV}{dt} = \dfrac{\pi h^2}{4}\dfrac{dh}{dt}$. When $\dfrac{dh}{dt} = -1$, we have $\dfrac{dV}{dt} = -\dfrac{\pi h^2}{4}$. when $h = 6$, we have $\dfrac{dV}{dt} = -9\pi$, so the water is draining from the tank at 9π cubic feet per minute.

3.11.38 The volume of the upper tank is $V_u = \frac{1}{3}\pi r^2 h$ with $\dfrac{h}{r} = \dfrac{5}{4}$, so $V_u = \dfrac{\pi}{3}\dfrac{16}{25}h^3$. We have $\dfrac{dV_u}{dt} = \dfrac{16\pi}{25}h^2\dfrac{dh}{dt}$, and we are given that $\dfrac{dh}{dt} = -0.5$ meters per minute. If $h = 3$, we have $\dfrac{dV_u}{dt} = -\dfrac{144\pi}{50}$ meters per minute.

The volume of the lower tank is given by $V_l = 16\pi h_l$, so $\dfrac{dV_l}{dt} = 16\pi\dfrac{dh_l}{dt} = \dfrac{144\pi}{50}$, so $\dfrac{dh_l}{dt} = \dfrac{9}{50}$ meters per minute.

Now suppose that $h = 1$. Then $\dfrac{dV_u}{dt} = \dfrac{16\pi}{50}$ meters per minute. Then $\dfrac{dV_l}{dt} = 16\pi\dfrac{dh_l}{dt} = \dfrac{16\pi}{50}$, so $\dfrac{dh_l}{dt} = \dfrac{1}{50}$ meters per minute.

3.11.39 The volume of a segment of water of height h within a hemisphere of radius 10 is given by $V = \dfrac{1}{3}\pi h^2(30 - h) = 10\pi h^2 - \dfrac{1}{3}\pi h^3$. We have that $\dfrac{dV}{dt} = 20\pi h\dfrac{dh}{dt} - \pi h^2\dfrac{dh}{dt}$. We are given that $\dfrac{dV}{dt} = 3\,\text{m}^3/\text{min}$, so when $h = 5$ we have $3 = (100\pi - 25\pi)\dfrac{dh}{dt}$, so $\dfrac{dh}{dt} = \dfrac{3}{75\pi} = \dfrac{1}{25\pi}$ meters per minute.

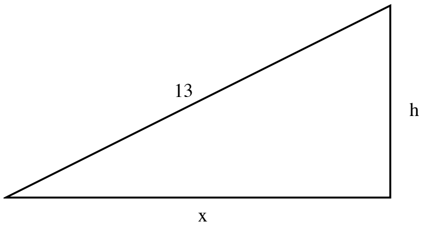
3.11.40 Let r be the radius of the exposed surface of the water of height h at time t. Consider the right triangle with legs of length r and $10 - h$ (from the center of the sphere measured down to where the water level is). The hypotenuse is given by the radius of the sphere, which is 10. By the Pythagorean theorem, we have

$$r^2 + (10 - h)^2 = 10^2,$$

which can be written as $r^2 + 100 - 20h + h^2 = 100$, so $r^2 + h^2 = 20h$. So $20\dfrac{dh}{dt} = 2h\dfrac{dh}{dt} + 2r\dfrac{dr}{dt}$. When $h = 5$, we have $10 \cdot \dfrac{1}{25\pi} = 5 \cdot \dfrac{1}{25\pi} + 5\sqrt{3}\dfrac{dr}{dt}$, so $\dfrac{dr}{dt} = \dfrac{\sqrt{3}}{75\pi}$. The surface area is given by $S = \pi r^2$, so $\dfrac{dS}{dt} = 2\pi r\dfrac{dr}{dt}$, so at this moment it is given by $\dfrac{dS}{dt} = 2\pi \cdot 5\sqrt{3} \cdot \dfrac{\sqrt{3}}{75\pi} = \dfrac{2}{5}$ square meters per minute.

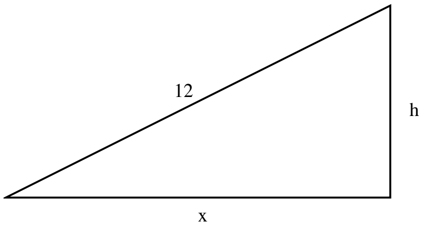
3.11.41

Let h be the vertical distance from the ground to the top of the ladder, and let x be the horizontal distance from the wall to the bottom of the ladder. By the Pythagorean Theorem, we have that $x^2 + h^2 = 169$. Thus, $2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$, so $\frac{dh}{dt} = -\frac{x}{h}\frac{dx}{dt}$, and we are given that $\frac{dx}{dt} = 0.5$ feet per second. At $x = 5$ we have $h = \sqrt{169-25} = 12$ feet. Thus, $\frac{dh}{dt} = -\frac{5}{12}\cdot\frac{1}{2} = -\frac{5}{24}$ feet per second. So the top of the ladder slides down the wall at $\frac{5}{24}$ feet per second.



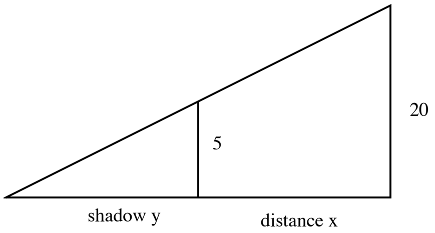
3.11.42

Let h be the vertical distance from the ground to the top of the ladder, and let x be the horizontal distance from the wall to the bottom of the ladder. By the Pythagorean Theorem, we have that $x^2 + h^2 = 144$. Thus, $2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$. We are given that $\frac{dx}{dt} = 0.2$ feet per second. We are seeking the configuration when $\frac{dh}{dt} = -0.2$ feet per second. This occurs when $0.2x - 0.2h = 0$, or $x = h$. At this point in time, the triangle is forming a 45-45-90 triangle with $x = h = 6\sqrt{2}$.



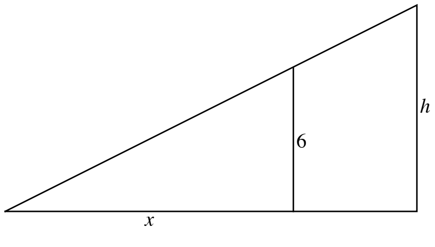
3.11.43

By similar triangles, $\frac{x+y}{20} = \frac{y}{5}$, so $x + y = 4y$, so $x = 3y$, and $\frac{dx}{dt} = 3\frac{dy}{dt}$. Because we are given that $\frac{dx}{dt} = -8$, we have $\frac{dy}{dt} = -\frac{8}{3}$ feet per second. The tip of her shadow is therefore moving at $-8 - \frac{8}{3} = -\frac{32}{3}$ feet per second.



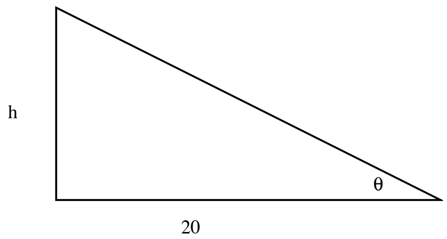
3.11.44

Let h be the height of the shadow of the 6-ft tall man who is x feet from the light. Then we have $\frac{x}{6} = \frac{15}{h}$ or $x = \frac{90}{h}$. So $\frac{dx}{dt} = -\frac{90}{h^2}\frac{dh}{dt}$. When he is 9 feet from the light, the height of his shadow is $h = \frac{90}{9} = 10$ ft and his walking speed is $\frac{dx}{dt} = -\frac{90}{10^2}(-2) = 1.8$ ft/s.



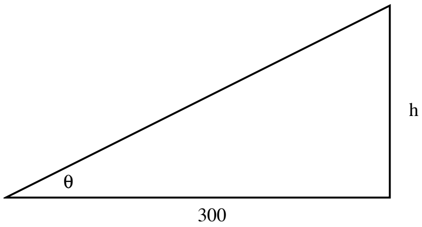
3.11.45

Let h be the vertical distance between the point on the elevator shaft positioned directly opposite the observer and the point on the elevator shaft that the observer is observing. So $h > 0$ corresponds to $\theta > 0$ and $h < 0$ corresponds to $\theta < 0$. We have $\frac{h}{20} = \tan\theta$, so $\frac{1}{20}\frac{dh}{dt} = \sec^2\theta\frac{d\theta}{dt}$. We are given that $\frac{dh}{dt} = 5\,\text{m/s}$. At $h = -10$, we have $\tan\theta = -.5$, so $\sec^2\theta = 1 + \tan^2\theta = 1 + (.5)^2 = 1.25$. So $\frac{d\theta}{dt} = \frac{1}{20 \cdot 1.25} \cdot 5 = \frac{1}{5}$ radian per second.
When $h = 20$, we have that $\tan\theta = 1$, so $\sec^2\theta = 1 + 1^2 = 2$, and thus $\frac{d\theta}{dt} = \frac{1}{20 \cdot 2} \cdot 5 = \frac{1}{8}$ radian per second.



3.11.46

Let h be the height of the balloon at time t. We have $\tan\theta = \frac{h}{300}$. Then $(\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{300}\frac{dh}{dt}$, so $\frac{d\theta}{dt} = \frac{\cos^2\theta}{300} \cdot \frac{d\theta}{dt}$. When the balloon is 400 feet off the ground, $\frac{d\theta}{dt} = \frac{1}{300}\left(\frac{3}{5}\right)^2 \cdot 20 = 0.024$ radians/s. (Note that when $h = 400$ the hypotenuse of the pictured triangle is 500, so $\cos\theta = \frac{3}{5}$).



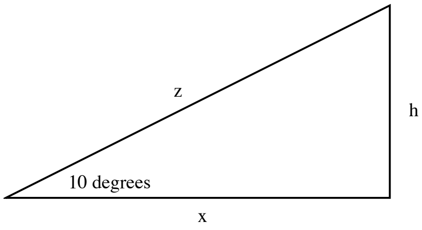
3.11.47 Let α be the angle between the line of sight to the bottom of the screen and the line of sight to the point 3 feet below where the floor and the wall meet. Note that $\cot\alpha = \frac{x}{3}$ and $\cot(\alpha + \theta) = \frac{x}{10}$, so $\alpha = \cot^{-1}(\frac{x}{3})$ and $\alpha + \theta = \cot^{-1}\left(\frac{x}{10}\right)$. Thus, $\theta = \cot^{-1}\left(\frac{x}{10}\right) - \cot^{-1}\left(\frac{x}{3}\right)$. So

$$\frac{d\theta}{dt} = -\frac{10x'}{100 + x^2} + \frac{3x'}{9 + x^2},$$

and at $x = 30$ feet, and with $\frac{dx}{dt} = 3$ feet per second, we have $\frac{d\theta}{dt} = -\frac{30}{1000} + \frac{9}{909} \approx -0.0201$ radians per second.

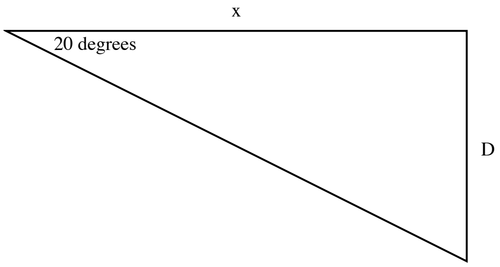
3.11.48

Let x be the distance the shadow has traveled, h the altitude of the jet, and z the line of flight of the jet. We have that $\frac{dz}{dt} = 550\,\text{mi/hr}$ and $h = z \cdot \sin(10°) \approx 0.174z$, so $\frac{dh}{dt} = 0.174\frac{dz}{dt} = 95.51\,\text{mi/hr}$. Also, $x = z \cdot \cos(10°) \approx 0.985z$, so $\frac{dx}{dt} = 0.985\frac{dz}{dt} = 541.64\,\text{mi/hr}$. So the shadow is moving at about 541.64 miles per hour.



3.11.49

Let x be the distance the surface ship has traveled and D the depth of the submarine. We have $\frac{dx}{dt} = 10\,\text{km/hr}$. Note that $\frac{D}{x} = \tan 20°$, so $D = x \cdot \tan 20° \approx 0.364x$. We have $\frac{dD}{dt} = 0.364\frac{dx}{dt} = 3.64\,\text{km/hr}$. The depth of the submarine is increasing at a rate of 3.64 km/hr.



3.11.50 Let θ be the angle RLP where L represents the lighthouse and R represents the point on the land where the light is currently hitting. Let s be the distance from the point P to the point R. We are given that $\frac{d\theta}{dt} = \frac{2\pi}{15}$ radians per second. Note that $\tan\theta = \frac{s}{500}$, so $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{500}\frac{ds}{dt}$. When the light is at point Q, $\tan\theta = \frac{2}{5}$, so $\sec^2\theta = 1 + \frac{4}{25} = \frac{29}{25}$. Then

$$\frac{ds}{dt} = 500 \cdot \frac{2\pi}{15} \cdot \frac{29}{25} = \frac{232\pi}{3}\ \text{m/s}.$$

The beam moves more slowly when R is near P, and more quickly when it is further away from P.

3.11.51

a. Let A be the point where the dragster started, let B be the point where camera 1 is located and let $C = y(t)$ be the position of the car at time t. Let θ be angle ABC. Note that $\tan\theta = \frac{y}{50}$, so $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{50}\frac{dy}{dt}$. At time $t = 2$, we have that $\tan^2\theta = 4$, so $\sec^2\theta = \tan^2\theta + 1 = 5$. So $\frac{dy}{dt} = 5 \cdot 50 \cdot .75 = 187.5$ feet per second.

b. Let D be the point where camera 2 is located, and let ϕ be angle ADC. The $\phi = \tan^{-1}\left(\frac{y}{100}\right)$, so $\frac{d\phi}{dt} = \frac{1}{100(1 + \left(\frac{y}{100}\right)^2} \cdot \frac{dy}{dt}$. After 2 seconds, we know that $y = 100$ and $\frac{dy}{dt} = 187.5$. Thus $\frac{d\phi}{dt} = \frac{100}{20,000} \cdot 187.5 = .9375$ radians per second.

3.11.52

a. We have that the radius of the reel is 2 inches, so if L is the length and R is the number of revolutions, that $L = 4\pi R$.

b. Because $L = 4\pi R$, we have $\frac{dL}{dt} = 4\pi\frac{dR}{dt}$, so $\frac{dL}{dt} = 4\pi \cdot 1.5 = 6\pi$ inches per second.

3.11.53

a. $P = \frac{dE}{dt} = \frac{1}{2}v^2\frac{dm}{dt}$.

b. $P = \frac{1}{2}v^2\frac{dm}{dt} = \frac{1}{2}v^2(\rho Av) = \frac{1}{2}\rho Av^3$.

c. $P = \frac{1}{2}\rho Av^3 = \frac{1}{2}(1.23)(\pi)(3^2)(10^3) \approx 388.7$ W

d. $17388.7(0.25) \approx 4347.2$ W.

3.11.54 Because $V = 0.5$ when $P = 50$, we must have $0.5 = \frac{k}{50}$, so $k = 50(0.5) = 25$. Then $\frac{dV}{dt} = -\frac{25}{P^2}\frac{dP}{dt}$, so

$$\frac{dP}{dt} = -\frac{P^2}{25}\frac{dV}{dt} = -\frac{50^2}{25}(0.15) = -15\,\text{kPa/min}.$$

3.11.55 Let θ be the angle between the hands of the clock, and D the distance between the tips of the hands. By the Law of Cosines, $D^2 = 2.5^2 + 3^2 - 15\cos\theta$. So $2D\frac{dD}{dt} = 15\sin\theta\frac{d\theta}{dt}$. At 9:00 AM, we have $D^2 = 6.25 + 9$, so $D = \sqrt{15.25}$. Also, $\theta = \pi/2$ so $\sin\theta = 1$. Thus, $\frac{dD}{dt} = \frac{15}{2\sqrt{15.25}}\frac{d\theta}{dt}$. Now $\frac{d\theta}{dt} = \frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}$ where $\frac{d\theta_1}{dt}$ is the angular change of the minute hand and $\frac{d\theta_2}{dt}$ is the angular change of the hour hand. We have $\frac{d\theta_1}{dt} = \frac{\pi}{30}$ radians per minute and $\frac{d\theta_2}{dt} = \frac{\pi}{360}$ radians per minute, so $\frac{d\theta}{dt} = \frac{11\pi}{360}$ radians per minute. Thus $\frac{dD}{dt} = \frac{15}{2\sqrt{15.25}} \cdot \frac{11\pi}{360} \approx 0.18436$ meters per minute, or about 11.06 meters per hour.

3.11.56 At time t, the boat traveling west has gone $20t$ miles while the boat traveling southwest has gone $15t$ miles. Let D be the distance between the boats; the line between the boats forms the third side of a triangle. Then by the Law of Cosines,

$$D^2 = (20t)^2 + (15t)^2 - 2 \cdot 20t \cdot 15t\cos(\pi/4) = 625t^2 - 600t^2 \cdot \frac{\sqrt{2}}{2} = (625 - 300\sqrt{2})t^2.$$

Thus,

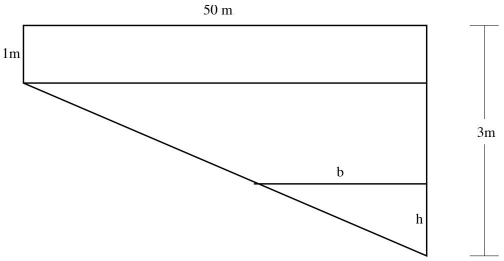
$$D = \left(625 - 30\sqrt{2}\right)^{1/2} t,$$

so that

$$\frac{dD}{dt} = \left(625 - 30\sqrt{2}\right)^{1/2} \approx 14.168\,\text{mph}.$$

3.11.57

By similar triangles, $\frac{2}{50} = \frac{h}{b}$, so $b = 25h$. Also, $A = \frac{1}{2}bh = 12.5h^2$, so the volume for $0 \le h \le 2$ is $V(h) = 12.5 \cdot h^2 \cdot 20 = 250h^2$. For $2 < h \le 3$, $V(h) = 250 \cdot 2^2 + 50 \cdot 20 \cdot (h - 2) = 1000h - 1000$. When $t = 250$ minutes, then $V = 250\,\text{min} \cdot 1\,\text{m}^3/\text{min} = 250\,\text{m}^3$. So $V(h) = 250h^2 = 250$, so $h = 1\,\text{m}$. At that time $\frac{dV}{dt} = 500h\frac{dh}{dt} = 500 \cdot 1 \cdot \frac{dh}{dt} = 1\,\text{m}^3/\text{min}$. So $\frac{dh}{dt} = \frac{1}{500}\,\text{m/min} = 0.002\,\text{m/min} = 2\,\text{mm/min}$.



Fill time: The volume of the entire swimming pool is 2000 cubic meters, so at 1 cubic meter per minute, it will take 2000 minutes.

3.11.58 Let l be the length of a side of triangle, and let x be the line segment from a vertex to the midpoint of the opposite side. Then $\sin(\pi/3) = \frac{x}{l}$, so $l = \frac{2}{\sqrt{3}}x$. Now $A = \frac{xl}{2} = \frac{x^2}{\sqrt{3}}$. Thus $\frac{dA}{dt} = \frac{2x}{\sqrt{3}}\frac{dx}{dt}$, and when $x = 0$, this quantity is zero.

3.11.59 By the Law of Sines, $\frac{\sin\theta}{s} = \frac{\sin\left(\frac{3\pi}{4} - \theta\right)}{2}$, so

$$2\sin\theta = s\sin\left(\frac{3\pi}{4} - \theta\right) = s\left(\sin\left(\frac{3\pi}{4}\right)\cos\theta - \cos\left(\frac{3\pi}{4}\right)\sin\theta\right).$$

We have

$$\begin{aligned}2\sin\theta &= \frac{\sqrt{2}}{2}s(\sin\theta+\cos\theta)\\ 2\tan\theta &= \frac{\sqrt{2}}{2}s(\tan\theta+1)\\ \tan\theta &= \frac{(\sqrt{2}/2)\cdot s}{2-(\sqrt{2}/2)s} = \frac{\sqrt{2}s}{4-\sqrt{2}s}\\ \theta &= \tan^{-1}\left(\frac{\sqrt{2}s}{4-\sqrt{2}s}\right).\end{aligned}$$

Thus, $\frac{d\theta}{dt} = \frac{\sqrt{2}\cdot\frac{ds}{dt}}{4-2\sqrt{2}s+s^2}$. When $\frac{ds}{dt}=15$ and $s=7.5$ we arrive at $\frac{d\theta}{dt}=0.54$ radians per hour.

3.11.60 Let s be the distance the ship has traveled. By the Law of Sines, $\frac{\sin\theta}{s} = \frac{\sin\left(\frac{3\pi}{4}-\theta\right)}{1.5}$, so $1.5\sin\theta = s\cdot\left(\sin\left(\frac{3\pi}{4}\right)\cos\theta - \cos\left(\frac{3\pi}{4}\right)\sin\theta\right)$.We have

$$\begin{aligned}\sin\theta &= \frac{\sqrt{2}}{3}s(\sin\theta+\cos\theta)\\ \tan\theta &= \frac{\sqrt{2}}{3}s(\tan\theta+1)\\ \tan\theta &= \frac{(\sqrt{2}/3)\cdot s}{1-(\sqrt{2}/3)s} = \frac{\sqrt{2}s}{3-\sqrt{2}s}\\ \theta &= \tan^{-1}\left(\frac{\sqrt{2}s}{3-\sqrt{2}s}\right).\end{aligned}$$

Thus, $\frac{d\theta}{dt} = \frac{3\sqrt{2}\cdot\frac{ds}{dt}}{9-6\sqrt{2}s+4s^2}$. At noon, $2s^2=1.5^2$, so $s=\frac{3}{\sqrt{8}}$. At 1:30 pm $s = 18+\frac{3}{\sqrt{8}} \approx 19.06\,\text{mi}$, and $\frac{ds}{dt} = 12\,\text{mi/hr}$, so $\frac{d\theta}{dt} \approx 0.04$ radians per hour.

3.11.61 Let x be the distance the eastbound boat has traveled at time t and let s be the distance the northeastbound boat has traveled. Note the diagram shown. By the Law of Sines, $\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{s} = \frac{\sin\left(\frac{\pi}{4}+\theta\right)}{x}$. Thus,

$$x\left(\sin\left(\frac{\pi}{2}\right)\cos\theta - \cos\left(\frac{\pi}{2}\right)\sin\theta\right) =$$
$$s\left(\sin\left(\frac{\pi}{4}\right)\cos\theta + \cos\left(\frac{\pi}{4}\right)\sin\theta\right)$$

So

$$x\cos\theta = \frac{\sqrt{2}}{2}\cdot s\cdot\cos\theta + \frac{\sqrt{2}}{2}\cdot s\cdot\sin\theta,$$
$$x = \frac{\sqrt{2}}{2}\cdot s + \frac{\sqrt{2}}{2}\cdot s\cdot\tan\theta,$$

and thus $\tan\theta = \frac{x-\frac{\sqrt{2}}{2}s}{\frac{\sqrt{2}}{2}s} = \frac{\sqrt{2}x-s}{s}$, and therefore
$\theta = \tan^{-1}\left(\frac{\sqrt{2}x-s}{s}\right)$.

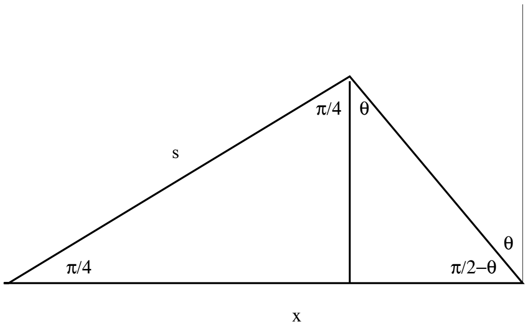
We have

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{\sqrt{2}x-s}{s}\right)^2} \cdot \frac{\left(\sqrt{2}\left(\frac{dx}{dt}\right) - \left(\frac{ds}{dt}\right)\right)\cdot s - \left(\sqrt{2}x - s\right)\cdot \frac{ds}{dt}}{s^2} = \frac{\sqrt{2}\left(s\frac{dx}{dt} - x\frac{ds}{dt}\right)}{s^2 + (\sqrt{2}x-s)^2}.$$

At time t, we have $s(t) = 15t$ and $x(t) = 12t$. Note that

$$s\frac{dx}{dt} - x\frac{ds}{dt} = 15t \cdot 12 - 12t \cdot 15 = 0.$$

Thus $\theta' = 0$ for every value of t, so that the angle is constant.



3.11.62

Let D be the distance from the bottom center of the Ferris wheel to the cart. Note that $\tan\theta = \dfrac{D}{20}$, so $\theta = \tan^{-1}\left(\dfrac{D}{20}\right)$, and

$$\frac{d\theta}{dt} = \frac{20\cdot\frac{dD}{dt}}{400+D^2} = \frac{20}{400+D^2}\cdot\frac{dD}{dt}.$$

Let α be the angle pictured. By the Law of Cosines,

$$D^2 = 5^2 + 5^2 - 2\cdot 5\cdot 5\cdot\cos\alpha = 50 - 50\cos\alpha.$$

So $2D\dfrac{dD}{dt} = 50\dfrac{d\alpha}{dt}\cdot\sin\alpha$, and solving for $\dfrac{dD}{dt}$ gives

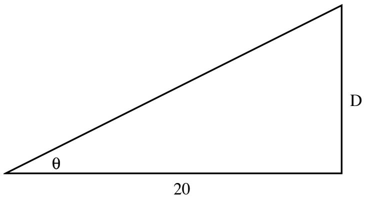
$$\frac{dD}{dt} = \frac{25\sin\alpha}{D}\cdot\frac{d\alpha}{dt}.$$

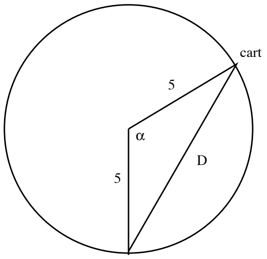
At $t = 40$ seconds (which is $\dfrac{2}{3}$ minutes), we have $\alpha = \dfrac{2\pi}{3}$, so that $D^2 = 50 - 50\cos(2\pi/3) = 75$, and thus $D = 5\sqrt{3}$. Also $\sin\alpha = \dfrac{\sqrt{3}}{2}$. Finally, we are given that $\dfrac{d\alpha}{dt} = \pi$ radians per minute. Then we have

$$\frac{dD}{dt} = \frac{25\cdot\frac{\sqrt{3}}{2}}{5\sqrt{3}}\cdot\pi = \frac{5}{2}\pi.$$

Finally, we have

$$\frac{d\theta}{dt} = \frac{20}{400+75}\cdot\frac{5}{2}\pi = \frac{2\pi}{19} \approx 0.331\,\text{radians per second.}$$





3.11.63

a. The volume of the water in the tank (as a function of h – the depth of the water in the tank) is given by 5 times the area of the segment of water in a cross-sectional circle. For a tank of radius 1, the formula for such a segment is $\cos^{-1}(1-h)-(1-h)\sqrt{2h-h^2}$. Thus the volume of the water in the tank is given by $V=5(\cos^{-1}(1-h)-(1-h)\sqrt{2h-h^2})$. We have

$$\begin{aligned}\frac{dV}{dt}&=5\cdot\left(-\frac{1}{\sqrt{1-(1-h)^2}}\cdot\left(-\frac{dh}{dt}\right)+\frac{dh}{dt}\sqrt{2h-h^2}-\frac{(1-h)^2}{\sqrt{2h-h^2}}\frac{dh}{dt}\right)\\&=5\left(\sqrt{2h-h^2}+\frac{1-(1-h)^2}{\sqrt{2h-h^2}}\right)\frac{dh}{dt}\\&=5\left(\frac{2h-h^2+1-1+2h-h^2}{\sqrt{2h-h^2}}\right)\frac{dh}{dt}\\&=5\left(\frac{2(2h-h^2)}{\sqrt{2h-h^2}}\right)\frac{dh}{dt}\\&=10\sqrt{2h-h^2}\cdot\frac{dh}{dt}\end{aligned}$$

When $h=0.5$, we have $-\frac{3}{2}=\frac{dV}{dt}=5\sqrt{3}\frac{dh}{dt}$, so $\frac{dh}{dt}=-\frac{\sqrt{3}}{10}$ meters per hr.

b. The surface area of the water is given by $S=5\cdot 2\sqrt{2h-h^2}$. So $\frac{dS}{dt}=10\cdot\frac{2-2h}{2\sqrt{2h-h^2}}\cdot\frac{dh}{dt}$, so at $h=.5$, we have $\frac{5}{\sqrt{3/4}}\cdot -\frac{\sqrt{3}}{10}=-1$ square meter per hr.

3.11.64 Let r be the distance from the point on the highway perpendicular to the searchlight to the right-hand edge of the beam, and let l be the distance from that point to the left-hand edge of the beam. Then $w=l-r$. We have that

$$r=100\tan\left(\theta-\frac{\pi}{32}\right)\quad\text{and}\quad l=100\tan\left(\theta+\frac{\pi}{32}\right).$$

Thus

$$\begin{aligned}\frac{dw}{dt}&=\frac{dl}{dt}-\frac{dr}{dt}\\&=100\left(\sec^2\left(\theta+\frac{\pi}{32}\right)-\sec^2\left(\theta-\frac{\pi}{32}\right)\right)\cdot\frac{d\theta}{dt}\\&=100\left(\tan^2\left(\theta+\frac{\pi}{32}\right)-1-\left(\tan^2\left(\theta-\frac{\pi}{32}\right)-1\right)\right)\cdot\frac{d\theta}{dt}\\&=100\left(\tan^2\left(\theta+\frac{\pi}{32}\right)-\tan^2\left(\theta-\frac{\pi}{32}\right)\right)\cdot\frac{d\theta}{dt}.\end{aligned}$$

With $\theta'=\frac{\pi}{6}$ radians per second and $\theta=\frac{\pi}{3}$, we have

$$\frac{dw}{dt}=100\left(\tan^2\left(\frac{35\pi}{96}\right)-\tan^2\left(\frac{29\pi}{96}\right)\right)\cdot\frac{\pi}{6}\approx 153.081\ \text{meters per second.}$$

Chapter Three Review

1

a. False. This function is not differentiable at $x=-\frac{1}{2}$. It is possible for a function to be continuous at a point and not differentiable at that point.

b. False. For example, $f(x) = x^2 + 3$ and $g(x) = x^2 + 100$ have the same derivative, but aren't the same function.

c. False. For example, $\frac{d}{dx}|e^{-x}| = \frac{d}{dx}e^{-x} = -e^{-x} \neq |-e^{-x}|$.

d. False. For example, the function $f(x) = |x|$ has no derivative at 0, but there is no vertical tangent there.

e. True. For example, a ball dropping from a high tower has acceleration due to gravity which is negative, but it is speeding up as it falls because the velocity (which is negative also) is in the same direction as the acceleration.

2

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + 2(x+h) + 9 - (x^2+2x+9)}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2+2x+2h+9-x^2-2x-9}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2+2h}{h} = \lim_{h\to 0} \frac{h(2x+h+2)}{h} \\ &= \lim_{h\to 0}(2x+h+2) = 2x+2. \end{aligned}$$

3

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2+5} - \frac{1}{x^2+5}}{h} = \\ &= \lim_{h\to 0} \frac{x^2+5-(x^2+2xh+h^2+5)}{h((x+h)^2+5)(x^2+5)} \\ &= \lim_{h\to 0} \frac{-h(2x+h)}{h((x+h)^2+5)(x^2+5)} \\ &= \lim_{h\to 0} \frac{-(2x+h)}{((x+h)^2+5)(x^2+5)} \\ &= -\frac{2x}{(x^2+5)^2}. \end{aligned}$$

4

$$\begin{aligned} h'(t) &= \lim_{x\to t} \frac{h(x)-h(t)}{x-t} = \lim_{x\to t} \frac{\sqrt{3x+5}-\sqrt{3t+5}}{x-t} \\ &= \lim_{x\to t} \frac{(\sqrt{3x+5}-\sqrt{3t+5})}{(x-t)} \cdot \frac{(\sqrt{3x+5}+\sqrt{3t+5})}{(\sqrt{3x+5}+\sqrt{3t+5})} \\ &= \lim_{x\to t} \frac{3x+5-(3t+5)}{(x-t)(\sqrt{3x+5}+\sqrt{3t+5}} \\ &= \lim_{x\to t} \frac{3(x-t)}{(x-t)(\sqrt{3x+5}+\sqrt{3t+5})} \\ &= \lim_{x\to t} \frac{3}{\sqrt{3x+5}+\sqrt{3t+5}} = \frac{3}{2\sqrt{3t+5}}. \end{aligned}$$

5 $\frac{d}{dx}(\tan^3 - 3\tan x + 3x) = 3\tan^2 x \sec^2 x - 3\sec^2 x + 3 = 3\sec^2 x(\tan^2 x - 1) + 3 = 3(\tan^2 x + 1)(\tan^2 x - 1) + 3 = 3\tan^4 x - 3 + 3 = 3\tan^4 x$.

6

$$\frac{d}{dx}\left(\frac{x}{\sqrt{1-x^2}}\right) = \frac{\sqrt{1-x^2} - x\left(\frac{-2x}{2\sqrt{1-x^2}}\right)}{1-x^2} = \frac{\frac{1-x^2}{\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}}}{1-x^2} = \frac{\frac{1}{\sqrt{1-x^2}}}{1-x^2} = \frac{1}{(1-x^2)^{3/2}}.$$

7

$$\frac{d}{dx}(x^4 - \ln(x^4+1)) = 4x^3 - \frac{4x^3}{x^4+1} = \frac{4x^3(x^4+1)}{x^4+1} - \frac{4x^3}{x^4+1} = \frac{4x^7+4x^3-4x^3}{x^4+1} = \frac{4x^7}{x^4+1}.$$

8

$$\begin{aligned}\frac{d}{dx}\left(\ln\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right) &= \frac{d}{dx}\left(\ln(1-\sqrt{x}) - \ln(1+\sqrt{x})\right)\\ &= \frac{-1}{2\sqrt{x}(1-\sqrt{x})} - \frac{1}{2\sqrt{x}(1+\sqrt{x})}\\ &= \frac{-1}{2\sqrt{x}}\left(\frac{1}{1-\sqrt{x}} + \frac{1}{1+\sqrt{x}}\right)\\ &= \frac{-1}{2\sqrt{x}}\left(\frac{1+\sqrt{x}+1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right)\\ &= \frac{-1}{\sqrt{x}}\left(\frac{1}{1-x}\right) = \frac{1}{\sqrt{x}(x-1)}.\end{aligned}$$

9 $y' = 2x^2 + 2\pi x + 7$.

10 $y' = 16x^3 \ln x + \dfrac{4x^4}{x} - 4x^3 = 16x^3 \ln x + 4x^3 - 4x^3 = 16x^3 \ln x$.

11 $y' = 2^x(\ln 2)$.

12 $y' = 2\sqrt{2}x^{\sqrt{2}-1}$.

13 $y' = e^{2\theta} \cdot 2 = 2e^{2\theta}$.

14 $y' = 2x^{3/2} + \frac{3}{2}(2x-3)x^{1/2} = 5x^{3/2} - \frac{9}{2}x^{1/2} = \sqrt{x}\left(5x - \frac{9}{2}\right)$.

15 $y' = \dfrac{3}{2}\sqrt{1+x^4}(4x^3) = 6x^3\sqrt{1+x^4}$.

16 $y' = 2\sqrt{x^2-2x+2} + 2x \cdot \dfrac{1}{2\sqrt{x^2-2x+2}}(2x-2) = 2\left(\sqrt{x^2-2x+2} + \dfrac{x^2-x}{\sqrt{x^2-2x+2}}\right) = \dfrac{4x^2-6x+4}{\sqrt{x^2-2x+2}}$.

17 $y' = 10t \sin t + 5t^2 \cos t$.

18 $y' = 5 + 3\sin^2 x \cos x + 3x^2 \cos x^3$.

19 $y' = -e^{-x}(x^2+2x+2) + e^{-x}(2x+2) = -e^{-x}(x^2+2x+2-2x-2) = -x^2e^{-x}$.

20 $y' = \dfrac{(\ln x + a)\frac{1}{x} - (\ln x)\frac{1}{x}}{(\ln x + a)^2} = \dfrac{a}{x(\ln x + a)^2}$.

21 $y' = \dfrac{((\sec 2w + 1)2\sec 2w \tan 2w - 2\sec 2w \sec 2w \tan 2w)}{(\sec 2w + 1)^2} = \dfrac{2\sec 2w \tan 2w}{(\sec 2w + 1)^2}$.

22

$$\begin{aligned}y' &= \frac{1}{3}\left(\frac{\sin x}{\cos x + 1}\right)^{-2/3}\left(\frac{(\cos x + 1)\cos x - (\sin x)(-\sin x)}{(\cos x + 1)^2}\right)\\ &= \frac{1}{3}\left(\frac{\sin x}{\cos x + 1}\right)^{-2/3}\left(\frac{\cos^2 x + \cos x + \sin^2 x}{(\cos x + 1)^2}\right)\\ &= \frac{1}{3}\left(\frac{(\cos x + 1)^{2/3}}{\sin^{2/3} x}\right)\left(\frac{\cos x + 1}{(\cos x + 1)^2}\right)\\ &= \frac{1}{3\sin^{2/3} x(1+\cos x)^{1/3}}.\end{aligned}$$

23 $y' = \dfrac{3}{\sec 3x} \sec 3x \tan 3x = 3\tan 3x.$

24 $y' = \dfrac{-7\csc 7x \cot 7x - 7\csc^2 7x}{\csc 7x + \cot 7x} = \dfrac{-7\csc 7x(\cot 7x + \csc 7x)}{\csc 7x + \cot 7x} = -7\csc 7x.$

25 $y' = 100(5t^2+10)^{99}(10t) = 1000t(5t^2+10)^{99}.$

26 $y' = e^{\sin x + 2x + 1} \cdot (\cos x + 2).$

27 $y' = \dfrac{1}{\sin x^3} \cdot \cos x^3 \cdot 3x^2 = \dfrac{3x^2 \cos x^3}{\sin x^3} = 3x^2 \cot x^3.$

28 $y' = e^{\tan x}\sec^2 x(\tan x - 1) + e^{\tan x}(\sec^2 x) = e^{\tan x}\sec^2 x(\tan x - 1 + 1) = e^{\tan x}\sec^2 x \tan x.$

29 $y' = \dfrac{1}{(\sqrt{t^2-1})^2+1} \cdot \dfrac{1}{2}(t^2-1)^{-1/2} \cdot 2t = \dfrac{2t}{2t^2\sqrt{t^2-1}} = \dfrac{1}{t\sqrt{t^2-1}}.$

30 Let $y = x^{\sqrt{x+1}}$. Then $\ln y = \sqrt{x+1}\ln x$. Then

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{1}{2\sqrt{x+1}} \cdot \ln x + \frac{\sqrt{x+1}}{x} \\ &= \frac{\ln x}{2\sqrt{x+1}} + \frac{\sqrt{x+1}}{x} \\ &= \frac{x\ln x}{2x\sqrt{x+1}} + \frac{2(x+1)}{2x\sqrt{x+1}} \\ &= \frac{x\ln x + 2x + 2}{2x\sqrt{x+1}}.\end{aligned}$$

Thus

$$\frac{dy}{dx} = y \cdot \frac{x\ln x + 2x + 2}{2x\sqrt{x+1}} = x^{\sqrt{x+1}}\left(\frac{x\ln x + 2x + 2}{2x\sqrt{x+1}}\right).$$

31 $y' = (4\sec^2(\theta^2+3\theta+2)) \cdot (2\theta+3) = (8\theta+12)\sec^2(\theta^2+3\theta+2).$

32 $y' = 5\csc^4 3x \cdot (-\csc 3x \cot 3x) \cdot 3 = -15\csc^5 3x \cot 3x.$

33 $y' = \dfrac{w^5 \cdot \frac{1}{w} - \ln w \cdot 5w^4}{w^{10}} = \dfrac{w^4(1-5\ln w)}{w^{10}} = \dfrac{1-5\ln w}{w^6}.$

34 $y' = \dfrac{e^{as} \cdot 1 - s \cdot e^{as} \cdot a}{e^{2as}} = \dfrac{e^{as}(1-as)}{e^{2as}} = \dfrac{1-as}{e^{as}}.$

35 $y' = \dfrac{(8u+1)(8u+1) - (4u^2+u)(8)}{(8u+1)^2} = \dfrac{64u^2+16u+1-32u^2-8u}{(8u+1)^2} = \dfrac{32u^2+8u+1}{(8u+1)^2}.$

36 $y' = -3\left(\dfrac{3t^2-1}{3t^2+1}\right)^{-4} \cdot \dfrac{(3t^2+1)(6t)-(3t^2-1)(6t)}{(3t^2+1)^2} = -36 \cdot \dfrac{(3t^2+1)^2 \cdot t}{(3t^2-1)^4}.$

37 $y' = \sec^2(\sin\theta) \cdot \cos\theta.$

38 $y' = \dfrac{4}{3}\left(\dfrac{v}{v+1}\right)^{1/3}\left(\dfrac{(v+1)-v}{(v+1)^2}\right) = \dfrac{4}{3}\left(\dfrac{v}{v+1}\right)^{1/3}\left(\dfrac{1}{(v+1)^2}\right) = \dfrac{4v^{1/3}}{3(v+1)^{7/3}}.$

39 $y' = \cos\sqrt{\cos^2 x + 1} \cdot \dfrac{1}{2\sqrt{\cos^2 x+1}} \cdot -2\cos x \sin x = -\dfrac{\cos\sqrt{\cos^2 x+1}\cos x \sin x}{\sqrt{\cos^2 x + 1}}.$

40 $y' = e^{\sin(\cos x)} \cdot \cos(\cos x) \cdot (-\sin x) = -\sin x \cos(\cos x) e^{\sin(\cos x)}$.

41 $y' = \frac{d}{dt} \ln \sqrt{e^t + 1} = \frac{d}{dt}\left(\frac{1}{2}\ln(e^t + 1)\right) = \frac{1}{2} \cdot \frac{1}{e^t + 1} \cdot e^t = \frac{e^t}{2(e^t + 1)}$.

42 $y' = 1 \cdot e^{-10x} + x(-10e^{-10x}) = e^{-10x}(1 - 10x)$.

43 $y' = 2x + 2\tan^{-1}(\cot x) + 2x \cdot \frac{1}{1 + \cot^2 x} \cdot (-\csc^2 x) = 2x + 2\tan^{-1}(\cot x) + \frac{2x}{\csc^2 x} \cdot (-\csc^2 x) = 2\tan^{-1}(\cot x)$.

44 $y' = \frac{1}{2}(1-x^4)^{-1/2}(-4x^3)+2x\sin^{-1} x^2+x^2\frac{1}{\sqrt{1-x^4}}\cdot 2x = \frac{-2x^3}{\sqrt{1-x^4}}+2x\sin^{-1} x^2+\frac{2x^3}{\sqrt{1-x^4}} = 2x\sin^{-1} x^2$.

45 $y' = \ln^2 x + x \cdot 2\ln x \cdot \left(\frac{1}{x}\right) = \ln x \cdot (\ln x + 2)$.

46 $y' = 6e^{6x} \sin x + e^{6x} \cos x = e^{6x}(6 \sin x + \cos x)$.

47 $y' = 2^{x^2 - x} \cdot \ln 2 \cdot (2x - 1)$.

48 $y' = 10^{\sin x} \ln 10 \cdot \cos x + 10 \sin^9 x \cos x = \cos x(10^{\sin x} \ln 10 + 10 \sin^9 x)$.

49 Let $y = (x^2 + 1)^{\ln x}$. Then $\ln y = \ln x \ln(x^2 + 1)$. Then

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \frac{1}{x}\ln(x^2 + 1) + (\ln x)\frac{1}{x^2 + 1} \cdot 2x \\ &= \frac{\ln(x^2 + 1)}{x} + \frac{2x \ln x}{x^2 + 1}, \end{aligned}$$

so

$$\frac{dy}{dx} = (x^2 + 1)^{\ln x}\left(\frac{\ln(x^2 + 1)}{x} + \frac{2x \ln x}{x^2 + 1}\right).$$

50

$$\begin{aligned} \frac{d}{dx}x^{\cos 2x} = \frac{d}{dx}e^{(\cos 2x)\ln x} &= e^{(\cos 2x)\ln x}\left((-2\sin 2x)\ln x + (\cos 2x) \cdot \frac{1}{x}\right) \\ &= x^{\cos 2x}\left((-2\sin 2x)\ln x + \frac{\cos 2x}{x}\right). \end{aligned}$$

51 $y' = \frac{1}{\sqrt{1 - \left(\frac{1}{x}\right)^2}} \cdot -\frac{1}{x^2} = -\frac{1}{|x|\sqrt{x^2 - 1}}$.

52 $y' = \frac{1}{(x + 8)\ln 3}$.

53 $y' = 6\cot^{-1} 3x - \frac{18x}{9x^2 + 1} + \frac{18x}{9x^2 + 1} = 6\cot^{-1} 3x$.

54 $y' = 4x\cos^{-1} x + 2x^2 \cdot \frac{-1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - x^2}} = 4x\cos^{-1} x + \frac{1 - 2x^2}{\sqrt{1 - x^2}}$.

55 Differentiate implicitly. $1 = -\sin(x - y)(1 - y')$, so $y' - 1 = \frac{1}{\sin(x - y)}$, so $y' = \frac{1}{\sin(x - y)} + 1 = \csc(x - y) + 1$.

56 Differentiate implicitly. $1 \cdot y^4 + x \cdot 4y^3y' + 4x^3y + x^4y' = 0$, so $(4xy^3 + x^4)y' = -y^4 - 4x^3y$. Then $y' = \frac{-y(y^3 + 4x^3)}{x(4y^3 + x^3)}$.

57 Because

$$y' = \frac{(1+\sin x)y'e^y - e^y\cos x}{(1+\sin x)^2},$$

collecting terms gives

$$y'\left(1 - \frac{e^y}{1+\sin x}\right) = -\frac{e^y\cos x}{(1+\sin x)^2},$$

so

$$y'(1-y) = -\frac{\cos x}{1+\sin x}\cdot y.$$

Thus $y' = -\dfrac{y\cos x}{(1-y)(1+\sin x)}$. This can also be written as $y' = \dfrac{y\cos x}{e^y - 1 - \sin x}$.

58 $\cos x\cos(y-1) - (\sin x)y'\sin(y-1) = 0$, so $y' = \cot x\cot(y-1)$.

59 $y'\sqrt{x^2+y^2} + y\cdot\dfrac{x+yy'}{\sqrt{x^2+y^2}} = 0$, and thus $y'\left(\sqrt{x^2+y^2} + \dfrac{y^2}{\sqrt{x^2+y^2}}\right) = -\dfrac{xy}{\sqrt{x^2+y^2}}$. This can be written as $y'\left(\dfrac{x^2+2y^2}{\sqrt{x^2+y^2}}\right) = -\dfrac{xy}{\sqrt{x^2+y^2}}$, so $y' = -\dfrac{xy}{x^2+2y^2}$.

60 Use logarithmic differentiation. $\ln y = 3\ln(x^2+1) - 8\ln(x^4+7) - 7\ln(2x+1)$. Then

$$\frac{1}{y}\frac{dy}{dx} = \frac{6x}{x^2+1} - \frac{32x^3}{x^4+7} - \frac{14}{2x+1}.$$

So

$$\frac{dy}{dx} = \frac{(x^2+1)^3}{(x^4+7)^8(2x+1)^7}\left(\frac{6x}{x^2+1} - \frac{32x^3}{x^4+7} - \frac{14}{2x+1}\right).$$

61 Use logarithmic differentiation. $\ln y = 10\ln(3x+5) + \frac{1}{2}\ln(x^2+5) - 50\ln(x^3+3)$. Then

$$\frac{1}{y}\frac{dy}{dx} = \frac{30}{3x+5} + \frac{x}{x^2+5} - \frac{150x^2}{x^3-3}.$$

So

$$\frac{dy}{dx} = \frac{(3x+5)^{10}\sqrt{x^2+5}}{(x^3+1)^{50}}\left(\frac{30}{3x+5} + \frac{x}{x^2+5} - \frac{150x^2}{x^3-3}\right).$$

62 $f'(x) = \dfrac{1}{1+(4x^2)^2}\cdot 8x = \dfrac{8x}{1+16x^4}$. So $f'(1) = \dfrac{8}{17}$.

63 $f'(x) = \sec^{-1}x + \dfrac{1}{\sqrt{x^2-1}}$. So $f'(2/\sqrt{3}) = \dfrac{\pi}{6} + \sqrt{3}$.

64 $f'(x) = \dfrac{1}{1+e^{-2x}}\cdot\left(-e^{-x}\right) = -\dfrac{1}{e^{-x}+e^x}$. So $f'(0) = -\dfrac{1}{2}$.

65 $\dfrac{d}{dx}x^{1/x} = \dfrac{d}{dx}e^{\frac{\ln x}{x}} = e^{\frac{\ln x}{x}}\cdot\left(\dfrac{1-\ln x}{x^2}\right) = x^{1/x}\left(\dfrac{1-\ln x}{x^2}\right)$. So $\dfrac{d}{dx}x^{1/x}\Big|_{x=1} = 1\cdot\dfrac{1-0}{1^2} = 1$.

66 $y' = 2xe^{x^2+1}$, so $y'' = 2e^{x^2+1} + 2x(2xe^{x^2+1}) = (4x^2+2)e^{x^2+1}$.

67 $y' = 2^x x\ln 2 + 2^x = 2^x(x\ln 2+1)$, so $y'' = 2^x\ln 2(x\ln 2+1) + 2^x\ln 2 = 2^x\ln 2(x\ln 2+2)$.

68 $y' = \dfrac{(x+1)3-(3x-1)}{(x+1)^2} = \dfrac{4}{(x+1)^2}$. Thinking of y' as $4(x+1)^{-2}$, we have

$$y'' = -8(x+1)^{-3} = -\frac{8}{(x+1)^3}.$$

69 $y' = \dfrac{x^2 \cdot \frac{1}{x} - 2x\ln x}{x^4} = \dfrac{1-2\ln x}{x^3}$. Then

$$y'' = \frac{x^3(\frac{-2}{x}) - (1-2\ln x)3x^2}{x^6} = \frac{-2x^2 - 3x^2 + 6x^2\ln x}{x^6} = \frac{6\ln x - 5}{x^4}.$$

70 Differentiating implicitly: $1 + y'\cos y = y'$, so $y'(\cos y - 1) = -1$, so $y' = \dfrac{1}{1-\cos y}$. Then $y'' = -(1-\cos y)^{-2}(\sin y)y' = -\dfrac{1}{(1-\cos y)^2}\cdot \sin y \cdot \dfrac{1}{1-\cos y} = -\dfrac{\sin y}{(1-\cos y)^3} = \dfrac{\sin y}{(\cos y - 1)^3}$.

71 Differentiating implicitly: $y + xy' + 2yy' = 0$, so $y'(x+2y) = -y$, so $y' = -\dfrac{y}{x+2y}$. Then $y'' = -\dfrac{(x+2y)y' - y(1+2y')}{(x+2y)^2} = \dfrac{y - xy'}{(x+2y)^2} = \dfrac{y + x\left(\frac{y}{x+2y}\right)}{(x+2y)^2} = \dfrac{y(x+2y)+xy}{(x+2y)^3} = \dfrac{2xy+2y^2}{(x+2y)^3} = \dfrac{2}{(x+2y)^3}$.

Note that for the last equality we are using the fact (from the original equation) that $xy + y^2 = 1$.

72 $y' = 2^{3x-6}3\ln 2$ so $y'(2) = 3\ln 2$. The equation of the tangent line is $y - 1 = 3\ln 2(x-2)$, or $y = (3\ln 2)x - 6\ln 2 + 1$.

73 $y' = 9x^2 + \cos x$. At $x = 0$, $y' = 1$. So the tangent line is given by $y - 0 = 1(x-0)$, or $y = x$.

74 $y' = \dfrac{4(x^2+3) - 8x^2}{(x^2+3)^2}$, so $y'(1) = \dfrac{1}{2}$. The tangent line is given by $y - 1 = \dfrac{1}{2}(x-1)$, or $y = \dfrac{1}{2}x + \dfrac{1}{2}$.

75 $y' + \dfrac{y + xy'}{2\sqrt{xy}} = 0$. At the point $(1,4)$, we have $y' + \dfrac{4+y'}{4} = 0$, so $y' = -\dfrac{4}{5}$. The tangent line is given by $y - 4 = -\dfrac{4}{5}(x-1)$, or $y = -\dfrac{4}{5}x + \dfrac{24}{5}$.

76 $2xy + x^2y' + 3y^2y' = 0$. At the point $(2,1)$ we have $4 + 4y' + 3y' = 0$, so $y' = -\dfrac{4}{7}$. The tangent line is given by $y - 1 = -\dfrac{4}{7}(x-2)$, or $y = -\dfrac{4}{7}x + \dfrac{15}{7}$.

77 $\dfrac{d}{dx}[x^2f(x)] = 2xf(x) + x^2f'(x)$.

78 $\dfrac{d}{dx}\sqrt{\dfrac{f(x)}{g(x)}} = \dfrac{1}{2\sqrt{\frac{f(x)}{g(x)}}}\cdot\dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$.

79 $\dfrac{d}{dx}\left(\dfrac{xf(x)}{g(x)}\right) = \dfrac{(f(x)+xf'(x))g(x) - xf(x)g'(x)}{g(x)^2}$.

80 $\dfrac{d}{dx}f\left(\sqrt{g(x)}\right) = f'\left(\sqrt{g(x)}\right)\cdot\dfrac{1}{2\sqrt{g(x)}}\cdot g'(x)$.

81

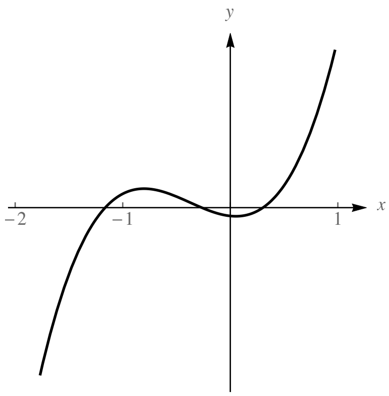
a. This has (D) as its derivative. Note that it consists of two pieces each of which are linear with the same slope. So its derivative is constant – but at $x = 2$ the derivative doesn't exist. We can easily know that this is true because the function isn't continuous at $x = 2$, so it can't be differentiable there.

b. This has (C) as its derivative. The slope of the tangent line is positive for $x < 2$ and negative for $x > 2$ and doesn't exist at $x = 2$. Also, near $x = 2$ the slope is near zero.

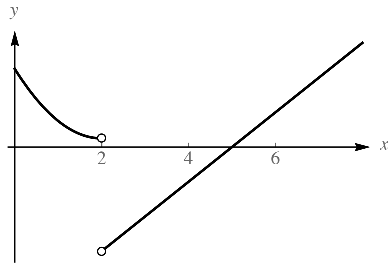
c. This has (B) as its derivative. Note that the slope of the tangent line is always positive, and gets infinitely steep at $x = 2$.

d. This has (A) as its derivative. Note that the slope of the tangent line is positive for $x < 2$, negative for $x > 2$, and is infinitely steep at $x = 2$ where the cusp occurs.

82



83



84

a. $5f'(1) + 3g'(1) = 5(3) + 3\left(\frac{1}{2}\right) = \frac{33}{2}$.

b. $f'(1)g(1) + f(1)g'(1) = 3(1) + 2\left(\frac{1}{2}\right) = 4$.

c. $\dfrac{g(3)f'(3) - f(3)g'(3)}{g(3)^2} = \dfrac{2(-2) - 3\left(\frac{1}{2}\right)}{2^2} = -\dfrac{11}{8}$.

d. $f'(f(4))f'(4) = f'(1)f'(4) = 3(-2) = -6$.

e. $g'(f(1))f'(1) = g'(2)f'(1) = \frac{1}{2}(3) = \frac{3}{2}$.

85

a. $\dfrac{d}{dx}\left[f(x) + 2g(x)\right]_{x=3} = f'(3) + 2g'(3) = 9 + 2 \cdot 9 = 27$.

b. $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]_{x=1} = \dfrac{g(1)f'(1) - f(1)g'(1)}{g(1)^2} = \dfrac{9(7) - 3(5)}{9^2} = \dfrac{48}{81} = \dfrac{16}{27}$.

c. $\left.\dfrac{d}{dx}(f(x)g(x))\right|_{x=3} = f'(3)g(3) + f(3)g'(3) = 9(7) + 1(9) = 72$.

d. $\left.\dfrac{d}{dx}\,(f(x))^3\right|_{x=5} = 3f(5)^2f'(5) = 3(9)^2 \cdot 5 = 1215$.

e. $(g^{-1})'(7) = \dfrac{1}{g'(3)} = \dfrac{1}{9}$.

86 $\left.\left(f^{-1}(x)\right)'\right|_{x=f(0)} = \dfrac{1}{f'(0)} = -\dfrac{1}{(0+1)^2} = -1$.

87 Note that for $x = 2$, we have $y = \sqrt{8+2-1} = 3$. $\left.\left(f^{-1}(x)\right)'\right|_{x=f(2)} = \dfrac{1}{f'(2)} = \dfrac{1}{\frac{3(2^2)+1}{2\sqrt{2^3+2-1}}} = \dfrac{6}{13}$.

88 $\left(f^{-1}(x)\right)' = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{12}$.

89 If $f(x) = x^{-1/3}$, then $f^{-1}(x) = x^{-3}$. So $\left(f^{-1}\right)'(x) = -3x^{-4}$ for $x \neq 0$.

90

a. $\frac{d}{dx}\left.(xf(x))\right|_{x=2} = \left.(f(x)+xf'(x))\right|_{x=2} = f(2)+2f'(2) = 5+2\cdot 3 = 11.$

b. $\frac{d}{dx}\left.(f(x^2))\right|_{x=1} = \left.(2xf'(x^2))\right|_{x=1} = 2f'(1) = 2.$

c. $\frac{d}{dx}\left.(f(f(x)))\right|_{x=1} = f'(f(1))\cdot f'(1) = f'(3)\cdot 1 = 4.$

91

a. Because $f^{-1}(7) = 3$, we have $\left(f^{-1}\right)'(7) = \frac{1}{f'(3)} = \frac{1}{4}$.

b. Because $f^{-1}(3) = 1$, we have $\left(f^{-1}\right)'(3) = \frac{1}{f'(1)} = 1$.

c. $\left(f^{-1}\right)'(f(2)) = \frac{1}{f'(2)} = \frac{1}{3}$.

92 The value of $f(3)$ is the same as the value of the tangent line to f at $x = 3$, so it is $4(3) - 10 = 2$. The value of $f'(3)$ is the slope of the tangent line to $y = f(x)$ at $x = 3$, so it is 4. The value of $g(5)$ is the same as the value of the tangent line to g at $x = 5$, so it is $30 - 27 = 3$. The value of $g'(5)$ is the slope of the tangent line to $y = g(x)$ at $x = 5$, so it is 6.

93 $\left.\frac{d}{dx}(f(g(x)))\right|_{x=5} = f'(g(5))g'(5) = f'(3)g'(5) = 4(6) = 24$. Also, $f(g(5)) = f(3) = 2$. So the tangent line is given by $y - 2 = 24(x-5)$, or $y = 24x - 118$.

94

a. $\frac{ds}{dt} = v(t) = 27 - 3t^2$. This is 0 for $t > 0$ only for $t = 3$. On the interval $(0,3)$ $v(t) > 0$ and on $(3,\infty)$ $v(t) < 0$. So the object is stationary at $t = 3$, is moving to the right for $0 < t < 3$, and is moving to the left for $t > 3$.

b. $v(2) = 27 - 12 = 15$ ft/s. $a(t) = \frac{dv}{dt} = -6t$, so $a(2) = -12\,\text{ft/s}^2$.

c. $a(3) = -6(3) = -18\,\text{ft/s}^2$.

d. Note that for $t > 0$ the acceleration is always negative. So the speed is decreasing when the object is moving to the right, which occurs for $0 < t < 3$.

95

a. $\frac{ds}{dt} = v(t) = 96 - 12t$, so $v(1) = 84$ ft/s.

b. $v(t) = 12$ when $96 - 12t = 12$, or $12t = 84$, so $t = 7$ s.

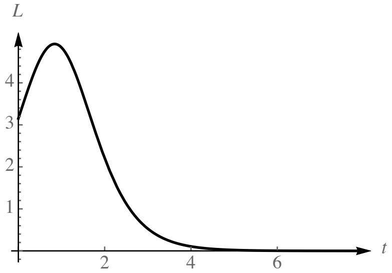
c. $v(t) = 0$ when $96 - 12t = 0$, or $t = 8$ s. The height at $t = 8$ is $s(8) = 96(8) - 6(64) = 384$ ft.

d. $s(t) = 0$ when $96t - 6t^2 = 0$, or $t(16-t) = 0$, so $t = 0$ and $t = 16$. The velocity at time $t = 16$ is $v(16) = -96$, so it strikes the ground with a speed of 96 ft/s.

96

a. We can write $L(t) = 11.94(1+4e^{-1.65t})^{-1}$, so $L'(t) = -11.94(1+4e^{-1.65t})^{-2}((-1.65)4e^{-1.65t}) = \frac{78.804e^{-1.65t}}{(1+4e^{-1.65t})^2}$. Then $L'(1) \approx 4.8$. The culmen of a one-week-old Indian Spotted Owlet is growing at almost 5 mm/week.

b. The culmen grows quickly at first, reaching a maximum growth rate just before the first week of life, and then the growth rate gradually decreases over the next 4 weeks. By the fifth week, the growth rate is approximately 0 which means it has reached its adult size.



97

a. $A(20) = 40,000(1.0075^{240} - 1) \approx \$200,366$.
$A'(t) = 40,000(1.0075^{12t})\ln(1.0075) \cdot 12 \approx 3586.57(1.0075^{12t})$. $A'(20) \approx \$21,552$ dollars/year.

b. $A(t) = 100,000$ when $40,000(1.0075^{12t} - 1) = 100,000$, which occurs when

$$1.0075^{12t} - 1 = \frac{100,000}{40,000} = 2.5,$$

so when

$$1.0075^{12t} = 3.5.$$

This occurs when $12t\ln(1.0075) = \ln 3.5$, or $t = \dfrac{\ln 3.5}{12\ln(1.0075)} \approx 13.97 \approx 14$ years. At that time we have $A'(13.97) \approx \$12,551$ per year.

98

a. The instantaneous rate of change is $Q'(t) = -1.386e^{-0.0693t}$ mg/hr.

b. At $t = 0$ hours, we have $Q'(0) = -1.386$, so the amount of antibiotic is decreasing at a rate of 1.386 mg/hr. At $t = 2$ hours, we have $Q'(2) = -1.386e^{-0.1386} \approx -1.207$, so the amount of antibiotic is decreasing at a rate of about 1.207 mg/hr.

c. $\lim_{t\to\infty} Q(t) = 20 \lim_{t\to\infty} e^{-0.0693t} = 0$. In the long run, the antibiotic is all used up. $\lim_{t\to\infty} Q'(t) = -1.386 \lim_{t\to\infty} e^{-0.0693t} = 0$. The rate of change of the amount of antibiotic in the bloodstream also goes to zero as $t \to \infty$.

99

a. Average growth is $\dfrac{p(60) - p(50)}{10} = 2.7$ million people per year.

b. The curve is pretty straight between $t = 50$ and $t = 60$, so the secant line between these two points is approximately as steep as the tangent line at a point in between.

c. A reasonable estimate to the instantaneous grow rate at 1985 would be the slope of the secant line between $t = 80$ and $t = 90$. This is $\dfrac{p(90) - p(80)}{10} = 2.217$ million people per year.

100

a. The graph has the steepest slope at about $t = 18$. At this point the rate is about $\dfrac{N(20) - N(16)}{4} = \dfrac{3500 - 1900}{4} = 400$ bacteria per hour.

b. It is smallest at $t = 0$ or $t = 36$, where it is about $\dfrac{N(36) - N(32)}{4} \approx \dfrac{4900 - 4800}{4} = 25$ bacteria per hour.

c. The average growth rate over $[0, 36]$ is $\dfrac{N(36) - N(0)}{36} \approx \dfrac{4900 - 400}{36} = \dfrac{4500}{36} = 125$ bacteria per hour.

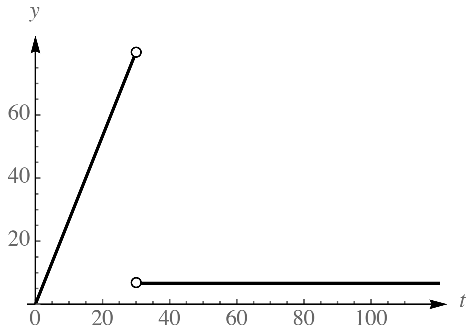
101

a. $v(15) \approx \frac{400-200}{5} = 40$ meters per second.

b. Because the graph is a straight line for $t \geq 30$, $v(70) = \frac{D(90)-D(60)}{30} = \frac{1600-1400}{30} = \frac{20}{3}$ meters per second. The points at 60 and 90 were chosen because it is easier to detect the function values at those points using the given grid.

c. The average velocity is $\dfrac{D(90) - D(20)}{70} \approx \dfrac{1600 - 550}{70} = 15$ meters per second.

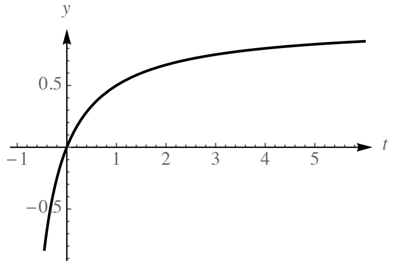
d.



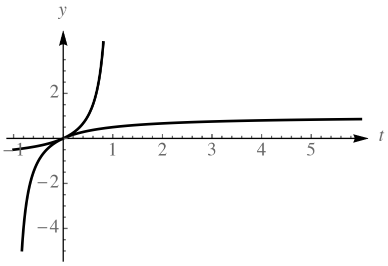
e. The parachute was deployed.

102

a.



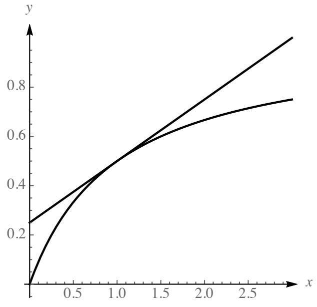
b.



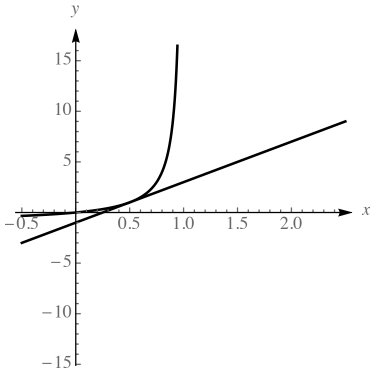
If $y = \dfrac{x}{x+1}$, then $yx + y = x$, so $y = x - yx$, and $y = x(1-y)$, so $x = \dfrac{y}{1-y}$. The inverse function is given by $f^{-1}(x) = \dfrac{x}{1-x}$.

c. $\left(f^{-1}(x)\right)' = \dfrac{1-x+x}{(1-x)^2} = \dfrac{1}{(1-x)^2}$. So $\left(f^{-1}\right)\left(\frac{1}{2}\right) = 4$.

d1.



d2.



103 We are looking for values of x so that $y'(x) = 0$. We have $y' = \sqrt{6-x} - \frac{x}{2\sqrt{6-x}}$, and this quantity is zero when $2(6-x) - x = 0$, or $12 - 3x = 0$, so when $x = 4$. So at the point $(4, 4\sqrt{2})$ there is a horizontal tangent line. There is a vertical tangent line at $x = 6$, because $\lim\limits_{x\to 6^-} y'(x) = -\infty$.

104 With $a = \frac{\pi}{4}$, $f(x) = \sin^2(x)$ we have

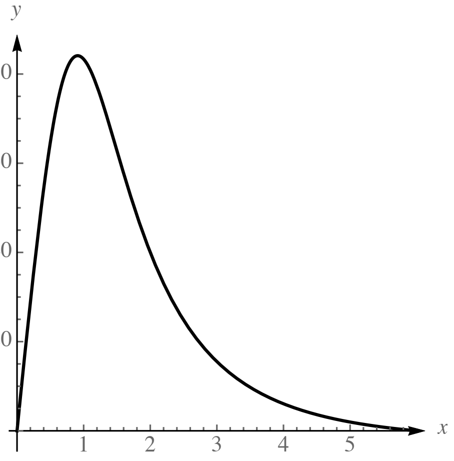
$$
\begin{aligned}
f'\left(\frac{\pi}{4}\right) &= \lim_{h\to 0} \frac{f(\pi/4+h) - f(\pi/4)}{h} = \lim_{h\to 0} \frac{\sin^2(\pi/4+h) - (1/2)}{h} \\
&= 2\sin(\pi/4)\cos(\pi/4) = 2(\sqrt{2}/2)(\sqrt{2}/2) = 1.
\end{aligned}
$$

Note that we used the fact that $\frac{d}{dx}\sin^2(x) = 2\sin x\cos x$ in the middle of this derivation.

105 Let $a = 5$ and $f(x) = \tan(\pi\sqrt{3x-11})$. Note that $f'(5) = \frac{3\pi}{2}\frac{\sec^2(2\pi)}{2} = \frac{3\pi}{4}$.

So $\lim\limits_{x\to 5} \frac{f(x) - f(5)}{x-5} = \lim\limits_{x\to 5} \frac{\tan(\pi\sqrt{3x-11}) - 0}{x-5} = f'(5) = \frac{3\pi}{4}$.

106



a. The probe climbs quickly, and achieves a maximum height of about 84.1 at about $t = 0.91$.

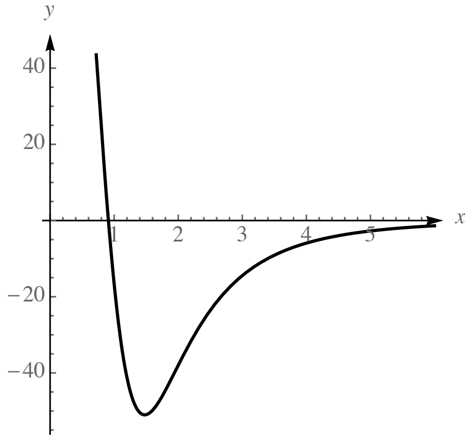
b.

$$v(t) = s'(t) = \frac{(t^3+2)(300-100t)-(300t-50t^2)(3t^2)}{(t^3+2)^2}$$

$$= \frac{300t^3-100t^4+600-200t-900t^3+150t^4}{(t^3+2)^2}$$

$$= \frac{50t^4-600t^3-200t+600}{(t^3+2)^2}.$$

c.



The maximum velocity is attained at $t = 0$.

107

a. The average cost is $\frac{C(3000)}{3000} = \frac{1025000}{3000} \approx \$341.67$. The marginal cost is $C'(3000) = -0.04(3000) + 400 = \$280$.

b. The average cost of producing 3000 lawnmowers is \$341.67 per mower. The cost of producing the 3001st lawnmower is approximately \$280.

108

a. The marginal cost is given by $C'(x) = -0.0003x^2 + 0.1x + 60$, so $C'(400) = \$52$. The average cost of producing 400 fly rods is $\frac{C(400)}{400} = \$66$.

b. The average cost of producing 400 fly rods is \$66 per fly rod. The cost of producing the 401st fly rod is approximately \$52.

109

a. The average growth rate is $\frac{p(50)-p(0)}{50} = \frac{407500-80000}{50} = 6550$ people per year.

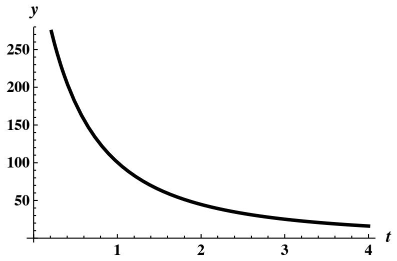
b. The growth rate in 1990 is $p'(40) = -5.1(40^2) + 144 \cdot 40 + 7200 = 4800$ people per year.

110

a. $v(t) = \pi \cdot 4^2 \cdot \frac{8t}{t+1} = \frac{128\pi t}{t+1}$ cubic cm.

b. $v'(t) = 128\pi \cdot \frac{(t+1)-t}{(t+1)2} = \frac{128\pi}{(t+1)^2}$.

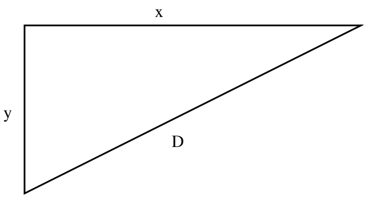
c.



Because the rate of change of volume is strictly positive, the volume function must be increasing for $t > 0$.

111

Let x be the distance the eastbound boat has traveled, and y the distance the southbound boat has traveled. By the Pythagorean Theorem, $D^2 = x^2 + y^2$, so $2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$, so $\frac{dD}{dt} = \frac{x \cdot x' + y \cdot y'}{D}$. We are given that $x' = 40$, $y' = 30$, and at $t = .5$ hours, we have $x = 20$, $y = 15$, and $D = 25$. Thus, $\frac{dD}{dt} = \frac{20 \cdot 40 + 30 \cdot 15}{25} = 50$ mph.



112 $V = \frac{4}{3}\pi r^3 = \frac{\pi d^3}{6}$, so $V' = \frac{\pi d^2 d'}{2}$. With $V' = 10\,\text{cm}^3/\text{min}$ and $d = 5$ cm, we have $d' = \frac{20}{25\pi} = \frac{4}{5\pi}$ cm/min.

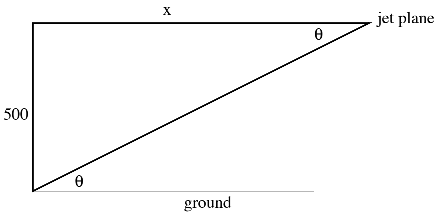
113 Let h be the elevation of the balloon, and s the length of the rope. We have $h = s\sin(65°)$, so $h' = s'\sin(65°) = -5 \cdot \sin(65°) \approx -4.53$ feet per second.

114 $\frac{r}{h} = \frac{2}{3}$, so $r = \frac{2}{3}h$. $V = \frac{1}{3}\pi r^2 h = \frac{4}{27}\pi h^3$. So $\frac{dV}{dt} = \frac{4}{9}\pi h^2 \frac{dh}{dt}$. When $h = 2$, $\frac{dV}{dt} = 2$, so $\frac{dh}{dt} = \frac{2}{\frac{4}{9}\pi \cdot 4} = \frac{9}{8\pi}$ feet per minute.

115

Let x be the distance the jet has flown since it went over the spectator. Let θ be the angle of elevation between the ground and the line from the spectator to the jet. Note that θ is also the angle pictured, and that $\cot\theta = \frac{x}{500}$. Thus, $\theta = \cot^{-1}\left(\frac{x}{500}\right)$. We are given that $x' = 450\,\text{mph} = 660\,\text{ft/sec}$.

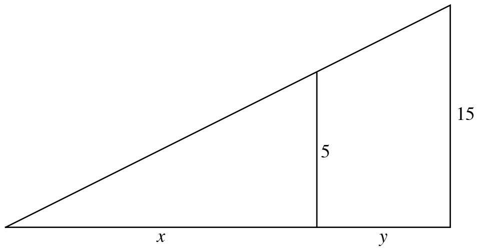
$\theta' = -\frac{x'}{500 \cdot \left(1 + \left(\frac{x}{500}\right)^2\right)} = -\frac{500x'}{250,000 + x^2}$. After 2 seconds, $x = 1320$ feet, so at this time $\theta' = -\frac{500 \cdot 660}{250,000 + (1320)^2} \approx -0.166$ radians per second.



116 Let D be the distance the man is from the billboard, and let α be the angle between his eye level and the line of sight to the bottom of the billboard, and let θ be the angle between his line of sight to the bottom of the billboard and his line of sight to the top of the billboard. We have that $\cot\alpha = \frac{D}{4}$, so $\alpha = \cot^{-1}\left(\frac{D}{4}\right)$. Also, $\cot(\alpha + \theta) = \frac{D}{19}$, so $\theta = \cot^{-1}\left(\frac{D}{19}\right) - \alpha = \cot^{-1}\left(\frac{D}{19}\right) - \cot^{-1}\left(\frac{D}{4}\right)$.

So $\theta' = -\frac{19D'}{361 + D^2} + \frac{4D'}{16 + D^2}$. We are given that $D' = -2$ feet per second, so at $D = 30$ we have $\theta' \approx 0.03 - 0.009 = 0.021$ radians per second.

117 Let x equal the length of her shadow and let y equal her distance from the pole. Using similar triangles, we have $\frac{x}{5} = \frac{x+y}{15}$, which implies that $3x = x + y$ or $x = \frac{y}{2}$. It follows that the rate at which her shadow is increasing is $\frac{dx}{dt} = \frac{1}{2}\frac{dy}{dt} = \frac{1}{2}(3) = 1.5$ ft/s.



118

a. $f'(a) = \lim_{x\to a} \frac{x^2 - a^2}{x - a} = \lim_{x\to a} \frac{(x-a)(x+a)}{x-a} = \lim_{x\to a}(x+a) = 2a.$

b. $f'(a) = \lim_{h\to 0} \frac{b(a+h)^2 + c(a+h) + d - ba^2 - ca - d}{h} = \lim_{h\to 0} \frac{2bah + bh^2 + ch}{h} = \lim_{h\to 0}(2ba + bh + c) = 2ab + c.$

119

a. $\left(f^{-1}\right)'\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{f'\left(\frac{\pi}{4}\right)} = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \sqrt{2}.$

b. $\frac{d}{dx}\sin^{-1}(x)\Big|_{x=1/\sqrt{2}} = \frac{1}{\sqrt{1-(1/2)}} = \frac{1}{\sqrt{1/2}} = \sqrt{2}.$

120

a. Note that $f'(x) = 2x$, so $f'\left(\frac{x+y}{2}\right) = 2\cdot\frac{x+y}{2} = x+y$. The quantity $\frac{f(x)-f(y)}{x-y}$ can be written as $\frac{x^2-y^2}{x-y} = \frac{(x-y)(x+y)}{x-y} = x+y$, so these quantities are equal for $x \neq y$.

b. Yes. Note that $f'(x) = 2ax$, so $f'\left(\frac{x+y}{2}\right) = 2a\cdot\frac{x+y}{2} = a(x+y)$. The quantity $\frac{f(x)-f(y)}{x-y}$ can be written as $\frac{ax^2-ay^2}{x-y} = a\cdot\frac{(x-y)(x+y)}{x-y} = a(x+y)$, so these quantities are equal for $x \neq y$.

c. The line through $(x, f(x))$ and $(y, f(y))$ is parallel to the tangent line at the midpoint between x and y.

d. No. For example, consider $a = 1$, $x = 0$, and $y = 1$. Note that $f'(x) = 3x^2$. Then $f'\left(\frac{x+y}{2}\right) = f'(1/2) = 3/4$. On the other hand, $\frac{f(x)-f(y)}{x-y} = \frac{1-0}{1-0} = 1$.

Chapter 4

Applications of the Derivative

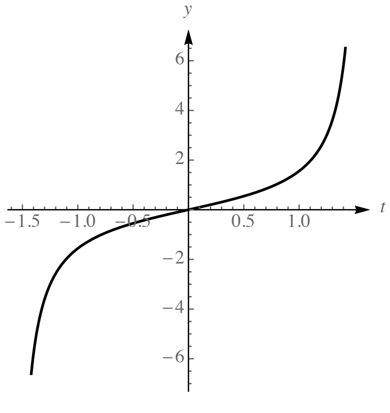
4.1 Maxima and Minima

4.1.1 A number $M = f(c)$ where $c \in [a, b]$ with the property that $f(x) \leq M$ for all $x \in [a, b]$ is an absolute maximum for f on $[a, b]$, and a number $m = f(d)$ where $d \in [a, b]$ with the property that $f(x) \geq m$ for all $x \in [a, b]$ is an absolute minimum for f on $[a, b]$.

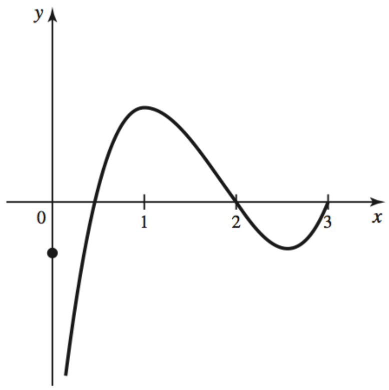
4.1.2 A number $M = f(c)$ is a local maximum for f if there is an interval (r, s) containing c so that $f(x) \leq M$ for all $x \in (r, s)$. A number $m = f(d)$ is a local minimum for f if there is an interval (r, s) containing d so that $f(x) \geq m$ for all $x \in (r, s)$.

4.1.3 The function must be a continuous function defined on a closed interval.

4.1.4 The tangent function on the interval $(-\pi/2, \pi/2)$ is continuous but has no maximum or minimum.

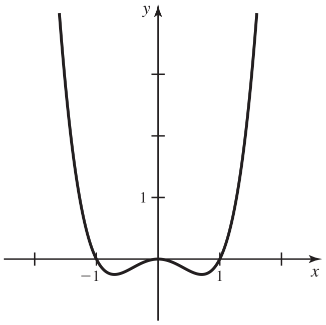


4.1.5 The function shown has no absolute minimum on $[0,3]$ because $\lim_{x\to 0^+} f(x) = -\infty$. It has an absolute maximum near $x=1$ and a local minimum near $x=2.5$.

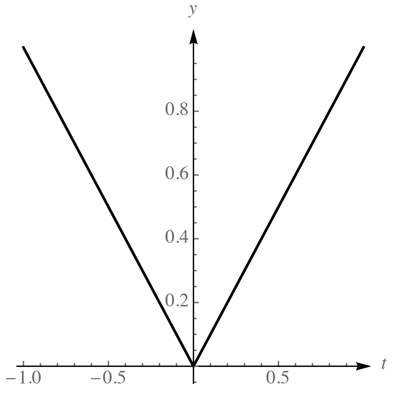


4.1.6 An interior point c of the domain of f at which $f'(c)=0$ or $f'(c)$ doesn't exist is a critical point of f.

4.1.7 Note the existence of a horizontal tangent line at $x=0$ where the maximum occurs.



4.1.8 Note the minimum at $x=0$ where $f'(x)$ does not exist. Note that $\lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ is different for $h \to 0^-$ and $h \to 0^+$.



4.1.9 First find all the critical points by seeking all points x in the domain of f so that $f'(x)=0$ or $f'(x)$ doesn't exist. Now compare the y-values of all of these points, together with the y-values of the endpoints. The largest y-value from among these is the maximum, and the smallest is the minimum.

4.1.10 If a is an endpoint of the given interval, and $f(a) \le f(x)$ for all x in the interval, then $f(a)$ is the absolute minimum. This happens, for example, for a line of positive slope defined on an interval $[a,b]$ – the y-value at the left endpoint is the smallest y-value over the interval.

4.1.11 $y = h(x)$ has an absolute maximum at $x = b$ and an absolute minimum at $x = c_2$.

4.1.12 $y = f(x)$ has an absolute maximum at $x = c$ and no absolute minimum.

4.1.13 $y = g(x)$ has no absolute maximum, but has an absolute minimum at $x = a$.

4.1.14 $y = g(x)$ has an absolute maximum at $x = a$ and an absolute minimum at $x = c$.

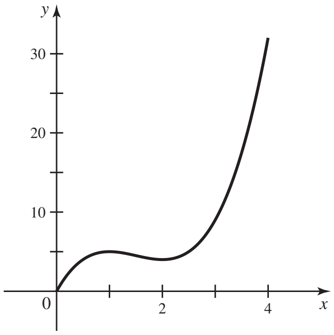
4.1.15 $y = f(x)$ has an absolute maximum at $x = b$ and an absolute minimum at $x = a$. It has local maxima at $x = p$ and $x = r$, and local minima at $x = q$ and $x = s$.

4.1.16 $y = f(x)$ has an absolute maximum at $x = p$, and an absolute minimum at $x = a$. It has local minima at $x = q$ and $x = s$, and local maxima at $x = r$ and $x = p$.

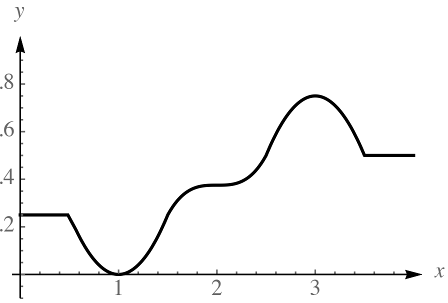
4.1.17 $y = g(x)$ has an absolute minimum at $x = b$ and an absolute maximum at $x = p$. It has local maxima at $x = p$ and $x = r$. It has a local minimum at $x = q$.

4.1.18 $y = h(x)$ has an absolute maximum at $x = p$ and an absolute minimum at $x = u$. It has local maxima at $x = p$, $x = r$ and $x = t$. It has local minima at $x = q$, $x = s$, and $x = u$.

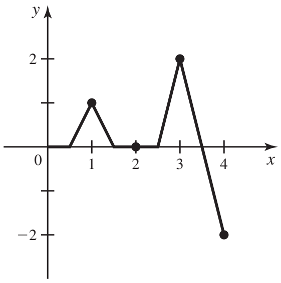
4.1.19 Note the horizontal tangent lines at 1 and 2, and the minimum at 0 and the maximum at 4.



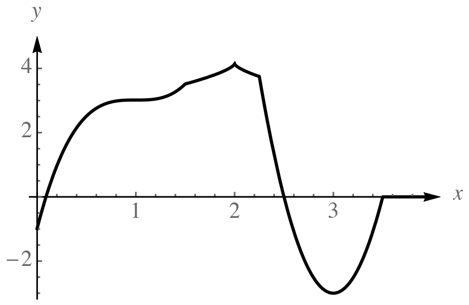
4.1.20 Note the minimum at $x = 1$, the maximum at $x = 3$, and the horizontal tangent lines at 1, 2, and 3.



4.1.21 Note the horizontal tangent line at $x = 2$, and the "corners" at $x = 1$ and $x = 3$. Also note the absolute maximum at $x = 3$ and the absolute minimum at $x = 4$.



4.1.22 Note the maximum at 2, and the minimum at 3. Note also the horizontal tangent lines at $x = 1$ and $x = 3$, and the sharp "corner" at $x = 2$.



4.1.23 $f'(x) = 6x - 4$, which is zero when $x = \frac{2}{3}$. So that is the only critical point.

4.1.24 $f'(x) = \frac{3}{8}x^2 - \frac{1}{2}$, which is zero when $3x^2 - 4 = 0$, which occurs for $x = \pm\frac{2}{\sqrt{3}}$.

4.1.25 $f'(x) = x^2 - 9$, which is zero for $x = \pm 3$. So the critical points are $x = \pm 3$.

4.1.26 $f'(x) = x^3 - x^2 - 6x = x(x^2 - x - 6) = x(x-3)(x+2)$, which is zero for $x = 0, 3$, and -2. So the critical points are $x = 0, 3$, and -2.

4.1.27 $f'(x) = 9x^2 + 3x - 2 = (3x+2)(3x-1)$, which is zero for $x = -\frac{2}{3}$ and $x = \frac{1}{3}$. So the critical points are $x = -\frac{2}{3}$ and $x = \frac{1}{3}$.

4.1.28 $f'(x) = 4x^4 - 9x^2 = x^2(4x^2 - 9) = x^2(2x+3)(2x-3)$, which is zero for $x = 0$ and $x = \pm\frac{3}{2}$. So the critical points are $x = \pm\frac{3}{2}$.

4.1.29 $f'(x) = 3x^2 - 4a^2$, which is zero for $3x^2 = 4a^2$, or $x^2 = \frac{4a^2}{3}$. So the critical points are $x = \pm\frac{2a}{\sqrt{3}}$.

4.1.30 $f'(x) = 1 - \frac{5}{1+x^2}$, which is zero for $1 + x^2 = 5$, so $x = \pm 2$. So those are the critical points.

4.1.31 $f'(t) = \frac{(t^2+1)(1) - t(2t)}{(t^2+1)^2} = \frac{1-t^2}{(t^2+1)^2}$. This quantity is zero exactly when $1 - t^2 = 0$, so $t = 1$ and $t = -1$ are the critical points.

4.1.32 $f'(x) = 60x^4 - 60x^2 = 60x^2(x^2 - 1)$, which is zero on the given interval when $x = \pm 1$ and when $x = 0$, so those are the critical points.

4.1.33 $f'(x) = \frac{e^x - e^{-x}}{2}$, which is zero when $e^x = e^{-x}$ or $x = -x$, so only for $x = 0$.

4.1.34 $f'(x) = \cos x \cos x - \sin x \sin x$, which is zero when $\sin^2 x = \cos^2 x$, so when $\sin x = \cos x$ or $\sin x = -\cos x$. This occurs when $x = \frac{\pi}{4} + k\frac{\pi}{2}$ where k is an integer.

4.1.35 $f'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{x-1}{x^2}$. There is a critical point at $x = 1$.

4.1.36 $f'(t) = 2t - \frac{4t}{t^2+1} = \frac{2t^3 - 2t}{t^2+1} = \frac{2t(t+1)(t-1)}{t^2+1}$. This is zero for $t = 0, \pm 1$.

4.1.37 $f'(x) = 2x\sqrt{x+5} + x^2 \cdot \frac{1}{2\sqrt{x+5}} = \frac{4x(x+5)}{2\sqrt{x+5}} + \frac{x^2}{2\sqrt{x+5}} = \frac{5x^2+20x}{2\sqrt{x+5}}$. This is zero when $5x^2+20x = 5x(x+4)$ is zero, which occurs for $x = 0$ and $x = -4$. The critical points are $x = 0$ and $x = -4$.

4.1.38 $f'(x) = \frac{1}{\sqrt{1-x^2}}\cos^{-1}(x) + \sin^{-1}(x) \cdot \left(-\frac{1}{\sqrt{1-x^2}}\right) = \frac{\cos^{-1}(x) - \sin^{-1}(x)}{\sqrt{1-x^2}}$. This is zero when $\sin^{-1}(x) = \cos^{-1}(x)$, which occurs for $x = \frac{\sqrt{2}}{2}$.

4.1.39 $f'(x) = \sqrt{x-a} + \frac{x}{2\sqrt{x-a}} = \frac{2x-2a+x}{2\sqrt{x-a}} = \frac{3x-2a}{2\sqrt{x-a}}$. This expression is zero when $x = \frac{2a}{3}$; however, that number is not in the domain of f if $a > 0$. However, if $a < 0$, then $\frac{2a}{3}$ is in the domain, and thus gives a critical point.

4.1.40 $f'(x) = \frac{\sqrt{x-a} - \frac{x}{2\sqrt{x-a}}}{x-a} = \frac{\sqrt{x-a} - \frac{x}{2\sqrt{x-a}}}{x-a} \cdot \frac{2\sqrt{x-a}}{2\sqrt{x-a}} = \frac{2x-2a-x}{2(x-a)^{3/2}} = \frac{x-2a}{2(x-a)^{3/2}}$. This is zero when $x = 2a$, so there is a critical point at $x = 2a$ for $a > 0$.

4.1.41 $f'(t) = t^4 - a^4$, which is zero when $t^4 = a^4$, or $|t| = a$. So there are critical points at $t = a$ and at $t = -a$.

4.1.42 $f'(x) = 3x^2 - 6ax + 3a^2 = 3(x-a)^2$. So the point $x = a$ is a critical point.

4.1.43 $f'(x) = 2x$, which is zero for $x = 0$. We have that $f(-2) = -6$, $f(0) = -10$, and $f(3) = -1$, so the maximum value of f on this interval is -1 and the minimum is -10.

4.1.44 $f'(x) = \frac{4}{3}(x+1)^{1/3}$, which is zero for $x = -1$. So $(-1, 0)$ is the only critical point. Checking endpoints and critical points we have that $f(-9) = (-8)^{4/3} = 16$, $f(-1) = 0$, and $f(7) = 8^{4/3} = 16$, so the maximum value of f on this interval is 16 and the minimum is 0.

4.1.45 $f'(x) = 3x^2 - 6x = 3x(x-2)$. This is zero for $x = 0$ and $x = 2$. Checking endpoints and critical points we have $f(-1) = -4$, $f(0) = 0$, $f(2) = -4$, and $f(3) = 0$. So there is an absolute maximum of 0 at $x = 0$ and $x = 3$, and an absolute minimum of -4 at $x = -1$ and $x = 2$.

4.1.46 $f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-2)(x-1)$, which is zero for $x = 0$, $x = 1$, and $x = 2$. Checking endpoints and critical points we have $f(-1) = 9$, $f(0) = 0$, $f(1) = 1$, $f(2) = 0$, and $f(3) = 9$. There is an absolute maximum of 9 at $x = 3$ and $x = -1$, and an absolute minimum of 0 at $x = 0$ and $x = 2$.

4.1.47 $f'(x) = 15x^4 - 75x^2 + 60 = 15(x^4 - 5x^2 + 4) = 15(x^2-4)(x^2-1) = 15(x-2)(x+2)(x-1)(x+1)$. So the critical points are $x = 1, -1, 2, -2$. Checking the endpoints and critical points we have $f(-2) = -16$, $f(-1) = -38$, $f(0) = 0$, $f(1) = 38$, $f(2) = 16$, and $f(3) = 234$. There is an absolute minimum of -38 at $x = -1$, and an absolute maximum of 234 at $x = 3$.

4.1.48 $f'(x) = 2e - 2x$, which is zero for $x = e$. Checking endpoints and critical points we have $f(0) = 0$, $f(e) = e^2$, and $f(2e) = 0$, so there is an absolute maximum of e^2 at $x = e$ and an absolute minimum of 0 at $x = 0$ and $x = 2e$.

4.1.49 $f'(x) = -2\cos x \sin x$, which is zero for $x = 0$, $x = \pi/2$, and $x = \pi$. Because there are endpoints at $x = 0$ and $x = \pi$, only $(\pi/2, 0)$ is a critical point. We have that $f(0) = 1$, $f(\pi/2) = 0$, and $f(\pi) = 1$, so the maximum value of f on this interval is 1 and the minimum is 0.

4.1.50 $f'(x) = \frac{(x^2+3)^2 - x \cdot 2(x^2+3) \cdot 2x}{(x^2+3)^4} = \frac{(x^2+3) - 4x^2}{(x^2+3)^3} = \frac{3-3x^2}{(x^2+3)^3}$, which is zero for $x = \pm 1$. We have that $f(-2) = -\frac{2}{49} \approx -0.041$, $f(2) = \frac{2}{49} \approx 0.041$, $f(\pm 1) = \pm\frac{1}{16} = \pm 0.0625$. The absolute maximum of f on the given interval is 0.0625 at $x = 1$ and the absolute minimum is -0.0625 at $x = -1$.

4.1.51 $f'(x) = 3\cos 3x$, which is zero when

$$3x = \ldots, -\pi/2,\ \pi/2,\ 3\pi/2, \ldots,$$

so when $x = \ldots, -\pi/6,\ \pi/6, \pi/2, \ldots$. The only such values on the given interval are $x = -\pi/6$ and $x = \pi/6$. We have

$$f(-\pi/4) = -\sqrt{2}/2 \approx -0.707,$$

$f(-\pi/6) = -1$, $f(\pi/6) = 1$, and $f(\pi/3) = 0$, so the absolute maximum of f is 1 at $x = \pi/6$ and the absolute minimum is -1 at $x = -\pi/6$.

4.1.52 $f'(x) = 3 \cdot \dfrac{2}{3}x^{-1/3} - 1 = \dfrac{2}{\sqrt[3]{x}} - 1 = \dfrac{2 - \sqrt[3]{x}}{\sqrt[3]{x}}$, which is zero for $x = 8$. Also, f' doesn't exist for $x = 0$ (even though 0 is in the domain of f). We have $f(8) = 4$, and $f(0) = 0$ and $f(27) = 0$. So the absolute maximum of f on this interval is 4 at $x = 8$ and the absolute minimum is 0 at $x = 0$ and $x = 27$.

4.1.53 Let $y = (2x)^x$, so that $\ln y = x\ln(2x)$. Then $\dfrac{1}{y}y' = \ln(2x) + \dfrac{x}{2x} \cdot 2 = 1 + \ln(2x)$. Thus $y' = (2x)^x(1 + \ln(2x))$. This quantity is zero when $1 + \ln(2x) = 0$, which occurs when $\ln(2x) = -1$, or $x = \dfrac{1}{2e} \approx 0.184$. We have $f(0.1) \approx 0.851$, $f\left(\dfrac{1}{2e}\right) = e^{-(1/2e)} \approx 0.832$, and $f(1) = 2$. So the absolute minimum is $e^{-(1/2e)}$ and the absolute maximum is 2.

4.1.54 $f'(x) = e^{1-x/2} + xe^{1-x/2} \cdot \left(-\dfrac{1}{2}\right) = e^{1-x/2}\left(\dfrac{2-x}{2}\right)$. Because the exponential function is never zero, this expression is zero only when $x = 2$. So $x = 2$ is the only critical point. We have $f(0) = 0$ and $f(2) = 2$, and $f(5) \approx 1.12$. So the absolute maximum of f on this interval is 2 and the absolute minimum is 0.

4.1.55 $f'(x) = 2x - \dfrac{1}{\sqrt{1-x^2}} = \dfrac{2x\sqrt{1-x^2} - 1}{\sqrt{1-x^2}}$. This is zero on $(-1, 1)$ when the numerator is zero, which is when $2x\sqrt{1-x^2} = 1$, so when $(4x^2)(1-x^2) = 1$, or $4x^4 - 4x^2 + 1 = 0$. This factors as $(2x^2-1)(2x^2-1) = 0$, so we have solutions for $x = \pm\sqrt{1/2}$. We have $f(-1) = 1 + \pi$, $f(-1/\sqrt{2}) = \dfrac{1}{2} + \dfrac{3\pi}{4}$, $f(1/\sqrt{2}) = \dfrac{1}{2} + \dfrac{\pi}{4}$, and $f(1) = 1$. So the maximum for f is $1 + \pi$ and the minimum is 1.

4.1.56 $f'(x) = \sqrt{2-x^2} + x\left(-\dfrac{2x}{2\sqrt{2-x^2}}\right) = \dfrac{2-2x^2}{\sqrt{2-x^2}}$. This is zero on $(-\sqrt{2}, \sqrt{2})$ when $2 - 2x^2 = 0$, which occurs for $x = \pm 1$. $f(-\sqrt{2}) = 0 = f(\sqrt{2})$, $f(-1) = -1$, and $f(1) = 1$, so the absolute maximum of f is 1 and the absolute minimum is -1.

4.1.57 $f'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = 6(x-4)(x-1)$. This is zero at $x = 4$ and $x = 1$. $f(1) = 11$ and $f(4) = -16$. At the endpoints we have $f(0) = 0$ and $f(5) = -5$. The absolute maximum is 11 and the absolute minimum is -16.

4.1.58 $f'(x) = e^x - 2$, which is 0 for $e^x = 2$, or $x = \ln 2$. Checking endpoints and critical points, we have $f(0) = 1$, $f(\ln 2) = 2 - 2\ln 2 \approx 0.61$, and $f(2) = e^2 - 4 \approx 3.39$. So the absolute maximum is $e^2 - 4$ at $x = 2$ and the absolute minimum is $2 - 2\ln 2$ at $x = \ln 2$.

4.1.59 $f'(x) = 4x^2 + 10x - 6 = 2(2x^2 + 5x - 3) = 2(x+3)(2x-1)$. This is zero when $x = -3$ and when $x = 1/2$. $f(-3) = 27$ and $f(1/2) = -19/12$. At the endpoints we have $f(-4) = 56/3 \approx 18.7$, and $f(1) = 1/3$. The absolute maximum is 27 and the absolute minimum is $-19/12$.

4.1.60 $f'(x) = 12x^5 - 60x^3 + 48x = 12x(x^4 - 5x^2 + 4) = 12x(x^2-4)(x^2-1) = 12x(x+2)(x-2)(x+1)(x-1)$. This is zero for $x = 0$, $x = \pm 2$, and $x = \pm 1$. The critical points occur at 0 and ± 1, since the endpoints are ± 2. $f(\pm 2) = -16$, $f(\pm 1) = 11$, and $f(0) = 0$. The absolute maximum is 11 and the absolute minimum is -16.

4.1.61 $f'(x) = \dfrac{(x^2+9)^5 - x\cdot 5(x^2+9)^4(2x)}{(x^2+9)^{10}} = \dfrac{(x^2+9)^4(x^2+9-10x^2)}{(x^2+9)^{10}} = \dfrac{9-9x^2}{(x^2+9)^6} = \dfrac{9(1-x)(1+x)}{(x^2+9)^2}$. This expression is zero for $x = \pm 1$. Checking the endpoints and critical points gives $f(-2) = -\dfrac{2}{13^5}$, $f(-1) = -\dfrac{1}{10^5}$, $f(1) = \dfrac{1}{10^5}$ and $f(2) = \dfrac{2}{13^5}$. The absolute maximum is $\dfrac{1}{100000}$ at $x = 1$ and the absolute minimum is $-\dfrac{1}{100000}$ at $x = -1$.

4.1.62 $f'(x) = \dfrac{1}{2\sqrt{x}}\cdot(x^2/5-4) + \sqrt{x}\left(\dfrac{2x}{5}\right) = \dfrac{x^2-20}{10\sqrt{x}} + \dfrac{4x^2}{10\sqrt{x}} = \dfrac{x^2-4}{2\sqrt{x}}$. On the given domain, this expression is zero only for $x = 2$. $f(0) = 0$, $f(2) = -\dfrac{16\sqrt{2}}{5} \approx -4.5255$, and $f(4) = -\dfrac{8}{5}$. So the absolute maximum is 0 and the absolute minimum is $-\dfrac{16\sqrt{2}}{5} \approx -4.5255$.

4.1.63 $f'(x) = \sec x \tan x$ which is zero when $\tan x = 0$ (since $\sec x$ is never zero.) So we are looking for where $\dfrac{\sin x}{\cos x} = 0$, which is when $\sin x = 0$, which is at $x = 0$. $f(-\pi/4) = \sqrt{2} = f(\pi/4)$ and $f(0) = 1$. So the absolute maximum for f is $\sqrt{2}$ and the absolute minimum is 1.

4.1.64 $f'(x) = \dfrac{1}{3}\cdot x^{-2/3}\cdot(x+4) + x^{1/3} = \dfrac{x+4}{3x^{2/3}} + \dfrac{3x}{3x^{2/3}} = \dfrac{4x+4}{3x^{2/3}}$. This expression is zero when $x = -1$, and is undefined when $x = 0$ (although 0 is in the domain of f.) So $(0,0)$ and $(-1,-3)$ are the critical points. $f(-27) = 69$, $f(-1) = -3$, $f(0) = 0$, and $f(27) = 93$. So the absolute maximum is 93 and the absolute minimum is -3.

4.1.65 $f'(x) = 3x^2e^{-x} + x^3\cdot(-e^{-x}) = e^{-x}\cdot\left(3x^2 - x^3\right) = e^{-x}\cdot x^2\cdot(3-x)$. This expression is zero when $x = 0$ and when $x = 3$, so $(0,0)$ and $(3,(27/e^3))$ are the critical points. $f(-1) = -e$, $f(0) = 0$, $f(3) = \dfrac{27}{e^3} \approx 1.344$, and $f(5) \approx 0.8422$. So the absolute maximum of f on the given interval is about 1.344, and the absolute minimum is $-e \approx -2.718$.

4.1.66 $f'(x) = \ln(x/5) + x\cdot\dfrac{5}{x}\cdot\dfrac{1}{5} = \ln(x/5) + 1$. This expression is zero when $\ln(x/5) = -1$, or $x = 5e^{-1} = \dfrac{5}{e} \approx 1.8394$. $f(0.1) \approx -0.391$, $f\left(\dfrac{5}{e}\right) = -\dfrac{5}{e} \approx -1.8394$, and $f(5) = 0$. So the absolute maximum of f is 0, and the absolute minimum is $-\dfrac{5}{e} \approx -1.8394$.

4.1.67 $f'(x) = x^{2/3}(-2x) + (4-x^2)\cdot\dfrac{2}{3\sqrt[3]{x}} = \dfrac{-6x^2+8-2x^2}{3\sqrt[3]{x}} = \dfrac{8-8x^2}{3\sqrt[3]{x}}$. This quantity is zero when $x = \pm 1$ and it doesn't exist at $x = 0$. So there are critical points at $(0,0)$, $(-1,3)$, and $(1,3)$. Checking the endpoints, we have $f(\pm 2) = 0$. So the absolute minimum is 0 at $x = \pm 2$ and $x = 0$, and the absolute maximum is 3 at $x = \pm 1$.

4.1.68 $f'(t) = \dfrac{(t^2+1)\cdot 3 - 3t(2t)}{(t^2+1)^2} = \dfrac{3-3t^2}{(t^2+1)^2}$. This quantity is zero when $t = \pm 1$. So there are critical points at $t = \pm 1$. Checking endpoints and critical points we see that $f(-2) = -\dfrac{6}{5}$ and $f(2) = \dfrac{6}{5}$, $f(1) = \dfrac{3}{2}$ and $f(-1) = -\dfrac{3}{2}$. There is an absolute maximum of $\dfrac{3}{2}$ at $x = 1$ and an absolute minimum of $-\dfrac{3}{2}$ at $x = -1$.

4.1.69

a. The wind either passes through the blade without slowing down or the turbine blades slow down the wind so that $0 \le v_2 \le v_1$, and because $v_1 > 0$, $0 \le \dfrac{v_2}{v_1} \le 1$.

b. $R(1) = \frac{1}{2}(1+1)(1-1) = 0$. Given that $r = 1$, it is the case that $v_1 = v_2$ which means that the wind does not slow down after passing through the blades and therefore no power is captured by the turbine blades. It stands to reason that the blades should slow down the wind, at least a little bit, and therefore it seems unlikely or impossible for $r = 1$.

c. $R(0) = \frac{1}{2}$. A value of $r = 0$ means that $v_2 = 0$, so that the wind turbine stops the wind stream, which is unlikely or impossible to occur.

d. Finding the derivative of R and setting it equal to zero, we have

$$R'(r) = \frac{1}{2}((1-r^2) + (1+r)(-2r)) = -\frac{1}{2}(3r^2 + 2r - 1) = -\frac{1}{2}(3r-1)(r+1) = 0.$$

Given that $R(0) = \frac{1}{2}$, $R\left(\frac{1}{3}\right) = \frac{1}{2}\left(\frac{4}{3}\right)\left(\frac{8}{9}\right) = \frac{16}{27} \approx 0.593$, R is maximized on $[0,1]$ when $r = \frac{1}{3}$. At best, a wind turbine can extract about 59.3% of the available power from the wind stream.

4.1.70

a. Equating the two power equations, we have $v^2\rho A(v_1 - v_2) = \frac{1}{2}\rho v A(v_1^2 - v_2^2)$, or

$$v(v_1 - v_2) = \frac{1}{2}(v_1^2 - v_2^2) = \frac{1}{2}(v_1 - v_2)(v_1 + v_2).$$

Dividing through by $v_1 - v_2$ gives

$$v = \frac{1}{2}(v_1 + v_2).$$

b. By replacing v by $\frac{v_1 + v_2}{2}$ in the power equation $P = \frac{1}{2}\rho v A(v_1^2 - v_2^2)$, we have

$$P = \frac{1}{2}\rho\left(\frac{v_1 + v_2}{2}\right)A(v_1^2 - v_2^2) = \frac{\rho A}{4}(v_1 + v_2)(v_1^2 - v_2^2).$$

c.

$$\begin{aligned}\frac{P}{P_0} &= \frac{\frac{\rho A}{4}(v_1 + v_2)(v_1^2 - v_2^2)}{\frac{\rho A v_1^3}{2}} \\ &= \frac{(v_1 + v_2)(v_1^2 - v_2^2)}{2v_1^3} = \frac{1}{2}\cdot\frac{v_1 + v_2}{v_1}\cdot\frac{v_1^2 - v_2^2}{v_1^2} \\ &= \frac{1}{2}\left(1 + \frac{v_2}{v_1}\right)\left(1 - \left(\frac{v_2}{v_1}\right)^2\right) = \frac{1}{2}(1+r)(1-r^2) = R(r).\end{aligned}$$

4.1.71 $s'(t) = 32 - 4t^3$ which is 0 when $t^3 = 8$, so $t = 2$. Because $s(0) = 0$, $s(2) = 48$, and $s(3) = 15$, the object is furthest to the right at time $t = 2$.

4.1.72 $S'(x) = 4x - \frac{200}{x^2} = \frac{4x^3 - 200}{x^2}$, which is zero when $x^3 = 50$, so for $x = \sqrt[3]{50} \approx 3.684$. This critical point does indeed yield a minimum (which can be determined via a graphing calculator, or by techniques in an upcoming section). So the minimum surface area is given by $S(\sqrt[3]{50}) \approx 81.433$, when the box has dimensions $\sqrt[3]{50} \times \sqrt[3]{50} \times \sqrt[3]{50}$.

4.1.73 The stone will reach its maximum height when its velocity is zero, which occurs at the only critical point for this inverted parabola. We have that $v(t) = s'(t) = -32t + 64$, which is zero when $t = 2$. The height at this time is $s(2) = 256$, the maximum height.

4.1.74

a. $R'(x) = -120x + 300$, which is zero when $x = 2.5$. This is the only critical number.

b. The maximum must occur at either an endpoint or a critical point. Note that $R(0) = 0$, $R(2.5) = 375$, and $R(5) = 0$, so the maximum revenue is \$375, which occurs when the price is \$2.50.

4.1.75

a. Note that $P(n) = 50n - .5n^2 - 100$, so $P'(n) = 50 - n$, which is zero when $n = 50$. It is clear that this is a maximum, since the graph of P is an inverted parabola.

b. Given a domain of $[0, 45]$, since the only critical point is not in the domain, the maximum must occur at an endpoint. Because $P(0) = -100$, and $P(45) = \$1137.50$, he should take 45 people on the tour.

4.1.76 $P(x) = 2x + \dfrac{128}{x}$, $x > 0$, so $P'(x) = 2 - \frac{128}{x^2}$, which is zero when $x^2 = 64$, or when $x = 8$. So $(8, 32)$ is the only critical point. This does turn out to be a minimum, so the dimensions of the rectangle with minimal perimeter are 8×8.

4.1.77

a. False. The derivative $f'(x) = \dfrac{1}{2\sqrt{x}}$ is never zero, and the function has no critical points.

b. False. For example, the function $f(x) = \begin{cases} \sin x & \text{if } -5 \le x \le 0, \\ -8 & \text{if } 0 < x \le 5 \end{cases}$ is not continuous on $[-5, 5]$, but has an absolute maximum of 1.

c. False. For example, the function $f(x) = (x-2)^3$ satisfies $f'(2) = 0$, but it has neither a maximum nor a minimum at $x = 2$.

d. True. This follows from the theorems in this section.

4.1.78

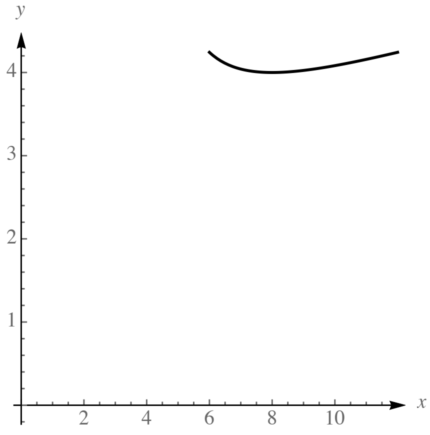
a.

$$\begin{aligned} f'(x) &= \frac{\sqrt{x-4} \cdot 1 - x \cdot \frac{1}{2\sqrt{x-4}} \cdot 1}{x-4} \\ &= \frac{\sqrt{x-4} \cdot 1 - x \cdot \frac{1}{2\sqrt{x-4}} \cdot 1}{x-4} \cdot \frac{2\sqrt{x-4}}{2\sqrt{x-4}} \\ &= \frac{2x - 8 - x}{2(x-4)^{3/2}} = \frac{x-8}{2(x-4)^{3/2}}. \end{aligned}$$

This expression is 0 when $x = 8$. So $(8, 4)$ is the only critical point.

b. $f(6) = 3\sqrt{2} = f(12)$, and $f(8) = 4$. Note that $3\sqrt{2} \approx 4.24 > 4$. So the absolute maximum of f on this interval is $3\sqrt{2}$, and the absolute minimum is 4.

c.

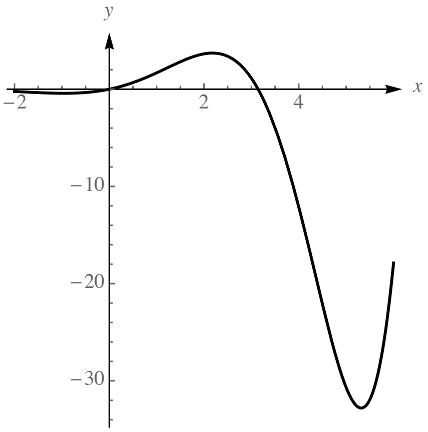


4.1.79

a. $f'(x) = 2^x \cdot \ln 2 \cdot \sin x + 2^x \cos x = 2^x((\ln 2) \cdot \sin x + \cos x)$. Because 2^x is never zero, this expression is zero only when $(\ln 2) \cdot \sin x + \cos x = 0$, or $(\ln 2) \cdot \tan x = -1$, or $\tan x = \left(-\dfrac{1}{\ln 2}\right)$. So one solution is $x = \tan^{-1}\left(-\dfrac{1}{\ln 2}\right) \approx -0.9647$. And since the tangent function is periodic with period π, we also have solutions at approximately $-0.9647 + \pi \approx 2.1769$, and $-0.9647 + 2\pi \approx 5.3185$. These are the only solutions on the given interval.

b. $f(-2) \approx -0.2273$, $f(-0.9647) \approx -0.4211$, $f(2.1769) \approx 3.7164$, $f(5.3185) \approx -32.7968$, and $f(6) \approx -17.8826$. Thus the absolute maximum is about 3.7164 and the absolute minimum is about -32.7968.

c.

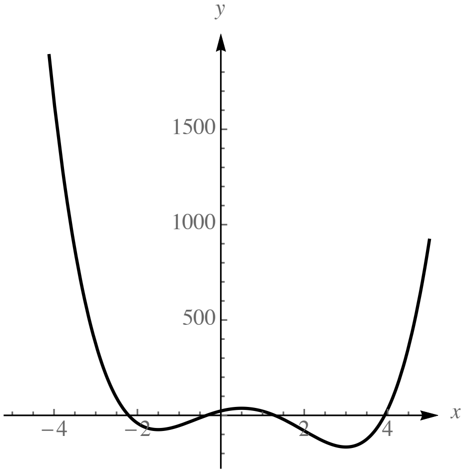


4.1.80

a. $f'(x) = 24x^3 - 48x^2 - 90x + 54 = 6(4x^3 - 8x^2 - 15x + 9)$, which can be written as $6(2x+3)(2x-1)(x-3)$. Note: to get this factorization, we used the rational root theorem to establish candidates for roots, then used trial-and-error to find that $x = 3$ was one of the roots, which means that $(x-3)$ is one of the factors of f'. Then we used long division to determine that $f'(x) = 6(x-3)(4x^2+4x-3)$, then factored the quadratic. After factoring, it is clear that the roots of f' are 3, $-\dfrac{3}{2}$, and $\dfrac{1}{2}$, so these are the locations of the critical points.

b. From the graph, it appears that there is a local minimum at $x = -\dfrac{3}{2}$, a local maximum at $x = \dfrac{1}{2}$, and a local minimum at $x = 3$.

c. The local minimum at $x = 3$ is also an absolute minimum. The value of the absolute minimum is -166 and the value of the absolute maximum (which occurs at the left endpoint $x = -5$) is 4378.

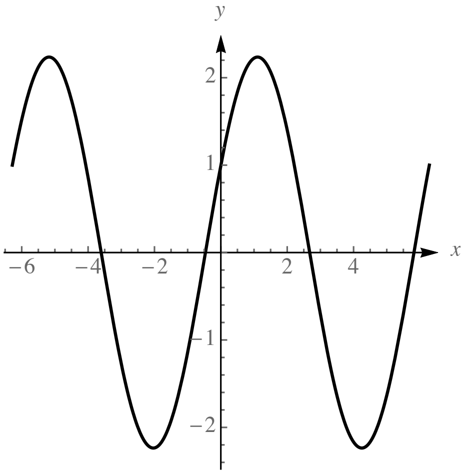


4.1.81

a. $f'(\theta) = 2\cos\theta - \sin\theta$, which is zero when $\tan\theta = 2$. So one critical point occurs at $\theta = \tan^{-1}(2) \approx 1.107$. And since the tangent function is periodic with period π, there are will also be solutions at this number plus or minus integer multiples of π. On the given interval, these are located at approximately $1.107 - 2\pi \approx -5.176$, at $1.107 - \pi \approx -2.034$, and at $1.107 + \pi \approx 4.249$.

b. From the graph, it appears that there is a local minimum at about $\theta = -2.034$ and at $\theta = 4.249$, and there is a local maximum at about $\theta = -5.176$, and at about $\theta = 1.107$.

c. From the graph, it appears that the local minimum at about $\theta = -2.034$ is also an absolute minimum, as is the one at $\theta = 4.249$. The local maximum at about $\theta = -5.176$, and at about $\theta = 1.107$ are also absolute maxima. The value of the absolute maximum appears to be about 2.24 and the value of the absolute minimum appears to be about -2.24.

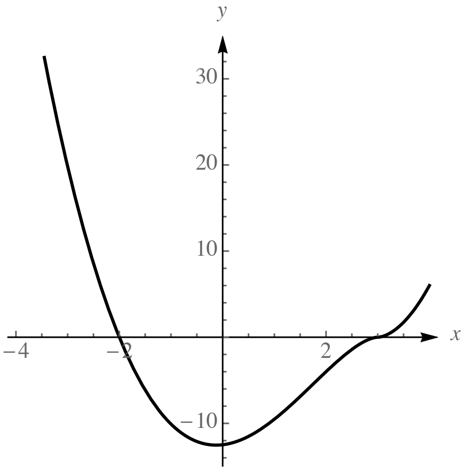


4.1.82

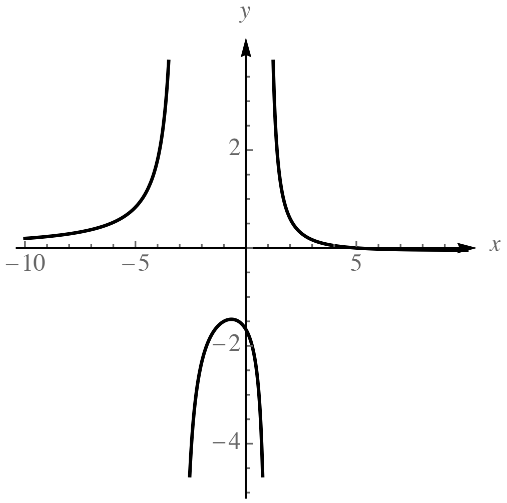
a. $f'(x) = (x-3)^{5/3} + (x+2)\cdot\frac{5}{3}(x-3)^{2/3} = \frac{(x-3)^{2/3}}{3}(3x - 9 + 5x + 10) = \frac{(x-3)^{2/3}}{3}(8x + 1)$.This is zero when $x = 3$ and when $x = -\frac{1}{8}$. There are critical points at $x = -\frac{1}{8}$ and at $x = 3$.

b. From the graph, it appears that there is a local minimum of about -12.52 at $x = -\frac{1}{8}$.

c. The local minimum mentioned above is also an absolute minimum. The absolute maximum occurs at the left endpoint $x = -4$, where the value of f is about 51.23.



4.1.83



a. $h'(x) = \dfrac{(x^2+2x-3)(-1)-(5-x)(2x+2)}{(x^2+2x-3)^2} = \dfrac{x^2-10x-7}{(x^2+2x-3)^2}$. By the quadratic formula, the numerator is zero (making the quotient zero) when $x = \dfrac{10 \pm \sqrt{100-4(-7)}}{2} = 5 \pm \dfrac{1}{2}\sqrt{128} = 5 \pm 4\sqrt{2}$. Note that $5+4\sqrt{2}$ isn't in the domain, so the only critical point is at $x = 5-4\sqrt{2}$.

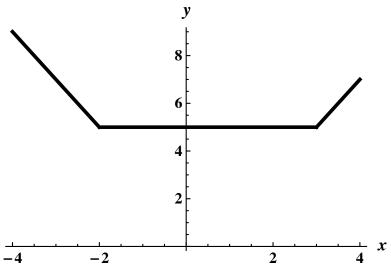
b. From the graph, it appears that the one critical point mentioned above yields a local maximum.

c. The function has no absolute maximum and no absolute minimum on the given interval.

4.1.84

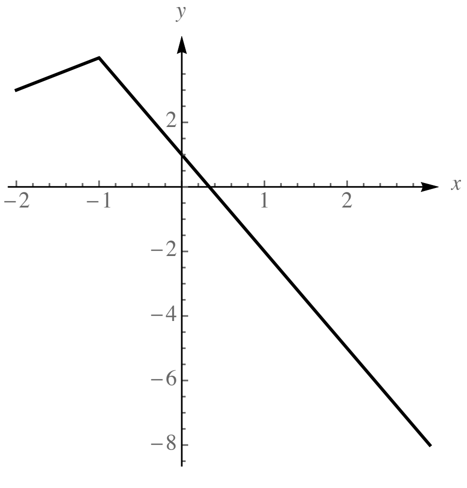
Note that

$$f(x) = \begin{cases} 3-x-x-2 = 1-2x & \text{if } -4 \le x \le -2, \\ 3-x+x+2 = 5 & \text{if } -2 \le x \le 3, \\ x-3+x+2 = 2x-1 & \text{if } 3 \le x \le 4. \end{cases}$$



There is an absolute maximum of 9 and an absolute minimum of 5. The absolute maximum occurs at $x = -4$, and the absolute minimum occurs at all of the values of x between -2 and 3.

4.1.85



Note that

$$g(x) = \begin{cases} 3-x+2x+2 = x+5 & \text{if } -2 \le x \le -1, \\ 3-x-2x-2 = 1-3x & \text{if } -1 \le x \le 3. \end{cases}$$

There is an absolute maximum of 4 and an absolute minimum of -8. The absolute maximum occurs at $x = -1$, and the absolute minimum occurs at a $x = 3$.

4.1.86

a. $f'(x) = 2x$, so $f'(a) = 2a$ is the slope at $x = a$. The slope of the line perpendicular to the tangent line is $-\frac{1}{2a}$.

b. We are looking for a line through (a, a^2) with slope $-\frac{1}{2a}$, so the equation is given by $y = -\frac{1}{2a}(x - a) + a^2$.

c. To find B's position, we find where the parabola and the line from the last part of this problem intersect. So we seek the solution to $x^2 = -\frac{1}{2a}x + \frac{1}{2} + a^2$, or $x^2 + \frac{1}{2a}x + (-a^2 - \frac{1}{2}) = 0$. By the quadratic formula, we find that $x = \frac{-2a^2 - 1}{2a}$ is the desired point. The other root is $x = a$ which corresponds to point A.

d. $F(a) = \left(a - \left(\frac{-2a^2-1}{2a}\right)\right)^2 + \left(a^2 - \frac{(2a^2+1)^2}{4a^2}\right)^2$
$= \frac{(4a^2+1)^2}{4a^2} + \left(\frac{4a^4 - 4a^4 - 4a^2 - 1}{4a^2}\right)^2 = \frac{64a^6 + 48a^4 + 12a^2 + 1}{16a^4} = \frac{(4a^2+1)^3}{16a^4}$.

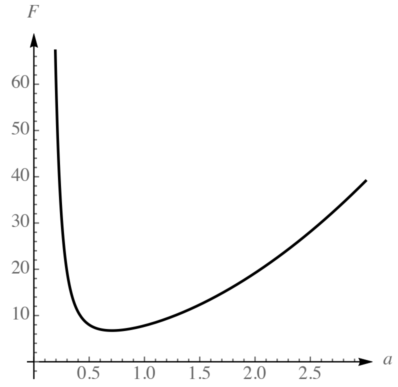
e.

$$\begin{aligned} F'(a) &= \frac{16a^4 \cdot 3(4a^2+1)^2 \cdot 8a - (4a^2+1)^3 \cdot 64a^3}{256a^8} \\ &= \frac{64a^3 \cdot (4a^2+1)^2 \left(6a^2 - (4a^2+1)\right)}{256a^8} \\ &= \frac{(4a^2+1)^2 \cdot (2a^2-1)}{4a^5}. \end{aligned}$$

The critical point of F for $a > 0$ occurs at $a = \sqrt{0.5}$.

f. The value of F at the critical point is $F(\sqrt{0.5}) = \frac{27}{4}$. The points are at $A = (\sqrt{0.5}, 0.5)$ and $B = (-\sqrt{2}, 2)$.

g.



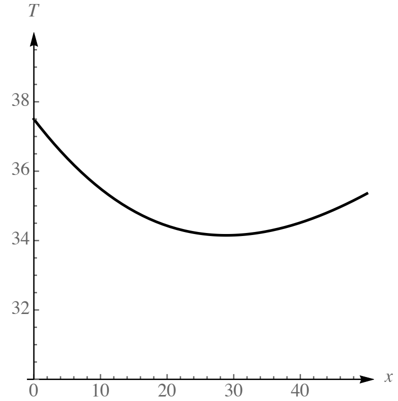
4.1.87

a. Because distance is rate times time, the time will be distance over rate. The swim distance is given by $\sqrt{2500 + x^2}$ meters, so the time for swimming is $\frac{\sqrt{2500+x^2}}{2}$. For running, the distance is $50 - x$, so the time is $\frac{50-x}{4}$. Thus we have $T(x) = \frac{\sqrt{2500+x^2}}{2} + \frac{50-x}{4}$.

b. $T'(x) = \frac{1}{2} \cdot \frac{1}{2}\left(x^2+2500\right)^{-1/2} \cdot 2x - \frac{1}{4} = \frac{x}{2\sqrt{x^2+2500}} - \frac{1}{4}$. This expression is zero when $\frac{x^2}{x^2+2500} = \frac{1}{4}$, so when $4x^2 = x^2 + 2500$, which occurs when $x^2 = \frac{2500}{3}$. So $x = \sqrt{\frac{2500}{3}} \approx 28.868$.

c. $T(0) = 37.5$, $T(28.868) \approx 34.151$, and $T(50) = 25\sqrt{2} \approx 35.355$. The absolute minimum occurs at the only critical point. The minimal crossing time is approximately 34.151 seconds.

d.



4.1.88 Because a parabola either opens up and has a minimum at its vertex, or open down and has a maximum at its vertex, it will always have exactly one extreme value. We have $f'(x) = 2ax + b$, which is zero when $x = -\frac{b}{2a}$, so that one critical point is the location of the vertex which gives the extreme point.

4.1.89

a. Note that since there is a local extreme value at 2 for f and since f is differentiable everywhere, we must have $f'(2) = 0$.

$g(2) = 2f(2) + 1 = 1$.

$h(2) = 2f(2) + 2 + 1 = 3$.

$g'(2) = 2 \cdot f'(2) + f(2) = 0$.

$h'(2) = 2f'(2) + f(2) + 1 = 1$.

b. h doesn't, since its derivative isn't zero at $x = 2$. However g might: for example, if $f(x) = (x-2)^2$ then $g(x) = x(x-2)^2 + 1$ has a local minimum at $x = 2$.

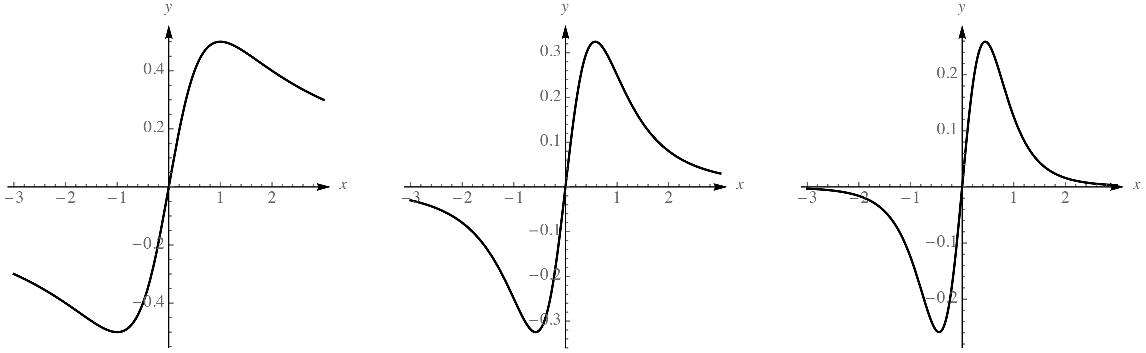
4.1.90

a. $f(-x) = -\frac{x}{((-x)^2+1)^n} = -\frac{x}{(x^2+1)^n} = -\frac{x}{(x^2+1)^n} = -f(x)$.

b. $f'(x) = \frac{(x^2+1)^n - x \cdot n(x^2+1)^{n-1} \cdot 2x}{(x^2+1)^{2n}} = \frac{(x^2+1)^{n-1}\left[x^2+1-2x^2n\right]}{(x^2+1)^{2n}} = \frac{1-(2n-1)x^2}{(x^2+1)^{n+1}}$. This quantity is zero when $x^2 = \frac{1}{2n-1}$, so $x = \pm\sqrt{\frac{1}{2n-1}}$ is a critical point.

c. The maximum value occurs at the positive critical number. The value of the maximum is given by $\frac{\sqrt{\frac{1}{2n-1}}}{\left(\left(\sqrt{\frac{1}{2n-1}}\right)^2+1\right)^n} = \frac{1}{\sqrt{2n-1}} \cdot \left(1 - \frac{1}{2n}\right)^n$. As $n \to \infty$, this quantity has limit 0.

d. Here are the graphs for $n = 1$, $n = 2$, and $n = 3$.



4.1.91

a. If $f(c)$ is a local maximum, then when x is near c but not equal to c, $f(c) \geq f(x)$, so $f(x) - f(c) \leq 0$.

b. When x is near to c but a little bigger than c, $x - c > 0$. So in this case, $\dfrac{f(x) - f(c)}{x - c} \leq 0$, since the numerator is negative (or 0) and the denominator is positive.

Thus, $\displaystyle\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = f'(c) \leq 0$

c. When x is near to c but a little smaller than c, $x - c < 0$. So in this case, $\dfrac{f(x) - f(c)}{x - c} \geq 0$, since the numerator is negative (or 0) and the denominator is negative, making the quotient positive (or 0).

Thus, $\displaystyle\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = f'(c) \geq 0$.

d. From the above, we have that $f'(c) \leq 0$ and $f'(c) \geq 0$, so $f'(c) = 0$.

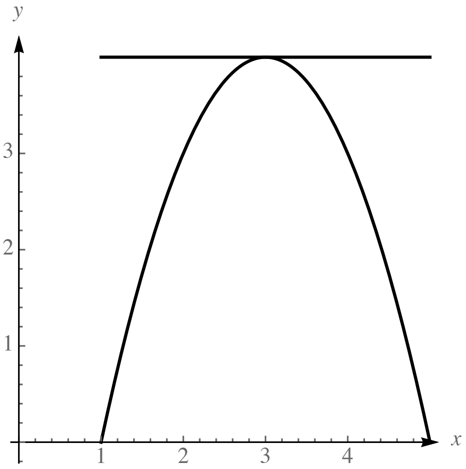
4.1.92

a. Because of the symmetry about the y-axis for an even function, a minimum at $x = c$ will correspond to a minimum at $x = -c$ as well.

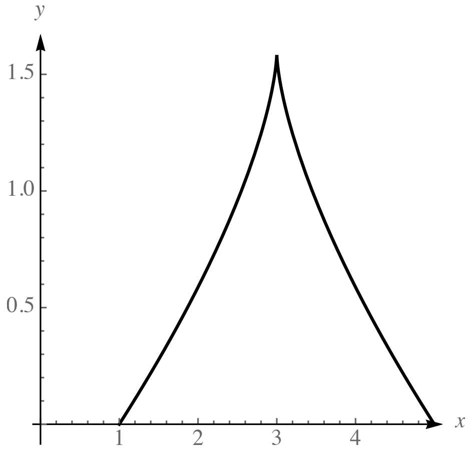
b. Because of the symmetry about the origin, a minimum at $x = c$ will correspond to a maximum at $x = -c$. It is helpful to think about the symmetry about the origin as being the result of flipping about the y-axis and then flipping about the x-axis.

4.2 Mean Value Theorem

4.2.1 If f is a continuous function on the closed interval $[a, b]$ and is differentiable on (a, b) and the slope of the secant line that joins $(a, f(a))$ and $(b, f(b))$ is zero, then there is at least one value c in (a, b) at which the slope of the line tangent to f at $(c, f(c))$ is also zero.

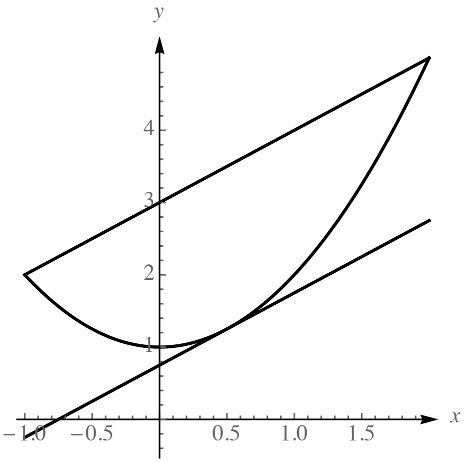


4.2.2 We seek a continuous function over an interval for which it isn't true that there is a horizontal tangent line to the function.



4.2.3 The function $f(x) = |x|$ is not differentiable at 0.

4.2.4 If f is a continuous function on the closed interval $[a, b]$ and is differentiable on (a, b), then there is at least one value c in (a, b) at which the slope of the line tangent to f at $(c, f(c))$ is equal to the slope of the secant line that joins $(a, f(a))$ and $(b, f(b))$.



4.2.5

a. It appears that at $c = 1$ the tangent line is parallel to the secant line shown.

b. $f'(x) = \frac{x}{2}$. The slope of the secant line is $\frac{f(4)-f(-2)}{4-(-2)} = \frac{5-2}{6} = \frac{1}{2}$. We have $\frac{x}{2} = \frac{1}{2}$ at $x = 1$, as conjectured.

4.2.6

a. It appears that at $c = 1$ the tangent line is parallel to the secant line shown.

b. $f'(x) = \frac{1}{\sqrt{x}}$. The slope of the secant line is $\frac{f(4)-f(0)}{4-0} = \frac{4-0}{4-0} = 1$. We have $\frac{1}{\sqrt{x}} = 1$ at $x = 1$, as conjectured.

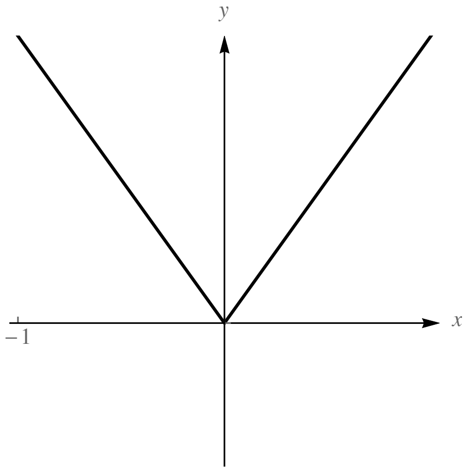
4.2.7

a. It appears that at a point a little bigger than 1 (but less than 1.5) the tangent line is parallel to the secant line shown, and then again at a point between -1 and -1.5.

b. $f'(x) = \frac{5x^4}{16}$. The slope of the secant line is $\frac{f(2)-f(-2)}{2-(-2)} = \frac{2-(-2)}{2-(-2)} = 1$. We have $\frac{5x^4}{16} = 1$ when $x^4 = \frac{16}{5}$, or $c = \pm\frac{2}{\sqrt[4]{5}} \approx \pm 1.34$.

4.2.8 The average rate of change of f on the interval $[-10, 10]$ is $\frac{f(10)-f(-10)}{10-(-10)} = \frac{10^3-(-10)^3}{20} = 100$. We wish to find a point x in $(-10, 10)$ such that $f'(x) = 100$, or equivalently $3x^2 = 100$, which gives $x = \pm 10/\sqrt{3}$.

4.2.9 We seek a function over an interval for which it isn't true that there is a tangent line parallel to the secant line between the endpoints.



4.2.10 $f'(x) = \frac{2}{3\sqrt[3]{x}}$, while $\frac{f(8)-f(-1)}{8-(-1)} = \frac{4-1}{9} = \frac{1}{3}$. In order for $f'(x) = \frac{1}{3}$, we would need $\frac{2}{3\sqrt[3]{x}} = \frac{1}{3}$. This occurs only when $\sqrt[3]{x} = 2$, or $x = 8$. However, $x = 8$ is not in the interval $(-1, 8)$. This does violate the MVT because f is not differentiable at $x = 0$.

4.2.11 The function f is differentiable on $[0, 1]$ and $f(0) = f(1) = 0$, so Rolle's theorem applies. We wish to find a point x in $(0, 1)$ such that $f'(x) = 0$; we have $f'(x) = (x-1)^2 + 2x(x-1) = (x-1)(3x-1)$, so $x = 1/3$ satisfies the conclusion of Rolle's theorem.

4.2.12 The function f is differentiable on $[0, \pi/2]$ and $f(0) = f(\pi/2) = 0$, so Rolle's theorem applies. We wish to find a point x in $(0, \pi/2)$ such that $f'(x) = 0$; we have $f'(x) = 2\cos 2x$, so $x = \pi/4$ satisfies the conclusion of Rolle's theorem.

4.2.13 The function f is differentiable on $[\pi/8, 3\pi/8]$ and $f(\pi/8) = f(3\pi/8) = 0$, so Rolle's theorem applies. We wish to find a point x in $(\pi/8, 3\pi/8)$ such that $f'(x) = 0$; we have $f'(x) = -4\sin 4x$, so $x = \pi/4$ satisfies the conclusion of Rolle's theorem.

4.2.14 The function f is not differentiable at $x = 0$, so Rolle's theorem does not apply.

4.2.15 The function f is not differentiable at $x = 0$, so Rolle's theorem does not apply.

4.2.16 The function f is continuous on $[-2, 4]$ and differentiable on $(-2, 4)$, and $f(-2) = f(4) = 0$, so Rolle's theorem does apply. $f'(x) = 3x^2 - 4x - 8$, which is zero at $x = \frac{4\pm\sqrt{16+96}}{6} = \frac{4\pm\sqrt{112}}{6} = 2\left(\frac{1\pm\sqrt{7}}{3}\right)$. Both of these values lie between -2 and 4, so both satisfy the conclusion of Rolle's theorem.

4.2.17 g is continuous on $[-1, 3]$ and differentiable on $(-1, 3)$, and $g(-1) = 0 = g(3)$, so Rolle's theorem does apply. $g'(x) = 3x^2 - 2x - 5 = (x + 1)(3x - 5)$. This is zero for $x = -1$ (which is not on $(-1, 3)$) and for $x = 5/3$ (which is on $(-1, 3)$.) So $x = 5/3$ satisfies the conclusion of Rolle's theorem.

4.2.18 h is continuous on $[-a, a]$ and differentiable on $(-a, a)$, and $h(-a) = e^{-a^2} = h(a)$, so Rolle's theorem does apply. $h'(x) = -2xe^{-x^2}$, which is zero at $x = 0$. So $x = 0$ satisfies the conclusion of Rolle's theorem.

4.2.19 The average rate of change of the temperature from 3.2 km to 6.1 km is $\dfrac{-10.3 - 8.0}{6.1 - 3.2} \approx -6.3^\circ/\text{km}$. Based on this, we cannot conclude that the lapse rate exceeds the critical value of $7^\circ/$ km.

4.2.20 The average acceleration over the 4.45 seconds is $\dfrac{330 - 0}{4.45 - 0} \approx 74.2$mi/hr/s, so at some point during the race, the maximum acceleration of the drag racer is at least 74 mi/hr/s.

4.2.21

a. The function f is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$. so the Mean Value Theorem applies.

b. The average rate of change of f on $[-1, 2]$ is $\dfrac{f(2) - f(-1)}{2 - (-1)} = \dfrac{3 - 6}{3} = -1$. We wish to find a point c in $(-1, 2)$ such that $f'(c) = -1$, or equivalently $-2c = -1$ which gives $c = 1/2$.

4.2.22

a. The function f is differentiable on $(0, 1)$ and continuous on $[0, 1]$ so the Mean Value Theorem applies.

b. The average rate of change of f on $[0, 1]$ is $\dfrac{f(1) - f(0)}{1 - 0} = \dfrac{-1 - 0}{1} = -1$.

   We wish to find a point c in $(0, 1)$ such that $f'(c) = -1$, or equivalently $3c^2 - 4c = -1$. This can be written as $0 = 3c^2 - 4c + 1 = (3c - 1)(c - 1)$, which has solutions $c = \dfrac{1}{3}$ and $c = 1$. Only $c = \dfrac{1}{3}$ is in the interval $(0, 1)$.

4.2.23

a. The MVT does not apply because f is not differentiable at $x = 0$. To see this note that $\displaystyle\lim_{h\to 0^+} \frac{f(0 + h) - f(0)}{h} = 1$ and $\displaystyle\lim_{h\to 0^-} \frac{f(0 + h) - f(0)}{h} = -2$.

4.2.24

a. The MVT does not apply because f is not continuous at $x = 1$ (because 1 isn't even in the domain of f).

4.2.25

a. The function f is differentiable on $(0, 1)$ and continuous on $[0, 1]$ so the Mean Value Theorem applies.

b. The average rate of change of f on $[0, 1]$ is $\dfrac{f(1) - f(0)}{1 - 0} = \dfrac{e - 1}{1} = e - 1$. We wish to find a point c in $(0, 1)$ such that $f'(c) = e - 1$, or equivalently $e^c = e - 1$ which gives $c = \ln(e - 1) \approx 0.541$.

4.2.26

a. The function f is differentiable on $(1, e)$ and continuous on $(1, e)$ so the Mean Value Theorem applies.

b. The average rate of change of f on $[1, e]$ is $\dfrac{f(e) - f(1)}{e - 1} = \dfrac{\ln e + \ln 2 - \ln 2}{e - 1} = \dfrac{1}{e - 1}$. We wish to find a point c in $(1, e)$ such that $f'(c) = 1/(e - 1)$, or equivalently $1/c = 1/(e - 1)$ which gives $c = e - 1 \approx 1.71828$.

4.2.27

a. f is continuous on $[-\pi/2, \pi/2]$ and differentiable on $(-\pi/2, \pi/2)$, so the Mean Value Theorem applies.

b. The average rate of change of f on $[-\pi/2, \pi/2]$ is $\dfrac{f(\pi/2) - f(-\pi/2)}{\pi/2 - (-\pi/2)} = \dfrac{1 - (-1)}{\pi} = \dfrac{2}{\pi}$. We wish to find a point c in $(-\pi/2, \pi/2)$ such that $f'(c) = \dfrac{2}{\pi}$, or equivalently $\cos c = \dfrac{2}{\pi}$ which gives $c \approx \pm 0.881$ (using a root finder).

4.2.28

a. f is continuous on $[0, \pi/4]$ and differentiable on $(0, \pi/4)$, so the Mean Value Theorem applies.

b. The average rate of change of f on $[0, \pi/4]$ is $\dfrac{f(\pi/4) - f(0)}{\pi/4 - 0} = \dfrac{1 - 0}{\pi/4} = \dfrac{4}{\pi}$. We wish to find a point c in $(0, \pi/4)$ such that $f'(c) = \dfrac{4}{\pi}$, or equivalently $\sec^2 c = \dfrac{4}{\pi}$ which gives $c \approx 0.482$ (using a root finder).

4.2.29

a. The function f is differentiable on $(0, 1/2)$ and continuous on $[0, 1/2]$ so the Mean Value Theorem applies.

b. The average rate of change of f on $[0, 1/2]$ is $\dfrac{f(1/2) - f(0)}{\frac{1}{2} - 0} = \dfrac{\frac{\pi}{6} - 0}{\frac{1}{2}} = \dfrac{\pi}{3}$.

   We wish to find a point c in $(0, 1/2)$ such that $f'(c) = \pi/3$, or equivalently $\dfrac{1}{\sqrt{1 - c^2}} = \dfrac{\pi}{3}$, so $c = \sqrt{1 - \dfrac{9}{\pi^2}}$.

4.2.30

a. The function f is differentiable on $(1, 3)$ and continuous on $[1, 3]$ so the Mean Value Theorem applies.

b. The average rate of change of f on $[1, 3]$ is $\dfrac{f(3) - f(1)}{3 - 1} = \dfrac{\frac{10}{3} - 2}{2} = \dfrac{2}{3}$. We wish to find a point c in $(1, 3)$ such that $f'(c) = 2/3$, or equivalently $1 - \dfrac{1}{c^2} = \dfrac{2}{3}$, so $c = \sqrt{3}$.

4.2.31

a. The Mean Value Theorem does not apply because the function f is not differentiable at $x = 0$.

b. Even though the Mean Value Theorem doesn't apply, it still happens to be the case that there are numbers c between -8 and 8 where the tangent line has slope $\frac{f(8) - f(-8)}{8 - (-8)} = \frac{1}{2}$. This occurs where $\frac{2}{3}c^{-2/3} = 1/2$, which gives $c = \pm\frac{8}{9} \cdot \sqrt{3}$.

4.2.32

a. The function f is differentiable on $(-1, 2)$ and continuous on $[-1, 2]$ so the Mean Value Theorem applies.

b. The average rate of change of f on $[-1, 2]$ is $\frac{f(2) - f(-1)}{2 - (-1)} = \frac{\frac{1}{2} - (-1)}{3} = \frac{1}{2}$. We wish to find a point c in $(-1, 2)$ such that $f'(c) = 1/2$, or equivalently $\frac{2}{(c+2)^2} = \frac{1}{2}$, so $c = 0$.

4.2.33

a. False. The function f is not differentiable at $x = 0$.

b. True. If $f(x) - g(x) = c$ is constant, then $f'(x) - g'(x) = 0$.

c. False. If $f'(x) = 0$ then we can conclude that $f(x) = c$ for some constant.

4.2.34

a. Let $f(x) = \tan^{-1} x + \tan^{-1}(1/x)$. Then $f'(x) = \frac{1}{1+x^2} + \frac{1}{1+(1/x)^2}\left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$ for all $x \neq 0$. By Theorem 4.5, f is constant on $(-\infty, 0)$ and $(0, \infty)$.

b. Because $f(1) = \tan^{-1} 1 + \tan^{-1} 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$, and f is constant, we must have that $f(x) = \frac{\pi}{2}$ on $(0, \infty)$.

c. Similarly, $f(-1) = \tan^{-1}(-1) + \tan^{-1}(-1) = -\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$. Because f is constant on $(-\infty, 0)$, we must have $f(x) = -\frac{\pi}{2}$ there. Combining this result with the result of part b allows us to conclude that

$$f(x) = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0. \end{cases}$$

4.2.35

a.

$$\frac{d}{dx}\tan^{-1}(2/x^2) = \frac{1}{1+4/x^4} \cdot -\frac{4}{x^3} = \frac{1}{1+4/x^4} \cdot -\frac{4}{x^3} \cdot \frac{x}{x} = -\frac{4x}{x^4+4}.$$

$$\begin{aligned}\frac{d}{dx}\left(\tan^{-1}(x+1) - \tan^{-1}(x-1)\right) &= \frac{1}{1+(x+1)^2} - \frac{1}{1+(x-1)^2} \\ &= \frac{x^2 - 2x + 2 - (x^2 + 2x + 2)}{1 + (x+1)^2 + (x-1)^2 + (x^2-1)^2} \\ &= -\frac{4x}{1 + x^2 + 2x + 1 + x^2 - 2x + 1 + x^4 - 2x^2 + 1} \\ &= -\frac{4x}{x^4+4}.\end{aligned}$$

Because these two functions have the same derivative, they differ by a constant. So for any x, $\tan^{-1}(2/x^2) - \left(\tan^{-1}(x+1) - \tan^{-1}(x-1)\right)$ is a constant.

b. Because the two function in part (a) differ by a constant, we can compute the constant by evaluating for a specific number x. Choosing $x = 1$, we have $\tan^{-1} 2 - \left(\tan^{-1} 2 - \tan^{-1} 0\right) = 0$, so the constant is 0, and we have
$$\tan^{-1}(2/x^2) = \left(\tan^{-1}(x+1) - \tan^{-1}(x-1)\right).$$

4.2.36 Observe that $\ln 2x = \ln 2 + \ln x$ and $\ln 10x^2 = \ln 10 + \ln x^2$, so the pairs $f(x), g(x)$ and $h(x), p(x)$ have the same derivative.

4.2.37 The functions $h(x)$ and $p(x)$ have the same derivative as $f(x)$ because they differ from $f(x)$ by a constant.

4.2.38 One example of a function f with $f'(x) = x+1$ is $f(x) = \dfrac{x^2}{2} + x$; therefore the most general function with derivative $x+1$ is $f(x) = \dfrac{x^2}{2} + x + C$ where C is a constant.

4.2.39 The secant line between the endpoints has slope $\dfrac{f(4) - f(-4)}{4 - (-4)} = \dfrac{4-1}{8} = \dfrac{3}{8}$. The slope of the tangent line to the graph appears to have this value at approximately -2.5 and at about 2.6. These are eyeballed estimates, so your personal estimate may differ.

4.2.40 Because $f(3) = f(1) \approx 2$, the average rate of change of f on $[1,3]$ is $\dfrac{f(3) - f(1)}{3-1} = 0$. However, there is no point in $(1,3)$ such that the slope of the tangent to f at that point is zero. This does not contradict the Mean Value Theorem because f is not differentiable everywhere on $(1,3)$; in particular, it is not differentiable at $x = 2$.

4.2.41 Because $f(1) \approx 2$ and $f(3) \approx 2$, the average rate of change of f on $[1,3]$ is $\dfrac{f(3) - f(1)}{3-1} = 0$. However, the tangent to f between $x = 1$ and $x = 2$ is the graph of f itself, which has slope 2, and the tangent to f between $x = 2$ and $x = 3$ is also the graph of f, which has slope 1. So there is no point in $(1,3)$ where the tangent line has slope 0. This does not contradict the Mean Value Theorem because f is not continuous everywhere on $[1,3]$, nor differentiable everywhere on $(1,3)$ both of these hypotheses fail at x = 2.

4.2.42

a. The average temperature gradient from $h = 0$ to $h = 1.1$ m is $\dfrac{-2 - (-16)}{1.1 - 0} \approx 12.7°/\text{m}$, so by the Mean Value Theorem the temperature gradient must equal $12.7°/\text{m}$ somewhere in the snowpack, and the formation of a weak layer is likely.

b. The average temperature gradient from $h = 0$ to $h = 1.4$ m is $\dfrac{-1 - (-12)}{1.4 - 0} \approx 7.86°/\text{m}$. While it is still possible that the temperature gradient exceeds $10°/\text{m}$ somewhere in the snowpack, one may suspect that the formation of a weak layer is not likely in this case.

c. If the surface temperature and temperature at the bottom of the snowpack are both roughly constant, then the temperature gradient will be larger in areas where the snowpack is less deep.

d. If all layers of the snowbank are the same temperature, then the temperature gradient is 0 and a weak layer is not likely to form.

4.2.43 The average speed of the car over the 28 minute period $(= 28/60 \text{ hr})$ is $\dfrac{30-0}{28/60} \approx 64.3\text{mi/hr}$, so the officer can conclude by the Mean Value Theorem that at some point the car exceeded the speed limit.

4.2.44 The average speed of the car over the 30 minute period $(= 1/2 \text{ hr})$ is exactly 60 mi/hr. But because the car started from rest, the average speed for the first few seconds of the trip is less than 60 mi/hr, and therefore the average speed for the remainder of the trip must exceed 60 mi/hr, and the officer can conclude that the driver exceeded the speed limit.

4.2.45 The runner's average speed is $6.2/(32/60) \approx 11.6$ mi /hr. By the Mean Value Theorem, the runner's speed was 11.6 mi/hr at least once. By the intermediate value theorem, all speeds between 0 and 11.6 mi/hr were reached. Because the initial and final speed was 0 mi/hr, the speed of 11 mi/hr was reached at least twice.

4.2.46 For linear functions $f(x)$ the average rate of change of f on any interval $[a, b]$ is the same as the slope $f'(c)$ for *any* point c in (a, b).

4.2.47 Observe that

$$\frac{f(b)-f(a)}{b-a} = \frac{A(b^2-a^2)+B(b-a)}{b-a} = A(a+b)+B$$

and $f'(c) = 2Ac+B$, so the point c that satisfies the conclusion of the Mean Value Theorem is $c = (a+b)/2$.

4.2.48

a. Observe that $\dfrac{f(b)-f(a)}{b-a} = \dfrac{b^2-a^2}{b-a} = (a+b)$ and $f'(c) = 2c$, so the point c that satisfies the conclusion of the Mean Value Theorem is $c = (a+b)/2$.

b. Observe that

$$\frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = \frac{a-b}{ab(b-a)} = -\frac{1}{ab}$$

and $f'(c) = -1/c^2$, so the point c that satisfies the conclusion of the Mean Value Theorem is $c = \sqrt{ab}$.

4.2.49 Note that $f'(x) = 2\tan x \sec^2 x$ and $g'(x) = 2\sec x \sec x \tan x = 2\tan x \sec^2 x$, so $f'(x) = g'(x)$. This implies that $f-g$ is a constant, which also follows from the trigonometric identity $\sec^2 x = \tan^2 x + 1$.

4.2.50 Bolt's average speed during the race was $\dfrac{100}{9.58}$ m/s $= \dfrac{100}{9.58} \cdot \dfrac{3600}{1000}$ km/hr ≈ 37.58 km/hr, so by the Mean Value Theorem he must have exceeded 37 km/hr during the race.

4.2.51 Let $f(x) = \sec^{-1} x$ and $g(x) = \cos^{-1}\left(\dfrac{1}{x}\right)$. Then

$$f'(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

and

$$g'(x) = -\frac{1}{\sqrt{1-(1/x)^2}} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{|x|^2\sqrt{1-(1/x)^2}} = \frac{1}{|x|\sqrt{x^2}\sqrt{1-(1/x)^2}} = \frac{1}{|x|\sqrt{x^2-1}}.$$

Because $f'(x) = g'(x)$ for $x > 0$, we must have that $f(x) - g(x) = C$ for $x > 0$. Let $x = 1$. Then $C = f(1) - g(1) = 0 - 0 = 0$, so $f(x) - g(x) = 0$, so $f(x) = g(x)$ for $x > 0$. Similarly, $f(x) - g(x) = C$ for $x < 0$, and letting $x = -1$ gives $f(x) - g(x) = 0$ so $f(x) = g(x)$ for $x < 0$. Thus $\sec^{-1} x = \cos^{-1}\left(\dfrac{1}{x}\right)$ for all $x \neq 0$.

4.2.52

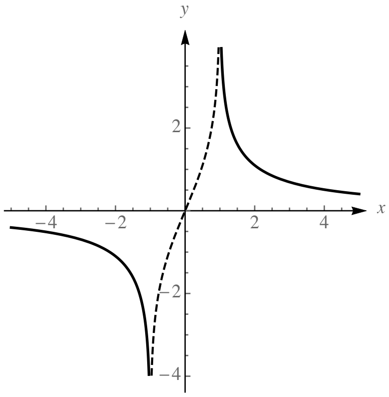
a.

$$f'(x) = \frac{(x-1)}{(x+1)}\left(\frac{(x-1)-(x+1)}{(x-1)^2}\right) = -\frac{2}{(x+1)(x-1)} = -\frac{2}{x^2-1} \text{ for all } x \text{ such that } |x| > 1.$$

Also,

$$\begin{aligned} g'(x) &= \frac{(1-x)}{(x+1)}\left(\frac{(1-x)-(x+1)(-1)}{(1-x)^2}\right) = \frac{2}{(x+1)(1-x)} = -\frac{2}{(x+1)(x-1)} \\ &= -\frac{2}{x^2-1} \text{ for all } x \text{ such that } |x| > 1. \end{aligned}$$

b.



The solid curve is $f(x)$ and the dashed curve is $g(x)$.

c. f' and g' do not equal each other on a common interval I, so Theorem 4.6 is not violated.

4.2.53 By the Mean Value Theorem, there is a number c in $(2,4)$ so that $\dfrac{f(4)-f(2)}{4-2} = f'(c)$, or $f(4)-7 = 2f'(c)$. Because $f'(c) < 2$, we must have $f(4)-7 < 4$, which implies that $f(4) < 11$.

4.2.54 Suppose $x > 0$. By the Mean Value Theorem, there is a number c in $(0,x)$ so that $\dfrac{f(x)-f(0)}{x-0} = f'(c) > 1$. This implies that $\dfrac{f(x)}{x} > 1$, so $f(x) > x$.

4.2.55 Let $a > 0$ and let $f(x) = \sqrt{1+x}$ on $[0,a]$. By the Mean Value Theorem, there is a number c in $(0,a)$ so that $\dfrac{f(a)-f(0)}{a-0} = f'(c)$, which implies that

$$\frac{\sqrt{1+a}-1}{a} = \frac{1}{2\sqrt{1+c}}.$$

Because $\sqrt{1+c} > 1$ for $c > 0$, $\dfrac{1}{2\sqrt{1+c}} < \dfrac{1}{2}$. So

$$\frac{\sqrt{1+a}-1}{a} = \frac{1}{2\sqrt{1+c}} < \frac{1}{2}$$

so

$$\frac{\sqrt{1+a}-1}{a} < \frac{1}{2}.$$

Rewriting this inequality gives

$$1+\frac{a}{2} > \sqrt{1+a}$$

for $a > 0$.

4.2.56

a. Let $f(x) = \sin x$. If $a = b$, then the inequality $|\sin a - \sin b| \le |a-b|$ holds because both sides of the inequality are 0. Now assume $a < b$. By the Mean Value Theorem on (a,b), we know that there exists a number c in (a,b) so that

$$\frac{f(b)-f(a)}{b-a} = f'(c) = \cos c.$$

Because $|\cos c| \le 1$, we have

$$|f(b)-f(a) \le |b-a|$$

so

$$|\sin b - \sin a| \leq |b - a|.$$

For the case $a > b$, a similar argument using the Mean Value Theorem on the interval (b, a) shows that $|\sin a - \sin b| \leq |a - b|$, which can be written $|\sin b - \sin a| \leq |b - a|$. Thus this inequality holds for all a and b.

b. Using the above inequalty, let $b = 0$. Then

$$|\sin b - \sin a| = |\sin a| \leq |b - a| = |a|.$$

So $|\sin a| \leq |a|$ for all real numbers a.

4.2.57

a. If $g(x) = x$ then $g'(x) = 1$ and hence $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f(b) - f(a)}{b - a} = \dfrac{f'(c)}{g'(c)} = f'(c)$.

b. We have $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{0 - (-1)}{6 - 2} = \dfrac{1}{4}$; $\dfrac{f'(c)}{g'(c)} = \dfrac{2c}{4} = \dfrac{c}{2}$; so $c = 1/2$.

4.2.58 Observe that f' is positive and decreasing for $x > a$ (because $f'' < 0$ for $x > a$). Fix some $b > a$ and let $a < x < b$. Then by the Mean Value Theorem $\dfrac{f(x) - f(a)}{x - a} = f'(c)$ for some c in (a, x). We have $f'(c) > f'(b) > 0$, so therefore if $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists, we must have $f'(a) \geq f'(b) > 0$. On the other hand f' is negative for $x < a$ so the Mean Value Theorem implies $\dfrac{f(x) - f(a)}{x - a} < 0$ for $x < a$, and so if $f'(a)$ exists we must have $f'(a) \leq 0$. This gives a contradiction, so we conclude that $f'(a)$ does not exist.

More generally, if f' and f'' both change signs at some point a, then one of the functions $f(x)$, $-f(x)$,$f(-x)$ or $-f(-x)$ satisfies the hypotheses above, and so $f'(a)$ does not exist.

4.3 What Derivatives Tell Us

4.3.1 If f' is positive on an interval, f is increasing on that interval. If f' is negative on an interval, f is decreasing on that interval.

4.3.2 The First Derivative Test can be used to tell whether or not a critical point is a local maximum or minimum, as follows: If $(c, f(c))$ is a critical point, we investigate the sign of f' for points that are just to the left and just to the right of c. If the sign of f' changes from positive to negative, then f is changing from increasing to decreasing at c, so there is a local maximum at c. If the signs of f' are changing from negative to positive, then f is changing from decreasing to increasing at c, so there is a local minimum at c. If the signs of f' are the same on either side of c, then there is neither kind of local extremum at $x = c$.

Note that if we find *all* of the critical points of f, and if the domain of f is an interval or union of intervals, then the critical points naturally divide up the domain into intervals on which we can check the sign of f' and look for places where the sign changes.

4.3.3

a. $x - 3 = 0$ for $x = 3$, so $x = 3$ is a critical point of f.

b. $f'(x) > 0$ for $x > 3$, so f is increasing on $(3, \infty)$. $f' < 0$ on $(-\infty, 3)$, so f is decreasing on $(-\infty, 3)$.

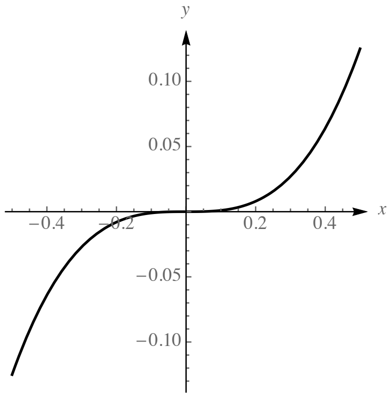
4.3.4

a. $(x - 1)(x - 2) = 0$ for $x = 1$ and $x = 2$, so these are the critical points.

b. $f' > 0$ on $(-\infty, 1)$ and on $(2, \infty)$, so f is increasing there. $f' < 0$ on $(1, 2)$ so f is decreasing there.

4.3.5

One such example is $f(x) = x^3$ at $x = 0$.



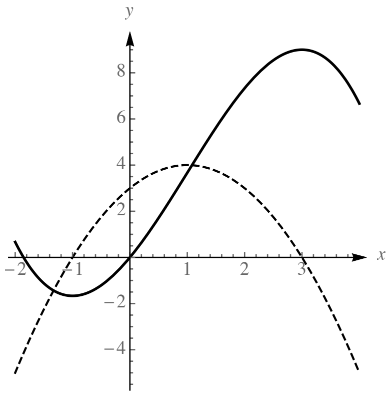
4.3.6

a. $g'(x) = 0$ for $x = 1$ and $x = 2$, so those are the critical points.

b. $g' > 0$ on $(1, 2)$, so g is increasing there. $g' < 0$ on $(0, 1)$ and on $(2, 3)$, so g is decreasing there.

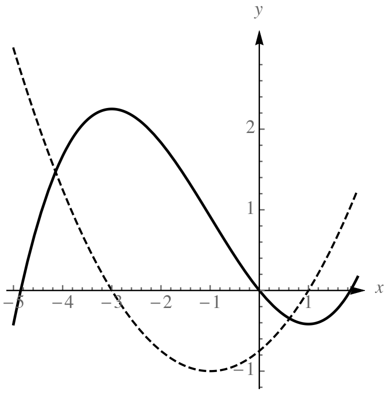
c. By the First Derivative Test, g has a local maximum at $x = 2$ and a local minimum at $x = 1$.

4.3.7



f' is dotted and possible f is solid.

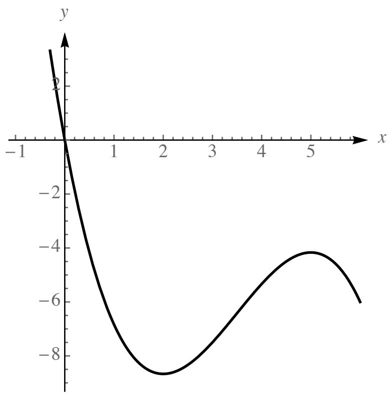
4.3.8



f' is dotted and possible f is solid.

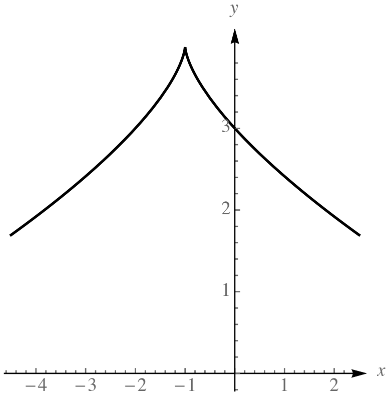
4.3.9

Such a function would be decreasing until $x = 2$, then increasing until $x = 5$, and then decreasing again after that.



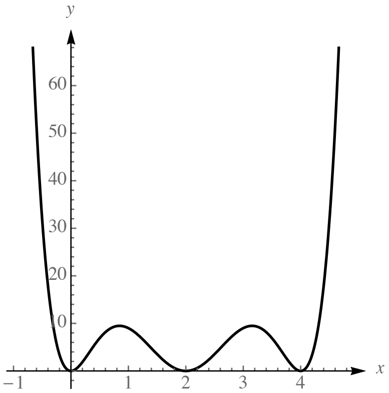
4.3.10

Such a function would be increasing on $(-\infty, -1)$, and decreasing on $(-1, \infty)$. It should have a point of non-differentiability at $x = -1$.



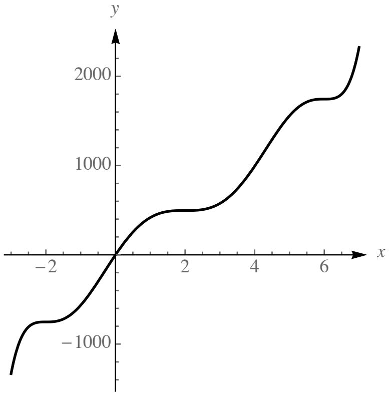
4.3.11

Such a function has extrema (minima) at 0 and 4, where the y value is zero. The function should never go below the x axis.



4.3.12

Such a function is never decreasing, but is flat at -2, 2, and 6.



4.3.13

a. $g'' > 0$ on $(-\infty, 2)$ so g is concave up there; $g'' < 0$ on $(2, \infty)$, so g is concave down there.

b. There is an inflection point at $x = 2$.

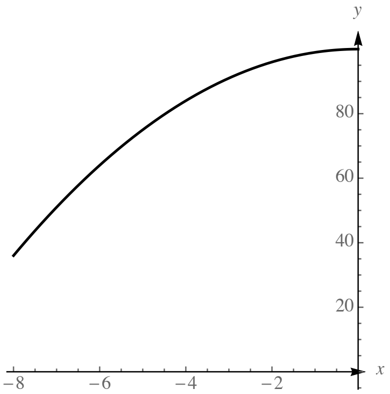
4.3.14

a. $g'' > 0$ on $(-3, 0)$ and $(2, 3)$, so g is concave up on those intervals. $g'' < 0$ on $(-4, 3)$ and $(0, 2)$, so g is concave down on those intervals.

b. There are inflection points at $x = -3$, $x = 0$, and $x = 2$.

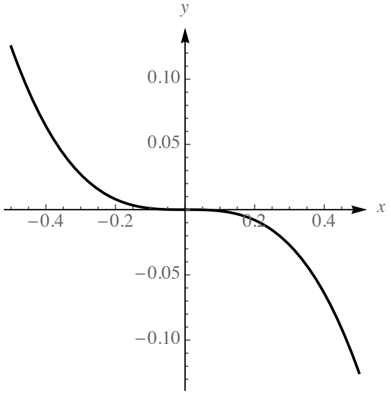
4.3.15

Yes, for example, consider $f(x) = 100 - x^2$ on the interval $(-8, 0)$. It is above the x axis, increasing, and concave down on that interval.



4.3.16

The second derivative is positive to the left of the inflection point, and negative to the right.



4.3.17 $f(x) = x^4$ has this property at 0. Note that $f''(x) = 12x^2$, which is 0 at $x = 0$, but the function doesn't have an inflection point there.

4.3.18 Because the Second Derivative Test is inconclusive, the First Derivative Test should be used in this case.

4.3.19 $f'(x) = -2x$, which is zero exactly when $x = 0$. On $(-\infty, 0)$ we note that $f' > 0$, so that f is increasing on this interval. On $(0, \infty)$, we note that $f' < 0$, so f is decreasing on this interval.

4.3.20 $f'(x) = 2x$, which is zero exactly when $x = 0$. On $(-\infty, 0)$ we note that $f' < 0$, so that f is decreasing on this interval. On $(0, \infty)$, we note that $f' > 0$, so f is increasing on this interval.

4.3.21 $f'(x) = 2(x - 1)$, which is zero exactly when $x = 1$. On $(-\infty, 1)$ we note that $f' < 0$, so that f is decreasing on this interval. On $(1, \infty)$, we note that $f' > 0$, so f is increasing on this interval.

4.3.22 $f'(x) = 3x^2 + 4$, which is always positive, because it is always 4 or greater. So f is increasing on $(-\infty, \infty)$.

4.3.23 $f'(x) = x^2 - 5x + 4 = (x-4)(x-1)$, which is 0 for $x = 1$ and $x = 4$. On $(-\infty, 1)$ $f' > 0$ so f is increasing; on $(1, 4)$ $f' < 0$ so f is decreasing; and on $(4, \infty)$ $f' > 0$ so f is increasing.

4.3.24 $f'(x) = -x^2 + x + 2 = -(x^2 - x - 2) = -(x-2)(x+1)$, which is 0 for $x = 2$ and $x = -1$. On $(-\infty, -1)$ $f' < 0$ so f is decreasing; on $(-1, 2)$ $f' > 0$ so f is increasing; on $(2, \infty)$ $f' < 0$ so f is decreasing.

4.3.25 $f'(x) = 1 - 2x$, which is 0 when $x = 1/2$. On $(-\infty, 1/2)$ $f' > 0$ so f is increasing on this interval, while on $(1/2, \infty)$ $f' < 0$, so f is deceasing on this interval.

4.3.26 $f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-2)(x-1)$, which is 0 when x is 0, 1, or 2. On $(-\infty, 0)$ $f' < 0$ so f is decreasing. On $(0, 1)$, $f' > 0$ so f is increasing, on $(1, 2)$ $f' < 0$ so f is decreasing, and on $(2, \infty)$ $f' > 0$ so f is increasing.

4.3.27 $f'(x) = -x^3 + 3x^2 - 2x = -x(x^2 - 3x + 2) = -x(x-1)(x-2)$. This is zero when $x = 0$, $x = 1$, and $x = 2$. Note that $f'(-1) > 0$, and $f'(1.5) > 0$, while $f'(.5) < 0$, and $f'(3) < 0$. So f is increasing on $(-\infty, 0)$ and on $(1, 2)$, while it is decreasing on $(0, 1)$ and on $(2, \infty)$.

4.3.28 $f'(x) = 10x^4 - 15x^3 + 5x^2 = 5x^2(2x^2 - 3x + 1) = 5x^2(x-1)(2x-1)$. This is zero when $x = 0$, $x = 1$ and $x = 1/2$. Note that $f'(-1) > 0$, and $f'(1/4) > 0$. Because the given function is continuous, we can combine the intervals and conclude that f is increasing on $(-\infty, 1/2)$. Also, $f'(3/4) < 0$, so f is decreasing on $(1/2, 1)$, and $f'(2) > 0$, so f is increasing on $(1, \infty)$.

4.3.29 $f'(x) = 2x \ln x^2 + x^2 \cdot \dfrac{1}{x^2} \cdot 2x = 2x(\ln x^2 + 1)$. This is undefined for $x = 0$, and when $\ln x^2 + 1 = 0$. For $x > 0$, this occurs when $2\ln x + 1 = 0$, which occurs when $\ln x = -\dfrac{1}{2}$, or $x = \dfrac{1}{\sqrt{e}}$. By symmetry, we also have that $f'(x)$ is zero for $x = -\dfrac{1}{\sqrt{e}}$. Note that $\dfrac{1}{\sqrt{e}} \approx 0.6$, and that $f'(-1) < 0$, $f'(-1/2) > 0$, $f'(1/2) < 0$, and $f'(1) > 0$. Thus, f is decreasing on $(-\infty, -1/\sqrt{e})$ and on $(0, 1/\sqrt{e})$, and is increasing on $(-1/\sqrt{e}, 0)$ and on $(1/\sqrt{e}, \infty)$.

4.3.30 $f'(x) = \dfrac{(e^{2x}+1)e^x - e^x(2e^{2x})}{(e^{2x}+1)^2} = \dfrac{e^x(1 - e^{2x})}{(e^{2x}+1)^2}$. This is zero when $e^{2x} = 1$, which occurs when $x = 0$. Note that $f'(-1) > 0$ and $f'(1) < 0$, so f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

4.3.31 $f'(x) = 2\sin x - 1$ which is 0 for $\sin x = 1/2$, or (on the given interval) $x = \pi/6, 5\pi/6$. On $(0, \pi/6)$ $f' < 0$ so f is decreasing; on $(\pi/6, 5\pi/6)$ $f' > 0$ so f is increasing; on $(5\pi/6, 2\pi)$ $f' < 0$ so f is decreasing.

4.3.32 $f'(x) = \sqrt{2}\cos x - 1$, which is 0 for $\cos x = 1/\sqrt{2}$, or (on the given interval) $x = \pi/4, 7\pi/4$. On $(0, \pi/4)$ $f' > 0$ so f is increasing; on $(\pi/4, 7\pi/4)$ $f' < 0$ so f is decreasing; on $(7\pi/4, 2\pi)$ $f' > 0$ so f is increasing.

4.3.33 $f'(x) = -9\sin 3x$, which is 0 for $3x = -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, \text{and } 3\pi$, which corresponds to $x = -\pi, -2\pi/3, -\pi/3, 0, \pi/3, 2\pi/3$ and π. Note that $f'(-5\pi/6) = 9 > 0$, $f'(-\pi/2) = -9 < 0$, $f'(-\pi/6) = 9 > 0$, $f'(\pi/6) = -9 < 0$, $f'(\pi/2) = 9 > 0$, and $f'(5\pi/6) = -9 < 0$. Thus f is increasing on $(-\pi, -2\pi/3)$, on $(-\pi/3, 0)$, and on $(\pi/3, 2\pi/3)$, while f is decreasing on $(-2\pi/3, -\pi/3)$, on $(0, \pi/3)$, and on $(2\pi/3, \pi)$.

4.3.34 $f'(x) = 2(\cos x)(-\sin x) = -\sin 2x$. This is 0 for $2x = -2\pi, -\pi, 0, \pi$, and 2π, which corresponds to $x = -\pi, -\pi/2, 0, \pi/2$, and π. Note that $f'(-3\pi/4) = -1 < 0$, $f'(-\pi/4) = 1 > 0$, $f'(\pi/4) = -1 < 0$, and $f'(3\pi/4) = 1 > 0$. So f is decreasing on $(-\pi, -\pi/2)$ and on $(0, \pi/2)$, and is increasing on $(-\pi/2, 0)$ and on $(\pi/2, \pi)$.

4.3.35 $f'(x) = (2/3)x^{-1/3}(x^2 - 4) + x^{2/3} \cdot 2x = \dfrac{2(x^2-4)}{3x^{1/3}} + \dfrac{2x^{5/3}}{1} \cdot \dfrac{3x^{1/3}}{3x^{1/3}} = \dfrac{8x^2 - 8}{3x^{1/3}} = \dfrac{8(x+1)(x-1)}{3x^{1/3}}$. This is zero for $x = \pm 1$ and is undefined for $x = 0$. Note that $f'(-8) = -84 < 0$, $f'(-1/8) = \frac{21}{4} > 0$, $f'(1/8) = -\frac{21}{4} < 0$, and $f''(8) = 84 > 0$. Thus f is decreasing on $(-\infty, -1)$ and on $(0, 1)$, while f is increasing on $(-1, 0)$ and on $(1, \infty)$.

4.3.36 $f'(x) = 2x\sqrt{9-x^2} + x^2 \cdot \dfrac{1}{2\sqrt{9-x^2}} \cdot (-2x) = \dfrac{2x(9-x^2)}{\sqrt{9-x^2}} + -\dfrac{x^3}{\sqrt{9-x^2}} = \dfrac{18x-3x^3}{\sqrt{9-x^2}} = \dfrac{3x(6-x^2)}{\sqrt{9-x^2}}$. This is zero when $x = 0$, and when $x = \pm\sqrt{6}$. Note that $\sqrt{6} \approx 2.4$. Also note that $f'(-2.7) > 0$, $f'(-1) < 0$, $f'(1) > 0$ and $f'(2.7) < 0$. Thus, f is increasing on $(-3, -\sqrt{6})$ and on $(0, \sqrt{6})$, and is decreasing on $(-\sqrt{6}, 0)$ and on $(\sqrt{6}, 3)$.

4.3.37 $f'(x) = \dfrac{1}{2\sqrt{9-x^2}} \cdot (-2x) + \dfrac{1}{3\sqrt{1-\frac{x^2}{9}}} = \dfrac{-x}{\sqrt{9-x^2}} + \dfrac{1}{\sqrt{9-x^2}} = \dfrac{1-x}{\sqrt{9-x^2}}$. This is 0 for $x = 1$ and is undefined at the endpoints of the domain which are 3 and -3. On $(-3, 1)$ $f' > 0$ so f is increasing; on $(1, 3)$ $f' < 0$ so f is decreasing.

4.3.38 $f'(x) = \ln x + \dfrac{x}{x} - 2 = \ln x - 1$. This is 0 for $\ln x = 1$, or $x = e$. On $(0, e)$ $f' < 0$ so f is decreasing; on (e, ∞) $f' > 0$ so f is increasing.

4.3.39 $f'(x) = -60x^4 + 300x^3 - 240x^2 = -60x^2(x^2-5x+4) = -60x^2(x-4)(x-1)$. This is 0 for $x = 0$, $x = 1$, and $x = 4$. Note that $f'(-1) = -600 < 0$, $f'(1/2) = -26.25 < 0$, $f'(2) = 480 > 0$, and $f'(5) = -6000 < 0$. Thus f is increasing on $(1, 4)$ and is decreasing on $(-\infty, 1)$ and on $(4, \infty)$.

4.3.40 $f'(x) = 2x - \dfrac{2}{x} = \dfrac{2(x^2-1)}{x}$ which is 0 for $x = 1$. Note that the domain of f is $(0, \infty)$ and that $f'(1/2) = -3 < 0$ and $f'(2) = 3 > 0$, so f is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

4.3.41 $f'(x) = -8x^3 + 2x = -2x(4x^2-1) = -2x(2x+1)(2x-1)$. This is zero for $x = 0$ and $x = \pm 1/2$. Note that $f'(-1) > 0$, $f'(-1/4) < 0$, $f'(1/4) > 0$, and $f'(1) < 0$, so f is increasing on $(-\infty, -1/2)$ and on $(0, 1/2)$, while it is decreasing on $(-1/2, 0)$ and on $(1/2, \infty)$.

4.3.42 $f'(x) = x^3 - 8x^2 + 15x = x(x^2-8x+15) = x(x-5)(x-3)$. This is zero when $x = 0$, $x = 3$, and $x = 5$. Note that $f'(-1) < 0$, $f'(1) > 0$, $f'(4) < 0$, and $f'(6) > 0$. Thus f is increasing on $(0, 3)$ and on $(5, \infty)$, while it is decreasing on $(-\infty, 0)$ and on $(3, 5)$.

4.3.43 We have $f'(x) = e^{-x^2/2} + xe^{-x^2/2} \cdot (-x) = (1-x^2)e^{-x^2/2}$. This is zero only when $x = \pm 1$. Note that $f'(-2) = -3e^{-2} < 0$, $f'(0) = 1 > 0$, and $f'(2) = -3e^{-2} < 0$. Thus f is decreasing on $(-\infty, -1)$ and on $(1, \infty)$, and is increasing on $(-1, 1)$.

4.3.44 $f'(x) = \dfrac{1}{1+\left(\frac{x}{x^2+2}\right)^2} \cdot \dfrac{(x^2+2) - x(2x)}{(x^2+2)^2} = \dfrac{1}{1+\left(\frac{x}{x^2+2}\right)^2} \cdot \dfrac{2-x^2}{(x^2+2)^2}$. Note that the first factor is always positive, so the expression is zero exactly when $2 - x^2 = 0$, so only at $\pm\sqrt{2}$. Note that $f'(0) > 0$ while $f'(\pm 2) < 0$, so f is increasing on $(-\sqrt{2}, \sqrt{2})$ and decreasing on $(-\infty, -\sqrt{2})$ and on $(\sqrt{2}, \infty)$.

4.3.45

a. $f'(x) = 2x$, so $x = 0$ is the only critical point.

b. Note that $f' < 0$ for $x < 0$ and $f' > 0$ for $x > 0$, so f has a local minimum of $f(0) = 3$ at $x = 0$.

c. Note that $f(-3) = 12$, $f(0) = 3$ and $f(2) = 7$, so the absolute maximum is 12 and the absolute minimum is 3.

4.3.46

a. $f'(x) = -2x - 1$, which exists everywhere and is zero only for $x = -1/2$, so that is the only critical point.

b. Note that $f'(-2) = 3 > 0$ and $f'(0) = -1 < 0$, so f has a local maximum of $f(-1/2) = 9/4$ at $x = -1/2$.

c. Note that $f(-4) = -10$ and $f(4) = -18$, so the absolute maximum is $9/4$ at $x = -1/2$ and the absolute minimum is -18 at $x = 4$.

4.3.47

a. $f'(x) = x \cdot \frac{1}{2}(4-x^2)^{-1/2} \cdot (-2x) + \sqrt{4-x^2} \cdot 1 = \frac{4-2x^2}{\sqrt{4-x^2}}$, which exists everywhere on $(-2, 2)$ and is zero only for $x = \pm\sqrt{2}$, so those are the only critical points.

b. Note that $f'(-1.5) < 0$, $f'(0) > 0$ and $f'(1.5) < 0$, so f has a local minimum of $f(-\sqrt{2}) = -2$ and a local maximum of $f(\sqrt{2}) = 2$.

c. Note that $f(-2) = 0 = f(2)$. So the absolute maximum is 2 at $x = \sqrt{2}$ and the absolute minimum is -2 at $x = -\sqrt{2}$.

4.3.48

a. $f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1)$, which exists everywhere and is 0 at $x = -2$ (an endpoint of the given domain) and $x = 1$.

b. Note that $f'(-1.5) < 0$ and $f'(2) > 0$, so f has a local minimum at $x = 1$ of $f(1) = -6$.

c. Note that $f(-2) = 21$ and $f[4] = 129$, so the absolute maximum of f on $[-2, 4]$ is 129 and the absolute minimum is -6.

4.3.49

a. $f'(x) = -3x^2 + 9$, which is zero when $9 = 3x^2$, or $x^2 = 3$. So the critical points are at $x = \pm\sqrt{3}$.

b. Note that $f'(-2) < 0$, $f'(0) > 0$, and $f'(2) < 0$, so there is a local minimum of $f(-\sqrt{3}) = -6\sqrt{3}$ and a local maximum of $f(\sqrt{3}) = 6\sqrt{3}$.

c. There is an absolute maximum of 28 at $x = -4$ and an absolute minimum of $-6\sqrt{3}$ at $x = -\sqrt{3}$.

4.3.50

a. $f'(x) = 10x^4 - 20x^3 - 30x^2 = 10x^2(x^2 - 2x - 3) = 10x^2(x+1)(x-3)$. This is zero for $x = 0$, $x = -1$, and $x = 3$.

b. Note that $f'(-2) > 0$, $f'(-.5) < 0$, $f'(1) < 0$, and $f'(4) > 0$. So there is a local maximum of $f(-1) = 7$ and a local minimum of $f(3) = -185$. There isn't any sort of extremum at $x = 0$.

c. The local minimum value of -185 is an absolute minimum, and the absolute maximum is $f(4) = 132$.

4.3.51

a. $f'(x) = x^{2/3} + (x-5) \cdot \frac{2}{3}x^{-1/3} = \frac{5x-10}{3x^{1/3}}$, which is undefined at $x = 0$ and is 0 at $x = 2$. So these are the two critical points.

b. Note that $f'(-1) > 0$ and $f'(1) < 0$, and $f'(3) > 0$ so f has a local maximum at $x = 0$ of $f(0) = 0$ and a local minimum at $x = 2$ of $-3\sqrt[3]{4} \approx -4.762$.

c. Note that $f(-5) = -10\sqrt[3]{25} \approx -29.24$, $f(0) = 0$, and $f(5) = 0$, so the absolute maximum of f on $[-5, 5]$ is 0 and the absolute minimum is $-10\sqrt[3]{25}$.

4.3.52 First note that even though the interval given is $[-4, 4]$, the function isn't defined at $x = \pm 1$, so we will assume that the given domain is $[-4, -1) \cup (-1, 1) \cup (1, 4]$.

a. $f'(x) = \frac{(x^2-1) \cdot 2x - (x^2)(2x)}{(x^2-1)^2} = -\frac{2x}{(x^2-1)^2}$, which is 0 only at $x = 0$.

b. Note that $f'(-2) > 0$ and $f'(-1/2) > 0$, and $f'(1/2) < 0$ and $f'(2) < 0$, so f is increasing on $(-4, -1)$ and on $(-1, 0)$, while it is decreasing on $(0, 1)$ and on $(1, 4)$. There is a local maximum of 0 at $x = 0$.

c. Because f becomes arbitrarily large as x approaches 1 from the left, and arbitrarily large in the negative sense as x approaches 1 from the right, it has no absolute extrema.

4.3.53

a. $f'(x) = \frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}} = \frac{2+\ln x}{2\sqrt{x}}$. This is defined everywhere on $(0, \infty)$ and is 0 only at $x = e^{-2}$.

b. Note that $f' < 0$ on $\left(0, \frac{1}{e^2}\right)$ and $f' > 0$ on $\left(\frac{1}{e^2}, \infty\right)$, so there is a local minimum at $x = \frac{1}{e^2}$.

c. Because there is only one critical point, the local minimum at $x = \frac{1}{e^2}$ yields an absolute minimum of $f(1/e^2) = -\frac{2}{e} \approx -0.736$. There is no absolute maximum because f increases without bound as $x \to \infty$.

4.3.54

a. $f'(x) = 1 - \frac{2}{1+x^2} = \frac{1+x^2}{1+x^2} - \frac{2}{1+x^2} = \frac{x^2-1}{x^2+1} = \frac{(x-1)(x+1)}{x^2+1}$. This is zero for $x = \pm 1$.

b. $f' > 0$ on $(-\sqrt{3}, -1)$ and on $(1, \sqrt{3})$ while $f' < 0$ on $(-1, 1)$. So there is a local maximum value of $f(-1) = -1 + \frac{\pi}{2} \approx 0.57$ at $x = -1$ and a local minimum value of $f(1) = 1 - \frac{\pi}{2} \approx -0.57$ at $x = 1$.

c. Checking endpoints, we have $f(-\sqrt{3}) = -\sqrt{3} + \frac{2\pi}{3} \approx 0.36$ and $f(\sqrt{3}) = \sqrt{3} - \frac{2\pi}{3} \approx -0.36$. So the absolute maximum of f is $f(-1) = -1 + \frac{\pi}{2} \approx 0.57$ at $x = -1$ and the absolute minimum value of f is $f(1) = 1 - \frac{\pi}{2} \approx -0.57$ at $x = 1$.

4.3.55 $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x)$, which is 0 only for $x = 1$. f is continuous on $(-\infty, \infty)$ and contains only one critical point. Note that $f' > 0$ for $x < 1$ and $f' < 0$ for $x > 1$. So there is a local maximum of $f(1) = 1/e$ at $x = 1$. The local maximum of $1/e$ at $x = 1$ is an absolute maximum. There is no absolute minimum, because the function is unbounded in the negative direction as $x \to -\infty$.

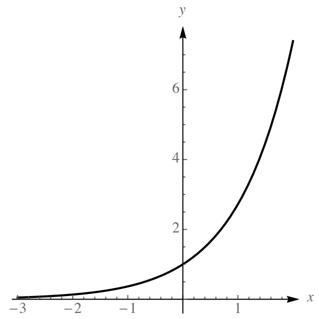
4.3.56 f is continuous on $(0, \infty)$. $f'(x) = 4 - \frac{1}{2x^{3/2}} = \frac{8x^{3/2}-1}{2x^{3/2}}$, which exists for $x > 0$. This quantity is 0 for $x = \frac{1}{4}$. So f has only one critical point on $(0, \infty)$. Also, $f' < 0$ on $(0, 1/4)$ and $f' > 0$ on $(1/4, \infty)$. So there is a local minimum of $f(1/4) = 3$ at $x = 1/4$. The local minimum of 3 at $x = 1/4$ is an absolute minimum. There is no absolute maximum, because the function is unbounded as $x \to \infty$.

4.3.57 A is continuous on $(0, \infty)$. $A'(r) = -\frac{24}{r^2} + 4\pi r = \frac{4\pi r^3 - 24}{r^2}$, which is 0 for $r = \sqrt[3]{6/\pi}$, so there is only one critical point on the stated interval. Note that $A' < 0$ on $(0, \sqrt[3]{6/\pi})$ and $A' > 0$ on $(\sqrt[3]{6/\pi}, \infty)$, so there is a local minimum of $A(\sqrt[3]{6/\pi}) = 36\sqrt[3]{\pi/6}$. The local minimum mentioned above is an absolute minimum. There is no absolute maximum, because A is unbounded as $r \to \infty$.

4.3.58 f is continuous on $(-\infty, 3)$. $f'(x) = -\frac{x}{2\sqrt{3-x}} + \sqrt{3-x} = \frac{-3x+6}{2\sqrt{3-x}}$, which is 0 only for $x = 2$, so there is only one critical point on the stated interval. Also, $f' > 0$ for $x < 2$ and $f' < 0$ on $(2, 3)$. Thus there is a local maximum of $f(2) = 2$ which is also an absolute maximum. There is no absolute minimum, because the function is unbounded in the negative direction as $x \to -\infty$.

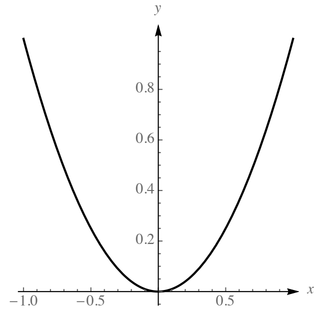
4.3.59

The function sketched should be increasing and concave up everywhere.



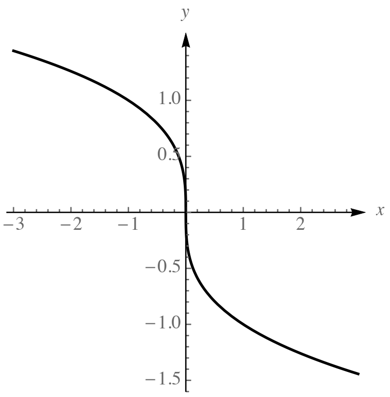
4.3.60

The function sketched should be concave up everywhere, decreasing for $x < 0$ and increasing for $x > 0$.



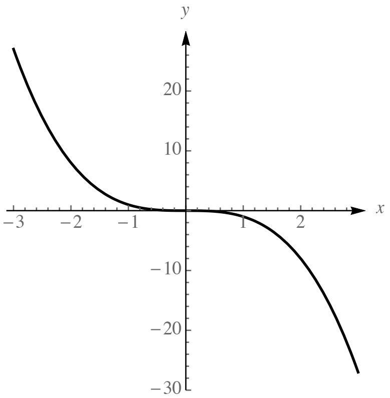
4.3.61

The function sketched should be decreasing everywhere, concave down for $x < 0$, and concave up for $x > 0$.



4.3.62

The function sketched should be decreasing everywhere, concave up for $x < 0$, and concave down for $x > 0$.



4.3.63 $f'(x) = 4x^3 - 6x^2$, so $f''(x) = 12x^2 - 12x = 12x(x-1)$. Note that f'' is zero when $x = 0$ and $x = 1$, so these are potential inflection points. Also note that $f''(-1) > 0$, $f''(0.5) < 0$, and $f''(2) > 0$, so f is concave up on $(-\infty, 0)$ and on $(1, \infty)$, and is concave down on $(0, 1)$. There are inflection points at $(0, 1)$ and $(1, 0)$.

4.3.64 $f'(x) = -4x^3 - 6x^2 + 24x$, so $f''(x) = -12x^2 - 12x + 24 = -12(x^2 + x - 2) = -12(x+2)(x-1)$. Note the f'' is zero for $x = -2$ and $x = 1$, so these are potential inflection points. Now note that $f''(-3) < 0$, $f''(0) > 0$, and $f''(2) < 0$. Thus f is concave up on $(-2, 1)$ and is concave down on $(-\infty, -2)$ and on $(1, \infty)$. There are inflection points at $(-2, 48)$ and $(1, 9)$.

4.3.65 $f'(x) = 20x^3 - 60x^2$, and $f''(x) = 60x^2 - 120x = 60x(x-2)$. This is 0 for $x = 0$ and for $x = 2$. Note that $f''(-1) > 0$, $f''(1) < 0$, and $f''(3) > 0$. So f is concave up on $(-\infty, 0)$, concave down on $(0, 2)$, and concave up on $(2, \infty)$. There are inflection points at $x = 0$ and $x = 2$.

4.3.66 $f'(x) = -1(1+x^2)^{-2} \cdot 2x = -\frac{2x}{(1+x^2)^2}$. $f''(x) = \dfrac{(-2)(1+x^2)^2 - (-2x) \cdot 2 \cdot (1+x^2) \cdot 2x}{(1+x^2)^4} = \dfrac{6x^2 - 2}{(1+x^2)^3}$. Note that f'' is 0 for $x = \pm\sqrt{1/3}$. Also, $f''(-1) > 0$, $f''(0) < 0$, and $f''(1) > 0$, so f is concave up on $(-\infty, \sqrt{1/3})$, concave down on $(-\sqrt{1/3}, \sqrt{1/3})$, and concave up on $(\sqrt{1/3}, \infty)$. There are inflection points at $x = \pm\sqrt{1/3}$.

4.3.67 $f'(x) = e^x(x-3) + e^x = e^x(x-3+1) = e^x(x-2)$. $f''(x) = e^x(x-2) + e^x = e^x(x-2+1) = e^x(x-1)$. Note that f'' is zero only at $x = 1$. Also note that $f''(0) < 0$ and $f''(2) > 0$, so f is concave down on $(-\infty, 1)$ and is concave up on $(1, \infty)$. The point $(1, -2e)$ is an inflection point.

4.3.68 $f'(x) = 4x\ln x + 2x^2 \cdot \dfrac{1}{x} - 10x = 4x\ln x - 8x$. $f''(x) = 4\ln x + 4x \cdot \frac{1}{x} - 8 = 4\ln x - 4$. Note that f'' is zero when $\ln x = 1$, which occurs for $x = e$. Also note that $f''(1) < 0$ and $f''(4) > 0$, so f is concave down on $(0, e)$ and is concave up on (e, ∞). There is an inflection point at $(e, -3e^2)$.

4.3.69 $g'(t) = \dfrac{6t}{3t^2+1}$, and $g''(t) = \dfrac{(3t^2+1) \cdot 6 - 6t(6t)}{(3t^2+1)^2} = \dfrac{6 - 18t^2}{(3t^2+1)^2}$. Note that g'' is 0 for $t = \pm\sqrt{1/3}$. Also, $g''(-1) < 0$, $g''(0) > 0$, and $g''(1) < 0$, so g is concave down on $(-\infty, -\sqrt{1/3})$ and on $(\sqrt{1/3}, \infty)$, and is concave up on $(-\sqrt{1/3}, \sqrt{1/3})$. There are inflection points at $t = \pm\sqrt{1/3}$.

4.3.70 $g'(x) = \dfrac{1}{3\sqrt[3]{(x-4)^2}}$, and $g''(x) = -\dfrac{2}{9\sqrt[3]{(x-4)^5}}$. Note that g'' is never zero, but is undefined at $x = 4$. On $(-\infty, 4)$ we have $g'' > 0$ so g is concave up, and on $(4, \infty)$ we have $g'' < 0$, so g is concave down. There is an inflection point at $(4, 0)$.

4.3.71 $f'(x) = -xe^{-x^2/2}$, and $f''(x) = (-x)(-xe^{-x^2/2}) + e^{-x^2/2} \cdot -1 = e^{-x^2/2}(x^2 - 1)$. $f''(x)$ is 0 for $x = \pm 1$. Also $f'' > 0$ on $(-\infty, -1)$ and on $(1, \infty)$, so f is concave up there, while on $(-1, 1)$ f is concave down because $f'' < 0$ on that interval. The inflection points are at $(\pm 1, e^{-1/2})$.

4.3.72 $p'(x) = 4x^3e^x + x^4e^x + 1$ and $f''(x) = 12x^2e^x + 4x^3e^x + 4x^3e^x + x^4e^x = x^2e^x(12 + 4x + 4x + x^2) = x^2e^x(x^2 + 8x + 12) = x^2e^x(x + 6)(x + 2)$. So $p'' = 0$ for $x = -6, -2, 0$. Checking the sign of p'' shows that $p'' > 0$ on $(-\infty, -6)$ and on $(-2, 0)$, and on $(0, \infty)$, so p is concave up on these intervals, while $p'' < 0$ on $(-6, -2)$ so p is concave down there. The inflection points occur at $x = -6$ and $x = -2$.

4.3.73 $f'(x) = \sqrt{x}/x + (\ln x)\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{2 + \ln x}{2\sqrt{x}}$. $f''(x) = \dfrac{2\sqrt{x}/x - (2 + \ln x)/\sqrt{x}}{(2\sqrt{x})^2} = -\dfrac{\ln x}{4\sqrt{x^3}}$. Note that f'' is 0 only at $x = 1$. On $(0, 1)$ we note that $f'' > 0$ so f is concave up, and on $(1, \infty)$ we note that $f'' < 0$ so f is concave down. There is an inflection point at $(1, 0)$.

4.3.74 $h'(t) = -2\sin 2t$ and $h''(t) = -4\cos 2t$, which on the stated domain is 0 when $2t = \pi/2$, and $3\pi/2$, which means for $t = \pi/4$, and $3\pi/4$. $h'' < 0$ on $(0, \pi/4)$ and on $(3\pi/4, \pi)$, so h is concave down on those intervals, while $h'' > 0$ on $(\pi/4, 3\pi/4)$, so h is concave up on that interval. There are inflection points at $t = \pi/4$, and $t = 3\pi/4$.

4.3.75 $g'(t) = 15t^4 - 120t^3 + 240t^2$, and $g''(t) = 60t^3 - 360t^2 + 480t = 60t(t - 2)(t - 4)$. Note that g'' is 0 for $t = 0, 2$, and 4. Note also that $g'' < 0$ on $(-\infty, 0)$ and on $(2, 4)$, so g is concave down on those intervals, while $g'' > 0$ on $(0, 2)$ and on $(4, \infty)$, so g is concave up there. There are inflection points at $t = 0, 2$, and 4.

4.3.76 $f'(x) = 8x^3 + 24x^2 + 24x - 1$, and $f''(x) = 24x^2 + 48x + 24 = 24(x + 1)^2$. Note that this quantity is always greater than 0 for $x \neq -1$, and is 0 only at $x = -1$. Thus f is concave up on $(-\infty, -1)$ and on $(-1, \infty)$, and because f and f' are continuous at -1, we can say that f is concave up on $(-\infty, \infty)$.

4.3.77 $f'(x) = 3x^2 - 6x = 3x(x - 2)$. This is zero when $x = 0$ and when $x = 2$, and these are the critical points. $f''(x) = 6x - 6$. Note that $f''(0) < 0$ and $f''(2) > 0$. Thus by the Second Derivative Test, there is a local maximum at $x = 0$ and a local minimum at $x = 2$.

4.3.78 $f'(x) = 12x - 3x^2 = 3x(4 - x)$. This is zero when $x = 0$ and when $x = 4$, and these are the critical points. $f''(x) = 12 - 6x$. Note that $f''(0) > 0$ and $f''(4) < 0$, so there is a local minimum at 0 and a local maximum at 4.

4.3.79 $f'(x) = -2x$, so $x = 0$ is a critical point. $f''(x) = -2$, so $f''(0) = -2$ and the critical point yields a local maximum.

4.3.80 $f'(x) = 3x^2 - 3x - 36 = 3(x^2 - x - 12) = 3(x - 4)(x + 3)$, which is zero for $x = 4$ and $x = -3$. $f''(x) = 6x - 3$. We have $f''(4) = 21 > 0$, so there is a local minimum at $x = 4$, while $f''(-3) = -21 < 0$, so there is a local maximum at $x = -3$.

4.3.81 $f'(x) = e^x(x - 7) + e^x = e^x(x - 6)$. This is zero when $x = 6$, and this is a critical point. $f''(x) = e^x(x - 6) + e^x = e^x(x - 5)$. Note that $f''(6) > 0$, so there is a local minimum at $x = 6$.

4.3.82 $f'(x) = e^x(x - 2)^2 + e^x \cdot 2(x - 2) = e^x(x - 2)(x - 2 + 2) = xe^x(x - 2)$. This is zero for $x = 0$ and $x = 2$. $f''(x) = e^x(x - 2) + xe^x(x - 2) + xe^x = e^x(x - 2 + x^2 - 2x + x) = e^x(x^2 - 2)$. We have $f''(0) = -2 < 0$ so there is a local maximum at $x = 0$, while $f''(2) = 2e^2 > 0$ so there is a local minimum at $x = 2$.

4.3.83 $f'(x) = 6x^2 - 6x = 6x(x - 1)$, so $x = 0$ and $x = 1$ are critical points. $f''(x) = 12x - 6$, so $f''(1) = 6 > 0$, so the critical point at $x = 1$ yields a local minimum. Also, $f''(0) = -6 < 0$, so the critical point at 0 yields a local maximum.

4.3.84 $f'(x) = \dfrac{(x + 1)e^x - e^x}{(x + 1)^2} = \dfrac{xe^x}{(x + 1)^2}$. This is zero for $x = 0$, so that is the only critical point. $f''(x) = \dfrac{(x + 1)^2(e^x + xe^x) - xe^x \cdot 2(x + 1)}{(x + 1)^4} = \dfrac{(x + 1)e^x((x + 1)^2 - 2x)}{(x + 1)^4} = \dfrac{e^x(x^2 + 1)}{(x + 1)^3}$. Because $f''(0) = 1 > 0$, there is a local minimum at $x = 0$.

4.3.85 $f'(x) = x^2 \cdot (-e^{-x}) + e^{-x} \cdot 2x = e^{-x}(2x - x^2)$, which is zero for $x = 0$ and $x = 2$, so these are the critical points. $f''(x) = e^{-x}(2 - 2x) + (2x - x^2)(-e^{-x}) = e^{-x}(2 - 4x + x^2)$. Note that $f''(0) = 2 > 0$, so there is a local minimum at $x = 0$. Also, $f''(2) = -2e^{-2} < 0$, so there is a local maximum at $x = 2$.

4.3.86

$$g'(x) = \frac{(2-12x^2)(4x^3) - x^4(-24x)}{(2-12x^2)^2} = \frac{-24x^5 + 8x^3}{(2-12x^2)^2} = 2\left(\frac{-3x^5 + x^3}{(1-6x^2)^2}\right).$$

g' is zero at 0 and at $\pm\frac{1}{\sqrt{3}}$. We have

$$\begin{aligned} g''(x) &= 2\left(\frac{(1-6x^2)^2(-15x^4+3x^2) - (-3x^5+x^3)(2(1-6x^2)\cdot(-12x))}{(1-6x^2)^4}\right) \\ &= 2\left(\frac{(1-6x^2)((1-6x^2)(-15x^4+3x^2) - (72x^6-24x^4))}{(1-6x^2)^4}\right) \\ &= 2\left(\frac{-15x^4+3x^2+90x^6-18x^4-72x^6+24x^4}{(1-6x^2)^3}\right) \\ &= 2\left(\frac{3x^2-9x^4+18x^6}{(1-6x^2)^3}\right) \\ &= 6\left(\frac{x^2-3x^4+6x^6}{(1-6x^2)^3}\right). \end{aligned}$$

Then $g''\left(\pm\frac{1}{\sqrt{3}}\right) = -\frac{4}{3}$, so these critical points give local maxima by the Second Derivative Test. However, $g''(0) = 0$, so the Second Derivative Test is inconclusive at $x = 0$. Using the First Derivative Test, note that $0.5 < \frac{1}{\sqrt{3}}$, and $g'(0.5) = 0.25 > 0$, and using the fact that g' is an odd function, $g'(-0.5) = -0.25 < 0$, so there is a local minimum for g at $x = 0$.

4.3.87 $f'(x) = 4x\ln x + 2x^2 \cdot \frac{1}{x} - 22x = 4x\ln x - 20x = 4x(\ln x - 5)$. This is zero for $x = e^5$, so that is the critical point. $f''(x) = 4\ln x + 4x \cdot \dfrac{1}{x} - 20 = 4\ln x - 16$. Note that $f''(e^5) > 0$, so there is a local minimum at e^5.

4.3.88 Note that $f(x)$ can be written as $f(x) = \dfrac{12}{7}x^{7/2} - 4x^{5/2}$. Thus $f'(x) = 6x^{5/2} - 10x^{3/2} = 2x^{3/2}(3x-5)$. This is zero on the given interval only at $x = 5/3$, so that is the critical point. $f''(x) = 15x^{3/2} - 15x^{1/2}$, and $f''(5/3) > 0$, so there is a local minimum at $x = 5/3$.

4.3.89 $p'(t) = 6t^2 + 6t - 36 = 6(t+3)(t-2)$, which is 0 at $t = -3$ and $t = 2$. Note that $p''(t) = 12t + 6$, so $p''(-3) = -30 < 0$ and $p''(2) = 30 > 0$, so there is a local maximum at $t = -3$ and a local minimum at $t = 2$.

4.3.90 $f'(x) = x^3 - 5x^2 - 8x + 48 = (x-4)^2(x+3)$. (This can be obtained by using trial-and-error to determine that $x = 4$ is a root, and then using long division of polynomials to see that $f'(x) = (x-4)(x^2 - x - 12)$.) Note that $f''(x) = 3x^2 - 10x - 8$, so $f''(-3) = 49 > 0$ and $f''(4) = 0$. So there is a local minimum at $x = -3$, but the test is inconclusive for $x = 4$. The first derivative test shows that there is neither a maximum nor a minimum at $x = 4$.

4.3.91 $f'(x) = 4(x+a)^3$, which is 0 for $x = -a$. Note that $f''(x) = 12(x+a)^2$, which is 0 at $x = -a$, so the test is inconclusive. The first derivative test shows that there is a local minimum at $x = -a$.

4.3.92 $f'(x) = 3x^2 - 26x - 9 = (3x+1)(x-9)$. This is zero for $x = 9$ and $x = -\dfrac{1}{3}$, so those are the critical points. $f''(x) = 6x - 26$. $f''(9) > 0$ so there is a local minimum at $x = 9$, while $f''(-1/3) < 0$, so there is a local maximum at $x = -\dfrac{1}{3}$.

4.3.93 $f'(x) = 24x^3 \ln x^2 + 6x^4 \left(\frac{2}{x}\right) - 28x^3 = 24x^3 \ln x^2 + 12x^3 - 28x^3 = 24x^3 \ln x^2 - 16x^3 = 8x^3(3\ln x^2 - 2)$. Note that 0 is not in the domain of f, so the only values of x that makes f' 0 are those for which $\ln x^2 = \frac{2}{3}$, or $x^2 = e^{2/3}$, so $x = \pm e^{1/3}$.

$$f''(x) = 24x^2(3\ln x^2 - 2) + 8x^3\left(\frac{6}{x}\right) = 24x^2(3\ln x^2 - 2 + 2) = 72x^2 \ln x^2.$$

A check of f'' evaluated at either $\pm e^{1/3}$ shows a positive value, so there are local minimums at both $x = e^{1/3}$ and $x = -e^{1/3}$.

4.3.94 $f'(x) = -6x^{-4} + 2x^{-3} = \frac{-6}{x^4} + \frac{2x}{x^4} = \frac{2(x-3)}{x^4}$. $x = 3$ is the only critical point.

$$f''(x) = 24x^{-5} - 6x^{-4} = \frac{24}{x^5} - \frac{6x}{x^5} = \frac{6(4-x)}{x^5}.$$

We have $f''(3) > 0$, so there is a local minimum at $x = 3$.

4.3.95

a. True. $f'(x) > 0$ implies that f is increasing, and $f''(x) < 0$ implies that f' is decreasing. So f is increasing, but at a decreasing rate.

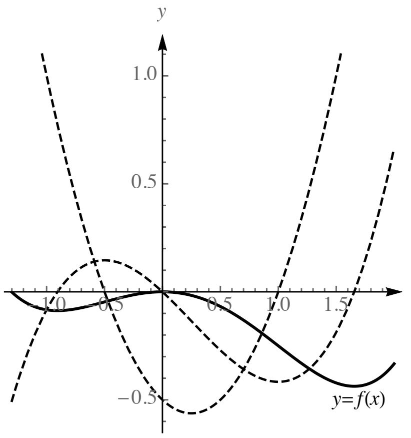
b. False. In fact, if $f'(c)$ exists and isn't zero, then there isn't any kind of local extrema at $x = c$.

c. True. In fact, if two functions differ by a constant, then all of their derivatives are the same.

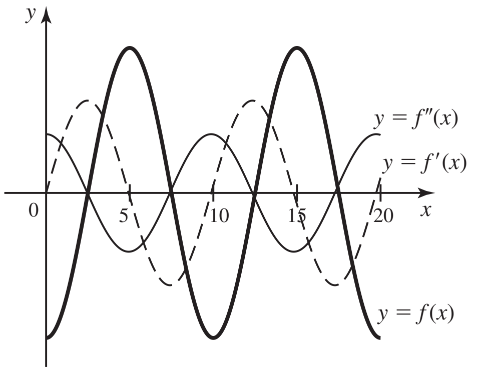
d. False. For example, consider $f(x) = x$ and $g(x) = x - 10$. Both are increasing, but $f(x)g(x) = x^2 - 10x$ is decreasing on $(-\infty, 5)$.

e. False. A continuous function with two local maxima must have a local minimum in between.

4.3.96

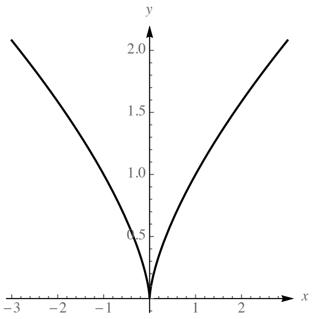


4.3.97

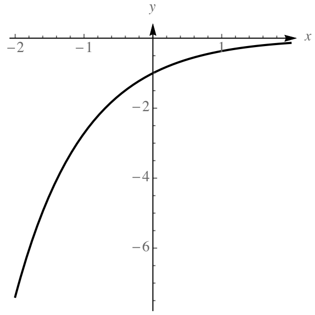


4.3.98

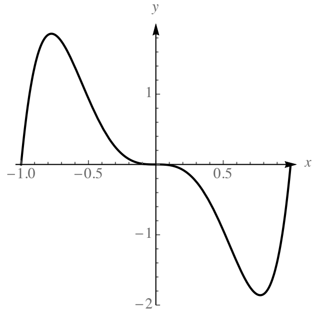
a. Not Possible. The closest thing would be a function like $f(x) = x^{2/3}$ which is positive and is concave down on $(-\infty, 0)$ and on $(0, \infty)$.



b. This is possible, for example $y = -e^{-x}$ has this property.



c. This is possible. See graphic pictured.



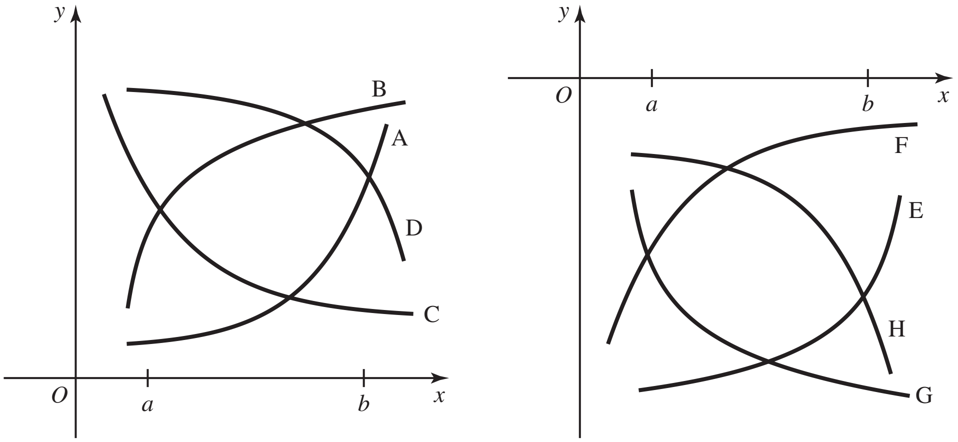
d. This is not possible. Between every pair of zeros, there must be a maximum or a minimum, so a continuous function with four zeros must have at least three local extrema.

4.3.99 The graphs match as follows: (a) – (f) – (g); (b) – (e) –(i); (c) – (d) –(h). Note that (a) is always increasing, so its derivative must be always positive, and (f) switches from decreasing to increasing at 0, so its derivative must be negative for $x < 0$ and positive for $x > 0$.

Note that (b) has three extrema where there are horizontal tangent lines, so its derivative must cross the x-axis three times, and (e) has two extrema, so its derivative must cross the x-axis two times.

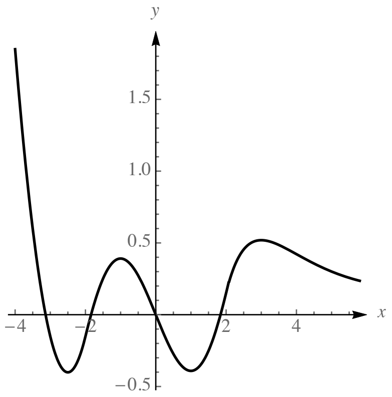
4.3.100 Note that C is increasing where B is positive, and is decreasing where B is negative, so it seems reasonable to assert that B is the derivative of C. Also, B is increasing where A is positive and decreasing where A is negative, so it is reasonable to assert that A is the derivative of B. So it appears that $C = f(x)$, $B = f'(x)$, and $A = f''(x)$.

4.3.101



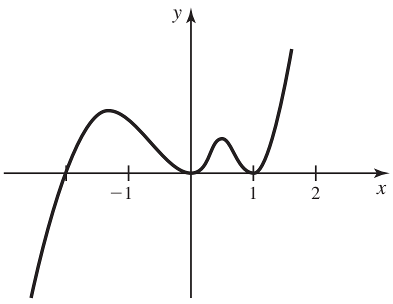
4.3.102

The graph sketched must be concave up on $(-\infty,-2)$ and on $(4,\infty)$, and must have a flat tangent line at $x=-1$, $x=1$, and $x=3$. A convenient way to ensure that $f''(-2)=f''(2)=0$ is to have inflection points occur there. The example to the right is only one possible such graph.



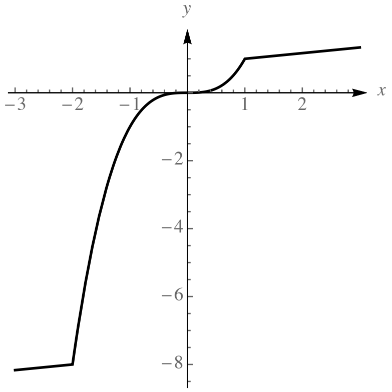
4.3.103

The graph sketched must have a flat tangent line at $x=-3/2$, $x=0$, and $x=1$, and must contain the points $(-2,0)$, $(0,0)$, and $(1,0)$. The example to the right is only one possible such graph.



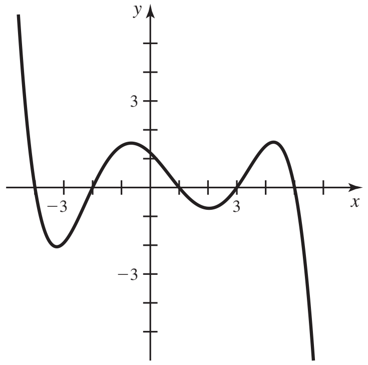
4.3.104

The graph sketched must be increasing everywhere and differentiable everywhere except at $x = -2$ and $x = 1$. We also have $f''(0) = 0$. The example to the right is only one possible such graph.

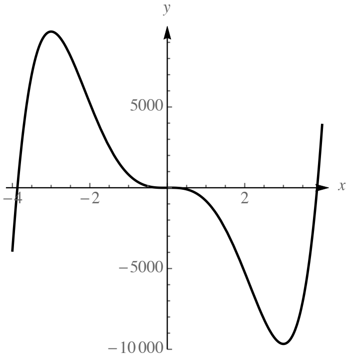


4.3.105

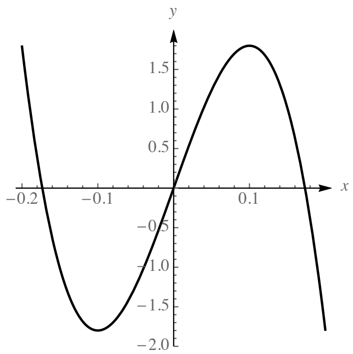
The graph sketched must be concave up on $(-\infty, -2)$ and on $(1, 3)$, and concave down on $(-2, 1)$ and on $(3, \infty)$. The example to the right is only one possible such graph.



4.3.106



It appears that there are 2 extreme values near $x = \pm 3$ and perhaps a flat tangent line corresponding to an inflection point at $(0, 0)$. However, note that $f'(x) = 300x^4 - 2703x^2 + 27 = 3(100x^4 - 901x^2 + 9) = 3(x^2 - 9)(100x^2 - 1) = 3(x - 3)(x + 3)(10x - 1)(10x + 1)$. It turns out that there are 4 critical points at $x = \pm 3$ and $x = \pm\frac{1}{10}$ (and there isn't one at $x = 0$). Below is a graph on the range $[-.2, .2]$, which clearly shows a maximum at $x = 0.1$ and a minimum at $x = -0.1$.



4.3.107

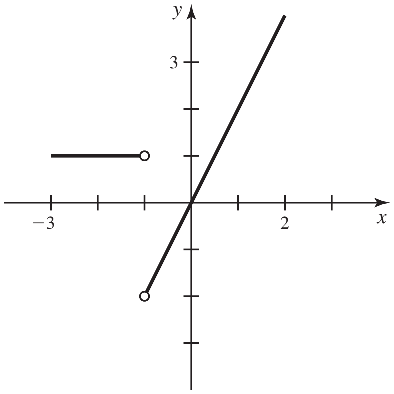
a. f is increasing on $(-2, 2)$. It is decreasing on $(-3, -2)$.

b. There are critical points of f at $x = -2$ and at $x = 0$. There is a local minimum at $x = -2$ and no extremum at $x = 0$.

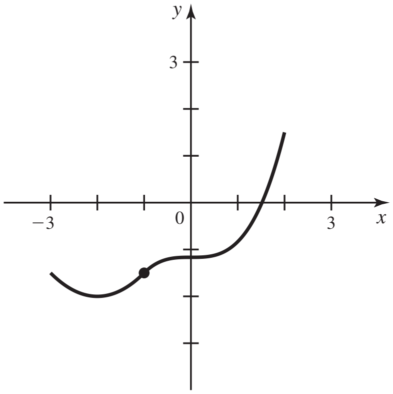
c. There are inflection points of f at $x = -1$ and at $x = 0$.

d. f is concave up on $(-3, -1)$ and on $(0, 2)$, while it is concave down on $(-1, 0)$.

e.



f.



4.3.108

$$r'(t) = \frac{(37.98e^{-2.2t}+1)^2 \cdot (-2.2) \cdot 10147.9e^{-2.2t} - 10147.9e^{-2.2t} 2(37.98e^{-2.2t}+1) \cdot (-2.2)(37.98)e^{-2.2t}}{(37.98e^{-2.2t}+1)^4}.$$

After some algebra, this can be written as

$$\frac{e^{-2.2t}(847,998.7358e^{-2.2t} - 22,325.39)}{(37.98e^{-2.2t}+1)^3}.$$

The numerator is 0 when

$$e^{-2.2t} = \frac{22,325.39}{847,993.7358}$$

or at about 1.7 weeks. Using the First Derivative Test it can be verfied that this value maximizes r. The owlet is growing at its fastest rate when it is 1.7 weeks old.

4.3.109

a. $E = \dfrac{dD}{dp} \cdot \dfrac{p}{D} = -10\dfrac{p}{500 - 10p} = \dfrac{p}{p - 50}.$

b. $E = \dfrac{12}{12-50} \cdot .045 = -1.42\%$.

c. If $D(p) = a - bp$, then $E(p) = -b \cdot \dfrac{p}{a-bp} = \dfrac{bp}{bp-a}$. So $E'(p) = \dfrac{(bp-a)b - bpb)}{(bp-a)^2} = -\dfrac{ab}{(bp-a)^2}$, which is less than 0 for $a, b > 0$ and $p \neq a/b$.

d. If $D(p) = \dfrac{a}{p^b}$, then $E(p) = -\dfrac{ab}{p^{b+1}} \cdot \dfrac{p}{a/p^b} = -b$.

4.3.110 The growth rate is given by the slope of the tangent line to the curve. Up to the inflection point, the curve is concave up, meaning that the slopes are increasing. After the inflection point, the curve is concave down, meaning that the slopes are decreasing. So the maximum slope occurs at the inflection point.

4.3.111

a. $\displaystyle\lim_{t\to\infty} \frac{300t^2}{t^2+30} \cdot \frac{1/t^2}{1/t^2} = \lim_{t\to\infty} \frac{300}{1+(30/t^2)} = 300.$

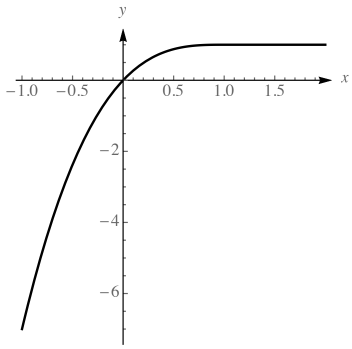
b. Note that $P'(t) = \frac{(t^2+30)(600t)-300t^2(2t)}{(t^2+30)^2} = \frac{18000t}{(t^2+30)^2}$. We want to maximize this, so we compute its derivative $P''(t) = \frac{(t^2+30)^2\cdot 18000 - 18000t\cdot 2(t^2+30)\cdot 2t}{(t^2+30)^4} = \frac{54000(10-t^2)}{(t^2+30)^3}$. This is 0 for $t = \sqrt{10}$, and an analysis of $P''(t)$ reveals that $P''(t) > 0$ for $t < \sqrt{10}$ and $P''(t) < 0$ for $t > \sqrt{10}$ so there is a local maximum for $P'(t)$ at $t = \sqrt{10}$.

c. Following the outline from the previous problem, we see that $P'(t) = \frac{2bKt}{(t^2+b)^2}$, and $P''(t) = \frac{2bK(b-3t^2)}{(t^2+b)^3}$. $P''(t)$ is 0 for $t = \sqrt{b/3}$, and the first derivative test reveals that this is a local maximum.

4.3.112 If f is concave up at $x = c$, then $f''(c) > 0$. This means that f' is increasing in a neighborhood near c. So for $x < c$, the slope of the function is less than the slope of the tangent line at c, and for $x > c$ it is greater than the slope of the tangent line. This means that the curve is "bending upward," away from its tangent line, so the tangent line is below the curve in a neighborhood near c.

4.3.113 $f'(x) = 4x^3 + 3ax^2 + 2bx + c$, and $f''(x) = 12x^2 + 6ax + 2b = 2(6x^2 + 3ax + b)$. Note that $f''(x) = 0$ exactly when $x = \frac{-3a\pm\sqrt{9a^2-24b}}{12}$. This represents no real solutions when $9a^2 - 24b < 0$, which occurs when $b > 3a^2/8$. When $b = 3a^2/8$, there is one root, but in this case the sign of f'' doesn't change at the double root $x = -a/4$, so there are no inflection points for f. In the case $b < 3a^2/8$, there are two roots of f'', both of which yield inflection points of f, as can be seen by the change in sign of f'' at its two roots.

4.3.114 One possible such function is $f(x) = \begin{cases} (x-1)^3 + 1 & \text{if } x \le 1 \\ 1 & \text{if } x > 1. \end{cases}$



Note that the derivative of f to the left of 1 is positive, and to the right it is 0 (so it is neither positive nor negative). This example highlights the fact that the statement of the First Derivative Test does not include all possible cases for what might be happening on either side of the critical point. The missing cases include:

- f' changes from negative to 0 as x increases through c (in which case f has a local minimum at c).
- f' changes from positive to 0 as x increases through c (in which case f has a local maximum at c).
- f' changes from 0 to positive as x increases through c (in which case f has a local maximum at c).
- f' changes from 0 to negative as x increases through c (in which case f has a local minimum at c).
- f' remains 0 as x increases through c (in which case f has both a local minimum and local maximum at c – f is constant near c.

4.3.115

a. $f'(x) = 3x^2 + 2ax + b$, which is 0 when $x = \dfrac{-2a \pm \sqrt{4a^2 - 12b}}{6} = \dfrac{-a \pm \sqrt{a^2 - 3b}}{3}$. These solutions represent distinct real numbers when $a^2 > 3b$. Let the two distinct roots be $r_1 < r_2$. Note that f' is negative on the interval (r_1, r_2) and positive on $(-\infty, r_1)$ and on (r_2, ∞) so there is a maximum at r_1 and a minimum at r_2.

b. If $a^2 < 3b$, then there are no real critical points, so there are no extreme values.

4.3.116 $f'(x) = 2ax + b$, and $f''(x) = 2a$. Note that $f''(x)$ is positive for $a > 0$ and negative for $a < 0$. So f is concave up for $a > 0$ and concave down for $a < 0$.

4.4 Graphing Functions

4.4.1 Because the intervals of increase and decrease and the intervals of concavity must be subsets of the domain, it is helpful to know what the domain is at the outset.

4.4.2 If a function is symmetric, then only one half the function needs to be graphed, and the information about the other half will follow immediately. Also, knowledge about symmetry can help catch mistakes.

4.4.3 No. Polynomials are continuous everywhere, so they have no vertical asymptotes. Also, polynomials in x always tend to $\pm\infty$ as $x \to \pm\infty$.

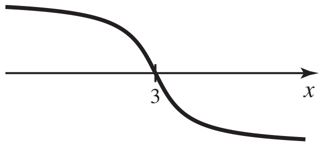
4.4.4 If a rational function is in simplified form (with no common factors in the numerator and denominator), then there is a vertical asymptote wherever the denominator is zero.

4.4.5 The maximum and minimum must occur at either an endpoint or a critical point. So to find the absolute maximum and minimum, it suffices to find all the critical points, and then compare the values of the function at those points and at the endpoints. The largest such value is the maximum and the smallest is the minimum.

4.4.6 For every polynomial $p(x)$, $\lim\limits_{x\to\pm\infty} p(x) = \pm\infty$.

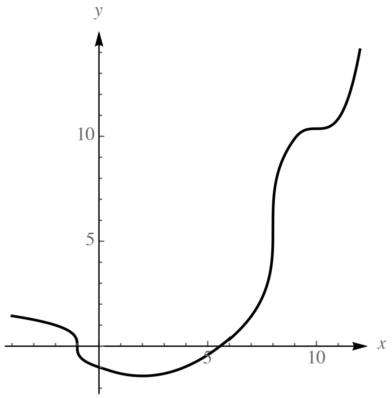
4.4.7

The function sketched should be decreasing and concave down for $x < 3$ and decreasing and concave up for $x > 3$.

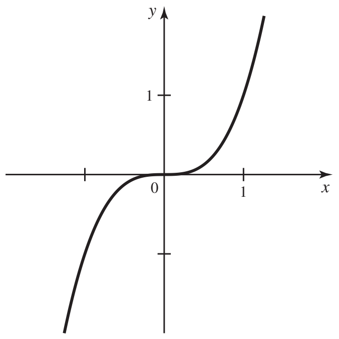


4.4.8

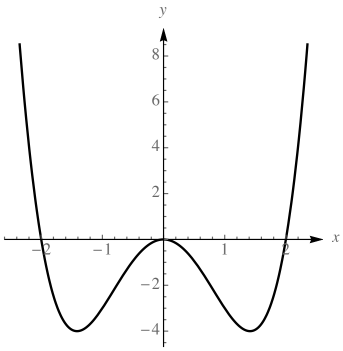
The function sketched should be decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$. It should be concave down on $(-\infty, -1)$ and on $(8, 10)$. It should be concave up on $(-1, 8)$ and on $(10, \infty)$.



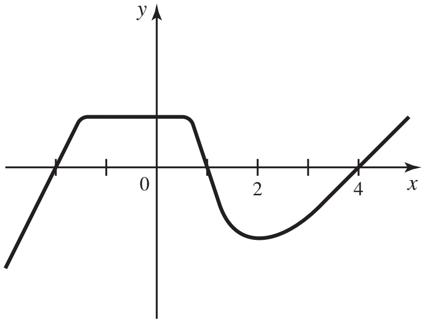
4.4.9



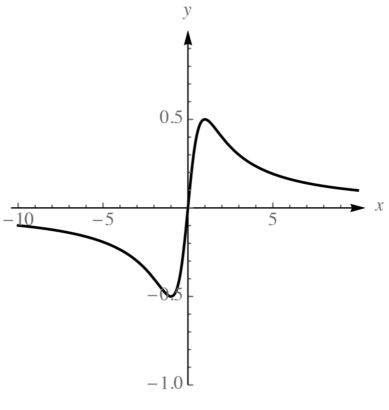
4.4.10



4.4.11



4.4.12



4.4.13

a. $f'(x) = 1 \cdot (x+3)^2 + (x-3)(2)(x+3) = (x+3)(x+3+2x-6) = (x+3)(3x-3) = 3(x-1)(x+3)$. Also, $f''(x) = 3(x+3) + 3(x-1) = 3x+9+3x-3 = 6x+6 = 6(x+1)$.

b. $f'(x) = 0$ for $x = 1$ and $x = -3$ (so those are the critical points), while $f''(x) = 0$ for $x = -1$, so that is the location of a potential inflection point.

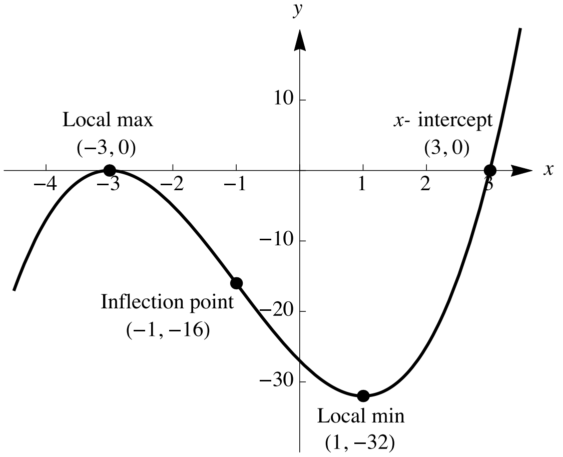
c. $f' > 0$ on $(-\infty, -3)$ and $(1, \infty)$, so f is increasing there. $f' < 0$ on $(-3, 1)$ so f is decreasing there.

d. $f'' > 0$ on $(-1, \infty)$ so f is concave up there; $f'' < 0$ on $(-\infty, -1)$, so f is concave down there.

e. f has a local max of $f(-3) = 0$ at $x = -3$ and a local minimum of $f(1) = -32$ at $x = 1$. The point $(-1, -16)$ is an inflection point.

f. The y-intercept is $f(0) = -27$. The x-intercepts are $x = 3, -3$.

g.



4.4.14

a. $f'(x) = 0$ for $x = 0$ (so that is the only critical point), while $f''(x = 0$ for $x = -1$, and $x = 1$, so those are the locations of potential inflection points.

b. $f' > 0$ on $(-\infty, 0)$, so f is increasing there. $f' < 0$ on $(0, \infty)$, so f is decreasing there.

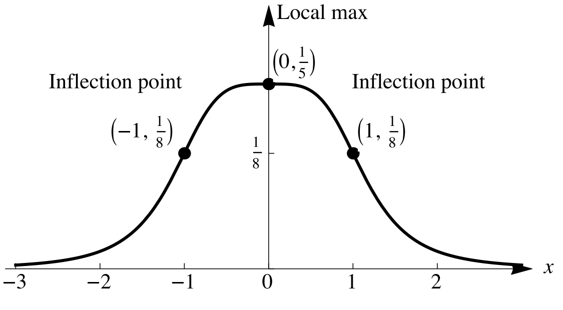
c. $f'' > 0$ on $(-\infty, -1)$ and $(1, \infty)$, so f is concave up there; $f'' < 0$ on $(-1, 1)$, so f is concave down there.

d. f has a local max of $f(0) = 1/5$ at $x = 0$. The points $(-1, 1/8)$ and $(1, 1/8)$ are inflection points.

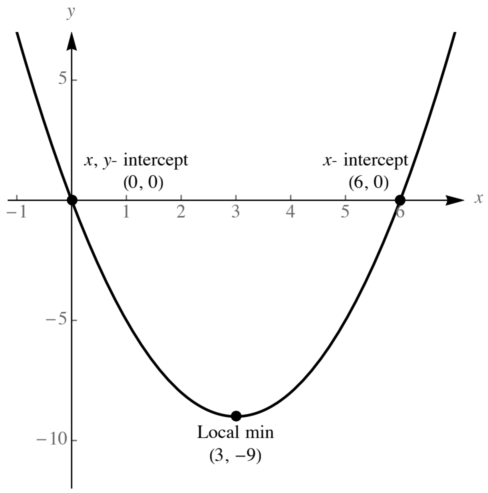
e. The y-intercept is $f(0) = 1/5$. There are no x-intercepts.

f. There are no vertical asymptotes. There s a horizontal asymptote of $y = 0$ because $\lim_{x\to\pm\infty} \dfrac{1}{3x^4+5} = 0$.

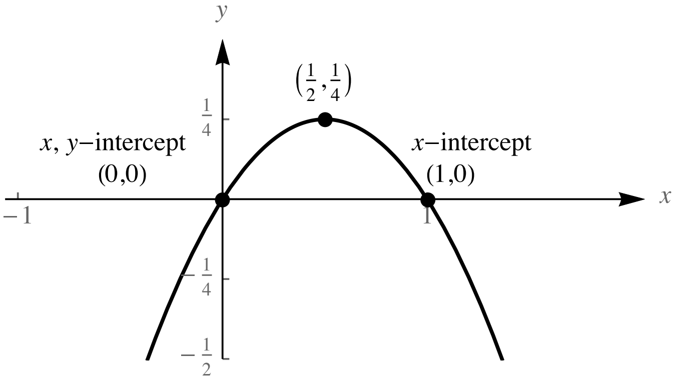
g.



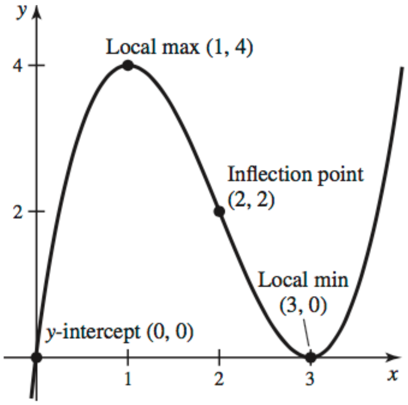
4.4.15 Because f is a polynomial, its domain is $(-\infty, \infty)$. There is neither even nor odd symmetry because $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. $f'(x) = 2x - 6 = 2(x - 3)$ which is zero for $x = 3$. We have $f' > 0$ on $(3, \infty)$ so f is increasing there, while $f' < 0$ on $(-\infty, 3)$, so f is decreasing there, and there is a local minimum at $x = 3$. $f''(x) = 2$ which is always positive, so f is concave up on $(-\infty, \infty)$ and there are no inflection points. The local minimum value is $f(3) = -9$, the y-intercept is $(0, 0)$, which is also an x-intercept, and $(6, 0)$ is also an x-intercept.



4.4.16 Because f is a polynomial, its domain is $(-\infty, \infty)$. There is neither even nor odd symmetry because $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. $f'(x) = 1 - 2x$ which is zero for $x = 1/2$. We have $f' > 0$ on $(-\infty, 1/2)$ so f is increasing there, while $f' < 0$ on $(1/2, -\infty)$, so f is decreasing there, and there is a local maximum at $x = 1/2$. $f''(x) = -2$ which is always negative, so f is concave down on $(-\infty, \infty)$ and there are no inflection points. The local maximum value is $f(1/2) = 1/4$, the y-intercept is $(0, 0)$, which is also an x-intercept, and $(1, 0)$ is also an x-intercept.

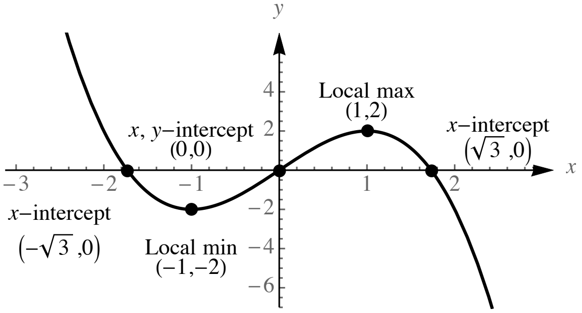


4.4.17 The domain of f is $(-\infty, \infty)$, and there is no symmetry. The y intercept is $f(0) = 0$, and the x-intercepts are 0 and 3 because $f(x) = x(x-3)^2$. $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-3)(x-1)$. This is zero when $x = 1$ and $x = 3$. Note that $f'(0) > 0$, $f'(2) < 0$ and $f'(4) > 0$, so f is increasing on $(-\infty, 1)$ and on $(3, \infty)$. It is decreasing on $(1, 3)$. Note that $f''(x) = 6x - 12$ which is zero at $x = 2$. Because $f''(1) < 0$ and $f''(3) > 0$, we conclude that f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$. There is an inflection point at $(2, 2)$, a local maximum at $(1, 4)$ and a local minimum at $(3, 0)$.

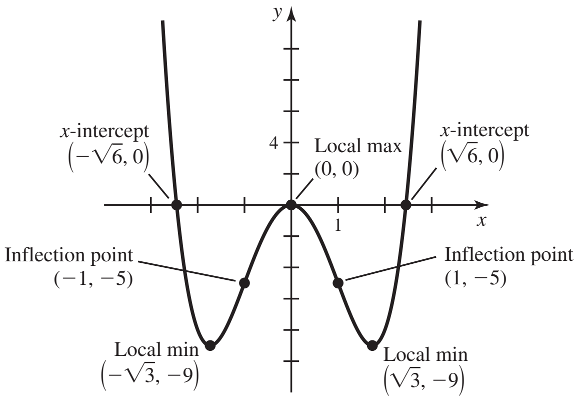


4.4.18 The domain of f is $(-\infty, \infty)$, and there is odd symmetry, because $f(-x) = 3(-x) - (-x)^3 = -(3x - x^3) = -f(x)$. The y intercept is $f(0) = 0$, and the x-intercepts are 0 and $\pm\sqrt{3}$ because $f(x) = x(3 - x^2)$. $f'(x) = 3 - 3x^2 = 3(1 - x^2)$, which is zero for $x = \pm 1$. Note that $f'(-2) < 0$, $f'(0) > 0$, and $f'(2) < 0$, so f is decreasing on $(-\infty, -1)$ and on $(1, \infty)$, and is increasing on $(-1, 1)$. There is a local minimum at $(-1, -2)$ and a local maximum at $(1, 2)$.

$f''(x) = -6x$, which is zero at $x = 0$. Note that $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$, so f is concave up on $(-\infty, 0)$ and is concave down on $(0, \infty)$. There is an inflection point at $(0, 0)$.

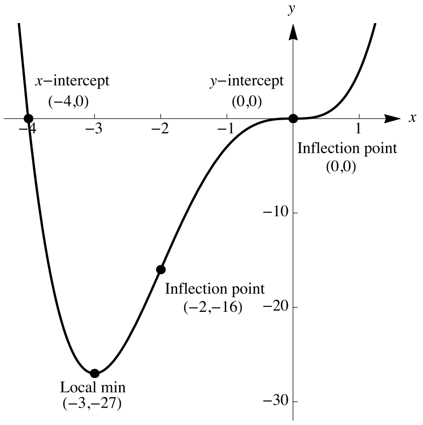


4.4.19 The domain of f is $(-\infty, \infty)$, and there is even symmetry, because $f(-x) = f(x)$. $f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$. This is 0 when $x = \pm\sqrt{3}$ and when $x = 0$. $f''(x) = 12x^2 - 12 = 12(x^2 - 1)$, which is 0 when $x = \pm 1$. Note that $f'(-2) < 0$, $f'(-1) > 0$, $f'(1) < 0$, and $f'(2) > 0$. So f is decreasing on $(-\infty, -\sqrt{3})$ and on $(0, \sqrt{3})$. It is increasing on $(-\sqrt{3}, 0)$ and on $(\sqrt{3}, \infty)$. There is a local maximum of 0 at $x = 0$ and local minima of -9 at $x = \pm\sqrt{3}$. Note also that $f''(x) > 0$ for $x < -1$ and for $x > 1$ and $f''(x) < 0$ for $-1 < x < 1$, so there are inflection points at $x = \pm 1$. Also, f is concave down on $(-1, 1)$ and concave up on $(-\infty, -1)$ and on $(1, \infty)$. There is a y-intercept at $f(0) = 0$ and x-intercepts where $f(x) = x^4 - 6x^2 = x^2(x^2 - 6) = 0$, which is at $x = \pm\sqrt{6}$ and $x = 0$.



4.4.20 The domain of f is $(-\infty, \infty)$, and there is no symmetry. $f'(x) = 4x^3 + 12x^2 = 4x^2(x+3)$. This is zero for $x = 0$ and $x = -3$. $f' > 0$ on $(-3, 0)$ and on $(0, \infty)$ so f is increasing there, while $f' < 0$ on $(-\infty, -3)$, so f is decreasing there. There is a local minimum at $x = -3$ with y-value $f(-3) = -27$.

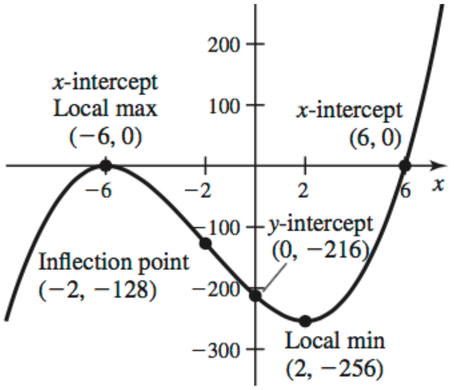
$f''(x) = 12x^2 + 24x = 12x(x+2)$ which is zero for $x = 0$ and $x = -2$. $f'' > 0$ on $(-\infty, -2)$ and on $(0, \infty)$, so f is concave up there, while $f'' < 0$ on $(-2, 0)$, so f is concave down there. There are inflection points at $(-2, -16)$ and $(0, 0)$. In addition to the x- and y-intercept at $(0, 0)$ there is an x-intercept at $x = -4$.



4.4.21 The domain of f is $(-\infty, \infty)$, and there is no symmetry. The y- intercept is $f(0) = -216$. The x-intercepts are 6 and -6.

$f'(x) = (x+6)^2 + (x-6)2(x+6) = (x+6)(x+6+2x-12) = (x+6)(3x-6) = 3(x+6)(x-2)$. The critical numbers are -6 and 2. Note that $f'(-7) > 0$, $f'(-2) < 0$, and $f'(3) > 0$, so f is increasing on $(-\infty, -6)$ and on $(2, \infty)$. It is decreasing on $(-6, 2)$. There is a local maximum of 0 at -6 and a local minimum of -256 at $x = 2$.

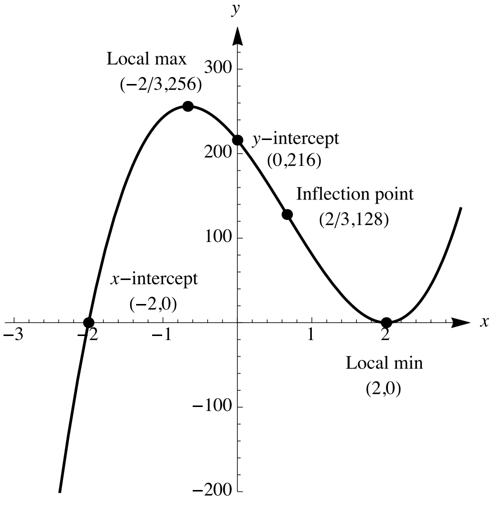
$f''(x) = 3(x-2) + 3(x+6) = 3(x-2+x+6) = 3(2x+4) = 6(x+2)$, which is zero for $x = -2$. Note that $f''(x) < 0$ for $x < -2$ and $f''(x) > 0$ for $x > -2$, so f is concave down on $(-\infty, -2)$ and concave up on $(-2, \infty)$. The point $(-2, -128)$ is an inflection point.



4.4.22 The domain of f is $(-\infty, \infty)$ and there is no symmetry. The y- intercept is $f(0) = 216$ and the x-intercepts are ± 2.

$f'(x) = 27 \cdot 2(x-2)(x+2) + 27(x-2)^2 = 27(x-2)(2x+4+x-2) = 27(x-2)(3x+2)$. This is zero for $x = 2$ and $x = -2/3$. Note that $f'(-1) > 0$, $f'(0) < 0$, and $f'(3) > 0$. Thus f is increasing on $(-\infty, -2/3)$ and on $(2, \infty)$, and is decreasing on $(-2/3, 2)$. There is a local maximum of 256 at $x = -2/3$ and a local minimum of 0 at $x = 2$.

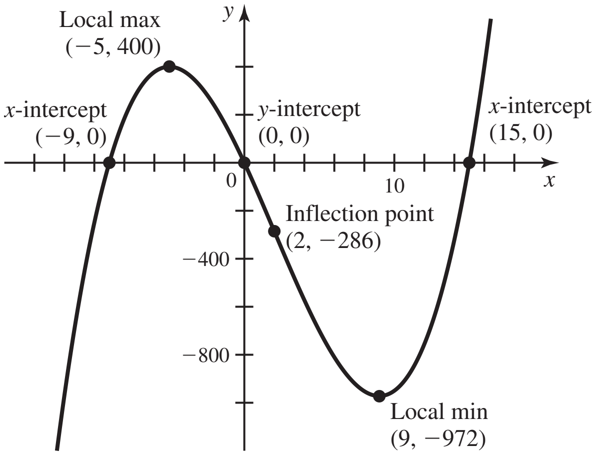
$f''(x) = 27(3x+2) + 27(x-2) \cdot 3 = 27(3x+2+3x-6) = 27(6x-4) = 54(3x-2)$. This is zero when $x = 2/3$. Note that $f''(x) < 0$ for $x < 2/3$ and $f''(x) > 0$ for $x > 2/3$, so f is concave down on $(-\infty, 2/3)$ and concave up on $(2/3, \infty)$. There is an inflection point at $(2/3, 128)$.



4.4.23 The domain of f is $(-\infty,\infty)$ and there is no symmetry. There are no asymptotes because f is a polynomial.

$f'(x) = 3x^2 - 12x - 135 = 3(x-9)(x+5)$, which is 0 for $x = 9$ and $x = -5$. $f'(x) > 0$ on $(-\infty,-5)$ and on $(9,\infty)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-5,9)$, so f is decreasing on that interval. There is a local maximum at $x = -5$ and a local minimum at $x = 9$.

$f''(x) = 6x-12$, which is 0 for $x = 2$. $f''(x) > 0$ on $(2,\infty)$, so f is concave up on that interval. $f''(x) < 0$ on $(-\infty,2)$, so f is concave down on that interval. There is a point of inflection at $x = 2$. The y-intercept is 0 and the x-intercepts are -9 and 15.



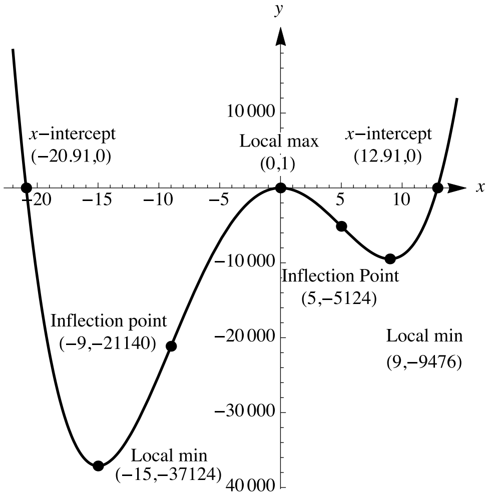
4.4.24 The domain of f is $(-\infty,\infty)$ and there is no symmetry. There are no asymptotes because f is a polynomial.

$f'(x) = 4x^3 + 24x^2 - 540x = 4x(x-9)(x+15)$, which is 0 for $x = 9$, $x = -15$, and $x = 0$.

$f'(x) > 0$ on $(-15,0)$ and on $(9,\infty)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-\infty,-15)$ and on $(0,9)$, so f is decreasing on those intervals. There is a local maximum at $x = 0$ and local minima at $x = -15$ and $x = 9$.

$f''(x) = 12x^2 + 48x - 540 = 12(x-5)(x+9)$, which is 0 for $x = 5$ and $x = -9$.

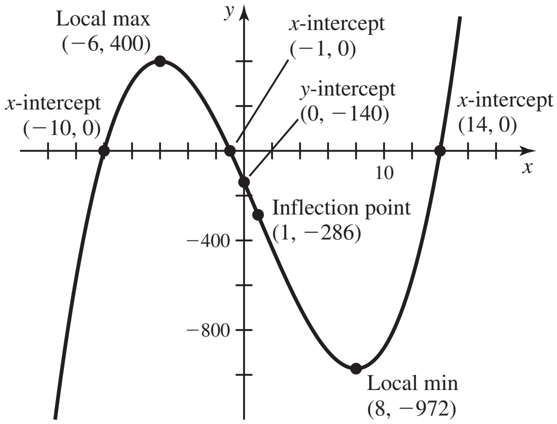
$f''(x) > 0$ on $(-\infty,-9)$ and on $(5,\infty)$, so f is concave up on those intervals. $f''(x) < 0$ on $(-9,5)$ so f is concave down on that interval. There are points of inflection at $x = -9$ and $x = 5$. The y-intercept is 1 and the x-intercepts are ≈ -20.912 and ≈ 12.911. There are also x-intercepts at approximately ± 0.061.



4.4.25 The domain of f is $(-\infty, \infty)$ and there is no symmetry. There are no asymptotes because f is a polynomial.

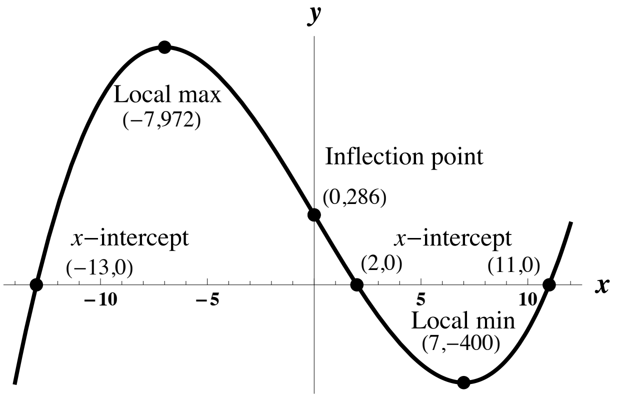
$f'(x) = 3x^2 - 6x - 144 = 3(x+6)(x-8)$, which is 0 for $x = -6$ and $x = 8$. $f'(x) > 0$ on $(-\infty, -6)$ and on $(8, \infty)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-6, 8)$, so f is decreasing on that interval. There is a local maximum at $x = -6$ and a local minimum at $x = 8$.

$f''(x) = 6x - 6$, which is 0 for $x = 1$. $f''(x) > 0$ on $(1, \infty)$, so f is concave up on that interval. $f''(x) < 0$ on $(-\infty, 1)$, so f is concave down on that interval. There is a point of inflection at $x = 1$. The y-intercept is -140 and the x-intercepts are at -10, -1, and 14.



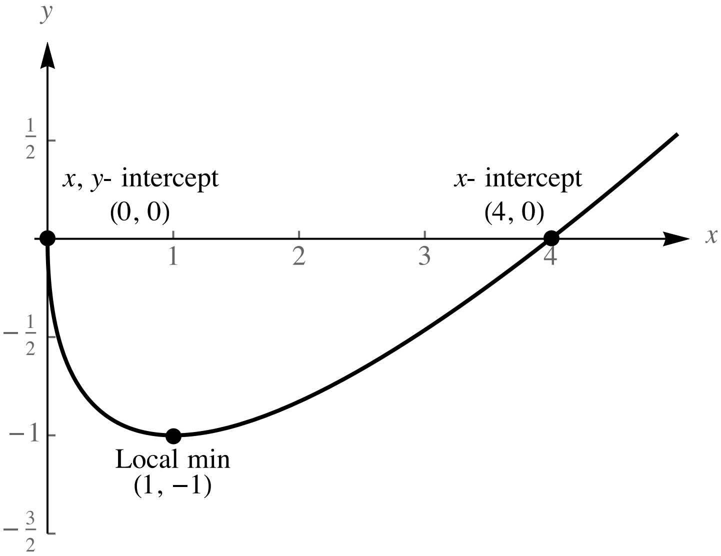
4.4.26 The domain of f is $(-\infty, \infty)$ and there is no symmetry. There are no asymptotes because f is a polynomial. $f'(x) = 3x^2 - 147 = 3(x+7)(x-7)$, which is 0 for $x = 7$ and $x = -7$. $f'(x) > 0$ on $(-\infty, -7)$ and on $(7, \infty)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-7, 7)$, so f is decreasing on that interval. There is a local maximum at $x = -7$ and a local minimum at $x = 7$.

$f''(x) = 6x$, which is 0 for $x = 0$. $f''(x) > 0$ on $(0, \infty)$, so f is concave up on that interval. $f''(x) < 0$ on $(-\infty, 0)$, so f is concave down on that interval. There is a point of inflection at $x = 0$. The y-intercept is 286 and the x-intercepts are at -13, 2, and 11.



4.4.27 The domain of f is $[0,\infty)$ and there is no symmetry. $f'(x) = 1 - \frac{1}{\sqrt{x}} = \frac{\sqrt{x}-1}{\sqrt{x}}$ which is zero for $x = 1$. $f' > 0$ on $(1,\infty)$, so f is increasing there, while $f' < 0$ on $(0,1)$, so f is decreasing there. There is a local minimum of -1 at $x = 1$.

If we write $f'(x)$ as $f'(x) = 1 - x^{-1/2}$, then we see that $f''(x) = \frac{1}{2}x^{-3/2} = \frac{1}{2x^{3/2}}$. This is never zero, and is always positive. So f is concave up and has no inflection points. The x- and y-intercept is $(0,0)$ and there is another x-intercept at $(4,0)$.

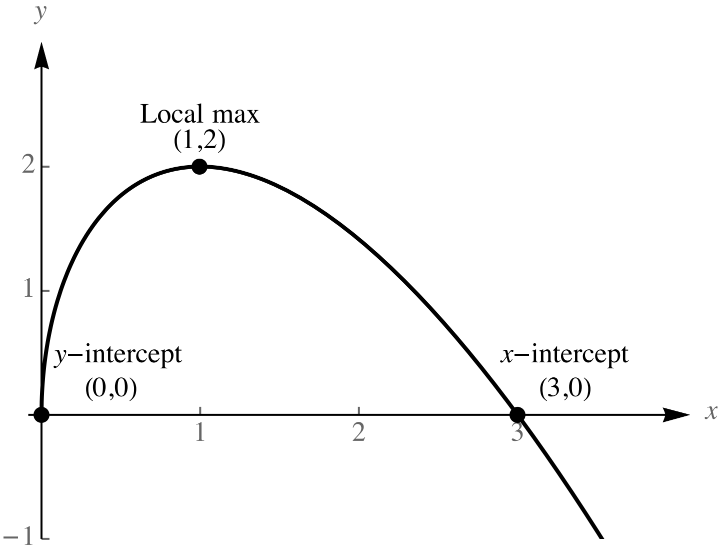


4.4.28 The domain of f is $[0,\infty)$ and there is no symmetry. $f'(x) = \frac{3}{2\sqrt{x}} - \frac{3\sqrt{x}}{2} = \frac{3(1-x)}{2\sqrt{x}}$ which is zero for $x = 1$. $f' < 0$ on $(1,\infty)$, so f is decreasing there, while $f' > 0$ on $(0,1)$, so f is increasing there. There is a local maximum of 2 at $x = 1$.

If we write $f'(x)$ as $f'(x) = \frac{3}{2}x^{-1/2} - \frac{3}{2}x^{1/2}$, then we see that

$$f''(x) = -\frac{3}{4}x^{-3/2} - \frac{3}{4}x^{-1/2} = -\frac{3}{4}\left(\frac{1}{x^{3/2}} + \frac{1}{x^{1/2}}\right) = -\frac{3}{4}\left(\frac{x+1}{x^{3/2}}\right),$$

which is never zero on the domain of f, and is always negative. So f is always concave down and has no inflection points. The x- and y- intercept is $(0,0)$, and another x-intercept is $(3,0)$.



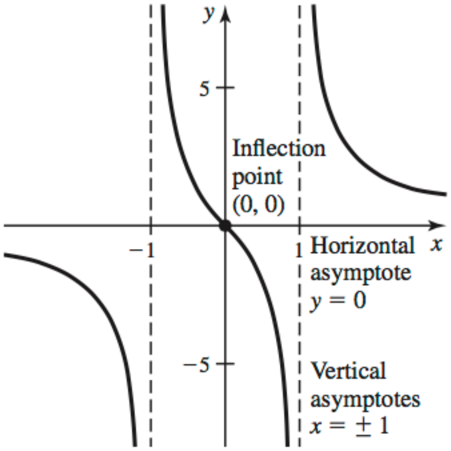
4.4.29 The domain of f is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$, and there is odd symmetry, because $f(-x) = \dfrac{3(-x)}{(-x)^2-1} = -\dfrac{3x}{x^2-1} = -f(x)$. The only intercept is $(0,0)$ which is both the y- and x-intercept.

Note that $\lim\limits_{x\to-1^+} f(x) = \infty$ and $\lim\limits_{x\to-1^-} f(x) = -\infty$, so there is a vertical asymptote at $x=-1$. Also, $\lim\limits_{x\to1^+} f(x) = \infty$ and $\lim\limits_{x\to1^-} f(x) = -\infty$, so there is a vertical asymptote at $x = 1$

Note that $\lim\limits_{x\to\pm\infty} \dfrac{3x}{x^2-1}\cdot\dfrac{1/x^2}{1/x^2} = \lim\limits_{x\to\pm\infty} \dfrac{3/x}{1-(1/x^2)} = 0$, so $y = 0$ is a horizontal asymptote.

$f'(x) = \dfrac{(x^2-1)\cdot 3 - 3x\cdot 2x}{(x^2-1)^2} = \dfrac{-3x^2-3}{(x^2-1)^2} = -\dfrac{3(x^2+1)}{(x^2-1)^2}$. This is never 0, and is in fact negative wherever it is defined. Thus, f is decreasing on $(-\infty,-1)$, on $(-1,1)$, and on $(1,\infty)$. There are no extrema.

$f''(x) = \dfrac{(x^2-1)^2(-6x)+3(x^2+1)\cdot 2\cdot(x^2-1)\cdot 2x}{(x^2-1)^4} = \dfrac{-6x^3+6x+12x^3+12x}{(x^2-1)^3} = \dfrac{6x(x^2+3)}{(x^2-1)^3}$. This is 0 for $x = 0$. The point $(0,0)$ is an point of inflection, because it is an interior point on the domain, and the second derivative changes from positive to negative there. The other concavity changes take place at the asymptotes. Note that f is concave down on $(-\infty,-1)$ and on $(0,1)$, and concave up on $(-1,0)$ and on $(1,\infty)$.

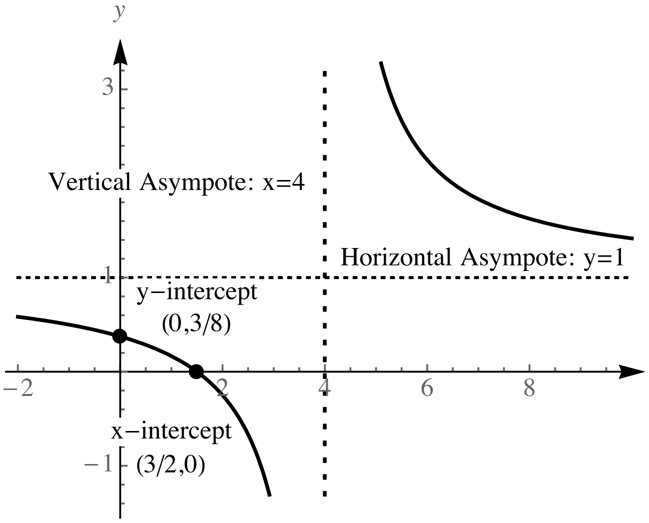


4.4.30 The domain of f is $(-\infty,4)\cup(4,\infty)$, and there is no symmetry.

Because $\lim_{x\to\pm\infty} \frac{2x-3}{2x-8}\cdot\frac{1/x}{1/x} = \lim_{x\to\pm\infty}\frac{2-(3/x)}{2-(8/x)} = \frac{2}{2} = 1$, there is a horizontal asymptote of $y = 1$. Also, because $\lim_{x\to 4^-} f(x) = -\infty$ and $\lim_{x\to 4^+} f(x) = \infty$, there is a vertical asymptote at $x = 4$. $f'(x) = \frac{(2x-8)\cdot 2-(2x-3)\cdot 2}{(2x-8)^2} = -\frac{10}{(2x-8)^2}$. This is never 0. $f''(x) = \frac{40}{(2x-8)^3}$ which is also never 0.

Note that $f'(x) < 0$ on $(-\infty, 4)$ and on $(4, \infty)$. So f is decreasing on $(-\infty, 4)$ and on $(4, \infty)$. There are no extrema.

Note also that $f''(x) < 0$ for $x < 4$, and $f''(x) > 0$ for $x > 4$, so f is concave down on $(-\infty, 4)$ and is concave up on $(4, \infty)$. There are no inflection points because the only change in concavity occurs at the vertical asymptote. The x-intercept is $x = 3/2$ and the y-intercept is $f(0) = 3/8$.



4.4.31 The domain of f is $(-\infty, 2) \cup (2, \infty)$, and there is no symmetry. Note that $\lim_{x\to 2^+} f(x) = \infty$ and $\lim_{x\to 2^-} f(x) = -\infty$, so there is a vertical asymptote at $x = 2$. There isn't a horizontal asymptote, because $\lim_{x\to\pm\infty} f(x) = \pm\infty$.

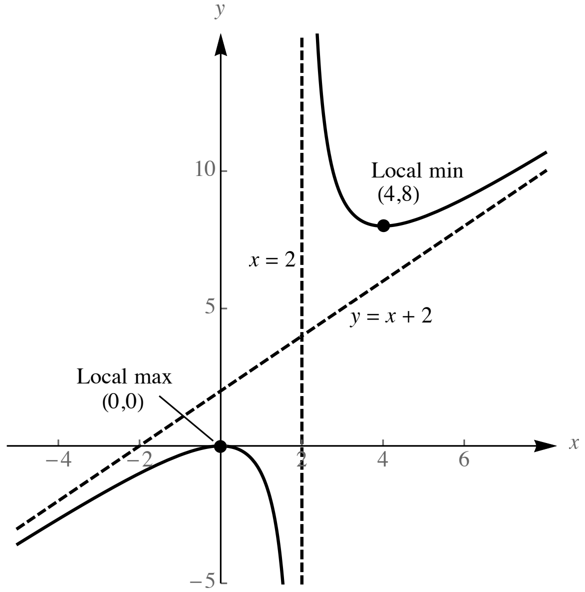
$f'(x) = \frac{(x-2)\cdot 2x - x^2}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}$. This is 0 when $x = 4$ and when $x = 0$.

$f''(x) = \frac{(x-2)^2(2x-4)-(x^2-4x)\cdot 2\cdot(x-2)}{(x-2)^4} = \frac{8}{(x-2)^3}$. This is never 0.

Note that $f'(-1) > 0$, $f'(1) < 0$, $f'(3) < 0$ and $f'(5) > 0$. So f is decreasing on $(0, 2)$ and on $(2, 4)$. It is increasing on $(-\infty, 0)$ and on $(4, \infty)$. There is a local maximum of 0 at $x = 0$ and a local minimum of 8 at $x = 4$.

Note that $f''(x) > 0$ for $x > 2$ and $f''(x) < 0$ for $x < 2$, So f is concave up on $(2, \infty)$ and concave down on $(-\infty, 4)$. There are no inflection points, because the only change in concavity occurs at a vertical asymptote. The only intercept is $(0, 0)$.

By long division, we can write f as $x + 2 + \frac{4}{x-2}$. Therefore the line $y = x + 2$ is a slant asymptote.



4.4.32 The domain of f is $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$, and there is even symmetry because $f(-x) = \dfrac{(-x)^2}{(-x)^2-4} = \dfrac{x^2}{x^2-4} = f(x)$.

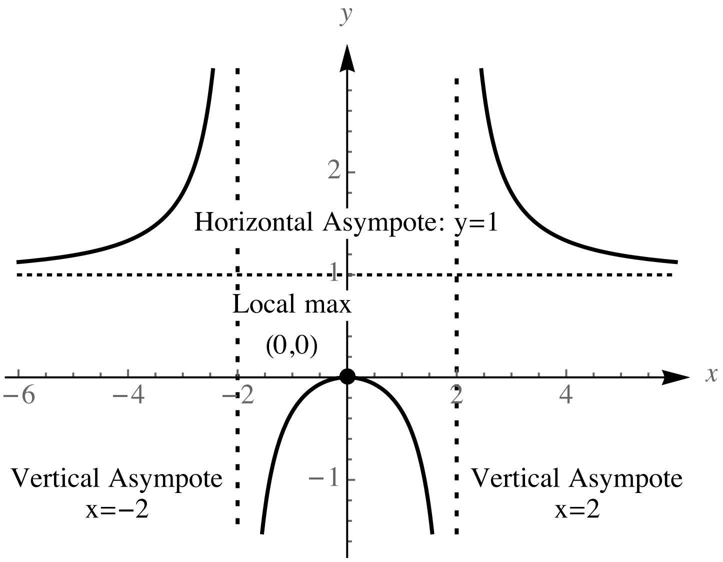
Because $\lim\limits_{x\to\pm\infty} \dfrac{x^2}{x^2-4}\cdot\dfrac{1/x^2}{1/x^2} = \lim\limits_{x\to\pm\infty} \dfrac{1}{1-(4/x^2)} = 1$, there is a horizontal asymptote at $y=1$. Also, because $\lim_{x\to-2^-} f(x) = \infty$, $\lim_{x\to-2^+} f(x) = -\infty$, $\lim_{x\to2^-} f(x) = -\infty$ and $\lim_{x\to2^+} f(x) = \infty$, there are vertical asymptotes at $x=-2$ and $x=2$.

$f'(x) = \dfrac{(x^2-4)\cdot 2x - x^2\cdot 2x}{(x^2-4)^2} = -\dfrac{8x}{(x^2-4)^2}$. This is 0 when $x=0$.

$f''(x) = \dfrac{(x^2-4)^2(-8)-(-8x)\cdot 2\cdot(x^2-4)\cdot 2x}{(x^2-4)^4} = \dfrac{8(3x^2+4)}{(x^2-4)^3}$, which is never 0.

Note that $f'(x) > 0$ on $(-\infty,-2)$ and on $(-2,0)$, while $f'(x) < 0$ on $(0,2)$ and on $(2,\infty)$. So f is increasing on $(-\infty,-2)$ and on $(0,2)$, and is decreasing on $(0,2)$ and on $(2,\infty)$. There is a local maximum of 0 at $x=0$.

Note also that $f''(x) > 0$ for $x < -2$, and $f''(x) > 0$ for $x > 2$, while $f''(x) < 0$ for $-2 < x < 2$. So f is concave up on $(-\infty,-2)$ and on $(2,\infty)$, while it is concave down on $(-2,2)$. There are no inflection points because the only changes in concavity occur at asymptotes.



4.4.33 The domain of f is $(-\infty,-1/2)\cup(-1/2,\infty)$, and there is no symmetry.

Because $\lim_{x\to\pm\infty}\frac{x^2+12}{2x+1}\cdot\frac{1/x}{1/x}=\lim_{x\to\pm\infty}\frac{x+(12/x)}{2+(1/x)}=\pm\infty$, there is no horizontal asymptote. However, there is a slant asymptote of $y=\frac{x}{2}-\frac{1}{4}$, because we can write f as $f(x)=\frac{x}{2}-\frac{1}{4}+\frac{49/4}{2x+1}$ by long division.

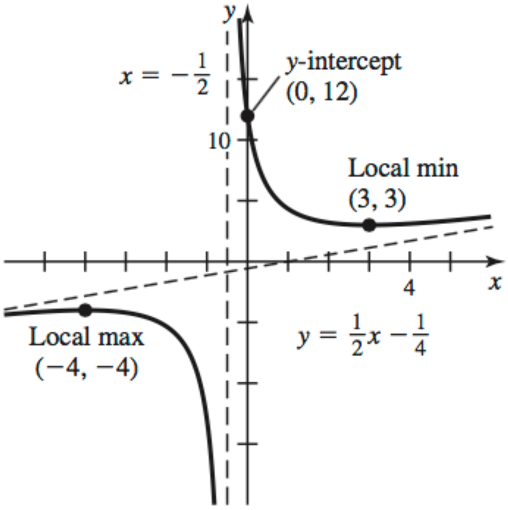
Also, because $\lim_{x\to(-1/2)^-}f(x)=-\infty$ and $\lim_{x\to(-1/2)^+}f(x)=\infty$, there is a vertical asymptote at $x=-1/2$.

$f'(x)=\frac{(2x+1)\cdot 2x-(x^2+12)\cdot 2}{(2x+1)^2}=\frac{2x^2+2x-24}{(2x+1)^2}=\frac{2(x+4)(x-3)}{(2x+1)^2}$. This is 0 for $x=-4$ and $x=3$.

$f''(x)=\frac{(2x+1)^2(4x+2)-(2x^2+2x-24)\cdot 2(2x+1)\cdot 2}{(2x+1)^4}=\frac{98}{(2x+1)^3}$, which is never 0.

Note that $f'(x)>0$ on $(-\infty,-4)$ and on $(3,\infty)$. So f is increasing on $(-\infty,-4)$ and on $(3,\infty)$. Also, $f'(x)<0$ on $(-4,-1/2)$ and on $(-1/2,3)$. So f is decreasing on those intervals. There is a local maximum of -4 at $x=-4$ and a local minimum of 3 at $x=3$.

Note also that $f''(x)<0$ for $x<-1/2$, and $f''(x)>0$ for $x>-1/2$, so f is concave down on $(-\infty,-1/2)$ and is concave up on $(-1/2,\infty)$. There are no inflection points because the only change in concavity occurs at the vertical asymptote. There are no x-intercepts because $x^2+12>0$ for all x, and the y-intercept is $f(0)=12$.



4.4.34 The domain of f is $(-\infty,\infty)$. There is odd symmetry because

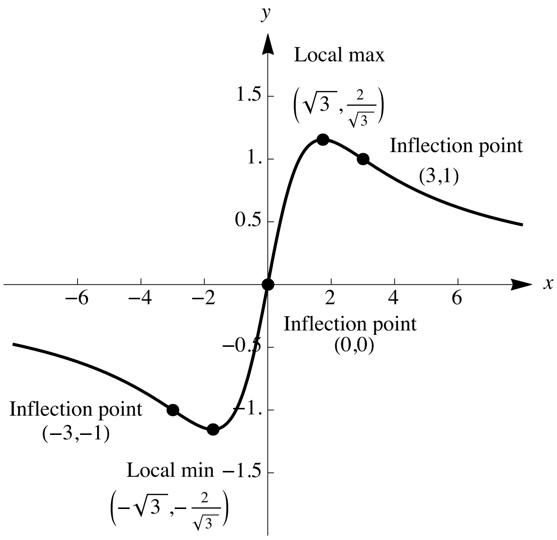
$$f(-x) = \frac{-4x}{(-x)^2+3} = -\frac{4x}{x^2+3} = -f(x).$$

The only intercept is $(0,0)$. We have $\displaystyle\lim_{x\to\infty}\frac{4x}{x^2+3} = \lim_{x\to\infty}\frac{4/x}{1+3/x^2} = 0$, so $y=0$ is a horizontal asymptote.

$f'(x) = \dfrac{(x^2+3)4-4x(2x)}{(x^2+3)^2} = \dfrac{4(3-x^2)}{(x^2+3)^2}$. This is zero for $x=\pm\sqrt{3}$. $f'>0$ on $(-\sqrt{3},\sqrt{3})$ so f is increasing there, while $f'<0$ on $(-\infty,-\sqrt{3})$ and on $(\sqrt{3},\infty)$, so f is decreasing there. There is therefore a local maximum at $x=\sqrt{3}$ and a local minimum at $x=-\sqrt{3}$. The y value of the local maximum is $\dfrac{2}{\sqrt{3}}$ and for the minimum it is $-\dfrac{2}{\sqrt{3}}$

$$\begin{aligned}
f''(x) &= \frac{(x^2+3)^2(-8x)-4(3-x^2)(2)(x^2+3)(2x)}{(x^2+3)^4}\\
&= \frac{-8x(x^2+3)(x^2+3+2(3-x^2))}{(x^2+3)^4}\\
&= \frac{-8x(9-x^2)}{(x^2+3)^3}\\
&= \frac{8x(x-3)(x+3)}{(x^2+3)^3}.
\end{aligned}$$

This is zero for $x=-3,0,3$, $f''<0$ on $(-\infty,-3)$ and on $(0,3)$ so f is concave down there, while $f'>0$ on $(-3,0)$ and on $(3,\infty)$, so f is concave up there. There are inflection points at $(-3,-1)$, $(0,0)$, and $(3,1)$.



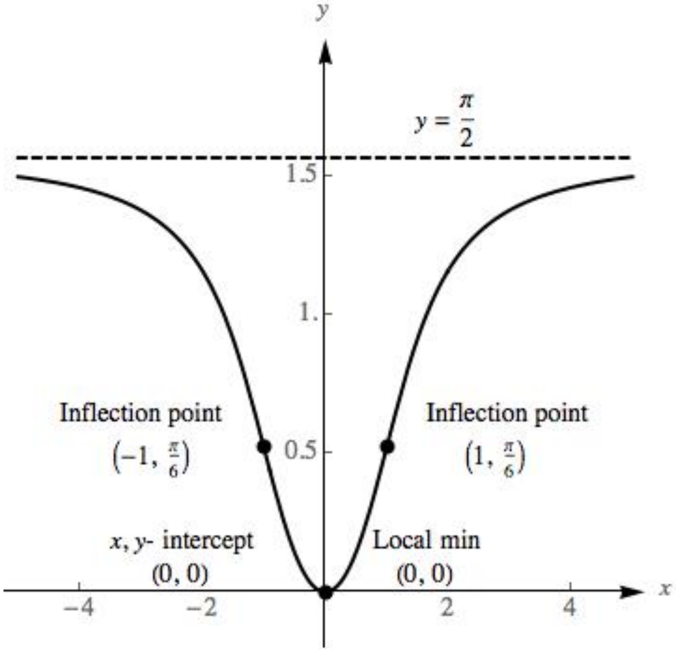
4.4.35 The domain of f is $(-\infty,\infty)$. There is even symmetry because

$$f(-x) = \tan^{-1}\left(\frac{(-x)^2}{\sqrt{3}}\right) = \tan^{-1}\left(\frac{x^2}{\sqrt{3}}\right) = f(x).$$

The only intercept is $(0,0)$. $f'(x) = \dfrac{1}{x^4/3+1}\cdot\dfrac{2x}{\sqrt{3}} = \dfrac{2\sqrt{3}x}{x^4+3}$. This is zero for $x=0$, and $f'>0$ on $(0,\infty)$ so f is increasing there, while $f'<0$ on $(-\infty,0)$ so f is deceasing there. There is a local minimum of 0 at $x=0$.

$f''(x) = \dfrac{(x^4+3)(2\sqrt{3}) - 2\sqrt{3}x(4x^3)}{(x^4+3)^2} = \dfrac{2\sqrt{3}(3-3x^4)}{(x^4+3)^2} = \dfrac{6\sqrt{3}(1-x)(1+x)(1+x^2)}{(x^4+3)^2}$. This is zero for $x = 1$ and $x = -1$. $f'' < 0$ on $(-\infty, -1)$ and on $(1, \infty)$ so f is concave down there, while $f'' > 0$ on $(-1, 1)$, so f is concave up there. There are inflection points at $(1, \pi/6)$ and $(-1, \pi/6)$.

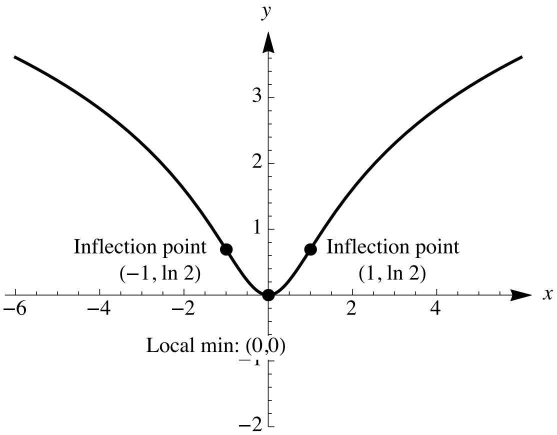
$\lim_{x\to\pm\infty} f(x) = \dfrac{\pi}{2}$, so $y = \dfrac{\pi}{2}$ is a horizontal asymptote.



4.4.36 The domain of f is $(-\infty, \infty)$, and f is symmetric about the y-axis, because $f(-x) = \ln((-x)^2+1) = f(x)$. The only intercept is $(0, 0)$. Note that $\lim_{x\to\pm\infty} \ln(x^2+1) = \infty$, so there are no horizontal asymptotes.

We have $f'(x) = \dfrac{2x}{1+x^2}$, which is zero for $x = 0$. Note that f' is negative when x is negative and positive when x is positive, so f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, and there is a local (in fact, absolute) minimum at $(0, 0)$.

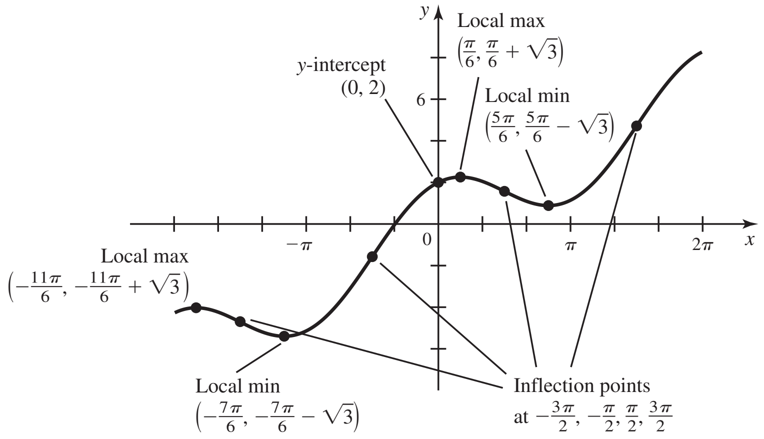
$f''(x) = \dfrac{(1+x^2)\cdot 2 - 2x \cdot 2x}{(1+x^2)^2} = \dfrac{2(1-x^2)}{(1+x^2)^2}$. This is zero for $x = \pm 1$. Also note that $f''(-2) < 0$, $f''(0) > 0$, and $f''(2) < 0$, so f is concave down on $(-\infty, -1)$ and on $(1, \infty)$, and is concave up on $(-1, 1)$. There are inflection points at $(-1, \ln 2)$ and $(1, \ln 2)$.



4.4.37 The domain of f is given to be $[-2\pi, 2\pi]$, and there is no symmetry, and no vertical asymptotes. There are no horizontal asymptotes to consider on this restricted domain.

$f'(x) = 1 - 2\sin x$. This is 0 when $\sin x = 1/2$, which occurs on the given interval for $x = -11\pi/6$, $-7\pi/6$, $\pi/6$, and $5\pi/6$. $f''(x) = -2\cos x$, which is 0 for $x = -3\pi/2, -\pi/2, \pi/2$, and $3\pi/2$.

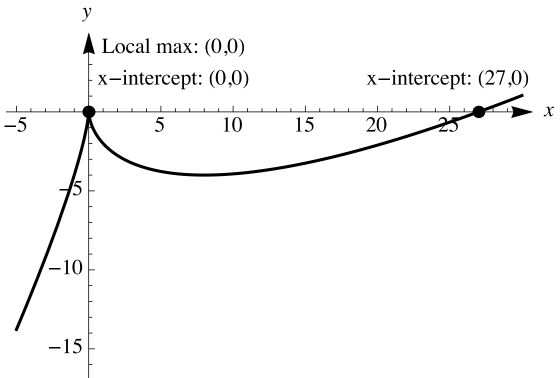
Note that $f'(x) > 0$ on $(-2\pi, -11\pi/6)$, and on $(-7\pi/6, \pi/6)$, and on $(5\pi/6, 2\pi)$. So f is increasing on those intervals, while $f'(x) < 0$ on $(-11\pi/6, -7\pi/6))$ and on $(\pi/6, 5\pi/6)$, so f is decreasing there. f has local maxima at $x = -11\pi/6$ and at $x = \pi/6$ and local minima at $x = -7\pi/6$ and at $x = 5\pi/6$. Note also that $f''(x) < 0$ on $(-2\pi, -3\pi/2)$ and on $(-\pi/2, \pi/2)$ and on $(3\pi/2, 2\pi)$, so f is concave down on those intervals, while $f''(x) > 0$ on $(-3\pi/2, -\pi/2)$ and on $(\pi/2, 3\pi/2)$, so f is concave up there and there are inflection points at $x = \pm 3\pi/2$ and $x = \pm\pi/2$. The y-intercept is $f(0) = 2$ and the x-intercept is at approximately -1.030.



4.4.38 The domain of f is $(-\infty, \infty)$. There are no asymptotes. There are x-intercepts at $(0, 0)$ and $(27, 0)$.

$f'(x) = 1 - \frac{2}{\sqrt[3]{x}} = \frac{\sqrt[3]{x}-2}{\sqrt[3]{x}}$. This is undefined at $x = 0$, and is equal to zero at $x = 8$. Note that $f'(-1) > 0$, $f'(1) < 0$, and $f'(27) > 0$, so f is increasing on $(-\infty, 0)$, decreasing on $(0, 8)$, and increasing on $(8, \infty)$. There is a local maximum at $(0, 0)$ and a local minimum at $(8, -4)$.

$f''(x) = \frac{2}{3\sqrt[3]{x^4}}$, which is never zero, but is undefined at $x = 0$. Because this is always positive, f is concave up on $(-\infty, 0)$ and on $(0, \infty)$. There are no inflection points.

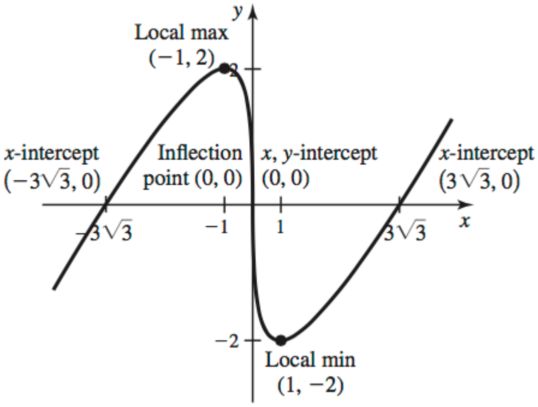


4.4.39 The domain of f is $(-\infty, \infty)$. There are no asymptotes. There are x-intercepts at $(0, 0)$ and $(\pm 3\sqrt{3}, 0)$. f does have odd symmetry, because $f(-x) = -x - 3((-x)^{1/3}) = -(x - 3x^{1/3}) = -f(x)$.

$f'(x) = 1 - \dfrac{1}{\sqrt[3]{x^2}} = \dfrac{\sqrt[3]{x^2} - 1}{\sqrt[3]{x^2}}$. This is undefined at $x = 0$, and is equal to zero at ± 1. Note that $f'(-2) > 0$, $f'(-1/2) < 0$, $f'(1/2) < 0$, $f'(2) > 0$. Thus, f is increasing on $(-\infty, -1)$ and on $(1, \infty)$. Because f is continuous at 0 (even though f' doesn't exist there), we can combine the intervals $(-1, 0)$ and

$(0,1)$ and state that f is decreasing on $(-1,1)$. There is a local maximum at $(-1,2)$ and a local minimum at $(1,-2)$.

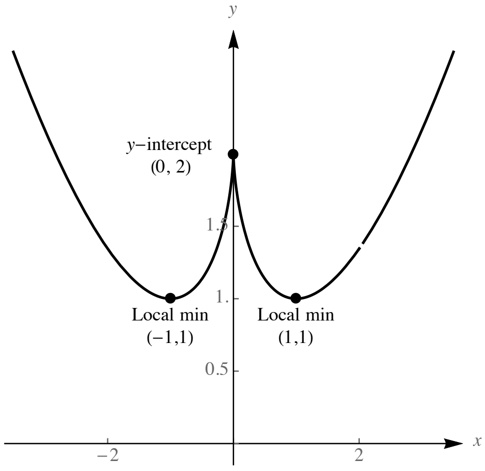
$f''(x) = \dfrac{2}{3\sqrt[3]{x^5}}$, which is never zero, but is undefined at $x=0$. Note that $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so f is concave down on $(-\infty,0)$ and is concave up on $(0,\infty)$. There is an inflection point at $(0,0)$.



4.4.40 The domain of f is $(-\infty,\infty)$. There are no asymptotes, nor x-intercepts. f does have even symmetry, because $f(-x) = 2-2(-x)^{2/3}+(-x)^{4/3} = 2-2x^{2/3}+x^{4/3} = f(x)$.

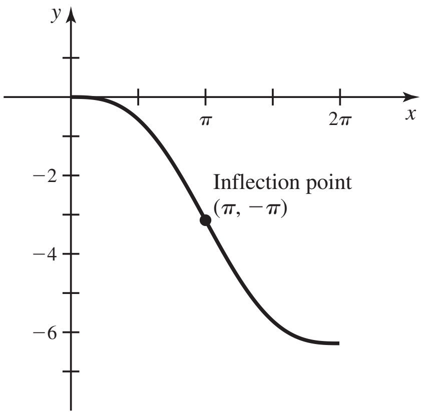
$f'(x) = -\dfrac{4}{3}x^{-1/3}+\dfrac{4}{3}x^{1/3} = -\dfrac{4}{3}\left(\dfrac{1}{x^{1/3}}-\dfrac{x^{1/3}}{1}\right) = -\dfrac{4}{3}\left(\dfrac{1-x^{2/3}}{x^{1/3}}\right)$. The numerator is zero for $x=\pm 1$, so these are critical points, and the denominator is zero at 0, and 0 is in the domain of the original function, so 0 is also a critical point. $f' > 0$ on $(-1,0)$ and on $(1,\infty)$ so f is increasing there, while $f' < 0$ on $(-\infty,-1)$ and on $(0,1)$, so f is decreasing there. There is a local max of 2 at $x=0$ and local minimums of 1 at $x=\pm 1$.

Working from $f'(x) = -\dfrac{4}{3}x^{-1/3}+\dfrac{4}{3}x^{1/3}$, we have $f''(x) = \dfrac{4}{9}x^{-4/3}+\dfrac{4}{9}x^{-2/3} = \dfrac{4(x^{2/3}+1)}{9x^{4/3}}$. $f'' > 0$ on $(-\infty,0)$ and on $(0,\infty)$, so there are no inflection points.



4.4.41 The domain of f is given to be $[0,2\pi]$, so questions about symmetry and horizontal asymptotes aren't relevant. There are no vertical asymptotes.

$f'(x) = \cos x - 1$. This is never 0 on $(0,2\pi)$. $f''(x) = -\sin x$, which is 0 on the given interval only for $x=\pi$. Note that $f'(x) < 0$ on $(0,2\pi)$, so f is decreasing on the given interval and there are no relative extrema. Note also that $f''(x) < 0$ on $(0,\pi)$ and $f''(x) > 0$ on $(\pi,2\pi)$, so f is concave down on $(0,\pi)$ and is concave up on $(\pi,2\pi)$, and there is an inflection point at $x=\pi$. The only intercept is the origin $(0,0)$.



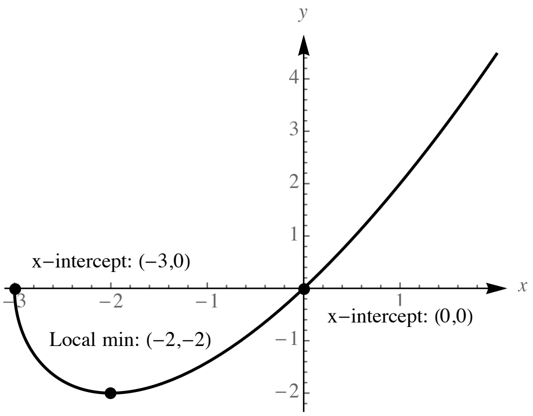
4.4.42 The domain of f is $[-3, \infty)$, and there is no symmetry, and there are no vertical asymptotes. Note that $\lim\limits_{x\to\infty} f(x) = \infty$, so there are no horizontal asymptotes.

$f'(x) = x \cdot (1/2) \cdot (x+3)^{-1/2} + (x+3)^{1/2} = \dfrac{3x+6}{2\sqrt{x+3}}$. This is 0 for $x = -2$.

$f''(x) = \dfrac{2(x+3)^{1/2} \cdot 3 - (3x+6)(x+3)^{-1/2}}{4(x+3)} = \dfrac{3x+12}{4(x+3)^{3/2}}$. The numerator is zero for $x = -4$, but that number isn't in the domain of f.

Note that $f'(x) < 0$ on $(-3, -2)$ and $f'(x) > 0$ on $(-2, \infty)$, so f is decreasing on $(-3, -2)$ and increasing on $(-2, \infty)$, and there is a local (and absolute) minimum of -2 at $x = -2$.

Note also that $f''(x) > 0$ for all x in the domain of f, so f is concave up on its domain, and there are no inflection points. The function is equal to zero at the x-intercepts $(-3, 0)$ and $(0, 0)$, and the latter is also the y-intercept.

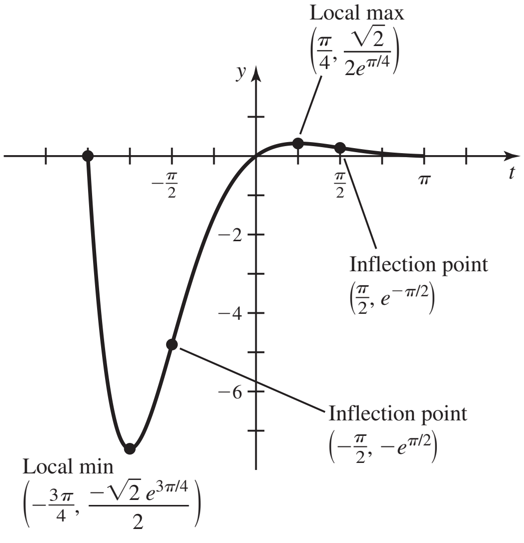


4.4.43 The domain of g is given to be $[-\pi, \pi]$, and there is neither symmetry nor vertical asymptotes. Because the domain is finite, questions about horizontal asymptotes are not relevant.

$g'(t) = e^{-t}\cos t + \sin t \cdot (-e^{-t}) = e^{-t}(\cos t - \sin t)$. This is 0 on the given interval for $t = -3\pi/4$ and $t = \pi/4$. $g''(t) = e^{-t}(-\sin t - \cos t) + (\cos t - \sin t)(-e^{-t}) = -2e^{-t}\cos t$, which is 0 for $t = -\pi/2$ and $t = \pi/2$.

Note that $g'(t) < 0$ on $(-\pi, -3\pi/4)$ and on $(\pi/4, \pi)$, so g is decreasing on those intervals. On $(-3\pi/4, \pi/4)$ we have $g'(t) > 0$ and so g is increasing. There is a local minimum of about -7.460 at $t = -3\pi/4$ and a local maximum of about 0.322 at $t = \pi/4$.

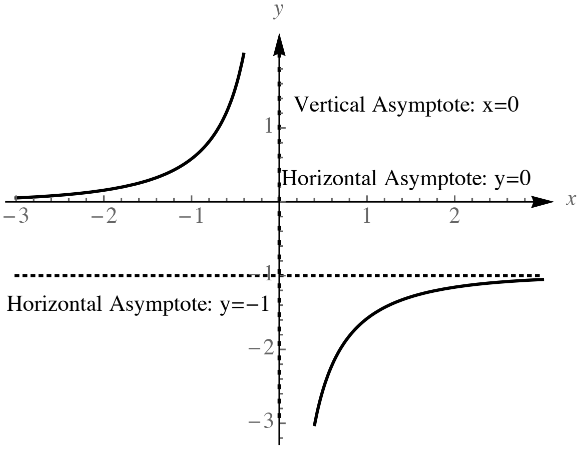
Note also that $g''(t) > 0$ on $(-\pi, -\pi/2)$ and on $(\pi/2, \pi)$, while $g''(t) < 0$ on $(-\pi/2, \pi/2i)$, so g is concave down on $(-\pi/2, \pi/2)$ and is concave up on $(\pi/2, \pi)$ and on $(-\pi, -\pi/2)$. There are inflection points at $t = \pm\pi/2$. The origin is both the $y-$intercept and an x-intercept. The endpoints are x-intercepts as well.



4.4.44 The domain of f is $(-\infty, 0) \cup (0, \infty)$, and f has no symmetry. Note that $\lim_{x\to\infty} \frac{1}{e^{-x}-1} = \frac{1}{0-1} = -1$, so $y = -1$ is a horizontal asymptote as $x \to \infty$. Also, $\lim_{x\to-\infty} \frac{1}{e^{-x}-1} = 0$, so $y = 0$ is a horizontal asymptote as $x \to -\infty$.

$f'(x) = -(e^{-x}-1)^{-2}(-e^{-x}) = \frac{e^{-x}}{(e^{-x}-1)^2}$, which is never 0, and is positive on $(-\infty, 0)$ and on $(0, \infty)$, so f is increasing on $(-\infty, 0)$ and is increasing on $(0, \infty)$. Thus f has no extrema.

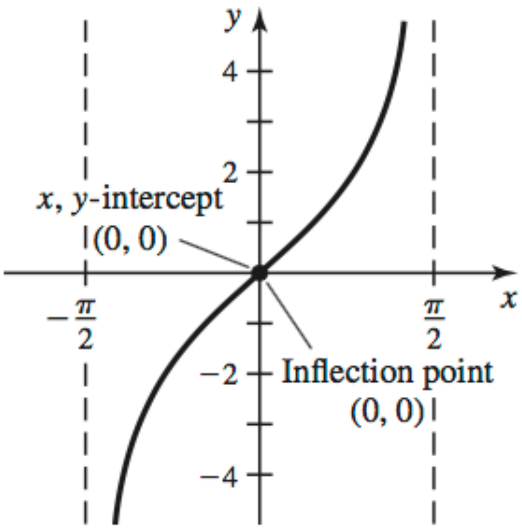
$f''(x) = \frac{(e^{-x}-1)^2(-e^{-x}) - e^{-x} \cdot 2 \cdot (e^{-x}-1)(-e^{-x})}{(e^{-x}-1)^4} = \frac{e^{-x}(e^{-x}+1)}{(e^{-x}-1)^3}$, which is never 0. It is positive on $(-\infty, 0)$ and negative on $(0, \infty)$, so there are no inflection points as the only change in concavity occurs at the vertical asymptote, where the concavity changes from concave up (for $x < 0$) to concave down (for $x > 0$).



4.4.45 Note that $f(-x) = -x + \tan(-x) = -x - \tan x = -(x + \tan x) = -f(x)$, so f has odd symmetry. f has vertical asymptotes at $x = \pm\pi/2$, because the tangent function increases or decreases without bound as x approaches these values.

$f'(x) = 1 + \sec^2 x$ which is always greater than 0. Thus f is increasing on each interval on which it is defined, and it has no extrema. $f''(x) = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x$. This is 0 at $x = 0$.

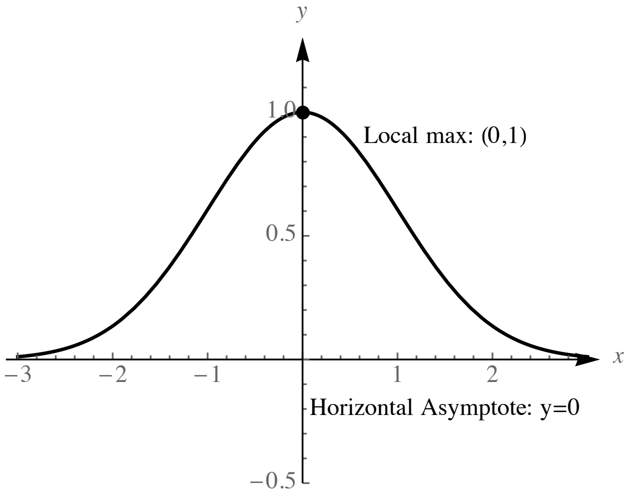
Note that $f''(x)$ is positive on $(0, \pi/2)$, so f is concave up there. Also, $f''(x)$ is negative on $(-\pi/2, 0)$, so f is concave down there. There is a point of inflection at $(0, 0)$.



4.4.46 The domain of f is $(-\infty, \infty)$. Note that $f(-x) = e^{-(-x)^2/2} = e^{-x^2/2} = f(x)$, so f has even symmetry. There are no vertical asymptotes, but $\lim_{x\to\pm\infty} \frac{1}{e^{x^2}} = 0$, so the x-axis is a horizontal asymptote.

$f'(x) = -xe^{-x^2/2}$, which is 0 only for $x = 0$, and is positive on $(-\infty, 0)$ and is negative on $(0, \infty)$, so f is increasing on $(-\infty, 0)$ and is decreasing on $(0, \infty)$, so there is a local maximum which is actually an absolute maximum of $f(0) = 1$ at $x = 0$.

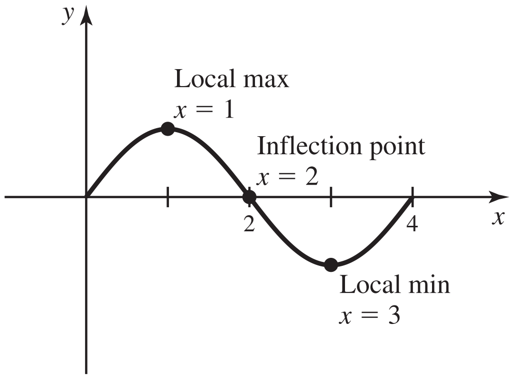
$f''(x) = -x \cdot (-xe^{-x^2/2}) + e^{-x^2/2}(-1) = e^{-x^2/2}(x^2 - 1)$ which is 0 only for $x = \pm 1$. Note that $f''(x) < 0$ on $(-1, 1)$ (so f is concave down there) and $f''(x) > 0$ on $(-\infty, -1)$ and on $(1, \infty)$, where f is concave up. There are inflection points at $x = \pm 1$, and there are no x-intercepts. There is a y-intercept at $(0, 1)$.



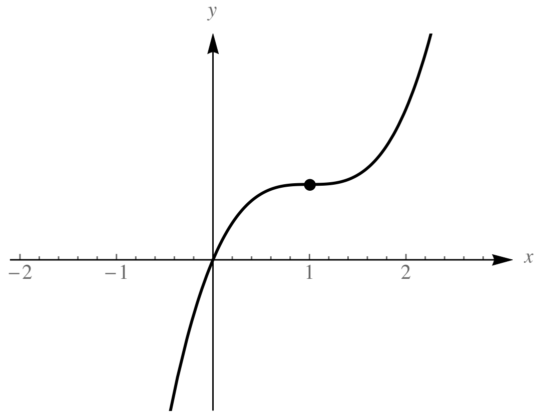
4.4.47 $f'(x)$ is 0 at $x = 1$ and $x = 3$.

$f'(x) > 0$ on $(0, 1)$ and on $(3, 4)$, so f is increasing on those intervals. $f'(x) < 0$ on $(1, 3)$, so f is decreasing on that interval. There is a local maximum at $x = 1$ and a local minimum at $x = 3$.

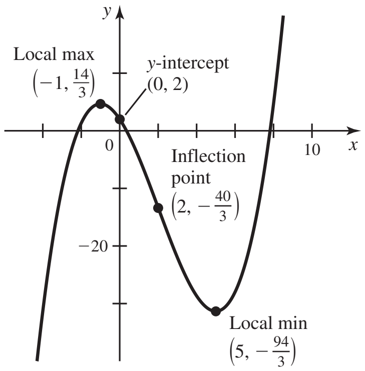
$f''(x)$ changes sign at $x = 2$ from negative to positive, so $x = 2$ is an inflection point where the concavity of f changes from down to up. An example of such a function is sketched.



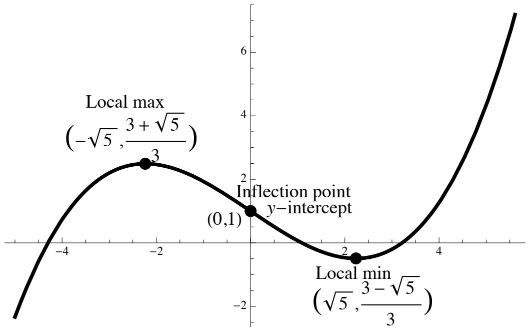
4.4.48 $f'(x) > 0$ on $(-\infty, 1)$ and on $(1, \infty)$, so f should be increasing on both of those intervals. There should be an inflection point at $x = 1$, because the 2nd derivative changes from negative to positive there, so f should change from concave down to concave up at that point.



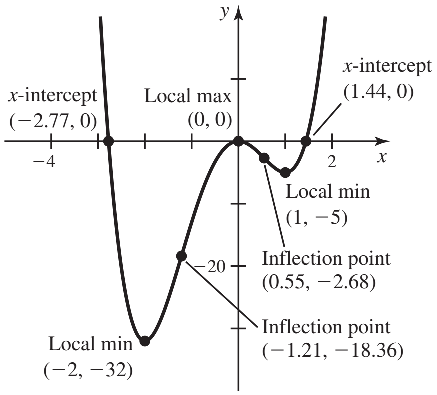
4.4.49 The domain of f is $(-\infty, \infty)$, and there is no symmetry. $f'(x) = x^2 - 4x - 5 = (x - 5)(x + 1)$. This is 0 when $x = -1, 5$. $f''(x) = 2x - 4$, which is 0 when $x = 2$. Note that $f'(-2) > 0$, $f'(0) < 0$, and $f'(6) > 0$. So f is increasing on $(-\infty, -1)$ and on $(5, \infty)$. It is decreasing on $(-1, 5)$. There is a local maximum of $14/3$ at $x = -1$ and a local minimum of $-94/3$ at $x = 5$. Note also that $f''(x) < 0$ for $x < 2$ and $f''(x) > 0$ for $x > 2$, so there is an inflection point at $(2, -40/3)$, and f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$. The y intercept is $f(0) = 2$.



4.4.50 The domain of f is $(-\infty, \infty)$, and there is no symmetry. $f'(x) = \frac{1}{5}x^2 - 1$. This is 0 when $x = \pm\sqrt{5}$. $f''(x) = \frac{2}{5}x$, which is 0 when $x = 0$. Note that $f'(-3) > 0$, $f'(0) < 0$, and $f'(3) > 0$. So f is increasing on $(-\infty, -\sqrt{5})$ and on $(\sqrt{5}, \infty)$. It is decreasing on $(-\sqrt{5}, \sqrt{5})$. There is a local maximum of $\frac{3+2\sqrt{5}}{3}$ at $x = -\sqrt{5}$ and a local minimum of $\frac{3-2\sqrt{5}}{3}$ at $x = \sqrt{5}$. Note also that $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so there is an inflection point at the y-intercept $(0,1)$, and f is concave down on $(-\infty,0)$ and concave up on $(0,\infty)$.

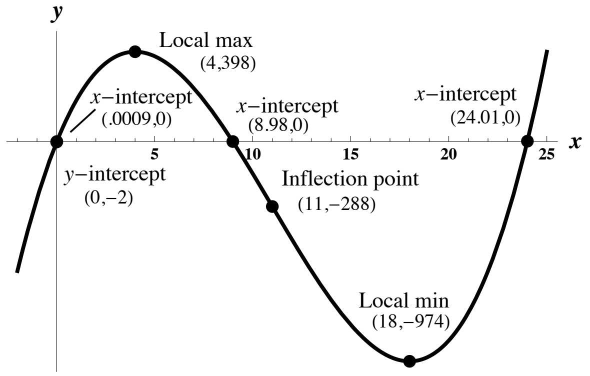


4.4.51 The domain of f is $(-\infty,\infty)$, and there is no symmetry. $f'(x) = 12x^3+12x^2-24x = 12x(x+2)(x-1)$. This is 0 when $x=-2$, when $x=1$, and when $x=0$. $f''(x) = 36x^2+24x-24 = 12(3x^2+2x-2)$, which is 0 when $x = \frac{-1\pm\sqrt{7}}{3}$. These values are at approximately -1.215 and 0.549. Note that $f'(-3) < 0$, $f'(-1) > 0$, $f'(.5) < 0$, and $f'(2) > 0$. So f is decreasing on $(-\infty,-2)$ and on $(0,1)$. It is increasing on $(-2,0)$ and on $(1,\infty)$. There is a local maximum of 0 at $x=0$ and a local minimum of -32 at $x=-2$ and a local minimum of -5 at $x=1$. Let $r_1 < r_2$ be the two roots of $f''(x)$ mentioned above. Note that $f''(x) > 0$ for $x < r_1$ and for $x > r_2$ and $f''(x) < 0$ for $r_1 < x < r_2$, so there are inflection points at $x = r_1$ and at $x = r_2$. Also, f is concave down on (r_1, r_2) and concave up on $(-\infty, r_1)$ and on (r_2,∞). There is a y-intercept at $f(0) = 0$ and x-intercepts where $f(x) = 3x^4+4x^3-12x^2 = x^2(3x^2+4x-12) = 0$, which is at $x = \frac{-2\pm2\sqrt{10}}{3}$ and $x = 0$.



4.4.52 The domain of f is $(-\infty,\infty)$, and there is no symmetry. $f'(x) = 3x^2-66x+216 = 3(x-4)(x-18)$. This is 0 when $x=4$ and when $x=18$. $f''(x) = 6x-66 = 6(x-11)$, which is 0 when $x=11$. Note that $f'(0) > 0$, $f'(10) < 0$, and $f'(20) > 0$. Thus f is decreasing on $(4,18)$, while it is increasing on $(-\infty,4)$ and on $(18,\infty)$.

There is a local maximum of 398 at $x=4$ and a local minimum of -974 at $x=18$. Note also that $f''(x) < 0$ for $x < 11$, and $f''(x) > 0$ for $x > 11$, so there is an inflection point at $x=11$, and f is concave up on $(11,\infty)$ and is concave down on $(-\infty,11)$.



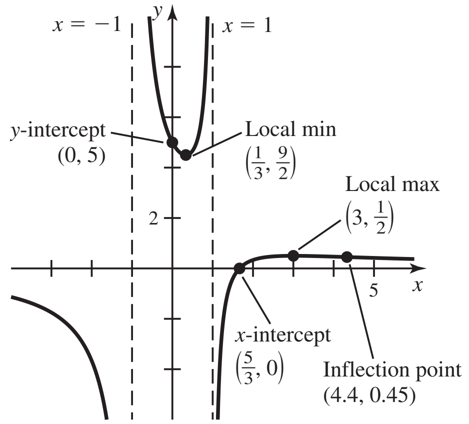
4.4.53 The domain of f is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$, and there is no symmetry. Note that $\lim_{x\to -1^+} f(x)=\infty$ and $\lim_{x\to -1^-} f(x)=-\infty$, so there is a vertical asymptote at $x=-1$. Also, $\lim_{x\to 1^+} f(x)=-\infty$ and $\lim_{x\to 1^-} f(x)=\infty$, so there is a vertical asymptote at $x=1$

Note that $\lim_{x\to\pm\infty} \frac{3x-5}{x^2-1}\cdot\frac{1/x^2}{1/x^2}=\lim_{x\to\pm\infty}\frac{3/x}{1-(1/x^2)}=0$, so $y=0$ is a horizontal asymptote.

$f'(x)=\frac{(x^2-1)\cdot 3-(3x-5)\cdot 2x}{(x^2-1)^2}=\frac{-3x^2+10x-3}{(x^2-1)^2}=\frac{(-3x+1)(x-3)}{(x^2-1)^2}$. This is 0 when $x=3$ and when $x=1/3$. $f''(x)=\frac{(x^2-1)^2(-6x+10)-(-3x^2+10x-3)(2)(x^2-1)\cdot 2x}{(x^2-1)^4}=\frac{2(3x^3-15x^2+9x-5)}{(x^2-1)^3}$. This is 0 for $x\approx 4.405$. Let r be this root of $f''(x)$.

Note that $f'(-2)<0$, $f'(-1/2)<0$, $f'(1/2)>0$, $f'(2)>0$ and $f'(4)<0$. So f is decreasing on $(-\infty,-1)$, on $(-1,1/3)$ and on $(3,\infty)$. It is increasing on $(1/3,1)$ and on $(1,3)$. There is a local maximum of $1/2$ at $x=3$ and a local minimum of $9/2$ at $x=1/3$.

Note that $f''(x)<0$ for $x<-1$ and $f''(x)<0$ for $1<x<r$, while $f''(x)>0$ for $-1<x<1$, and for $x>r$. Thus f is concave up on $(-1,1)$ and on (r,∞) and concave down on $(-\infty,-1)$ and on $(1,r)$. There is an inflection point at r. There is a y-intercept at $f(0)=5$ and an x-intercept at $(5/3,0)$.



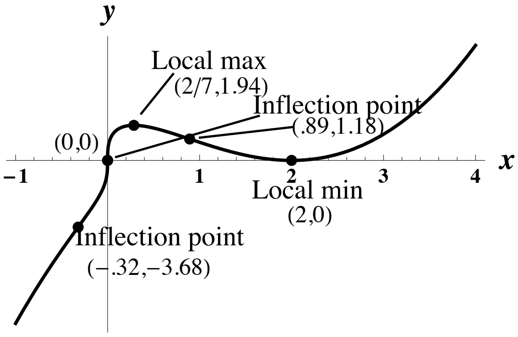
4.4.54 The domain of f is $(-\infty,\infty)$, and there is no symmetry, and no vertical asymptotes. Note that $\lim_{x\to\pm\infty} f(x)=\pm\infty$, so there are no horizontal asymptotes.

$f'(x)=x^{1/3}\cdot 2\cdot(x-2)+(x-2)^2(1/3)x^{-2/3}=\frac{(x-2)(7x-2)}{3x^{2/3}}$. This is 0 for $x=2$ and $x=2/7$, and does not exist for $x=0$.

$f''(x)=\frac{3x^{2/3}(14x-16)-(7x^2-16x+4)(2x^{-1/3})}{9x^{4/3}}=\frac{4(7x^2-4x-2)}{9x^{5/3}}$. The numerator of this last expression has two roots, which we will call r_1 and r_2. Note that $r_1\approx -0.320$ and $r_2\approx 0.892$. Note also that f'' doesn't exist for $x=0$.

$f'(x) > 0$ on $(-\infty, 0)$ and $(0, 2/7)$ and $(2, \infty)$, so f is increasing on those intervals, while $f'(x) < 0$ on $(2/7, 2)$, so f is decreasing there. f has a local maximum at $x = 2/7$ and a local minimum of 0 at $x = 2$.

Note also that $f''(x) < 0$ for $x < r_1$ and for $0 < x < r_2$, so f is concave down on $(-\infty, r_1)$ and on $(0, r_2)$. However, $f''(x) > 0$ for $r_1 < x < 0$ and for $x > r_2$, so f is concave up on $(r_1, 0)$ and on (r_2, ∞). There are inflection points at each of r_1, r_2, and 0. The inflection point $(0, 0)$ serves also as the x- and y- intercept.



4.4.55

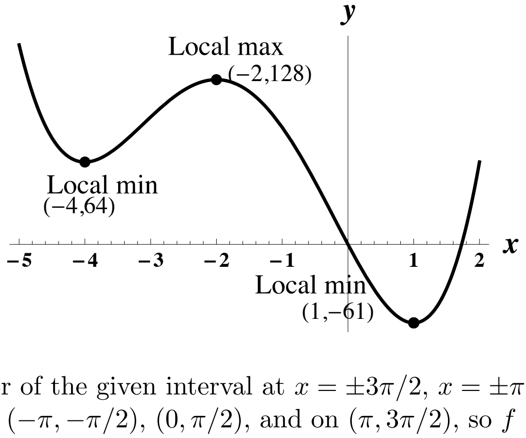
a. False. Maxima and minima can also occur at points where $f'(x)$ doesn't exist. Also, it is possible to have a zero of f' which doesn't correspond to an extreme point.

b. False. Inflection points can also occur at points where $f''(x)$ doesn't exist, and a zero of f'' might not correspond to an inflection point.

c. False. For example, $f(x) = \dfrac{(x^2 - 9)(x^2 - 16)}{(x + 3)(x - 4)}$ doesn't have a vertical asymptote at $x = -3$ or $x = 4$.

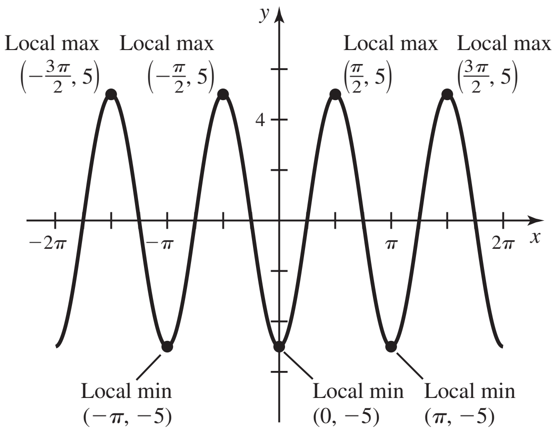
d. True. The limit of a rational function as $x \to \infty$ is a finite number when the degree of the denominator is greater than or equal to that of the numerator. If they both have the same degree, the limit is the ratio of the leading coefficients, and this is also true of the limit as $x \to -\infty$. In the case where the denominator has greater degree than the numerator, the limit is 0 as $x \to -\infty$ and as $x \to \infty$.

4.4.56 $f'(x)$ is 0 at $x = -4$, $x = -2$, and $x = 1$. $f'(x) > 0$ on $(-4, -2)$ and on $(1, \infty)$, so f is increasing there, while $f'(x) < 0$ on $(-\infty, -4)$ and on $(-2, 1)$, so f is decreasing on those intervals. There must be a local maximum at $x = -2$ and local minimums at $x = -4$ and $x = 1$. An example of such a function is sketched.

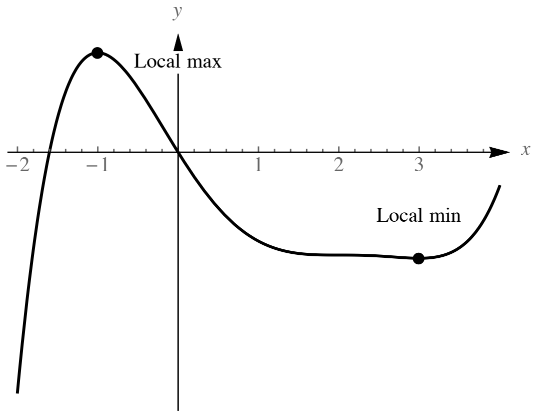


4.4.57 $f'(x)$ is 0 on the interior of the given interval at $x = \pm 3\pi/2$, $x = \pm\pi$, $x = \pm\pi/2$, and at $x = 0$.

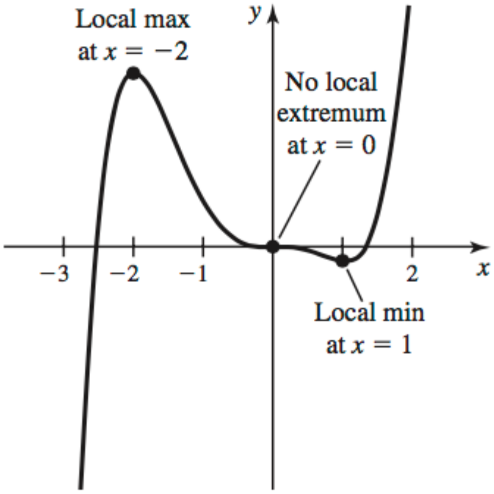
$f'(x) > 0$ on $(-2\pi, -3\pi/2)$, $(-\pi, -\pi/2)$, $(0, \pi/2)$, and on $(\pi, 3\pi/2)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-3\pi/2, -\pi)$, $(-\pi/2, 0)$, $(\pi/2, \pi)$, and on $(3\pi/2, 2\pi)$, so f is decreasing on those intervals. There are local maxima at $x = \pm 3\pi/2$ and $x = \pm\pi/2$, and local minima at $x = 0$ and at $x = \pm\pi$. An example of such a function is sketched.



4.4.58 $f'(x)$ is 0 at -1, 2, and 3. On the interval $(-\infty, -1)$, f' is positive (so f is increasing), and likewise on the interval $(3, \infty)$. On the interval $(-1, 2)$ and on $(2, 3)$, f' is negative (so f is decreasing). There is a local maximum at $x = -1$ and a local minimum at $x = 3$.



4.4.59 $f'(x)$ is 0 at $x = 0$, $x = -2$, and $x = 1$. $f'(x) > 0$ on $(-\infty, -2)$ and on $(1, \infty)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-2, 0)$ and on $(0, 1)$, so f is decreasing on those intervals. There is a local maximum at $x = -2$ and a local minimum at $x = 1$. There isn't an extremum at $x = 0$.

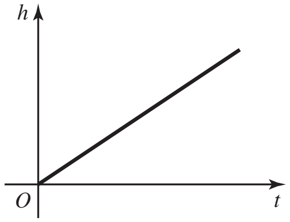


4.4.60 Let $f(x) = \dfrac{\ln x}{x}$. $f'(x) = \dfrac{1 - \ln x}{x^2}$, which is 0 at $x = e$. Note that $f'(x) > 0$ on $(0, e)$ and $f'(x) < 0$ on (e, ∞), so $f(x)$ has its maximal value at $x = e$. Thus, $f(\pi) < f(e)$, so $\dfrac{\ln \pi}{\pi} < \dfrac{1}{e}$, so $\ln \pi < \pi/e$, so $e \ln \pi < \pi$. Thus $\ln \pi^e < \pi$, and so $\pi^e < e^\pi$.

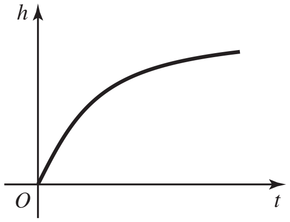
4.4.61

a.

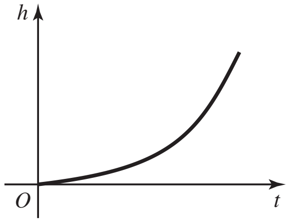
(a)



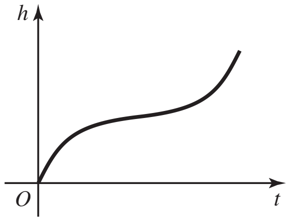
(b)



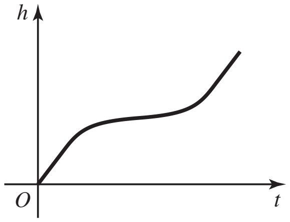
(c)



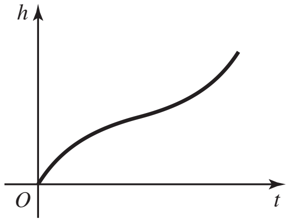
(d)



(e)



(f)



b. The water is being poured in at a constant rate, so the depth is always increasing, so $y = h(t)$ is an increasing function.

c. (a) No concavity

(b) Always concave down.

(c) Always concave up.

(d) Concave down for for the first half and concave up for the second half.

(e) At the beginning, in the middle, and at the end, there is no concavity. In the lower middle it is concave down and in the upper middle it will be concave up.

(f) This is concave down for the first half, and concave up for the second half.

d. (a) $h'(t)$ is constant, so there is no local max/min.

(b) $h'(t)$ is maximal at $t = 0$.

(c) $h'(t)$ is maximal at $t = 10$.

(d) $h'(t)$ is maximal at $t = 0$ and $t = 10$.

(e) $h'(t)$ is maximal on the first and last straight parts of $h(t)$.

(f) $h'(t)$ is maximal at $t = 0$ and $t = 10$.

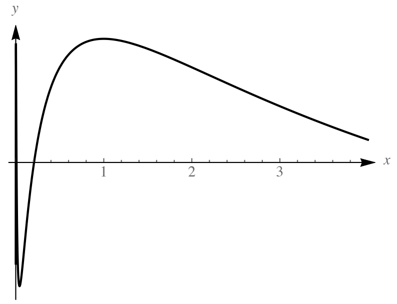
4.4.62

a. $f'(x) = -\sin(\ln x) \cdot \dfrac{1}{x} = -\dfrac{\sin(\ln x)}{x}$, which is 0 when $\ln x = k\pi$ for an integer k, which occurs for $x = e^{k\pi}$. On the given interval, this occurs for $x = 1$, $x = e^{-\pi}$, $x = e^{-2\pi}$, ...

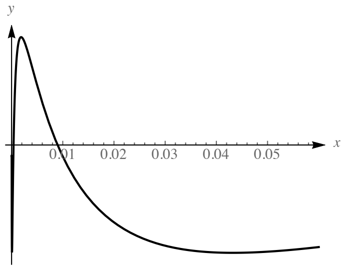
b. $f''(x) = \dfrac{x(-\cos(\ln x) \cdot (1/x)) - (-\sin(\ln x))}{x^2} = \dfrac{\sin(\ln x) - \cos(\ln x)}{x^2}$. This is 0 when $\ln x = \dfrac{4k+1}{4}\pi$ where k is an integer. For our domain, this occurs for $x = e^{\pi/4}$, $x = e^{-3\pi/4}$, $x = e^{-7\pi/4}$, ...

c. Using a computer algebra system, the three smallest zeroes on $(0.1, \infty)$ are at ≈ 0.208, ≈ 4.81, and ≈ 111.318.

d.

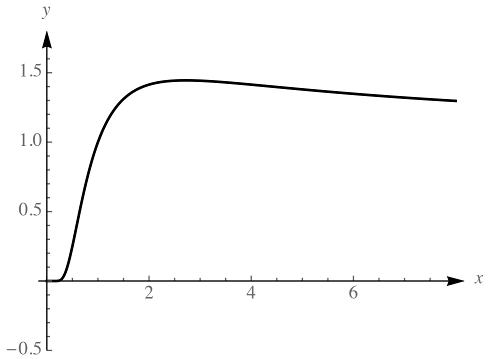


Here it is shown graphed on $(0, 4)$.

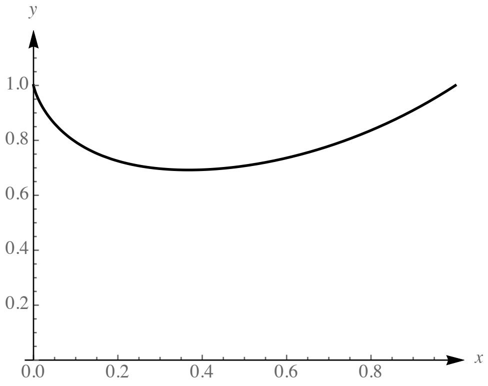


Here it is shown graphed on $(0, .06)$.

4.4.63 f can be written as $f(x) = e^{(\ln x)/x}$. $f'(x) = e^{(\ln x)/x}\left(\dfrac{1 - \ln x}{x^2}\right)$. This is 0 for $x = e$, and is positive on $(0, e)$ and negative on (e, ∞). There is a local maximum at $x = e$ of $e^{1/e}$.

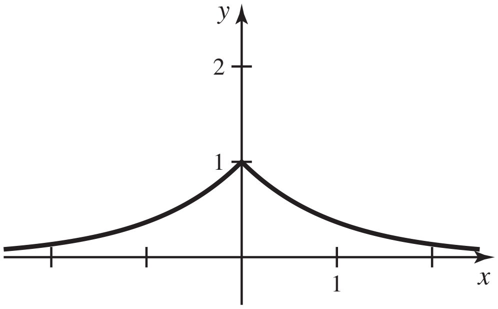


4.4.64 f can be written as $f(x) = e^{x \ln x}$. $f'(x) = e^{x \ln x}(1 + \ln x)$. This is 0 for $x = 1/e$, and is positive on $(1/e, \infty)$ and negative on $(0, 1/e)$. There is a local minimum at $x = 1/e$ of $1/(e^{1/e})$.



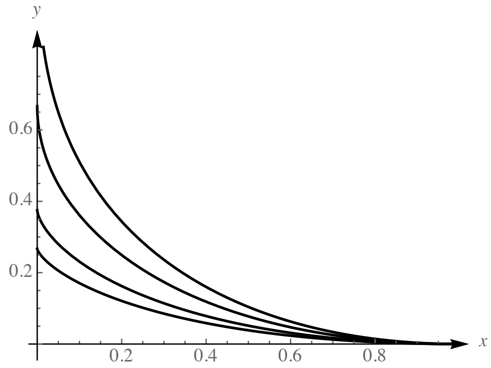
4.4.65

If $f''(x) > 0$ on $(-\infty, 0)$ and on $(0, \infty)$, then $f'(x)$ is increasing on both of those intervals. But if there is a local max at 0, the function f must be switching from increasing to decreasing there. This means that f' must be switching from positive to negative. But if f' is switching from positive to negative, but increasing, there must be a cusp at $x = 0$, so $f'(0)$ does not exist.



4.4.66

As s increases, the man reaches the dog faster.



The curves shown (from top to bottom) are for $s = 1.5, 2, 3, 4$.

4.4.67 The domain of f is given to be $[-2\pi, 2\pi]$. There are no vertical asymptotes. Note that $f(-x) = \dfrac{(-x)(\sin(-x))}{((-x)^2+1)} = \dfrac{x\sin x}{x^2+1} = f(x)$, f has even symmetry. Questions about horizontal asymptotes aren't relevant because the given domain is an interval with finite length.

$f'(x) = \dfrac{(x^2+1)(x\cos x + \sin x) - x\sin x \cdot (2x)}{(x^2+1)^2}$, which can be simplified to

$$\frac{x(x^2+1)\cos x + (1-x^2)\sin x}{(x^2+1)^2},$$

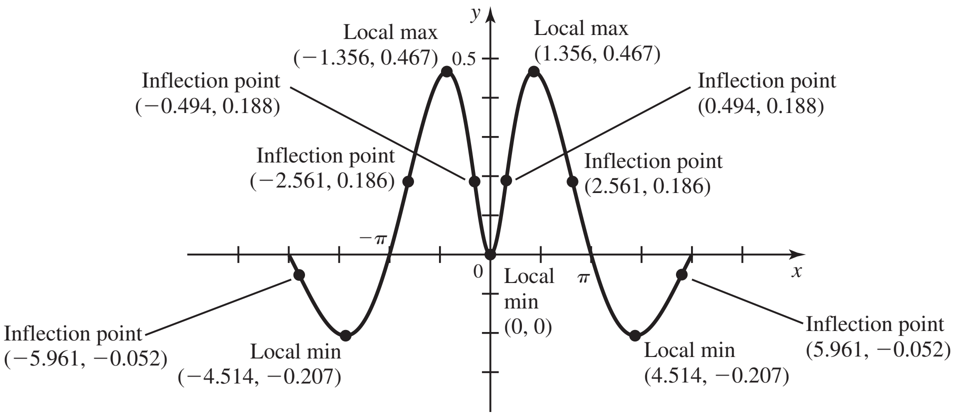
and with the aid of a computer algebra system, the roots of this expression can be found to be approximately ±4.514 and ±1.356, as well as $x = 0$. We will call the non-zero roots $\pm r_1$ and $\pm r_2$ where $0 < r_1 < r_2$. Note that $f'(x) < 0$ on $(-2\pi, -r_2)$ and on $(-r_1, 0)$ and (r_1, r_2), so f is decreasing there, while $f'(x) > 0$ on $(-r_2, -r_1)$, on $(0, r_1)$, and on $(r_2, 2\pi)$, so f is increasing on these intervals. There are local maxima at $x = \pm r_1$ and local minima at $x = 0$ and at $x = \pm r_2$.

$f''(x)$ has numerator $(x^2+1)^2((x^3+x)(-\sin x) + \cos x(3x^2+1) + (1-x^2)(\cos x) + \sin x(-2x)) - ((x^3+x)\cos x + (1-x^2)\sin x)(4x)(x^2+1)$ and denominator $(x^2+1)^4$. This simplifies to

$$f''(x) = \frac{(-x^5 - 7x)\sin x + (-2x^4 + 2)\cos x}{(x^2+1)^3},$$

which is 0 at approximately ±5.961 and ±2.561 and ±0.494. We will call these 6 roots $\pm r_3$, $\pm r_4$ and $\pm r_5$ where $0 < r_3 < r_4 < r_5$. Note that $f''(x) < 0$ on $(-2\pi, -r_5)$ and on $(-r_4, -r_3)$, and on (r_3, r_4), and on $(r_5, 2\pi)$, so f is concave down on these intervals, while $f''(x) > 0$ on $(-r_5, -r_4)$, and on $(-r_3, r_3)$, and on (r_4, r_5), so f is concave up on these intervals. There are points of inflection at each of $\pm r_3$, $\pm r_4$, and $\pm r_5$.

There is an x-intercept at $(0, 0)$, which is also the y-intercept, as well as x-intercepts at $\pm 2\pi$.



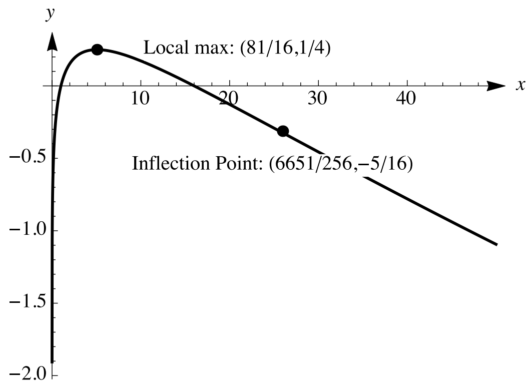
4.4.68 The domain of f is $[0, \infty)$ and there is no symmetry. There are no asymptotes.

$f'(x) = (3/4)x^{-3/4} - (1/2)x^{-1/2} = \frac{3-2x^{1/4}}{4x^{3/4}}$, which is 0 for $x = \frac{81}{16}$.

$f'(x) > 0$ on $(0, 81/16)$, so f is increasing on that interval. $f'(x) < 0$ on $(81/16, \infty)$, so f is decreasing on that interval. There is a local maximum at 81/16 (which also gives an absolute maximum.)

$f''(x) = (-9/6)x^{-7/4} + (1/4)x^{-3/2} = \frac{4x^{1/4}-9}{16x^{7/4}}$, which is 0 at $x = 6561/256$.

$f''(x) > 0$ on $(6561/256, \infty)$, so f is concave up on that interval. $f''(x) < 0$ on $(0, 6561/256)$, so f is concave down on that interval. There is a point of inflection at $x = 6561/256$.

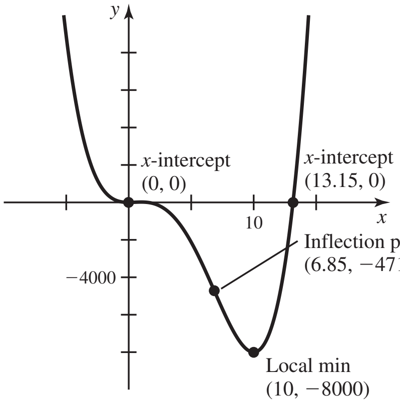


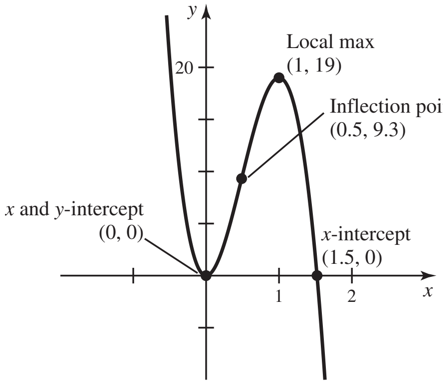
4.4.69 The domain of f is $(-\infty, \infty)$ and there is no symmetry. There are no asymptotes because f is a polynomial.

$f'(x) = 12x^3 - 132x^2 + 120x = 12x(x-10)(x-1)$, which is 0 for $x = 0$, $x = 1$, and $x = 10$.

$f'(x) > 0$ on $(0, 1)$ and on $(10, \infty)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-\infty, 0)$ and on $(1, 10)$, so f is decreasing on those intervals. There is a local maximum at $x = 1$, and local minima at $x = 0$ and at $x = 10$.

$f''(x) = 36x^2 - 264x + 120 = 12(3x^2 - 22x + 10)$. This is 0 at approximately $x = .487$ and $x = 6.846$. Let these two roots be r_1 and r_2 with $r_1 < r_2$. $f''(x) > 0$ on $(-\infty, r_1)$ and on (r_2, ∞), so f is concave up on those intervals. $f''(x) < 0$ on (r_1, r_2), so f is concave down on that interval. There are points of inflection at $x = r_1$ and $x = r_2$.





4.4.70 The domain of f is $(0, 3/2) \cup (3/2, 2)$ and there is no symmetry. There is a vertical asymptote at $x = 3/2$, because $\lim_{x\to 3/2} f(x) = -\infty$.

$$f'(x) = \frac{(1+\sin\pi x)\cos(\pi x)\cdot\pi - \sin(\pi x)\cos(\pi x)\cdot\pi}{(1+\sin(\pi x))^2} = \frac{\pi\cos\pi x}{(1+\sin\pi x)^2},$$

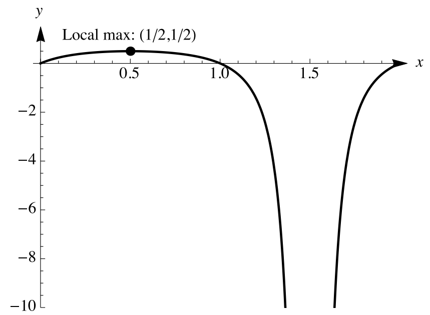
which is 0 for $x = 1/2$.

$f'(x) > 0$ on $(0, 1/2)$ and on $(3/2, 2)$, so f is increasing on those intervals. $f'(x) < 0$ on $(1/2, 3/2)$, so f is decreasing on that interval. There is a local maximum at $x = 1/2$.

$$f''(x) = \frac{(1+\sin\pi x)^2(-\pi^2)(\sin\pi x) - 2\pi^2\cos^2\pi x(1+\sin\pi x)}{(1+\sin\pi x)^4} = \frac{-\pi^2(2-\sin\pi x)(1+\sin\pi x)}{(1+\sin\pi x)^3}.$$

This expression isn't 0 on the given interval.

$f''(x) < 0$ on both $(0, 3/2)$ and on $(3/2, 2)$, so f is concave down on those intervals. There are no points of inflection.



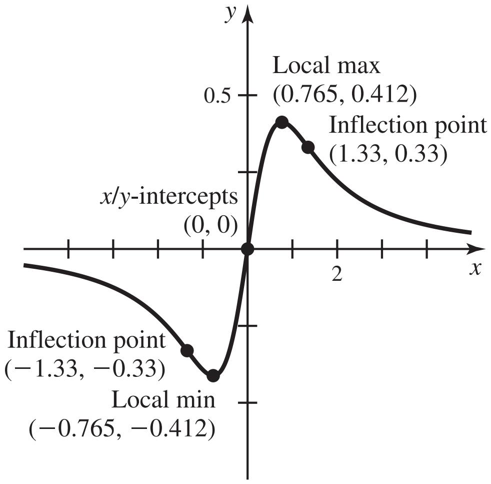
4.4.71 The domain of f is $(-\infty, \infty)$. There are no vertical asymptotes. Note that $f(-x) = \dfrac{\tan^{-1}(-x)}{(-x)^2+1} = -\dfrac{\tan^{-1}(x)}{x^2+1} = -f(x)$, so f has odd symmetry. Because $\displaystyle\lim_{x\to\pm\infty} \tan^{-1}(x) = \pm\pi/2$, $\displaystyle\lim_{x\to\pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote.

$f'(x) = \dfrac{(x^2+1)(1/(x^2+1)) - \tan^{-1}x\cdot 2x}{(x^2+1)^2} = \dfrac{1-2x\tan^{-1}x}{(x^2+1)^2}$. Using a computer algebra system shows that the numerator has two roots at approximately ± 0.765. Let the roots be $\pm r_1$ where $r_1 > 0$. Note that $f'(x) < 0$ on $(-\infty, -r_1)$ and on (r_1, ∞), so f is decreasing there, while $f'(x) > 0$ on $(-r_1, r_1)$, so f is increasing on that interval. There is a local minimum at $-r_1$ and a local maximum at r_1.

$$\begin{aligned} f''(x) &= \frac{(x^2+1)^2\left[(-2x)(1/(x^2+1)) - \tan^{-1}x\cdot 2\right] - (1-2x\tan^{-1}x)\cdot 2(x^2+1)(2x)}{(x^2+1)^4} \\ &= \frac{(6x^2-2)\tan^{-1}x - 6x}{(x^2+1)^3}. \end{aligned}$$

Again, using a computer algebra system reveals roots at approximately ±1.330 in addition to the root at 0. Let the non-zero roots of the numerator be $\pm r_2$ where $r_2 > 0$. We see that $f''(x) < 0$ on $(-\infty, -r_2)$, and on $(0, r_2)$, so f is concave down on those intervals, while $f''(x) > 0$ on $(-r_2, 0)$ and on (r_2, ∞), so f is concave up on those intervals, and there are points of inflection at $-r_2$, 0, and r_2.

There is an x-intercept at $(0, 0)$, which is also the y-intercept.



4.4.72 The domain of f is $(-\infty, \infty)$. Note that $f(-x) = \frac{\sqrt{4(-x)^2+1}}{(-x)^2+1} = \frac{\sqrt{4x^2+1}}{x^2+1} = f(x)$, so f has even symmetry.

Note that $\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{x^2+1} \cdot \frac{\sqrt{(1/x^4)}}{1/x^2} = \lim_{x\to\infty} \frac{\sqrt{(4/x^2)+(1/x^4)}}{1+(1/x^2)} = 0$, so $y = 0$ is a horizontal asymptote as $x \to \infty$, and by symmetry it is a horizontal asymptote as $x \to -\infty$ as well.

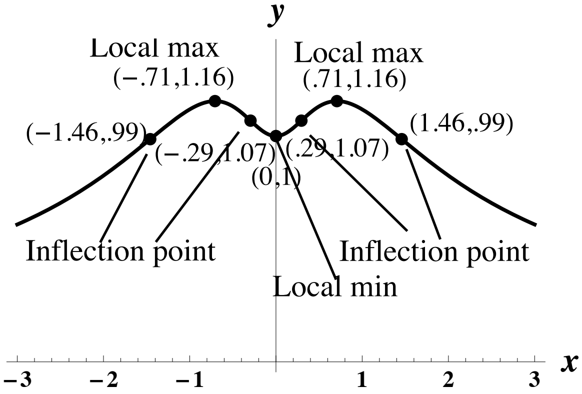
$$\begin{aligned} f'(x) &= \frac{(x^2+1)(4x)(4x^2+1)^{-1/2} - \sqrt{4x^2+1}(2x)}{(x^2+1)^2} \\ &= \frac{4x^3+4x-8x^3-2x}{(x^2+1)^2 \cdot \sqrt{4x^2+1}} \\ &= \frac{2x-4x^3}{(x^2+1)^2 \cdot \sqrt{4x^2+1}} \\ &= \frac{2x(1-2x^2)}{(x^2+1)^2 \cdot \sqrt{4x^2+1}}. \end{aligned}$$

This expression is 0 for $x = 0$ and $x = \pm\sqrt{1/2}$. Note that $f'(x) > 0$ on the interval $(-\infty, -\sqrt{1/2})$ and on $(0, \sqrt{1/2})$, so f is increasing on those intervals, while $f'(x) < 0$ on $(-\sqrt{1/2}, 0)$ and on $(\sqrt{1/2}, \infty)$, so f is decreasing on those intervals. There is a local minimum at $(0, 1)$ and local maxima at approximately $(\pm\sqrt{1/2}, 1.16)$.

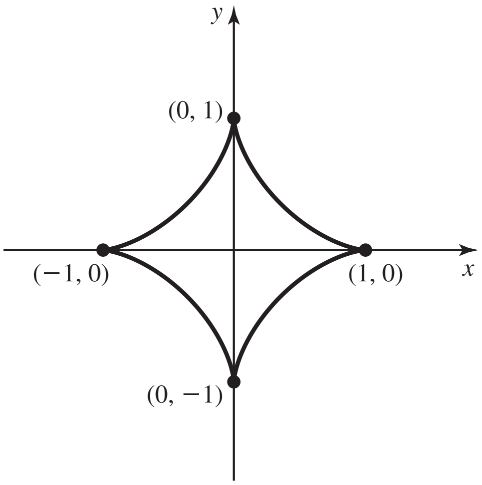
$f''(x)$ has numerator $(4x^2+1)^{1/2}(x^2+1)^2(2-12x^2)$
$-(2x-4x^3)((4x^2+1)^{1/2}(2)(x^2+1)(2x) + (x^2+1)^2(4x)(4x^2+1)^{-1/2}$ and denominator $(4x^2+1)(x^2+1)^4$. When simplified, this yields

$$f''(x) = \frac{2(16x^6-30x^4-9x^2+1)}{(4x^2+1)^{3/2}(x^2+1)^3}.$$

Using a computer algebra system, the roots of the polynomial are determined to be approximately ±1.458 and ±.295. We will refer to these roots as $\pm r_1$ and $\pm r_2$ where $0 < r_1 < r_2$. Note that $f''(x) < 0$ on $(-r_2, -r_1)$ and on (r_1, r_2), so f is concave down there, while $f''(x) > 0$ on $(-\infty, -r_2)$, and on $(-r_1, r_1)$, and on (r_2, ∞), so f is concave up on these intervals, and there are inflection points at each of $\pm r_1$ and $\pm r_2$. The y-intercept is $(0, 1)$.



4.4.73 The equation is valid only for $|x| \le 1$ and $|y| \le 1$. Using implicit differentiation, we have $(2/3)x^{-1/3} + (2/3)y^{-1/3}y' = 0$, so $y' = \frac{-y^{1/3}}{x^{1/3}}$. This is 0 for $y = 0$ (in which case $x = \pm 1$) and doesn't exist for $x = 0$ (in which case $x = \pm 1$.) In the first quadrant the curve is decreasing, in the 2nd it is increasing, in the 3rd it is decreasing, and in the 4th it is increasing. Differentiating y' yields $y'' = \frac{x^{1/3}(-1/3)y^{-2/3}y' + y^{1/3}(1/3)(x^{-2/3})}{x^{2/3}} = \frac{y^{2/3} + x^{2/3}}{3x^{4/3}y^{1/3}}$, which is positive when y is positive and negative when y is negative, so the curve is concave up in the first and 2nd quadrants, and concave down in the 3rd and 4th.

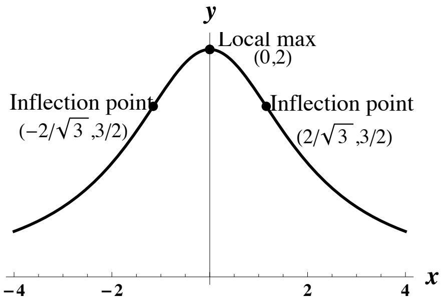


4.4.74 The domain of f is $(-\infty, \infty)$ and f has even symmetry, because $f(-x) = f(x)$.
Note that $\lim_{x\to\infty} \frac{8}{x^2+4} = 0$, so $y = 0$ is a horizontal asymptote. $f'(x) = -\frac{16x}{(x^2+4)^2}$, which is negative for $x > 0$ and positive for $x < 0$, so f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$, and there is a local maximum of 2 at $x = 0$.

$$f''(x) = \frac{(x^2+4)^2(-16) - (-16x)(2)(x^2+4)(2x)}{(x^2+4)^4}$$

$$= -\frac{16(4-3x^2)}{(x^2+4)^3},$$

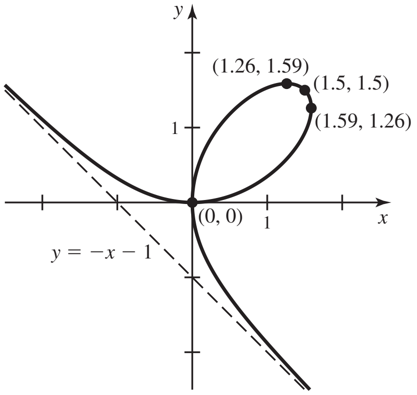
which is 0 for $x = \pm\sqrt{4/3}$. Note that $f'' < 0$ on $(-2/\sqrt{3}, 2/\sqrt{3})$, and is positive elsewhere, so f is concave up on $(-\infty, -2/\sqrt{3})$ and on $(2/\sqrt{3}, \infty)$, and is concave down on $(-2/\sqrt{3}, 2/\sqrt{3})$. There are inflection points at $x = \pm 2/\sqrt{3}$.



4.4.75 First note that the expression is symmetric when x and y are switched, so the curve should be symmetric about the line $y = x$. Also, if $y = x$, then $2x^3 = 3x^2$, so either $x = 0$ or $x = 3/2$, so this is where the curve intersects the line $y = x$.

Differentiating implicitly yields $3x^2 + 3y^2y' = 3xy' + 3y$, so $y' = \dfrac{y - x^2}{y^2 - x}$. This is 0 when $y = x^2$, but this occurs on the curve when $x^3 + x^6 = 3x^3$, which yields $x = 0$ (and $y = 0$), or $x^3 = 2$, so $x = \sqrt[3]{2} \approx 1.260$. Note also that the derivative doesn't exist when $x = y^2$, which again yields $(0, 0)$ and $y^6 + y^3 = 3y^3$, or $y = \sqrt[3]{2}$. So there should be a flat tangent line at approximately $(1.260, 1.587)$ and a vertical tangent line at about $(1.587, 1.260)$.

Differentiating again and solving for y'' yields $y''(x) = \dfrac{2xy\left(x^3 - 3xy + y^3 + 1\right)}{\left(x - y^2\right)^3} = \dfrac{2xy}{(x - y^2)^3}$. In the first quadrant, when $x > y^2$, the curve is concave up, when $x < y^2$, the curve is concave down. In both the 2nd and 4th quadrants, the curve is concave up.



4.4.76 Note that the curve requires $0 \leq x < 1$.

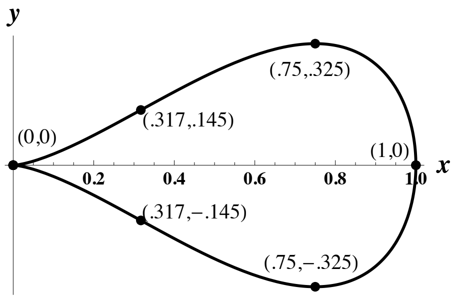
Note also that the curve is symmetric about the x-axis, so we can just consider the first quadrant, and obtain the rest by reflection.

Differentiating implicitly and solving for y' yields $y'(x) = \dfrac{3x^2 - 4x^3}{2y}$. This quantity is positive on the first quadrant for $0 < x < 3/4$ and negative for $3/4 < x < 1$, so f is increasing in the first quadrant for $0 < x < 3/4$ and decreasing for $3/4 < x < 1$. There is a maximum at $x = 3/4$.

Differentiating again and solving for y'' yields

$$y''(x) = \frac{-x\left((3 - 4x)^2x^3 + 12(2x - 1)y^2\right)}{4y^3}.$$

Rewriting and simplifying yields $y''(x) = \dfrac{x^4(8x^2 - 12x + 3)}{4y^3}$ which is positive in the first quadrant for $0 < x < r_1$ where $r_1 \approx .317$, and negative for $r_1 < x < 1$. So the function in the first quadrant is concave up for $0 < x < r_1$ and concave down for $r_1 < x < 1$, and there is a point of inflection at r_1.



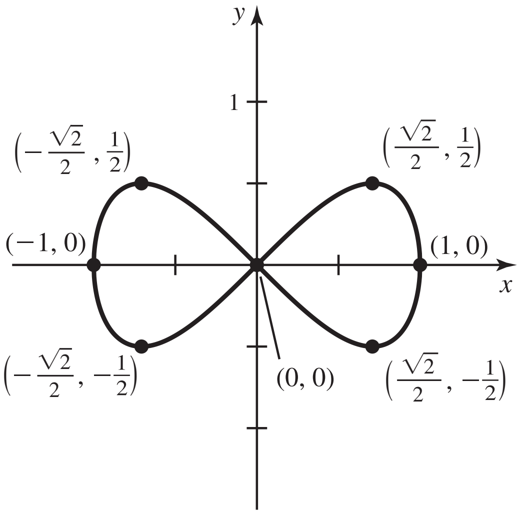
4.4.77 Note that the curve requires $-1 \leq x < 1$.

Note also that the curve is symmetric about both the x-axis and the y-axis, so we can just consider the first quadrant, and obtain the rest by reflection.

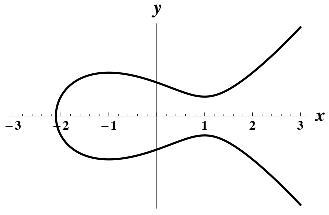
Differentiating implicitly yields $4x^3 - 2x + 2yy' = 0$, so $y' = \dfrac{x - 2x^3}{y}$. This is 0 in the first quadrant for $x = \sqrt{2}/2$. Note also that there is a vertical tangent line at the point $(1, 0)$. The derivative is positive on $(0, \sqrt{2}/2)$ and negative on $(\sqrt{2}/2, 1)$, so in the first quadrant the curve is increasing on that first interval and decreasing on the second.

Differentiating again and solving for y'' (and rewriting) yields $y''(x) = \dfrac{x^4(2x^2 - 3)}{y}$, which is negative in the first quadrant for $0 < x < 1$, so this curve is concave down in the first quadrant.

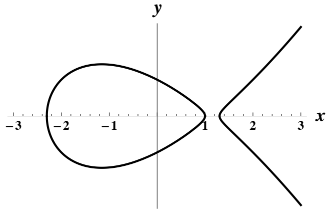
The rest of the curve can be found by reflection.



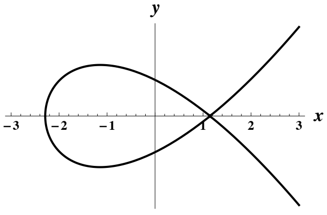
4.4.78



a.

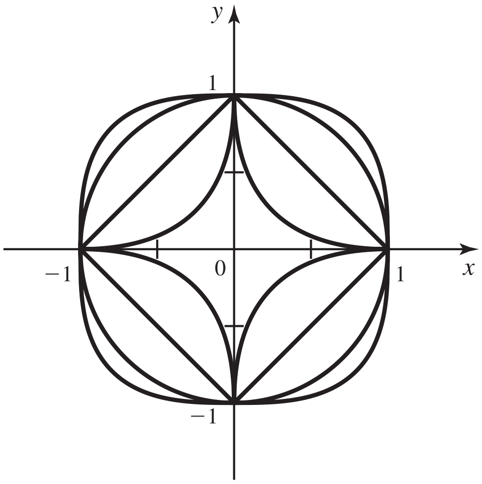


b.



c. This occurs for $a \approx 3.931$.

4.4.79 As n increases, the curves retain their symmetry, but move "outward." That is, the curves enclose a greater area. It appears that the figures approach the 2×2 square centered at the origin with sides parallel to the coordinate axes.



4.5 Optimization Problems

4.5.1 . . . objective. . . constraints . . . .

4.5.2 The constraints are used to express all but one of the variables in terms of one independent variable.

4.5.3 The constraint is $x + y = 10$, so we can express $y = 10 - x$ or $x = 10 - y$. Therefore the objective function can be expressed $Q = x^2(10 - x)$ or $Q = (10 - y)^2y$.

4.5.4 The minimum will occur at one of the endpoints of the closed interval.

4.5.5

a. From the constraint, we have $y = 100 - 10x$. Substituting this into the objective function, we have $P(x) = x(100 - 10x) = 100x - 10x^2$.

b. $P'(x) = 100 - 20x$, which is zero for $x = 5$. Because $P''(x) = -20$, P is concave down everywhere, so the critical point $x = 5$ yields an absolute maximum. The value of $P(5)$ is $100(5) - 10(25) = 250$.

4.5.6

a. Because $y = \dfrac{50}{x}$, we can write $S = x + 2\left(\dfrac{50}{x}\right) = x + \dfrac{100}{x}$.

b. $S'(x) = 1 - \dfrac{100}{x^2}$, which is zero for $x = 10$. $S''(x) = \dfrac{200}{x^3}$, so $S''(10) > 0$ so there is a local minimum at $x = 10$. Because $x = 10$ is the only critical point, this is actually an absolute minimum. The value of $S(10) = 20$.

4.5.7 Let x and y be the two non-negative numbers. The constraint is $x + y = 23$, which gives $y = 23 - x$. The objective function to be maximized is the product of the numbers, $P = xy$. Using $y = 23 - x$, we have $P = xy = x(23 - x) = 23x - x^2$. Now x must be at least 0, and cannot exceed 23 (otherwise $y < 0$). Therefore we need to maximize $P(x) = 23x - x^2$ for $0 \le x \le 23$. The critical points of the objective function satisfy $P'(x) = 23 - 2x = 0$, which has the solution $x = 23/2$. To find the absolute maximum of P, we check the endpoints of $[0, 23]$ and the critical point $x = 23/2$. Because $P(0) = P(23) = 0$ and $P(23/2) = (23/2)^2$, the absolute maximum occurs when $x = y = 23/2$.

4.5.8 Let a and b be the two non-negative numbers. The constraint is $a + b = 23$, which gives $b = 23 - a$. The objective function to be maximized/minimized is the quantity $Q = a^2 + b^2$. Using $b = 23 - a$, we have $Q = a^2 + b^2 = a^2 + (23 - a)^2 = 2a^2 - 46a + 529$. Now a must be at least 0, and cannot exceed 23 (otherwise $b < 0$). Therefore we need to maximize $Q(a) = 2a^2 - 46a + 529$ for $0 \le a \le 23$. The critical points of the objective function satisfy $Q'(a) = 4a - 46 = 0$, which has the solution $a = 23/2$. To find the absolute maximum/minimum of Q, we check the endpoints of $[0, 23]$ and the critical point $a = 23/2$. Observe that $Q(0) = Q(23) = 529$ and $Q(23/2) = 529/2$, so the absolute maximum occurs when $a, b = 0, 23$ or $23, 0$ and the absolute minimum occurs when $a = b = 23/2$.

4.5.9 Let x and y be the two positive numbers. The constraint is $xy = 50$, which gives $y = 50/x$. The objective function to be minimized is the sum of the numbers, $S = x + y$. Using $y = 50/x$, we have $S = x + y = x + \frac{50}{x}$. Now x can be any positive number, so we need to maximize $S(x) = x + 50/x$ on the interval $(0, \infty)$. The critical points of the objective function satisfy $S'(x) = 1 - \frac{50}{x^2} = 0$, which has the solution $x = \sqrt{50} = 5\sqrt{2}$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. Therefore the numbers with minimum sum are $x = 5\sqrt{2}$ and $y = \frac{50}{5\sqrt{2}} = \frac{10}{\sqrt{2}} = 5\sqrt{2}$, so $x = y = 5\sqrt{2}$.

4.5.10 We seek to maximize $P = xy$ subject to the constraint $y = 12 - 3x$. Substituting gives $P = x(12 - 3x) = 12x - 3x^2$. Then $P'(x) = 12 - 6x$, which is zero for $x = 2$. Because $P'(x) > 0$ for $0 < x < 2$ and $P'(x) < 0$ for $x > 2$, we have a maximum at $x = 2$. When $x = 2$, we have $y = 12 - 3x = 6$. So the two numbers are 2 and 6.

4.5.11 Let x and y be the dimensions of the rectangle. The perimeter is $2x + 2y$, so the constraint is $2x + 2y = 10$, which gives $y = 5 - x$. The objective function to be maximized is the area of the rectangle, $A = xy$. Thus we have $A = xy = x(5 - x) = 5x - x^2$. We have $x, y \geq 0$, which also implies $x \leq 5$ (otherwise $y < 0$). Therefore we need to maximize $A(x) = 5x - x^2$ for $0 \leq x \leq 5$. The critical points of the objective function satisfy $A'(x) = 5 - 2x = 0$, which has the solution $x = 5/2$. To find the absolute maximum of A, we check the endpoints of $[0, 5]$ and the critical point $x = 5/2$. Because $A(0) = A(5) = 0$ and $A(5/2) = 25/4$, the absolute maximum occurs when $x = y = 5/2$, so width = length = 5/2 m.

4.5.12 Let x and y be the dimensions of the rectangle. The perimeter is $2x + 2y$, so the constraint is $2x + 2y = P$, which gives $y = P/2 - x$. The objective function to be maximized is the area of the rectangle, $A = xy$. Thus we have $A = xy = x(P/2 - x) = (P/2)x - x^2$. We have $x, y \geq 0$, which also implies $x \leq P/2$ (otherwise $y < 0$). Therefore we need to maximize $A(x) = (P/2)x - x^2$ for $0 \leq x \leq P/2$. The critical points of the objective function satisfy $A'(x) = P/2 - 2x = 0$, which has the solution $x = P/4$. To find the absolute maximum of A, we check the endpoints of $[0, P/2]$ and the critical point $x = P/4$. Because $A(0) = A(P/2) = 0$ and $A(P/4) = P^2/8 - P^2/16 = P^2/16$, the absolute maximum occurs when $x = y = P/4$, so width = length = $P/4$.

4.5.13 Let x and y be be the dimensions of the rectangle. The area is $xy = 100$, so the constraint is $y = 100/x$. The objective function to be minimized is the perimeter of the rectangle, $P = 2x + 2y$. Using $y = 100/x$, we have $P = 2x + 2y = 2x + \dfrac{200}{x}$. Because $xy = 100 > 0$ we must have $x > 0$, so we need to minimize $P(x) = 2x + 200/x$ on the interval $(0, \infty)$. The critical points of the objective function satisfy $P'(x) = 2 - \dfrac{200}{x^2} = 0$, which has the solution $x = 10$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. Therefore the dimensions of the rectangle with minimum perimeter are $x = 10$ and $y = \frac{100}{10} = 10$, so width = length = 10.

4.5.14 Let x and y be be the dimensions of the rectangle. The area is $xy = A$ and A is fixed, so the constraint is $xy = A$, which gives $y = A/x$. The objective function to be minimized is the perimeter of the rectangle, $P = 2x + 2y$. Using $y = A/x$, we have $P = 2x + 2y = 2x + \dfrac{2A}{x}$. Because $xy = A > 0$ we must have $x > 0$, so we need to minimize $P(x) = 2x + 2A/x$ on the interval $(0, \infty)$. The critical points of the objective function satisfy $P'(x) = 2 - \dfrac{2A}{x^2} = 0$, which has the solution $x = \sqrt{A}$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. Therefore the dimensions of the rectangle with minimum perimeter are $x = \sqrt{A}$ and $y = A/\sqrt{A} = \sqrt{A}$, so width = length = $\sqrt{A}$.

4.5.15 We seek to minimize $S = 2x+y$ subject to the constraint $y = 12/x$. Substituting gives $S = 2x+12/x$, so $S'(x) = 2 - 12/x^2$. This is zero when $x^2 = 6$, or $x = \sqrt{6}$. Note that for $0 < x < \sqrt{6}$ we have $S'(x) < 0$, and for $x > \sqrt{6}$ we have $S'(x) > 0$, so we have a minimum at $x = \sqrt{6}$. Note that when $x = \sqrt{6}$, we have $y = 12/x = 12/\sqrt{6} = 2\sqrt{6}$.

4.5.16

a. Let x and y be the lengths of the sides of the pen, with y the side parallel to the barn. Then the constraint is $2x + y = 200$, which gives $y = 200 - 2x$. The objective function to be maximized is the area of the pen, $A = xy$. Using $y = 200 - 2x$, we have $A = xy = x(200 - 2x) = 200x - 2x^2$. The length x must be at least 0, and cannot exceed 100 (otherwise $y < 0$). Therefore we need to maximize $A(x) = 200x - 2x^2$ for $0 \le x \le 100$. The critical points of the objective function satisfy $A'(x) = 200 - 4x = 0$, which has the solution $x = 50$. To find the absolute maximum of A, we check the endpoints of $[0, 100]$ and the critical point $x = 50$. Because $A(0) = A(100) = 0$ and $A(50) = 5000$, the absolute maximum occurs when $x = 50$ m and $y = 200 - 2 \cdot 50 = 100$ m.

b. Let x and y be the lengths of the sides of each individual rectangular pen, with y the side parallel to the barn. Then the constraint is $xy = 100$, which gives $y = 100/x$. The objective function to be minimized is the total amount of fencing required, which is $Q = 5x + 4y$. Using the constraint, we have $Q = 5x + 4y = 5x + \dfrac{400}{x}$. Now x can be any positive number, so so we need to minimize $Q(x) = 5x + 400/x$ on the interval $(0, \infty)$. The critical points of the objective function satisfy $Q'(x) = 5 - \dfrac{400}{x^2} = 0$, which has the solution $x = \sqrt{80} = 4\sqrt{5}$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. Therefore the dimensions which require the least fencing are $x = 4\sqrt{5}$ m and $y = 100/4\sqrt{5} = 5\sqrt{5}$ m.

4.5.17 Let the coordinates of the base of the rectangle be $(x, 0)$ and $(-x, 0)$ where $0 \le x \le 5$. Then the width of the rectangle is $2x$ and the height is $\sqrt{25 - x^2}$, so the area A is given by $A(x) = 2x\sqrt{25 - x^2}$. The critical points of this function satisfy $A'(x) = 2\sqrt{25 - x^2} + \dfrac{2x \cdot (-x)}{\sqrt{25 - x^2}} = \dfrac{2(25 - 2x^2)}{\sqrt{25 - x^2}} = 0$, which has unique solution $x = 5/\sqrt{2}$ in $(0, 5)$. We have $A(0) = A(5) = 0$, so the rectangle of maximum area has width $2x = 10/\sqrt{2}$ cm, height $y = \sqrt{25 - (25/2)} = 5/\sqrt{2}$ cm.

4.5.18 Let the coordinates of the base of the rectangle be $(x, 0)$ and $(-x, 0)$ where $0 \le x \le \sqrt{48}$. Then the width of the rectangle is $2x$ and the height is $48 - x^2$, so the area A is given by $A(x) = 2x(48 - x^2) = 2(48x - x^3)$. The critical points of this function satisfy $A'(x) = 2(48 - 3x^2) = 0$, which has unique solution $x = 4$ in $(0, \sqrt{48})$. Note that $A''(x) = 2(-6x) = -12x$ which is always negative on the domain interval, so $x = 4$ gives a maximum. The biggest rectangle has dimensions 8×32 and has area 256.

4.5.19 Let x be the length of the sides of the base of the box and y be the height of the box. The volume is $x \cdot x \cdot y = 8$, so the constraint is $x^2y = 8$, which gives $y = \dfrac{8}{x^2}$. The objective function to be minimized is the surface area S of the box, which consists of $2x^2$ (for the top and base) + $4xy$ (for the 4 sides); therefore $S = 2x^2 + 4xy$. Using $y = \dfrac{8}{x^2}$, we have $S = 2x^2 + 4xy = 2x^2 + 4x \cdot \dfrac{8}{x^2} = 2x^2 + \dfrac{32}{x}$. The base side length can be any $x > 0$, so we need to maximize $S(x) = 2x^2 + \dfrac{32}{x}$ on the interval $(0, \infty)$. The critical points of the objective function satisfy $S'(x) = 4x - \dfrac{32}{x^2} = 0$; clearing denominators gives $4x^3 = 32$ so $x^3 = 8$, so $x = 2$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. When $x = 2$ we have $y = \dfrac{8}{4} = 2$, so the box with the smallest surface area is $2 \times 2 \times 2$.

4.5.20 Let x be the length of the sides of the base of the box and y be the height of the box. The constraint is $2x+y = 108$, which gives $y = 108-2x$. The objective function to be maximized is the volume V of the box, which is given by $V = x \cdot x \cdot y = x^2y$. Using $y = 108 - 2x$, we have $V = x^2y = x^2(108 - 2x) = 108x^2 - 2x^3$. The length x must be at least 0, and cannot exceed $108/2 = 54$ (otherwise $y < 0$). Therefore we need to maximize $V(x) = 108x^2 - 2x^3$ for $0 \le x \le 54$. The critical points of the objective function satisfy $V'(x) = 216x - 6x^2 = 0$, which has solutions $x = 0$ and $x = 216/6 = 36$. To find the absolute maximum of V, we check the endpoints of $[0, 54]$ and the critical point $x = 36$. Because $V(0) = V(54) = 0$ and $V(36) = 36^3$, the absolute maximum occurs when $x = 36$ in and $y = 108 - 2 \cdot 36 = 36$ in.

4.5.21 Let x be the length of the sides of the base of the box and y be the height of the box. The volume of the box is $x \cdot x \cdot y = x^2y$, so the constraint is $x^2y = 16$, which gives $y = 16/x^2$. Let c be the cost per square foot of the material used to make the sides. Then the cost to make the base is $2cx^2$, the cost to make the 4 sides is $4cxy$, and the cost to make the top is $\frac{1}{2}cx^2$. The objective function to be minimized is the total cost, which is $C = 2cx^2 + 4cxy + \dfrac{1}{2}cx^2 = \dfrac{5}{2}cx^2 + 4cx \cdot \dfrac{16}{x^2} = c\left(\dfrac{5x^2}{2} + \dfrac{64}{x}\right)$. The base side length can be any $x > 0$, so we need to maximize $C(x) = c(5x^2/2 + 64/x)$ on the interval $(0, \infty)$. The critical points of the objective function satisfy $5x - \dfrac{64}{x^2} = 0$, which gives $x^3 = 64/5$ or $x = 4/\sqrt[3]{5}$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. Therefore the box with minimum cost has base $4/\sqrt[3]{5}$ ft by $4/\sqrt[3]{5}$ ft and height $y = 16/(4/\sqrt[3]{5})^2 = 5^{2/3}$ ft.

4.5.22 A point on the line $y = 3x + 4$ has the form $(x, 3x + 4)$, which has distance L to the origin given by $L^2 = x^2 + (3x + 4)^2 = 10x^2 + 24x + 16$. Because L is positive, it suffices to minimize L^2. The quadratic function $10x^2 + 24x + 16$ takes its minimum at $x = -24/20 = -6/5$, and the corresponding value of $y = 2/5$. Therefore the point closest to the origin on this line is $(-6/5, 2/5)$.

4.5.23 A point on the parabola $y = 1 - x^2$ has the form $(x, 1 - x^2)$, which has distance L to the point (1,1) given by $L^2 = (x - 1)^2 + \big(1 - (1 - x^2)\big)^2 = x^4 + x^2 - 2x + 1$. Because L is positive, it suffices to minimize L^2. The critical points of L^2 satisfy $\dfrac{dL^2}{dx} = 4x^3 + 2x - 2 = 2(2x^3 + x - 1) = 0$ This cubic equation has a unique root $x \approx 0.590$, so the point closest to $(1, 1)$ on this parabola is approximately $(0.590, 0.652)$.

4.5.24 The distance between (x, x^2) and $(18, 0)$ is $d(x) = \sqrt{(x^2 - 0)^2 + (x - 18)^2}$. Instead of working with the distance, we can instead work with the square of the distance, because these two functions have minima which occur at the same place. So consider

$$d(x)^2 = D(x) = (x^2)^2 + (x - 18)^2 = x^4 + x^2 - 36x + 324$$

$\dfrac{dD}{dx} = 4x^3 + 2x - 36 = 2(2x^3 + x - 18) = 2(x - 2)\left(2x^2 + 4x + 9\right)$, which has only one real root at $x = 2$. This critical point gives a minimum, so the closest point is $(2, 4)$. The distance at this point is $d(4) = \sqrt{16 + 256} = \sqrt{272} = 4\sqrt{17}$.

4.5.25 The distance between $(x, 3x)$ and $(50, 0)$ is $d(x) = \sqrt{(3x - 0)^2 + (x - 50)^2}$. Instead of working with the distance, we can instead work with the square of the distance, because these two functions have minima which occur at the same place. So consider

$$(d(x))^2 = D(x) = (3x)^2 + (x - 50)^2 = 9x^2 + x^2 - 100x + 2500 = 10(x^2 - 10x + 250).$$

$\dfrac{dD}{dx} = 10(2x - 10)$, which is zero for $x = 5$. Because $\dfrac{d^2D}{dx^2} = 20 > 0$, we see that the critical point at $x = 5$ is a minimum. So the minimum of D (and d) occurs at $x = 5$. The value of d at the point $(5, 15)$, is $d(5) = \sqrt{15^2 + (-45)^2} = 15\sqrt{10} \approx 47.4$.

4.5.26

a. Label the starting point, finishing point and transition point P as in Figure 4.62 in the text. In terms of the angle θ, the swimming distance is $2\sin(\theta/2)$ and the walking distance is $\pi - \theta$, as derived in

Example 3. So the time for the swimming leg is $\frac{\text{distance}}{\text{rate}} = \frac{2\sin(\theta/2)}{2} = \sin\frac{\theta}{2}$ and the time for the walking leg is $\frac{\text{distance}}{\text{rate}} = \frac{\pi-\theta}{4}$. The total travel time for the trip is the objective function $T(\theta) = \sin\frac{\theta}{2} + \frac{\pi-\theta}{4}, \quad 0 \le \theta \le \pi$. The critical points of T satisfy $\frac{dT}{d\theta} = \frac{1}{2}\cos\frac{\theta}{2} - \frac{1}{4} = 0 \quad \text{or} \quad \cos\frac{\theta}{2} = \frac{1}{2}$. Because $0 \le \theta/2 \le \pi/2$, the only solution is given by $\theta/2 = \pi/3$ or $\theta = 2\pi/3$. Evaluating the objective function at the critical point and the endpoints, we find that $T(2\pi/3) = \sqrt{3}/2 + \pi/12 \approx 1.128$ hr, $T(0) = \pi/4 \approx 0.785$ hr and $T(\pi) = 1$ hr. Therefore the minimum travel time is $T(0) \approx 0.785$ hr when the entire trip is done walking. The maximum travel time, corresponding to $\theta = 120^o$, is $T \approx 1.128$ hr.

b. In this case the time for the walking leg is $\frac{\text{distance}}{\text{rate}} = \frac{\pi-\theta}{1.5}$, so the total travel time for the trip is now given by $T(\theta) = \sin\frac{\theta}{2} + \frac{\pi-\theta}{1.5}, \quad 0 \le \theta \le \pi$. The critical points of T satisfy $\frac{dT}{d\theta} = \frac{1}{2}\cos\frac{\theta}{2} - \frac{1}{1.5} = 0 \quad \text{or} \quad \cos\frac{\theta}{2} = \frac{4}{3}$. Therefore in this case there are no critical points. Evaluating the objective function at the endpoints, we find that $T(0) = \pi/1.5 \approx 2.09$ hr and $T(\pi) = 1$ hr. Therefore the minimum travel time is $T(\pi) = 1$ hr when the entire trip is done swimming, and the maximum travel time is $T(0) \approx 2.09$ hr when the entire trip is done walking.

c. Denote the walking speed by $v > 0$. Then the time for the walking leg is $\frac{\text{distance}}{\text{rate}} = \frac{\pi-\theta}{v}$, so the total travel time for the trip is now given by $T(\theta) = \sin\frac{\theta}{2} + \frac{\pi-\theta}{v}, \quad 0 \le \theta \le \pi$. Observe that $\frac{d^2T}{d\theta^2} = -\frac{1}{4}\sin\frac{\theta}{2} < 0$ on the interval $(0,\pi)$. Therefore by the Second Derivative Test, the function T cannot have a local minimum in $(0,\pi)$, and so the minimum travel time must occur either at $\theta = 0$ (all walking) or $\theta = \pi$ (all swimming) in all cases. Evaluating the objective function at the endpoints, we find that $T(0) = \pi/v$ and $T(\pi) = 1$. In the case $v > \pi$ we have $T(0) < 1$ and the minimum corresponds to all walking; when $v < \pi$ we have $T(0) > 1$ and the minimum corresponds to all swimming (and when $v = \pi$ the travel time is 1 hr for both all walking and all swimming). Hence the minimum walking speed for which it is quickest to walk the entire distance is $v = \pi$ m/hr.

4.5.27

a. Let x be the distance from the point on the shoreline nearest to the boat to the point where the woman lands on shore; then the remaining distance she must travel on shore is $6 - x$. By the Pythagorean Theorem, the distance the woman must row is $\sqrt{x^2+16}$. So the time for the rowing leg is $\frac{\text{distance}}{\text{rate}} = \frac{\sqrt{x^2+16}}{2}$ and the time for the walking leg is $\frac{\text{distance}}{\text{rate}} = \frac{6-x}{3}$. The total travel time for the trip is the objective function $T(x) = \frac{\sqrt{x^2+16}}{2} + \frac{6-x}{3}$. We wish to minimize this function for $0 \le x \le 6$. The critical points of the objective function satisfy $T'(x) = \frac{x}{2\sqrt{x^2+16}} - \frac{1}{3} = 0$, which when simplified gives $5x^2 = 64$, so $x = 8/\sqrt{5}$ is the only critical point in $(0,6)$. From the First Derivative Test we see that T has a local minimum at this point, so $x = 8/\sqrt{5}$ must give the minimum value of T on $[0,6]$.

b. Let $v > 0$ be the woman's rowing speed. Then the total travel time is now given by $T(x) = \frac{\sqrt{x^2+16}}{v} + \frac{6-x}{3}$. The derivative of the objective function is $T'(x) = \frac{x}{v\sqrt{x^2+16}} - \frac{1}{3}$. If we try to solve the equation $T'(x) = 0$ as in part (a) above, we see that there is at most one solution $x > 0$. Therefore there can be at most one critical point of T in the interval $(0,6)$. Observe also that $T'(0) = -1/3 < 0$ so the absolute minimum of T on $[0,6]$ cannot occur at $x = 0$. So one of two things must happen: there is a unique critical point for T in $(0,6)$ which is the absolute minimum for T on $[0,6]$, and then $T'(6) > 0$;

or, T is decreasing on $[0,6]$, and then $T'(6) \leq 0$ (the quickest way to the restaurant is to row directly in this case). The condition $T'(6) \leq 0$ is equivalent to $\frac{6}{\sqrt{6^2+16}} \leq \frac{v}{3}$ which gives $v \geq 9/\sqrt{13}$ mi/hr.

4.5.28 Let x be the distance from the point on shore nearest the island to the point where the underwater cable meets the shore, and let y be the be the length of the underwater cable. By the Pythagorean Theorem, $y = \sqrt{x^2+3.5^2}$. The objective function to be minimized is the cost given by $C(x) = 2400\sqrt{x^2+3.5^2} + 1200\cdot(8-x) = 2400\sqrt{x^2+3.5^2} - 1200x + 9600$. We wish to minimize this function for $0 \leq x \leq 8$. The critical points of $C(x)$ satisfy $C'(x) = \frac{2400x}{\sqrt{x^2+3.5^2}} - 1200 = 1200\left(\frac{2x}{\sqrt{x^2+3.5^2}} - 1\right) = 0$, which we solve to obtain $x = 7\sqrt{3}/6$. By the First Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0,8]$. Therefore the optimal point on shore has distance $x = 7\sqrt{3}/6$ mi from the point on shore nearest the island, in the direction of the power station.

4.5.29 Let x be the distance from the point on shore nearest the island to the point where the underwater cable meets the shore, and let y be the be the length of the underwater cable. In terms of the angle θ in the figure, $\tan\theta = 3.5/x$ so $x = 3.5\cot\theta$, and $\sin\theta = 3.5/y$ so $y = 3.5\csc\theta$. The objective function to be minimized is the cost given by $C(\theta) = 2400\cdot 3.5\csc\theta + 1200\cdot(8-3.5\cot\theta) = 8400\csc\theta - 4200\cot\theta + 9600$. The angle θ must be between $\tan^{-1}(3.5/8)$ (≈ 0.412) and $\pi/2$. The critical points of $C(\theta)$ satisfy $C'(\theta) = 8400(-\csc\theta\cot\theta) - 4200(-\csc^2\theta) = 4200\csc^2\theta(1-2\cos\theta)$, which has unique solution $\theta = \pi/3$ in the interval under consideration. By the First Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[\tan^{-1}(3.5/8), \pi/2]$. Therefore the optimal point on shore has distance $x = 3.5\cot(\pi/3) = 7\sqrt{3}/6$ mi from the point on shore nearest the island, in the direction of the power station.

4.5.30 Let L be the ladder length and x be the distance between the foot of the ladder and the fence. The Pythagorean Theorem gives the relationship $L^2 = (x+4)^2 + b^2$, where b is the height of the top of the ladder. We see that $b/(x+4) = 10/x$ by similar triangles, which gives $b = 10(x+4)/x$. Substituting in the expression for L^2 above gives $L^2 = (x+4)^2 + 100\frac{(x+4)^2}{x^2} = (x+4)^2\left(1+\frac{100}{x^2}\right)$. It suffices to minimize L^2 for $x > 0$ because L and L^2 have the same local extrema (L is positive). We have $\frac{d}{dx}L^2 = (x+4)^2\left(-\frac{200}{x^3}\right) + 2(x+4)\left(1+\frac{100}{x^2}\right) = \frac{2(x+4)(x^3-400)}{x^3}$. Because $x > 0$, the only critical point is $x = \sqrt[3]{400} \approx 7.368$. By the First Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0,\infty)$. Substituting $x \approx 7.368$ in the expression for L^2 we find the length of the shortest ladder $L \approx 19.16$ ft.

4.5.31 Let L be the ladder length and x be the distance between the foot of the ladder and the fence. The Pythagorean Theorem gives the relationship $L^2 = (x+5)^2 + b^2$, where b is the height of the top of the ladder. We see that $b/(x+5) = 8/x$ by similar triangles, which gives $b = 8(x+5)/x$. Substituting in the expression for L^2 above gives $L^2 = (x+5)^2 + 64\frac{(x+5)^2}{x^2} = (x+5)^2\left(1+\frac{64}{x^2}\right)$. It suffices to minimize L^2 instead of L. However in this case x and b must satisfy $x, b \leq 20$. Solving $20 = 8(x+5)/x$ for x gives $x = 10/3$, so the condition $b \leq 20$ corresponds to $x \geq 10/3$, and we see that we must minimize L^2 for $10/3 \leq x \leq 20$. We have $\frac{d}{dx}L^2 = (x+5)^2\left(-\frac{128}{x^3}\right) + 2(x+5)\left(1+\frac{64}{x^2}\right) = \frac{2(x+5)(x^3-320)}{x^3}$. Because $x > 0$, the only critical point is $x = \sqrt[3]{320} \approx 6.840$. By the First Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[10/3, 20]$. Substituting $x \approx 6.840$ in the expression for L^2 we find the length of the shortest ladder $L \approx 18.220$ ft.

4.5.32 Let x be the length of the piece of wire used to make the circle; then $60-x$ is the length of the piece used to make the square. Let r be the radius of the circle and s the side length of the square. The circle has circumference $2\pi r$ so we have $x = 2\pi r$ or $r = x/2\pi$; the square has perimeter $4s$ so $60 - x = 4s$

which gives $s = (60 - x)/4$. The objective function to be maximized/minimized is the combined area of the circle and square given by $A = \pi r^2 + s^2 = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{60-x}{4}\right)^2 = \left(\frac{1}{4\pi} + \frac{1}{16}\right)x^2 - \frac{15}{2}x + 225$. The critical points of this function satisfy $A'(x) = \left(\frac{1}{2\pi} + \frac{1}{8}\right)x - \frac{15}{2} = \left(\frac{4+\pi}{8\pi}\right)x - \frac{15}{2} = 0$, which has unique solution $x = 60\pi/(4+\pi) \approx 26.39$. By the First (or Second) Derivative test, this critical point gives a local minimum, which by Theorem 4.5 must be the absolute minimum of A over the interval $[0, 60]$. So the area is minimized by using 26.394 cm of wire for the circle and 33.606 cm of wire for the square. The maximum area must therefore occur at the endpoints, and because $A(0) = 225$, $A(60) \approx 286.479$, the maximum area occurs when all 60 cm of wire is used to make the circle.

4.5.33 If we remove a sector of angle θ from a circle of radius 20, the remaining circumference is $2\pi\cdot 20 - \theta\cdot 20 = 20(2\pi - \theta)$, so the base of the cone formed has radius $r = \frac{20(2\pi-\theta)}{2\pi} = \frac{10(2\pi-\theta)}{\pi}$. As θ varies from 0 to 2π, the radius ranges from 0 to 20, but all possible cones formed have side length 20. The height h of the cone is given by the Pythagorean Theorem: $h^2 + r^2 = 20^2$, so $h = \sqrt{400 - r^2}$. The volume of the cone given by $V = \frac{\pi}{3}r^2 h = \frac{\pi}{3}r^2\sqrt{400 - r^2}$.

Thus

$$\begin{aligned} V'(r) &= \frac{2\pi}{3}r\sqrt{400-r^2} + \frac{\pi}{3}r^2(400-r^2)^{-1/2}\cdot\frac{1}{2}\cdot(-2r) \\ &= \frac{\pi}{3}\cdot\frac{2r(400-r^2) - r^3}{\sqrt{400-r^2}} \\ &= \frac{\pi}{3}\cdot\frac{800r - 3r^3}{\sqrt{400-r^2}}. \end{aligned}$$

The only positive critical point occurs where $r = \sqrt{\frac{800}{3}} = 20\sqrt{\frac{2}{3}}$. An application of the First Derivative Test shows that this is a maximum. So

$$h = \sqrt{400 - \left(20\sqrt{\frac{2}{3}}\right)^2} = \sqrt{400 - 400\cdot\frac{2}{3}} = 20\sqrt{\frac{1}{3}}.$$

4.5.34 Let r and h be the radius and height of the cone; then we have the constraint $r^2 + h^2 = 3^2 = 9$, which gives $r^2 = 9 - h^2$. The objective function to be maximized is the volume of the cone, given by $V = \frac{\pi}{3}r^2 h = \frac{\pi}{3}(9 - h^2)h = \frac{\pi}{3}(9h - h^3)$. Because $r, h \geq 0$ we must have $0 \leq h \leq 3$. Therefore we need to maximize $V(h)$ over $[0, 3]$. The critical points of $V(h)$ satisfy $V'(h) = \frac{\pi}{3}(9 - 3h^2) = \pi(3 - h^2) = 0$, so $h = \sqrt{3}$ is the only critical point in $[0, 3]$. Because $V(0) = V(3) = 0$, the cone of maximum volume has height $h = \sqrt{3}$ and radius $r = \sqrt{6}$.

4.5.35

a. Let r and h be the radius and height of the can. The volume of the can is $V = \pi r^2 h$, which gives the constraint $\pi r^2 h = 354$ or $h = 354/(\pi r^2)$. The objective function to be minimized is the surface area, which consists of $2\pi r^2$ (for the top and bottom of the can) and $2\pi rh$ (for the side of the can). Therefore the objective function to be minimized is $A = 2\pi r^2 + 2\pi rh = 2\pi\left(r^2 + r\left(\frac{354}{\pi r^2}\right)\right) = 2\pi\left(r^2 + \frac{354}{\pi r}\right)$. We need to minimize $A(r)$ for $r > 0$. The critical points of $A(r)$ satisfy $A'(r) = 2\pi\left(2r - \frac{354}{\pi r^2}\right) = 0$, which gives $r = \sqrt[3]{(177/\pi)} \approx 3.834$ cm. The corresponding value of h is $h = \frac{354}{\pi r^2} = \frac{354r}{\pi r^3} = 2r \cdot \frac{177}{\pi r^3} = 2r$, so $h = 2\sqrt[3]{(177/\pi)} \approx 7.667$ cm. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$.

b. We modify the objective function in part (a) above to account for the fact that the top and bottom of the can have double thickness: $A = 4\pi r^2 + 2\pi rh = 2\pi\left(2r^2 + r\left(\frac{354}{\pi r^2}\right)\right) = 4\pi\left(r^2 + \frac{177}{\pi r}\right)$. We need to minimize $A(r)$ for $r > 0$. The critical points of $A(r)$ satisfy $A'(r) = 4\pi\left(2r - \frac{177}{\pi r^2}\right) = 0$, which gives $r = \sqrt[3]{(177/2\pi)} \approx 3.043$ cm. The corresponding value of h is $h = \frac{354}{\pi r^2} = \frac{354r}{\pi r^3} = 4r \cdot \frac{177}{2\pi r^3} = 4r$, so $h = 4\sqrt[3]{(177/2\pi)} \approx 12.171$ cm. These dimensions are closer to those of a real soda can.

4.5.36 If we fill the pot with just enough water to cover the marble, the water in the pot will have height $2r$. Because the pot has radius 4, the water and marble together have volume $\pi \cdot 4^2 \cdot 2r = 32\pi r$. The marble has volume $4\pi r^3/3$, so the volume of water needed to cover the marble is $V(r) = 32\pi r - \frac{4}{3}\pi r^3$. The critical points of this function satisfy $V'(r) = 32\pi - 4\pi r^2 = 0$, which has unique solution $r = \sqrt{8} = 2\sqrt{2}$ cm. By the First (or Second) Derivative test, this critical point gives a local maximum, which by Theorem 4.5 must be the absolute maximum of A over the interval $[0, 4]$.

4.5.37 Let x and y be the dimensions of the flower garden; the area of the flower garden is 30, so we have the constraint $xy = 30$ which gives $y = 30/x$. The dimensions of the garden and borders are $x + 4$ and $y + 2$, so the objective function to be minimized for $x > 0$ is $A = (x+4)(y+2) = (x+4)\left(\frac{30}{x} + 2\right) = 2x + \frac{120}{x} + 38$. The critical points of $A(x)$ satisfy $A'(x) = 2 - \frac{120}{x^2} = 0$, which has unique solution $x = \sqrt{60} = 2\sqrt{15}$. By the First (or Second) Derivative test, this critical point gives a local minimum, which by Theorem 4.5 must be the absolute minimum of A over $(0, \infty)$. The corresponding value of y is $30/2\sqrt{15} = \sqrt{15}$, so the dimensions are $\sqrt{15}$ by $2\sqrt{15}$ m.

4.5.38

a. Suppose the side on the x-axis extends to the point $(a, 0)$ and the side on the y-axis to $(0, b)$. Then $b = 10 - 2a$ and the rectangle has area $A = ab = a(10 - 2a) = 10a - 2a^2$. We must have $0 \le a \le 5$ to ensure that both $a, b \ge 0$. The critical points of $A(a)$ satisfy $A'(a) = 10 - 4a = 0$, which has unique solution $a = 5/2$. Because $A(0) = A(5) = 0$, $a = 5/2$ gives the maximum area. The corresponding b value is 5, and the maximum area is $25/2$.

b. Let $(a, 0)$ be the vertex on the x-axis and $(0, b)$ the vertex on the y-axis, and label the two vertices on the line $y = 10-2x$ as P and Q. The line joining $(a, 0)$ and $(0, b)$ must be parallel to the line $y = 10-2x$, which has slope -2. This gives the constraint $\frac{0-b}{a-0} = -\frac{b}{a} = -2$, so $b = 2a$. The segment joining $(a, 0)$ and $(0, b)$ has length $\sqrt{a^2 + b^2} = \sqrt{5}a$. The other side length of the rectangle can be found by observing that the triangle with vertices (0,0), $(a, 0)$, $(0, b)$ is similar to the triangle with vertices P, $(5, 0)$, $(a, 0)$

in that order, so the remaining side of the rectangle has length l satisfying $\frac{l}{5-a} = \frac{2a}{\sqrt{5}a}$; hence $l = \frac{2}{\sqrt{5}}(5-a)$. Therefore the area of the rectangle is given by $A(a) = \sqrt{5}a\frac{2}{\sqrt{5}}(5-a) = 10a - 2a^2$, which is the same function as in part (a) above. The maximum again occurs when $a = 5/2$, and the dimensions of the rectangle of maximum area are $5\sqrt{5}/2$ and $\sqrt{5}$.

4.5.39 Because the length of the box (in inches) is $18 - 2x$, the width is $9 - x$, and the height is x, the volume of the box is $V(x) = x(9-x)(18-2x) = 2x^3 - 36x^2 + 162x$, where $0 \le x \le 9$. Taking the derivative of V, we have $V'(x) = 6x^2 - 72x + 162 = 6(x^2 - 12x + 27) = 6(x-3)(x-9)$. Setting the derivative equal to 0 and solving for x, we find that $x = 3$ and $x = 9$. Because $V(0) = V(9) = 0$ and $V(3) = 216$, the box of maximum volume is $12 \times 6 \times 3$ and has a volume of 216 in^3.

4.5.40

a. The dimensions of the box are $5-2x$, $8-2x$, and x, so the volume is given by $V(x) = x(5-2x)(8-2x)$. The dimensions cannot be negative, so we must have $0 \le x \le 2.5$. The critical points of $V(x)$ satisfy

$$\begin{aligned} V'(x) &= (5-2x)(8-2x) + x(-2)(8-2x) + x(5-2x)(-2) \\ &= 40 - 10x - 16x + 4x^2 - 16x + 4x^2 - 10x + 4x^2 \\ &= 12x^2 - 52x + 40 = 4(3x-10)(x-1) = 0. \end{aligned}$$

Thus the critical points are $x = 0$ and $x = 1$ (the other root $x = 10/3$ isn't in the domain). We have $V(0) = V(2.5) = 0$, and $V(1) = 18$, so the maximum is 18 cubic feet.

b. In this case the dimensions of the box are $l - 2x$, $l - 2x$ and x, so the volume is given by $V(x) = x(l-2x)^2 = 4x^3 - 4lx^2 + l^2x$. The dimensions cannot be negative, so we must have $0 \le x \le l/2$. The critical points of $V(x)$ satisfy $V'(x) = 12x^2 - 8lx + l^2 = (6x - l)(2x - l) = 0$. This quadratic equation has roots $x = l/6$ and $l/2$, so the only critical point in $(0, l/2)$ is $x = l/6$. We have $V(0) = V(l/2) = 0$, so the maximum volume is $V(l/6) = 2l^3/27$.

4.5.41 Let x and y equal the width and height, respectively, of the rectangular part of the window. Then the perimeter is $x + 2y + \frac{\pi x}{2} = 20$ and the area of the window is $A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = xy + \frac{\pi x^2}{8}$. Solving the perimeter equation for y, we have $y = \frac{1}{2}\left(20 - x - \frac{\pi x}{2}\right) = 10 - \frac{x}{2} - \frac{\pi x}{4}$. By substitution, $A(x) = x\left(10 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{1}{8}\pi x^2 = \left(\frac{\pi}{8} - \frac{1}{2} - \frac{\pi}{4}\right)x^2 + 10x = \left(-\frac{1}{2} - \frac{\pi}{8}\right)x^2 + 10x$. Taking the derivative of A, we have $A'(x) = \left(-1 - \frac{2\pi}{8}\right)x + 10$, which is 0 for $x = \frac{10}{1 + 2\pi/8} = \frac{40}{4+\pi} \approx 5.6$ ft. At this x value, $y \approx 2.8$ ft. Because $A''\left(\frac{40}{4+\pi}\right) < 0$, the Second Derivative Test implies that the area is maximized when the dimensions of the rectangular pane are approximately 5.6 ft wide by 2.8 ft high.

4.5.42 Suppose that the base of the rectangle has dimension x and the side of the rectangle has dimension y. The semicircular pane has radius $x/2$, so the perimeter of the window is $P = 2y + x + \frac{1}{2} \cdot 2\pi \cdot \frac{x}{2} = 2y + \left(1 + \frac{\pi}{2}\right)x$ which gives the constraint $y = \frac{P}{2} - \left(\frac{1}{2} + \frac{\pi}{4}\right)x$. The rectangular pane has area xy and the semicircular pane has area $(1/2)\pi(x/2)^2$, so the amount of light transmitted through the window is proportional to $L = 2xy + \frac{\pi x^2}{8} = 2x\left(\frac{P}{2} - \left(\frac{1}{2} + \frac{\pi}{4}\right)x\right) + \frac{\pi x^2}{8} = Px - \left(1 + \frac{3\pi}{8}\right)x^2$. Because $x, y \ge 0$ we must have $0 \le x \le P/(1 + (\pi/2)) \approx 0.389P$. The quadratic function $L(x)$ has maximum at $x = \frac{P}{2 + \frac{3\pi}{4}} = \frac{4P}{8 + 3\pi} \approx 0.230P$, which is in the interval under consideration. The corresponding value of y is

$$y = \frac{P}{2} - \left(\frac{1}{2} + \frac{\pi}{4}\right)\frac{4P}{8+3\pi} = \frac{4+\pi}{16+6\pi}P \approx 0.205P.$$

4.5.43 The radius r and height h of the barrel satisfy the constraint $r^2 + h^2 = d^2$, which we can rewrite as $r^2 = d^2 - h^2$. The volume of the barrel is given by $V = \pi r^2 h = \pi(d^2 - h^2)h = \pi(d^2 h - h^3)$. The height h must satisfy $0 \le h \le d$, so we need to maximize $V(h)$ on the interval $[0, d]$. The critical points of V satisfy $V'(h) = \pi(d^2 - 3h^2) = 0$. The only critical point in $(0, d)$ is $h = d/\sqrt{3}$, which gives the maximum volume because at the endpoints $V(0) = V(d) = 0$. The corresponding r value satisfies $r^2 = d^2 - d^2/3 = 2d^2/3$, so $r = \sqrt{2}d/\sqrt{3}$ and we see that the ratio r/h that maximizes the volume is $\sqrt{2}$.

4.5.44 The objective function to be minimized is the average number of tests required, given by $A(x) = N\left(1 - q^x + \frac{1}{x}\right)$ where x is the group size, $N = 10{,}000$ and $q = 0.95$. We may assume that $1 \le x \le 10{,}000$. The critical points of this function satisfy $A'(x) = N\left(-(\ln q)q^x - \frac{1}{x^2}\right) = 0$, which is equivalent to the equation $(0.95)^{-x} + \ln(0.95)x^2 = 0$. Using a numerical solver, we find that this equation has one root between 5 and 6, and one between 132 and 133. By the First Derivative Test, we see that the smaller of these roots gives a local minimum and the larger a local maximum. Therefore the minimum value of $A(x)$ for $1 \le x \le 10,000$ occurs either at the smaller root or at the endpoint 10,000. The group size x must be an integer, so the possible optimal choices are $x = 5, 6$ and 10,000; comparing the value of $A(x)$ at these points shows that $x = 5$ is the optimal group size.

4.5.45 Let r and h be the radius and height of the cylinder. The distance d from the centroid of the cylinder (the midpoint of the cylinder's axis of rotation) to any point on the top or bottom edge satisfies $d^2 = r^2 + \left(\frac{h}{2}\right)^2$ so the constraint is $r^2 + (h/2)^2 = R^2$. The volume of the cylinder is given by $V = \pi r^2 h = \pi\left(R^2 - \left(\frac{h}{2}\right)^2\right)h = \pi\left(R^2 h - \frac{h^3}{4}\right)$. Because $r, h \ge 0$ we must have $0 \le h \le 2R$. We wish to maximize $V(h)$ on this interval. The critical points of $V(h)$ satisfy $V'(h) = \pi\left(R^2 - \frac{3h^2}{4}\right) = 0$ which gives $h = 2R/\sqrt{3}$, and from the constraint we obtain $r = \sqrt{2}R/\sqrt{3}$. The volume $V(h) = 0$ at the endpoints $h = 0$ and $h = 2R$, so the maximum volume must occur at this critical point.

4.5.46 Let x be the number of tickets sold. The cost per ticket is $30 - 0.25x$, and the fixed expenses are 200, so the profit is $P(x) = x(30 - 0.25x) - 200 = -0.25x^2 + 30x - 200$. We wish to maximize this function for $20 \le x \le 70$. The critical points of $P(x)$ satisfy $P'(x) = -0.5x + 30 = 0$ so $x = 60$ is the only critical point. By the First (or Second) Derivative Test, this critical point corresponds to a local maximum, and by Theorem 4.5, this solitary local maximum is also the absolute maximum on the interval $[20, 70]$. Therefore the profit is maximized by selling 60 tickets.

4.5.47 Let R and H be the radius and height of the larger cone and let r and h be the radius and height of the smaller inscribed cone. The region that lies above the smaller cone inside the larger cone is a cone with radius r and height $H - h$; by similar triangles we have $\frac{H-h}{r} = \frac{H}{R}$ so $h = \frac{H}{R}(R - r)$. The volume of the smaller cone is $V = \frac{\pi}{3}r^2 h = \frac{\pi H}{3R}\left(Rr^2 - r^3\right)$, which we must maximize over $0 \le r \le R$. The critical points of $V(r)$ satisfy $V'(r) = \frac{\pi H}{3R}\left(2Rr - 3r^2\right) = 0$ which has unique solution $r = 2R/3$ in $(0, R)$. Because $V(r) = 0$ at the endpoints $r = 0$ and $r = R$, the smaller cone with maximum volume has radius $r = 2R/3$ and height $h = H/3$, so the optimal ratio of the heights is 3:1.

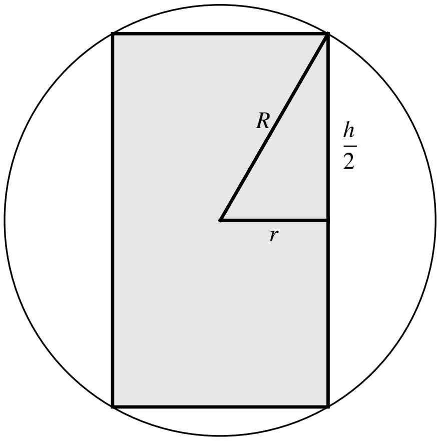
4.5.48 The objective function is the volume of the cylinder, which is $V = \pi r^2 h$. By using the right triangle in the diagram below, we see that $r^2 = R^2 - \frac{h^2}{4}$. Thus we can write $V = \pi\left(R^2 - \frac{h^2}{4}\right)h = \pi\left(R^2 h - \frac{h^3}{4}\right)$, which is valid for $0 < h < 2R$. Differentiating leads to $V' = \pi\left(R^2 - \frac{3h^2}{4}\right)$. This is zero for $h^2 = \frac{4R^2}{3}$, or $h = \frac{2R}{\sqrt{3}}$. Using the Second Derivative Test, it can be verified that this critical point yields a maximum for

V. At this value of h, we have

$$r^2 = R^2 - \frac{h^2}{4} = R^2 - \frac{4R^2}{12} = \frac{8R^2}{12} = \frac{2R^2}{3},$$

so $r = \dfrac{\sqrt{2}R}{\sqrt{3}}$. The corresponding volume is

$$V = \pi r^2 h = \frac{\pi(2R^2)(2R)}{3\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}.$$



4.5.49

a. We find $g(0) = 0$, $g(40) = 30$ and $g(60) = 25$ miles per gallon. The value at $v = 0$ is reasonable because when a car first starts moving it needs a lot of power from its engine, so the gas mileage is very low. The decline from 30 to 25 mi/gal as v increases from 40 mi/hr to 60 mi/hr reflects the fact that gas mileage tends to decrease at speeds over 55 mi/hr.

b. The quadratic function $g(v) = (85v - v^2)/60$ takes its maximum value at $v = 85/2 = 42.5$ mi/hr.

c. At speed v the amount of gas needed to drive one mile is $1/g(v)$ and the time it takes is $1/v$. Hence the cost of gas for one mile is $p/g(v)$ and the cost for the driver is w/v, and so the cost for L miles is $C(v) = Lp/g(v) + Lw/v$.

d. We have $C(v) = 400\left(\dfrac{4}{g(v)} + \dfrac{20}{v}\right) = 1600\left(\dfrac{1}{g(v)} + \dfrac{5}{v}\right)$. The critical points of $C(v)$ satisfy $\dfrac{g'(v)}{g(v)^2} + \dfrac{5}{v^2} = 0$, which simplifies to $v^2 g'(v) + 5g(v)^2 = 0$. Substituting the formula for $g(v)$ above and using $g'(v) = (85 - 2v)/60$, we can factor out v^2 and reduce to the quadratic equation $v^2 - 194v + 8245 = 0$, which has roots $v \approx 62.883, 131.117$. The First (or Second) Derivative Test shows that $C(v)$ has a local minimum at $v \approx 62.9$, which is the unique critical point for $0 \le v \le 131$. Therefore the cost is minimized at this value of v.

e. Because L is a constant factor in the cost function $C(v)$, changing L will not change the critical points of $C(v)$.

f. The critical points of $C(v)$ now satisfy the equation $\dfrac{4.2g'(v)}{g(v)^2} + \dfrac{20}{v^2} = 0$, which simplifies to $4.2v^2 g'(v) + 20g(v)^2 = 0$. As above, substituting the formula for $g(v)$ above and using $g'(v) = (85 - 2v)/60$, we can factor out v^2 and reduce to the quadratic equation $v^2 - 195.2v + 8296 = 0$, which has roots $v \approx 62.532, 132.668$. As in part (d), the minimum cost occurs for $v \approx 62.532$, slightly less than the speed in part (d).

g. The critical points of $C(v)$ now satisfy the equation $\frac{4g'(v)}{g(v)^2}+\frac{15}{v^2}=0$, which simplifies to $4v^2g'(v)+15g(v)^2=0$. As above, substituting the formula for $g(v)$ above and using $g'(v)=(85-2v)/60$, we can factor out v^2 and reduce to the quadratic equation $v^2-202v+8585=0$, which has roots $v\approx 60.800, 141.200$. As in part (d), the minimum cost occurs for $v\approx 60.8$, less than the speed in part (d).

4.5.50

a. The dog runs distance $z-y$ and swims distance $\sqrt{x^2+y^2}$. Using the fact that time is distance/speed, we see that the total time it takes the dog to get to the tennis ball is $T(y)=\frac{z-y}{r}+\frac{\sqrt{x^2+y^2}}{s}$.

b. The critical points of $T(y)$ satisfy $T'(y)=-\frac{1}{r}+\frac{y}{s\sqrt{x^2+y^2}}=0$, which simplifies to $\frac{\sqrt{x^2+y^2}}{y}=\frac{r}{s}$ so $\left(\frac{r^2}{s^2}-1\right)y^2=x^2$, so $y=\frac{x}{\sqrt{r/s+1}\sqrt{r/s-1}}$. The First (or Second) Derivative Test shows that a local minimum occurs for this value of y, which by Theorem 4.5 must give the absolute minimum of $T(y)$ for $y>0$. If $y\le z$ then $T(y)$ is minimized for this value of y; otherwise, the minimum occurs at $y=z$ (all swimming).

c. The ratio is $\frac{y}{x}=\frac{1}{\sqrt{8+1}\sqrt{8-1}}=\frac{1}{\sqrt{63}}$.

d. Elvis's optimal ratio is $\frac{y}{x}\approx\frac{1}{\sqrt{7.033+1}\sqrt{7.033-1}}\approx 0.144$, so Elvis appears to know calculus!

4.5.51 The viewing angle θ is given by $\theta=\cot^{-1}\left(\frac{x}{10}\right)-\cot^{-1}\left(\frac{x}{3}\right)$, and we wish to maximize this function for $x>0$. The critical points satisfy $\theta'(x)=-\frac{1}{1+\left(\frac{x}{10}\right)^2}\cdot\frac{1}{10}-(-)\frac{1}{1+\left(\frac{x}{3}\right)^2}\cdot\frac{1}{3}=\frac{3}{x^2+3^2}-\frac{10}{x^2+10^2}=0$ which simplifies to $3(x^2+100)=10(x^2+9)$ or $x^2=30$. Therefore $x=\sqrt{30}\approx 5.477$ ft is the only critical point in $(0,\infty)$. By the First (or Second) Derivative Test, this critical point corresponds to a local maximum, and by Theorem 4.5, this solitary local maximum must be the absolute maximum on the interval $(0,\infty)$.

4.5.52 The two cables joined to the ceiling each have length $\sqrt{x^2+1}$ by the Pythagorean Theorem, and the vertical cable has length $6-x$. The objective function to be minimized is the total length of the three cables, given by $L(x)=2\sqrt{x^2+1}+6-x$. Because the lengths cannot be negative, we must have $0\le x\le 6$. The critical points of $L(x)$ satisfy $L'(x)=\frac{2x}{\sqrt{x^2+1}}-1=0$, which occurs when $3x^2=1$, so $x=1/\sqrt{3}=\sqrt{3}/3$ is the unique critical point in $(0,6)$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0,6]$. Therefore the cables should be joined at distance $x=\sqrt{3}/3$ m below the ceiling.

4.5.53 Let x be the distance between the point and the weaker light source; then $12-x$ is the distance to the stronger light source. The intensity is proportional to $I(x)=\frac{1}{x^2}+\frac{2}{(12-x)^2}$, so we can take this as our objective function to be minimized for $0<x<12$. The critical points of $I(x)$ satisfy $I'(x)=-\frac{2}{x^3}+\frac{4}{(12-x)^3}=0$ which gives $\left(\frac{12-x}{x}\right)^3=2$, or $\frac{12-x}{x}=\sqrt[3]{2}$, or $x=\frac{12}{\sqrt[3]{2}+1}\approx 5.310$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0,12)$. Therefore the intensity is weakest at the point $12/(\sqrt[3]{2}+1)\approx 5.310$ m from the weaker source.

4.5.54

a. Setting $h(18) = 10$ gives $\dfrac{-32(18^2)}{v^2}+18+8 = 10$, so $\dfrac{32(18)^2}{v^2} = 16$, so $v^2 = 2(18)^2$, and $v = 18\sqrt{2} \approx 25.46$ ft/s.

b. Using the result above, we have $h(x) = -\dfrac{4x^2}{81} + x + 8$. We know that the center of the hoop is located at the point (18,10) and because the radius of the hoop is 0.75 feet, the front of the hoop is located at the point (17.25,10).

In this case, the distance formula is

$$s = \sqrt{(x-17.25)^2 + \left(-\frac{4x^2}{81} + x + 8 - 10\right)^2} = \sqrt{(x-17.25)^2 + \left(-\frac{4x^2}{81} + x - 2\right)^2}.$$

c. A women's basketball has a radius of slightly less than 4.62 inches. To determine the value of x at which the ball reaches its closest distance from the rim, we calculate the derivative of the function underneath the square root, set it equal to 0 and solve for x:

$$2(x-17.25) + 2\left(-\frac{4x^2}{81} + x - 2\right)\left(-\frac{8}{81}x + 1\right) = 0$$

. Using a root-finder, we find that $x \approx 17.52$, which means that the center of the ball is at its closest point to the front of the rim when $x \approx 17.52$ ft. Plugging this value of x into the formula for s, we find that $s \approx 0.45$ ft or about 5.42 inches. So at its closest point, the center of the ball is 5.42 inches from the front of the rim and because this distance is greater than the radius of the basketball, the ball will not hit the front of the rim.

d. A men's basketball has a radius of about 4.75 inches, which is less than the value of s found in part (c). So the men's ball will not hit the front of the rim either.

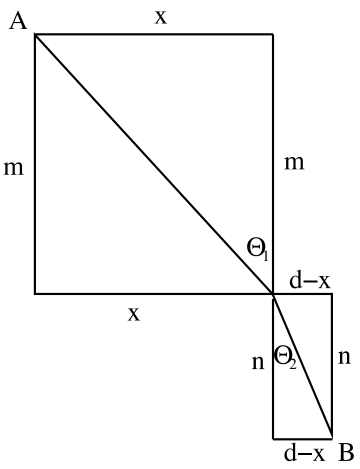
4.5.55

a. Let x, $d-x$ be the distances from the point where the rope meets the ground to the poles of height m, n respectively. Then the rope has length $L(x) = \sqrt{x^2+m^2} + \sqrt{(d-x)^2+n^2}$. We wish to minimize this function for $0 \le x \le d$. The critical points of $L(x)$ satisfy $L'(x) = \dfrac{x}{\sqrt{x^2+m^2}} - \dfrac{d-x}{\sqrt{(d-x)^2+n^2}} = 0$, which is equivalent to $\dfrac{x}{\sqrt{x^2+m^2}} = \dfrac{d-x}{\sqrt{(d-x)^2+n^2}}$, or in terms of the angles θ_1 and θ_2 in the figure, $\sec\theta_1 = \sec\theta_2$ and therefore $\theta_1 = \theta_2$. Observe that $L'(0) < 0$ and $L'(d) > 0$, so the minimum value of $L(x)$ must occur at some $x \in (0, d)$. There must be exactly one critical point, because as x ranges from 0 to d, θ_1 decreases and θ_2 increases, and so $\theta_1 = \theta_2$ can occur for at most one value of x.

b. Because the speed of light is constant, travel time is minimized when distance is minimized, which we saw in part (a) occurs when $\theta_1 = \theta_2$.

4.5.56

Let x, d, m, n be the distances labeled in the figure below. Then using time = distance/speed, we see that the time for light to travel from A to B is $T(x) = \dfrac{\sqrt{x^2+m^2}}{v_1} + \dfrac{\sqrt{(d-x)^2+n^2}}{v_2}$. We wish to minimize this function for $0 \le x \le d$. The critical points of $T(x)$ satisfy $T'(x) = \dfrac{x}{v_1\sqrt{x^2+m^2}} - \dfrac{d-x}{v_2\sqrt{(d-x)^2+n^2}} = 0$, which is equivalent to $\dfrac{x}{v_1\sqrt{x^2+m^2}} = \dfrac{d-x}{v_2\sqrt{(d-x)^2+n^2}}$, or in terms of the angles θ_1 and θ_2 in the figure, $\dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$ (Snell's Law). Observe that $T'(0) < 0$ and $T'(d) > 0$, so the minimum value of $T(x)$ must occur at some $x \in (0, d)$. There must be exactly one critical point, because as x ranges from 0 to d, $\sin\theta_1$ decreases and $\sin\theta_2$ increases, so Snell's Law can hold for at most one value of x.



4.5.57 Let h be the height of the cylindrical tower and r the radius of the dome. The cylinder has volume $\pi r^2 h$, and the hemispherical dome has volume $2\pi r^3/3$ (half the volume of a sphere of radius r). The total volume is 750, so we have the constraint $\pi r^2 h + \dfrac{2\pi r^3}{3} = 750$ which gives $h = \dfrac{750}{\pi r^2} - \dfrac{2r}{3}$. We must have $h \ge 0$, which is equivalent to $r \le \sqrt[3]{1125/\pi}$. The objective function to be maximized is the cost of the metal to make the silo, which is proportional to the surface area of the cylinder ($= 2\pi r h$) plus 1.5 times the surface area of the hemisphere ($= 2\pi r^2$). So we can take as objective function

$$C = 2\pi r h + 1.5 \cdot 2\pi r^2 = 2\pi r\left(\frac{750}{\pi r^2} - \frac{2r}{3}\right) + 3\pi r^2 = \frac{1500}{r} + \frac{5}{3}\pi r^2.$$

The critical points of $C(r)$ satisfy $C'(r) = -\dfrac{1500}{r^2} + \dfrac{10}{3}\pi r = 0$, which gives $\pi r^3 = 450$ and hence $r = \sqrt[3]{450/\pi}$. The corresponding value of h is

$$h = \frac{750}{\pi r^2} - \frac{2r}{3} = \frac{750r}{\pi r^3} - \frac{2r}{3} = \left(\frac{750}{450} - \frac{2}{3}\right) r = r.$$

By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0, \sqrt[3]{1125/\pi}]$. Therefore the dimensions that minimize the cost are $r = h = \sqrt[3]{450/\pi}$ m.

4.5.58

a. Using the figure, observe that $\sin\theta = \dfrac{l_3}{l_2}$ and $\cot\theta = \dfrac{l_4 - l_1}{l_3}$. It follows that $l_2 = l_3 \csc\theta$ and $l_1 = l_4 - l_3 \cot\theta$. So by substitution, we have

$$T = k\left(\frac{l_1}{r_1^4} + \frac{l_2}{r_2^4}\right) = k\left(\frac{l_4 - l_3\cot\theta}{r_1^4} + \frac{l_3\csc\theta}{r_2^4}\right).$$

b. Differentiating gives

$$\frac{dT}{d\theta} = k\left(\frac{l_3}{r_1^4}\csc^2\theta - \frac{l_3}{r_2^4}\csc\theta\cot\theta\right) = k\left(\frac{l_3}{r_1^4}\csc^2\theta - \frac{l_3}{r_2^4}\cdot\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}\right) = kl_3\csc^2\theta\left(\frac{1}{r_1^4} - \frac{\cos\theta}{r_2^4}\right).$$

For $0 < \theta < \pi/2$, this can only be zero for $\cos\theta = \dfrac{r_2^4}{r_1^4} = \left(\dfrac{r_2}{r_1}\right)^4$. To verify that this corresponds to a minimum using the First Derivative Test, we rewrite the derivative in a different form:

$$\frac{dT}{d\theta} = kl_3\csc^2\theta\left(\frac{1}{r_1^4} - \frac{\cos\theta}{r_2^4}\right) = \frac{kl_3\csc^2\theta}{r_2^4}\cdot\left(\frac{r_2^4}{r_1^4} - \cos\theta\right).$$

Suppose that $\theta^* < \pi/2$ is the solution to $\cos\theta = \left(\frac{r_2}{r_1}\right)^4$. If $\theta < \theta^*$, then $\cos\theta > \cos\theta^* = \left(\frac{r_2}{r_1}\right)^4$, which implies that $\frac{dT}{d\theta} < 0$. Similarly, if $\theta > \theta^*$, then $\frac{dT}{d\theta} > 0$. So by the First Derivative Test, T has a local minimum at $\cos\theta = \left(\frac{r_2}{r_1}\right)^4$. Because the local minimum corresponds to the only critical point in $(0, \pi/2)$, it is an absolute minimum.

c. We have $r_2 = 0.85r_1$, which implies that $\cos\theta = 0.85^4$. Using a calculator, we find that $\theta \approx 1.02$ rad which is about 58.5°.

4.5.59

a. $f'(x) = 2(x - a_1) + 2(x - a_2) = 4x - 2(a_1 + a_2)$. This is zero for $x = \frac{a_1 + a_2}{2}$. Because $f'(x) < 0$ for $x < \frac{a_1 + a_2}{2}$ and $f'(x) > 0$ for $x > \frac{a_1 + a_2}{2}$, we have a minimum at $x = \frac{a_1 + a_2}{2}$.

b. $f'(x) = 2(x - a_1) + 2(x - a_2) + 2(x - a_3) = 6x - 2(a_1 + a_2 + a_3)$. This is zero for $x = \frac{a_1 + a_2 + a_3}{3}$. Because $f'(x) < 0$ for $x < \frac{a_1 + a_2 + a_3}{3}$ and $f'(x) > 0$ for $x > \frac{a_1 + a_2 + a_3}{3}$, we have a minimum at $x = \frac{a_1 + a_2 + a_3}{3}$.

c. $f'(x) = 2\sum_{k=1}^{n}(x - a_k) = 2nx - 2\sum_{k=1}^{n} a_k$. This is zero when $x = \frac{\sum_{k=1}^{n} a_k}{n}$. An application of the First Derivative Test shows that this value of x yields a minimum.

4.5.60 We have $x = 100\tan\theta$, so the rate at which the beam sweeps along the highway is

$$\frac{dx}{dt} = 100\sec^2\theta\frac{d\theta}{dt} = 100\sec^2\theta \cdot \frac{\pi}{6} = \frac{50\pi}{3}\sec^2\theta.$$

The beam meets the highway provided that the angle θ satisfies $-\pi/2 < \theta < \pi/2$. The function $\sec^2\theta$ is unbounded on this interval, and so has no maximum. The minimum value occurs at $\theta = 0$, because everywhere else $\sec^2\theta > 1$. Therefore the minimum rate is $50\pi/3 \approx 52.360$ m/s, and there is no maximum rate.

4.5.61 The cross-section is a trapezoid with height $3\sin\theta$; the larger of the parallel sides has length $3 + 2 \cdot 3\cos\theta = 3 + 6\cos\theta$ and the smaller parallel side has length 3. The area of this trapezoid is given by

$$A(\theta) = \frac{1}{2}\left(3 + (3 + 6\cos\theta)\right) \cdot 3\sin\theta = 9(1 + \cos\theta)\sin\theta = 9\left(\sin\theta + \frac{\sin 2\theta}{2}\right),$$

using the identity $\sin 2\theta = 2\sin\theta\cos\theta$. We wish to maximize this function for $0 \le \theta \le \pi/2$. The critical points of $A(\theta)$ satisfy $\cos\theta + \cos 2\theta = \cos\theta + 2\cos^2\theta - 1 = 0$, using the identity $\cos 2\theta = 2\cos^2\theta - 1$. Therefore $x = \cos\theta$ satisfies the quadratic equation $2x^2 + x - 1 = 0$, which has roots $x = 1/2$ and -1. So the only critical point in $(0, \pi/2)$ is $\theta = \cos^{-1}(1/2) = \pi/3$, which by the First (or Second) Derivative Test and Theorem 4.5 gives the maximum area.

4.5.62

a. Gliding is more efficient if $S > 0$; substituting $m = 200$ in the equation for $S(m, \theta)$ gives

$$S > 0 \iff 8.46 \cdot 200^{2/3} - 1.36 \cdot 200\tan\theta > 0 \iff \tan\theta < 1.064$$

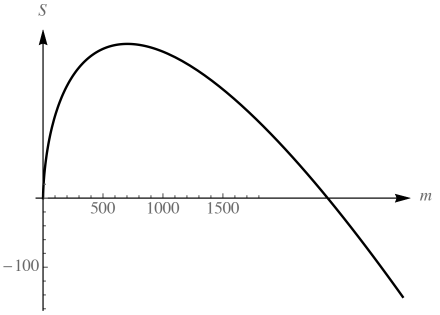
so $\theta < \tan^{-1} 1.064 \approx 46.768°$.

b. We solve $S(m,\theta)=0$ for θ, which gives $\tan\theta \approx 6.22m^{-1/3}$, so $\theta = g(m) \approx \tan^{-1}(6.22m^{-1/3})$. This is a decreasing function of body mass.

c. Because $\theta = g(m)$ is a decreasing function of body mass, larger gliders have a smaller selection of glide angles for which gliding is more efficient than walking.

d.

Gliding is more efficient when $S(m, 25°) > 0$, which is true if and only if $8.46m^{2/3} > 1.36m\tan 25°$, which is true if and only if $m^{1/3} < \dfrac{8.46}{1.36\tan 25°} \approx 13.34$, which again is true if and only if $m < 2374$ g.



e. We have $\frac{d}{dm}S(m,\theta) = 5.64m^{-1/3} - 1.36\tan\theta$, which is 0 for $m^* \approx 71.32\cot^3\theta$. For $\theta = 25°$ this gives $m^* \approx 703$ g. The First Derivative Test shows that m^* is a local maximum, which by Theorem 4.5 is the absolute maximum for $m \geq 0$.

f. Because $\cot\theta$ is a decreasing function on $0 < \theta < 90°$, we see that m^* decreases with increasing θ.

g. From part (b) we have $g(10^6) \approx \tan^{-1}(6.22m^{-1/3}) \approx 3.56°$, so any angle $\theta < 3.56°$.

4.5.63 Let the radius of the Ferris wheel have length r, and let α be the angle the specific seat on the Ferris wheel makes with the center of the wheel (see the figure in the text). This point has coordinates $(r\cos\alpha, r + r\sin\alpha)$ so the distance from the seat to the base of the wheel is

$$d = \sqrt{r^2\cos^2\alpha + r^2(1+\sin\alpha)^2} = \sqrt{2}r\sqrt{1+\sin\alpha}.$$

Therefore the observer's angle satisfies $\tan\theta = \dfrac{r\sqrt{2}}{20}\sqrt{1+\sin\alpha}$. Think of θ and α as functions of time t and differentiate: $\sec^2\theta\dfrac{d\theta}{dt} = \dfrac{r\sqrt{2}}{20}\cdot\dfrac{\cos\alpha}{2\sqrt{1+\sin\alpha}}\dfrac{d\alpha}{dt} = \dfrac{\pi r\sqrt{2}}{40}\cdot\dfrac{\cos\alpha}{\sqrt{1+\sin\alpha}}$. Therefore

$$\frac{d\theta}{dt} = \frac{\pi r\sqrt{2}}{40}\,\frac{\cos^2\theta\cos\alpha}{\sqrt{1+\sin\alpha}}.$$

Observe that $\left|\dfrac{d\theta}{dt}\right| = \dfrac{\pi r\sqrt{2}}{40}\dfrac{\cos^2\theta|\cos\alpha|}{\sqrt{1+\sin\alpha}}\dfrac{\sqrt{1-\sin\alpha}}{\sqrt{1-\sin\alpha}}$, which can be written as $\dfrac{\pi r\sqrt{2}}{40}\cos^2\theta\sqrt{1-\sin\alpha}$. When the seat on the Ferris wheel is at its lowest point we have $\theta = 0$ and $\alpha = -\pi/2$, which gives $\cos^2\theta = 1$ and $\sqrt{1-\sin\alpha} = \sqrt{2}$. At any other point on the wheel we have $\cos^2\theta \leq 1$ and $\sqrt{1-\sin\alpha} < \sqrt{2}$, so θ is changing most rapidly when the seat is at its lowest point.

4.5.64 Let x and y be the base and height of the triangle that is folded over, and z the height of point P above the base (see figure in the text). The Pythagorean Theorem gives $z^2 = x^2 - (a-x)^2 = 2ax - a^2$, so $z = \sqrt{2ax - a^2}$. The Pythagorean Theorem also gives $(y - \sqrt{2ax-a^2})^2 + a^2 = y^2$, which can be simplified to $2y\sqrt{2ax-a^2} = 2ax$, so $y = ax/\sqrt{2ax-a^2}$. The length L of the crease satisfies $L^2 = x^2 + y^2 = x^2 + \dfrac{a^2x^2}{2ax-a^2} = x^2\left(1 + \dfrac{a^2}{2ax-a^2}\right) = \dfrac{x^3}{x - \frac{a}{2}}$. Because L is positive, it suffices to minimize the function L^2 over $a/2 < x \leq a$. We have $\dfrac{dL^2}{dx} = \dfrac{3x^2}{x-\frac{a}{2}} - \dfrac{x^3}{(x-\frac{a}{2})^2} = \dfrac{x^2}{x-\frac{a}{2}}\left(3 - \dfrac{x}{x-\frac{a}{2}}\right)$, and we solve $3(x - a/2) = x$

to obtain $x = 3a/4$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(a/2, a]$. Substituting in the equation for L^2 above, we find that the shortest crease has length $L = 3\sqrt{3}a/4$, and the height of the point P is $z = a/\sqrt{2}$. (The corresponding value of $y = 3\sqrt{2}a/4 \approx 1.061a$; so the height b of the rectangle must satisfy $b > 1.061a$ to be able to form the minimal crease.)

4.5.65 The critical points of the function $a(\theta)$ satisfy

$$a'(\theta) = \omega^2 r \left(-\sin\theta - \frac{2r\sin 2\theta}{L}\right) = -\omega^2 r \sin\theta \left(1 + \frac{4r\cos\theta}{L}\right) = 0,$$

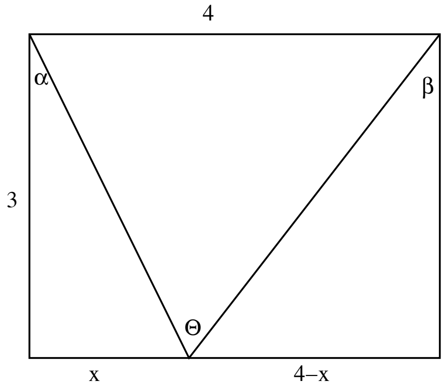
using the identity $\sin 2\theta = 2\sin\theta\cos\theta$. There are two cases to consider separately: (a) $0 < L < 4r$ and (b) $L \geq 4r$. In case (a) the critical points in $[0, 2\pi]$ are $\theta = 0, \pi, 2\pi$ and also $\theta = \cos^{-1}(-L/(4r))$ and $2\pi - \cos^{-1}(-L/(4r))$. Comparing the values of $a(\theta)$ at these points shows that the maximum acceleration occurs at $\theta = 0$ and 2π and the minimum occurs at $\theta = \cos^{-1}(-L/(4r))$ and $2\pi - \cos^{-1}(-L/(4r))$. (There is a local maximum at $\theta = \pi$.) In case (b) the only critical points are $\theta = 0, \pi$ and 2π, and comparing the values of $a(\theta)$ at these points shows that the maximum acceleration occurs at $\theta = 0$ and 2π as in case (a), whereas the minimum occurs at $\theta = \pi$ in this case.

4.5.66 Let α and β be the angles labeled below (see figure). Then $\left(\frac{\pi}{2} - \alpha\right) + \theta + \left(\frac{\pi}{2} - \beta\right) = \pi$ so $\theta = \alpha + \beta$. We have $\tan\alpha = x/3$ and $\tan\beta = (4 - x)/3$, so we can express θ in terms of x as $\theta(x) = \tan^{-1}\left(\frac{x}{3}\right) + \tan^{-1}\left(\frac{4-x}{3}\right)$. We wish to maximize this function for $0 \leq x \leq 4$. The critical points of $\theta(x)$ satisfy

$$\theta'(x) = \frac{1}{1 + \left(\frac{x}{3}\right)^2} \cdot \frac{1}{3} + \frac{1}{1 + \left(\frac{4-x}{3}\right)^2} \cdot \left(-\frac{1}{3}\right)$$

which can be written as $\frac{3}{x^2+9} - \frac{3}{(4-x)^2+9}$.

This is equal to zero for $x^2 = (4 - x)^2$ and because $x, 4 - x \geq 0$ we must have $x = 4 - x$, so $x = 2$ is the only critical point. We compare $\theta(x)$ at $x = 2$ and the endpoints $x = 0, 4$: $\theta(2) = 2\tan^{-1}\left(\frac{2}{3}\right) \approx 1.176$, $\theta(0) = \theta(4) = \tan^{-1}\left(\frac{4}{3}\right) \approx 0.937$. Therefore the maximum angle occurs when $x = 2$.



4.5.67

a. Using the Pythagorean Theorem, we find that the height of this triangle is 2. Let x be the distance from the point P to the base of the triangle; then the distance from P to the top vertex is $2 - x$ and the distance to each of the base vertices is $\sqrt{x^2 + 4}$, again by the Pythagorean Theorem. Therefore the sum of the distances to the three vertices is given by $S(x) = 2\sqrt{x^2 + 4} + 2 - x$. We wish to minimize this function for $0 \le x \le 2$. The critical points of $S(x)$ satisfy $S'(x) = \dfrac{2x}{\sqrt{x^2+4}} - 1 = 0$, which has unique solution $x = 2/\sqrt{3}$ in $(0,2)$. By the First Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0,2]$. Therefore the optimal location for P is $2/\sqrt{3}$ units above the base.

b. In this case the objective function to be minimized is $S(x) = 2\sqrt{x^2+4} + h - x$ where $0 \le x \le h$. Exactly as above, we find that the only critical point $x > 0$ is $x = 2/\sqrt{3}$. This will give the absolute minimum on $[0,h]$ as long as $h \ge 2/\sqrt{3}$. When $h < 2/\sqrt{3}$, $S(x)$ is decreasing on $[0,h]$ and the minimum occurs at the endpoint $x = h$.

4.5.68 Let r and h be the radius and height of both the cylinder and cones. The surface area of each cone is $\pi r\sqrt{r^2+h^2}$ and the surface area of the cylinder is $2\pi rh$, so we have the constraint $2\pi r\sqrt{r^2+h^2} + 2\pi rh = A$, which we rewrite as $h + \sqrt{r^2+h^2} = \dfrac{A}{2\pi r}$. Square to obtain $h^2 + 2h\sqrt{r^2+h^2} + r^2 + h^2 = \left(\dfrac{A}{2\pi r}\right)^2$, and substitute $\sqrt{r^2+h^2} = A/(2\pi r) - h$ in this equation to obtain $h^2 + 2h\left(\dfrac{A}{2\pi r} - h\right) + r^2 + h^2 = \left(\dfrac{A}{2\pi r}\right)^2$. Solving for h yields $h = \dfrac{\pi r}{A}\left(\dfrac{A^2}{4\pi^2 r^2} - r^2\right) = \dfrac{A}{4\pi r} - \dfrac{\pi r^3}{A}$.

We must have $h \ge 0$, which is equivalent to the condition $r \le \sqrt{A}/\sqrt{2\pi}$. So the possible r under consideration satisfy $0 \le r \le \sqrt{A}/\sqrt{2\pi}$. The objective function to be maximized is the combined volume of the cylinder and cones, which is given by

$$V = \pi r^2 h + 2\cdot\frac{\pi}{3}r^2 h = \frac{5\pi}{3}r^2 h = \frac{5\pi}{3}r^2\left(\frac{A}{4\pi r} - \frac{\pi r^3}{A}\right) = \frac{5A}{12}r - \frac{5\pi^2}{3A}r^5.$$

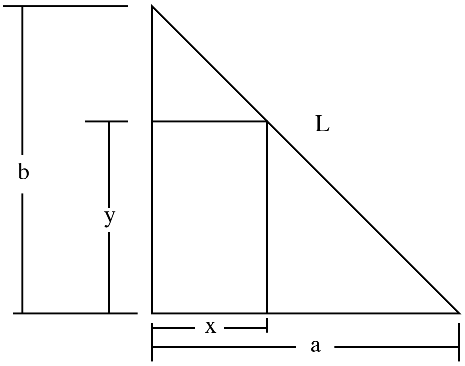
The critical points of $V(r)$ satisfy $V'(r) = \dfrac{5A}{12} - \dfrac{25\pi^2}{3A}r^4 = 0$, which has unique positive solution $r = \sqrt{A}/(\sqrt[4]{20}\sqrt{\pi})$. To find the corresponding value of h, observe that $\pi r^3/A = A/(20\pi r)$, so $h = \dfrac{A}{4\pi r} - \dfrac{\pi r^3}{A} = \dfrac{A}{4\pi r} - \dfrac{A}{20\pi r} = \dfrac{A}{5\pi r}$ which gives $h = \sqrt{A}\sqrt[4]{20}/(5\sqrt{\pi})$. Note that $V(r) = 0$ at the endpoints of the interval $[0, \sqrt{A}/\sqrt{2\pi}]$, so the maximum volume must occur at the values of r and h given above.

4.5.69 Let x be the distance between the point on the track nearest your initial position to the point where you catch the train. If you just catch the back of the train, then the train will have travelled $x + 1/3$ miles, which will require time $T = \dfrac{\text{distance}}{\text{rate}} = \dfrac{x+\frac{1}{3}}{20}$. The distance you must run is $\sqrt{x^2 + 1/(16)}$, so your running speed must be $v = \dfrac{\text{distance}}{\text{time}} = \dfrac{20\sqrt{x^2+\frac{1}{16}}}{x+\frac{1}{3}}$. We wish to minimize this function for $x \ge 0$. The derivative of $v(x)$ can be written $v'(x) = \left(\dfrac{x}{x^2+\frac{1}{16}} - \dfrac{1}{x+\frac{1}{3}}\right)v(x)$, so the critical points of $v(x)$ satisfy $\dfrac{x}{x^2+\frac{1}{16}} = \dfrac{1}{x+\frac{1}{3}}$ so $x\left(x+\dfrac{1}{3}\right) = x^2 + \dfrac{1}{16}$ which gives $x = 3/16$ mi. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0,\infty)$. The minimum running speed is $v\left(\dfrac{3}{16}\right) = \dfrac{20\sqrt{\left(\frac{3}{16}\right)^2+\frac{1}{16}}}{\frac{3}{16}+\frac{1}{3}} = \dfrac{60\sqrt{9+16}}{9+16} = 12$ mph.

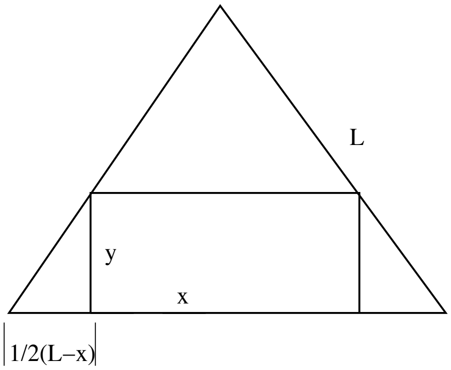
4.5.70

a. Let a and b be the side lengths of a particular right triangle with hypotenuse L, and let x and y be the side lengths of an inscribed rectangle (see figure). Then using similar triangles we see that

$y/(a-x) = b/a$, so $y = (b/a)(a-x)$. The area to be maximized is $A(x) = \frac{b}{a}x(a-x)$ over $0 \le x \le a$. This function has unique critical point $x = a/2$ and is 0 at the endpoints $x = 0$ and $x = a$; hence the maximum occurs when $x = a/2$. The other side of the rectangle has length $y = b/2$, so the maximum area is $A(a/2) = ab/4$. Note: We could also consider inscribing the rectangle so that one side rests on the hypotenuse. The maximum area is also $ab/4$ using this configuration. Now consider all possible right triangles with hypotenuse L. The side lengths a and b must satisfy $a, b \ge 0$ and $a^2 + b^2 = L^2$, which gives $b = \sqrt{L^2 - a^2}$. For each triangle, the largest area of an inscribed rectangle is $A = \dfrac{ab}{4} = \dfrac{a}{4}\sqrt{L^2 - a^2}$, which now we must maximize over $0 \le a \le L$. This function has unique critical point $a = L/\sqrt{2}$, and is 0 at the endpoints $a = 0$ and $a = L$. The constraint gives $b = a$, so the optimal triangle is an isosceles right triangle, and the largest inscribed rectangle is a square with side length $L/(2\sqrt{2})$ and area $L^2/8$.

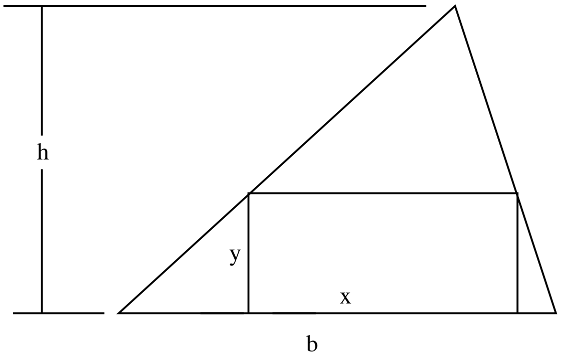


b. Let x and y be the dimensions of a rectangle inscribed in an equilateral triangle with side length L (see figure). Then $\tan\dfrac{\pi}{3} = \dfrac{y}{\frac{1}{2}(L-x)} = \sqrt{3}$ so $y = \dfrac{\sqrt{3}}{2}(L-x)$. The area to be maximized is $A = xy = \dfrac{\sqrt{3}}{2}x(L-x)$ over $0 \le x \le L$. This function has unique critical point $x = L/2$, and is 0 at the endpoints $x = 0$ and $x = L$; hence the maximum occurs when $x = L/2$. The other side of the rectangle has length $y = L\sqrt{3}/4$, so the maximum area is $A = L^2\sqrt{3}/8$.



c. Let a and b be the (non-hypotenuse) side lengths of a right triangle; then as shown in part (a) above, the inscribed rectangle of maximum area has side lengths $a/2$ and $b/2$ and area $ab/4 = A/2$.

d. Let b and h be the base and height of the triangle, assume the angles to the base are both less than or equal to 90°, and let x and y be the side lengths of an inscribed rectangle (see figure). Then by similar triangles $\dfrac{h-y}{h} = \dfrac{x}{b}$ so $y = h\left(1 - \dfrac{x}{b}\right)$. The rectangle has area $xy = (h/b)x(b-x)$, and the maximum value of this function over $0 \le x \le b$ occurs at $x = b/2$, which gives $y = h/2$ and area $= bh/4 = A/2$ if the triangle has area A. (Note that if a rectangle is inscribed in a triangle, then two of the vertices of the rectangle must lie on the same side of the triangle, so the rectangle must rest on one of the sides of the triangle, and therefore the angles to that side must both be less than or equal to 90°. So the maximum area of an inscribed rectangle is $A/2$ in all cases.)



4.5.71

a. Let r and h be the radius and height of the inscribed cylinder. The region that lies above the cylinder inside the cone is a cone with radius r and height $H-h$; by similar triangles we have $\dfrac{H-h}{r} = \dfrac{H}{R}$ so $h = \dfrac{H}{R}(R-r)$. The volume of the cylinder is $V = \pi r^2 h = \dfrac{\pi H}{R}\left(Rr^2 - r^3\right)$, which we must maximize over $0 \le r \le R$. The critical points of $V(r)$ satisfy $V'(r) = \dfrac{\pi H}{R}\left(2Rr - 3r^2\right) = 0$, which has unique solution $r = 2R/3$ in $(0, R)$. Because $V(r) = 0$ at the endpoints $r = 0$ and $r = R$, the cylinder with maximum volume has radius $r = 2R/3$, height $h = H/3$ and volume $V = \pi r^2 h = \dfrac{4\pi}{27}R^2 H = \dfrac{4}{9}\cdot\dfrac{\pi}{3}R^2 H$; i.e. 4/9 the volume of the cone.

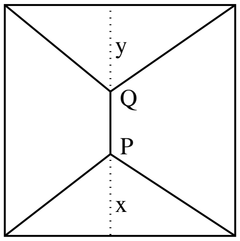
b. The lateral surface area of the cylinder is $A = 2\pi r h = 2\pi r \cdot \dfrac{H}{R}(R-r) = \dfrac{2\pi H}{R} r(R-r)$. This function takes its maximum over $0 \le r \le R$ at $r = R/2$, so the cylinder with maximum lateral surface area has dimensions $r = R/2$ and $h = H/2$.

4.5.72

a. Referring to the diagram in the text, note that if we drop a perpendicular from the 150° angle, the trapezoid is divided into a 30–60–90 triangle and an $x \times y$ rectangle. The slanted side has length $y \sec 60^\circ = 2y$, and the base of the trapezoid has length $x + 2y\cos 30^\circ = x + \sqrt{3}y$. The perimeter of the trapezoid is 1000, so we get the constraint $P = 2x + (3+\sqrt{3})y = 1000$, which gives $x = 500 - (3+\sqrt{3})y/2$. The objective function to be maximized is the area of the trapezoid, which is $A = \dfrac{1}{2}\left(x + (x + \sqrt{3}y)\right)y = (x + \dfrac{\sqrt{3}}{2}y)y = (500 - \dfrac{3}{2}y)y = 500y - \dfrac{3}{2}y^2$. Because we need both $x, y \ge 0$, we also must have $y \le 1000/(3+\sqrt{3}) \approx 211.325$. The maximum value of the quadratic function $A(y)$ occurs at $y = 500/3 \approx 166.667$ ft, which is in the interval under consideration; the corresponding value of x is $250(3 - \sqrt{3}/3) \approx 105.662$ ft.

b. In this case we do not use fencing for the slanted side with length $2y$, so we modify the constraint to be $2x + (1+\sqrt{3})y = 1000$, which gives $x = 500 - (1+\sqrt{3})y/2$. The area of the trapezoid is $A = \dfrac{1}{2}\left(x + (x+\sqrt{3}y)\right)y = (x + \dfrac{\sqrt{3}}{2}y)y = (500 - \dfrac{1}{2}y)y = 500y - \dfrac{1}{2}y^2$. Because we need both $x, y \ge 0$ we also must have $y \le 1000/(1+\sqrt{3}) \approx 366.025$. But the maximum value of the quadratic function $A(y)$ occurs at $y = 500$, which is outside the interval under consideration. Hence $A(y)$ is increasing over the interval $[0, 1000/(1+\sqrt{3})]$ and the maximum area occurs when $y = 1000/(1+\sqrt{3}) \approx 366.025$ ft and $x = 0$.

4.5.73 Following the hint, place two points P and Q above the midpoint of the base of the square, at distances x and y to the sides (see figure), where $0 \le x, y \le 1/2$. Then join the bottom vertices of the square to P, the upper vertices to Q and join P to Q. This road system has total length $L = 2\sqrt{x^2 + \frac{1}{4}} + 2\sqrt{y^2 + \frac{1}{4}} + (1 - x - y) = 1 + \left(\sqrt{4x^2+1} - x\right) + \left(\sqrt{4y^2+1} - y\right)$. We can minimize the contributions from x and y separately; the critical points of the function $f(x) = \sqrt{4x^2+1} - x$ satisfy $f'(x) = \dfrac{4x}{\sqrt{4x^2+1}} - 1 = 0$ which gives $\sqrt{4x^2+1} = 4x$, so $12x^2 = 1$ and $x = 1/(2\sqrt{3})$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0, 1/2]$. The minimum value of $f(x)$ on this interval is $f(1/(2\sqrt{3})) = \sqrt{3}/2$, so the shortest road system has length $L = 1 + 2 \cdot \dfrac{\sqrt{3}}{2} = 1 + \sqrt{3} \approx 2.732$.



4.5.74

a. Let x be the diameter of the smaller semicircle joining points A and B; then the other smaller semicircle has diameter $1 - x$. The area of a semicircle with diameter d is $\pi d^2/8$, so the area of the arbelos is given by
$$A(x) = \frac{\pi}{8}\left(1 - x^2 - (1-x)^2\right) = \frac{\pi}{8}\left(2x - 2x^2\right) = \frac{\pi}{4}x(1-x).$$
The quadratic function $x(1-x)$ takes its maximum at $x = 1/2$, so the largest area is obtained when we position point B at the center of the larger semicircle.

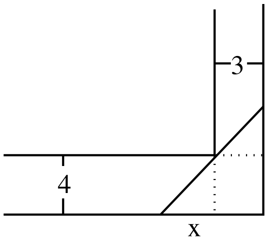
b. Point B has distance $|x - 1/2|$ to the center of the larger semicircle, so the length l of the segment BD can be found using the Pythagorean Theorem: $l^2 = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2 = x(1-x)$, so $l = \sqrt{x(1-x)}$ and a circle with diameter l has area $\pi\left(\frac{l}{2}\right)^2 = \frac{\pi}{4}x(1-x)$, which is the area of the arbelos.

4.5.75 A point on the curve $y = \sqrt{x}$ has the form $(x, \sqrt{x})$, which has distance L to the point $(p, 0)$ given by $L^2 = (x-p)^2 + \sqrt{x}^2 = x^2 + (1-2p)x + p^2$. Because L is positive, it suffices to minimize L^2 for $x \ge 0$. This quadratic function takes its minimum at $x = -(1-2p)/2 = p - 1/2$, so in case (i) the minimum occurs at the point $(p - 1/2, \sqrt{p - 1/2})$ and in case (ii) there are no critical points for $x > 0$, the function L^2 is increasing on $[0, \infty)$ so the minimum occurs at $(0, 0)$.

4.5.76

a. The length of the longest pole that can be carried around the corner is equal to the shortest length of a segment that joins the outer walls of the corridor while touching the inner corner (see figure). Using similar triangles, we can express this length in terms of the length x in the figure:
$$L(x) = \sqrt{x^2 + 4^2} + \frac{3}{x}\sqrt{x^2 + 4^2} = \left(1 + \frac{3}{x}\right)\sqrt{x^2 + 16}.$$
We wish to minimize this function for $x > 0$. The critical points of $L(x)$ satisfy
$$L'(x) = \left(1 + \frac{3}{x}\right)\frac{x}{\sqrt{x^2+16}} - \frac{3}{x^2}\sqrt{x^2+16} = 0$$

which simplifies to $(x^2 + 3x)x = 3(x^2 + 16)$ or $x^3 = 48$. This gives a unique critical point $x = \sqrt[3]{48}$ in the interval $(0, \infty)$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. The length of the longest pole is therefore $L(\sqrt[3]{48}) \approx 9.866$ ft.



b. For this case we replace 4 with a and 3 with b in the objective function from part (a) above, so we need to minimize the function $L(x) = \left(1 + \frac{b}{x}\right)\sqrt{x^2 + a^2}$ for $x > 0$. The critical points of $L(x)$ satisfy

$$L'(x) = \left(1 + \frac{b}{x}\right)\frac{x}{\sqrt{x^2 + a^2}} - \frac{b}{x^2}\sqrt{x^2 + a^2} = 0$$

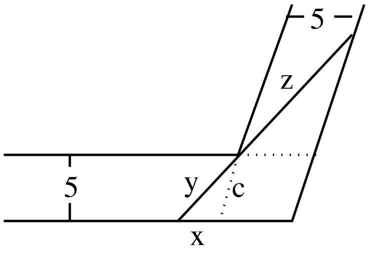
which simplifies to $(x^2 + bx)x = b(x^2 + a^2)$ or $x^3 = ba^2$. As above, this gives a unique critical point $x = \sqrt[3]{ba^2}$ in the interval $(0, \infty)$ which minimizes $L(x)$. The length of the longest pole is therefore

$$L(a^{\frac{2}{3}}b^{\frac{1}{3}}) = \left(1 + b^{\frac{2}{3}}a^{-\frac{2}{3}}\right)\sqrt{a^{\frac{4}{3}}b^{\frac{2}{3}} + a^2} = \left(1 + b^{\frac{2}{3}}a^{-\frac{2}{3}}\right)a^{\frac{2}{3}}\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

c. As above, we need to minimize the length $L = y + z$ as shown in the figure for $x > 0$. Note that $c = 5\csc 60° = \frac{10}{\sqrt{3}}$ and by the law of cosines, $y^2 = x^2 + c^2 - 2cx\cos 120° = x^2 + cx + c^2$. By similar triangles $z/y = c/x$, so we have $L(x) = y + z = \left(1 + \frac{c}{x}\right)\sqrt{x^2 + cx + c^2}$. The critical points of $L(x)$ satisfy

$$L'(x) = \left(1 + \frac{c}{x}\right)\frac{2x + c}{2\sqrt{x^2 + cx + c^2}} - \frac{c}{x^2}\sqrt{x^2 + cx + c^2} = 0,$$

which simplifies to $(x^2+cx)(2x+c) = 2c(x^2+cx+c^2)$ or $2x^3+cx^2-c^2x-2c^3 = (x-c)(2x^2+3cx+2c^2) = 0$. The quadratic equation $2x^2+3cx+2c^2 = 0$ has discriminant $-7c^2 < 0$, so $x = c$ is the only critical point in $(0, \infty)$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. The length of the longest pole is therefore $L(c) = 2\sqrt{3}c = 20$ ft.



d. Imagine a pole in any position in this corridor, and form a right triangle by dropping a segment perpendicular to the floor from the highest point on the pole down to the height of the lowest point on the pole. The base of this triangle can be no longer than the maximum length for the two-dimensional corridor in part (b), and the height is at most 8, so the maximum length L is given by combining the result from part (b) with the Pythagorean Theorem:

$$L = \sqrt{64 + \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^3}.$$

4.5.77 Let the angle of the cuts with the horizontal be ϕ_1 and ϕ_2, where $\phi_1 + \phi_2 = \theta$. The volume of the notch is proportional to $\tan\phi_1 + \tan\phi_2 = \tan\phi_1 + \tan(\theta - \phi_1)$, so it suffices to minimize the objective function $V(\phi_1) = \tan\phi_1 + \tan(\theta - \phi_1)$ for $0 \le \phi_1 \le \theta$. The critical points of $V(\phi_1)$ satisfy $\sec^2\phi_1 - \sec^2(\theta - \phi_1) = 0$, which is equivalent to the condition $\cos\phi_1 = \cos(\theta - \phi_1)$. This is satisfied if $\phi_1 = \theta - \phi_1$ which gives $\phi_1 = \theta/2$. There are no other solutions in $(0, \theta)$, because $\cos\phi_1$ is decreasing and $\cos(\theta - \phi_1)$ is increasing on $(0, \theta)$ and therefore can intersect at most once. So the only critical point occurs when $\phi_1 = \phi_2 = \theta/2$, and the First Derivative Test shows that this critical point is a local minimum; by Theorem 4.5, this must be the absolute minimum on $[0, \theta]$.

4.5.78 The radius r of a circle inscribed in a triangle is given by the formula $r = 2A/P$, where A is the area and P the perimeter of the triangle. Let x be the length of the base of the isosceles triangle. The height is then $\sqrt{1 - (x/2)^2}$ by the Pythagorean Theorem, and therefore the area is given by

$$A = \frac{1}{2}x\sqrt{1 - \frac{x^2}{4}} = \frac{1}{4}x\sqrt{4 - x^2}.$$

The perimeter is $x + 2$, so the radius of the inscribed triangle is

$$r = \frac{2A}{P} = \frac{1}{2} \cdot \frac{x\sqrt{4 - x^2}}{x + 2}.$$

The possible x values here satisfy $0 \le x \le 2$, so we need to maximize the function $r(x)$ on $[0, 2]$. We have

$$r'(x) = \frac{1}{2}\left(\frac{\sqrt{4 - x^2}}{x + 2} + \frac{x}{x + 2}\left(\frac{-x}{\sqrt{4 - x^2}}\right) - \frac{x\sqrt{4 - x^2}}{(x + 2)^2}\right),$$

which simplifies to

$$r'(x) = \frac{4 - 2x - x^2}{2(x + 2)\sqrt{4 - x^2}}.$$

Thus the critical points satisfy $x^2 + 2x - 4 = 0$. This equation has roots $-1 \pm \sqrt{5}$, so the only critical point in $(0, 2)$ is $x = \sqrt{5} - 1$. Because $r(0) = r(2) = 0$, the maximum radius must occur at $x = \sqrt{5} - 1$. For this value of x we have

$$r = \frac{\sqrt{2}}{8}(\sqrt{5} - 1)^{5/2} \approx 0.300, \quad \pi r^2 = \frac{\pi}{32}(\sqrt{5} - 1)^5 \approx 0.283.$$

4.5.79 Let x and y be the lengths of the sides of the pen, with y the side parallel to the barn. The diagonal has length $\sqrt{x^2 + y^2}$, by the Pythagorean Theorem. Therefore the constraint is $2x + y + \sqrt{x^2 + y^2} = 200$, which we rewrite as $2x + y = 200 - \sqrt{x^2 + y^2}$. Square both sides to obtain $4x^2 + 4xy + y^2 = 40{,}000 - 400\sqrt{x^2 + y^2} + x^2 + y^2$, which simplifies to $3x^2 + 4xy = 40{,}000 - 400\sqrt{x^2 + y^2}$. Now substitute $\sqrt{x^2 + y^2} = 200 - 2x - y$ in this equation and simplify to obtain $(3x - 200)(x - 200) = 4(100 - x)y$ so $y = \frac{(3x-200)(x-200)}{4(100-x)}$. The objective function to be maximized is the area of the pen, $A = xy$. Using the expression above for y in terms of x, we have

$$A = xy = \frac{x(3x - 200)(x - 200)}{4(100 - x)} = -\frac{1}{4} \cdot \frac{x(3x - 200)(x - 200)}{(x - 100)}.$$

The length x must be at least 0, and because the diagonal is at least as long as x, we must have $3x \le 200$; so x cannot exceed $200/3$. Therefore we need to maximize the function $A(x)$ defined above for $0 \le x \le 200/3$. We have

$$\begin{aligned} A'(x) &= -\frac{1}{4} \cdot \left(\frac{(3x - 200)(x - 200)}{(x - 100)} + \frac{x \cdot 3(x - 200)}{(x - 100)} + \frac{x(3x - 200)}{(x - 100)} - \frac{x(3x - 200)(x - 200)}{(x - 100)^2}\right) \\ &= \left(\frac{1}{x} + \frac{3}{3x - 200} + \frac{1}{x - 200} - \frac{1}{x - 100}\right)A(x). \end{aligned}$$

Because $A(x) > 0$ for $0 < x < 200/3$, the critical points of the objective function satisfy $\frac{1}{x} + \frac{3}{3x - 200} + \frac{1}{x - 200} = \frac{1}{x - 100}$ which when simplified gives the equation $6x^3 - 1700x^2 + 160{,}000x - 4{,}000{,}000 = 0$. Using a numerical solver, we find that this equation has exactly one solution in the interval $(0, 200/3)$, which is $x \approx 38.81$. To find the absolute maximum of A, we check the endpoints of $[0, 200/3]$ and the critical point $x \approx 38.814$. We have $A(0) = A(200/3) = 0$, so the absolute maximum occurs when $x \approx 38.814$ m; using the formula for y in terms of x above gives $y \approx 55.030$ m.

4.5.80 The dimensions of the box are $l - 2x$, $L - 2x$, and x, so the volume is given by

$$V(x) = x(l-2x)(L-2x) = 4x^3 - 2(l+L)x^2 + lLx.$$

The dimensions cannot be negative, so we must have $0 \le x \le l/2$ (because we are letting $L \to \infty$, we may assume that $l \le L$). The critical points of $V(x)$ satisfy

$$V'(x) = 12x^2 - 4(l+L)x + lL = 0,$$

and this quadratic equation has roots

$$x = \frac{L + l \pm \sqrt{L^2 - lL + l^2}}{6}.$$

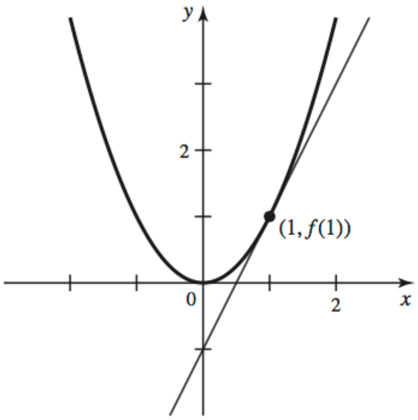
Now $V(x)$ is a cubic polynomial with roots $x = 0, l/2, L/2$, and so has exactly one critical point between 0 and $l/2$, which gives the maximum of $V(x)$ on the interval $[0, l/2]$. Because $V'(0) > 0$, this critical point is given by the smaller root of the quadratic above:

$$\begin{aligned} x &= \frac{L + l - \sqrt{L^2 - lL + l^2}}{6} \\ &= \frac{(L + l - \sqrt{L^2 - lL + l^2})}{6} \frac{(L + l + \sqrt{L^2 - lL + l^2})}{(L + l + \sqrt{L^2 - lL + l^2})} \\ &= \frac{(L+l)^2 - (L^2 - lL + l^2)}{6(L + l + \sqrt{L^2 - lL + l^2})} \\ &= \frac{3lL}{6(L + l + \sqrt{L^2 - lL + l^2})} \\ &= \frac{l}{2\left(1 + \frac{l}{L} + \sqrt{1 - \frac{l}{L} + \frac{l^2}{L^2}}\right)} \end{aligned}$$

(for the last step, divide all terms by L). As $L \to \infty$ with l fixed, $l/L \to 0$ so the size x of the corner squares that maximizes the volume has limit $l/4$ as $L \to \infty$.

4.6 Linear Approximation and Differentials

4.6.1



4.6.2 The derivative of a function is 0 at a local maximum, so the linear approximation is a horizontal line.

4.6.3 If f is differentiable at the point, then near that point, f is approximately linear, so the function nearly coincides with the tangent line at that point.

4.6.4 The change in $y = f(x)$ may be approximated by the formula $\Delta y \approx f'(x)\Delta x$.

4.6.5 $L(x) = f(1) + f'(1)(x-1) = 2 + 3(x-1) = 3x - 1$, so $f(1.1) \approx L(1.1) = 3(1.1) - 1 = 2.3$.

4.6.6 $L(x) = 5x - 3$, so $f(2.01) \approx 5(2.01) - 3 = 7.05$.

4.6.7 $L(x) = f(4) + f'(4)(x-4) = 3 + 2(x-4)$. So $f(3.85) \approx L(3.85) = 3 + 2(3.85 - 4) = 3 - .3 = 2.7$.

4.6.8 $L(x) = f(5) + f'(5)(x-5) = 10 - 2(x-5)$. So $f(5.1) \approx L(5.1) = 10 - 2(.1) = 9.8$.

4.6.9 The relationship is given by $dy = f'(x)dx$, which is the linear approximation of the change Δy in $y = f(x)$ corresponding to a change dx in x.

4.6.10 The differential dy is precisely the change in the linear approximation to f, which is an approximation of the change in f for small changes dx in x.

4.6.11 $\Delta y \approx f'(a)\Delta x$, so $f'(a) \approx \dfrac{\Delta y}{\Delta x}$. Therefore $f'(5) \approx \dfrac{f(5.01) - f(5)}{5.01 - 5} = \dfrac{0.25}{0.01} = 25$.

4.6.12 $\Delta y \approx f'(a)\Delta x$, so $f'(a) \approx \dfrac{\Delta y}{\Delta x}$. Therefore $f'(6) \approx \dfrac{f(5.99) - f(6)}{5.99 - 6} = \dfrac{-0.002}{-0.01} = 0.2$.

4.6.13 The approximate average speed is $L(-1) = 60 - (-1) = 61$ miles per hour. The exact speed is $\dfrac{3600}{59}$ miles per hour which is about 61.02 miles per hour.

4.6.14 The approximate average speed is $L(3) = 60 - (3) = 57$ miles per hour. The exact speed is $\dfrac{3600}{63}$ miles per hour which is about 57.14 miles per hour.

4.6.15 Let $T(x) = \dfrac{60D}{60 + x}$. Then $T'(x) = -\dfrac{60D}{(60+x)^2}$, so $T'(0) = -\dfrac{D}{60}$. The linear approximation is given by $L(x) = T(0) - \dfrac{D}{60}(x - 0)$, or $L(x) = D\left(1 - \dfrac{x}{60}\right)$.

4.6.16 With $D = 45$, we have $T(2) \approx L(2) = 45(1 - \dfrac{2}{60}) = 45 - (3/2) = 43.5$ minutes. The exact time required is $T(2) = \dfrac{60 \cdot 45}{60 + 2} \approx 43.55$ minutes.

4.6.17 With $D = 80$, we have $T(-3) \approx L(-3) = 80(1 - \dfrac{-3}{60}) = 80 + 4 = 84$ minutes. The exact time required is $T(-3) = \dfrac{60 \cdot 80}{60 - 3} \approx 84.211$ minutes.

4.6.18 With $D = 93$, we have $T(3) \approx L(3) = 93(1 - \dfrac{3}{60}) = 93 - \dfrac{93}{20} = 88.35$ minutes. The exact time required is $T(3) = \dfrac{60 \cdot 93}{60 + 3} \approx 88.57$ minutes.

4.6.19 $f'(x) = 8x + 1$, so $f'(1) = 9$.

$$L(x) = f(1) + f'(1)(x-1) = 5 + 9(x-1) = 9x - 4.$$

4.6.20 $f'(x) = 3x^2 - 5$, so $f'(2) = 7$.

$$L(x) = f(2) + f'(2)(x-2) = 1 + 7(x-2) = 7x - 13.$$

4.6.21 $g'(t) = \dfrac{2}{2\sqrt{2t+9}} = \dfrac{1}{\sqrt{2t+9}}$, so $g'(-4) = 1$.

$$L(t) = g(-4) + g'(-4)(t+4) = 1 + 1(t+4) = t+5.$$

4.6.22 $h'(w) = \dfrac{5}{2\sqrt{5w-1}}$, so $h'(1) = \dfrac{5}{4}$.

$$L(w) = h(1) + h'(1)(w-1) = 2 + \frac{5}{4}(w-1) = \frac{5}{4}w + \frac{3}{4}.$$

4.6.23 $f'(x) = 3e^{3x-6}$, so $f'(2) = 3$.

$$L(x) = f(2) + f'(2)(x-2) = 1 + 3(x-2) = 3x-5.$$

4.6.24 $f'(x) = 9(2/3)(4x+11)^{-1/3}(4) = \dfrac{24}{(4x+11)^{1/3}}$, so $f'(4) = \dfrac{24}{3} = 8$.

$$L(x) = f(4) + f'(4)(x-4) = 81 + 8(x-4) = 8x+49.$$

4.6.25

a. Note that $f(a) = f(2) = 8$ and $f'(a) = -2a = -4$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = 8 + (-4)(x-2) = -4x+16.$$

b. We have $f(2.1) \approx L(2.1) = 7.6$.

c. The percentage error is $100 \cdot \dfrac{|7.6 - 7.59|}{7.59} \approx 0.13\%$.

4.6.26

a. Note that $f(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$ and $f'(a) = \cos a = \sqrt{2}/2$, so the linear approximation has equation $y = L(x) = f(a) + f'(a)(x-a) = \sin\frac{\pi}{4} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) = \frac{\sqrt{2}}{2}\left(x + 1 - \frac{\pi}{4}\right)$.

b. We have $f(0.75) \approx L(0.75) \approx 0.68$.

c. The percentage error is $100 \cdot \dfrac{\left|\frac{\sqrt{2}}{2}\left(1.75 - \frac{\pi}{4}\right) - \sin 0.75\right|}{\sin 0.75} \approx 0.064\%$.

4.6.27

a. Note that $f(a) = f(0) = \ln 1 = 0$ and $f'(a) = 1/(1+a) = 1$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = x.$$

b. We have $f(0.9) \approx L(0.9) = 0.9$.

c. The percentage error is $100 \cdot \dfrac{|0.9 - \ln 1.9|}{|\ln 1.9|} \approx 40\%$.

4.6.28

a. Note that $f(a) = \dfrac{a}{a+1} = \dfrac{1}{1+1} = \dfrac{1}{2}$ and $f'(a) = \dfrac{1}{(1+a)^2} = \dfrac{1}{4}$, so the linear approximation has equation $y = L(x) = f(a) + f'(a)(x-a) = \dfrac{1}{2} + \dfrac{1}{4}(x-1) = \dfrac{1}{4}(x+1)$.

b. We have $f(1.1) \approx L(1.1) = 0.525$.

c. The percentage error is $100 \cdot \dfrac{|0.525 - (1.1/2.1)|}{1.1/2.1} \approx 0.23\%$.

4.6.29

a. Note that $f(a) = f(0) = \cos 0 = 1$ and $f'(a) = -\sin a = 0$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x - a) = 1.$$

b. We have $f(-0.01) \approx L(-0.01) = 1$.

c. The percentage error is $100 \cdot \dfrac{|1 - \cos(-0.01)|}{\cos(-0.01)} \approx 0.005\%$.

4.6.30

a. Note that $f(a) = e^a = e^0 = 1$ and $f'(a) = e^a = 1$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x - a) = 1 + x.$$

b. We have $f(0.05) \approx L(0.05) = 1.05$.

c. The percentage error is $100 \cdot \dfrac{|1.05 - e^{0.05}|}{e^{0.05}} \approx 0.12\%$.

4.6.31

a. Note that $f(a) = 8^{-1/3} = 1/2$ and $f'(a) = (-1/3)(8)^{-4/3} = -1/48$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x - a) = \frac{1}{2} + \frac{-1}{48}x.$$

b. We have $f(-0.1) \approx L(-0.1) \approx 0.50208333$

c. The percentage error is $100 \cdot \dfrac{|7.9^{-1/3} - .050208333|}{(7.9)^{-1/3}} \approx 0.003\%$.

4.6.32

a. Note that $f(a) = \sqrt[4]{81} = 3$ and $f'(a) = \frac{1}{4}(81)^{-3/4} = \frac{1}{108}$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x - a) = 3 + \frac{1}{108}(x - 81).$$

b. We have $f(85) \approx L(85) = 3 + \frac{4}{108} = 3 + \frac{1}{27} \approx 3.04$.

c. The percentage error is $100 \cdot \dfrac{|3.04 - \sqrt[4]{85}|}{\sqrt[4]{85}} \approx 0.12\%$.

4.6.33

a. Note that $f(a) = f(0) = 1$ and $f'(a) = -1/(1 + a)^2 = -1$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x - a) = 1 - x.$$

b. The linear approximation to $1/1.1$ is $\dfrac{1}{1.1} \approx L(0.1) = 0.9$.

c. The percentage error is $100 \cdot \dfrac{|0.9 - \frac{1}{1.1}|}{\frac{1}{1.1}} = 1\%$.

4.6.34

a. Note that $f(a) = \cos(\pi/4) = \sqrt{2}/2$ and $f'(a) = -\sin(\pi/4) = -\sqrt{2}/2$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(1 + \frac{\pi}{4} - x\right).$$

b. The linear approximation to $\cos(0.8)$ is $\cos(0.8) \approx L(0.8) = \dfrac{\sqrt{2}}{2}\left(0.2 + \dfrac{\pi}{4}\right) \approx 0.697$.

c. The percentage error is $100 \cdot \dfrac{\left|\frac{\sqrt{2}}{2}\left(0.2 + \frac{\pi}{4}\right) - \cos(0.8)\right|}{\cos(0.8)} \approx 0.011\%$.

4.6.35

a. Note that $f(a) = f(0) = 1$ and $f'(a) = -e^{-a} = -1$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = 1 - x.$$

b. The linear approximation to $e^{-0.03}$ is $e^{-0.03} \approx L(0.03) = 0.97$.

c. The percentage error is $100 \cdot \dfrac{|0.97 - e^{-0.03}|}{e^{-0.03}} \approx 0.046\%$.

4.6.36

a. Note that $f(a) = \tan 0 = 0$ and $f'(a) = \sec^2 0 = 1$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = x.$$

b. The linear approximation to $\tan 3^\circ$ is $\tan 3^\circ = \tan \dfrac{\pi}{60} \approx L\left(\dfrac{\pi}{60}\right) = \dfrac{\pi}{60} \approx 0.052$ Note that we must convert 3° to radians before applying the linear approximation formula.

c. The percentage error is $100 \cdot \dfrac{\left|\dfrac{\pi}{60} - \tan\dfrac{\pi}{60}\right|}{\dfrac{\pi}{60}} \approx 0.091\%$.

4.6.37 Let $f(x) = 1/x$, $a = 200$. Then $f(a) = 0.005$ and $f'(a) = -1/a^2 = -0.000025$, so the linear approximation to f near $a = 200$ is

$$L(x) = f(a) + f'(a)(x-a) = 0.005 - 0.000025(x - 200).$$

Therefore

$$\frac{1}{203} = f(203) \approx L(203) = 0.004925.$$

4.6.38 Let $f(x) = \tan x$, $a = 0$. Then $f(a) = 0$ and $f'(a) = \sec^2 a = 1$, so the linear approximation to f near $a = 0$ is

$$L(x) = f(a) + f'(a)(x-a) = x.$$

Therefore

$$\tan(-2^\circ) = \tan\left(\frac{-\pi}{90}\right) = f\left(\frac{-\pi}{90}\right) \approx L\left(\frac{-\pi}{90}\right) \approx -0.0349.$$

Note that we must convert -2° to radians before applying the linear approximation formula.

4.6.39 Let $f(x) = \sqrt{x}$, $a = 144$. Then $f(a) = 12$ and $f'(a) = 1/(2\sqrt{a}) = 1/24$, so the linear approximation to f near $a = 144$ is

$$L(x) = f(a) + f'(a)(x - a) = 12 + \frac{1}{24}(x - 144).$$

Therefore

$$\sqrt{146} = f(146) \approx L(146) = \frac{145}{12}.$$

4.6.40 Let $f(x) = x^{1/3}$, $a = 64$. Then $f(a) = 4$ and $f'(a) = (1/3)a^{-2/3} = 1/48$, so the linear approximation to f near $a = 144$ is

$$L(x) = f(a) + f'(a)(x - a) = 4 + \frac{1}{48}(x - 64).$$

Therefore

$$\sqrt[3]{65} = f(65) \approx L(65) = \frac{193}{48}.$$

4.6.41 Let $f(x) = \ln x$, $a = 1$. Then $f(a) = 0$ and $f'(a) = 1/a = 1$, so the linear approximation to f near $a = 1$ is

$$L(x) = f(a) + f'(a)(x - a) = x - 1.$$

Therefore

$$\ln(1.05) = f(1.05) \approx L(1.05) = 0.05.$$

4.6.42 Let $f(x) = \sqrt{x}$, $a = 0.16$ (note that $5/29 \approx 0.17$). Then $f(a) = 0.4$ and $f'(a) = 1/(2\sqrt{a}) = 1.25$, so the linear approximation to f near $a = 0.16$ is

$$L(x) = f(a) + f'(a)(x - a) = 0.4 + 1.25(x - 0.16).$$

Therefore

$$\sqrt{5/29} = f(5/29) \approx L(5/29) \approx 0.416.$$

4.6.43 Let $f(x) = e^x$, $a = 0$. Then $f(a) = 1$ and $f'(a) = e^a = 1$, so the linear approximation to f near $a = 0$ is

$$L(x) = f(a) + f'(a)(x - a) = 1 + x.$$

Therefore

$$e^{0.06} = f(0.06) \approx L(0.06) \approx 1.060.$$

4.6.44 Let $f(x) = 1/\sqrt{x}$, $a = 121$. Then $f(a) = 1/11$ and $f'(a) = -1/(2a^{3/2}) = -1/2662$, so the linear approximation to f near $a = 121$ is

$$L(x) = f(a) + f'(a)(x - a) = \frac{1}{11} - \frac{1}{2662}(x - 121).$$

Therefore

$$\frac{1}{\sqrt{119}} = f(119) \approx L(119) \approx 0.0917.$$

4.6.45 Let $f(x) = 1/\sqrt[3]{x}$, $a = 512$. Then $f(a) = 1/8$ and $f'(a) = -1/(3a^{4/3}) = -1/12{,}288$, so the linear approximation to f near $a = 512$ is

$$L(x) = f(a) + f'(a)(x - a) = \frac{1}{8} - \frac{1}{12{,}288}(x - 512).$$

Therefore

$$\frac{1}{\sqrt[3]{510}} = f(510) \approx L(510) = \frac{769}{6144} \approx 0.1252.$$

4.6.46 Let $f(x) = \cos x$, $a = \pi/6$ $(= 30°)$. Then $f(a) = \sqrt{3}/2$ and $f'(a) = -\sin a = -1/2$, so the linear approximation to f near $a = 0$ is

$$L(x) = f(a) + f'(a)(x-a) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right).$$

Therefore

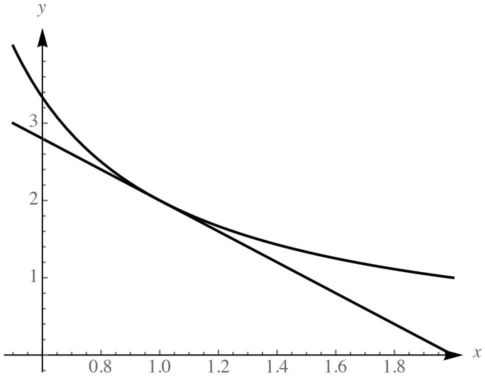
$$\cos 31° = \cos\frac{31\pi}{180} = f\left(\frac{31\pi}{180}\right) \approx L\left(\frac{31\pi}{180}\right) \approx 0.857.$$

Note that we must convert 31° to radians before applying the linear approximation formula.

4.6.47

a. With $f(x) = \dfrac{2}{x}$ and $a = 1$, we have $f(a) = 2$ and $f'(a) = -\frac{2}{a^2} = -2$. Thus the linear approximation to $f(x)$ at $x = 1$ is $L(x) = f(1) + f'(1)(x-1) = 2 + -2(x-1) = -2x + 4$.

b. A plot of f with L:



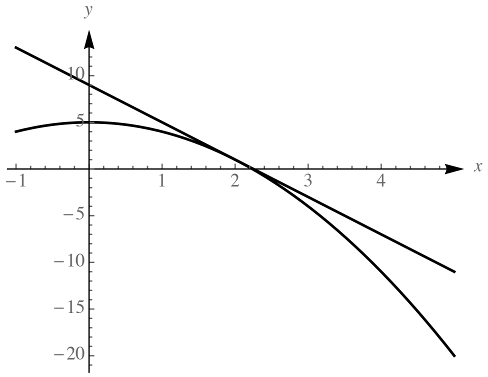
c. The linear approximation in part (b) appears to be an underestimate everywhere, because it lies below the graph of f.

d. Because $f'(x) = -\dfrac{2}{x^2}$, we have $f''(x) = \dfrac{4}{x^3}$, so that $f''(1) > 0$, and f is concave up at $x = 1$. This is consistent with L being an underestimate near $x = 1$.

4.6.48

a. With $f(x) = 5 - x^2$ and $a = 2$, we have $f(a) = 1$ and $f'(a) = -2a = -4$. Thus the linear approximation to f at $x = 2$ is $L(x) = f(2) + f'(2)(x-2) = 1 + -4(x-2) = -4x + 9$.

b. A plot of f with L:



c. The linear approximation in part (b) appears to be an overestimate everywhere, because it lies above the graph of f

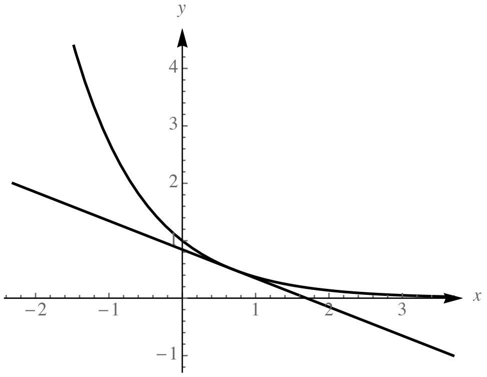
d. Because $f'(x) = -2x$, we have $f''(x) = -2$, so that $f''(x) < 0$ for all values of x and f is therefore concave down everywhere. This is consistent with L being an overestimate.

4.6.49

a. With $f(x) = e^{-x}$ and $a = \ln 2$, we have $f(a) = \frac{1}{2}$ and $f'(a) = -e^{-a} = -\frac{1}{2}$. Thus the linear approximation to f at $x = \ln 2$ is

$$L(x) = f(\ln 2) + f'(\ln 2)(x - \ln 2) = \frac{1}{2} + -\frac{1}{2}(x - \ln 2) = -\frac{1}{2}x + \frac{1}{2}(1 + \ln 2).$$

b. A plot of f with L:



c. The linear approximation in part (b) appears to be an underestimate everywhere, because it lies below the graph of f

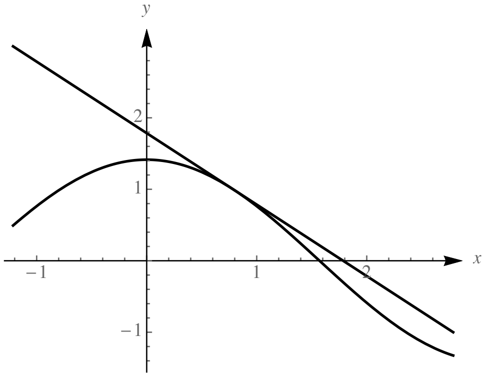
d. Because $f'(x) = -e^{-x}$, we have $f''(x) = e^{-x}$, so that $f''(x) > 0$ for all values of x and f is therefore concave up everywhere. This is consistent with L being an underestimate.

4.6.50

a. With $f(x) = \sqrt{2}\cos x$ and $a = \frac{\pi}{4}$, we have $f(a) = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1$ and $f'(a) = -\sqrt{2}\sin(\pi/4) = -1$. Thus the linear approximation to f at $x = a$ is

$$L(x) = f(a) + f'(a)(x - a) = 1 + -1\left(x - \frac{\pi}{4}\right) = -x + 1 + \frac{\pi}{4}.$$

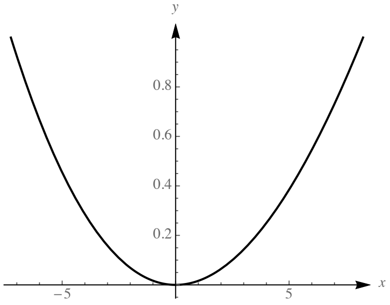
b. A plot of f with L:



c. The linear approximation in part (b) appears to be an overestimate everywhere, because it lies above the graph of f

d. Because $f'(x) = -\sqrt{2}\sin x$, we have $f''(x) = -\sqrt{2}\cos x$, so that $f''(a) < 0$ for x near $a = \frac{\pi}{4}$ and f is therefore concave down near $x = a$. This is consistent with L being an overestimate.

4.6.51 $E(x) = |L(x) - s(x)| = \left|60 - x - \dfrac{3600}{60 + x}\right|$. A graph is shown below, for x from -7.26 to 8.26.



$E(x) \leq 1$ when $-7.26 \leq x \leq 8.26$, which corresponds to driving times for 1 mi from about 53 s to 68 s. Therefore, $L(x)$ gives approximations to $s(x)$ that are within 1 mi/hr of the true value when you drive 1 mile in t seconds, where $53 < t < 68$.

4.6.52

a. We have $P = T/V$ with V held constant, which is a linear function of T. Hence $\Delta P = \frac{\Delta T}{V} = \frac{0.05}{V}$. Because this is greater than 0, the pressure increases.

b. If T is held constant then $dP = -(T/V^2)dV$, and the approximate change in pressure is $\Delta P \approx dP = -0.1\dfrac{T}{V^2} < 0$, so the pressure decreases.

c. We have $T = PV$ with P held constant, which is a linear function of V. Hence $\Delta T = P\Delta V = 0.1P > 0$, so the temperature increases.

4.6.53

a. True. Note that $f(0) = 0$ and $f'(0) = 0$, so the linear approximation at 0 is in fact $L(x) = 0$.

b. False. The function $f(x) = |x|$ is not differentiable at $x = 0$, so there is no good linear approximation at 0.

c. True. For linear functions, the linear approximation at any point and the function are equal.

d. True. Note that $f'(x) = \frac{1}{x}$, so that $f''(x) = \frac{-1}{x^2}$, and $f''(e) < 0$, so f is concave down near $x = e$. Thus L is an overestimate of f.

4.6.54 Because $T = \dfrac{D}{\text{speed}}$, we have $T = \dfrac{D}{60 + x}$ hours which is $\dfrac{60D}{60 + x}$ minutes.

4.6.55 Note that $V'(r) = 4\pi r^2$, so $\Delta V \approx V'(a)\Delta r = 4\pi a^2\Delta r$. Substituting $a = 5$ and $\Delta r = 0.1$ gives $\Delta V \approx 4\pi \cdot 25 \cdot 0.1 = 10\pi \approx 31.416$ ft^3.

4.6.56 Note that $P'(z) = -100e^{-z/10}$, so $\Delta P \approx P'(a)\Delta z = -100e^{-a/10}\Delta z$. Substituting $a = 2$ and $\Delta z = 0.01$ gives $\Delta P \approx -100e^{-0.2} \cdot 0.01 = -0.819$.

4.6.57 Note that V is a linear function of h with $V'(h) = \pi r^2 = 400\pi$, so $\Delta V = V'(a)\Delta r = 400\pi\Delta r$. Substituting $\Delta r = -0.1$ gives $\Delta V = -40\pi \approx -125.664$ cm^3.

4.6.58 Note that $V'(r) = 2\pi rh/3 = 8\pi r/3$, so $\Delta V \approx V'(a)\Delta h = \dfrac{8\pi a}{3}\Delta h$. Substituting $a = 3$ and $\Delta h = 0.05$ gives $\Delta V \approx 8\pi \cdot (0.05) = 0.4\pi \approx 1.257$ cm^3.

4.6.59 Note that $S'(r) = \pi\sqrt{r^2+h^2} + \pi r \cdot \dfrac{r}{\sqrt{r^2+h^2}} = \pi\dfrac{2r^2+h^2}{\sqrt{r^2+h^2}}$, so $\Delta S \approx S'(a)\Delta r = \pi\dfrac{2a^2+h^2}{\sqrt{a^2+h^2}}\Delta r$. Substituting $h = 6$, $a = 10$ and $\Delta r = -0.1$ gives $\Delta S \approx \pi\dfrac{236}{\sqrt{136}}(-0.1) = \dfrac{-59\pi}{5\sqrt{34}} \approx -6.358$ m^2.

4.6.60 Note that $F'(r) = -0.02r^{-3}$, so $\Delta F \approx F'(a)\Delta r = -0.02a^{-3}\Delta r$. Substituting $a = 20$ and $\Delta r = 1$ gives $\Delta F \approx -0.02 \cdot 20^{-3} \cdot 1 = -2.5 \cdot 10^{-6}$.

4.6.61 We have $f'(x) = 2$, so $dy = 2\,dx$.

4.6.62 We have $f'(x) = 2\sin x\cos x$, so $dy = 2\sin x\cos x\,dx$.

4.6.63 We have $f'(x) = -3/x^4$, so $dy = -\dfrac{3}{x^4}dx$.

4.6.64 We have $f'(x) = 2e^{2x}$, so $dy = 2e^{2x}\,dx$.

4.6.65 We have $f'(x) = a\sin x$, so $dy = a\sin x\,dx$.

4.6.66 We have $f'(x) = \dfrac{(4-x)\cdot 1 - (4+x)\cdot(-1)}{(4-x)^2} = \dfrac{8}{(x-4)^2}$, so $dy = \dfrac{8}{(x-4)^2}\,dx$.

4.6.67 We have $f'(x) = 9x^2 - 4$, so $dy = (9x^2-4)\,dx$.

4.6.68 We have $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$, so $dy = \dfrac{1}{\sqrt{1-x^2}}\,dx$.

4.6.69 We have $f'(x) = \sec^2 x$, so $dy = \sec^2 x\,dx$.

4.6.70 We have $f'(x) = -1/(1-x) = 1/(x-1)$, so $dy = \dfrac{1}{x-1}\,dx$.

4.6.71 Note that $f(a) = f(8) = 2$ and $f'(a) = (1/3)a^{-2/3} = 1/12$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = 2 + \frac{1}{12}(x-8) = \frac{x}{12} + \frac{4}{3}.$$

x	Linear approx	Exact value	Percent error
8.1	$2.008\overline{3}$	2.00829885	1.717×10^{-3}
8.01	$2.0008\overline{3}$	2.000832986	1.734×10^{-5}
8.001	$2.00008\overline{3}$	2.00008333	1.736×10^{-7}
8.0001	$2.000008\overline{3}$	2.00000833	1.736×10^{-9}
7.9999	$1.999991\overline{6}$	1.999991667	1.736×10^{-9}
7.999	$1.99991\overline{6}$	1.999916663	1.736×10^{-7}
7.99	$1.9991\overline{6}$	1.999166319	1.738×10^{-5}
7.9	$1.991\overline{6}$	1.991631701	1.756×10^{-3}

The percentage errors become extremely small as x approaches 8 In fact, each time we decrease Δx by a factor of 10, the percentage error decreases by a factor of 100.

4.6.72 Note that $f(a) = f(0) = 1$ and $f'(a) = -1(1+a)^2 = -1$, so the linear approximation has equation $y = L(x) = f(a) + f'(a)(x-a) = 1 - x$.

x	Linear approx	Exact value	Percent error
0.1	0.9	$0.\overline{90}$	1
0.01	0.99	$0.\overline{9900}$	1×10^{-2}
0.001	0.999	$0.\overline{999000}$	1×10^{-4}
0.0001	0.9999	$0.\overline{99990000}$	1×10^{-6}
-0.0001	1.0001	$1.\overline{0001}$	1×10^{-6}
-0.001	1.001	$1.\overline{001}$	1×10^{-4}
-0.01	1.01	$1.\overline{01}$	1×10^{-2}
$-.1$	1.1	$1.\overline{1}$	1

The percentage errors become extremely small as x approaches 0. In fact, each time we decrease Δx by a factor of 10, the percentage error decreases by a factor of 100.

4.6.73

a. The linear approximation near $x = 1$ is more accurate for f because the rate at which f' is changing at 1 is smaller than the rate at which g' is changing at 1. The graph of f bends away from the linear function more slowly than the graph of g.

b. The larger the value of $|f''(a)|$, the greater the deviation of the curve $y = f(x)$ from the tangent line at points near $x = a$.

4.7 L'Hôpital's Rule

4.7.1 If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then we say $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is of indeterminate form $\dfrac{0}{0}$.

4.7.2 In general, limits with the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ can have *any* value, and so cannot be evaluated by direct substitution.

4.7.3 Take the limit of the quotient of the derivatives of the numerator and denominator.

4.7.4

a. Let $f(x) = 3\sin x$ and $g(x) = x$. Then $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$, but $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = \lim_{x\to 0} \dfrac{3\sin x}{x} = 3$.

b. Let $f(x) = 4\sin x$ and $g(x) = x$. Then $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$, but $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = \lim_{x\to 0} \dfrac{4\sin x}{x} = 4$.

4.7.5

a. Let $f(x) = x^2$ and $g(x) = \dfrac{1}{x^2}$. Then $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = \infty$, but $\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} x^2 \cdot \dfrac{1}{x^2} = \lim_{x\to 0} 1 = 1$.

b. Let $f(x) = 2x^2$ and $g(x) = \dfrac{1}{x^2}$. Then $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = \infty$, but $\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} 2x^2 \cdot \dfrac{1}{x^2} = \lim_{x\to 0} 2 = 2$.

4.7.6

a. L'Hôpital's Rule is not needed. This is not an indeterminate form. $\lim_{x\to 0} \dfrac{\sin x}{x^2 + 2x + 1} = \dfrac{0}{1} = 0$.

b. L'Hôpital's Rule is needed. $\lim\limits_{x\to 0}\dfrac{\sin x}{x^3+2x}=\lim\limits_{x\to 0}\dfrac{\cos x}{3x^2+2}=\dfrac{1}{2}$.

4.7.7 If $\lim\limits_{x\to a} f(x)g(x)$ has the form $0\cdot\infty$, then $\lim\limits_{x\to a}\dfrac{f(x)}{1/g(x)}$ has the indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

4.7.8 A simple example is $\lim\limits_{x\to 0}\dfrac{1/x^2}{1/x^2}$.

4.7.9 $\lim\limits_{x\to 0}(\tan^{-1}x)\left(\dfrac{1}{5x}\right)=\lim\limits_{x\to 0}\dfrac{\tan^{-1}x}{5x}=\lim\limits_{x\to 0}\dfrac{1/(1+x^2)}{5}=\dfrac{1}{5}$.

4.7.10 By l'Hôpital's rule :

$$\lim_{x\to 2}\frac{x^3-3x^2+2x}{x-2}=\lim_{x\to 2}\frac{3x^2-6x+2}{1}=2.$$

By a method from a previous chapter:

$$\lim_{x\to 2}\frac{x^3-3x^2+2x}{x-2}=\lim_{x\to 2}\frac{x(x^2-3x+2)}{x-2}=\lim_{x\to 2}\frac{x(x-2)(x-1)}{x-2}=\lim_{x\to 2}x(x-1)=2.$$

4.7.11 If $\lim\limits_{x\to a} f(x)=1$ and $\lim\limits_{x\to a} g(x)=\infty$, then $f(x)^{g(x)}\to 1^\infty$ as $x\to a$, which is meaningless; so direct substitution does not work.

4.7.12 First, evaluate $L=\lim\limits_{x\to a} g(x)\ln f(x)$, which can usually be handled by L'Hôpital's rule. Then we have $\lim\limits_{x\to a} f(x)^{g(x)}=e^L$.

4.7.13 This means $\lim\limits_{x\to\infty}\dfrac{g(x)}{f(x)}=0$.

4.7.14 This means $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=M$ where $0<M<\infty$.

4.7.15 By Theorem 4.14, we have $\ln x, x^3, 2^x, x^x$ in order of increasing growth rates.

4.7.16 By Theorem 4.14, we have $\ln x^{10}, x^{100}, 10^x, x^x$ in order of increasing growth rates.

4.7.17 L'Hôpital's rule gives $\lim\limits_{x\to 2}\dfrac{x^2-2x}{8-6x+x^2}=\lim\limits_{x\to 2}\dfrac{2x-2}{-6+2x}=\dfrac{2}{-2}=-1$.

4.7.18 L'Hôpital's rule gives $\lim\limits_{x\to -1}\dfrac{x^4+x^3+2x+2}{x+1}=\lim\limits_{x\to -1}\dfrac{4x^3+3x^2+2}{1}=-4+3+2=1$.

4.7.19 $\lim\limits_{x\to 1}\dfrac{x^2+2x}{x+3}=\dfrac{1^2+2}{1+3}=\dfrac{3}{4}$. Note that this is not an indeterminate form.

4.7.20 $\lim\limits_{x\to 0}\dfrac{e^x-1}{2x+5}=\dfrac{1-1}{5}=\dfrac{0}{5}=0$. Note that this is not an indeterminate form.

4.7.21 L'Hôpital's rule gives $\lim\limits_{x\to 1}\dfrac{\ln x}{4x-x^2-3}=\lim\limits_{x\to 1}\dfrac{1/x}{4-2x}=\dfrac{1}{2}$.

4.7.22 L'Hôpital's rule gives $\lim\limits_{x\to 0}\dfrac{e^x-1}{x^2+3x}=\lim\limits_{x\to 0}\dfrac{e^x}{2x+3}=\dfrac{1}{3}$.

4.7.23 Apply L'Hôpital's rule three times:

$$\lim_{x\to\infty}\frac{3x^4-x^2}{6x^4+12}=\lim_{x\to\infty}\frac{12x^3-2x}{24x^3}=\lim_{x\to\infty}\frac{36x^2-2}{72x^2}=\lim_{x\to\infty}\frac{72x}{144x}=\frac{1}{2}.$$

4.7.24 Apply L'Hôpital's rule three times:

$$\lim_{x\to\infty}\frac{4x^3-2x^2+6}{\pi x^3+4}=\lim_{x\to\infty}\frac{12x^2-4x}{3\pi x^2}=\lim_{x\to\infty}\frac{24x-4}{6\pi x}=\lim_{x\to\infty}\frac{24}{6\pi}=\frac{4}{\pi}.$$

4.7.25 L'Hôpital's rule gives $\lim_{x\to e}\frac{\ln x-1}{x-e}=\lim_{x\to e}\frac{1/x}{1}=\frac{1}{e}$.

4.7.26 L'Hôpital's rule gives $\lim_{x\to1}\frac{4\tan^{-1}x-\pi}{x-1}=\lim_{x\to1}\frac{4/(1+x^2)}{1}=2$.

4.7.27 Apply L'Hôpital's rule:

$$\lim_{x\to0^+}\frac{1-\ln x}{1+\ln x}=\lim_{x\to0^+}\frac{-1/x}{1/x}=\lim_{x\to0^+}(-1)=-1.$$

4.7.28 Apply L'Hôpital's rule once, then simplify:

$$\lim_{x\to0^+}\frac{x-3\sqrt{x}}{x-\sqrt{x}}=\lim_{x\to0^+}\frac{1-3/(2\sqrt{x})}{1-1/\sqrt{x}}=\lim_{x\to0^+}\frac{(1-3/(2\sqrt{x}))}{(1-1/\sqrt{x})}\cdot\frac{2\sqrt{x}}{2\sqrt{x}}=\lim_{x\to0^+}\frac{2\sqrt{x}-3}{2\sqrt{x}-2}=\frac{3}{2}.$$

4.7.29 L'Hôpital's rule gives $\lim_{x\to0}\frac{3\sin4x}{5x}=\lim_{x\to0}\frac{12\cos4x}{5}=\frac{12}{5}$.

4.7.30 L'Hôpital's rule gives $\lim_{x\to2\pi}\frac{x\sin x+x^2-4\pi^2}{x-2\pi}=\lim_{x\to2\pi}\frac{x\cos x+\sin x+2x}{1}=2\pi+0+4\pi=6\pi$.

4.7.31 L'Hôpital's rule gives $\lim_{u\to\pi/4}\frac{\tan u-\cot u}{u-\pi/4}=\lim_{u\to\pi/4}\frac{\sec^2u+\csc^2u}{1}=2+2=4$.

4.7.32 L'Hôpital's rule gives $\lim_{z\to0}\frac{\tan4z}{\tan7z}=\lim_{z\to0}\frac{4\sec^24z}{7\sec^27z}=\frac{4}{7}$.

4.7.33 Apply L'Hôpital's rule twice: $\lim_{x\to0}\frac{1-\cos3x}{8x^2}=\lim_{x\to0}\frac{3\sin3x}{16x}=\lim_{x\to0}\frac{9\cos3x}{16}=\frac{9}{16}$.

4.7.34 Observe that $\lim_{x\to0}\frac{\sin^23x}{x^2}=\left(\lim_{x\to0}\frac{\sin3x}{x}\right)^2$, and apply L'Hôpital's rule to obtain $\lim_{x\to0}\frac{\sin3x}{x}=\lim_{x\to0}\frac{3\cos3x}{1}=3$. Therefore $\lim_{x\to0}\frac{\sin^23x}{x^2}=9$.

4.7.35 Apply L'Hôpital's rule twice:

$$\lim_{x\to\pi}\frac{\cos x+1}{(x-\pi)^2}=\lim_{x\to\pi}\frac{-\sin x}{2(x-\pi)}=\lim_{x\to\pi}\frac{-\cos x}{2}=\frac{1}{2}.$$

4.7.36 Apply L'Hôpital's rule twice:

$$\lim_{x\to0}\frac{e^x-x-1}{5x^2}=\lim_{x\to0}\frac{e^x-1}{10x}=\lim_{x\to0}\frac{e^x}{10}=\frac{1}{10}.$$

4.7.37 Apply L'Hôpital's rule three times:

$$\lim_{x\to\pi/2^-}\frac{\tan x}{3/(2x-\pi)}=\lim_{x\to\pi/2^-}\frac{\sec^2x}{-6/(2x-\pi)^2}=-(1/6)\lim_{x\to\pi/2^-}\frac{(2x-\pi)^2}{\cos^2x}=(-1/6)\lim_{x\to\pi/2^-}\frac{4(2x-\pi)}{2\cos x(-\sin x)}=$$

$$(-1/6)\lim_{x\to\pi/2^-}\frac{8x-4\pi}{-\sin(2x)}=(-1/6)\lim_{x\to\pi/2^-}\frac{8}{-2\cos(2x)}=-\frac{2}{3}.$$

4.7.38 Applying L'Hôpital's rule gives:

$$\lim_{x\to\infty}\frac{e^{3x}}{3e^{3x}+5}=\lim_{x\to\infty}\frac{3e^{3x}}{9e^{3x}}=\frac{3}{9}=\frac{1}{3}.$$

4.7.39 Apply L'Hôpital's rule twice:

$$\lim_{x\to0}\frac{e^x-\sin x-1}{x^4+8x^3+12x^2}=\lim_{x\to0}\frac{e^x-\cos x}{4x^3+24x^2+24x}=\lim_{x\to0}\frac{e^x+\sin x}{12x^2+48x+24}=\frac{1}{24}.$$

4.7.40 Apply L'Hôpital's rule three times:

$$\lim_{x\to0}\frac{\sin x-x}{7x^3}=\lim_{x\to0}\frac{\cos x-1}{21x^2}=\lim_{x\to0}\frac{-\sin x}{42x}=\lim_{x\to0}\frac{-\cos x}{42}=-\frac{1}{42}.$$

4.7.41 L'Hôpital's rule gives:

$$\lim_{x\to\infty}\frac{e^{1/x}-1}{1/x}=\lim_{x\to\infty}\frac{e^{1/x}(-1/x^2)}{(-1/x^2)}=\lim_{x\to\infty}e^{1/x}=1.$$

4.7.42 Apply L'Hôpital's rule twice:

$$\lim_{x\to\infty}\frac{\tan^{-1}x-\pi/2}{1/x}=\lim_{x\to\infty}\frac{1/(1+x^2)}{(-1/x^2)}=\lim_{x\to\infty}\frac{-x^2}{x^2+1}=\lim_{x\to\infty}\frac{-2x}{2x}=-1.$$

4.7.43 Apply L'Hôpital's rule twice:

$$\lim_{x\to-1}\frac{x^3-x^2-5x-3}{x^4+2x^3-x^2-4x-2}=\lim_{x\to-1}\frac{3x^2-2x-5}{4x^3+6x^2-2x-4}=\lim_{x\to-1}\frac{6x-2}{12x^2+12x-2}=4.$$

4.7.44 L'Hôpital's rule gives $\lim_{x\to1}\frac{x^n-1}{x-1}=\lim_{x\to1}\frac{nx^{n-1}}{1}=n.$

4.7.45 Applying L'Hôpital's rule twice gives:

$$\lim_{x\to\infty}\frac{\ln(3x+5)}{\ln(7x+3)+1}=\lim_{x\to\infty}\frac{3/(3x+5)}{7/(7x+3)}=\frac{3}{7}\lim_{x\to\infty}\frac{7x+3}{3x+5}=\frac{3}{7}\lim_{x\to\infty}\frac{7}{3}=1.$$

4.7.46 Applying L'Hôpital's rule numerous times gives:

$$\lim_{x\to\infty}\frac{\ln(3x+5e^x)}{\ln(7x+3e^{2x})}=\lim_{x\to\infty}\frac{(3+5e^x)/(3x+5e^x)}{(7+6e^{2x})/(7x+3e^{2x})}=\lim_{x\to\infty}\frac{3+5e^x}{3x+5e^x}\cdot\lim_{x\to\infty}\frac{7x+3e^{2x}}{7+6e^{2x}}=$$

$$\lim_{x\to\infty}\frac{5e^x}{3+5e^x}\cdot\lim_{x\to\infty}\frac{7+6e^{2x}}{12e^{2x}}=\lim_{x\to\infty}\frac{5e^x}{5e^x}\cdot\lim_{x\to\infty}\frac{12e^{2x}}{24e^{2x}}=\frac{1}{2}.$$

4.7.47 L'Hôpital's rule gives $\lim_{v\to3}\frac{v-1-\sqrt{v^2-5}}{v-3}=\lim_{v\to3}\frac{1-\frac{v}{\sqrt{v^2-5}}}{1}=-\frac{1}{2}.$

4.7.48 L'Hôpital's rule gives $\lim_{y\to2}\frac{y^2+y-6}{\sqrt{8-y^2}-y}=\lim_{y\to2}\frac{2y+1}{-\frac{y}{\sqrt{8-y^2}}-1}=-\frac{5}{2}.$

4.7.49 Apply L'Hôpital's rule twice:

$$\lim_{x\to2}\frac{x^2-4x+4}{\sin^2\pi x}=\lim_{x\to2}\frac{2x-4}{2\pi(\sin\pi x)(\cos\pi x)}=\lim_{x\to2}\frac{2x-4}{\pi\sin2\pi x}=\lim_{x\to2}\frac{2}{2\pi^2\cos2\pi x}=\frac{2}{2\pi^2}=\frac{1}{\pi^2}.$$

4.7.50 L'Hôpital's rule gives $\lim_{x\to 2} \frac{(3x+2)^{1/3}-2}{x-2} = \lim_{x\to 2} \frac{(3x+2)^{-2/3}}{1} = 8^{-2/3} = \frac{1}{4}$. (Notice that this limit is the derivative of $(3x+2)^{1/3}$ at $x=2$.)

4.7.51 Applying L'Hôpital's rule twice gives:

$$\lim_{x\to\infty} \frac{x^2-\ln(2/x)}{3x^2+2x} = \lim_{x\to\infty} \frac{2x+(1/x)}{6x+2} = \lim_{x\to\infty} \frac{2-1/x^2}{6} = \frac{2}{6} = \frac{1}{3}.$$

4.7.52 Observe that $\lim_{x\to 1+} \left(\frac{1}{x-1} - \frac{1}{\sqrt{x-1}}\right) = \lim_{x\to 1+} \frac{1-\sqrt{x-1}}{x-1} = \infty$.

4.7.53 Observe that $\lim_{x\to 0} x\csc x = \lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{1}{\cos x} = 1$, by L'Hôpital's rule.

4.7.54 Observe that $\lim_{x\to 1^-} (1-x)\tan\left(\frac{\pi x}{2}\right) = \lim_{x\to 1^-} \frac{1-x}{\cot\left(\frac{\pi x}{2}\right)} = \lim_{x\to 1^-} \frac{-1}{-\left(\frac{\pi}{2}\right)\csc^2\left(\frac{\pi x}{2}\right)} = \frac{2}{\pi}$ by L'Hôpital's rule.

4.7.55 By L'Hôpital's rule: $\lim_{x\to 0} \frac{\sin 7x}{\sin 6x} = \lim_{x\to 0} \frac{7\cos 7x}{6\cos 6x} = \frac{7}{6}$.

4.7.56 By L'Hôpital's rule: $\lim_{x\to\infty} \frac{e^{1/x}-1}{\sin(1/x)} = \lim_{x\to\infty} \frac{e^{1/x}\cdot(-1/x^2)}{\cos(1/x)(-1/x^2)} = \frac{1}{1} = 1$.

4.7.57 Observe that $\lim_{x\to(\pi/2)^-} \left(\frac{\pi}{2}-x\right)\sec x = \lim_{x\to(\pi/2)^-} \frac{\pi/2-x}{\cos x} = \lim_{x\to(\pi/2)^-} \frac{-1}{-\sin x} = 1$ by L'Hôpital's rule.

4.7.58 Observe that $\lim_{x\to 0^+} \sin x\sqrt{\frac{1-x}{x}} = \lim_{x\to 0^+} \sin x\sqrt{\frac{x(1-x)}{x^2}} = \lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \sqrt{x(1-x)} = 1\cdot 0 = 0$, where we use L'Hôpital's rule for $\lim_{x\to 0^+} \sin x/x = 1$.

4.7.59

$$\begin{aligned}\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) &= \lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} \\ &= \lim_{x\to 1^+} \frac{1/x-1}{\ln x + (x-1)(1/x)} \\ &= \lim_{x\to 1^+} \frac{1/x-1}{\ln x + 1 - 1/x} = \lim_{x\to 1^+} \frac{-1/x^2}{1/x+1/x^2} = -\frac{1}{2},\end{aligned}$$

where L'Hôpital's rule was used twice.

4.7.60

$$\begin{aligned}\lim_{x\to 1^-} \left(\frac{x}{x-1} - \frac{x}{\ln x}\right) &= \lim_{x\to 1^-} \frac{x\ln x - x^2 + x}{(x-1)\ln x} \\ &= \lim_{x\to 1^-} \frac{\ln x + 1 - 2x + 1}{\ln x + \frac{x-1}{x}} \\ &= \lim_{x\to 1^-} \frac{\ln x - 2x + 2}{\ln x + 1 - 1/x} = \lim_{x\to 1^-} \frac{1/x-2}{1/x+1/x^2} = -\frac{1}{2},\end{aligned}$$

where L'Hôpital's rule was used twice.

4.7.61 Observe that $\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{\cos x}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x\cos x - \sin x}{x\sin x}$. Apply L'Hôpital's rule twice:

$$\begin{aligned}\lim_{x\to 0}\frac{x\cos x - \sin x}{x\sin x} &= \lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}\\ &= -\lim_{x\to 0}\frac{x\sin x}{\sin x + x\cos x} = -\lim_{x\to 0}\frac{\sin x + x\cos x}{\cos x + \cos x - x\sin x} = -\frac{0}{2} = 0.\end{aligned}$$

4.7.62 Observe that

$$\lim_{\theta\to(\pi/2)^-}(\tan\theta - \sec\theta) = \lim_{\theta\to(\pi/2)^-}\left(\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}\right) = \lim_{\theta\to(\pi/2)^-}\frac{\sin\theta - 1}{\cos\theta}.$$

By L'Hôpital's rule

$$\lim_{\theta\to(\pi/2)^-}\frac{\sin\theta - 1}{\cos\theta} = \lim_{\theta\to(\pi/2)^-}\frac{\cos\theta}{-\sin\theta} = \frac{0}{-1} = 0.$$

4.7.63 Make the substitution $x = \frac{1}{t}$. Then

$$\begin{aligned}\lim_{x\to\infty}(x^2 - \sqrt{x^4 + 16x^2}) &= \lim_{t\to 0^+}\left(\frac{1}{t^2} - \sqrt{\frac{1}{t^4} + \frac{16}{t^2}}\right)\\ &= \lim_{t\to 0^+}\frac{1 - \sqrt{1 + 16t^2}}{t^2} = \lim_{t\to 0^+}\frac{\frac{-32t}{2\sqrt{1+16t^2}}}{2t}\\ &= \lim_{t\to 0^+}\frac{-8}{\sqrt{1 + 16t^2}} = -8.\end{aligned}$$

4.7.64 Observe that

$$\lim_{x\to\infty}\left(x - \sqrt{x^2 + 4x}\right) = \lim_{x\to\infty} x\left(1 - \sqrt{1 + 4/x}\right).$$

Make the change of variables $t = 1/x$:

$$\lim_{x\to\infty} x\left(1 - \sqrt{1 + 4/x}\right) = \lim_{t\to 0+}\frac{1 - \sqrt{1 + 4t}}{t} = \lim_{t\to 0+}\frac{-\frac{4}{2\sqrt{1 + 4t}}}{1} = -2.$$

4.7.65 Observe that

$$(\sqrt{x-2} - \sqrt{x-4})\cdot\frac{\sqrt{x-2} + \sqrt{x-4}}{\sqrt{x-2} + \sqrt{x-4}} = \frac{x - 2 - (x-4)}{\sqrt{x-2} + \sqrt{x-4}} = \frac{2}{\sqrt{x-2} + \sqrt{x-4}},$$

so

$$\lim_{x\to\infty}\sqrt{x-2} - \sqrt{x-4} = \lim_{x\to\infty}\frac{2}{\sqrt{x-2} + \sqrt{x-4}} = 0.$$

4.7.66 Let $t = \frac{1}{x}$. Then

$$\lim_{x\to\infty} x^2\ln\left(\cos\frac{1}{x}\right) = \lim_{t\to 0^+}\frac{\ln(\cos t)}{t^2} = \lim_{t\to 0^+}\frac{-\tan t}{2t} = \lim_{t\to 0^+}\frac{-\sec^2 t}{2} = -\frac{1}{2}.$$

4.7.67 Use the identity $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$; then

$$\lim_{n\to\infty}\frac{1 + 2 + \cdots + n}{n^2} = \lim_{n\to\infty}\frac{n(n+1)}{2n^2} = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}.$$

4.7.68 Let $y = x^{1/\ln x}$. Then $\ln y = \ln x^{1/\ln x} = 1$, so $\lim_{x\to 0^+}\ln y = 1$, so $\lim_{x\to 0^+} y = e^1 = e$.

4.7.69

$$\lim_{x\to\infty}\frac{\log_2 x}{\log_3 x}=\lim_{x\to\infty}\frac{\frac{1}{x\ln 2}}{\frac{1}{x\ln 3}}=\frac{\ln 3}{\ln 2}.$$

4.7.70 Note that $\log_2 x=\ln x/\ln 2$ and $\log_3 x=\ln x/\ln 3$; therefore

$$\lim_{x\to\infty}(\log_2 x-\log_3 x)=\lim_{x\to\infty}\left(\frac{1}{\ln 2}-\frac{1}{\ln 3}\right)\ln x=\infty.$$

4.7.71 By L'Hôpital's rule, $\lim\limits_{x\to 6}\dfrac{(5x+2)^{1/5}-2}{x^{-1}-6^{-1}}=\lim\limits_{x\to 6}\dfrac{(5x+2)^{-4/5}}{-x^{-2}}=-\dfrac{9}{4}$.

4.7.72

$$\lim_{x\to\pi/2}(\pi-2x)\tan x=\lim_{x\to\pi/2}\frac{\pi-2x}{\cot x}=\lim_{x\to\pi/2}\frac{-2}{-\csc^2 x}=\lim_{x\to\pi/2}2\sin^2 x=2.$$

4.7.73 Make the substitution $t=1/x$; then

$$\lim_{x\to\infty}x^3\left(\frac{1}{x}-\sin\frac{1}{x}\right)=\lim_{t\to 0+}\frac{t-\sin t}{t^3}=\lim_{t\to 0+}\frac{1-\cos t}{3t^2}=\lim_{t\to 0+}\frac{\sin t}{6t}=\lim_{t\to 0+}\frac{\cos t}{6}=\frac{1}{6},$$

using L'Hôpital's rule.

4.7.74 Make the substitution $t=1/x$; then

$$\lim_{x\to\infty}(x^2e^{1/x}-x^2-x)=\lim_{t\to 0+}\frac{e^t-1-t}{t^2}=\lim_{t\to 0+}\frac{e^t-1}{2t}=\lim_{t\to 0+}\frac{e^t}{2}=\frac{1}{2},$$

using L'Hôpital's rule.

4.7.75 Note that $\ln x^{2x}=2x\ln x$, so we evaluate

$$L=\lim_{x\to 0^+}2x\ln x=2\lim_{x\to 0^+}\frac{\ln x}{1/x}=2\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=2\lim_{x\to 0^+}(-x)=0$$

by L'Hôpital's rule. Therefore $\lim\limits_{x\to 0^+}x^{2x}=e^L=1$.

4.7.76 Note that $\ln(1+4x)^{3/x}=\frac{3\ln(1+4x)}{x}$, so we evaluate

$$L=\lim_{x\to 0}\frac{3\ln(1+4x)}{x}=3\lim_{x\to 0}\frac{4/(1+4x)}{1}=12$$

by L'Hôpital's rule. Therefore $\lim\limits_{x\to 0}(1+4x)^{3/x}=e^L=e^{12}$.

4.7.77 Note that $\ln(\tan\theta)^{\cos\theta}=\cos\theta\ln\tan\theta$, so we evaluate

$$L=\lim_{\theta\to\pi/2^-}\cos\theta\ln\tan\theta=\lim_{\theta\to\pi/2^-}\frac{\ln\tan\theta}{\sec\theta}.$$

L'Hôpital's rule gives

$$\lim_{\theta\to\pi/2^-}\frac{\ln\tan\theta}{\sec\theta}=\lim_{\theta\to\pi/2^-}\frac{\sec^2\theta/\tan\theta}{\sec\theta\tan\theta}=\lim_{\theta\to\pi/2^-}\frac{\sec\theta}{\tan^2\theta}=\lim_{\theta\to\pi/2^-}\frac{\cos\theta}{\sin^2\theta}=0,$$

so $\lim\limits_{\theta\to\pi/2-}(\tan\theta)^{\cos\theta}=e^L=1$.

4.7.78 Note that $\ln(\sin\theta)^{\tan\theta} = \tan\theta \ln\sin\theta$, so we evaluate

$$L = \lim_{\theta\to 0+} \tan\theta\ln\sin\theta = \lim_{\theta\to 0+} \frac{\ln\sin\theta}{\cot\theta}.$$

L'Hôpital's rule gives

$$\lim_{\theta\to 0+} \frac{\ln\sin\theta}{\cot\theta} = \lim_{\theta\to 0+} \frac{\cos\theta/\sin\theta}{-\csc^2\theta} = -\lim_{\theta\to 0+} \cos\theta\sin\theta = 0,$$

so $\lim\limits_{\theta\to 0+} (\tan\theta)^{\cos\theta} = e^L = 1$.

4.7.79 Note that $\ln(1+x)^{\cot x} = \cot x\ln(1+x)$, so we evaluate

$$L = \lim_{x\to 0+} \cot x\ln(1+x) = \lim_{x\to 0+} \frac{\ln(1+x)}{\tan x} = \lim_{x\to 0+} \frac{1/(1+x)}{\sec^2 x} = \lim_{x\to 0+} \frac{\cos^2 x}{1+x} = 1$$

by L'Hôpital's rule. Therefore $\lim\limits_{x\to 0^+} (1+x)^{\cot x} = e^L = e$.

4.7.80 Note that $\ln(1+a/x)^x = x\ln(1+a/x)$, so we evaluate

$$L = \lim_{x\to\infty} x\ln(1+a/x) = \lim_{x\to\infty} \frac{\ln(1+a/x)}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{1+a/x}\cdot\frac{-a}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{a}{1+a/x} = a$$

by L'Hôpital's rule. Therefore $\lim\limits_{x\to\infty} (1+a/x)^x = e^L = e^a$.

4.7.81 Note that $\ln(e^{ax}+x)^{1/x} = \frac{1}{x}\ln(e^{ax}+x)$, so we evaluate

$$L = \lim_{x\to 0} \frac{\ln(e^{ax}+x)}{x} = \lim_{x\to 0} \frac{(ae^{ax}+1)/(e^{ax}+x)}{1} = \frac{(a+1)/1}{1} = a+1.$$

Therefore, $\lim\limits_{x\to 0} (e^{ax}+x)^{1/x} = e^{a+1}$.

4.7.82 Note that $\ln(1+10/z^2)^{z^2} = z^2\ln(1+10/z^2)$, so we evaluate

$$L = \lim_{z\to\infty} z^2\ln\left(1+\frac{10}{z^2}\right) = \lim_{z\to\infty} \frac{\ln\left(1+\frac{10}{z^2}\right)}{\frac{1}{z^2}} = \lim_{z\to\infty} \frac{\frac{1}{1+\frac{10}{z^2}}\cdot\frac{-20}{z^3}}{-\frac{2}{z^3}} = \lim_{z\to\infty} \frac{10z^2}{z^2+10} = 10$$

by L'Hôpital's rule. Therefore $\lim\limits_{z\to\infty} \left(1+\dfrac{10}{z^2}\right)^{z^2} = e^L = e^{10}$.

4.7.83 Note that $\ln(x+\cos x)^{1/x} = (\ln(x+\cos x))/x$, so we evaluate

$$L = \lim_{x\to 0} \frac{\ln(x+\cos x)}{x} = \lim_{x\to 0} \frac{(x+\cos x)^{-1}(1-\sin x)}{1} = 1$$

by L'Hôpital's rule. Therefore $\lim\limits_{x\to 0} (x+\cos x)^{1/x} = e^L = e$.

4.7.84 Observe that

$$\lim_{\delta\to 2m\pi} \frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)} = \left(\lim_{\delta\to 2m\pi} \frac{\sin(N\delta/2)}{\sin(\delta/2)}\right)^2,$$

and

$$\lim_{\delta\to 2m\pi} \frac{\sin(N\delta/2)}{\sin(\delta/2)} = \lim_{\delta\to 2m\pi} \frac{(N/2)\cos(N\delta/2)}{(1/2)\cos(\delta/2)} = \pm N,$$

so $\lim\limits_{\delta\to 2m\pi} \dfrac{\sin^2(N\delta/2)}{\sin^2(\delta/2)} = N^2$.

4.7.85

a. After each year the balance increases by the factor $1+r$; therefore the balance after t years is $B(t) = P(1+r)^t$.

b. Let $L = \lim_{m\to\infty} mt\ln\left(1+\frac{r}{m}\right) = t\lim_{m\to\infty}\frac{\ln\left(1+\frac{r}{m}\right)}{1/m}$. By L'Hôpital's rule we have

$$L = t\lim_{m\to\infty}\frac{\frac{1}{1+\frac{r}{m}}\left(\frac{-r}{m^2}\right)}{-1/m^2} = t\lim_{m\to\infty}\frac{r}{1+\frac{r}{m}} = rt,$$

so $\lim_{m\to\infty} B(t) = Pe^L = Pe^{rt}$.

4.7.86 Observe that $\lim_{x\to\infty}\frac{2x^3-x^2+1}{5x^3+2x} = \frac{2}{5}$ by Theorem 2.7. We can also use L'Hôpital's rule:

$$\lim_{x\to\infty}\frac{2x^3-x^2+1}{5x^3+2x} = \lim_{x\to\infty}\frac{6x^2-2x}{15x^2+2} = \lim_{x\to\infty}\frac{12x-2}{30x} = \lim_{x\to\infty}\frac{12}{30} = \frac{2}{5}.$$

4.7.87 By factoring, we have

$$\lim_{x\to 0}\frac{(e^x-1)(e^x+5)}{(e^x+1)(e^x-1)} = \lim_{x\to 0}\frac{e^x+5}{e^x+1} = \frac{6}{2} = 3.$$

Or, using L'Hôpital's rule, we have

$$\lim_{x\to 0}\frac{2e^{2x}+4e^x}{2e^{2x}} = \frac{2+4}{2} = 3.$$

4.7.88 Note that

$$\ln\left(\frac{\sin x}{x}\right)^{1/x^2} = \frac{\ln\sin x - \ln x}{x^2},$$

so we evaluate

$$L = \lim_{x\to 0}\frac{\ln\sin x-\ln x}{x^2} = \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}-\frac{1}{x}}{2x} = \lim_{x\to 0}\frac{x\cos x-\sin x}{2x^2\sin x}$$

by L'Hôpital's rule. Next, observe that

$$\lim_{x\to 0}\frac{x\cos x-\sin x}{2x^2\sin x} = \lim_{x\to 0}\frac{\cos x - x\sin x-\cos x}{4x\sin x+2x^2\cos x} = \lim_{x\to 0}\frac{-\frac{\sin x}{x}}{\frac{4\sin x}{x}+2\cos x} = -\frac{1}{6}$$

using L'Hôpital's rule and $\lim_{x\to 0}\sin x/x = 1$. Therefore $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^L = e^{-1/6}$.

4.7.89

$$\lim_{x\to 1}\frac{x\ln x - x+1}{x\ln^2 x} = \lim_{x\to 1}\frac{\ln x + x\cdot\frac{1}{x}-1}{\ln^2 x+2x\ln x\cdot\frac{1}{x}} = \lim_{x\to 1}\frac{\ln x}{\ln^2 x + 2\ln x} = \lim_{x\to 1}\frac{1}{\ln x+2} = \frac{1}{2},$$

where the first equality follows from L'Hôpital's rule and the penultimate follows by dividing the numerator and denominator by $\ln x$.

4.7.90

$$\begin{aligned}
\lim_{x\to1}\frac{x\ln x+\ln x-2x+2}{x^2\ln^3 x} &= \lim_{x\to1}\frac{\ln x+x\cdot\frac{1}{x}+\frac{1}{x}-2}{2x\ln^3 x+x^2\cdot 3\ln^2 x\cdot\frac{1}{x}}\\
&= \lim_{x\to1}\frac{\ln x+\frac{1}{x}-1}{2x\ln^3 x+3x\ln^2 x}\\
&= \lim_{x\to1}\frac{\frac{1}{x}-\frac{1}{x^2}}{2\ln^3 x+6x\ln^2 x\cdot\frac{1}{x}+3\ln^2 x+6x\ln x\cdot\frac{1}{x}}\\
&= \lim_{x\to1}\frac{\frac{-1}{x^2}+\frac{2}{x^3}}{6\ln x\cdot\frac{1}{x}+12\ln x\cdot\frac{1}{x}+6\ln x\cdot\frac{1}{x}+\frac{6}{x}}\\
&= \frac{1}{6},
\end{aligned}$$

where each equality except the second follows from L'Hôpital's rule.

4.7.91 Let $z=\ln x^{\frac{1}{1+\ln x}}$. Then $z=\frac{\ln x}{1+\ln x}$, and $\lim_{x\to0^+} z=\lim_{x\to0^+}\frac{\ln x}{1+\ln x}=\lim_{x\to0^+}\frac{1/x}{1/x}=\lim_{x\to0^+}1=1$. Then $\lim_{x\to0^+}e^z=\lim_{x\to0^+}x^{\frac{1}{1+\ln x}}=e^1=e$.

4.7.92 The given limit can be written as $\lim_{n\to\infty}\frac{\ln\left(n\sin(1/n)\right)}{\frac{1}{n^2}}$, and by L'Hôpital's rule this is equal to

$$\lim_{n\to\infty}\frac{1}{n\sin(1/n)}\cdot\frac{\sin(1/n)+n\cos(1/n)\left(\frac{-1}{n^2}\right)}{\frac{-2}{n^3}}=\lim_{n\to\infty}\left(-\frac{n^2}{2}+\frac{n\cot(1/n)}{2}\right).$$

This is exactly $\frac{1}{2}$ times $\lim_{n\to\infty}(n\cot(1/n)-n^2)$.

Let $z=\frac{1}{n}$. Then as $n\to\infty$ we have $z\to0^+$. Then

$$\begin{aligned}
\frac{1}{2}\lim_{n\to\infty}(n\cot(1/n)-n^2) &= \frac{1}{2}\lim_{z\to0^+}\frac{\cot z}{z}-\frac{1}{z^2}\\
&= \frac{1}{2}\lim_{z\to0^+}\frac{z\cos z-\sin z}{z^2\sin z}\\
&= \frac{1}{2}\lim_{z\to0^+}\frac{\cos z-z\sin z-\cos z}{2z\sin z+z^2\cos z}\\
&= \frac{1}{2}\lim_{z\to0^+}\frac{-\sin z}{2\sin z+z\cos z}\\
&= \frac{1}{2}\lim_{z\to0^+}\frac{-\cos z}{2\cos z+\cos z-z\sin z}\\
&= \frac{1}{2}\left(-\frac{1}{2+1-0}\right)=\frac{1}{2}\left(-\frac{1}{3}\right)=-\frac{1}{6},
\end{aligned}$$

where the third and fifth equalities follow from L'Hôpital's rule.

4.7.93 Apply L'Hôpital's rule: $\lim_{x\to0}\frac{a^x-b^x}{x}=\lim_{x\to0}\frac{(\ln a)a^x-(\ln b)b^x}{1}=\ln a-\ln b$.

4.7.94 Note that $\ln(1+ax)^{b/x}=b\ln(1+ax)/x$, so we evaluate

$$L=\lim_{x\to0}\frac{b\ln(1+ax)}{x}=b\lim_{x\to0}\frac{\frac{a}{1+ax}}{1}=ab$$

by L'Hôpital's rule. Therefore $\lim_{x\to0}(1+ax)^{b/x}=e^L=e^{ab}$.

4.7.95 By Theorem 4.14, $e^{0.01x}$ grows faster than x^{10} as $x \to \infty$.

4.7.96 Observe that $\lim\limits_{x\to\infty} \dfrac{x^2 \ln x}{(\ln x)^2} = \lim\limits_{x\to\infty} \dfrac{x^2}{\ln x} = \infty$, so $x^2 \ln x$ grows faster than $(\ln x)^2$ as $x \to \infty$.

4.7.97 Note that $\ln x^{20} = 20 \ln x$, so $\ln x^{20}$ and $\ln x$ have comparable growth rates as $x \to \infty$.

4.7.98 Make the substitution $y = \ln x$; then $y \to \infty$ if $x \to \infty$ and $\lim\limits_{x\to\infty} \dfrac{\ln(\ln x)}{\ln x} = \lim\limits_{y\to\infty} \dfrac{\ln y}{y} = 0$. Therefore $\ln x$ grows faster than $\ln(\ln x)$ as $x \to \infty$.

4.7.99 By Theorem 4.14, x^x grows faster than 100^x as $x \to \infty$.

4.7.100 Observe that $\lim\limits_{x\to\infty} \dfrac{x^2 \ln x}{x^3} = \lim\limits_{x\to\infty} \dfrac{\ln x}{x} = 0$, so x^3 grows faster than $x^2 \ln x$ as $x \to \infty$.

4.7.101 By Theorem 4.14, 1.00001^x grows faster than x^{20} as $x \to \infty$.

4.7.102 Observe that $\ln \sqrt{x} = (\ln x)/2$. Note that $\lim\limits_{x\to\infty} \dfrac{\ln^2 x}{(\ln x)/2} = \lim\limits_{x\to\infty} 2 \ln x = \infty$, so $\ln^2 x$ grows faster than $\ln \sqrt{x}$ as $x \to \infty$.

4.7.103 Note that $\lim\limits_{x\to\infty} \dfrac{e^{x^2}}{e^{10x}} = \lim\limits_{x\to\infty} e^{x^2 - 10x} = \infty$, so e^{x^2} grows faster than e^{10x} as $x \to \infty$.

4.7.104 Observe that $\lim\limits_{x\to\infty} \dfrac{x^{x/10}}{e^{x^2}} = \lim\limits_{x\to\infty} \left(\dfrac{x^{1/10}}{e^x}\right)^x = 0$ by Theorem 4.14, so e^{x^2} grows faster than $x^{x/10}$ as $x \to \infty$.

4.7.105

a. False. $\lim\limits_{x\to 2} x^2 - 1 = 3$, so L'Hôpital's rule does not apply. In fact, $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-1} = \dfrac{0}{3} = 0$.

b. False. L'Hôpital's rule does not say $\lim\limits_{x\to a} f(x)g(x) = \lim\limits_{x\to a} f'(x) \lim\limits_{x\to a} g'(x)$. In fact, $\lim\limits_{x\to 0} x \sin x = 0 \cdot 0 = 0$.

c. False. This limit has the form $0^\infty = 0$.

d. False. This limit has the indeterminate form 1^∞ which is not always 1.

e. True. $\ln x^{100} = 100 \ln x$.

f. True. Note that $\lim\limits_{x\to\infty} \dfrac{e^x}{2^x} = \lim\limits_{x\to\infty} \left(\dfrac{e}{2}\right)^x = \infty$ because $e/2 > 1$.

4.7.106

The domain of g is $(0, \infty)$, and there is no symmetry, and there are no vertical asymptotes. Note that $\lim\limits_{x\to\infty} g(x) = \infty$, so there are no horizontal asymptotes. To see the behavior as $x \to 0^+$, we compute

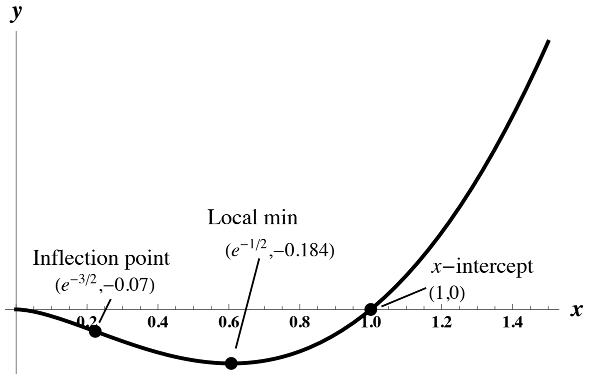
$$\lim_{x\to 0^+} \frac{\ln x}{x^{-2}} = \lim_{x\to 0^+} \frac{1/x}{-2x^{-3}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0.$$

$g'(x) = x^2 \cdot \dfrac{1}{x} + \ln x \cdot 2x = x(1 + 2 \ln x)$. This is 0 for $x = e^{-1/2}$.

$g''(x) = x(2/x) + (1 + 2 \ln x) = 3 + 2 \ln x$. This is 0 for $x = e^{-3/2}$.

Note that $g'(x) < 0$ on $(0, e^{-1/2})$ and $g'(x) > 0$ on $(e^{-1/2}, \infty)$, so g is decreasing on $(0, e^{-1/2})$ and increasing on $(e^{-1/2}, \infty)$, and there is a local (and absolute) minimum at $x = e^{-1/2}$.

Note also that $g''(x) < 0$ for $0 < x < e^{-3/2}$ and $g''(x) > 0$ for $x > e^{-3/2}$, so g is concave down on $(0, e^{-3/2})$ and concave up on $(e^{-3/2}, \infty)$ and there is an inflection point at $x = e^{-3/2}$. The only x-intercept is $(1, 0)$.



4.7.107

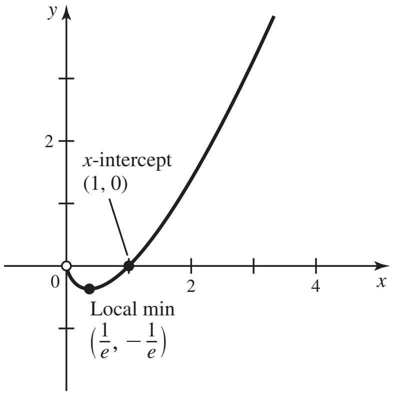
The domain of f is $(0,\infty)$, so questions about symmetry aren't relevant. There are no asymptotes. To see the behavior as $x \to 0^+$ we compute

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{1/x}{-x^{-2}} = \lim_{x\to 0^+} -x = 0.$$

$f'(x) = x \cdot 1/x + \ln x = 1 + \ln x$. This is 0 for $x = 1/e$. Note that $f'(x) < 0$ for $0 < x < 1/e$ and $f'(x) > 0$ for $x > 1/e$, so f is decreasing on $(0, 1/e)$ and increasing on $(1/e,\infty)$ and there is a local minimum (which is also an absolute minimum) at $x = 1/e$.

$f''(x) = 1/x$, which is always positive on the domain, so f is concave up on its domain and there are no inflection points.

There is an x-intercept at $x = 1$.



4.7.108

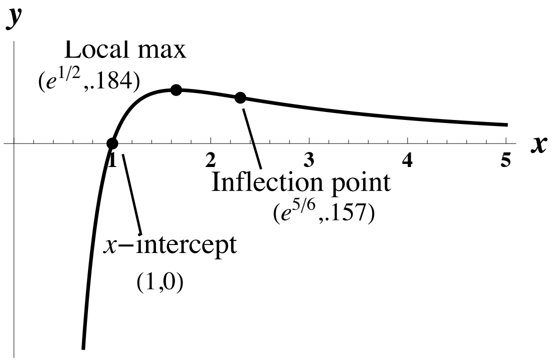
The domain of f is $(0,\infty)$. Note that $\lim_{x\to 0^+} \frac{\ln x}{x^2} = -\infty$, so $x = 0$ is a vertical asymptote. Questions about symmetry aren't relevant. We have

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0,$$

so $y = 0$ is a horizontal asymptote.

$f'(x) = \dfrac{x^2 \cdot (1/x) - \ln x \cdot 2x}{x^4} = \dfrac{1 - 2\ln x}{x^3}$, which is 0 for $x = e^{1/2}$. Note that $f'(x) > 0$ for $0 < x < e^{1/2}$ and $f'(x) < 0$ for $x > e^{1/2}$, so f is increasing on $(0, e^{1/2})$ and decreasing on $(e^{1/2},\infty)$, and there is a local maximum (which is actually an absolute maximum) at $x = e^{1/2}$ of about .184.

$f''(x) = \dfrac{x^3(-2/x) - (1 - 2\ln x) \cdot 3x^2}{x^6} = \dfrac{6\ln x - 5}{x^4}$, which is 0 for $x = e^{5/6}$. Note that $f''(x) < 0$ for $0 < x < e^{5/6}$ and $f''(x) > 0$ for $x > e^{5/6}$, so there is a point of inflection at $e^{5/6}$ where the concavity changes from down (to the left) to up (to the right).



4.7.109

The domain of p is $(-\infty, \infty)$. There are no vertical asymptotes. Note that $p(-x) = -xe^{-(-x)^2/2} = -(xe^{-x^2/2}) = -p(x)$, so p has odd symmetry. Note that

$$\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} \frac{x}{e^{x^2/2}} = \lim_{x\to\pm\infty} \frac{1}{xe^{x^2/2}} = 0,$$

so $y = 0$ is a horizontal asymptote.

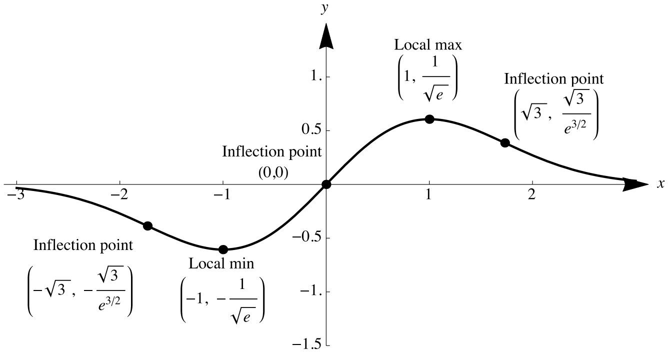
$$p'(x) = x \cdot (-xe^{-x^2/2}) + e^{-x^2/2} \cdot 1 = e^{-x^2/2}(1 - x^2) = e^{-x^2/2}(1 - x)(1 + x).$$

This is 0 for $x = \pm 1$. Note that $p'(x) < 0$ on $(-\infty, -1)$ and on $(1, \infty)$, so p is decreasing on those intervals, and $p'(x) > 0$ on $(-1, 1)$, so p is increasing on that interval. There is a local maximum at $x = 1$ and a local minimum at $x = -1$.

$$p''(x) = e^{-x^2/2}(-2x) + (1 - x^2) \cdot (-x)e^{-x^2/2} = e^{-x^2/2}x(x^2 - 3),$$

which is 0 at $x = 0$ and at $x = \pm\sqrt{3}$. Note that $p''(x) > 0$ on $(-\sqrt{3}, 0)$ and on $(\sqrt{3}, \infty)$, so p is concave up on those intervals, while $p''(x) < 0$ on $(-\infty, -\sqrt{3})$ and on $(0, \sqrt{3})$, so p is concave down on those intervals. There are inflection points at each of $x = \pm\sqrt{3}$ and at $x = 0$.

There is an x-intercept at $(0, 0)$, which is also the y-intercept.



4.7.110 Note that

$$\sqrt{n}\log_2 n << n\log_2 n << n(\log_2 n)^2 << n^{3/2}.$$

For the last relation, we note that $n^{3/2}/n(\log_2 n)^2 = n^{1/2}/(\log_2 n)^2 \to \infty$ as $n \to \infty$. Therefore the ranking in order of least to most efficient is A, C, B, D.

4.7.111 L'Hôpital's rule gives

$$\lim_{x\to\infty} \frac{\sqrt{ax + b}}{\sqrt{cx + d}} = \lim_{x\to\infty} \frac{a}{\sqrt{ax + b}} \cdot \frac{\sqrt{cx + d}}{c} = \frac{a}{c} \lim_{x\to\infty} \frac{\sqrt{cx + d}}{\sqrt{ax + b}},$$

which is the same form as the original limit, so L'Hôpital's rule fails in this case. We can evaluate this limit as follows: first observe that

$$\lim_{x\to\infty} \frac{ax+b}{cx+d} = \frac{a}{c}$$

by L'Hôpital's rule; therefore

$$\lim_{x\to\infty} \frac{\sqrt{ax+b}}{\sqrt{cx+d}} = \lim_{x\to\infty} \sqrt{\frac{ax+b}{cx+d}} = \sqrt{\frac{a}{c}}.$$

4.7.112 Observe that

$$(ax - \sqrt{a^2x^2 - bx}) \cdot \frac{(ax + \sqrt{a^2x^2 - bx})}{(ax + \sqrt{a^2x^2 - bx})} = \frac{bx}{(ax + \sqrt{a^2x^2 - bx})} = \frac{b}{(a + \sqrt{a^2 - b/x})},$$

so

$$\lim_{x\to\infty} (ax - \sqrt{a^2x^2 - bx}) = \frac{b}{2a}.$$

4.7.113 Let $t = b^x$, then $x = \ln t / \ln b$ and we have

$$\lim_{x\to\infty} \frac{x^p}{b^x} = \lim_{t\to\infty} \frac{\ln^p t}{t \ln^p b} = 0,$$

by Theorem 4.14.

4.7.114 Observe that $\lim\limits_{x\to\infty} \dfrac{a^x}{b^x} = \lim\limits_{x\to\infty} \left(\dfrac{a}{b}\right)^x = \infty$, because $a/b > 1$.

4.7.115 Note that $\log_a x = \ln x / \ln a$, so

$$\frac{\log_a x}{\log_b x} = \frac{\ln b}{\ln a},$$

and therefore $\log_a x$ and $\log_b x$ grow at a comparable rate as $x \to \infty$.

4.7.116 We have

$$b^n << n! << n^n$$

as $n \to \infty$ for any $b > 1$. To see this, observe that

$$\lim_{n\to\infty} \frac{n!}{b^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}\, n^n}{(be)^n} = \infty$$

and

$$\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}}{e^n} = 0,$$

by Theorem 4.14.

4.7.117 The triangle ABP has base $1 - \cos\theta$ and height $\sin\theta$, so its area is

$$f(\theta) = \frac{1}{2} \sin\theta(1 - \cos\theta).$$

The sector OBP has area $\theta/2$, and the triangle OBP has base 1 and height $\sin\theta$; therefore

$$g(\theta) = \frac{1}{2}(\theta - \sin\theta).$$

We have

$$\lim_{\theta\to 0} \frac{g(\theta)}{f(\theta)} = \lim_{\theta\to 0} \frac{\theta - \sin\theta}{\sin\theta(1 - \cos\theta)} = \lim_{\theta\to 0} \frac{\theta - \sin\theta}{\sin\theta - (1/2)\sin 2\theta}.$$

Three applications of L'Hôpital's rule gives

$$\lim_{\theta\to 0} \frac{g(\theta)}{f(\theta)} = \lim_{\theta\to 0} \frac{1 - \cos\theta}{\cos\theta - \cos 2\theta} = \lim_{\theta\to 0} \frac{\sin\theta}{2\sin 2\theta - \sin\theta} = \lim_{\theta\to 0} \frac{\cos\theta}{4\cos 2\theta - \cos\theta} = \frac{1}{3}.$$

4.7.118 The function isn't defined for $a = 0$ Assume $a > 0$. Then by L'Hôpital's rule,

$$\lim_{x\to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = \lim_{x\to a} \frac{\frac{2a^3-4x^3}{2\sqrt{2a^3x-x^4}} - \frac{1}{3}a^{5/3}x^{-2/3}}{-\frac{3}{4}a^{1/4}x^{-1/4}} = \frac{-a - \frac{a}{3}}{-\frac{3}{4}} = \frac{16}{9}a.$$

If $a < 0$, then the limit of the denominator of the given function satisfies $\lim\limits_{x\to a} a - \sqrt[4]{ax^3} = \lim\limits_{x\to a} a - \sqrt[4]{a^4} = \lim\limits_{x\to a} a - |a| = 2a$. So this is not an indeterminate form. The numerator has limit $\lim\limits_{x\to a} \sqrt{a^4} - a^2 = 0$. So the given limit is $\begin{cases} \dfrac{16a}{9} & \text{if } a > 0 \\ 0 & \text{if } a < 0. \end{cases}$

4.7.119 Note that

$$\ln\left(1 + \frac{a}{x}\right)^x = \frac{\ln(1 + a/x)}{1/x}$$

so we evaluate

$$L = \lim_{x\to\infty} \frac{\ln(1 + a/x)}{1/x} = \lim_{x\to\infty} \frac{1}{1 + a/x} \cdot -\frac{a}{x^2} \cdot \frac{1}{-1/x^2} = \lim_{x\to\infty} \frac{a}{1 + a/x} = a$$

by L'Hôpital's rule. Therefore

$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^L = e^a.$$

4.7.120 Because $x \to \infty$, eventually $x > 2b$. Then

$$\frac{x^x}{b^x} > \frac{(2b)^x}{b^x} = 2^x \to \infty$$

as $x \to \infty$, so x^x grows faster than b^x as $x \to \infty$.

4.7.121

a. Observe that $\lim\limits_{x\to\infty} \dfrac{b^x}{e^x} = \lim\limits_{x\to\infty} \left(\dfrac{b}{e}\right)^x$. This limit is ∞ exactly when $b > e$.

b. Observe that $\lim\limits_{x\to\infty} \dfrac{e^{ax}}{e^x} = \lim\limits_{x\to\infty} e^{(a-1)x}$. This limit is ∞ exactly when $a > 1$.

4.8 Newton's Method

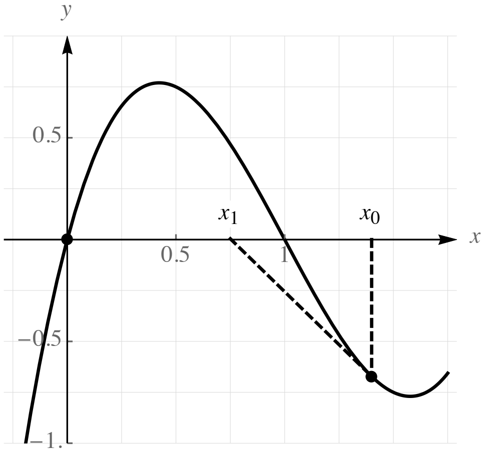
4.8.1 Newton's method generates a sequence of x-intercepts of lines tangent to the graph of f to approximate the roots of f.

4.8.2 To get the x_{n+1} iterate from x_n, one puts x_n into the recursive formula and computes x_{n+1}. Thus, starting with x_0, a sequence $x_0, x_1, x_2, x_3, \ldots$ is generated.

4.8.3 Starting at $x_0 = 3$ on the x-axis, travel with your eye downward until you hit the point on the curve at $(3, -3)$. Then follow the tangent line at that point until you hit the x-axis, and you will be at the point $x_1 = 2$. Repeating, we start at the point $(2, 0)$ and go down to the point on the curve which is $(2, -1)$. Then following the tangent line until it hits the x-axis again, we find ourselves at $x_2 = 1$. Repeating, we go down to the point on the curve at about $(1, -1/3)$, and then follow the tangent line until hitting the x-axis at $x_3 = 0$.

4.8.4 Starting at $x_0 = 3$ on the x-axis, travel with your eye downward until you hit the point on the curve at $(3, -5)$. Then follow the tangent line at that point until you hit the x-axis, and you will be at the point $x_1 = -3$. Repeating, we start at the point $(-3, 0)$ and go up to the point on the curve which is $(-3, 3)$. Then following the tangent line until it hits the x-axis again, we find ourselves at $x_2 = 2$. Repeating, we go down to the point on the curve at about $(2, -3.9)$, and then follow the tangent line until hitting the x-axis at $x_3 = -1$.

4.8.5 It is hard to find these exactly by hand, but you should find (approximately) $x_1 = 0.75$.



4.8.6

a. The line tangent to the graph of f at $x = 2$ does not intersect the x-axis because it is horizontal.

b. The expression $x_0 - \dfrac{f(x_0)}{f'(x_0)}$ is undefined because $f'(x_0) = 0$.

4.8.7 Generally, if two successive Newton approximations agree in their first p digits, then those approximations have p digits of accuracy. The method is terminated when the desired accuracy is reached.

4.8.8 Because $f'(x_n) = 2x_n$, we have

$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n} = \frac{2x_n^2 - x_n^2 + 5}{2x_n} = \frac{x_n^2 + 5}{2x_n}.$$

4.8.9 Because $f'(x_n) = 2x_n$, we have

$$x_{n+1} = x_n - \frac{x_n^2 - 6}{2x_n} = \frac{2x_n^2 - x_n^2 + 6}{2x_n} = \frac{x_n^2 + 6}{2x_n}.$$

$x_1 = 2.5$ and $x_2 = 2.45$.

4.8.10 Because $f'(x_n) = 2x_n - 2$, we have

$$x_{n+1} = x_n - \frac{x_n^2 - 2x_n - 3}{2x_n - 2} = \frac{2x_n^2 - 2x_n - (x_n^2 - 2x_n - 3)}{2x_n - 2} = \frac{x_n^2 + 3}{2x_n - 2}.$$

$x_1 = 3.5$ and $x_2 = 3.05$.

4.8.11 Because $f'(x_n) = -e^{-x_n} - 1$, we have

$$x_{n+1} = x_n - \frac{e^{-x_n} - x_n}{-e^{-x_n} - 1} = \frac{(-e^{-x_n} - 1)x_n - (e^{-x_n} - x_n)}{-e^{-x_n} - 1} = \frac{-e^{-x_n}x_n - e^{-x_n}}{-e^{-x_n} - 1} = \frac{x_n + 1}{e^{x_n} + 1}.$$

$x_1 = 0.564382$ and $x_2 = 0.567142$.

4.8.12 Because $f'(x_n) = 3x_n^2$, we have

$$x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2} = \frac{3x_n^3 - (x_n^3 - 2)}{3x_n^2} = \frac{2x_n^3 + 2}{3x_n^2}.$$

$x_1 = 1.5$ and $x_2 = 1.2963$.

4.8.13 Because $f'(x_n) = 2x_n$, we have

$$x_{n+1} = x_n - \frac{x_n^2 - 10}{2x_n} = \frac{2x_n^2 - (x_n^2 - 10)}{2x_n} = \frac{x_n^2 + 10}{2x_n}.$$

n	x_n
0	3.0
1	3.16667
2	3.16228
3	3.16228

4.8.14 Because $f'(x_n) = 3x_n^2 + 2x_n$, we have

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^3 + x_n^2 + 1}{3x_n^2 + 2x_n} \\ &= \frac{x_n(3x_n^2 + 2x_n) - (x_n^3 + x_n^2 + 1)}{3x_n^2 + 2x_n} \\ &= \frac{2x_n^3 + x_n^2 - 1}{3x_n^2 + 2x_n}. \end{aligned}$$

n	x_n
0	-1.5
1	-1.46667
2	-1.46557
3	-1.46557

4.8.15 Because $f'(x_n) = \cos(x_n) + 1$, we have

$$x_{n+1} = x_n - \frac{\sin x_n + x_n - 1}{\cos x_n + 1}.$$

n	x_n
0	0.5
1	0.51096
2	0.51097
3	0.51097

4.8.16 Because $f'(x_n) = e^{x_n} + 1$, we have

$$x_{n+1} = x_n - \frac{e^{x_n} + x - 5}{e^{x_n} + 1}.$$

n	x_n
0	1.6
1	1.33912
2	1.30697
3	1.30656
4	1.30656

4.8.17 Because $f'(x_n) = \sec^2(x_n) - 2$, we have

$$x_{n+1} = x_n - \frac{\tan x_n - 2x_n}{\sec^2(x_n) - 2}.$$

n	x_n
0	1.2
1	1.16934
2	1.16561
3	1.16556
4	1.16556

4.8.18 Because $f'(x_n) = \ln(x_n+1) + \frac{x_n}{x_n+1}$, we have

$$x_{n+1} = x_n - \frac{x_n \ln(x_n+1) - 1}{\ln(x_n+1) + \frac{x_n}{x_n+1}}$$

n	x_n
0	1.7
1	1.27574
2	1.24027
3	1.23998
4	1.23998

4.8.19 Because $f'(x_n) = \frac{-1}{\sqrt{1-x_n^2}} - 1$, we have

$$x_{n+1} = x_n - \frac{\cos^{-1}(x_n) - x_n}{\frac{-1}{\sqrt{1-x_n^2}} - 1}$$

n	x_n
0	0.75
1	0.73915
2	0.73909
3	0.73909

4.8.20 A preliminary sketch of f indicates that there are two roots on $[0, 2\pi]$, near $x = 1.5.$ and $x = 5.5$ The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{\cos x_n - (x_n/7)}{-\sin x_n - (1/7)}.$$

We have:

n	x_n
0	5.5
1	5.63692
2	5.65202
3	5.65222
4	5.65222

n	x_n
0	1.5
1	1.37412
2	1.37333
3	1.37333

The roots are approximately 5.65222 and 1.37333.

4.8.21 A preliminary sketch of f indicates that there are two roots, near $x = -0.4.$ and $x = 1.3$ The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{\cos(2x_n) - x_n^2 + 2x_n}{-2\sin(2x_n) - 2x_n + 2}.$$

We have:

n	x_n
0	-0.4
1	-0.337825
2	-0.335412
3	-0.335408
4	-0.335408

n	x_n
0	1.3
1	1.33256
2	1.33306
3	1.33306

The roots are approximately -0.335408 and 1.33306.

4.8.22 A preliminary sketch of f indicates that there are two roots on $[0, 8]$, near $x = 6.1$ and $x = 6.9$ The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{x_n/6 - \sec x_n}{1/6 - \sec x_n \tan x_n}.$$

We have:

n	x_n
0	6.1
1	6.10098
2	6.10099
3	6.10099

n	x_n
0	6.9
1	6.79197
2	6.76435
3	6.7627
4	6.7627

The roots are approximately 6.10099 and 6.7627.

4.8.23 A preliminary sketch of f indicates that there is one root, near $x = 0.2$ The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{e^{-x_n} - (x_n + 4)/5}{-e^{-x_n} - 1/5}.$$

We have:

n	x_n
0	0.2
1	0.179122
2	0.179295
3	0.179295

The root is approximately 0.179295.

4.8.24 A preliminary sketch of f indicates that there are three roots, near $x = -1$. $x = -0.7$ and $x = 1.2$. In fact, -1 is a root because $f(-1) = (-1/5) - (-1/4) - (1/20) = 0$.
The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{x_n^5/5 - x_n^3/4 - 1/20}{x_n^4 - 3x_n^2/4}.$$

We have:

n	x_n
0	-0.7
1	-0.683234
2	-0.683551
3	-0.683551

n	x_n
0	1.2
1	1.18424
2	1.18355
3	1.18355

The roots are -1 and approximately -0.683551 and 1.18355.

4.8.25 A preliminary sketch of f indicates that there are two roots, near $x = .5$ and near $x = 3$. The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{\ln x_n - x_n^2 + 3x_n - 1}{(1/x_n) - 2x_n + 3}.$$

We have:

n	x_n
0	0.5
1	0.610787
2	0.620655
3	0.620723
4	0.620723

n	x_n
0	3
1	3.03698
2	3.03645
3	3.03645

The roots are approximately 0.620723 and 3.03645.

4.8.26 A preliminary sketch of f indicates that there are three roots; two near $x = 0$ and one near $x = 100$. To avoid division by 0, we use initial estimates of $x_0 = \pm 0.1$ and $x_0 = 100$.

The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{x_n^3 - 100x_n^2 + 1}{3x_n^2 - 200x_n}.$$

We have:

n	x_n
0	-0.1
1	-0.0999501
2	-0.0999501

n	x_n
0	0.1
1	0.10005
2	0.10005

n	x_n
0	100
1	99.9999
2	99.9999

The roots are approximately -0.0999501, 0.10005, and 99.9999.

4.8.27 A preliminary sketch of the two curves seems to indicate that they intersect once near $x = 2$.

Let $f(x) = \sin x - x/2$. Then $f'(x_n) = \cos x_n - 1/2$. The Newton's method formula becomes

$$x_{n+1} = x_n - \frac{\sin x_n - x_n/2}{\cos x_n - 1/2}.$$

If we use an initial estimate of $x_0 = 2$, we obtain $x_1 = 1.901$, $x_2 = 1.89551$, $x_3 = 1.89549$ and $x_4 = 1.89549$, so the point of intersection appears to be at approximately $x = 1.89549$.

4.8.28 A preliminary sketch of the two curves seems to indicate that they intersect once near $x = 2$.

Let $f(x) = e^x - x^3$. Then $f'(x_n) = e^{x_n} - 3x_n^2$. The Newton's method formula becomes

$$x_{n+1} = x_n - \frac{e^{x_n} - x_n^3}{e^{x_n} - 3x_n^2}.$$

If we use an initial estimate of $x_0 = 2$, we obtain $x_1 = 1.8675$, $x_2 = 1.85725$, $x_3 = 1.85718$, so the point of intersection appears to be at approximately $x = 1.857$.

4.8.29 A preliminary sketch of the two curves seems to indicate that they intersect three times, once between -2.5 and -2, once between 0 and $1/2$, and once between 1.5 and 2.

Let $f(x) = 4 - x^2 - 1/x$. Then $f'(x_n) = -2x_n + (1/x_n^2)$. The Newton's method formula becomes

$$x_{n+1} = x_n - \frac{4 - x_n^2 - (1/x_n)}{-2x_n + (1/x_n^2)}.$$

If we use an initial estimate of $x_0 = -2.25$, we obtain $x_1 = -2.11843$, $x_2 = -2.11491$, $x_3 = -2.11491$, so there appears to be a point of intersection near $x = -2.115$.

If we use an initial estimate of $x_0 = .25$, we obtain $x_1 = .254032$, $x_2 = .254102$, so there appears to be a point of intersection near $x = .254$.

If we use an initial estimate of $x_0 = 1.75$, we obtain $x_1 = 1.86535$, $x_2 = 1.86081$, so there appears to be another point of intersection near $x = 1.86$.

4.8.30 A preliminary sketch of the two curves seems to indicate that they intersect once, near $x = 1.5$.

Let $f(x) = x^3 - (x^2 + 1)$. Then $f'(x_n) = 3x_n^2 - 2x_n$. The Newton's method formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 - 1}{3x_n^2 - 2x_n}.$$

If we use an initial estimate of $x_0 = 1.5$, we obtain $x_1 = 1.46667$, $x_2 = 1.46557$, $x_3 = 1.46557$, so there appears to be a point of intersection near $x = 1.46557$.

4.8.31 A preliminary sketch of the two curves seems to indicate that they intersect twice, once just to the right of 0, and once between 2 and 2.5.

Let $f(x) = 4\sqrt{x} - (x^2 + 1)$. Then $f'(x_n) = 2/\sqrt{x_n} - 2x_n$. The Newton's method formula becomes

$$x_{n+1} = x_n - \frac{4\sqrt{x_n} - (x_n^2 + 1)}{2/\sqrt{x_n} - 2x_n}.$$

If we use an initial estimate of $x_0 = .1$, we obtain $x_1 = .0583788$, $x_2 = .0629053$, $x_3 = .0629971$, so there appears to be a point of intersection near $x = .06299$.

If we use an initial estimate of $x_0 = 2.25$, we obtain $x_1 = 2.23026$, $x_2 = 2.23012$, $x_3 = 2.23012$, so there appears to be a point of intersection near $x = 2.23012$.

4.8.32 A preliminary sketch of the two curves seems to indicate that they intersect twice, once just to the right of 0, and once between 1 and 1.5

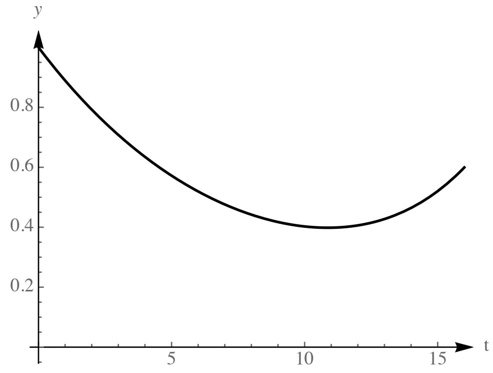
Let $f(x) = \ln x - (x^3 - 2)$. Then $f'(x_n) = 1/x_n - 3x_n^2$. The Newton's method formula becomes

$$x_{n+1} = x_n - \frac{\ln x_n - (x_n^3 - 2)}{1/x_n - 3x_n^2}.$$

If we use an initial estimate of $x_0 = .1$, we obtain $x_1 = .13045$, $x_2 = .13557$, $x_3 = .135674$, so there appears to be a point of intersection near $x = .13567$.

If we use an initial estimate of $x_0 = 1.4$, we obtain $x_1 = 1.32111$, $x_2 = 1.31501$, $x_3 = 1.31498$, so there appears to be a point of intersection near $x = 1.31498$.

4.8.33 The tumor decreases in size, then it starts growing again. Because the initial size of the tumor is 1, we need to find when $V(t) = \frac{1}{2}$. Using Newton's Method on the function $g(t) = V(t) - 1/2$, with an initial estimate of $t_0 = 6$, we find $t_1 = 6.40065$, $t_2 = 6.41774$, $t_3 = 6.41777$, $t_4 = 6.41777$. So the tumor reaches half its original size after about 6.4 days.



4.8.34 The derivative of V is $V'(t) = -0.120384e^{-0.1216t} + 0.00239e^{0.239t}$. Applying Newton's Method to this function to find where it is zero with an initial estimate of 8 yields

n	t_n
0	8
1	11.1212
2	10.8729
3	10.8691
4	10.8691

So the second treatment should be given after about 10.9 days.

4.8.35

a. We need $A = 1,000,000$, so $\dfrac{10,000((1+r)^{30}-1)}{r} = 1,000,000$. Multiplying both sides by r yields $10,000((1+r)^{30}-1) = 1,000,000r$, or

$$1,000,000r - 10,000(1+r)^{30} + 10,000 = 0.$$

b. Using Newton's Method with an initial estimate of 0.05 gives

n	r_n
0	0.05
1	0.121455
2	0.0969022
3	0.0811171
4	0.0743737
5	0.073191
6	0.0731527

The interest rate required is about 7.3%.

4.8.36

a. Let $f(r) = 2500\left(1+\dfrac{r}{12}\right)^{60} - 3200$. Applying Newton's method with an initial estimate of $r_0 = 0.05$ yields

n	r_n
0	0.05
1	0.04947
2	0.04947

So the interest rate should be about 4.95%.

b. We need $2500\left(1+\dfrac{r}{12}\right)^{60} = 3200$, so $\left(1+\dfrac{r}{12}\right)^{60} = \dfrac{3200}{2500}$, so $1+\dfrac{r}{12} = \left(\dfrac{32}{25}\right)^{1/60}$, so $r = 12\left(\dfrac{32}{25}\right)^{1/60} - 12 \approx 0.04947$.

4.8.37

a. We are seeking the time t when first $y(t) = 2.5e^{-t}\cos 2t$ is zero. This occurs first for $t = \pi/4$.

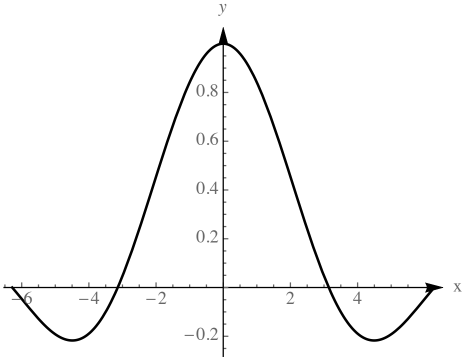
b. We are seeking the minimum value for y. We have $y'(t) = -2.5e^{-t}\cos 2t + 2.5e^{-t}(-2\sin 2t) = -2.5e^{-t}(\cos 2t + 2\sin 2t)$. This is zero when $\cos 2t = -2\sin 2t$, or $\tan 2t = \frac{-1}{2}$. Let $f(t) = \tan 2t + \frac{1}{2}$. If we apply Newton's method to $f(t)$ with a starting point of $t_0 = 1$, we obtain a root of 1.33897 after five iterations. An application of the First Derivative Test shows that there is a local minimum for y at this number. The displacement at this time is -0.586107. This local minimum is in fact an absolute minimum.

c. The second time that $y(t) = 2.5e^{-t}\cos 2t$ is zero is when $2t = \frac{3\pi}{2}$, or $t = 3\pi/4$.

d. Following our work in part b, we look for a root of $f(t) = \tan 2t + \frac{1}{2}$ that is bigger than 1.33897. From the graph, we are looking near $t = 3$. Applying Newton's method to $f(t)$ with an initial value of $x_0 = 3$ gives a root 2.90977 after three iterations. Applying the First Derivative Test, we see that there is a local maximum of 0.12181 at $x = 2.90977$.

4.8.38

a.



b. $f'(x) = \dfrac{x\cos x - \sin x}{x^2}$. This is zero when $x\cos x - \sin x = 0$, or $x - \tan x = 0$. Let $g(x) = x - \tan x$. If we apply Newton's method to g with a starting value of $x_0 = 4.5$, we obtain a root at 4.49341 after 2 iterations. The First Derivative Test confirms that there is a local minimum of about -0.217234 at $x = 4.49341$. Applying Newton's method to g with a starting value of 7.8 yields the root 7.72525 after 5 iterations. The First Derivative Test confirms that there is a local maximum of about .128375 at $x = 7.2525$.

4.8.39

Let $p(x) = x^4 - 7$. Because $p'(x_n) = 4x_n^3$, we have

$$x_{n+1} = x_n - \frac{x_n^4 - 7}{4x_n^3} = \frac{3}{4}x_n + \frac{7}{4}\cdot\frac{1}{x_n^3}.$$

We use an initial guess of $x_0 = 1.7$.

n	x_n
0	1.7
1	1.6312
2	1.6266
3	1.62658
4	1.62658

It appears that $\sqrt[4]{7} \approx 1.62658$.

4.8.40

Let $p(x) = x^3 - 2$. Because $p'(x_n) = 3x_n^2$, we have

$$x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2} = \frac{2}{3}x_n + \frac{2}{3}\cdot\frac{1}{x_n^2}.$$

We use an initial guess of $x_0 = 1.2$.

n	x_n
0	1.2
1	1.26296
2	1.25993
3	1.25992
4	1.25992

It appears that $\sqrt[3]{2} \approx 1.25992$.

4.8.41

Let $p(x) = x^3 + 9$. Because $p'(x_n) = 3x_n^2$, we have

$$x_{n+1} = x_n - \frac{x_n^3 + 9}{3x_n^2} = \frac{2}{3}x_n - \frac{3}{x_n^2}.$$

We use an initial guess of $x_0 = -2$.

n	x_n
0	−2.0
1	−2.08333
2	−2.08008
3	−2.08008

It appears that $\sqrt[3]{-9} \approx -2.08008$.

4.8.42 Let $p(x) = x^5 + 67$. Because $p'(x_n) = 5x_n^4$, we have

$$x_{n+1} = x_n - \frac{x_n^5 + 67}{5x_n^4} = \frac{4}{5}x_n - \frac{67}{5} \cdot \frac{1}{x_n^4}.$$

We use an initial guess of $x_0 = -2$.

n	x_n
0	-2.0
1	-2.4375
2	-2.3296
3	-2.31864
4	-2.31854
5	-2.31854

It appears that $\sqrt[5]{-67} \approx -2.31854$.

4.8.43

$$f'(x) = \frac{-x\sin x - \cos x}{x^2},$$

which is zero when $x\sin x + \cos x = 0$. Note that $f'(1) < 0$ and $f'(\pi) > 0$, so there must be a local minimum on the interval $(1, \pi)$. Let $g(x) = x\sin x + \cos x$. Then $g'(x_n) = \sin x_n + x_n\cos x_n - \sin x_n = x_n\cos x_n$, and the Newton's method formula becomes

$$x_{n+1} = x_n - \frac{x_n\sin x_n + \cos x_n}{x_n\cos x_n}.$$

If we use an initial estimate of $x_0 = 2.5$, we obtain $x_1 = 2.84702$, $x_2 = 2.79918$, $x_3 = 2.79839$, $x_4 = 2.79839$, so the smallest local minimum of f on $(0, \infty)$ occurs at approximately 2.79839.

4.8.44 $f'(x) = 12x^3 + 24x^2 + 24x + 48 = 12(x^3 + 2x^2 + 2x + 4)$. We are seeking values of x so that $x^3 + 2x^2 + 2x + 4 = 0$. Using Newton's method with an initial estimate of -1.5, we obtain $x_1 = -2.27273$, $x_2 = -2.04023$, $x_3 = -2.00104$, and $x_4 \approx -2$. We realize that $x = -2$ is a root of $f'(x)$. Using long division (by $x + 2$), we see that $f'(x) = 12(x + 2)(x^2 + 2)$, so $f'(x)$ has only the one root of -2.

Note that $f'(-3) < 0$ and $f'(0) > 0$, so there is a local (in fact, absolute) minimum at $x = -2$.

4.8.45 $f'(x) = 9x^4 - 30x^3 + 7x^2 + 60x$ and $f''(x) = 36x^3 - 90x^2 + 14x + 60 = 2(18x^3 - 45x^2 + 7x + 30)$. We are seeking roots of $f''(x)$. If we apply Newton's method to f'' we obtain the recursion

$$x_{n+1} = x_n - \frac{36x_n^3 - 90x_n^2 + 14x_n + 60}{108x_n^2 - 180x_n + 14}.$$

Starting with an initial estimate of $x_0 = 1$, we obtain $x_1 = 1.34483$, $x_2 = 1.45527$, $x_3 = 1.49284$, $x_4 = 1.49974$, and $x_5 \approx 1.5$. We check directly that 1.5 is a root of f'', so $2x - 3$ is a factor of f'', and using long division, we see that $f''(x) = 2(2x - 3)(9x^2 - 9x - 10) = 2(2x - 3)(3x + 2)(3x - 5)$. So the potential inflection points of f are located at $x = -2/3$, $x = 3/2$, and $x = 5/3$. A check of the sign of f'' on the various intervals confirms that these are all the locations of inflection points.

4.8.46 The domain of f is $(-\infty, 0) \cup (0, \infty)$.

$$f'(x) = \frac{xe^x - e^x}{x^2} = \frac{e^x(x - 1)}{x^2}.$$

This is 0 for $x = 1$. Note that $f'(1/2) < 0$ and $f'(2) > 0$, so there is a local minimum at $x = 1$. The point $(1, e)$ is the only extreme point. However, the questions calls for a solution via Newton's Method, so we attempt to solve the equation $\frac{xe^x - e^x}{x^2} = 0$ for x via Newton's Method. The recursion is given by

$$x_{n+1} = x_n - \frac{(x_n e^{x_n} - e^{x_n})/x_n^2}{e^{x_n}\left(x_n^2 - 2x_n + 2\right)/x_n^3}$$

. Starting with an initial estimate of $x_0 = 0.5$, we obtain $x_1 = 0.7$, $x_2 = 0.89266$, $x_3 = 0.98739$, $x_4 = 0.999839$, and $x_5 \approx 1$.

4.8.47

a. True.

b. False. The quadratic formula gives exact values.

c. False. It sometime fails depending on factors such as the shape of the curve and the closeness of the initial estimate.

4.8.48 A preliminary sketch suggests there are 3 intersection points, near $x = -0.5$, $x = 0.5$, and $x = 3$. Using Newton's Method on the function $g(x) = f(x) - x = x^3 - 3x^2 + x + 1 - x = x^3 - 3x^2 + 1$ gives.

n	x_n
0	-0.5
1	-0.53333
2	-0.53209
3	-0.53209

n	x_n
0	0.5
1	0.66666
2	0.65277
3	0.65270
4	0.65270

n	x_n
0	3.0
1	2.88888
2	2.87945
3	2.87939
4	2.87939

So the fixed points are approximately -0.53209, 0.65270, and 2.87939.

4.8.49 A preliminary sketch suggests there is 1 intersection point, near $x = 0.75$. Using Newton's Method on the function $g(x) = f(x) - x = \cos x - x$ gives.

n	x_n
0	0.75
1	0.73911
2	0.73908
3	0.73908

The fixed point is approximately 0.73908.

4.8.50 Let $g(x) = f(x) - x = \tan(x/2) - x$. Fixed points of f are roots of g. The Newton's method recursion for g is given by

$$x_{n+1} = x_n - \frac{\tan(x_n/2) - x_n}{(1/2)\sec^2(x_n/2) - 1}.$$

A preliminary sketch of g indicates that there are three roots, near $x = 0$ and $x = \pm 2.3$

n	x_n
0	-1.2
1	0.739493
2	-0.0887636
3	0.000116976
4	≈ 0

n	x_n
0	-2.3
1	-2.33281
2	-2.33113
3	-2.33112
4	-2.33112
5	-2.33112

n	x_n
0	2.3
1	2.33281
2	2.33113
3	2.33112
4	2.33112
5	2.33112

The fixed points are 0 and approximately ± 2.33112.

4.8.51 Let $g(x) = 2x\cos x - x$. Fixed points of f are roots of g. Clearly $x = 0$ is a root of g. The Newton's method recursion for g is given by

$$x_{n+1} = x_n - \frac{2x_n \cos x_n - x_n}{2\cos x_n - 2x\sin x_n - 1}.$$

A preliminary sketch of g indicates that there is only one nonzero root on $[0, 2]$, near $x = 1$. We have:

n	x_n
0	1
1	1.0503
2	1.04721
3	1.0472
4	1.0472

The fixed points are 0 and approximately 1.0472.

4.8.52

a. $f'(x) = \dfrac{1+x^2-x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$. Then the expression $x - \dfrac{f(x)}{f'(x)}$ can be written as

$$x - \frac{x}{1+x^2} \cdot \frac{(1+x^2)^2}{1-x^2} = x - \frac{x+x^3}{1-x^2} = \frac{2x^3}{x^2-1}.$$

Then note that starting with $x_0 = \dfrac{1}{\sqrt{3}}$, we have

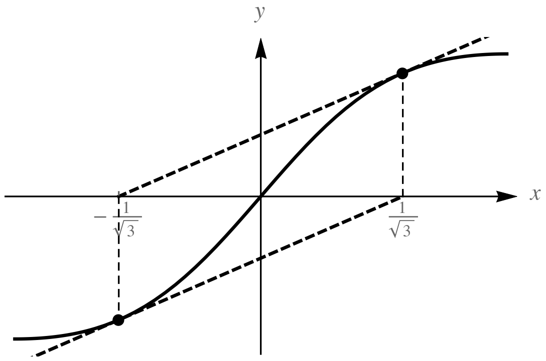
$$x_1 = \frac{\frac{2}{3\sqrt{3}}}{\frac{-2}{3}} = -\frac{1}{\sqrt{3}},$$

while

$$x_2 = \frac{\frac{-2}{3\sqrt{3}}}{\frac{-2}{3}} = \frac{1}{\sqrt{3}}.$$

b. By repetition, $x_3 = -\dfrac{1}{\sqrt{3}}$ and $x_4 = \dfrac{1}{\sqrt{3}}$, $x_5 = \dfrac{1}{\sqrt{3}}$, etc.

c.



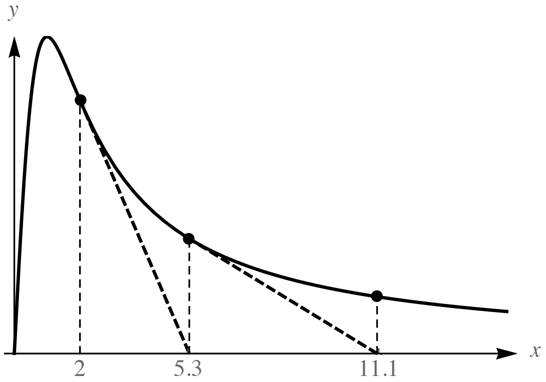
d. The sequence $x_0, x_1, x_2, \ldots$ doesn't converge at all because it just cycles between $\dfrac{1}{\sqrt{3}}$ and $-\dfrac{1}{\sqrt{3}}$.

4.8.53

a.

n	x_n
0	2
1	5.33333
2	11.0553
3	22.2931

b.



c. The tangent lines intersect the x-axis farther and farther away from the root r.

4.8.54

a. If r is a root of $x^2 - a$, then $r^2 - a = 0$, so $r^2 = a$, and $|r| = \sqrt{a}$, so either $r = \sqrt{a}$ or $r = -\sqrt{a}$. If we also insist that $r > 0$, then $r = \sqrt{a}$.

b. The Newton's method recursion is

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{2x_n^2 - (x_n^2 - a)}{2x_n} = \frac{x_n^2 + a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

c. Because $3^2 = 9 < 13$ and $4^2 = 16 > 13$, a good starting value for $\sqrt{13}$ would be a number between 3 and 4 (but closer to 4), like 3.6.

Because $8^2 = 64 < 73$ and $9^2 = 81 > 73$, a good starting value for $\sqrt{73}$ would be a number between 8 and 9, like 8.5.

d. The first chart is for $\sqrt{13}$ and the second is for $\sqrt{73}$.

n	x_n
0	3.6
1	3.605555555556
2	3.60555127547
3	3.6055127546
4	3.6055127546
5	3.6055127546
6	3.6055127546
7	3.6055127546
8	3.6055127546
9	3.6055127546
10	3.6055127546

n	x_n
0	8.5
1	8.54411764706
2	8.54400374608
3	8.54400374532
4	8.54400374532
5	8.54400374532
6	8.54400374532
7	8.54400374532
8	8.54400374532
9	8.54400374532
10	8.54400374532

4.8.55

a. The Newton's method formula would be:

$$x_{n+1} = x_n - \frac{1/x_n - a}{-1/x_n^2} = x_n + x_n - ax_n^2 = 2x_n - ax_n^2 = (2 - ax_n)x_n.$$

b. The approximation to $1/7$ is 0.14285714.

n	x_n
0	0.1
1	0.13
2	0.1417
3	0.14284777
4	0.14285714
5	0.14285714

4.8.56

a. Note that $f(0) = 0$, and $f'(x) = 2e^{2\sin x}\cos x - 2$, so $f'(0) = 2e^0 - 2 = 0$. Also, $f''(x) = 4e^{2\sin(x)}\cos^2(x) - 2e^{2\sin(x)}\sin(x)$, so $f''(0) = 4 \neq 0$. Thus, 0 is a root of multiplicity 2 for f.

b. The first chart is for the traditional Newton's method, and the second is for the modified version. Clearly, x_3 for the modified method is much closer to the actual root.

n	x_n
0	0.1
1	0.0511487
2	0.0258883
3	0.0130263

n	x_n
0	0.1
1	0.00229741
2	0.00000131725
3	-3.07429×10^{-11}

c. In Example 4, the value of x_3 was 0.0171665. For the modified version, we obtain:

n	x_n
0	0.15
1	-0.010125
2	0.00000311391
3	-9.05817×10^{-17}

Clearly the modified version converges much more quickly to the root of multiplicity two at 0.

4.8.57 Let $f(\lambda) = \tan(\pi\lambda) - \lambda$. We are looking for the first three positive roots of f. A preliminary sketch indicates that they are located near 1.4, 2.4, and 3.4. The Newton's method recursion is given by

$$x_{n+1} = x_n - \frac{\tan(\pi x_n) - x_n}{\pi\sec^2(\pi x_n) - 1}.$$

We obtain the following results:

n	x_n
0	1.4
1	1.34741
2	1.30555
3	1.29121
4	1.29012
5	1.29011
6	1.29011

n	x_n
0	2.4
1	2.37876
2	2.37331
3	2.37305
4	2.37305

n	x_n
0	3.4
1	3.4101
2	3.40919
3	3.40918
4	3.40918

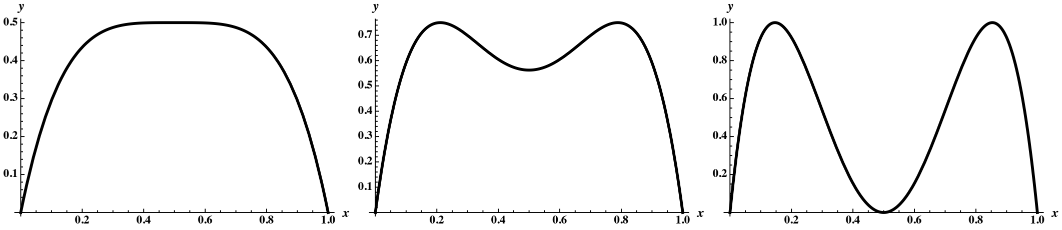
The first three positive eigenvalues are approximately 1.29011, 2.37305, and 3.40918.

4.8.58

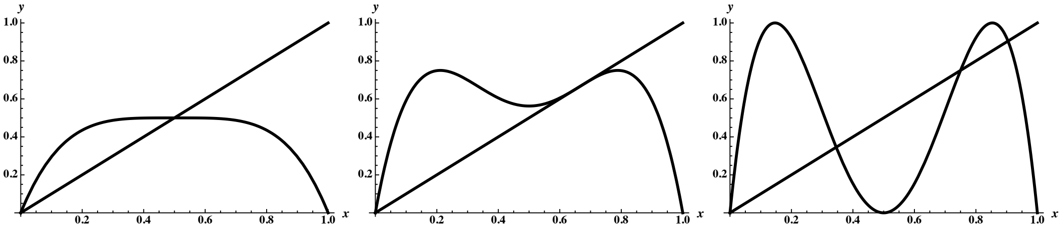
a. We are seeking solutions of $f(x) = ax(1-x) = x$. This can be written as $ax^2 + x(1-a) = 0$, or $x(ax + (1-a)) = 0$. The solutions of this equation are $x = 0$ and $x = \frac{a-1}{a}$. If $0 < a < 1$, this does not give a value of x in the range $(0,1)$. If $1 \le a \le 4$, we do get a fixed point $x = \frac{a-1}{a}$.

b. $g(x) = f(f(x)) = f(ax(1-x)) = a(ax(1-x))(1-ax(1-x)) = (a^2x - a^2x^2)(1 - ax + ax^2) = a^2x - a^2x^2 - a^3x^2 + a^3x^3 + a^3x^3 - a^3x^4 = a^2x - a^2x^2 - a^3x^2 + 2a^3x^3 - a^3x^4$. This is a fourth degree polynomial.

c. From left to right, with $a = 2$, then $a = 3$, then $a = 4$:



d. The graphs of $y = g(x)$ together with $y = x$ for $a = 2$, then $a = 3$, then $a = 4$.



When $a = 2$, we have $g(x) = -8x^4 + 16x^3 - 12x^2 + 4x$, so we are looking for a root of $g(x) - x = -8x^4 + 16x^3 - 12x^2 + 4x - x = -8x^4 + 16x^3 - 12x^2 + 3x = x(-8x^3 + 16x^2 - 12x + 3)$. Clearly $x = 0$ is one root, and the diagram indicates that $g(x) = x$ near $x = 0.5$. A quick check shows that $x = 0.5$ is a root of $g(x) - x$, so 0.5 is a fixed point of g.

When $a = 3$, we have $g(x) = -27x^4 + 54x^3 - 36x^2 + 9x$, so we are looking for a root of $g(x) - x = -27x^4 + 54x^3 - 36x^2 + 9x - x = -27x^4 + 54x^3 - 36x^2 + 8x = x(-27x^3 + 54x^2 - 36x + 8)$. Clearly $x = 0$ is one root, and the diagram indicates that $g(x) = x$ near $x = 0.6$. Applying Newton's method to $g(x) - x$ with an initial estimate of 0.6 yields a root of approximately $\overline{0.6} = 2/3$. A quick check shows that $2/3$ is a fixed point of g.

When $a = 4$, we have $g(x) = -64x^4 + 128x^3 - 80x^2 + 16x$, so we are looking for a root of $g(x) - x = -64x^4 + 128x^3 - 80x^2 + 16x - x = -64x^4 + 128x^3 - 80x^2 + 15x = x(-64x^3 + 128x^2 - 80x + 15)$. Clearly $x = 0$ is one root, and the diagram indicates that $g(x) = x$ near $x = 0.3$ and $x = 0.75$ and $x = 0.9$. Checking the value of $0.75 = 3/4$, we confirm that $g(3/4) = 3/4$. Applying Newton's method to $g(x) - x$ with an initial estimate of 0.3 yields a root of approximately 0.345492, and applying it with an initial estimate of 0.9 yields a root of approximately 0.904508.

Thus the fixed points of g with $a = 4$ are 0, 0.345492, 0.75, and 0.904508.

4.8.59 This problem can be solved (approximately) by setting up a computer or calculator program to run Newton's method, and then experimenting with different starting values. If this is done, it can be seen that any initial estimate between -4 and the local maximum at approximately -1.53 converges to the root at -2. Initial values between approximately -1.52 and -1.486 converge to the root at 3, while starting values between -1.485 and -1.475 converge to the root at -2. From -1.474 to approximately 0.841, starting values lead to convergence to the root at -1, while from 0.842 to 0.846 they lead to convergence to the root at -2. From about 0.847 to 0.862 they lead to convergence to the root at 3, while from 0.863 to the local minimum at 1.528 they lead to convergence to the root at -2. From about 1.528 to 4, the convergence is to the root at 3. Thus the approximate basis of convergence for -2 is $[-4,-1.53]\cup[-1.485,-1.475]\cup[0.842,0.846]$. For -1 the approximate basis of convergence is $[-1.474, 0.841]$, and for 3, it is $[-1.52,-1.486]\cup[0.847,0.862]\cup[1.53,4]$.

4.8.60 The recursion for $f(x)$ is

$$x_{n+1} = x_n - \frac{(x_n-1)^2}{2x_n-2} = \frac{2x_n^2-2x_n-(x_n^2-2x_n+1)}{2x_n-2} = \frac{x_n^2-1}{2(x_n-1)} = \frac{x_n+1}{2}.$$

The recursion for $g(x)$ is

$$y_{n+1} = y_n - \frac{y_n^2-1}{2y_n} = \frac{2y_n^2-(y_n^2-1)}{2y_n} = \frac{y_n^2+1}{2y_n}.$$

The comparison below shows that Newton's method converges much faster for $g(x)=x^2-1$. This is because it is steeper near the root $x=1$ – the value of $g'(1)=2$, while $f'(1)=0$. The flatness of f near 1 causes slow convergence.

n	x_n
0	2
1	1.5
2	1.25
3	1.125
4	1.0625
5	1.03125
6	1.01563
7	1.00781
8	1.00391
9	1.00195
10	1.00098

n	y_n
0	2
1	1.25
2	1.025
3	1.0003
4	≈ 1

4.9 Antiderivatives

4.9.1 Derivative, antiderivative.

4.9.2 C, where C is any constant.

4.9.3 $x+C$, where C is any constant.

4.9.4 By Theorem 4.11, if two functions have the same derivative then they differ by a constant.

4.9.5 $\dfrac{x^{p+1}}{p+1}+C$, where C is any real number and $p\neq -1$.

4.9.6 $a\sin^{-1}x+C$, where C is any constant.

4.9.7 $\ln|x| + C$, where C is any constant.

4.9.8 $\int a\cos x\,dx = a\sin x + C$ and $\int a\sin x\,dx = -a\cos x + C$

4.9.9 Observe that $F(-1) = 4 + C = 4$, so $C = 0$.

4.9.10 First, find the general solution $F(t)$, which is the family of all antiderivatives of $f(t)$. Then use the initial condition to find the specific value of the constant in the formula for $F(t)$.

4.9.11 The antiderivatives of $5x^4$ are $x^5 + C$. Check: $\frac{d}{dx}(x^5 + C) = 5x^4$.

4.9.12 The antiderivatives of $11x^{10}$ are $x^{11} + C$. Check: $\frac{d}{dx}(x^{11} + C) = 11x^{10}$.

4.9.13 The antiderivatives of $2\sin x + 1$ are $-2\cos x + x + C$. Check:

$$\frac{d}{dx}(-2\cos x + x + C) = -2(-\sin x) + 1 = 2\sin x + 1.$$

4.9.14 The antiderivatives of $-4\cos x - x$ are $-4\sin x - \frac{x^2}{2} + C$. Check:

$$\frac{d}{dx}\left(-4\sin x - \frac{x^2}{2} + C\right) = -4\cos x - x.$$

4.9.15 The antiderivatives of $3\sec^2 x$ are $3\tan x + C$. Check: $\frac{d}{dx}(3\tan x + C) = 3\sec^2 x$.

4.9.16 The antiderivatives of $\csc^2 s$ are $-\cot s + C$. Check: $\frac{d}{ds}(-\cot s + C) = \csc^2 s$.

4.9.17 The antiderivatives of $-2/y^3 = -2y^{-3}$ are $y^{-2} + C$. Check: $\frac{d}{dy}(y^{-2} + C) = -2y^{-3}$.

4.9.18 The antiderivatives of $-6z^{-7}$ are $z^{-6} + C$. Check: $\frac{d}{dz}(z^{-6} + C) = -6z^{-7}$.

4.9.19 The antiderivatives of e^x are $e^x + C$. Check: $\frac{d}{dx}(e^x + C) = e^x$.

4.9.20 The antiderivatives of y^{-1} are $\ln|y| + C$. Check: $\frac{d}{dy}(\ln|y| + C) = \frac{1}{y}$.

4.9.21 The antiderivatives of $\frac{1}{s^2+1}$ are $\tan^{-1} s + C$. Check: $\frac{d}{ds}(\tan^{-1}(s) + C) = \frac{1}{s^2+1}$.

4.9.22 The antiderivatives of π are $\pi t + C$. Check: $\frac{d}{dt}(\pi t + C) = \pi$.

4.9.23 $\int(3x^5 - 5x^9)\,dx = 3\cdot\frac{x^6}{6} - 5\cdot\frac{x^{10}}{10} + C = \frac{1}{2}x^6 - \frac{1}{2}x^{10} + C$. Check: $\frac{d}{dx}(\frac{1}{2}x^6 - \frac{1}{2}x^{10} + C) = 3x^5 - 5x^9$.

4.9.24 $\int(3u^{-2} - 4u^2 + 1)\,du = 3\cdot\frac{u^{-1}}{-1} - 4\cdot\frac{u^3}{3} + u + C = -\frac{4}{3}u^3 + u - \frac{3}{u} + C$. Check: $\frac{d}{du}(-\frac{4}{3}u^3 + u - \frac{3}{u} + C) = -4u^2 + 1 + 3u^{-2}$.

4.9.25 $\int\left(4\sqrt{x} - \frac{4}{\sqrt{x}}\right)dx = \int(4x^{1/2} - 4x^{-1/2})\,dx = 4\cdot\frac{x^{3/2}}{3/2} - 4\cdot\frac{x^{1/2}}{1/2} + C = \frac{8}{3}x^{3/2} - 8x^{1/2} + C$. Check: $\frac{d}{dx}(\frac{8}{3}x^{3/2} - 8x^{1/2} + C) = 4\sqrt{x} - \frac{4}{\sqrt{x}}$.

4.9.26 $\int \left(\frac{5}{t^2} + 4t^2\right) dt = 5 \cdot \frac{t^{-1}}{-1} + 4 \cdot \frac{t^3}{3} + C = \frac{4}{3}t^3 - \frac{5}{t} + C$. Check: $\frac{d}{dt}(\frac{4}{3}t^3 - \frac{5}{t} + C) = 4t^2 + \frac{5}{t^2}$.

4.9.27 $\int (5s+3)^2\, ds = \int (25s^2 + 30s + 9)\, ds = 25\frac{s^3}{3} + 30\frac{s^2}{2} + 9s + C = \frac{25s^3}{3} + 15s^2 + 9s + C$. Check: $\frac{d}{ds}\left(\frac{25s^3}{3} + 15s^2 + 9s + C\right) = 25s^2 + 30s + 9 = (5s+3)^2$.

4.9.28 $\int 5m(12m^3 - 10m)\, dm = \int (60m^4 - 50m^2)\, dm = 60 \cdot \frac{m^5}{5} - 50 \cdot \frac{m^3}{3} + C = 12m^5 - \frac{50}{3}m^3 + C$. Check: $\frac{d}{dm}(12m^5 - \frac{50}{3}m^3 + C) = 5m(12m^3 - 10m)$.

4.9.29 $\int (3x^{1/3} + 4x^{-1/3} + 6)\, dx = 3 \cdot \frac{3}{4}x^{4/3} + 4 \cdot \frac{3}{2}x^{2/3} + 6x + C = \frac{9}{4}x^{4/3} + 6x^{2/3} + 6x + C$. Check: $\frac{d}{dx}\left(\frac{9}{4}x^{4/3} + 6x^{2/3} + 6x + C\right) = 3x^{1/3} + 4x^{-1/3} + 6$.

4.9.30 $\int 6\sqrt[3]{x}\, dx = \int 6x^{1/3}\, dx = 6 \cdot \frac{3}{4}x^{4/3} + C = \frac{9}{2}x^{4/3} + C$. Check $\frac{d}{dx}\left(\frac{9}{2}x^{4/3} + C\right) = 6\sqrt[3]{x}$.

4.9.31 $\int (3x+1)(4-x)\, dx = \int (12x - 3x^2 + 4 - x)\, dx = \int (-3x^2 + 11x + 4)\, dx = -x^3 + \frac{11}{2}x^2 + 4x + C$. Check: $\frac{d}{dx}\left(\frac{9}{2}x^{4/3} + C\right) = 6\sqrt[3]{x}$.

4.9.32 $\int (4z^{1/3} - z^{-1/3})\, dz = 3z^{4/3} - \frac{3}{2}z^{2/3} + C$. Check: $\frac{d}{dz}\left(3z^{4/3} - \frac{3}{2}z^{2/3} + C\right) = 4z^{1/3} - z^{-1/3}$.

4.9.33 $\int (3x^{-4} + 2 - 3x^{-2})\, dx = -x^{-3} + 2x + 3x^{-1} + C$. Check: $\frac{d}{dx}\left(-x^{-3} + 2x + 3x^{-1} + C\right) = 3x^{-4} + 2 - 3x^{-2}$.

4.9.34 $\int r^{2/5}\, dr = \frac{5}{7}r^{7/5} + C$. Check: $\frac{d}{dr}\left(\frac{5}{7}r^{7/5} + C\right) = r^{2/5}$.

4.9.35 $\int \frac{4x^4 - 6x^2}{x}\, dx = \int \left(\frac{4x^4}{x} - \frac{6x^2}{x}\right) dx = \int (4x^3 - 6x)\, dx = x^4 - 3x^2 + C$. Check: $\frac{d}{dx}\left(x^4 - 3x^2 + C\right) = 4x^3 - 6x$.

4.9.36 $\int \frac{12t^8 - t}{t^{3/2}}\, dt = \int \left(\frac{12t^8}{t^{3/2}} - \frac{t}{t^{3/2}}\right) dt = \int \left(12t^{13/2} - t^{-1/2}\right) dt = \frac{24}{15}t^{15/2} - 2t^{1/2} + C = \frac{8}{5}t^{15/2} - 2\sqrt{t} + C$. Check: $\frac{d}{dt}\left(\frac{8}{5}t^{15/2} - 2\sqrt{t} + C\right) = 12t^{13/2} - t^{-1/2} = \frac{12t^8 - t}{t^{3/2}}$.

4.9.37 $\int \frac{x^2 - 36}{x - 6}\, dx = \int \frac{(x-6)(x+6)}{x-6}\, dx = \int (x+6)\, dx = \frac{x^2}{2} + 6x + C$. Check: $\frac{d}{dx}\left(\frac{x^2}{2} + 6x + C\right) = x + 6 = \frac{x^2 - 36}{x - 6}$.

4.9.38 $\int \frac{y^3 - 9y^2 + 20y}{y - 4}\, dy = \int \frac{y(y-4)(y-5)}{y-4}\, dy = \int y(y-5)\, dy = \int (y^2 - 5y)\, dy = \frac{y^3}{3} - \frac{5y^2}{2} + C$. Check: $\frac{d}{dy}\left(\frac{y^3}{3} - \frac{5y^2}{2} + C\right) = y^2 - 5y = \frac{(y^2 - 5y)(y-4)}{y-4} = \frac{y^3 - 9y^2 + 20y}{y-4}$.

4.9.39 $\int (\csc^2\theta + 2\theta^2 - 3\theta)\,d\theta = -\cot\theta + \frac{2}{3}\theta^3 - \frac{3}{2}\theta^2 + C$. Check: $\frac{d}{d\theta}\left(-\cot\theta + \frac{2}{3}\theta^3 - \frac{3}{2}\theta^2 + C\right) = \csc^2\theta + 2\theta^2 - 3\theta$.

4.9.40 $\int (\csc^2\theta + 1)\,d\theta = -\cot\theta + \theta + C$. Check: $\frac{d}{d\theta}(-\cot\theta + \theta + C) = \csc^2\theta + 1$.

4.9.41 $\int \frac{2 + 3\cos y}{\sin^2 y}\,dy = \int \left(\frac{2}{\sin^2 y} + \frac{3\cos y}{\sin^2 y}\right)\,dy = \int \left(2\csc^2 y + 3\cot y\csc y\right)\,dy = -2\cot y - 3\csc y + C$. Check: $\frac{d}{dy}(-2\cot y - 3\csc y + C) = -2(-\csc^2 y) - 3(-\csc y\cot y) = \frac{2}{\sin^2 y} + \frac{3\cos y}{\sin^2 y} = \frac{2 + 3\cos y}{\sin^2 y}$.

4.9.42 $\int \sin t(4\csc t - \cot t)\,dt = \int (4 - \cos t)\,dt = 4t - \sin t + C$. Check: $\frac{d}{dt}(4t - \sin t + C) = 4 - \cos t = \sin t\left(\frac{4}{\sin t} - \frac{\cos t}{\sin t}\right) = \sin t(4\csc t - \cot t)$.

4.9.43 $\int (\sec^2 x - 1)\,dx = \tan x - x + C$. Check: $\frac{d}{dx}(\tan x - x + C) = \sec^2 x - 1$.

4.9.44 $\int \frac{\sec^3 v - \sec^2 v}{\sec v - 1}\,dv = \int \frac{\sec^2 v(\sec v - 1)}{\sec v - 1}\,dv = \int \sec^2 v\,dv = \tan v + C$. Check: $\frac{d}{dv}(\tan v + C) = \sec^2 v = \frac{\sec^2 v(\sec v - 1)}{\sec v - 1} = \frac{\sec^3 v - \sec^2 v}{\sec v - 1}$.

4.9.45 $\int (\sec^2\theta + \sec\theta\tan\theta)\,d\theta = \tan\theta + \sec\theta + C$. Check: $\frac{d}{d\theta}(\tan\theta + \sec\theta + C) = \sec^2\theta + \sec\theta\tan\theta$.

4.9.46 $\int \frac{\sin\theta - 1}{\cos^2\theta}\,d\theta = \int (\sec\theta\tan\theta - \sec^2\theta)\,d\theta = \sec\theta - \tan\theta + C$. Check: $\frac{d}{d\theta}(\sec\theta - \tan\theta + C) = \sec\theta\tan\theta - \sec^2\theta = \frac{\sin\theta - 1}{\cos^2\theta}$.

4.9.47 $\int (3t^2 + 2\csc^2 t)\,dt = t^3 - 2\cot t + C$. Check: $\frac{d}{dt}(t^3 - 2\cot t + C) = 3t^2 + 2\csc^2 t$.

4.9.48 $\int \csc x(\cot x - \csc x)\,dx = \int (\csc x\cot x - \csc^2 x)\,dx = -\csc x + \cot x + C$. Check: $\frac{d}{dx}(-\csc x + \cot x + C) = \csc x\cot x - \csc^2 x = \csc x(\cot x - \csc x)$.

4.9.49 $\int \sec\theta(\tan\theta + \sec\theta + \cos\theta)\,d\theta = \int (\sec\theta\tan\theta + \sec^2\theta + 1)\,d\theta = \sec\theta + \tan\theta + \theta + C$. Check: $\frac{d}{d\theta}(\sec\theta + \tan\theta + \theta + C) = \sec\theta\tan\theta + \sec^2\theta + 1 = \sec\theta(\tan\theta + \sec\theta + \cos\theta)$.

4.9.50 $\int \frac{\csc^3 x + 1}{\csc x}\,dx = \int \frac{(\csc^3 x + 1)}{\csc x}\cdot\frac{\sin x}{\sin x}\,dx = \int (\csc^2 x + \sin x)\,dx = -\cot x - \cos x + C$. Check: $\frac{d}{dx}(-\cot x - \cos x + C) = \csc^2 x + \sin x = (\csc^2 x + \sin x)\cdot\frac{\csc x}{\csc x} = \frac{\csc^3 x + 1}{\csc x}$.

4.9.51 $\int \frac{1}{2y}\,dy = \frac{1}{2}\int y^{-1}\,dy = \frac{1}{2}\ln|y| + C$. Check: $\frac{d}{dy}\left(\frac{1}{2}\ln|y| + C\right) = \frac{1}{2y}$.

4.9.52 $\int \frac{e^{2t} - 1}{e^t - 1}\,dt = \int \frac{(e^t - 1)(e^t + 1)}{e^t - 1}\,dt = \int (e^t + 1)\,dt = e^t + t + C$. Check: $\frac{d}{dt}(e^t + t + C) = e^t + 1 = \frac{(e^t - 1)(e^t + 1)}{e^t - 1} = \frac{e^{2t} - 1}{e^t - 1}$.

4.9.53 $\int \frac{6}{\sqrt{4-4x^2}}\,dx = \frac{6}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx = 3\sin^{-1}x + C$. Check: $\frac{d}{dx}(3\sin^{-1}x + C) = \frac{3}{\sqrt{1-x^2}} = \frac{6}{2\sqrt{1-x^2}} = \frac{6}{\sqrt{4-4x^2}}$.

4.9.54 $\int \frac{v^3+v+1}{1+v^2}\,dv = \int \left(\frac{v^3+v}{1+v^2} + \frac{1}{1+v^2}\right)dv = \int \left(\frac{v(v^2+1)}{1+v^2} + \frac{1}{1+v^2}\right)dv = \int \left(v + \frac{1}{1+v^2}\right)dv = \frac{v^2}{2} + \tan^{-1}v + C$. Check: $\frac{d}{dv}\left(\frac{v^2}{2} + \tan^{-1}v + C\right) = v + \frac{1}{1+v^2} = \left(\frac{v(v^2+1)}{1+v^2} + \frac{1}{1+v^2}\right) = \frac{v^3+v+1}{1+v^2}$.

4.9.55 $\int \frac{4}{x\sqrt{x^2-1}}\,dx = 4\sec^{-1}|x| + C$. Check: $\frac{d}{dx}(4\sec^{-1}|x| + C) = \frac{4}{x\sqrt{x^2-1}}$.

4.9.56 $\int \frac{2}{25z^2+25}\,dz = \frac{2}{25}\int \frac{1}{z^2+1}\,dz = \frac{2}{25}\tan^{-1}z + C$. Check: $\frac{d}{dz}\left(\frac{2}{25}\tan^{-1}z + C\right) = \frac{2}{25}\cdot\frac{1}{z^2+1} = \frac{2}{25z^2+25}$.

4.9.57 $\int \frac{1}{x\sqrt{36x^2-36}}\,dx = \frac{1}{6}\int \frac{1}{x\sqrt{x^2-1}}\,dx = \frac{1}{6}\sec^{-1}|x| + C$. Check: $\frac{d}{dx}\left(\frac{1}{6}\sec^{-1}|x| + C\right) = \frac{1}{6}\cdot\frac{1}{x\sqrt{x^2-1}} = \frac{1}{x\sqrt{36x^2-36}}$.

4.9.58 $\int (49-49x^2)^{-1/2}\,dx = \frac{1}{7}\int \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{7}\sin^{-1}x + C$. Check: $\frac{d}{dx}\left(\frac{1}{7}\sin^{-1}x + C\right) = \frac{1}{7\sqrt{1-x^2}}\,dx = \frac{1}{\sqrt{49-49x^2}} = (49-49x^2)^{-1/2}$.

4.9.59 $\int \frac{t+1}{t}\,dt = \int \left(\frac{t}{t} + \frac{1}{t}\right)dt = \int \left(1 + \frac{1}{t}\right)dt = t + \ln|t| + C$. Check: $\frac{d}{dt}(t + \ln|t| + C) = 1 + \frac{1}{t} = \frac{t+1}{t}$.

4.9.60 $\int \frac{t^2-e^2t}{t+e^t}\,dt = \int \frac{(t-e^t)(t+e^t)}{t+e^t}\,dt = \int (t-e^t)\,dt = \frac{t^2}{2} - e^t + C$. Check: $\frac{d}{dt}\left(\frac{t^2}{2} - e^t + C\right) = t - e^t = \frac{(t-e^t)(t+e^t)}{t+e^t} = \frac{t^2-e^{2t}}{t+e^t}$.

4.9.61 $\int e^{x+2}\,dx = \int e^2e^x\,dx = e^2\int e^x\,dx = e^2e^x + C = e^{x+2} + C$. Check $\frac{d}{dx}(e^{x+2} + C) = e^{x+2}$.

4.9.62 $\int \frac{10t^5-3}{t}\,dt = \int \left(\frac{10t^5}{t} - \frac{3}{t}\right)dt = \int \left(10t^4 - \frac{3}{t}\right)dt = 2t^5 - 3\ln|t| + C$. Check: $\frac{d}{dt}(2t^5 - 3\ln|t| + C) = 10t^4 - \frac{3}{t} = \frac{10t^5-3}{t}$.

4.9.63 $\int \frac{e^{2w}-5e^w+4}{e^w-1}\,dw = \int \frac{(e^w-1)(e^w-4)}{e^w-1}\,dw = \int (e^w-4)\,dw = e^w - 4w + C$. Check: $\frac{d}{dw}(e^w - 4w + C) = e^w - 4 = \frac{(e^w-4)(e^w-1)}{e^w-1} = \frac{e^{2w}-5e^w+4}{e^w-1}$.

4.9.64 $\int \left(\sqrt[3]{x^2} + \sqrt{x^3}\right)dx = \int (x^{2/3} + x^{3/2})\,dx = \frac{3}{5}x^{5/3} + \frac{2}{5}x^{5/2} + C$. Check: $\frac{d}{dx}\left(\frac{3}{5}x^{5/3} + \frac{2}{5}x^{5/2} + C\right) = x^{2/3} + x^{3/2} = \sqrt[3]{x^2} + \sqrt{x^3}$.

4.9.65 $\int \frac{1+\sqrt{x}}{x}\,dx = \int (x^{-1}+x^{-1/2})\,dx = \ln|x| + 2x^{1/2} + C = \ln|x| + 2\sqrt{x} + C$. Check: $\frac{d}{dx}(\ln|x| + 2\sqrt{x} + C) = \frac{1}{x} + \frac{1}{\sqrt{x}} = \frac{1+\sqrt{x}}{x}$.

4.9.66 $\int \frac{16\cos^2 w - 81\sin^2 w}{4\cos w - 9\sin w}\,dw = \int \frac{(4\cos w - 9\sin w)(4\cos w + 9\sin w)}{4\cos w - 9\sin w}\,dw = \int (4\cos w + 9\sin w)\,dw = 4\sin w - 9\cos w + C$. Check: $\frac{d}{dw}(4\sin w - 9\cos w + C) = 4\cos w + 9\sin w = \frac{(4\cos w + 9\sin w)(4\cos w - 9\sin w)}{4\cos w - 9\sin w} = \frac{16\cos^2 w - 81\sin^2 w}{4\cos w - 9\sin w}$.

4.9.67 $\int \sqrt{x}(2x^6 - 4\sqrt[3]{x})\,dx = \int (2x^{13/2} - 4x^{5/6})\,dx = 2\cdot\frac{2}{15}x^{15/2} - 4\cdot\frac{6}{11}x^{11/6} + C = \frac{4}{15}x^{15/2} - \frac{24}{11}x^{11/6} + C$. Check: $\frac{d}{dx}\left(\frac{4}{15}x^{15/2} - \frac{24}{11}x^{11/6} + C\right) = 2x^{13/2} - 4x^{5/6} = \sqrt{x}(2x^6 - 4\sqrt[3]{x})$.

4.9.68 $\int \frac{2+x^2}{1+x^2}\,dx = \int \frac{(1+x^2)+1}{1+x^2}\,dx = \int \left(1 + \frac{1}{1+x^2}\right)\,dx = x + \tan^{-1}x + C$. Check: $\frac{d}{dx}(x + \tan^{-1}x + C) = 1 + \frac{1}{x^2+1} = \frac{2+x^2}{1+x^2}$.

4.9.69 We have $F(x) = \int (x^5 - 2x^2 + 1)\,dx = \frac{x^6}{6} - \frac{2x^3}{3} + x + C$; substituting $F(0) = 1$ gives $C = 1$, and thus $F(x) = \frac{x^6}{6} - \frac{2x^3}{3} + x + 1$.

4.9.70 $F(x) = \int (4\sqrt{x} + 6)\,dx = \frac{8}{3}x^{3/2} + 6x + C$; substituting $F(1) = 8$ gives $\frac{8}{3} + 6 + C = 8$, so $C = -\frac{2}{3}$, and thus $F(x) = \frac{8}{2}x^{3/2} + 6x - \frac{2}{3}$.

4.9.71 $F(x) = \int (8x^3 + \sin x)\,dx = 2x^4 - \cos x + C$; substituting $F(0) = 2$ gives $-1 + C = 2$, so $C = 3$, and thus $F(x) = 2x^4 - \cos x + 3$.

4.9.72 We have $F(t) = \int \sec^2 t\,dt = \tan t + C$; substituting $F(\pi/4) = 1$ gives $\tan\frac{\pi}{4} + C = 1 + C = 1$, so $C = 0$, and thus $F(t) = \tan t$, $-\pi/2 < t < \pi/2$.

4.9.73 We have $F(v) = \int \sec v \tan v\,dv = \sec v + C$; substituting $F(0) = 2$ gives $\sec 0 + C = 1 + C = 2$, so $C = 1$, and thus $F(v) = \sec v + 1$, $-\pi/2 < t < \pi/2$.

4.9.74 We have $F(u) = \int (2e^u + 3)\,du = 2e^u + 3u + C$; substituting $F(0) = 8$ gives $2 + 0 + C = 8$, so $C = 6$, and thus $F(u) = 2e^u + 3u + 6$.

4.9.75 We have $F(y) = \int \frac{3y^3+5}{y}\,dy = \int \left(\frac{3y^3}{y} + \frac{5}{y}\right)\,dy = \int \left(3y^2 + \frac{5}{y}\right)\,dy = y^3 + 5\ln|y| + C$; substituting $F(1) = 3$ gives $1 + 0 + C = 3$, so $C = 2$, and thus $F(y) = y^3 + 5\ln y + 2$, $y > 0$.

4.9.76 $F(\theta) = \int (2\sin\theta - 4\cos\theta)\,d\theta = -2\cos\theta - 4\sin\theta + C$; substituting $F(\pi/4) = 2$ gives $-\sqrt{2} - 2\sqrt{2} + C = 2$, so $C = 2 + 3\sqrt{2}$, and thus $F(\theta) = -2\cos\theta - 4\sin\theta + 2 + 3\sqrt{2}$.

4.9.77 We have $f(x) = \int (2x - 3)\,dx = x^2 - 3x + C$; substituting $f(0) = 4$ gives $C = 4$, so $f(x) = x^2 - 3x + 4$.

4.9.78 We have $g(x) = \int (7x^6 - 4x^3 + 12)\,dx = x^7 - x^4 + 12x + C$; substituting $g(1) = 24$ gives $1 - 1 + 12 + C = 24$, so $C = 12$, and thus $g(x) = x^7 - x^4 + 12x + 12$.

4.9.79 We have $g(x) = \int 7x\left(x^6 - \frac{1}{7}\right)\,dx = \int (7x^7 - x)\,dx = \frac{7}{8}x^8 - \frac{x^2}{2} + C$; substituting $g(1) = 2$ gives $\frac{7}{8} - \frac{1}{2} + C = 2$, so $C = \frac{13}{8}$, and thus $g(x) = \frac{7}{8}x^8 - \frac{x^2}{2} + \frac{13}{8}$.

4.9.80 $h(t) = \int (1 + 6\sin t)\,dt = t - 6\cos t + C$; substituting $h(\pi/3) = -3$ gives $\frac{\pi}{3} - 3 + C = -3$, so $C = -\frac{\pi}{3}$, and thus $h(t) = t - 6\cos t - \frac{\pi}{3}$.

4.9.81 $f(u) = \int 4(\cos u - \sin u)\,du = 4\sin u + 4\cos u + C$; substituting $f(\pi/2) = 0$ gives $4 + 0 + C = 0$, so $C = -4$, and thus $h(u) = 4\sin u + 4\cos u - 4$.

4.9.82 $p(t) = \int (10e^t + 70)\,dt = 10e^t + 70t + C$; substituting $p(0) = 100$ gives $10 + C = 100$, so $C = 90$, and thus $p(t) = 10e^t + 70t + 90$.

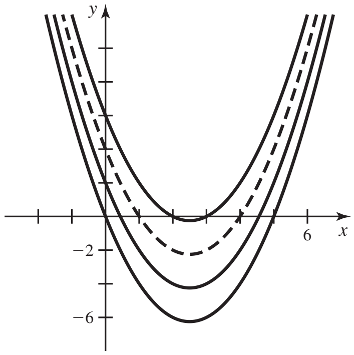
4.9.83 $y(t) = \int \left(\frac{3}{t} + 6\right)\,dt = 3\ln|t| + 6t + C$; substituting $y(1) = 8$ gives $0 + 6 + C = 8$, so $C = 2$, and thus $y(t) = 3\ln t + 6t + 2$, $t > 0$.

4.9.84 $u(x) = \int \frac{xe^{2x} + 4e^x}{xe^x}\,dx = \int \left(e^x + \frac{4}{x}\right)\,dx = e^x + 4\ln|x| + C$; substituting $u(1) = 0$ gives $e + C = 0$, so $C = -e$, and thus $u(x) = e^x + 4\ln x - e$, $x > 0$.

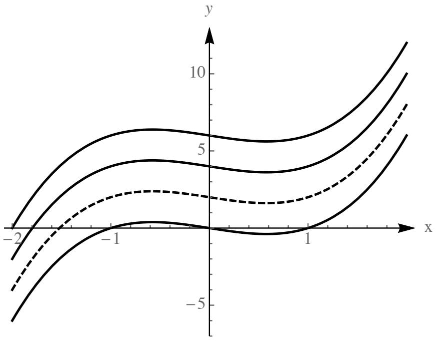
4.9.85 $y(\theta) = \int \frac{\sqrt{2}\cos^3\theta + 1}{\cos^2\theta}\,d\theta = \int \left(\frac{\sqrt{2}\cos^3\theta}{\cos^2\theta} + \frac{1}{\cos^2\theta}\right)\,d\theta = \int \left(\sqrt{2}\cos\theta + \sec^2\theta\right)\,d\theta = \sqrt{2}\sin\theta + \tan\theta + C$; substituting $y(\pi/4) = 3$ gives $1 + 1 + C = 3$, so $C = 1$, and thus $y(\theta) = \sqrt{2}\sin\theta + \tan\theta + 1$, $-\pi/2 < \theta < \pi/2$.

4.9.86 $v(x) = \int (4x^{1/3} + 2x^{-1/3})\,dx = 3x^{4/3} + 3x^{2/3} + C$, substituting $v(8) = 40$ gives $48 + 12 + C = 40$, so $C = -20$ and thus $v(x) = 3x^{4/3} + 3x^{2/3} - 20$, $x > 0$.

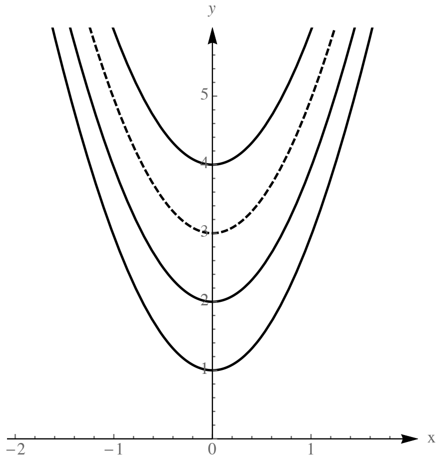
4.9.87 We have $f(x) = \int (2x - 5)\,dx = x^2 - 5x + C$; substituting $f(0) = 4$ gives $C = 4$, so $f(x) = x^2 - 5x + 4$.



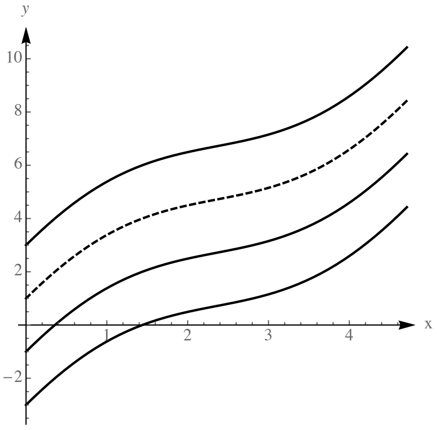
4.9.88 We have $f(x) = \int (3x^2 - 1)\,dx = x^3 - x + C$; substituting $f(1) = 2$ gives $C = 2$, so $f(x) = x^3 - x + 2$.



4.9.89 We have $f(x) = \int (3x + \sin x)\,dx = \frac{3}{2}x^2 - \cos x + C$; substituting $f(0) = 3$ gives $C = 4$, so $f(x) = \frac{3}{2}x^2 - \cos x + 4$.



4.9.90 We have $f(x) = \int (\cos x - \sin x + 2)\,dx = \sin x + \cos x + 2x + C$; substituting $f(0) = 1$ gives $C = 0$, so $f(x) = \sin x + \cos x + 2x$.



4.9.91 We have $s(t) = \int (2t + 4)\,dt = t^2 + 4t + C$; substituting $s(0) = 0$ gives $C = 0$, so $s(t) = t^2 + 4t$.

4.9.92 We have $s(t) = \int (e^t + 4)\,dt = e^t + 4t + C$; substituting $s(0) = 2$ gives $1 + C = 2$, so $C = 1$, and thus $s(t) = e^t + 4t + 1$

4.9.93 We have $s(t) = \int 2\sqrt{t}\,dt = 2 \cdot \frac{2}{3}t^{3/2} + C = \frac{4}{3}t^{3/2} + C$; substituting $s(0) = 1$ gives $C = 1$, so $s(t) = \frac{4}{3}t^{3/2} + 1$.

4.9.94 We have $s(t) = \int 2\cos t\,dt = 2\sin t + C$; substituting $s(0) = 0$ gives $C = 0$, so $s(t) = 2\sin t$.

4.9.95 We have $s(t) = \int (6t^2 + 4t - 10)\,dt = 2t^3 + 2t^2 - 10t + C$; substituting $s(0) = 0$ gives $C = 0$, so $s(t) = 2t^3 + 2t^2 - 10t$.

4.9.96 We have $s(t) = \int (4t + \sin t)\,dt = 2t^2 - \cos t + C$, substituting $s(0) = 0$ gives $-1 + C = 0$, so $C = 1$ and $s(t) = 2t^2 - \cos t + 1$.

4.9.97 $v(t) = \int a(t)\,dt = \int -32\,dt = -32t + C_1$. Because $v(0) = 20$, we have $0 + C_1 = 20$, so $v(t) = -32t + 20$.

$s(t) = \int v(t)\,dt = \int (-32t + 20)\,dt = -16t^2 + 20t + C_2$. Because $s(0) = 0$, we have $C_2 = 0$, and thus $s(t) = -16t^2 + 20t$.

4.9.98 $v(t) = \int a(t)\,dt = \int 4\,dt = 4t + C_1$. Because $v(0) = -3$, we have $C_1 = -3$.

$s(t) = \int v(t)\,dt = \int (4t - 3)\,dt = 2t^2 - 3t + C_2$. Because $s(0) = 2$, we have $C_2 = 2$, and thus $s(t) = 2t^2 - 3t + 2$.

4.9.99 $v(t) = \int a(t)\,dt = \int 0.2t\,dt = 0.1t^2 + C_1$. Because $v(0) = 0$, we have $C_1 = 0$.

$s(t) = \int v(t)\,dt = \int 0.1t^2\,dt = \frac{1}{30}t^3 + C_2$. Because $s(0) = 1$, we have $C_2 = 1$, and thus $s(t) = \frac{1}{30}t^3 + 1$.

4.9.100 $v(t) = \int a(t)\,dt = \int 2\cos t\,dt = 2\sin t + C_1$. Because $v(0) = 1$, we have $C_1 = 1$.

$s(t) = \int v(t)\,dt = \int (2\sin t + 1)\,dt = -2\cos t + t + C_2$. Because $s(0) = 0$, we have $-2 + 0 + C_2 = 0$, so $C_2 = 2$. Thus, $s(t) = -2\cos t + t + 2$.

4.9.101 $v(t) = \int a(t)\,dt = \int (2 + 3\sin t)\,dt = 2t - 3\cos t + C_1$. Because $v(0) = 1$, we have $0 - 3 + C_1 = 1$, so $C_1 = 4$.

$s(t) = \int v(t)\,dt = \int (2t - 3\cos t + 4)\,dt = t^2 - 3\sin t + 4t + C_2$. Because $s(0) = 10$, we have $C_2 = 10$, so $s(t) = t^2 + 4t - 3\sin t + 10$.

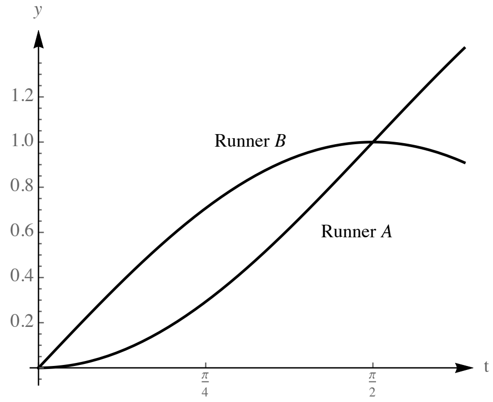
4.9.102 $v(t) = \int a(t)\,dt = \int (2e^t - 12)\,dt = 2e^t - 12t + C_1$. Because $v(0) = 1$, we have $2 + C_1 = 1$, so $C_1 = -1$.

$s(t) = \int v(t)\,dt = \int (2e^t - 12t - 1)\,dt = 2e^t - 6t^2 - t + C_2$. Because $s(0) = 0$, we have $C_2 = -2$, so $s(t) = 2e^t - 6t^2 - t - 2$.

4.9.103 $a(t) = 16$, so $v(t) = \int 16\,dt = 16t + C_1$. Because $v(0) = 0$, we have $C_1 = 0$, so $v(t) = 16t$. $s(t) = \int v(t)\,dt = \int 16t\,dt = 8t^2 + C_2$, where we can assume $s(0) = 0$ so $C_2 = 0$. Then the distance traveled in the first 5 seconds is $s(5) = 200$ feet.

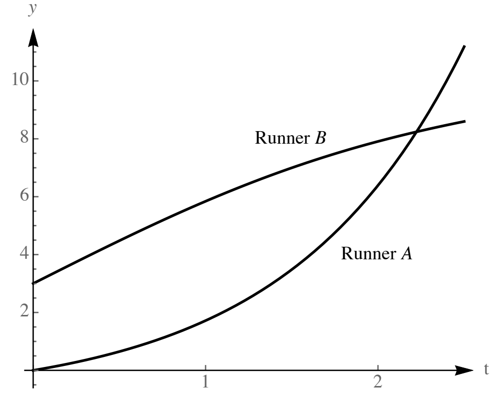
4.9.104 $a(t) = -10$ so $v(t) = \int -10\,dt = -10t + C_1$. Because $v(0) = 88$, we have $C_1 = 88$, so $v(t) = -10t + 88$. Then $s(t) = \int(-10t + 88)\,dt = -5t^2 + 88t + C_2$, where we can take $C_2 = 0$. The car travels $s(3) = -45 + 264 = 219$ ft in the 3 seconds.

4.9.105 Runner A has position function $s(y) = \int \sin t\,dt = -\cos t + C$; the initial condition $s(0) = 0$ gives $C = 1$, so $s(t) = 1 - \cos t$. Runner B has position function $S(t) = \int \cos t\,dt = \sin t + C$; the initial condition $S(0) = 0$ gives $C = 0$, so $S(t) = \sin t$. The smallest $t > 0$ where $s(t) = S(t)$ is $t = \pi/2$.



4.9.106 Runner A has position function $s(t) = \int e^t\,dt = e^t + C_1$, and because $s(0) = 0$ we have $s(t) = e^t - 1$.

Runner B has position function $S(t) = \int(2 + \cos t)\,dt = 2t + \sin t + C_2$, and because $S(0) = 3$ we have $C_2 = 3$, so $S(t) = 2t + \sin t + 3$. The smallest $t > 0$ where $s(t) = S(t)$ is $t \approx 2.22$.



4.9.107

a. We have $v(t) = -9.8t + v_0$ and $v_0 = 30$, so $v(t) = -9.8t + 30$.

b. The height of the softball above ground is given by $s(t) = \int(-9.8t + 30)\,dt = -4.9t^2 + 30t + s_0 = -4.9t^2 + 30t$.

c. The ball reaches its maximum height when $v(t) = -9.8t + 30 = 0$, which gives $t = 30/9.8 \approx 3.06$ s; the maximum height is $s(30/9.8) \approx 45.92$ m.

d. The ball strikes the ground when $s(t) = 0$ (and $t > 0$), which gives $t(30 - 4.9t) = 0$, so $t = 30/4.9 \approx 6.12$ s.

4.9.108

a. We have $v(t) = -9.8t + v_0$ and $v_0 = 30$, so $v(t) = -9.8t + 30$.

b. The height of the stone above ground is given by $s(t) = \displaystyle\int (-9.8t + 30)\,dt = -4.9t^2 + 30t + s_0 = -4.9t^2 + 30t + 200$.

c. The stone reaches its maximum height when $v(t) = -9.8t + 30 = 0$, which gives $t = 30/9.8 \approx 3.06$ s; the maximum height is $s(30/9.8) \approx 245.92$ m.

d. The stone strikes the ground when $s(t) = 0$ (and $t > 0$), which gives $-4.9t^2 + 30t + 200 = 0$, so $t \approx 10.15$ s.

4.9.109

a. We have $v(t) = -9.8t + v_0$ and $v_0 = 10$, so $v(t) = -9.8t + 10$.

b. The height of the payload above ground is given by $s(t) = \displaystyle\int (-9.8t + 10)\,dt = -4.9t^2 + 10t + s_0 = -4.9t^2 + 10t + 400$.

c. The payload reaches its maximum height when $v(t) = -9.8t + 10 = 0$, which gives $t = 10/9.8 \approx 1.02$ s; the maximum height is $s(10/9.8) \approx 405.10$ m.

d. The payload strikes the ground when $s(t) = 0$ (and $t > 0$), which gives $-4.9t^2 + 10t + 400 = 0$, so $t \approx 10.11$ s.

4.9.110

a. We have $v(t) = -9.8t + v_0$ and $v_0 = -10$, so $v(t) = -9.8t - 10$.

b. The height of the payload above ground is given by $s(t) = \displaystyle\int (-9.8t - 10)\,dt = -4.9t^2 - 10t + s_0 = -4.9t^2 - 10t + 400$.

c. Because $v(t) < 0$ for $t > 0$, the maximum height occurs at $t = 0$ and is the initial height 400 m.

d. The payload strikes the ground when $s(t) = 0$ (and $t > 0$), which gives $-4.9t^2 - 10t + 400 = 0$, so $t \approx 8.07$ s.

4.9.111

a. True, because $F'(x) = G'(x)$.

b. False; f is the derivative of F.

c. True; $\int f(x)\,dx$ is the most general antiderivative of $f(x)$, which is $F(x) + C$.

d. False; a function cannot have more than one derivative.

e. False; one can only conclude that $F(x)$ and $G(x)$ differ by a constant.

4.9.112 We have $F'(x) = \displaystyle\int 1\,dx = x + C$; $F'(0) = 3$ so $C = 3$. Then $F(x) = \displaystyle\int (x+3)\,dx = \frac{x^2}{2} + 3x + C$; $F(0) = 4$ so $C = 4$ and $F(x) = (1/2)x^2 + 3x + 4$.

4.9.113 We have $F'(x) = \displaystyle\int \cos x\,dx = \sin x + C$; $F'(0) = 3$ so $C = 3$. Then $F(x) = \displaystyle\int (\sin x + 3)\,dx = -\cos x + 3x + C$; $F(\pi) = 4$ gives $1 + 3\pi + C = 4$ so $C = 3 - 3\pi$ and $F(x) = -\cos x + 3x + 3 - 3\pi$.

4.9.114 We have $F''(x) = \int 4x\,dx = 2x^2 + C$; $F''(0) = 0$ so $C = 0$. Next $F'(x) = \int 2x^2\,dx = 2 \cdot \frac{x^3}{3} + C$; $F'(0) = 1$ so $C = 1$. Finally $F(x) = \int \left(\frac{2}{3}x^3 + 1\right) dx = \frac{2}{3} \cdot \frac{x^4}{4} + x + C$; $F(0) = 3$ so $C = 3$ and $F(x) = \frac{x^4}{6} + x + 3$.

4.9.115 We have $F''(x) = \int (672x^5 + 24x)\,dx = 672 \cdot \frac{x^6}{6} + 24 \cdot \frac{x^2}{2} + C$; $F''(0) = 0$ so $C = 0$. Next $F'(x) = \int (112x^6 + 12x^2)\,dx = 112 \cdot \frac{x^7}{7} + 12 \cdot \frac{x^3}{3} + C$; $F'(0) = 2$ so $C = 2$. Finally $F(x) = \int (16x^7 + 4x^3 + 2)\,dx = 16 \cdot \frac{x^8}{8} + 4 \cdot \frac{x^4}{4} + 2x + C$; $F(0) = 1$ so $C = 1$ and $F(x) = 2x^8 + x^4 + 2x + 1$.

4.9.116 The velocity is given by $v(t) = \int 2\sin t\,dt = -2\cos t + C$; $v(0) = 3$ implies $-2 + C = 3$, so $C = 5$. The position s is given by $s(t) = \int (-2\cos t + 5)\,dt = -2\sin t + 5t + C_1$; $s(0) = 0$ implies that $C_1 = 0$, so $s(t) = -2\sin t + 5t$.

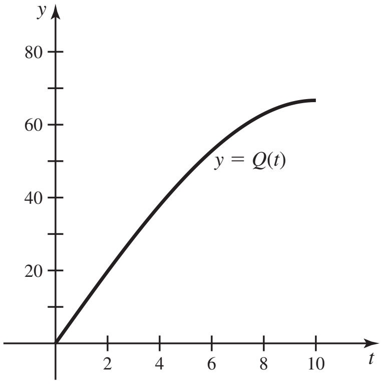
4.9.117

a. We have

$$Q(t) = \int 0.1(100 - t^2)\,dt = 0.1\left(100t - \frac{t^3}{3}\right) + C;$$

$Q(0) = 0$, so $C = 0$ and $Q(t) = 10t - t^3/30$ gal.

b.



c. $Q(10) = 200/3 \approx 67$ gal.

4.9.118 Object A has position function $s(t)$ given by $s(t) = \int 2at\,dt = at^2 + s_0$, and we are given $s_0 = 0$, so $s(t) = at^2$. Object B has position function $S(t)$ given by $S(t) = \int b\,dt = bt + S_0$, and we are given $S_0 = c > 0$, so $S(t) = bt + c$. Therefore A will overtake B when $s(t) = S(t)$, which gives the quadratic equation $at^2 - bt - c = 0$; this equation has a unique positive root given by $t = \frac{b + \sqrt{b^2 + 4ac}}{2a}$.

4.9.119 Check that $\frac{d}{dx}(2\sin\sqrt{x}) = 2\cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right) = \frac{\cos\sqrt{x}}{\sqrt{x}}$.

4.9.120 Check that $\frac{d}{dx}(\sqrt{x^2+1}) = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$.

4.9.121 Check that $\frac{d}{dx}\left(\frac{1}{3}\sin x^3\right) = \frac{1}{3}\cos x^3(3x^2) = x^2\cos x^3$.

4.9.122 Check that $\frac{d}{dx}\left(-\frac{1}{2(x^2-1)}\right) = -\frac{1}{2}\frac{d}{dx}(x^2-1)^{-1} = -\frac{1}{2}(-1)(x^2-1)^{-2}(2x) = \frac{x}{(x^2-1)^2}$.

Chapter Four Review

1

a. False. The point $(c, f(c))$ is a critical point for f, but is not necessarily a local maximum or minimum. Example: $f(x) = x^3$ at $c = 0$.

b. False. The fact that $f''(c) = 0$ does not necessarily imply that f changes concavity at c. Example: $f(x) = x^4$ at $c = 0$.

c. True. Both are antiderivatives of $2x$.

d. True. The function has a maximum on the closed interval determined by the two local minima, and the only way the maximum can occur at the endpoints is if the function is constant, in which case every point is a local max and min.

e. True. The slope of the linearization is given by $f'(0) = \cos(0) = 1$, and the line has y-intercept $(0, f(0)) = (0, 0)$.

f. False. For example, $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty} x = \infty$, but $\lim_{x\to\infty}(x^2 - x) = \infty$.

2

a. There is a local minimum at $(2, -3)$ and a local maximum at $(-1, 3)$.

b. The absolute minimum and maximum on $[-3, 3]$ occur at $(-3, -5)$ and $(-1, 3)$ respectively.

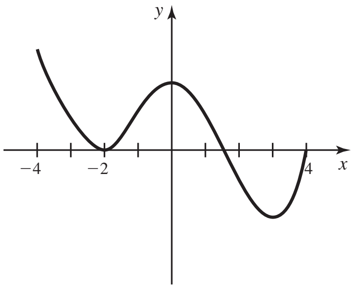
c. The inflection point has coordinates $(1/2, 0)$.

d. The function has zeros at $x \approx -2.2, 2.8$.

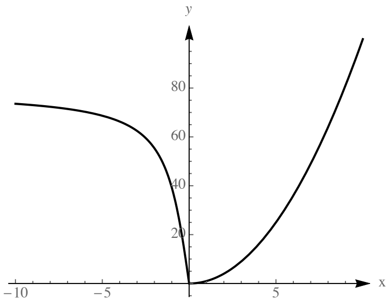
e. The function is concave up on the interval $(1/2, 3)$.

f. The function is concave down on the interval $(-3, 1/2)$.

3



4



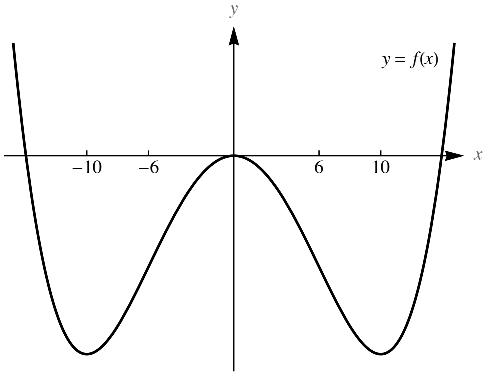
5

a. f' is zero for $x = 0$ and $x = \pm 10$, so those are the critical points. $f' > 0$ on $(-10, 0)$ and on $(10, \infty)$, so f is increasing on those intervals, while $f' < 0$ on $(-\infty, -10)$ and on $(0, 10)$, so f is decreasing on those intervals.

b. f'' is zero for $x = \pm 6$. $f'' > 0$ on $(-\infty, -6)$ and on $(6, \infty)$, so f is concave up on those intervals, while $f' < 0$ on $(-6, 6)$, so f is concave down on that interval.

c. There is a local minimum at $x = -10$ and $x = 10$, and a local maximum at $x = 0$.

d.



6 $f'(x) = 3x^2 - 12x = 3x(x-4)$, so f' is zero for $x = 0$ and $x = 4$. The critical points are $x = 0$ and $x = 4$. $f(0) = 0$ and $f(4) = -32$. Checking the endpoints, we have $f(-1) = -7$ and $f(5) = -25$. So the absolute maximum is 0 at $x = 0$ and the absolute minimum is -32 at $x = 4$.

7 $f'(x) = 12x^3 - 12x = 12x(x^2-1) = 12x(x-1)(x+1)$, which is zero for $x = 0, -1, 1$, so those are the critical points. $f(0) = 9$, $f(-1) = f(1) = 6$, and checking endpoints, we have $f(\pm 2) = 33$. So the absolute maximum is 33 at the endpoints and the absolute minimum is 6 at $x = \pm 1$.

8 $g'(x) = 4x^3 - 100x = 4x(x^2-25) = 4x(x-5)(x+5)$, which is zero for $x = 0, \pm 5$. However, -5 is not in the given domain, while 5 is an endpoint, so only $x = 0$ is a critical point. $g(0) = 0$, $g(5) = -625$, and $g(-1) = -49$. So the absolute maximum is 0 at $x = 0$ and the absolute minimum is -625 at $x = 5$.

9 $f'(x) = 6x^2 - 6x - 36 = 6(x-3)(x+2)$, which is zero for $x =$ and $x = -2$, so the critical points are $x = 3, -2$. Because $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, this function has no absolute max or min on $(-\infty, \infty)$.

10 $f'(x) = 3x^2 \ln x + x^3 \cdot \frac{1}{x} = x^2(3\ln x + 1)$. This is zero on the given domain when $3\ln x = -1$, or $x = e^{-1/3}$. Note that $f' < 0$ on $(0, e^{-1/3})$ and $f' > 0$ on $(e^{-1/3}, \infty)$ and $\lim_{x\to\infty} f(x) = \infty$. So there is no absolute maximum, but there is an absolute minimum of $f(e^{-1/3}) = -\frac{1}{3e}$ at $x = e^{-1/3}$.

11 $f'(x) = \frac{2x-2}{x^2-2x+2}$, which is zero for $x = 1$. The only critical point is $x = 1$. $f(0) = f(2) = \ln 2$, while $f(1) = 0$. The absolute maximum is $\ln 2$ at the endpoints, while the absolute minimum is 0 at $x = 1$.

12 $f'(x) = 2\cos 2x$, which is zero on the given interval for $x = \pm\pi/4$ and $x = \pm 3\pi/4$. We have $f(-\pi) = 3$, $f(-3\pi/4) = 4$, $f(-\pi/4) = 2$, $f(\pi/4) = 4$, $f(3\pi/4) = 2$ and $f(\pi) = 3$; therefore the absolute minimum and maximum values of f on $[-\pi, \pi]$ are 2 and 4 respectively.

13

$$\begin{aligned} g'(x) &= -\frac{1}{2}\cos x + \frac{1}{2}\cos x \cos x - \frac{1}{2}\sin x \sin x = \frac{1}{2}\left(-\cos x + \cos^2 x - (1 - \cos^2 x)\right) \\ &= \frac{1}{2}\left(2\cos^2 x - \cos x - 1\right) = \frac{1}{2}\left(2\cos x + 1\right)\left(\cos x - 1\right). \end{aligned}$$

This is zero on the interior of the given interval where $\cos x = -\frac{1}{2}$, which is $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$, so those are the critical points. We have $g(2\pi/3) = -\frac{3\sqrt{3}}{8}$ and $g(4\pi/3) = \frac{3\sqrt{3}}{8}$, while $g(0) = g(2\pi) = 0$. So the absolute minimum is $-\frac{3\sqrt{3}}{8}$ and the absolute maximum is $\frac{3\sqrt{3}}{8}$.

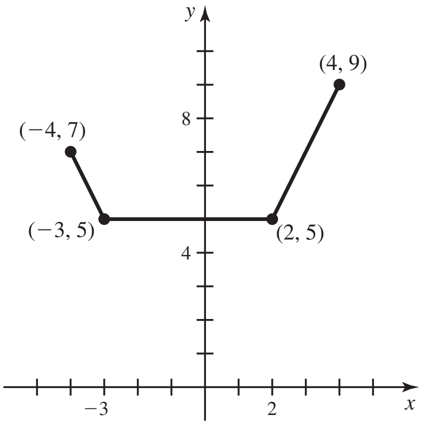
14 Observe that $f'(x) = 2x^{-1/2} - \frac{5}{2}x^{3/2}$, so the critical points satisfy $x^2 = 4/5$; hence $x = 2/\sqrt{5} \approx 0.894$ is the only critical point in the interval $(0, 4)$. We have $f(0) = 0$, $f(2/\sqrt{5}) \approx 3.026$, $f(4) = -24$; therefore the absolute minimum and maximum values are -24 and ≈ 3.026 respectively.

15 The critical points satisfy $f'(x) = 2\ln x + 2x \cdot \frac{1}{x} = 2\ln x + 2 = 0$, which has solution $x = 1/e$. The Second Derivative Test shows that this critical point is a local minimum, so by Theorem 4.5 the absolute minimum value on the interval $(0, \infty)$ is $f\left(\frac{1}{e}\right) = -\frac{2}{e} + 10$. Because $\lim_{x\to\infty} x\ln x = \infty$, this function does not have an absolute maximum on $(0, \infty)$.

16 The critical points satisfy $g'(x) = \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x = 0$. Note that $g'(0) = 0$, so $x = 0$ is a critical point, and this turns out to be the only critical point. This can be seen by noting that for $x < 0$ both $\frac{x}{\sqrt{1-x^2}}$ and $\sin^{-1} x$ are negative, so their sum is negative, and for $x > 0$ both $\frac{x}{\sqrt{1-x^2}}$ and $\sin^{-1} x$ are positive, so their sum is positive. Note that $g(-1) = \frac{\pi}{2} = g(1)$, and $g(0) = 0$, so the absolute maximum of g is $\frac{\pi}{2}$ and the absolute minimum is 0.

17

All points in the interval $[-3, 2]$ are critical points. The absolute max occurs at $(4, 9)$; there are no local maxima. All points $(x, 5)$ for x in the interval $[-3, 2]$ are absolute and local minima.



18

a. $f'(x) = \frac{(1+e^x)e^x - e^x \cdot e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$. This is never zero, and is always positive on $(-\infty, \infty)$, so f is increasing on $(-\infty, \infty)$.

b. $f''(x) = \frac{(1+e^x)^2 e^x - e^x(2)(1+e^x)e^x}{(1+e^x)^4} = \frac{(1+e^x)e^x - 2e^{2x}}{(1+e^x)^3} = \frac{e^x(1-e^x)}{(1+e^x)^3}$. This is zero when $e^x = 1$, or $x = 0$. $f'' > 0$ on $(-\infty, 0)$, so f is concave up there, while $f'' < 0$ on $(0, \infty)$, so f is concave down there.

19

a. $f'(x) = x^8 + 15x^4 - 16 = (x^4+16)(x^4-1) = (x^4+16)(x^2+1)(x^2-1) = (x^4+16)(x^2+1)(x+1)(x-1)$, which is zero for $x = \pm 1$. $f' > 0$ on $(-\infty, -1)$ and on $(1, \infty)$ so f is increasing there, while $f' < 0$ on $(-1, 1)$ so f is decreasing there.

b. $f''(x) = 8x^7 + 60x^3 = 4x^3(x^4 + 15)$, which is zero only for $x = 0$. $f'' < 0$ on $(-\infty, 0)$ so f is concave down there, while $f'' > 0$ on $(0, \infty)$ so f is concave up there.

20

a. Observe that the domain of f is $[-9,\infty)$.
$f'(x) = \sqrt{x+9} + \frac{x}{2\sqrt{x+9}} = \frac{2(x+9)}{2\sqrt{x+9}} + \frac{x}{2\sqrt{x+9}} = \frac{3(x+6)}{2\sqrt{x+9}}$. This is zero for $x=-6$. $f'>0$ on $(-6,\infty)$, so f is increasing there, while $f'<0$ on $(-9,-6)$, so f is decreasing there.

b. $f''(x) = \frac{3}{2}\left(\frac{\sqrt{x+9}-\frac{x+6}{2\sqrt{x+9}}}{x+9}\right) = \frac{3}{2}\left(\frac{\sqrt{x+9}-\frac{x+6}{2\sqrt{x+9}}}{x+9}\right)\cdot\frac{2\sqrt{x+9}}{2\sqrt{x+9}} = \frac{3}{4}\left(\frac{2x+18-x-6}{(x+9)^{3/2}}\right) = \frac{3(x+12)}{4(x+9)^{3/2}}$. This is always positive on the domain, so f is concave up on $(-9,\infty)$.

21 $f'(x) = 10x^4 - 40x^3 + 60x^2 + 1$, so $f''(x) = 40x^3 - 120x^2 + 120x = 40x(x^2-3x+3)$. The quadratic x^2-3x+3 has no real roots, so $x=0$ is the only possible inflection point. The sign of $f''(x)$ changes at $x=0$ so an inflection point occurs at $(0,1)$.

22 $f'(x) = 3x^5 + 5x^3 - 30x$, so $f''(x) = 15x^4 + 15x^2 - 30 = 15(x^4+x^2-2) = 15(x^2+2)(x^2-1) = 15(x^2+2)(x-1)(x+1)$. So the only potential location for inflection points are $x=\pm1$. An analysis of the sign of f'' shows that it is positive on $(-\infty,-1)$, negative on $(-1,1)$, and positive on $(1,\infty)$, so there are inflection points at both $x=1$ and $x=-1$.

23 $f'(x) = (x+a)^3 + (x-a)(3)(x+a)^2 = (x+a)^2(x+a+3x-3a) = (x+a)^2(4x-2a) = 2(x+a)^2(2x-a)$. This is zero for $x=\frac{a}{2}$ and $x=-a$, so those are the critical points. $f''(x) = 4(x+a)(2x-a)+2(x+a)^2(2) = 4(x+a)(2x-a+x+a) = 4(x+a)(3x) = 12x(x+a)$. This is zero for $x=0$ and $x=-a$, and an analysis of the sign of f'' shows that these are the locations of inflection points.

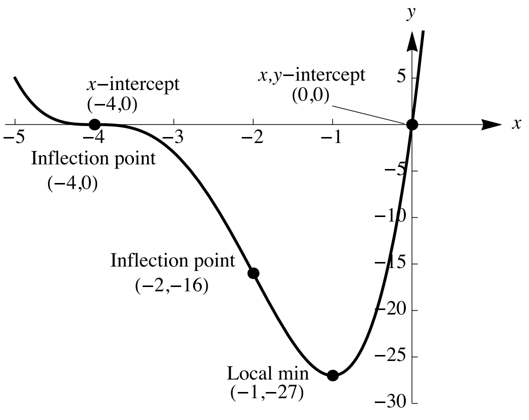
24 $f(x) = x(x+4)^3$ is a polynomial, so its domain is $(-\infty,\infty)$ and it has no asymptotes. $f(-x) = -x(4-x)^3 = x(x-4)^3$ which is neither $f(x)$ nor $-f(x)$, so there is neither even nor odd symmetry.

$$f'(x) = (x+4)^3 + x(3(x+4)^2) = (x+4)^2(x+4+3x) = 4(x+4)^2(x+1).$$

This is zero for $x=-1$ and $x=-4$. $f'<0$ on $(-\infty,-4)$ and on $(-4,-1)$, so f is decreasing there, while $f'(x)>0$ on $(-1,\infty)$, so f is increasing there. There is a local minimum of $f(-1)=-27$ at $x=-1$.

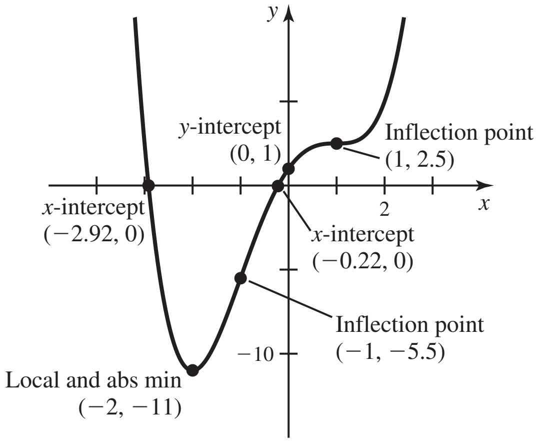
$$f''(x) = 8(x+4)(x+1) + 4(x+4)^2 = 4(x+4)(2x+2+x+4) = 4(x+4)(3x+6) = 12(x+4)(x+2).$$

This is zero for $x=-4$ and $x=-2$, so these are potential locations for inflection points. $f''(x)>0$ on $(-\infty,-4)$ and on $(-2,\infty)$, so f is concave up there, while $f''<0$ on $(-4,-2)$, so f is concave down there. There are inflection points at $(-4,0)$ and $(-2,-16)$. The x-intercepts are $(0,0)$ and $(-4,0)$. The y-intercept is $(0,0)$.



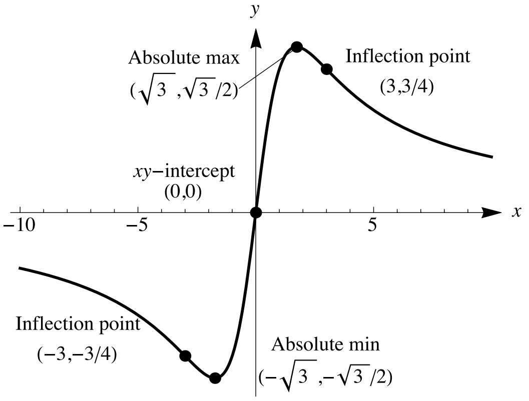
25 The derivatives of f are $f'(x) = 2x^3 - 6x + 4$ and $f''(x) = 6x^2 - 6$. Observe that $f'(x) = 2(x-1)^2(x+2)$, so we have critical points $x = 1, -2$. Solving $f''(x) = 6(x^2 - 1) = 0$ gives possible inflection points at $x = \pm 1$. Testing the sign of $f'(x)$ shows that f is decreasing on the interval $(-\infty, -2)$ and is increasing on $(-2, \infty)$. The First Derivative Test shows that a local minimum occurs at $x = -2$, and that the critical point $x = 1$ is neither a local max or min.

Testing the sign of $f''(x)$ shows that f is concave down on the interval $(-1, 1)$ and is concave up on the intervals $(-\infty, -1)$ and $(1, \infty)$. Therefore inflection points occur at $x = \pm 1$. Using a numerical solver, we see that the graph has x-intercepts at $x \approx -2.917$, -0.215. We also observe that $\lim_{x \to \pm\infty} f(x) = \infty$, so f has no absolute maximum and an absolute minimum at $x = -2$.



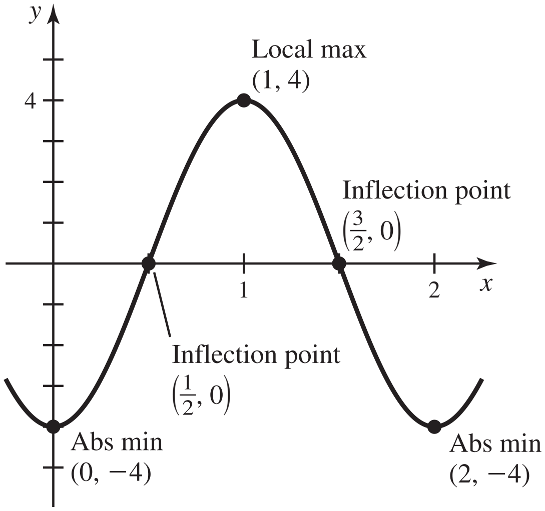
26 The derivatives of f are $f'(x) = 3 \cdot \dfrac{3 - x^2}{(x^2+3)^2}$ and $f''(x) = 6 \cdot \dfrac{x(x^2 - 9)}{(x^2+3)^3}$. Solving $f'(x) = 0$ gives critical points $x = \pm\sqrt{3}$, and solving $f''(x) = 0$ gives possible inflection points at $x = 0, \pm 3$. Testing the sign of $f'(x)$ shows that f is decreasing on the intervals $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$ and increasing on $(-\sqrt{3}, \sqrt{3})$. The First Derivative Test shows that a local minimum occurs at $x = -\sqrt{3}$ and a local maximum occurs at $x = \sqrt{3}$.

Testing the sign of $f''(x)$ shows that f is concave down on the intervals $(-\infty, -3)$ and $(0, 3)$ and concave up on the intervals $(-3, 0)$ and $(3, \infty)$. Therefore inflection points occur at $x = 0, \pm 3$. The graph has x-intercept at $x = 0$. We also observe that $\lim_{x \to \pm\infty} f(x) = 0$, so f has its absolute maximum and minimum at $x = \sqrt{3}, -\sqrt{3}$ respectively.



27 The derivatives of f are $f'(x) = -4\pi \sin[\pi(x-1)]$ and $f''(x) = -4\pi^2 \cos[\pi(x-1)]$. Solving $f'(x) = 0$ gives critical point $x = 1$, and solving $f''(x) = 0$ gives possible inflection points at $x = 1/2, 3/2$. Testing the sign of $f'(x)$ shows that f is decreasing on the interval $(1, 2)$ and increasing on $(0, 1)$. The First Derivative Test shows that a local maximum occurs at $x = 1$. By Theorem 4.5, this solitary local maximum must be the absolute maximum for f on the interval $[0, 2]$.

Testing the sign of $f''(x)$ shows that f is concave down on the interval $(1/2, 3/2)$ and concave up on the intervals $(0, 1/2)$ and $(3/2, 2)$. Therefore inflection points occur at $x = 1/2, 3/2$. These points are also the x-intercepts of the graph. We also observe that $f(0) = f(2) = -4$, so f takes its absolute minimum at these points.



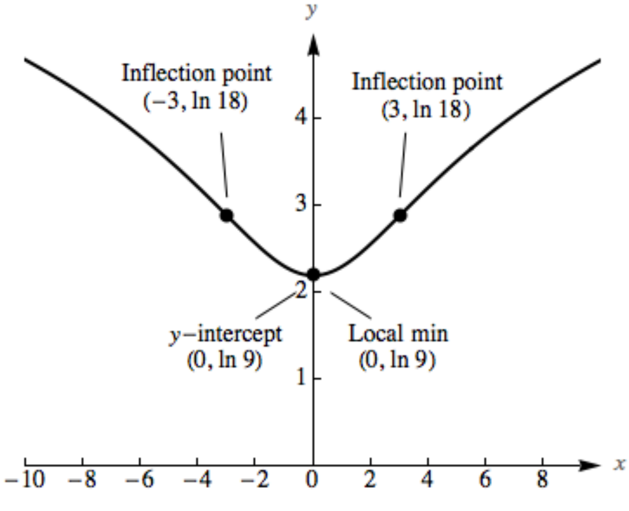
28 The domain of f is $(-\infty, \infty)$, and because $f(-x) = \ln((-x)^2 + 9) = \ln(x^2 + 9) = f(x)$, f has even symmetry.

$$f'(x) = \frac{1}{x^2+9} \cdot 2x = \frac{2x}{x^2+9},$$

which is zero for $x = 0$. $f' < 0$ on $(-\infty, 0)$ and $f' > 0$ on $(0, \infty)$ so f is decreasing on $(-\infty, 0)$ and is increasing on $(0, \infty)$. There is a local (and absolute) minimum of $\ln 9$ at $x = 0$.

$$f''(x) = \frac{(x^2+9)(2) - 2x(2x)}{(x^2+9)^2} = \frac{2(9-x^2)}{(x^2+9)^2},$$

which is zero for $x = \pm 3$. $f'' > 0$ on $(-3, 3)$, so f is concave up there, while $f'' < 0$ on $(-\infty, -3)$ and on $(3, \infty)$, so f is concave down there. There are inflection points at $(\pm 3, \ln 18)$. The y-intercept is $(0, \ln 9)$.



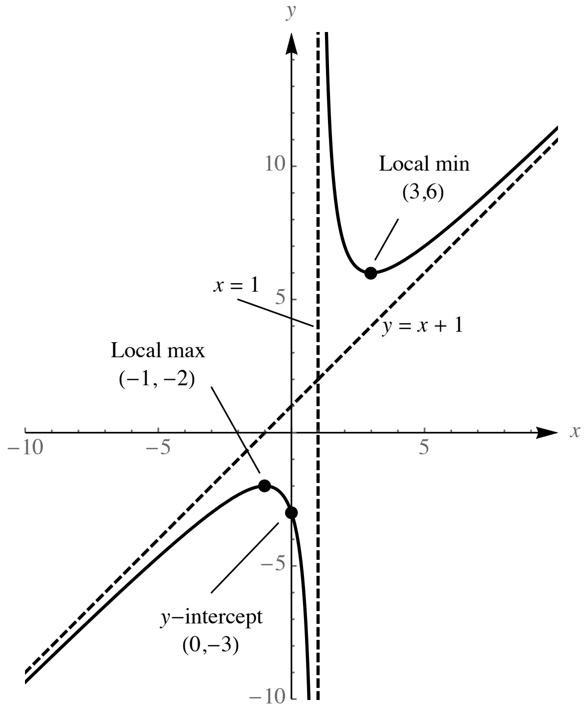
29 The domain of f is $(-\infty, 1) \cup (1, \infty)$. There is a vertical asymptote at $x = 1$. By long division, we can write $f(x) = x + 1 + \frac{4}{x-1}$, so $y = x + 1$ is a slant asymptote. $f(-x) = \frac{x^2+3}{-x-1} = -\frac{x^2+3}{x+1}$ which is not the same as either $f(x)$ or $-f(x)$, so f has neither even nor odd symmetry.

$$f'(x) = \frac{(x-1)(2x)-(x^2+3)}{(x-1)^2} = \frac{x^2-2x-3}{(x-1)^2} = \frac{(x+1)(x-3)}{(x-1)^2},$$

which is zero for $x=-1$ and $x=3$, so these are the critical points. $f'>0$ on $(-\infty,-1)$ and $(3,\infty)$, so f is increasing on those intervals, while $f'<0$ on $(-1,1)$ and on $(1,3)$ so f is decreasing on those intervals. There is a local minimum of 6 at $x=3$ and a local maximum of -2 at $x=-1$.

$$f''(x) = \frac{(x-1)^2(2x-2)-(x^2-2x-3)(2(x-1))}{(x-1)^4} = \frac{(2)(x-1)(x^2-2x+1-(x^2-2x-3))}{(x-1)^4} = \frac{8}{(x-1)^3}.$$

$f''>0$ on $(1,\infty)$ so f is concave up there, and $f''<0$ so f is concave down there. There are no inflection points as the only concavity change occurs at a vertical asymptote. The only intercept is the y-intercept at $f(0)=-3$.



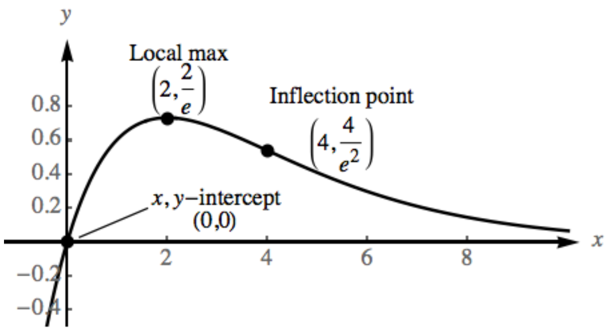
30 The domain of f is $(-\infty,\infty)$. $f(-x)=-xe^{x/2}$ which is neither $f(x)$ nor $-f(x)$, so this function has neither even nor odd symmetry. $\lim_{x\to\infty}\frac{x}{e^{x/2}} = \lim_{x\to\infty}\frac{1}{\frac{1}{2}e^{x/2}} = \lim_{x\to\infty}\frac{2}{e^{x/2}} = 0$, so $y=0$ is a horizontal asymptote as $x\to\infty$.

$$f'(x) = e^{-x/2} + x\left(\frac{1}{2}\right)(-e^{-x/2}) = e^{-x/2}\left(\frac{2-x}{2}\right),$$

which is zero for $x=2$. $f'>0$ on $(-\infty,2)$ so f is increasing there, while $f'<0$ on $(2,\infty)$, so f is decreasing there. There is a local (and absolute) maximum of $2/e$ at $x=2$.

$$f''(x) = -\frac{1}{2}e^{-x/2}\left(\frac{2-x}{2}\right) + e^{-x/2}\left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-x/2}\left(1-\frac{x}{2}+1\right) = e^{-x/2}\left(\frac{x-4}{4}\right),$$

which is zero for $x=4$. $f''>0$ on $(4,\infty)$, so f is concave up there, while $f''<0$ on $(-\infty,4)$ so f is concave down there. There is an inflection point at $(4,4/e^2)$. The only intercept is $(0,0)$ which is both an x- and a y-intercept.



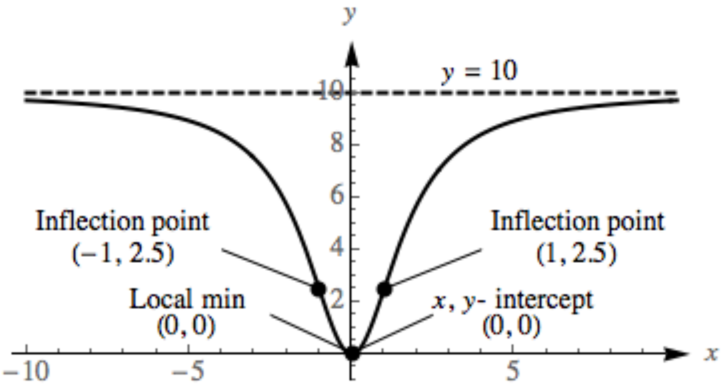
31 The domain of f is $(-\infty, \infty)$ and because $f(-x) = \dfrac{10(-x)^2}{(-x)^2+3} = f(x)$, f is symmetric with respect to the y-axis. Because $\lim\limits_{x\to\pm\infty} \dfrac{10x^2}{x^3+3} = \lim\limits_{x\to\pm\infty} \dfrac{20x}{2x} = 10$, the line $y = 10$ is a horizontal asymptote.

$$f'(x) = \frac{(x^2+3)(20x) - 10x^2(2x)}{(x^2+3)^2} = \frac{60x}{(x^2+3)^2},$$

which is zero for $x = 0$. $f' > 0$ on $(0, \infty)$, so f is increasing there, while $f' < 0$ on $(-\infty, 0)$, so f is decreasing there. There is a local (and absolute) minimum of 0 at $x = 0$.

$$f''(x) = \frac{(x^2+3)^2(60) - 60x(2)(x^2+3)(2x)}{(x^2+3)^4} = \frac{60(x^2+3)(x^2+3-4x^2)}{(x^2+3)^4} = \frac{180(1-x)(1+x)}{(x^2+3)^3},$$

which is zero for $x = -1$ and $x = 1$. $f'' < 0$ on $(-\infty, -1)$ and on $(1, \infty)$, so f is concave down there, while $f'' > 0$ on $(-1, 1)$, so f is concave up there. There are inflection points at $(\pm 1, 5/2)$. The only intercept is $(0, 0)$, which is both an x- and a y-intercept.



32 The domain of f is $(-\infty, \infty)$, and f has neither of our two types of symmetry.

$$f'(x) = \sqrt[3]{x+4} + \frac{x}{3\sqrt[3]{(x+4)^2}} = \frac{3(x+4)+x}{3\sqrt[3]{(x+4)^2}} = \frac{4(x+3)}{3\sqrt[3]{(x+4)^2}} = \frac{4(x+3)}{3(x+4)^{2/3}},$$

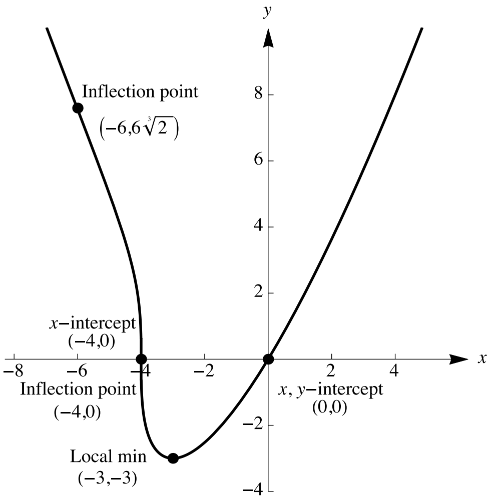
which is zero for $x = -3$ and is undefined for $x = -4$. Both $x = -3$ and $x = -4$ are critical points. $f'(x) < 0$ on $(-\infty, -4)$ and on $(-4, -3)$, so (using the fact that f is continuous at $x = -4$), f is decreasing on $(-\infty, -3)$, while $f' > 0$ on $(3, \infty)$, so f is increasing there. There is a local (and absolute) minimum of -3 at $x = -3$.

$$f''(x) = \frac{3(x+4)^{2/3}(4) - 4(x+3)(2)(x+4)^{-1/3}}{9\sqrt[3]{(x+4)^4}}.$$

This is perhaps best simplified by multiplying by $\dfrac{(x+4)^{1/3}}{(x+4)^{1/3}}$. Then we have

$$f''(x) = \frac{3(x+4)(4) - 8(x+3)}{9(x+4)^{5/3}} = \frac{4(x+6)}{9(x+4)^{5/3}},$$

which is zero for $x = -6$ and undefined for $x = -4$. $f'' > 0$ on $(-\infty, -6)$ and on $(-4, \infty)$, so f is concave up there, while $f'' < 0$ on $(-6, -4)$, so f is concave down there. There are inflection points at $(-4, 0)$ and at $(-6, 6\sqrt[3]{2})$. In addition to the intercept at $(-4, 0)$, there is the x- and y-intercept at $(0, 0)$.



33 Note that the domain of f is the interval $(0, \infty)$. The derivatives of f are

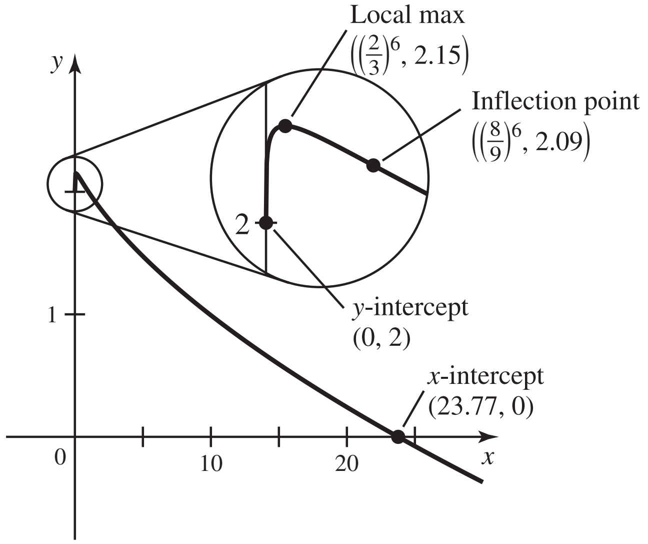
$$f'(x) = \frac{1}{3}x^{-2/3} - \frac{1}{2}x^{-1/2},$$

and

$$f''(x) = -\frac{2}{9}x^{-5/3} + \frac{1}{4}x^{-3/2}.$$

Solving $f'(x) = 0$ gives critical point $x = (2/3)^6$, and solving $f''(x) = 0$ gives a possible inflection point at $x = (8/9)^6$. Testing the sign of $f'(x)$ shows that f is increasing on the interval $(0, (2/3)^6)$ and decreasing on the interval $((2/3)^6, \infty)$. The First Derivative Test shows that a local maximum occurs at $x = (2/3)^6$ and By Theorem 4.5 this solitary local maximum must be the absolute maximum of f on the interval $(0, \infty)$.

Testing the sign of $f''(x)$ shows that f is concave down on the interval $(0, (8/9)^6)$ and concave up on the interval $((8/9)^6, \infty)$. Therefore an inflection point occurs at $x = (8/9)^6$. Using a numerical solver, we find that the graph has x-intercept at $x \approx 23.767$. Because $\lim_{x\to\infty} f(x) = -\infty$, f has no absolute minimum on the interval $(0, \infty)$.



34 The derivatives of f are

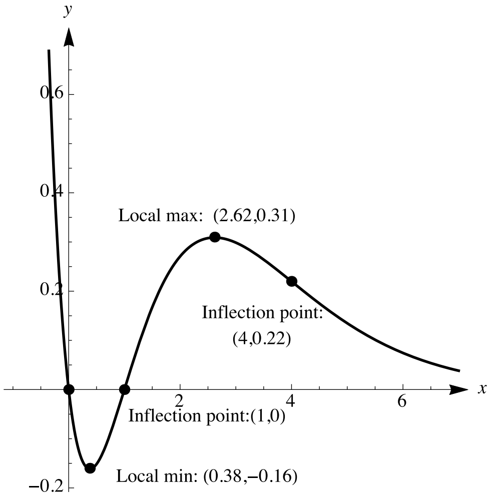
$$f'(x) = -(x^2 - 3x + 1)e^{-x},$$

and

$$f''(x) = (x^2 - 5x + 4)e^{-x}.$$

Solving $f'(x) = 0$ gives critical points $x \approx 0.382, 2.618$, and solving $f''(x) = 0$ gives possible inflection points at $x = 1, 4$. Testing the sign of $f'(x)$ shows that f is decreasing on the intervals $(-\infty, 0.382)$ and $(2.618, \infty)$ and increasing on $(0.382, 2.618)$. The First Derivative Test shows that a local minimum occurs at $x \approx 0.382$ and a local maximum occurs at $x \approx 2.618$.

Testing the sign of $f''(x)$ shows that f is concave up on the intervals $(-\infty, 1)$ and $(4, \infty)$ and concave down on the interval $(1, 4)$. Therefore inflection points occur at $x = 1, 4$. The graph has x-intercepts at $x = 0, 1$. Observe that $\lim\limits_{x\to-\infty} f(x) = \infty$, and $\lim\limits_{x\to\infty} f(x) = 0$; therefore f has no absolute max, and the absolute min occurs at $x \approx 0.382$.



35 The volume of the box (where x is measured in inches) is given by $V(x) = x(6-x)(12-x)$, for $0 \leq x \leq 6$. Using the product rule for three functions, we have

$$V'(x) = (6-x)(12-x)+x(-1)(12-x)+x(6-x)(-1) = 72-6x-12x+x^2-12x+x^2-6x+x^2 = 3(x^2-12x+24).$$

By the quadratic formula, this is zero on the domain for $x = 6 - 2\sqrt{3} \approx 2.5$. Using the Second Derivative Test, we have $V''(a) = 3(2a - 12)$ which is negative for $a \approx 2.5$, so the solitary critical point on the domain gives a maximum. Therefore, the largest box is approximately 2.5 × 3.5 × 9.5.

36 Let x be the distance from the fixed point P on the shore to the point where Hannah comes ashore, and let $T(x)$ be the total time for Hannah's trip to the beach house. Because time $= \dfrac{\text{distance}}{\text{rate}}$, we have

$$T(x) = \frac{\sqrt{x^2+1}}{2} + \frac{2-x}{6} \text{ for } 0 \le x \le 2.$$

Then $T'(x) = \dfrac{x}{2\sqrt{x^2+1}} - \dfrac{1}{6} = \dfrac{3x - \sqrt{x^2+1}}{6\sqrt{x^2+1}}$. This is zero when $3x = \sqrt{x^2+1}$, or $9x^2 = x^2+1$, so $x^2 = \dfrac{1}{8}$, so $x = \dfrac{1}{2\sqrt{2}}$. A quick check of the sign of T' on either side of this critical point shows that the critical point gives a minimum. So Hannah should come ashore at $\dfrac{1}{2\sqrt{2}} \approx 0.35$ miles down the beach and should make it home in approximately $T\left(\dfrac{1}{2\sqrt{2}}\right) \approx 0.805$ hours, which is about 48 minutes. So if she starts swimming at noon she should easily make it home by 1:00 pm (assuming she does her calculus correctly before she disembarks.)

37 Let x equal your distance from the louder speaker; then your distance from the other speaker is $100-x$. So we want to minimize the function $I(x) = \dfrac{3}{x^2} + \dfrac{1}{(100-x)^2}$, for $0 < x < 100$. Then

$$\begin{aligned} I'(x) &= -\frac{6}{x^3} + \frac{2}{(100-x)^3} = \frac{-6(100-x)^3 + 2x^3}{x^3(100-x)^3} \\ &= \left(\frac{-6000000 + 180000x - 1800x^2 + 6x^3 + 2x^3}{x^3(100-x)^3}\right) \\ &= \frac{8(x^3 - 225x^2 + 22,500x - 750,000)}{(x^3)(100-x)^3}. \end{aligned}$$

Using a computer algebra system or a root finder on a calculator, we find that $I' = 0$ for $x \approx 59$. Because $I'(1) < 0$ and $I'(99) > 0$, we see that the critical number yields a minimum for $0 < x < 100$. So you should stand 59 m away from the louder speaker.

38 Let $\theta = \alpha - \beta$ as pictured. $\tan\alpha = \dfrac{13}{x}$, and $\tan\beta = \dfrac{7}{x}$, so $\theta = \tan^{-1}\left(\dfrac{13}{x}\right) - \tan^{-1}\left(\dfrac{7}{x}\right)$. Then

$$\begin{aligned} \theta' &= \frac{-13}{x^2(1+\frac{13^2}{x^2})} - \frac{-7}{x^2(1+\frac{7^2}{x^2})} \\ &= \frac{-13}{x^2+169} + \frac{7}{x^2+49} \\ &= \frac{7(x^2+169) - 13(x^2+49)}{(x^2+169)(x^2+49)} \\ &= \frac{546 - 6x^2}{(x^2+169)(x^2+49)} = \frac{-6(x^2-91)}{(x^2+169)(x^2+49)}. \end{aligned}$$

This is zero for $x = \sqrt{91} \approx 9.5$. It can be verified with either the First Derivative Test or Second Derivative Test that this is a maximum. So he should shoot the puck at the goal when he is about 9.5 feet from the goal line.

39 The objective function to be maximized is the volume of the cone, given by $V = \pi r^2 h/3$. By the Pythagorean Theorem, r and h satisfy the constraint $h^2 + r^2 = 16$, which gives $r^2 = 16 - h^2$. Therefore

$$V(h) = \frac{\pi}{3}h(16-h^2) = \frac{\pi}{3}(16h - h^3).$$

We must maximize this function for $0 \le h \le 4$. The critical points of $V(h)$ satisfy $V'(h) = \dfrac{\pi}{3}(16 - 3h^2) = 0$, which has unique solution $h = 4/\sqrt{3} = 4\sqrt{3}/3$ in $(0,4)$. Because $V(0) = V(4) = 0$, $h = 4\sqrt{3}/3$ gives the maximum value of $V(h)$ on $[0,4]$. The corresponding value of r satisfies $r^2 = 16 - \dfrac{16}{3} = \dfrac{32}{3}$, so $r = \dfrac{4\sqrt{2}}{\sqrt{3}} = \dfrac{4\sqrt{6}}{3}$.

40 The rectangle has dimensions x and $\cos x$, so the objective function to be maximized is $A(x) = x\cos x$, where $0 \le x \le \pi/2$. The critical points of this function satisfy

$$A'(x) = \cos x - x\sin x = 0,$$

which can be solved numerically to obtain $x \approx 0.860$. Note that $A(0) = A(\pi/2) = 0$, so the maximum area occurs at $x \approx 0.860$; the dimensions of the largest rectangle are 0.860 and $\cos 0.860 \approx 0.652$, which gives maximum area ≈ 0.561.

41 We have that $xy = 98$, and we want to maximize $p = (y-2)(x-1) = (y-2)(98/y-1) = 98-y-196/y+2$. Note that $p'(y) = -1 + 196/y^2$, which is zero when $y^2 = 196$, so $y = \sqrt{196} = 14$. Also note that $p'(13) > 0$ and $p'(15) < 0$, so there is a local (in fact, absolute) maximum at $y = 14$. The value of x when $y = 14$ is $x = 98/14 = 7$.

42 A point on the graph of $y = \dfrac{5}{2} - x^2$ has the form $(x, \dfrac{5}{2} - x^2)$; the square of its distance to the origin is given by

$$Q(x) = x^2 + \left(\frac{5}{2} - x^2\right)^2 = x^4 - 4x^2 + \frac{25}{4},$$

which we can take as our objective function to be minimized. The critical points of $Q(x)$ satisfy

$$Q'(x) = 4x^3 - 8x = 4x(x^2 - 2) = 0,$$

which has solutions $x = 0$ and $x = \pm\sqrt{2}$. The First Derivative Test shows that $x = 0$ is a local maximum and $x = \pm\sqrt{2}$ are local minima. Note that $Q(x)$ takes the same value at $x = \pm\sqrt{2}$, so the absolute minimum of $Q(x)$ occurs at $x = \pm\sqrt{2}$ and the points closest to the origin on the graph are $(\pm\sqrt{2}, \frac{1}{2})$.

43 The area of the triangle is $\dfrac{1}{2}pq$, and the constraint is $\sqrt{p^2 + q^2} = 10$, or $p^2 + q^2 = 100$. So we can write the area of the triangle as $A(p) = (1/2)p\sqrt{100 - p^2}$. We have

$$A'(p) = (1/2)\sqrt{100 - p^2} + \frac{1}{2}p \cdot \frac{-p}{\sqrt{100 - p^2}} = \frac{100 - p^2 - p^2}{2\sqrt{100 - p^2}} = \frac{100 - 2p^2}{2\sqrt{100 - p^2}}.$$

This is zero for $p = \sqrt{50} = 5\sqrt{2}$. An application of the First Derivative Test shows that there is a local (in fact, absolute) maximum at this value of p. The value of q for this value of p is $\sqrt{100 - 50} = 5\sqrt{2}$ as well. So the area of the triangle is maximized when $p = q = 5\sqrt{2}$.

44 The volume of the cistern is $\pi r^2 h$, so our constraint is $\pi r^2 h = 50$, so $h = \dfrac{50}{\pi r^2}$. The area of the painted surface is given by

$$A = 2\pi rh + \pi r^2 = 2\pi r \cdot \frac{50}{\pi r^2} + \pi r^2 = \frac{100}{r} + \pi r^2.$$

Thus we have

$$A'(r) = \frac{-100}{r^2} + 2\pi r,$$

which is zero when $2\pi r = \dfrac{100}{r^2}$, or $r^3 = \dfrac{50}{\pi}$, so $r = \sqrt[3]{50/\pi}$. This is a minimum because $A'(r) < 0$ for $r < \sqrt[3]{50\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{50/\pi}$. The dimension of the cistern are $r = \sqrt[3]{50/\pi}$ and $h = \dfrac{50}{\pi r^2} = \sqrt[3]{50/\pi}$.

45

a. $f'(x) = \frac{2}{3}x^{-1/3}$, so $f'(27) = \frac{2}{3\sqrt[3]{27}} = \frac{2}{9}$. Thus

$$L(x) - 9 = \frac{2}{9}\left(x - 27\right),$$

or $L(x) = 9 + \frac{2}{9}(x - 27) = \frac{2}{9}x + 3$.

b. $f(29) \approx L(29) = 9 + \frac{4}{9} \approx 9.44$. This is an overestimate, because $f''(27) < 0$. Note that the calculator value of $f(29)$ is about 9.43913.

46

a. $f'(x) = \frac{1}{\sqrt{1-x^2}}$, so $f'(1/2) = \frac{2}{\sqrt{3}}$. Thus

$$L(x) - \pi/6 = \frac{2}{\sqrt{3}}(x - 1/2),$$

or $L(x) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}(x - 1/2)$.

b. $f(0.48) \approx L(0.48) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}(-.02) \approx 0.5005$. This is an underestimate because $f''(1/2) > 0$. The calculator value of $\sin^{-1}(0.48) \approx 0.500655$.

47 Let $f(x) = 1/x^2$ and let $a = 4$. Then $f'(x) = \frac{-2}{x^3}$ so $f'(4) = \frac{-2}{64} = \frac{-1}{32}$. The linearization is

$$L(x) = \frac{1}{16} + (-1/32)(x - 4).$$

Then $f(4.2) = 1/(4.2)^2 \approx L(4.2) = \frac{1}{16} - \frac{1}{32} \cdot \frac{2}{10} = 0.05625$.

48 Let $f(x) = \tan^{-1}(x)$ and $a = 1$. Then $f'(x) = \frac{1}{x^2+1}$ so $f'(1) = \frac{1}{2}$. The linearization is

$$L(x) = \frac{\pi}{4} + \frac{1}{2}(x - 1).$$

Then $f(1.05) = \tan^{-1}(1.05) \approx L(1.05) = \frac{\pi}{4} + \frac{1}{2} \cdot \frac{1}{20} \approx 0.8104$.

49 $\Delta h \approx h'(a)\Delta t = -32 \cdot 5(0.7) = -112$ feet.

50 $\Delta E \approx E'(a)\Delta M = 25000 \cdot 10^{(1.5)(7)} \cdot 1.5(\ln 10) \cdot 0.5 \approx 1.365 \times 10^{15}$ J.

51 The value of c appears to be between 2 and 3. We are seeking c so that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{10 - 0}{10 - 0} = 1,$$

so we want $\frac{\sqrt{10}}{2\sqrt{c}} = 1$. Thus $10 = 4c$ and $c = 2.5$, as projected.

52 The absolute value function is not differentiable at $x = 0$, so it is not differentiable on $(-1, 2)$, as required.

53

a. The average rate of change of $P(t)$ on the interval $[0, 8]$ is

$$\frac{P(8) - P(0)}{8 - 0} = \frac{800/9 - 0}{8} = \frac{100}{9}\text{cells/week.}$$

b. We solve

$$P'(t) = \frac{100}{(t+1)^2} = \frac{100}{9}$$

which gives $(t+1)^2 = 9$, so $t = 2$ weeks.

54

a. The average rate of change is $\dfrac{\text{change in growth}}{\text{elapsed time}} = \dfrac{15}{5} = 3$ cm per hour.

b. 3 cm per hour is equivalent to $30/3600 = 1/120$ mm per second. The Mean Value Theorem tells us that sometime between 10:00 a.m. and 3:00 p.m, there will be a time when the instantaneous growth rate is exactly $1/120$ mm per second.

55 It is possible to note that 1 is a root by inspection. Then by long division by $x - 1$, we have $f(x) = (x-1)(3x^2 - x - 1)$. We apply Newton's method to the function $g(x) = 3x^2 - x - 1$.

The Newton's method recursion is given by $x_{n+1} = x_n - \dfrac{3x_n^2 - x_n - 1}{6x_n - 1}$. Applying this recursion to the initial estimates of -0.5 and 0.8 yields:

n	x_n
0	-0.5
1	-0.4375
2	-0.434267
3	-0.434259
4	-0.434259

n	x_n
0	0.8
1	0.768421
2	0.767592
3	0.767592
4	0.767592

The roots are thus 1, and approximately -0.434259 and 0.767592.

56 The Newton's method recursion is given by $x_{n+1} = x_n - \dfrac{e^{-2x_n} + 2e^{x_n} - 6}{-2e^{-2x_n} + 2e^{x_n}}$. Applying this recursion to the initial estimates of -1 and 1 yields:

n	x_n
0	-1
1	-0.848685
2	-0.817331
3	-0.816164
4	-0.816162
5	-0.816162

n	x_n
0	1
1	1.08287
2	1.07918
3	1.07917
4	1.07917
5	1.07917

The roots are thus 1, and approximately -0.816162 and 1.07917.

57 First note that $f'(x) = 10x^4 - 18x^2 - 4$ and $f''(x) = 40x^3 - 36x = 4x(10x^2 - 9)$. This is clearly 0 when $x = 0$, and when $10x^2 - 9 = 0$. Applying Newton's method to the function $g(x) = 10x^2 - 9$ with initial estimates of -1 and 1 yields:

n	x_n
0	-1
1	-0.95
2	-0.948684
3	-0.948683
4	-0.948683
5	-0.948683

n	x_n
0	1
1	0.95
2	0.948684
3	0.948683
4	0.948683
5	0.948683

Checking the signs of $f''(x)$ on the appropriate intervals leads to the conclusion that these are all the locations of inflection points of f. So the inflection points of f are located at 0 and approximately ± 0.948683.

58 Observe that $\lim\limits_{x\to\infty} \dfrac{4x^4 - \sqrt{x}}{2x^4 + x^{-1}} = \lim\limits_{x\to\infty} \dfrac{4 - x^{-7/2}}{2 + x^{-5}} = \dfrac{4}{2} = 2$. We can also apply L'Hôpital's rule four times: $\lim\limits_{x\to\infty} \dfrac{4x^4 - x^{1/2}}{2x^4 + x^{-1}} = \lim\limits_{x\to\infty} \dfrac{96 + (15/16)x^{-7/2}}{48 + 24x^{-5}} = 2$.

59 By L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{2x^5 - x + 1}{5x^6 + x} = \lim_{x\to\infty} \frac{10x^4 - 1}{30x^5 + 1} = \lim_{x\to\infty} \frac{40x^3}{150x^4} = \lim_{x\to\infty} \frac{4}{15x} = 0.$$

60 L'Hôpital's rule gives $\lim\limits_{t\to 2} \dfrac{t^3 - t^2 - 2t}{t^2 - 4} = \lim\limits_{t\to 2} \dfrac{3t^2 - 2t - 2}{2t} = \dfrac{3}{2}$.

61 L'Hôpital's rule gives $\lim\limits_{t\to 0} \dfrac{1 - \cos 6t}{2t} = \lim\limits_{t\to 0} \dfrac{6 \sin 6t}{2} = 0$.

62 If we factor out x^2 from the numerator and denominator, we see that

$$\frac{5x^2 + 2x - 5}{\sqrt{x^4 - 1}} = \frac{5 + 2/x - 5/x^2}{\sqrt{1 - 1/x^4}} \to 5$$

as $x \to \infty$.

63 Observe that $\lim\limits_{\theta\to 0} \dfrac{3\sin^2 2\theta}{\theta^2} = 3\left(\lim\limits_{\theta\to 0} \dfrac{\sin 2\theta}{\theta}\right)^2$; L'Hôpital's rule gives $\lim\limits_{\theta\to 0} \dfrac{\sin 2\theta}{\theta} = \lim\limits_{\theta\to 0} \dfrac{2\cos 2\theta}{1} = 2$, so $\lim\limits_{\theta\to 0} \dfrac{3\sin^2 2\theta}{\theta^2} = 3 \cdot 2^2 = 12$.

64 First observe that

$$(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}) \frac{(\sqrt{x^2 + x + 1} + \sqrt{x^2 - x})}{(\sqrt{x^2 + x + 1} + \sqrt{x^2 - x})} = \frac{2x + 1}{(\sqrt{x^2 + x + 1} + \sqrt{x^2 - x})}.$$

Next, factor out x from both numerator and denominator to obtain

$$\frac{2x + 1}{(\sqrt{x^2 + x + 1} + \sqrt{x^2 - x})} = \frac{2 + 1/x}{(\sqrt{1 + 1/x + 1/x^2} + \sqrt{1 - 1/x})}.$$

This expression converges to $2/2 = 1$ as $x \to \infty$, so $\lim\limits_{x\to\infty} (\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}) = 1$.

65 Observe that $2\theta \cot 3\theta = 2\cos 3\theta \cdot \frac{\theta}{\sin 3\theta}$; $\lim\limits_{\theta\to 0} \dfrac{\theta}{\sin 3\theta} = \lim\limits_{\theta\to 0} \dfrac{1}{3\cos 3\theta} = \dfrac{1}{3}$ by L'Hôpital's rule, so

$$\lim_{\theta\to 0} 2\theta \cot 3\theta = 2 \cdot 1 \cdot \frac{1}{3} = \frac{2}{3}.$$

66 Apply L'Hôpital's rule twice: $\lim_{x\to 0}\dfrac{e^{-2x}-1+2x}{x^2}=\lim_{x\to 0}\dfrac{-2e^{-2x}+2}{2x}=\lim_{x\to 0}\dfrac{4e^{-2x}}{2}=2.$

67 Make the change of variables $x=1/y$; then $\ln^{10} y=(\ln y)^{10}=(-\ln x)^{10}=\ln^{10} x$, and so $\lim_{y\to 0^+}\dfrac{\ln^{10} y}{\sqrt{y}}=\lim_{x\to\infty}\sqrt{x}\ln^{10} x=\infty.$

68 Apply L'Hôpital's rule: $\lim_{\theta\to 0}\dfrac{3\sin 8\theta}{8\sin 3\theta}=\lim_{\theta\to 0}\dfrac{24\cos 8\theta}{24\cos 3\theta}=1.$

69 Apply L'Hôpital's rule twice:

$$\lim_{x\to 1}\frac{x^4-x^3-3x^2+5x-2}{x^3+x^2-5x+3}=\lim_{x\to 1}\frac{4x^3-3x^2-6x+5}{3x^2+2x-5}=\lim_{x\to 1}\frac{12x^2-6x-6}{6x+2}=0.$$

70 The function $\ln x^{100}=100\ln x$ grows more slowly than $\sqrt{x}$ as $x\to\infty$, so $\lim_{x\to\infty}\dfrac{\ln x^{100}}{\sqrt{x}}=0.$

71 $\lim_{x\to 0}\csc x\sin^{-1} x=\lim_{x\to 0}\dfrac{\sin^{-1} x}{\sin x}=\lim_{x\to 0}\dfrac{1/\sqrt{1-x^2}}{\cos x}=1$, by L'Hôpital's rule.

72 The function $\ln^3 x$ grows more slowly than $\sqrt{x}$ as $x\to\infty$, so $\lim_{x\to\infty}\dfrac{\ln^3 x}{\sqrt{x}}=0.$

73 Observe that $\lim_{x\to\infty}\dfrac{x+1}{x-1}=1$, by L'Hôpital's rule. Hence $\lim_{x\to\infty}\ln\left(\dfrac{x+1}{x-1}\right)=\ln 1=0.$

74 Let $z=(1+x)^{\ln x}$. Then $\ln z=(\ln x)\ln(x+1)$. Then

$$\begin{aligned}\lim_{x\to 0^+}\ln z&=\lim_{x\to 0^+}(\ln x)\ln(x+1)=\lim_{x\to 0^+}\frac{\ln(1+x)}{\ln^{-1} x}\\&=\lim_{x\to 0^+}\frac{1/(1+x)}{-(\ln x)^{-2}(1/x)}=\lim_{x\to 0^+}\frac{1}{1+x}\cdot\lim_{x\to 0^+}(x\ln x)^2=1\cdot\left(\lim_{x\to 0^+}x\ln x\right)^2.\end{aligned}$$

Now $\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\dfrac{\ln x}{1/x}=\lim_{x\to 0^+}\dfrac{1/x}{-1/x^2}=\lim_{x\to 0^+}-x=0.$ Thus,

$$\lim_{x\to 0^+}\ln z=0,$$

so

$$\lim_{x\to 0^+}(1+x)^{\ln x}=e^0=1.$$

75 Note that $\ln(\sin x)^{\tan x}=\tan x\ln\sin x$, so we evaluate $L=\lim_{x\to\pi/2^-}\tan x\ln\sin x=\lim_{x\to\pi/2^-}\dfrac{\ln\sin x}{\cot x}=\lim_{x\to\pi/2^-}\dfrac{\cot x}{-\csc^2 x}=\lim_{x\to\pi/2^-}(-\cos x\sin x)=0$ by L'Hôpital's rule. Therefore $\lim_{x\to\pi/2^-}(\sin x)^{\tan x}=e^L=1.$

76 Note that $\ln(\sqrt{x}+1)^{1/x}=\dfrac{\ln(\sqrt{x}+1)}{x}$, so we evaluate $L=\lim_{x\to\infty}\dfrac{\ln(\sqrt{x}+1)}{x}=\lim_{x\to\infty}\dfrac{\frac{1}{(\sqrt{x}+1)}\cdot\frac{1}{2\sqrt{x}}}{1}=0$ by L'Hôpital's rule. Therefore, $\lim_{x\to\infty}(\sqrt{x}+1)^{1/x}=e^L=e^0=1.$

77 Note that for $0<x<1$, we have $\ln x<0$, so $|\ln x|=-\ln x$. Then $|\ln x|^x=(-\ln x)^x$. Consider the natural logarithm of this quantity, $x\ln(-\ln x)=\dfrac{\ln(-\ln x)}{\frac{1}{x}}$. Using L'Hôpital's rule we have

$$\lim_{x\to 0^+}\frac{\ln(-\ln x)}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x\ln x}}{\frac{-1}{x^2}}=\lim_{x\to 0^+}\frac{-x}{\ln x}=0.$$

Thus $\lim_{x\to 0^+}|\ln x|^x=e^0=1.$

78 Note that $\ln x^{1/x} = \ln x/x$, so we evaluate $L = \lim_{x\to\infty} \frac{\ln x}{x} = 0$ because $\ln x$ grows more slowly than x as $x \to \infty$. Therefore $\lim_{x\to\infty} x^{1/x} = e^L = 1$.

79 Note that $\ln\left(1 - \frac{3}{x}\right)^x = x\ln\left(1 - \frac{3}{x}\right) = \frac{\ln(1-3/x)}{1/x}$, so we evaluate

$$L = \lim_{x\to\infty} \frac{\ln(1-3/x)}{1/x} = \lim_{x\to\infty} \frac{(1-3/x)^{-1}(3/x^2)}{-1/x^2} = -3$$

by L'Hôpital's rule. Therefore $\lim_{x\to\infty}\left(1 - \frac{3}{x}\right)^x = e^L = e^{-3}$.

80 Note that $\ln\left(\left(\frac{2}{\pi}\tan^{-1} x\right)^x\right) = x\ln\left(\frac{2}{\pi}\tan^{-1} x\right) = \frac{\ln\left(\frac{2}{\pi}\tan^{-1} x\right)}{1/x}$. We evaluate
$L = \lim_{x\to\infty} \frac{\ln\left(\frac{2}{\pi}\tan^{-1} x\right)}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{(2/\pi)\tan^{-1}(x)} \cdot \frac{2}{\pi(x^2+1)}}{-1/x^2} = \lim_{x\to\infty} \frac{-x^2}{x^2+1} \cdot \frac{1}{\tan^{-1}(x)} = -\frac{2}{\pi}$.
Thus, $\lim_{x\to\infty}\left(\frac{2}{\pi}\tan^{-1} x\right)^x = e^L = e^{-2/\pi}$.

81 Let $y = (x-1)^{\sin \pi x}$. Then $\ln y = \sin \pi x \ln(x-1) = \frac{\ln(x-1)}{\csc \pi x}$. Then

$$\begin{aligned}\lim_{x\to 1} \ln y &= \lim_{x\to 1} \frac{\ln(x-1)}{\csc \pi x} = \lim_{x\to 1} \frac{\frac{1}{x-1}}{-\pi \csc \pi x \cot \pi x} = \lim_{x\to 1} \frac{\sin^2 \pi x}{-\pi(x-1)\cos \pi x} \\ &= \lim_{x\to 1} \frac{1}{-\pi \cos \pi x} \cdot \lim_{x\to 1} \frac{\sin^2 \pi x}{x-1} = \frac{1}{\pi} \lim_{x\to 1} \frac{2\pi \sin \pi x \cos \pi x}{1} = \frac{1}{\pi} \cdot 0 = 0.\end{aligned}$$

Then $\lim_{x\to 1} y = \lim_{x\to 1} e^{\ln y} = e^0 = 1$.

82 By Theorem 4.14, 1.1^x grows faster than x^{100} as $x \to \infty$.

83 Observe that $\lim_{x\to\infty} \frac{x^{1/2}}{x^{1/3}} = \lim_{x\to\infty} x^{1/6} = \infty$, so $x^{1/2}$ grows faster than $x^{1/3}$ as $x \to \infty$.

84 Because $\log_{10} x = \ln x/\ln 10$, $\ln x$ and $\log_{10} x$ have comparable growth rates as $x \to \infty$.

85 By Theorem 4.14, $\sqrt{x}$ grows faster than $\ln^{10} x$ as $x \to \infty$.

86 Because $\ln x^2 = 2\ln x$, Theorem 4.14 shows that $10x$ grows faster than $\ln x^2$ as $x \to \infty$.

87 Observe that $\lim_{x\to\infty} \frac{e^x}{3^x} = \lim_{x\to\infty}\left(\frac{e}{3}\right)^x = 0$ because $e/3 < 1$. Therefore 3^x grows faster than e^x as $x \to \infty$.

88 Observe that $\lim_{x\to\infty} \frac{\sqrt{x^6+10}}{x^3} = \lim_{x\to\infty} \sqrt{1 + \frac{10}{x^6}} = 1$, so $\sqrt{x^6+10}$ and x^3 have comparable growth rates as $x \to \infty$.

89 Observe that $4^{x/2} = (4^{1/2})^x = 2^x$, so these functions are identical and hence have comparable growth rates as $x \to \infty$.

90 $\int (x^8 - 3x^3 + 1)\, dx = \frac{x^9}{9} - 3 \cdot \frac{x^4}{4} + x + C = \frac{x^9}{9} - \frac{3}{4}x^4 + x + C$.

91 $\int (2x+1)^2\, dx = \int (4x^2 + 4x + 1)\, dx = \frac{4}{3}x^3 + 2x^2 + x + C$.

92 $\displaystyle\int \frac{x^5-3}{x}\,dx = \int\left(\frac{x^5}{x}-\frac{3}{x}\right)dx = \int\left(x^4-\frac{3}{x}\right)dx = \frac{x^5}{5} - 3\ln|x| + C.$

93 $\displaystyle\int\left(\frac{1}{x^2}-\frac{2}{x^{5/2}}\right)dx = \int(x^{-2}-2x^{-5/2})\,dx = -x^{-1} - 2\cdot\left(-\frac{2}{3}\right)x^{-3/2} = -\frac{1}{x}+\frac{4}{3}x^{-3/2}+C.$

94 $\displaystyle\int\frac{x^4-2\sqrt{x}+2}{x^2}\,dx = \int(x^2-2x^{-3/2}+2x^{-2})\,dx = \frac{x^3}{3} - 2\cdot(-2x^{-1/2}) - 2x^{-1} + C = \frac{x^3}{3}+\frac{4}{\sqrt{x}}-\frac{2}{x}+C.$

95 $\displaystyle\int(1+3\cos\theta)\,d\theta = \theta + 3\sin\theta + C.$

96 $\displaystyle\int 2\sec^2\theta\,d\theta = 2\tan\theta + C.$

97 $\displaystyle\int\frac{dx}{1-\sin^2 x} = \int\frac{dx}{\cos^2 x} = \int\sec^2 x\,dx = \tan x + C.$

98 $\displaystyle\int\frac{2e^{2x}+e^x}{e^x}\,dx = \int(2e^x+1)\,dx = 2e^x + x + C.$

99 $\displaystyle\int\frac{12}{x}\,dx = 12\ln|x| + C.$

100 $\displaystyle\int\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C.$

101 $\displaystyle\int\frac{x^2}{x^4+x^2}\,dx = \int\frac{1}{x^2+1}\,dx = \tan^{-1}x + C.$

102 Note that $\displaystyle\frac{1+\tan\theta}{\sec\theta} = \cos\theta+\sin\theta$, so $\displaystyle\int\frac{1+\tan\theta}{\sec\theta}\,d\theta = \sin\theta - \cos\theta + C.$

103 $\displaystyle\int\left(\sqrt[4]{x^3}+\sqrt{x^5}\right)dx = \int(x^{3/4}+x^{5/2})\,dx = \frac{4}{7}x^{7/4}+\frac{2}{7}x^{7/2}+C.$

104 We have $\displaystyle f(x) = \int(3x^2-1)\,dx = x^3 - x + C$; $f(0) = C = 10$, so $f(x) = x^3 - x + 10$.

105 We have $\displaystyle f(t) = \int(\sin t + 2t)\,dt = -\cos t + t^2 + C$; $f(0) = -1 + C = 5$, so $C = 6$ and $f(t) = -\cos t + t^2 + 6$.

106 We have $\displaystyle f(t) = \int(t^2+t^{-2})\,dt = \frac{t^3}{3} - \frac{1}{t} + C$; $f(1) = 1/3 - 1 + C = C - 2/3 = 1$, so $C = 5/3$ and $f(t) = t^3/3 - 1/t + 5/3$.

107 Note that by long division, we can write $\displaystyle\frac{x^4-2}{1+x^2} = x^2 - 1 - \frac{1}{1+x^2}$. Therefore $\displaystyle f(x) = \int\left(x^2-1-\frac{1}{1+x^2}\right)dx = \frac{x^3}{3} - x - \tan^{-1}x + C$. Because $f(1) = \frac{2}{3}$, we have $\frac{1}{3} - \frac{\pi}{4} + C = -\frac{2}{3}$, so $C = \frac{\pi}{4}$. Thus $\displaystyle f(x) = \frac{x^3}{3} - x - \tan^{-1}x + \frac{\pi}{4}$.

108 The difference in the positions of the objects is given by the function

$$f(t) = 2\sin t - \sin(t-\pi/2) = 2\sin t + \cos t.$$

The objects meet when $f(t) = 0$, which occurs at $t = \pi - \tan^{-1}(1/2) \approx 2.673$ and $t = 2\pi - \tan^{-1}(1/2) \approx 5.820$. The objects are furthest apart when f attains its absolute max or min; the critical points of f satisfy

$$f'(t) = 2\cos t - \sin t = 0,$$

or $\tan t = 2$; this gives $t = \tan^{-1}(2) \approx 1.11$ and $t = \pi + \tan^{-1}2 \approx 4.25$. The value of f is $\approx \pm 2.24$ at these points, so the objects are furthest apart at these two times.

109 The velocity of the rocket is given by

$$v(t) = -9.8t + v_0 = -9.8t + 120,$$

and the position function of the rocket is

$$s(t) = \int (-9.8t + 120)\, dt = -4.9t^2 + 120t + s_0 = -4.9t^2 + 120t + 125.$$

The rocket reaches its maximum height when $v(t) = 0$, which occurs at $t = 120/9.8 \approx 12.24$ s; the maximum height is $s(120/9.8) \approx 859.69$ m. The rocket hits the ground when $s(t) = 0$; solving this quadratic equation gives $t \approx 25.49$ s.

110 Because $a(t) = 20$, we have $v(t) = \displaystyle\int a(t)\, dt = \int 20\ dt = 20t + C$. Because the car is starting from rest, $v(0) = 0$, and so $C = 0$. So $v(t) = 20t$. Then

$$s(t) = \int v(t)\, dt = \int 20t\, dt = 10t^2 + C_1,$$

but we can assume the car's initial position is 0, so $C_1 = 0$ and $s(t) = 10t^2$. Then $s(5) = 250$ ft.

111

a. We have $v(t) = -32t + v_0$ and $v_0 = 64$, so $v(t) = -32t + 64$.

b. The height of the ball above the river is given by $s(t) = \displaystyle\int (-32t + 64)\, dt = -16t^2 + 64t + s_0$., and because $s(0) = 128$, we have $s_0 = 128$. So $s(t) = -16t^2 + 64t + 128$.

c. The ball reaches its maximum height when $v(t) = -32t + 64 = 0$, which occurs for $t = 2$. The height at this time is $s(2) = 192$.

d. The ball strikes the river when $s(t) = 0$ (and $t > 0$), which gives $-16t^2 + 64t + 128 = -16(t^2 - 4t - 8) = 0$. Using the quadratic formula, we see that this occurs for $t = 2(1 + \sqrt{3})$. The velocity with which the ball strikes the river is therefore $v((2(1 + \sqrt{3})) = -64\sqrt{3} \approx -110.9$ ft per s.

112 We have $\ln(\ln(\ln x)) << \ln(\ln x) << \ln x$ as $x \to \infty$. For the first relation, make the change of variables $y = \ln(\ln(x))$: then

$$\lim_{x\to\infty} \frac{\ln(\ln(\ln x))}{\ln(\ln x)} = \lim_{y\to\infty} \frac{\ln y}{y} = 0;$$

and for the second, let $y = \ln x$:

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} = \lim_{y\to\infty} \frac{\ln y}{y} = 0.$$

113 For the second limit, let $y = x^2$:

$$\lim_{x\to 0} \frac{x^2}{1 - e^{-x^2}} = \lim_{y\to 0} \frac{y}{1 - e^{-y}} = \lim_{y\to 0} \frac{1}{e^{-y}} = 1$$

by L'Hôpital's rule. For the first, observe that for $x > 0$

$$\frac{x}{\sqrt{1 - e^{-x^2}}} = \left(\frac{x^2}{1 - e^{-x^2}} \right)^{1/2},$$

so $\displaystyle\lim_{x\to 0^+} \frac{x}{\sqrt{1 - e^{-x^2}}} = 1$ as well.

114 Observe that

$$\ln\left(\frac{a^r+b^r+c^r}{3}\right)^{1/r} = \frac{\ln(a^r+b^r+c^r)-\ln 3}{r}.$$

By L'Hôpital's rule, we have

$$L = \lim_{r\to 0}\frac{\ln(a^r+b^r+c^r)-\ln 3}{r} = \lim_{r\to 0}\frac{(a^r+b^r+c^r)^{-1}\left((\ln a)a^r+(\ln b)b^r+(\ln c)c^r\right)}{1}$$
$$= \frac{1}{3}(\ln a+\ln b+\ln c) = \ln(abc)^{1/3};$$

therefore $\lim\limits_{r\to 0}\left(\frac{a^r+b^r+c^r}{3}\right)^{1/r} = e^L = \sqrt[3]{abc}.$

115 Note that $\lim\limits_{x\to 0+} x^x = 1$, as shown in Section 4.6. Therefore $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}(x^x)^x = 1^0 = 1$, and $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} x^{(x^x)} = 0^1 = 0$.

116

a. Make the change of variables $y = x^n$ and apply L'Hôpital's rule twice:

$$\lim_{x\to 0}\frac{1-\cos x^n}{x^{2n}} = \lim_{y\to 0}\frac{1-\cos y}{y^2} = \lim_{y\to 0}\frac{\cos y}{2} = \frac{1}{2}.$$

b. Apply L'Hôpital's rule:

$$\lim_{x\to 0}\frac{1-\cos^n x}{x^2} = \lim_{x\to 0}\frac{n\cos^{n-1}x\sin x}{2x} = \frac{n}{2}\left(\lim_{x\to 0}\cos^{n-1}x\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right) = \frac{n}{2}\cdot 1\cdot 1 = \frac{n}{2}.$$

117

a. First, observe that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+a} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\left(1+\frac{1}{x}\right)^{a} = e\cdot 1 = e.$$

Therefore $\lim\limits_{x\to\infty}\ln g(x) = 1$. It suffices to determine whether $\ln g(x)-1$ is positive or negative as $x\to\infty$. To do this, consider

$$\lim_{x\to\infty} x(\ln g(x)-1) = \lim_{t\to 0}\frac{(1+at)\ln(1+t)-t}{t^2},$$

where we make the change of variables $t = 1/x$. This limit can be evaluated by using L'Hôpital's rule twice:

$$\lim_{t\to 0}\frac{(1+at)\ln(1+t)-t}{t^2} = a-\frac{1}{2}.$$

Therefore when $a > 1/2$ we have $g(x) > e$ as $x\to\infty$,

b. Using the analysis above, for $0 < a < 1/2$, $g(x) < e$ as $x\to\infty$. In the case $a = 1/2$ we consider the limit

$$\lim_{x\to\infty} x^2(\ln g(x)-1) = \lim_{t\to 0}\frac{(1+at)\ln(1+t)-t}{t^3},$$

which can be evaluated by using L'Hôpital's three times:

$$\lim_{t\to 0}\frac{(1+t/2)\ln(1+t)-t}{t^3} = \frac{1}{12}.$$

Therefore $g(x) > e$ as $x\to\infty$ in this case as well.

118

a. The domain is the interval $[-a, \infty)$.

b. Observe that $\lim\limits_{x\to -a^+} (a+x)^x = \lim\limits_{x\to -a^+} (a+x)^{a+x}(a+x)^{-a} = \lim\limits_{y\to 0+} y^y \lim\limits_{y\to 0+} y^{-a} = 1 \cdot \infty = \infty$. We also have $\lim\limits_{x\to\infty} (a+x)^x = \infty$ because $(a+x)^x > x^x$.

c. Using logarithmic differentiation, we find that $f'(x) = x(a+x)^{x-1} + \ln(a+x)(a+x)^x$.

d. Multiplying $f'(x)$ by $(a+x)^{1-x}$ shows that the critical point z for f satisfies the equation

$$z + (z+a)\ln(z+a) = 0,$$

which is equivalent to the equation

$$\ln(z+a) - \frac{a}{z+a} = -1.$$

The left side is an increasing function on $(-a, \infty)$ with range $(-\infty, \infty)$, so there exists a unique z satisfying this equation.

e. Graphical analysis shows that as $a \to \infty$, $z \to -\infty$ and $f(z) \to 0$.

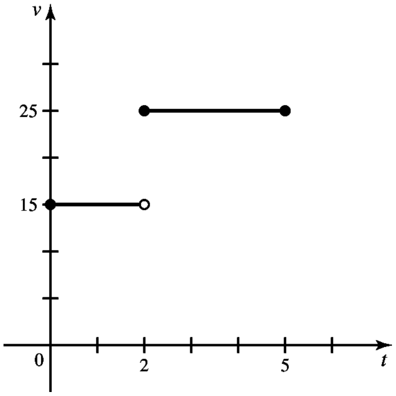
Chapter 5

Integration

5.1 Approximating Areas under Curves

5.1.1

In the first 2 seconds, the object moves $15 \cdot 2 = 30$ meters. In the next three seconds, the object moves $25 \cdot 3 = 75$ meters, so the total displacement is $75 + 30 = 105$ meters.



5.1.2 The area under the curve and above the t-axis, between $t = a$ and $t = b$ numerically represents the displacement.

5.1.3

a. The displacement is approximately $40(4) + 70(4) = 440$ ft.

b. The displacement is approximately $30(2) + 50(2) + 80(2) + 40(2) = 400$ ft.

5.1.4

a. The displacement is approximately $10(2) + 40(2) + 70(2) = 240$ ft.

b. The displacement is approximately $10(1) + 20(1) + 40(1) + 60(1) + 70(1) + 80(1) = 280$ ft.

5.1.5

a. The displacement is approximately $60(2) + 30(2) + 80(2) = 340$ ft.

b. The displacement is approximately $70(1) + 60(1) + 50(1) + 30(1) + 40(1) + 80(1) = 330$ ft.

5.1.6

a. The displacement is approximately $10(2) + 25(2) + 35(2) + 55(2) + 65(2) = 380$ ft.

b. The displacement is approximately $25(2)+35(2)+55(2)+65(2)+90(2)=540$ ft.

c. The displacement is approximately $20(2)+30(2)+45(2)+60(2)+75(2)=460$ ft.

5.1.7 Subdivide the interval from 0 to $\pi/2$ into subintervals. On each subinterval, pick a sample point (like the left endpoint, or the right endpoint, or the midpoint, for example) and call the first sample point x_1 and the second sample point x_2 and so on. For each sample point x_i, calculate the area of the rectangle which lies over the subinterval and has height $f(x_i)=\cos x_i$. Do this for each subinterval, and add the areas of the corresponding rectangles together. This will give an approximation to the area under the curve, with generally a better approximation occurring as n increases – where n is the number of subintervals used.

5.1.8 As the number of subintervals increases, the approximation to the area under the curve improves.

5.1.9 We have $\Delta x=\dfrac{7-1}{6}=1$. The left Riemann sum is given by $1(10+9+7+5+2+1)=34$ and the right Riemann sum is given by $1(9+7+5+2+1+0)=24$.

5.1.10 Using 3 subintervals, we have $\Delta x=\dfrac{6}{3}=2$. The left Riemann sum is $2(f(0)+f(2)+f(4))=2(1+6+9)=32$. The right Riemann sum is given by $2(f(2)+f(4)+f(6))=2(6+9+11)=52$. The midpoint Riemann sum is given by $2(f(1)+f(3)+f(5))=2(4+7+10)=42$.

5.1.11 Because the interval $[1,3]$ has length 2, if we subdivide it into 4 subintervals, each will have length $\Delta x=\dfrac{2}{4}=\dfrac{1}{2}$. The grid points will be $x_0=1$, $x_1=1+\Delta x=1.5$, $x_2=1+2\Delta x=2$, $x_3=1+3\Delta x=2.5$, and $x_4=1+4\Delta x=3$.

If we use the left-hand side of each subinterval, we will use 1, 1.5, 2, and 2.5.

If we use the right-hand side of each subinterval, we will use 1.5, 2, 2.5, and 3.

If we use the midpoint of each subinerval, we will use 1.25, 1.75, 2.25, and 2.75.

5.1.12

The left Riemann sum will be $\displaystyle\sum_{k=1}^{4} f(x_{k-1})\cdot 1=\sum_{k=1}^{4}(x_{k-1})^2$.

The right Riemann sum will be $\displaystyle\sum_{k=1}^{4} f(x_k)\cdot 1=\sum_{k=1}^{4}(x_k)^2$.

The midpoint Riemann sum will be $\displaystyle\sum_{k=1}^{4} f\left(\frac{x_{k-1}+x_k}{2}\right)\cdot 1=\sum_{k=1}^{4}\left(\frac{x_{k-1}+x_k}{2}\right)^2$.

5.1.13 It is an underestimate. If we use the right-hand side of each subinterval to determine the height of the rectangles, the height of each rectangle will be the minimum of f over the subinterval, so the sum of the areas of the rectangles will be less than the corresponding area under the curve.

5.1.14 It is an underestimate. If we use the left-hand side of each subinterval to determine the height of the rectangles, the height of each rectangle will be the minimum of f over the subinterval, so the sum of the areas of the rectangles will be less than the corresponding area under the curve.

5.1.15

a. On the first subinterval, the midpoint is 0.5, and $v(0.5)=1.75$. On the 2nd subinterval, the midpoint is 1.5 and $v(1.5)=7.75$. Continuing in this manner, we obtain the estimate to the displacement of

$$v(.5)\cdot 1+v(1.5)\cdot 1+v(2.5)\cdot 1+v(3.5)\cdot 1=1.75+7.75+19.75+37.75=67\,\text{ft}.$$

b. This time the midpoints are at 0.25, 0.75, 1.25 .... Each subinterval has length $\frac{1}{2}$. Thus, the estimate is given by

$$\begin{aligned}&v(0.25)\cdot 0.5+v(0.75)\cdot 0.5+v(1.25)\cdot 0.5+v(1.75)\cdot 0.5+v(2.25)\cdot 0.5+v(2.75)\cdot 0.5\\&+v(3.25)\cdot 0.5+v(3.75)\cdot 0.5\\&=0.5(1.1875+2.6875+5.6875+10.1875+16.1875+23.6875+32.6875+43.1875)\\&=0.5(135.5)=67.75\,\text{ft}..\end{aligned}$$

5.1.16

a. The midpoints are 2, 4, and 6. So the estimate is

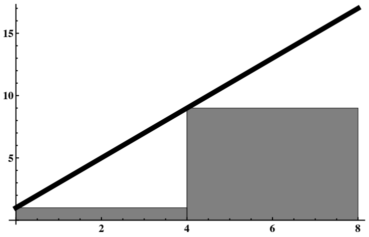
$$v(2)\cdot 2 + v(4)\cdot 2 + v(6)\cdot 2 \approx 37.085\,\text{ft}.$$

b. The midpoints are $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, $\frac{9}{2}$, $\frac{11}{2}$, and $\frac{13}{2}$. So the estimate is

$$v\left(\frac{3}{2}\right)+v\left(\frac{5}{2}\right)+v\left(\frac{7}{2}\right)+v\left(\frac{9}{2}\right)+v\left(\frac{11}{2}\right)+v\left(\frac{13}{2}\right)=\sqrt{15}+5+\sqrt{35}+3\sqrt{5}+\sqrt{55}+\sqrt{65}\approx 36.976\,\text{ft}.$$

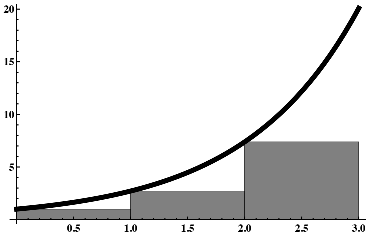
5.1.17

The left-hand grid points are 0 and 4. The length of each subinterval is $8/2 = 4$. So the left Riemann sum is given by $v(0)\cdot 4+v(4)\cdot 4 = 4\cdot(1+9) = 40$ m.



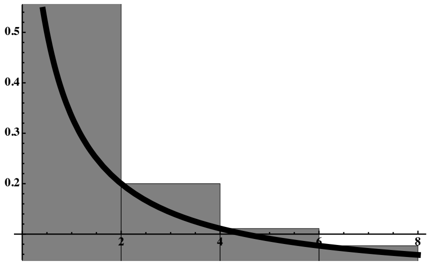
5.1.18

The left-hand grid points are 0, 1 and 2. The length of each subinterval is $3/3 = 1$. So the left Riemann sum is given by $v(0)\cdot 1 + v(1)\cdot 1 + v(2)\cdot 1 = 1 + e + e^2 \approx 11.1$ m.



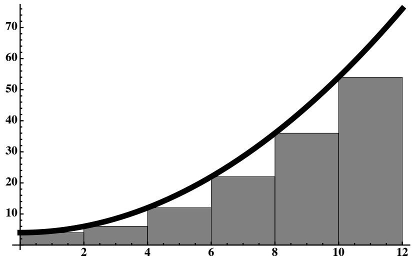
5.1.19

The left-hand grid points are 0, 2, 4, and 6. The length of each subinterval is 2. So the left Riemann sum is given by $v(0)\cdot 2+v(2)\cdot 2+v(4)\cdot 2 + v(6)\cdot 2 = 2\cdot\left(\frac{1}{1}+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}\right) \approx 2.776$ m.



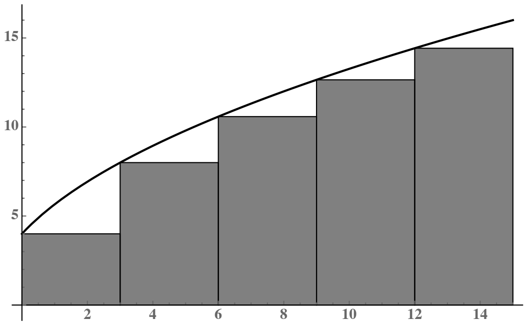
5.1.20

The left-hand grid points are 0, 2, 4, and 6, 8, and 10. The length of each subinterval is 2. So the left Riemann sum is given by $\sum_{k=1}^{6} v(2(k-1))\cdot 2 = 8+12+24+44+72+108 = 268$ ft.



5.1.21

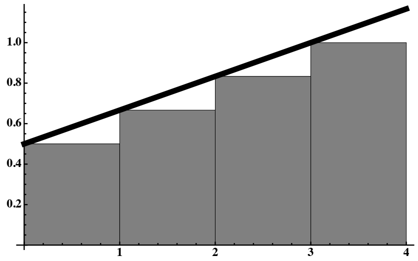
The left-hand grid points are 0, 3, 6, 9, and 12. The length of each subinterval is 3. So the left Riemann sum is given by $\sum_{k=1}^{5} v(3(k-1)) \cdot 3 = 12 + 24 + 12\sqrt{7} + 12\sqrt{10} + 12\sqrt{13} \approx 148.963$ mi.



5.1.22

The left-hand grid points are 0, 1, 2, and 3. The length of each subinterval is 1. So the left Riemann sum is given by

$$v(0)+v(1)+v(2)+v(3) = \frac{1}{2} + \frac{2}{3} + \frac{5}{6} + 1 = 3\,\text{m}.$$

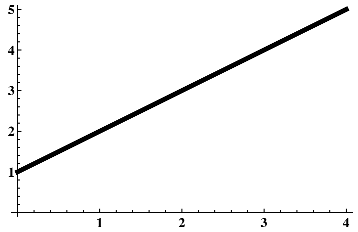


5.1.23 The left Riemann sum is given by $f(1) + f(2) + f(3) + f(4) + f(5) = 2 + 3 + 4 + 5 + 6 = 20$. The right Riemann sum is given by $f(2) + f(3) + f(4) + f(5) + f(6) = 3 + 4 + 5 + 6 + 7 = 25$.

5.1.24 The left Riemann sum is given by $f(1) + f(2) + f(3) + f(4) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$.

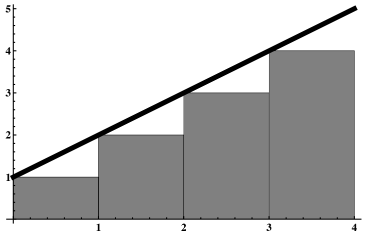
The right Riemann sum is given by $f(2) + f(3) + f(4) + f(5) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$.

5.1.25

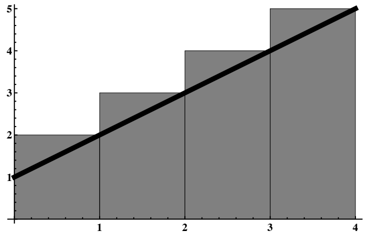


a.

b. We have $\Delta x = \frac{4-0}{4} = 1$. The grid points are $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$.



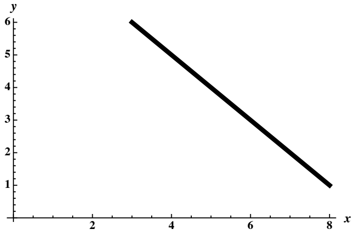
c.



d. The left Riemann sum is $1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 1 = 10$, which is an underestimate of the area under the curve. The right Riemann sum is $2 \cdot 1 + 3 \cdot 1 + 4 \cdot 1 + 5 \cdot 1 = 14$ which is an overestimate of the area under the curve.

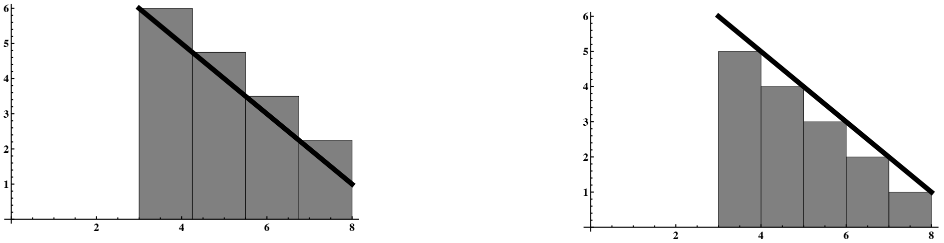
5.1.26

a.



b. We have $\Delta x = \dfrac{8-3}{5} = 1$. The grid points are $x_0 = 3$, $x_1 = 4$, $x_2 = 5$, $x_3 = 6$, $x_4 = 7$, and $x_5 = 8$.

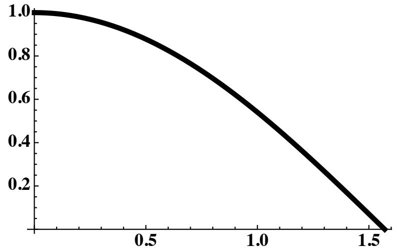
c.



d. The left Riemann sum is $6 \cdot 1 + 5 \cdot 1 + 4 \cdot 1 + 3 \cdot 1 + 2 \cdot 1 = 20$, which is an overestimate of the area under the curve. The right Riemann sum is $5 \cdot 1 + 4 \cdot 1 + 3 \cdot 1 + 2 \cdot 1 + 1 \cdot 1 = 15$ which is an underestimate of the area under the curve.

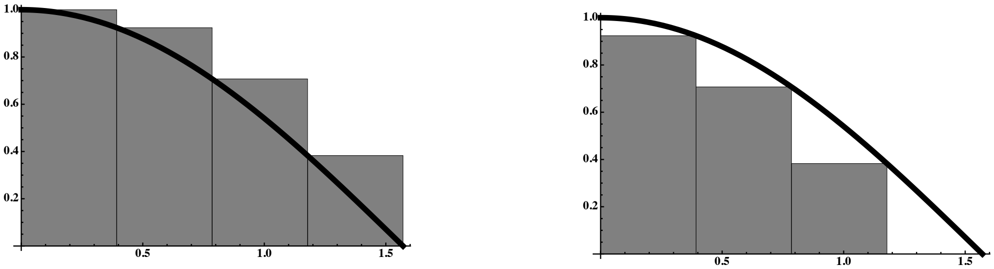
5.1.27

a.



b. We have $\Delta x = \dfrac{\pi/2 - 0}{4} = \dfrac{\pi}{8}$. The grid points are $x_0 = 0$, $x_1 = \dfrac{\pi}{8}$, $x_2 = \dfrac{\pi}{4}$, $x_3 = \dfrac{3\pi}{8}$, and $x_4 = \dfrac{\pi}{2}$.

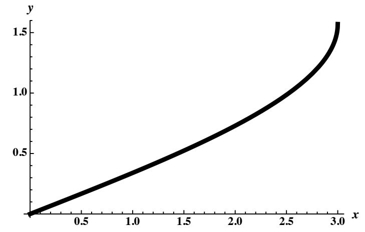
c.



d. The left Riemann sum is $1 \cdot \dfrac{\pi}{8} + \cos(\pi/8) \cdot \dfrac{\pi}{8} + \cos(\pi/4) \cdot \dfrac{\pi}{8} + \cos(3\pi/8) \cdot \dfrac{\pi}{8} \approx 1.185$, which is an overestimate of the area under the curve. The right Riemann sum is $\cos(\pi/8) \cdot \dfrac{\pi}{8} + \cos(\pi/4) \cdot \dfrac{\pi}{8} + \cos(3\pi/8) \cdot \dfrac{\pi}{8} + 0 \cdot \dfrac{\pi}{8} \approx$ 0.791 which is an underestimate of the area under the curve.

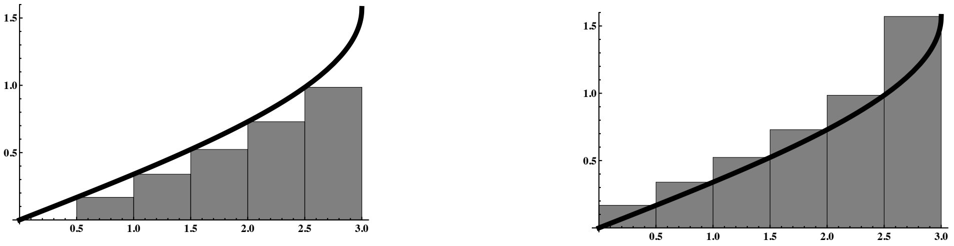
5.1.28

a.



b. We have $\Delta x = \dfrac{3-0}{6} = \dfrac{1}{2}$. The grid points are $x_0 = 0$, $x_1 = \dfrac{1}{2}$, $x_2 = 1$, $x_3 = \dfrac{3}{2}$, $x_4 = 2$, $x_5 = \dfrac{5}{2}$, and $x_6 = 3$.

c.



d. The left Riemann sum is

$$\frac{1}{2}\left(0 + \sin^{-1}(1/6) + \sin^{-1}(1/3) + \sin^{-1}(1/2) + \sin^{-1}(2/3) + \sin^{-1}(5/6)\right) \approx 1.373,$$

which is an underestimate of the area under the curve.

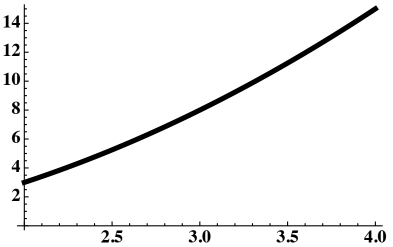
The right Riemann sum is

$$\frac{1}{2}\left(\sin^{-1}(1/6) + \sin^{-1}(1/3) + \sin^{-1}(1/2) + \sin^{-1}(2/3) + \sin^{-1}(5/6) + \sin^{-1} 1\right) \approx 2.158,$$

which is an overestimate of the area under the curve.

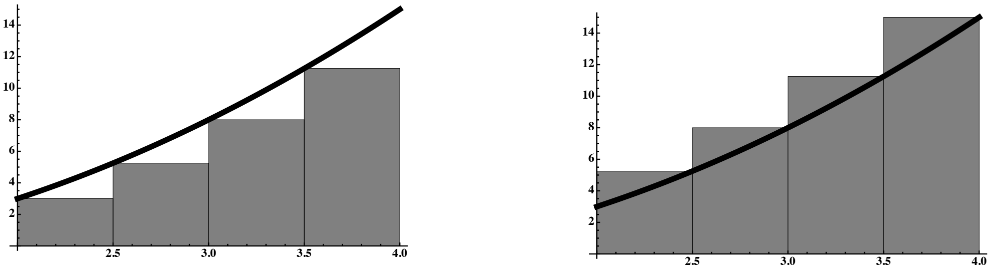
5.1.29

a.



b. We have $\Delta x = \dfrac{4-2}{4} = \dfrac{1}{2}$. The grid points are $x_0 = 2$, $x_1 = 2.5$, $x_2 = 3$, $x_3 = 3.5$, and $x_4 = 4$.

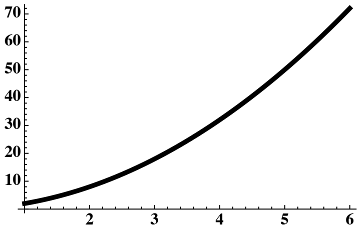
c.



d. The left Riemann sum is $(3+5.25+8+11.25)\cdot 0.5 = 13.75$, which is an underestimate of the area under the curve. The right Riemann sum is $(5.25+8+11.25+15)\cdot 0.5 = 19.75$ which is an overestimate of the area under the curve.

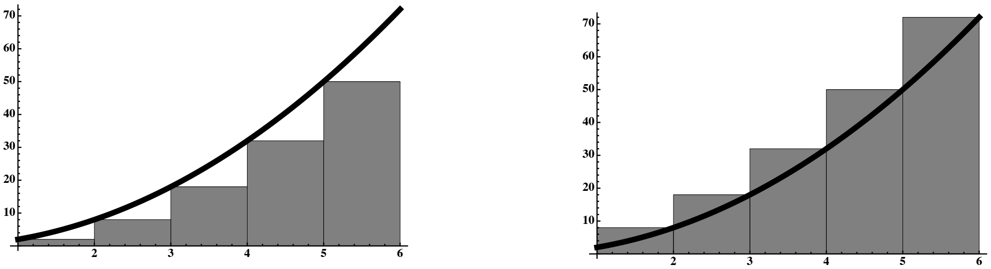
5.1.30

a.



b. We have $\Delta x = \dfrac{6-1}{5} = 1$. The grid points are $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 5$, and $x_5 = 6$.

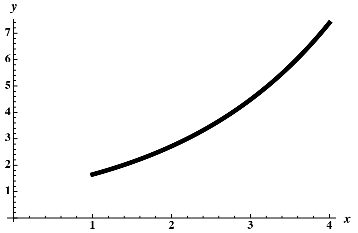
c.



d. The left Riemann sum is $2 + 8 + 18 + 32 + 50 = 110$, which is an underestimate of the area under the curve. The right Riemann sum is $8 + 18 + 32 + 50 + 72 = 180$ which is an overestimate of the area under the curve.

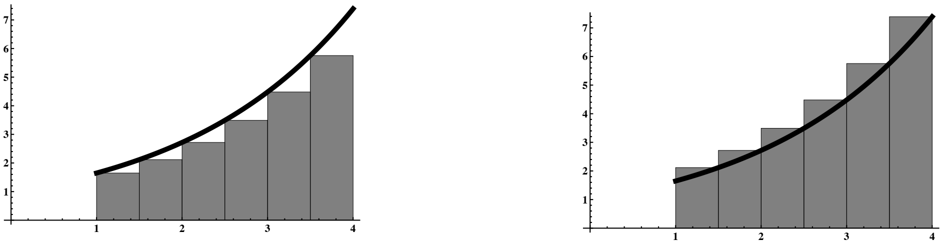
5.1.31

a.



b. We have $\Delta x = \dfrac{4-1}{6} = \dfrac{1}{2}$. The grid points are $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3$, $x_5 = 3.5$, and $x_6 = 4$.

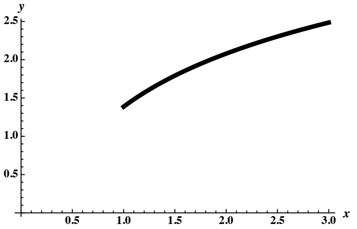
c.



d. The left Riemann sum is $\frac{1}{2}\left(e^{1/2} + e^{3/4} + e + e^{5/4} + e^{3/2} + e^{7/4}\right) \approx 10.105$, which is an underestimate of the area under the curve. The right Riemann sum is $\dfrac{1}{2}\left(e^{3/4} + e + e^{5/4} + e^{3/2} + e^{7/4} + e^{2}\right) \approx 12.975$ which is an overestimate of the area under the curve.

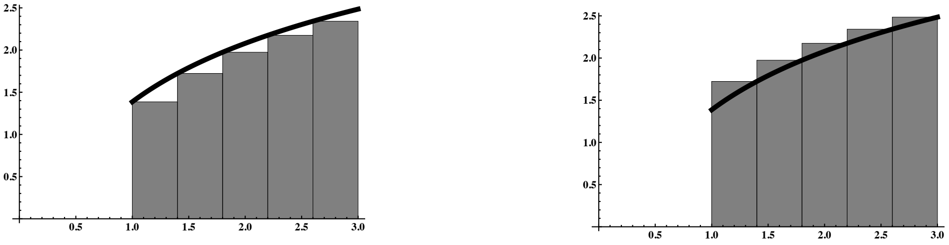
5.1.32

a.



b. We have $\Delta x = \dfrac{3-1}{5} = \dfrac{2}{5} = 0.4$. The grid points are $x_0 = 1$, $x_1 = 1.4$, $x_2 = 1.8$, $x_3 = 2.2$, $x_4 = 2.6$, and $x_5 = 3$.

c.



d. The left Riemann sum is

$$.4\,(\ln 4 + \ln 5.6 + \ln 7.2 + \ln 8.8 + \ln 10.4) \approx 3.840,$$

which is an underestimate of the area under the curve. The right Riemann sum is

$$0.4\,(\ln 5.6 + \ln 7.2 + \ln 8.8 + \ln 10.4 + \ln 12) \approx 4.279,$$

which is an overestimate of the area under the curve.

5.1.33 We have $\Delta x = 2$, so the midpoints are 1, 3, 5, 7, and 9. So the midpoint Riemann sum is $2(f(1) + f(3) + f(5) + f(7) + f(9)) = 670$.

5.1.34 We have $\Delta x = \pi/4$, so the midpoints are $\pi/8$, $3\pi/8$, $5\pi/8$, and $7\pi/8$. So the midpoint Riemann sum is $\frac{\pi}{4} \cdot (f(\pi/8) + f(3\pi/8) + f(5\pi/8) + f(7\pi/8)) \approx 2.013$.

5.1.35

a. The left Riemann sum is $315(6) + 350(6) + 365(6) + 370(6) + 350(6) = 10,500$ m. The right Riemann sum is $350(6) + 365(6) + 370(6) + 350(6) + 290(6) = 10,350$ m. Answers will vary.

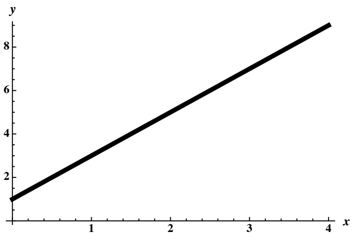
b. The left Riemann sum is more accurate.

c. More accurate approximations are obtained by increasing the number of subintervals.

5.1.36 Felix fell approximately $\dfrac{1}{2}(20)(200) = 2000$ m.

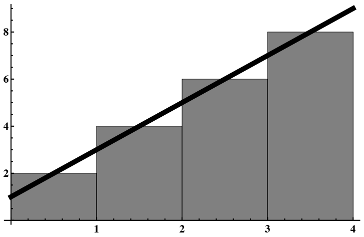
5.1.37

a.



b. We have $\Delta x = \dfrac{4-0}{4} = 1$. The gridpoints are $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$, so the midpoints are .5, 1.5, 2.5, and 3.5.

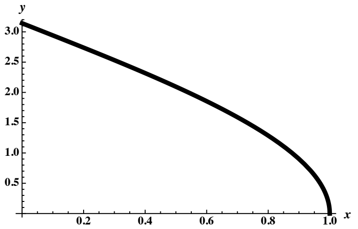
c.



d. The midpoint Riemann sum is $1(2+4+6+8) = 20$.

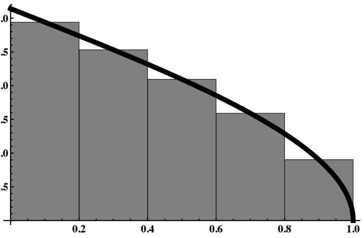
5.1.38

a.



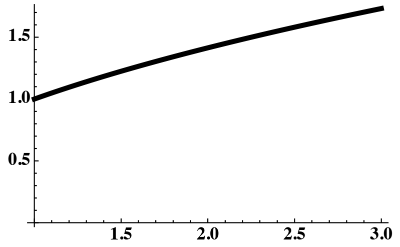
b. We have $\Delta x = \dfrac{1-0}{5} = 0.2$. The gridpoints are $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, and $x_5 = 1$, so the midpoints are 0.1, 0.3, 0.5, 0.7, and 0.9.

c.



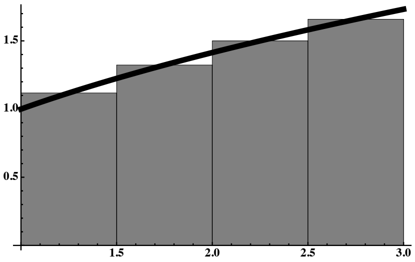
d. The midpoint Riemann sum is $.4(\cos^{-1} 0.1 + \cos^{-1} 0.3 + \cos^{-1} 0.5 + \cos^{-1} 0.7 + \cos^{-1} 0.9) \approx 2.012$.

5.1.39



a.

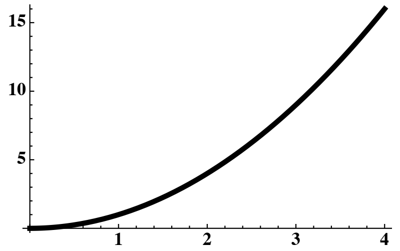
b. We have $\Delta x = \dfrac{3-1}{4} = \dfrac{1}{2}$. So the midpoints are 1.25, 1.75, 2.25, and 2.75.



c.

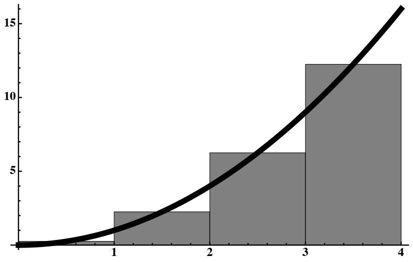
d. The midpoint Riemann sum is $.5(\sqrt{1.25}+\sqrt{1.75}+\sqrt{2.25}+\sqrt{2.75}) \approx 2.800$.

5.1.40



a.

b. We have $\Delta x = \dfrac{4-0}{4} = 1$. So the midpoints are 0.5, 1.5, 2.5, and 3.5.

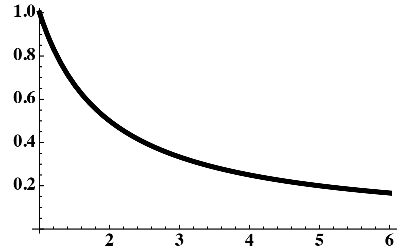


c.

d. The midpoint Riemann sum is $0.5^2+1.5^2+2.5^2+3.5^2 = 21$.

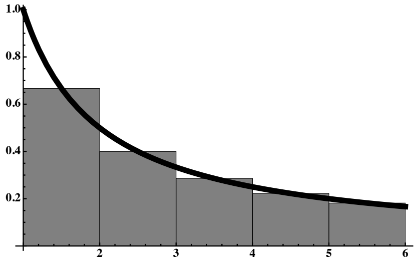
5.1.41

a.



b. We have $\Delta x = \dfrac{6-1}{5} = 1$. So the midpoints are 1.5, 2.5, 3.5, 4.5, and 5.5.

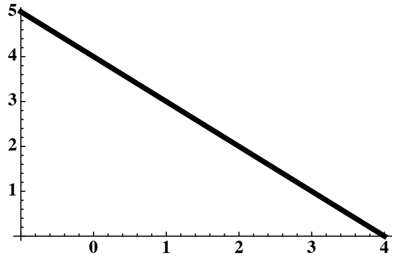
c.



d. The midpoint Riemann sum is $2/3 + 2/5 + 2/7 + 2/9 + 2/11 \approx 1.756$.

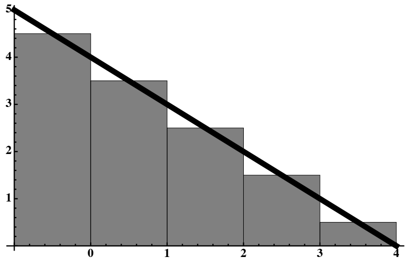
5.1.42

a.



b. We have $\Delta x = \dfrac{4-(-1)}{5} = 1$. So the midpoints are -0.5, 0.5, 1.5, 2.5, and 3.5.

c.



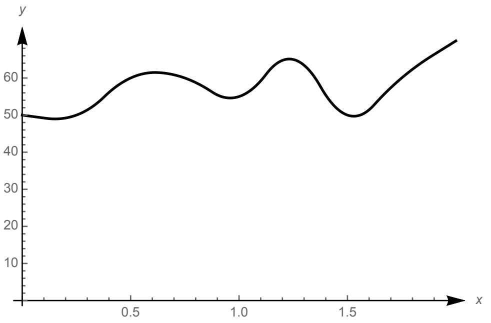
d. The midpoint Riemann sum is $(4-(-0.5)) + (4-0.5) + (4-1.5) + (4-2.5) + (4-3.5) = 20 - 7.5 = 12.5$.

5.1.43 Note that $\Delta x = \dfrac{2-0}{4} = .5$. So the left Riemann sum is given by $0.5(5+3+2+1) = 5.5$ and the right Riemann sum is given by $0.5(3+2+1+1) = 3.5$.

5.1.44 Note that $\Delta x = \dfrac{5-1}{8} = 0.5$. So the left Riemann sum is given by $0.5(0+2+3+2+2+1+0+2) = 6$ and the right Riemann sum is given by $0.5(2+3+2+2+1+0+2+3) = 7.5$.

5.1.45

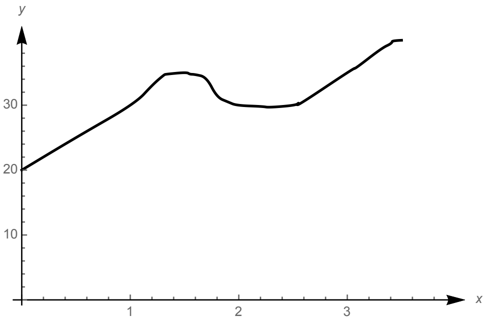
a.



b. With $n = 2$, we have $\Delta x = \dfrac{2-0}{2} = 1$, so the midpoints are .5 and 1.5. So the midpoint Riemann sum is $60 + 50 = 110$. For $n = 4$, we have $\Delta x = \dfrac{2-0}{4} = \dfrac{1}{2}$, so the midpoints are 0.25, 0.75, 1.25, and 1.75. The midpoint Riemann sum in this case is $0.5(50 + 60 + 65 + 60) = \dfrac{235}{2} = 117.5$.

5.1.46

a.



b. With $n = 2$, we have $\Delta x = \dfrac{4-0}{2} = 2$, so the midpoints are 1 and 3. So the midpoint Riemann sum is $2(30 + 35) = 130$. For $n = 4$, we have $\Delta x = \dfrac{4-0}{4} = 1$, so the midpoints are 0.5, 1.5, 2.5, and 3.5. The midpoint Riemann sum in this case is $25 + 35 + 30 + 40 = 130$.

5.1.47

a. $\displaystyle\sum_{k=1}^{5} k.$ b. $\displaystyle\sum_{k=1}^{6}(k+3).$ c. $\displaystyle\sum_{k=1}^{4} k^2.$ d. $\displaystyle\sum_{k=1}^{4}\frac{1}{k}.$

5.1.48

a. $\displaystyle\sum_{k=1}^{50}(2k-1).$ b. $\displaystyle\sum_{k=1}^{9}(5k-1).$ c. $\displaystyle\sum_{k=1}^{13}(5k-2).$ d. $\displaystyle\sum_{k=1}^{49}\frac{1}{k(k+1)}.$

5.1.49

a. $\displaystyle\sum_{k=1}^{10} k = 1 + 2 + 3 + \ldots + 10 = 55.$

b. $\displaystyle\sum_{k=1}^{6}(2k+1) = 3 + 5 + 7 + 9 + 11 + 13 = 48.$

c. $\displaystyle\sum_{k=1}^{4} k^2 = 1 + 4 + 9 + 16 = 30.$

d. $\displaystyle\sum_{n=1}^{5}(1+n^2) = 2 + 5 + 10 + 17 + 26 = 60.$

e. $\displaystyle\sum_{m=1}^{3}\frac{2m+2}{3} = \frac{4}{3} + \frac{6}{3} + \frac{8}{3} = 6.$

f. $\displaystyle\sum_{j=1}^{3}(3j-4) = -1 + 2 + 5 = 6.$

g. $\displaystyle\sum_{p=1}^{5}(2p+p^2) = 3 + 8 + 15 + 24 + 35 = 85.$

h. $\displaystyle\sum_{n=0}^{4}\sin\frac{n\pi}{2} = 0 + 1 + 0 + (-1) + 0 = 0.$

5.1.50

a. $\displaystyle\sum_{k=1}^{45} k = \frac{45 \cdot 46}{2} = 45 \cdot 23 = 1035.$

b. $\displaystyle\sum_{k=1}^{45}(5k-1) = 5\sum_{k=1}^{45} k - \sum_{k=1}^{45} 1 = 5 \cdot 1035 - 45 = 5130.$

c. $\displaystyle\sum_{k=1}^{75} 2k^2 = 2\sum_{k=1}^{75} k^2 = 2 \cdot \frac{75 \cdot 76 \cdot 151}{6} = 286,900.$

d. $\displaystyle\sum_{n=1}^{50}(1+n^2) = \sum_{n=1}^{50} 1 + \sum_{n=1}^{50} n^2 = 50 + \frac{50 \cdot 51 \cdot 101}{6} = 50 + 42925 = 42975.$

e. $\displaystyle\sum_{m=1}^{75} \frac{2m+2}{3} = \frac{2}{3}\sum_{m=1}^{75} m + \frac{2}{3}\sum_{m=1}^{75} 1 = \frac{2}{3} \cdot \frac{75 \cdot 76}{2} + \frac{2}{3} \cdot 75 = 1900 + 50 = 1950.$

f. $\displaystyle\sum_{j=1}^{20}(3j-4) = 3\sum_{j=1}^{20} j - \sum_{j=1}^{20} 4 = 3 \cdot \frac{20 \cdot 21}{2} - 20 \cdot 4 = 550.$

g. $\displaystyle\sum_{p=1}^{35}(2p+p^2) = 2\sum_{p=1}^{35} p + \sum_{p=1}^{35} p^2 = 2 \cdot \frac{35 \cdot 36}{2} + \frac{35 \cdot 36 \cdot 71}{6} = 16170.$

h. $\displaystyle\sum_{n=0}^{40}(n^2+3n-1) = \sum_{n=0}^{40} n^2 + \sum_{n=0}^{40} 3n - \sum_{n=0}^{40} 1 = \frac{40 \cdot 41 \cdot 81}{6} + 3 \cdot \frac{40 \cdot 41}{2} - 41 = 24559.$

5.1.51 Note that $\Delta x = \dfrac{1}{10}$, and $x_k = a + k\Delta x = \dfrac{k}{10}$.

a. The left Riemann sum is given by $\displaystyle\sum_{k=0}^{39} 3\sqrt{k/10} \cdot (1/10) \approx 15.6809.$

The right Riemann sum is given by $\displaystyle\sum_{k=1}^{40} 3\sqrt{k/10} \cdot (1/10) \approx 16.2809.$

The midpoint Riemann sum is given by $\displaystyle\sum_{k=0}^{39} 3\sqrt{(1/20)+(k/10)} \cdot (1/10) \approx 16.0055.$

b. It appears that the actual area is about 16.

5.1.52 Note that $\Delta x = \dfrac{1}{25}$, and $x_k = -1 + k\Delta x = -1 + (k/25)$. So $f(x_k) = (-1+(k/25))^2 + 1$.

a. The left Riemann sum is given by $\displaystyle\sum_{k=0}^{49}[(-1+(k/25))^2+1] \cdot (1/25) \approx 2.667.$

The right Riemann sum is given by $\displaystyle\sum_{k=1}^{50}[(-1+(k/25))^2+1] \cdot (1/25) \approx 2.667.$

The midpoint Riemann sum is given by $\displaystyle\sum_{k=0}^{49}[(-(49/50)+(k/25))^2+1] \cdot (1/25) \approx 2.666.$

b. It appears that the actual area is about $2 + 2/3$.

5.1.53 Note that $\Delta x = \frac{1}{25}$, and $x_k = 2 + k\Delta x = 2 + (k/25)$. So $f(x_k) = (2 + (k/25))^2 - 1$.

a. The left Riemann sum is given by $\sum_{k=0}^{74}[(2+(k/25))^2 - 1] \cdot (1/25) \approx 35.5808$.

The right Riemann sum is given by $\sum_{k=1}^{75}[(2+(k/25))^2 - 1] \cdot (1/25) \approx 36.2408$.

The midpoint Riemann sum is given by $\sum_{k=0}^{74}[((61/30)+(k/25))^2 - 1] \cdot (1/25) \approx 35.9996$.

b. It appears that the actual area is about 36.

5.1.54 Note that $\Delta x = \frac{\pi}{240}$, and $x_k = k\Delta x = (k\pi/240)$. So $f(x_k) = \cos(k\pi/120)$.

a. The left Riemann sum is given by $\sum_{k=0}^{59}\cos(k\pi/120) \cdot (\pi/240) \approx 0.507$.

The right Riemann sum is given by $\sum_{k=1}^{60}\cos(k\pi/120) \cdot (\pi/240) \approx 0.493$.

The midpoint Riemann sum is given by $\sum_{k=0}^{59}\cos((\pi/240) + k\pi/120) \cdot (\pi/240) \approx 0.500$.

b. It appears that the actual area is about 0.5.

5.1.55 $\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}$. $x_k = -1 + k\Delta x = -1 + \frac{2k}{n}$. The right Riemann sum is given by

$$\sum_{k=1}^{n} f(x_k)\Delta x = \frac{2}{n}\sum_{k=1}^{n}\left(12 - 3\left(-1+\frac{2k}{n}\right)^2\right).$$

n	Right Riemann Sum
10	21.96
30	21.9956
60	21.9989
80	21.9994

The sum approaches 22.

5.1.56 $\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}$. $x_k = -1 + k\Delta x = -1 + \frac{2k}{n}$. The right Riemann sum is given by

$$\sum_{k=1}^{n} f(x_k)\Delta x = \frac{2}{n}\sum_{k=1}^{n}\left(3\left(-1+\frac{2k}{n}\right)^2 + 1\right).$$

n	Right Riemann Sum
10	4.04
30	4.00444
60	4.00111
80	4.00063

The sum approaches 4.

5.1.57 $\Delta x = \frac{\pi - (-\pi)}{n} = \frac{2\pi}{n}$. $x_k = -\pi + k\Delta x = -\pi + \frac{2\pi k}{n}$. The right Riemann sum is given by

$$\sum_{k=1}^{n} f(x_k)\Delta x = \frac{2\pi}{n}\sum_{k=1}^{n}\left(\frac{1-\cos\left(-\pi + \frac{2\pi k}{n}\right)}{2}\right).$$

n	Right Riemann Sum
10	3.14159
30	3.14159
60	3.14159
80	3.14159

The sum approaches π.

5.1.58 $\Delta x = \frac{2-(-2)}{n} = \frac{4}{n}$. $x_k = -2 + k\Delta x = -2 + \frac{4k}{n}$. The right Riemann sum is given by

$$\sum_{k=1}^{n} f(x_k)\Delta x = \frac{4}{n}\ln 2\sum_{k=1}^{n}\left(2^{-2+\frac{4k}{n}} + 2^{-(-2+\frac{4k}{n})}\right).$$

n	Right Riemann Sum
10	7.54798
30	7.50534
60	7.50133
80	7.50075

The sum approaches 7.5.

5.1.59

a. True. Because the curve is a straight line, the region under the curve and over each subinterval is a trapezoid. The formula for the area of such a trapezoid over $[x_i, x_{i+1}]$ is $\frac{f(x_i) + f(x_{i+1})}{2} \cdot \Delta x = \frac{2x_i + 5 + 2x_{i+1} + 5}{2} \cdot \Delta x = (x_i + x_{i+1} + 5)\Delta x$ and the area given by using the midpoint formula is $f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x = (x_i + x_{i+1} + 5)\Delta x$. So the area under the curve is exactly given by the midpoint Riemann sum. Note that this holds for any straight line.

b. False. The left Riemann sum will underestimate the area under an increasing function.

c. True. The value of f at the midpoint will always be between the value of f at the endpoints, if f is monotonic increasing or monotonic decreasing.

5.1.60

a. Note that if $y = \sqrt{1-x^2}$, then $y^2 = 1 - x^2$, so $x^2 + y^2 = 1$, which represents a circle of radius one. Note that for the original function $y > 0$ for all x, so this represents the top semicircle.

b. We have $\sum_{k=1}^{25} f(x_k^*)\frac{2}{25}$ where x_k^* represents the midpoint of the kth subinterval. This sum is

$$\sum_{k=0}^{24}\sqrt{1-\left(-.96+\frac{2k}{25}\right)^2}\cdot\frac{2}{25} \approx 1.575.$$

c. We have $\sum_{k=1}^{75} f(x_k^*)\frac{2}{75}$ where x_k^* represents the midpoint of the kth subinterval. This sum is

$$\sum_{k=0}^{74}\sqrt{1-\left((-98\overline{6})+\frac{2k}{75}\right)^2}\cdot\frac{2}{75}\approx 1.572.$$

d. The sums approach $\pi/2$ as $n \to \infty$. Note that this confirms what we already know about the area of half a circle of radius one.

5.1.61 $\sum_{k=1}^{50}\left(\frac{2k}{25}+1\right)\cdot\frac{2}{25}=12.16.$

5.1.62 $\sum_{k=0}^{39} e^{k(\ln 2)/40}\cdot\frac{\ln 2}{40}\approx 0.991.$

5.1.63 $\sum_{k=0}^{31}\left(3+\frac{1}{8}+\frac{k}{4}\right)^3\cdot\frac{1}{4}\approx 3639.125.$

5.1.64 $\sum_{k=0}^{49}\left[1+\cos\left(\pi\left(\frac{1}{50}+\frac{k}{25}\right)\right)\right]\cdot\frac{1}{25}=2.$

5.1.65 This is the right Riemann sum for f on the interval $[1,5]$ for $n=4$.

5.1.66 This is the right Riemann sum for f on the interval $[2,6]$ for $n=4$.

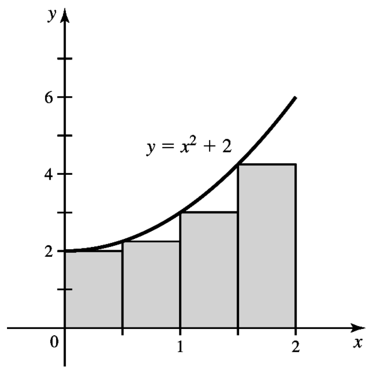
5.1.67 This is the midpoint Riemann sum for f on the interval $[2,6]$ for $n=4$.

5.1.68 This is the right Riemann sum for f on the interval $[1.5,5.5]$ for $n=8$.

5.1.69 For all of the calculations below, we have $\Delta x=\frac{1}{2}$, and grid points $x_0=0$, $x_1=0.5$, $x_2=1$, $x_3=2.5$, and $x_4=2$.

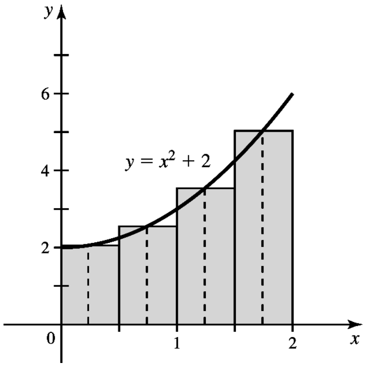
a.

The left Riemann sum is given by $\frac{1}{2}\left(f(0)+f(0.5)+f(1)+f(1.5)\right)$ which is equal to $\frac{1}{2}\left(2+2.25+3+4.25\right)=5.75$.



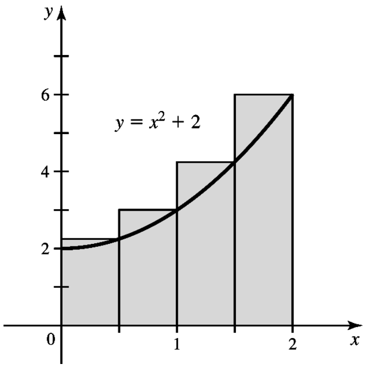
b.

The midpoint Riemann sum is given by $\frac{1}{2}(f(.25)+f(.75)+f(1.25)+f(1.75))$ which is equal to $\frac{1}{2}(2.0625+2.5625+3.5625+5.0625)=6.625$.



c.

The right Riemann sum is given by $\frac{1}{2}(f(0.5)+f(1)+f(1.5)+f(2))$ which is equal to $\frac{1}{2}(2.25+3+4.25+6)=7.75$.



5.1.70

a. The object's velocity decreases during the first second, then remains constant between time $t=1$ and $t=3$, and then steadily increases until $t=4$, and then stays constant after that.

b. The displacement is given by the area under the curve, which between $t=0$ and $t=3$ is 35, so the displacement is 35 meters.

c. Between $t=3$ and $t=5$ the area under the curve is 50, so the displacement is 50 meters.

d. Between $t=0$ and $t=4$ the displacement is 55, and between 4 and t for $t>4$, the displacement is $30(t-4)$. So the displacement between 0 and t for $t>4$ is $55+30(t-4)$.

5.1.71

a. During the first second, the velocity steadily increases from 0 to 20, then it remains constant until $t=3$. From $t=3$ until $t=5$ it steadily decreases, and then remains constant until $t=6$.

b. Between $t=0$ and $t=2$ the area under the curve is $\frac{1}{2}\cdot 1\cdot 20+1\cdot 20=30$.

c. Between $t=2$ and $t=5$ the displacement is the sum of the area of a rectangle with area 20 and a trapezoid with area 30, so the displacement is 50 meters.

d. Between $t=0$ and $t=5$ the displacement is 80. Between $t=5$ and any time $t\geq 5$ the displacement is $10(t-5)$ so the displacement between $t=0$ and $t\geq 5$ is $80+10(t-5)$.

5.1.72

a. Between 0 and 4, the area under the curve is given by $\frac{1}{2} \cdot 4000 \cdot 4 = 8000$ cubic feet.

b. Between 8 and 10, the area under the curve is given by $2 \cdot 5000 = 10,000$ cubic feet.

c. Between 4 and 6 the amount is 9500 cubic feet, which is more than between 0 and 4.

d. When we multiply $\text{ft}^3/\text{hr} \cdot \text{hr}$ the result is ft^3.

5.1.73

a. Between 0 and 5, the area under the curve is given by the area of a square of area 4 and the area of a trapezoid of area 10.5, so the total area is 14.5.

b. Between 5 and 10, the area under the curve is given by the area of a trapezoid of area 5.5 and the area of a rectangle of area $4 \cdot 6 = 24$, so the total area is 29.5.

c. The mass of the entire rod would be the total area under the curve from 0 to 10, which would be $14.5 + 29.5 = 44$ grams.

d. At $x = \dfrac{19}{3}$ there is a mass of 22 on each side. Note that from 0 to 6 the mass is 20 grams, so the center of mass is a little greater than 6.

5.1.74 If $0 \le t \le 1.5$, the displacement is $40t$. If $1.5 \le t \le 3$, the displacement is $60 + 50(t - 1.5)$.

$$\text{Thus, } d(t) = \begin{cases} 40t & \text{if } 0 \le t \le 1.5, \\ 50t - 15 & \text{if } 1.5 \le t \le 3. \end{cases}$$

5.1.75 If $0 \le t \le 2$, the displacement is $30t$. If $2 \le t \le 2.5$, the displacement is $60 + 50(t - 2)$. If $2.5 \le t \le 3$, the displacement is $85 + 44(t - 2.5)$.

$$\text{Thus, } d(t) = \begin{cases} 30t & \text{if } 0 \le t \le 2, \\ 50t - 40 & \text{if } 2 \le t \le 2.5 \\ 44t - 25 & \text{if } 2.5 \le t \le 3. \end{cases}$$

5.1.76 Using the left Riemann sum

$$\sum_{k=0}^{n-1} \left| 25 - \left(\frac{10k}{n}\right)^2 \right| \cdot \frac{10}{n},$$

we have

n	16	32	64
A_n	234.375	242.188	246.094

It appears that the areas are approaching 250.

5.1.77 Using the left Riemann sum

$$\sum_{k=0}^{n-1} \left| 1 - \left(-1 + \frac{3k}{n}\right)^3 \right| \cdot \frac{3}{n},$$

we have

n	16	32	64
A_n	4.33054	4.52814	4.63592

It appears that the areas are approaching 4.75.

5.1.78 Because the function f is constant, its value is c at each grid point. Thus the left Riemann sum is

$$\sum_{k=0}^{n-1} f(x_k)\cdot\frac{b-a}{n} = \sum_{k=0}^{n-1} c\cdot\frac{b-a}{n} = \frac{c(b-a)}{n}\cdot\sum_{k=0}^{n-1} 1 = \frac{c(b-a)}{n}\cdot n = c(b-a).$$

For the right Riemann sum we have

$$\sum_{k=1}^{n} f(x_k)\cdot\frac{b-a}{n} = \sum_{k=1}^{n} c\cdot\frac{b-a}{n} = \frac{c(b-a)}{n}\cdot\sum_{k=1}^{n} 1 = \frac{c(b-a)}{n}\cdot n = c(b-a).$$

For the midpoint Riemann sum we have

$$\sum_{k=0}^{n-1} f\left(a+\frac{b-a}{2n}+\frac{(b-a)k}{n}\right)\cdot\frac{b-a}{n} = \sum_{k=0}^{n-1} c\cdot\frac{b-a}{n} = \frac{c(b-a)}{n}\cdot\sum_{k=0}^{n-1} 1 = \frac{c(b-a)}{n}\cdot n = c(b-a).$$

So all three rules give the exact area of $(b-a)\cdot c$.

5.1.79 The midpoint Riemann sum gives

$$\sum_{k=0}^{n-1} f\left(a+\frac{b-a}{2n}+\frac{k(b-a)}{n}\right)\cdot\frac{b-a}{n} = \sum_{k=0}^{n-1}\left(m\left(a+\frac{b-a}{2n}+\frac{k(b-a)}{n}\right)+c\right)\cdot\frac{b-a}{n} =$$

$$m\cdot a\cdot n\cdot\frac{b-a}{n}+\frac{m(b-a)^2n}{2n^2}+\frac{(n-1)n}{2}\cdot\frac{m(b-a)^2}{n^2}+\frac{cn(b-a)}{n} =$$

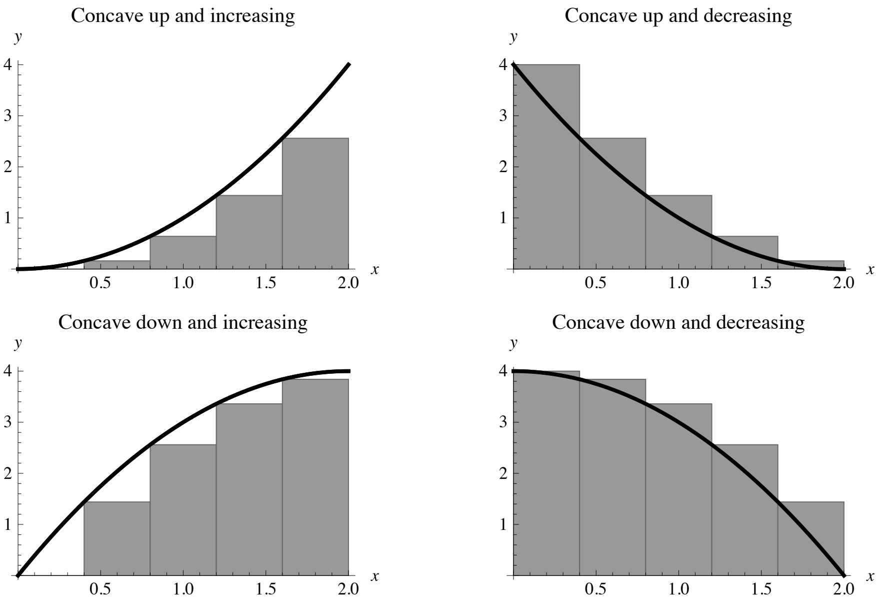
$$m\cdot a\cdot(b-a)+\frac{m(b-a)^2}{2n}+\frac{m(b-a)^2}{2}-\frac{m(b-a)^2}{2n}+c(b-a) =$$

$$m\cdot a\cdot(b-a)+\frac{m(b-a)^2}{2}+c(b-a) = (b-a)\cdot\left(\frac{m(a+b)}{2}+c\right).$$

This proves that the midpoint Riemann sum is independent of n. Because the region in question is a trapezoid, we know that the exact area is given by the width of the subinterval times the average value at the endpoints, which is

$$(b-a)\left(\frac{f(a)+f(b)}{2}\right) = (b-a)\left(\frac{ma+c+mb+c}{2}\right) = (b-a)\left(\frac{m(a+b)}{2}+c\right).$$

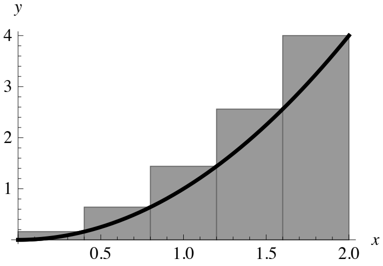
5.1.80 For a function that is concave up and increasing, each rectangle of the left Riemann sum will lie wholly below the curve, since the value of the function at the left edge of the rectangle will be smaller than at any other point in the rectangle. Thus this will be an underestimate. For a function that is concave up and decreasing, however, each rectangle will have its top edge above the curve, since the value of the function at the left edge will be larger than at any other point in the rectangle. Thus this will be an overestimate. For a function that is concave down and increasing, each rectangle of the left Riemann sum will lie wholly below the curve, since the value of the function at the left edge of the rectangle will be smaller than at any other point in the rectangle. Thus this will be an underestimate. Finally, for a function that is concave down and decreasing, each rectangle will have its top edge above the curve, since the value of the function at the left edge will be larger than at any other point in the rectangle. Thus this will be an overestimate. Graphs of each of the four situations are below:



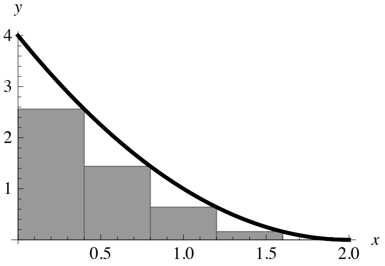
So the answer is

	Increasing on $[a, b]$	Decreasing on $[a, b]$
Concave up on $[a, b]$	Underestimate	Overestimate
Concave down on $[a, b]$	Underestimate	Overestimate

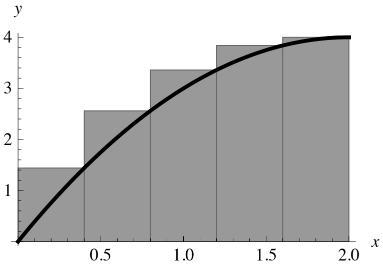
5.1.81 For a function that is concave up and increasing, each rectangle of the right Riemann sum will have its top edge above the curve, since the value of the function at the right edge of the rectangle will be larger than at any other point in the rectangle. Thus this will be an overestimate. For a function that is concave up and decreasing, however, each rectangle will lie wholly below the curve, since the value of the function at the right edge will be smaller than at any other point in the rectangle. Thus this will be an underestimate. For a function that is concave down and increasing, each rectangle of the right Riemann sum will have its top edge above the curve, since the value of the function at the right edge of the rectangle will be larger than at any other point in the rectangle. Thus this will be an overestimate. Finally, for a function that is concave down and decreasing, however, each rectangle will lie wholly below the curve, since the value of the function at the right edge will be smaller than at any other point in the rectangle. Thus this will be an underestimate. Graphs of each of the four situations are below:



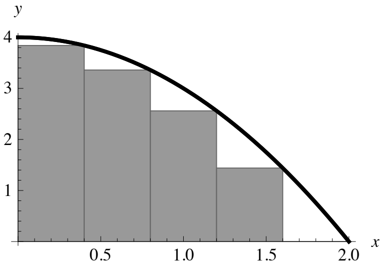
Concave up and increasing



Concave up and decreasing



Concave down and increasing



Concave down and decreasing

So the answer is

	Increasing on $[a, b]$	Decreasing on $[a, b]$
Concave up on $[a, b]$	Overestimate	Underestimate
Concave down on $[a, b]$	Overestimate	Underestimate

5.2 Definite Integrals

5.2.1 The net area is the difference between the area above the x-axis and below the curve, and below the x-axis and above the curve.

5.2.2 When the function is strictly above the x-axis, the net area is equal to the area. The net area differs from the area when the function dips below the x-axis so that the area below the x-axis and above the curve is nonzero.

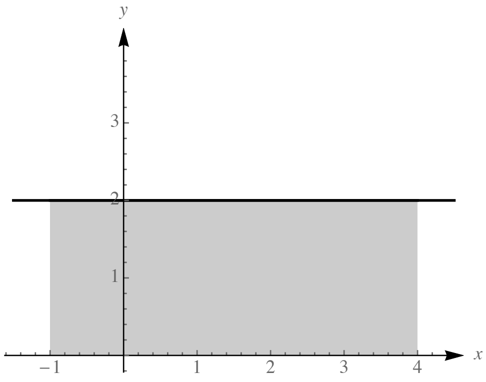
5.2.3 The left Riemann sum is: $40(1) + 30(1) + 20(1) + 0(1) - 10(1) - 20(1) = 60$. The right Riemann sum is $30(1) + 20(1) + 0(1) - 10(1) - 20(1) - 20(1) = 0$.

5.2.4 The left Riemann sum is: $-40(2)+0(2)+20(2)+50(2) = 60$. The right Riemann sum is $0(2)+20(2)+50(2) + 20(2) = 180$. The midpoint Riemann sum is $(-25)(2) + 10(2) + 30(2) + 45(2) = 120$.

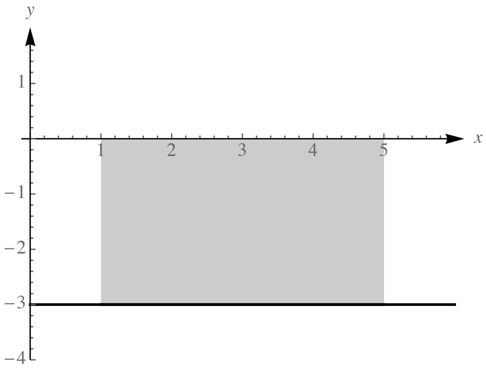
5.2.5 The left Riemann sum is: $2(2)-3(2)-5(2) = -12$. The right Riemann sum is $(-3)(2)-5(2)-1(2) = -18$. The midpoint Riemann sum is $-2(2) - 4(2) - 2(2) = -16$.

5.2.6 The left Riemann sum is: $(-5)(2) - 3(2) + 2(2) + 5(2) = -2$. The right Riemann sum is $(-3)2 + 2(2) + 5(2) - 1(2) = 6$. The midpoint Riemann sum is $-4(2) + 0(2) + 4(2) + 0(2) = 0$.

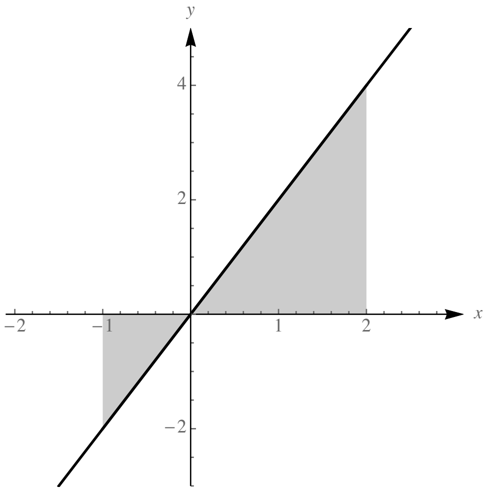
5.2.7 The integral represents the area of a 5 by 2 rectangle, so the value is 10.



5.2.8 The integral represents the negative of the area of a 4 by 3 rectangle, so the value is −12.



5.2.9 The integral represents the difference in areas of a triangle with base 2 and height 4 (so its area is 4) and one with base 1 and height 2 (so its area is 1), so the value of the integral is $4 - 1 = 3$.



5.2.10 $\int_1^3 (2f(x) - 4g(x))\,dx = 2\int_1^3 f(x)\,dx - 4\int_1^3 g(x)\,dx = 2(10) - 4(-20) = 100.$

$\int_3^1 (2f(x) - 4g(x))\,dx = -\int_1^3 (2f(x) - 4g(x))\,dx = -100.$

5.2.11 Because each of the functions $\sin x$ and $\cos x$ have the same amount of area above the x-axis as below between 0 and 2π, these both have value 0.

5.2.12 The greek letter $\sum$ and the integral sign $\int$ both remind us of the letter S, which stands for sum. The differential dx is analogous to Δx, helping us think of a small width. In both cases, the product of some form of $f(x)$ with either dx or Δx should make us think of an area – a height times a width. So both symbols are evocative of a sum of areas of rectangles, or a limit of such things.

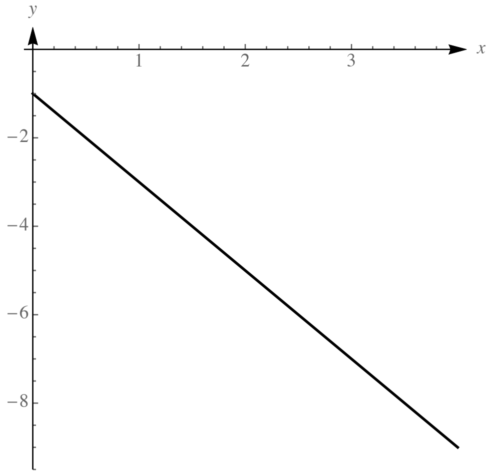
5.2.13 Because a region "from $x = a$ to $x = a$" has no width, its area is zero. This is akin to asking for the area of a one-dimensional object.

5.2.14 $\displaystyle\int_1^6 (2x^3 - 4x)\,dx = \int_1^6 2x^3\,dx - 4\int_1^6 x\,dx.$

5.2.15 This integral represents the area under $y = x$ between $x = 0$ and $x = a$, which is a right triangle. The length of the base of the triangle is a and the height is a, so the area is $\frac{1}{2} \cdot a^2$, so $\displaystyle\int_0^a x\,dx = \frac{a^2}{2}$.

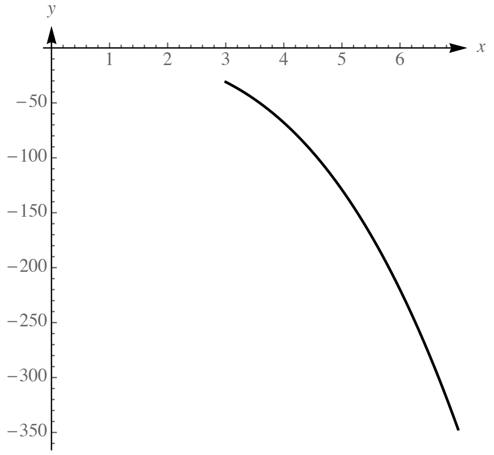
5.2.16 Because the function $|f|$ never goes below the x axis, the definite integral of $|f|$ does represent the area between $|f|$ and the x-axis. If this area is zero, then f must strictly lie on the x axis, so f must be the constant function with value 0.

5.2.17



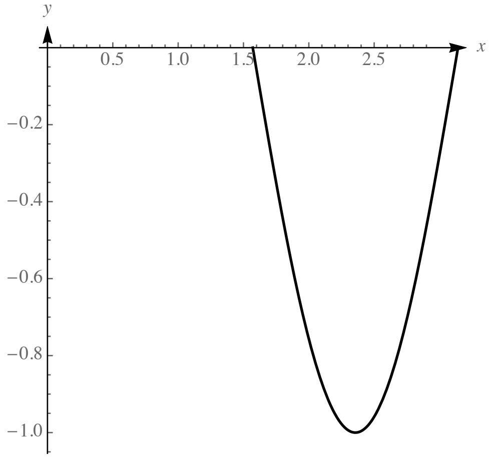
The left Riemann sum is $f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 = -1 - 3 - 5 - 7 = -16$.
The right Riemann sum is $f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 = -3 - 5 - 7 - 9 = -24$.
The midpoint Riemann sum is $f(.5)\cdot 1 + f(1.5)\cdot 1 + f(2.5)\cdot 1 + f(3.5)\cdot 1 = -2 - 4 - 6 - 8 = -20$.

5.2.18



We have $\Delta x = \dfrac{7-3}{4} = 1$. The left Riemann sum is $f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1 + f(6)\cdot 1 = -31 - 68 - 129 - 220 = -448$.
The right Riemann sum is $f(4)\cdot 1 + f(5)\cdot 1 + f(6)\cdot 1 + f(7)\cdot 1 = -68 - 129 - 220 - 347 = -764$.
The midpoint Riemann sum is $f(3.5)\cdot 1 + f(4.5)\cdot 1 + f(5.5)\cdot 1 + f(6.5)\cdot 1 = -46.875 - 95.125 - 170.375 - 278.625 = -591$.

5.2.19

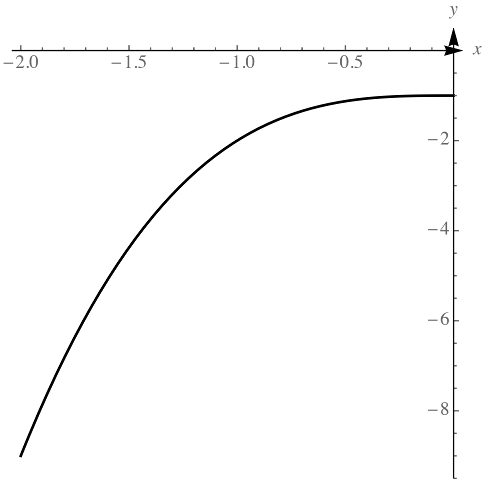


We have $\Delta x = \frac{\pi/2}{4} = \frac{\pi}{8}$. The left Riemann sum is $f(\pi/2)\cdot\frac{\pi}{8}+f(5\pi/8)\cdot\frac{\pi}{8}+f(6\pi/8)\cdot\frac{\pi}{8}+f(7\pi/8)\cdot\frac{\pi}{8} = \left(0-\sqrt{2}/2-1-\sqrt{2}/2\right)\cdot\frac{\pi}{8} = \frac{\pi}{8}\cdot(-1-\sqrt{2}) \approx -0.948$.

The right Riemann sum is $f(5\pi/8)\cdot\frac{\pi}{8} + f(6\pi/8)\cdot\frac{\pi}{8} + f(7\pi/8)\cdot\frac{\pi}{8} + f(\pi)\cdot\frac{\pi}{8} = \left(-\sqrt{2}/2-1-\sqrt{2}/2-0\right)\cdot\frac{\pi}{8} = \frac{\pi}{8}\cdot(-1-\sqrt{2}) \approx -0.948$.

The midpoint Riemann sum is $f(9\pi/16)\cdot\frac{\pi}{8} + f(11\pi/16)\cdot\frac{\pi}{8} + f(13\pi/16)\cdot\frac{\pi}{8} + f(15\pi/16)\cdot\frac{\pi}{8} \approx \frac{\pi}{8}\cdot(-0.382683-0.92388-0.92388-0.382683) \approx -1.026$.

5.2.20



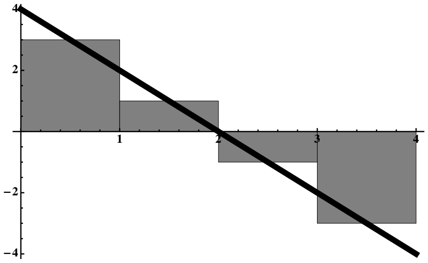
We have $\Delta x = \frac{2}{4} = \frac{1}{2}$. The left Riemann sum is $f(-2)\cdot\frac{1}{2}+f(-1.5)\cdot\frac{1}{2}+f(-1)\cdot\frac{1}{2}+f(-.5)\cdot\frac{1}{2} = .5(-9-4.375-2-1.125) = -8.25$.

The right Riemann sum is $f(-1.5)\cdot\frac{1}{2}+f(-1)\cdot\frac{1}{2}+f(-.5)\cdot\frac{1}{2}+f(0)\cdot\frac{1}{2} = .5(-4.375-2-1.125-1) = -4.25$.

The midpoint Riemann sum is $f(-1.75)\cdot\frac{1}{2} + f(-1.25)\cdot\frac{1}{2} + f(-.75)\cdot\frac{1}{2} + f(-.25)\cdot\frac{1}{2} = .5(-6.35938 - 2.95313 - 1.42188 - 1.10563) \approx -5.875$.

5.2.21

a.



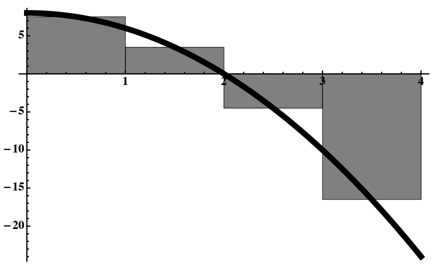
b. The left Riemann sum $\sum_{k=0}^{3} f(x_k)\cdot 1 = 4$.

The right Riemann sum $\sum_{k=1}^{4} f(x_k)\cdot 1 = -4$.

The midpoint Riemann sum $\sum_{k=1}^{4} f(x_k^*)\cdot 1 = 0$.

c. The rectangles whose height is $f(x_k)$ contribute positively when $x_k < 2$ and negatively when $x_k > 2$.

5.2.22

a. 

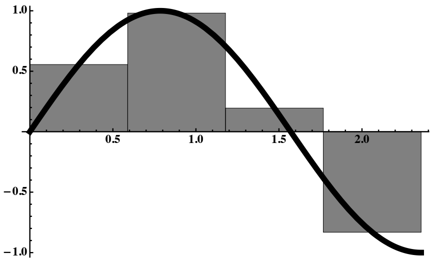
b. The left Riemann sum $\sum_{k=0}^{3} f(x_k) \cdot 1 = 4$.

The right Riemann sum $\sum_{k=1}^{4} f(x_k) \cdot 1 = -28$.

The midpoint Riemann sum $\sum_{k=1}^{4} f\left(x_k^*\right) \cdot 1 = -10$.

c. The rectangles whose height is $f(x_k)$ contribute positively when $x_k < 2$ and negatively when $x_k > 2$.

5.2.23

a. 

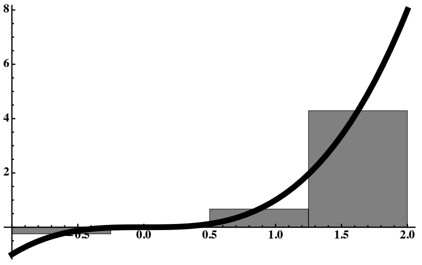
b. The left Riemann sum $\sum_{k=0}^{3} f(x_k) \cdot \frac{3\pi}{16} \approx 0.7353$.

The right Riemann sum $\sum_{k=1}^{4} f(x_k) \cdot \frac{3\pi}{16} \approx 0.146$.

The midpoint Riemann sum $\sum_{k=1}^{4} f\left(x_k^*\right) \cdot \frac{3\pi}{16} \approx 0.530$.

c. The rectangles whose height is $f(x_k)$ contribute positively when $x_k < \frac{\pi}{2}$ and negatively when $x_k > \frac{\pi}{2}$.

5.2.24

a. 

b. The left Riemann sum $\sum_{k=0}^{3} f(x_k) \cdot \frac{3}{4} \approx 0.797$.

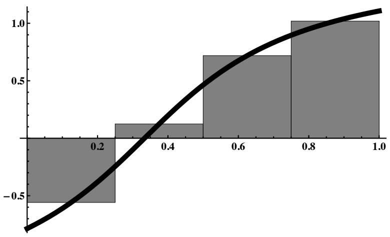
The right Riemann sum $\sum_{k=1}^{4} f(x_k) \cdot \frac{3}{4} \approx 7.547$.

The midpoint Riemann sum $\sum_{k=1}^{4} f\left(x_k^*\right) \cdot \frac{3}{4} \approx 3.539$.

c. The rectangles whose height is $f(x_k)$ contribute positively when $x_k > 0$ and negatively when $x_k < 0$.

5.2.25

a.



b. The left Riemann sum $\sum_{k=0}^{3} f(x_k) \cdot \frac{1}{4} \approx 0.082$.

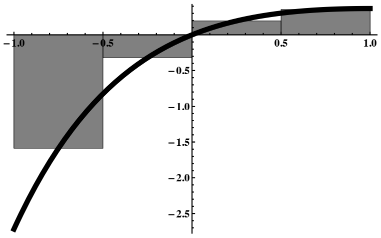
The right Riemann sum $\sum_{k=1}^{4} f(x_k) \cdot \frac{1}{4} \approx 0.555$.

The midpoint Riemann sum $\sum_{k=1}^{4} f\left(x_k^*\right) \cdot \frac{1}{4} \approx 0.326$.

c. The rectangles whose height is $f(x_k)$ contribute positively when $x_k > 1/3$ and negatively when $x_k < 1/3$.

5.2.26

a.



b. The left Riemann sum $\sum_{k=0}^{3} f(x_k) \cdot \frac{1}{2} \approx -1.620$.

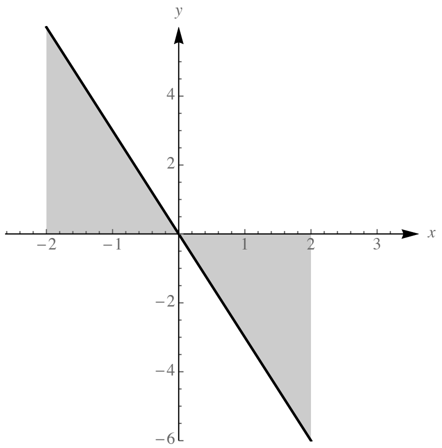
The right Riemann sum $\sum_{k=1}^{4} f(x_k) \cdot \frac{1}{2} \approx -0.077$.

The midpoint Riemann sum $\sum_{k=1}^{4} f\left(x_k^*\right) \cdot \frac{1}{2} \approx -0.6780$.

c. The rectangles whose height is $f(x_k)$ contribute positively when $x_k > 0$ and negatively when $x_k < 0$.

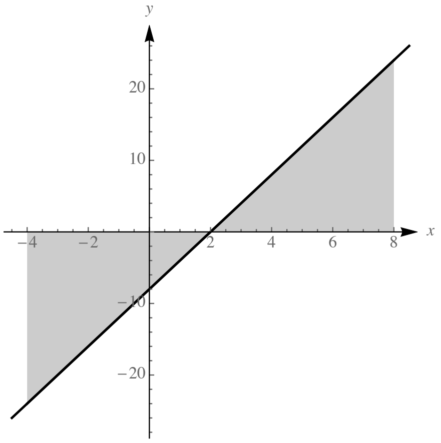
5.2.27

The region above the axis is a triangle with base 2 and height $f(-2) = 6$, and the region below the axis is a triangle with base 2 and height $-f(2) = 6$, so the net area is 0, and the area is 12.



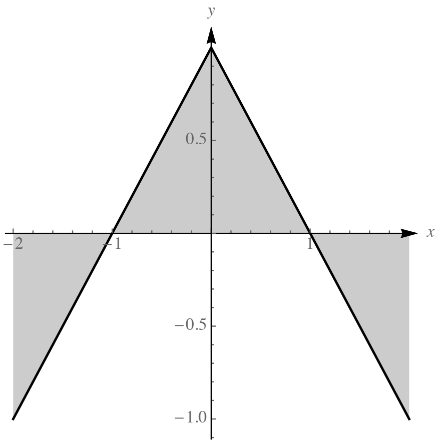
5.2.28

The region above the axis is a triangle with base $8-2=6$ and height $f(8)=24$, while the region below the axis is a triangle with base $2-(-4)=6$ and height $-f(-4)=24$, so the net area is 0, and the area is 144.



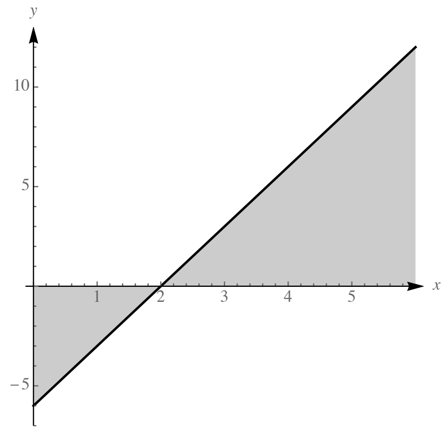
5.2.29

The region above the axis is a triangle with base 2 and height $f(0)=1$, while the region below the axis consists of two triangles each with base 1 and height 1, so the net area is 0, and the area is 2.



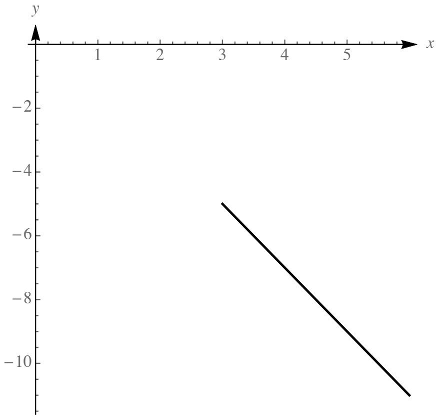
5.2.30

The region above the axis is a triangle with base $6-2=4$ and height $f(6)=12$, while the region below the axis is a triangle with base 2 and height $-f(-0)=6$, so the net area is $\frac{1}{2}\cdot 4\cdot 12-\frac{1}{2}\cdot 2\cdot 6=24-6=18$, while the area is 30.



5.2.31

a.



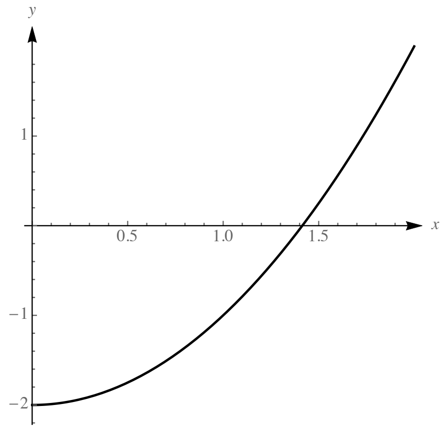
b. $\Delta x = \frac{1}{2}$, so the grid points are at 3, 3.5, 4, 4.5, 5, 5.5, and 6.

c. The left Riemann sum is $0.5(-5-6-7-8-9-10) = -22.5$. The right Riemann sum is $-.5(-6-7-8-9-10-11) = -25.5$.

d. The left Riemann sum overestimates the true value, while the right Riemann sum underestimates it.

5.2.32

a.



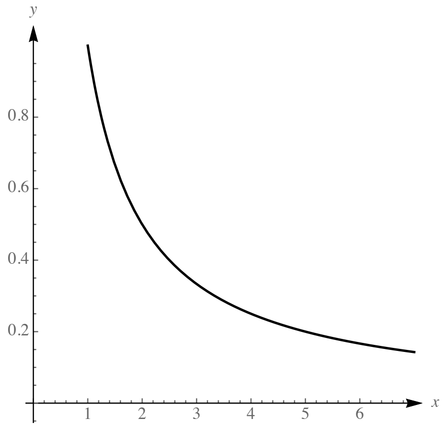
b. $\Delta x = \frac{1}{2}$, so the grid points are at 0, 0.5, 1, 1.5, and 2.

c. The left Riemann sum is $0.5(-2-1.75-1+0.25) = -2.25$. The right Riemann sum is $0.5(-1.75-1+.25+2) = -0.25$.

d. The left Riemann sum underestimates the true value, while the right Riemann sum overestimates it.

5.2.33

a.



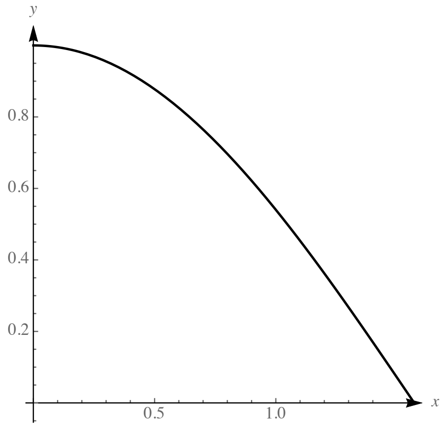
b. $\Delta x = 1$, so the grid points are at 1, 2, 3, 4, 5, 6, and 7.

c. The left Riemann sum is approximately $1 + 0.5 + 0.333333 + 0.25 + -.2 + 0.166666 = 2.45$. The right Riemann sum is approximately $.5 + 0.333333 + 0.25 + 0.2 + 0.166666 + 0.142857 \approx 1.593$.

d. The left Riemann sum overestimates the true value, while the right Riemann sum underestimates it.

5.2.34

a.



b. $\Delta x = \frac{\pi}{8}$, so the grid points are at 0, $\frac{\pi}{8}$, $\frac{\pi}{4}$, $\frac{3\pi}{8}$, and $\frac{\pi}{2}$.

c. The left Riemann sum is approximately

$$\frac{\pi}{8}(1 + 0.92388 + 0.707107 + 0.382683) \approx 1.183.$$

The right Riemann sum is about

$$\frac{\pi}{8}(0.92388 + 0.707107 + 0.382683 + 0) \approx 0.791.$$

d. The left Riemann sum overestimates the true value, while the right Riemann sum underestimates it.

5.2.35 This is $\int_0^2 (x^2+1)\,dx$.

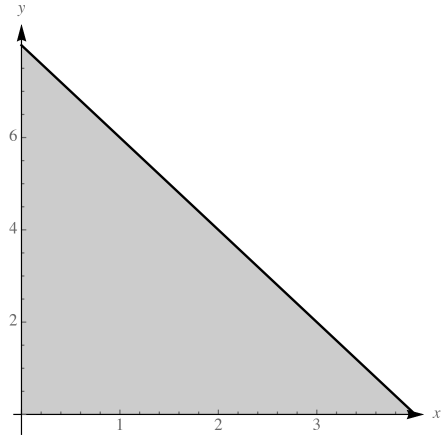
5.2.36 This is $\int_{-2}^2 (4-x^2)\,dx$.

5.2.37 This is $\int_1^2 x\ln(x)\,dx$.

5.2.38 This is $\int_{-2}^2 |x^2-1|\,dx$.

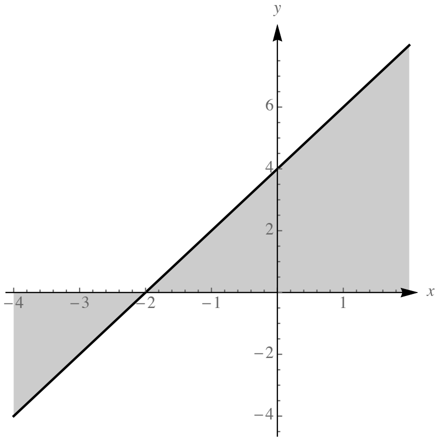
5.2.39

The region in question is a triangle with base 4 and height 8, so the area is $\frac{1}{2}\cdot 8\cdot 4 = 16$, and this is therefore the value of the definite integral as well.



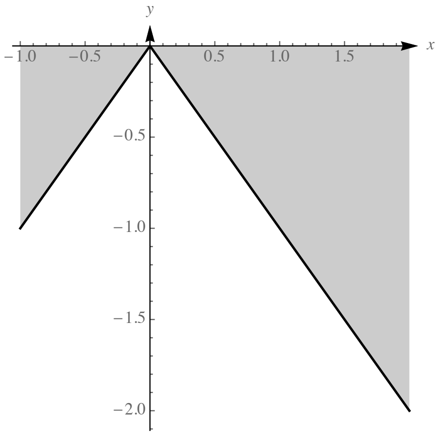
5.2.40

The region in question is a triangle with base 4 and height 8, above the axis, and a triangle with base 2 and height 4 below the axis, so the net area is $\frac{1}{2} \cdot 4 \cdot 8 - \frac{1}{2} \cdot 2 \cdot 4 = 16 - 4 = 12$.



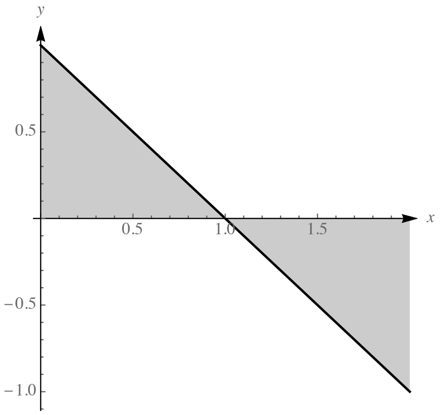
5.2.41

The region consists of two triangles, both below the axis. One has base 1 and height 1, the other has base 2 and height 2, so the net area is $-\dfrac{1}{2} \cdot 1 \cdot 1 - \dfrac{1}{2} \cdot 2 \cdot 2 = -2.5$.



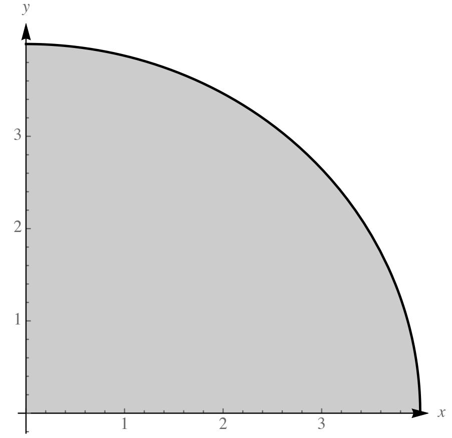
5.2.42

The region consists of two triangles of equal area, one of which is above the axis and one below, so the net area is 0.



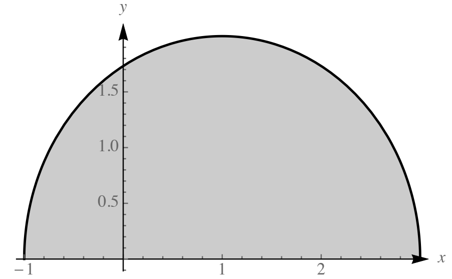
5.2.43

The region consists of a quarter circle of radius 4, situated above the axis. So the net area is $\frac{\pi \cdot 4^2}{4} = 4\pi$.



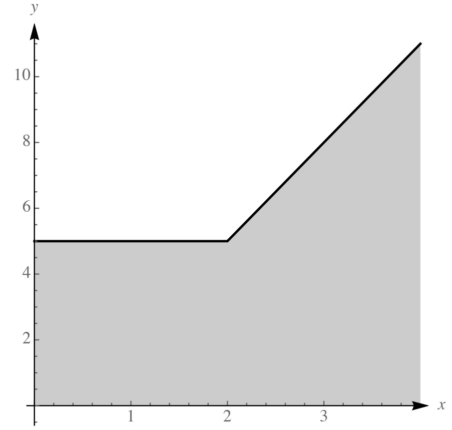
5.2.44

The region consists of a semicircle situated above the axis, of radius 2. The area is thus $\frac{4\pi}{2} = 2\pi$.



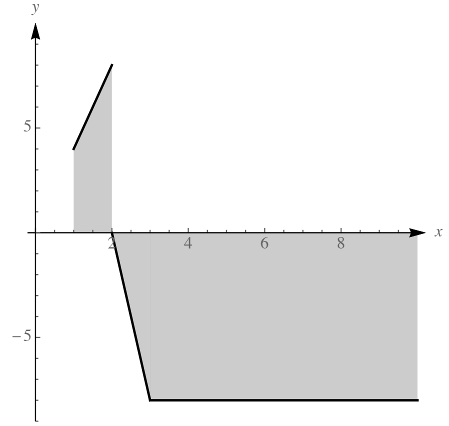
5.2.45

The region consists of a rectangle of area 10 above the axis, and a trapezoid of area 16 above the axis, so the net area is $10 + 16 = 26$.



5.2.46

The region consists of a trapezoid of area 6 above the axis, a triangle of area 4 below the axis, and a rectangle of area 56 below the axis. So the net area is $6 - 4 - 56 = -54$.



5.2.47 $\displaystyle\int_0^{\pi} x\sin x\,dx = A(R_1) + A(R_2) = 1 + \pi - 1 = \pi.$

5.2.48 $\displaystyle\int_0^{3\pi/2} x\sin x\,dx = A(R_1) + A(R_2) - A(R_3) = 1 + \pi - 1 - \pi - 1 = -1.$

5.2.49 $\displaystyle\int_0^{2\pi} x\sin x\,dx = A(R_1) + A(R_2) - A(R_3) - A(R_4) = 1 + \pi - 1 - \pi - 1 - 2\pi + 1 = -2\pi.$

5.2.50 $\displaystyle\int_{\pi/2}^{2\pi} x\sin x\,dx = A(R_2) - A(R_3) - A(R_4) = \pi - 1 - \pi - 1 - 2\pi + 1 = -2\pi - 1.$

5.2.51

a. $\displaystyle\int_4^0 3x(4-x)\,dx = -\int_0^4 3x(4-x)\,dx = -32.$

b. $\displaystyle\int_0^4 x(x-4)\,dx = -\frac{1}{3}\int_0^4 3x(4-x)\,dx = -\frac{1}{3}\cdot 32 = -\frac{32}{3}.$

c. $\displaystyle\int_4^0 6x(4-x)\,dx = -2\cdot\int_0^4 3x(4-x)\,dx = -2\cdot 32 = -64.$

d. $\displaystyle\int_0^8 3x(4-x)\,dx = \int_0^4 3x(4-x)\,dx + \int_4^8 3x(4-x)\,dx = 32 + \int_4^8 3x(4-x)\,dx.$ It is not possible to evaluate the given integral from the information given.

5.2.52

a. $\displaystyle\int_1^4 (-3f(x))\,dx = -3\int_1^4 f(x)\,dx = -3\cdot 8 = -24.$

b. $\displaystyle\int_1^4 3f(x)\,dx = 3\int_1^4 f(x)\,dx = 3\cdot 8 = 24.$

c. $\displaystyle\int_6^4 12f(x)\,dx = -12\int_4^6 f(x)\,dx = -12\left(\int_1^6 f(x)\,dx - \int_1^4 f(x)\,dx\right) = -12(5-8) = 36.$

d. $\displaystyle\int_4^6 3f(x)\,dx = 3\left(\int_1^6 f(x)\,dx - \int_1^4 f(x)\,dx\right) = 3(5-8) = -9.$

5.2.53

a. $\int_0^3 5f(x)\,dx = 5\int_0^3 f(x)\,dx = 5\cdot 2 = 10.$

b. $\int_3^6 (-3g(x))\,dx = -3\int_3^6 g(x)\,dx = -3\cdot 1 = -3.$

c. $\int_3^6 (3f(x)-g(x))\,dx = 3\int_3^6 f(x)\,dx - \int_3^6 g(x)\,dx = 3(-5)-1 = -16.$

d. $\int_6^3 [f(x)+2g(x)]\,dx = -\left[\int_3^6 f(x)\,dx + 2\int_3^6 g(x)\,dx\right] = -[-5+2\cdot 1] = 3.$

5.2.54

a. $\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx = 6+(-8) = -2.$

b. $\int_0^5 |f(x)|\,dx = \int_0^2 f(x)\,dx - \int_2^5 f(x)\,dx = 6+8 = 14.$

c. $\int_2^5 4|f(x)|\,dx = 4\int_2^5 (-f(x))\,dx = 4\cdot 8 = 32.$

d. $\int_0^5 (f(x)+|f(x)|)\,dx = \int_0^2 (f(x)+f(x))\,dx + \int_2^5 f(x)-f(x)\,dx = 2\int_0^2 f(x)\,dx = 2\cdot 6 = 12.$

5.2.55

a. $\int_1^4 3f(x)\,dx = 3\int_1^4 f(x)\,dx = 3\cdot\left(\int_1^6 f(x)\,dx - \int_4^6 f(x)\,dx\right) = 3\cdot(10-5) = 15.$

b. $\int_1^6 (f(x)-g(x))\,dx = \int_1^6 f(x)\,dx - \int_1^6 g(x)\,dx = 10-5 = 5.$

c. $\int_1^4 (f(x)-g(x))\,dx = \int_1^4 f(x)\,dx - \int_1^4 g(x)\,dx = \left(\int_1^6 f(x)\,dx - \int_4^6 f(x)\,dx\right) - 2 = (10-5)-2 = 3.$

d. $\int_4^6 (g(x)-f(x)\,dx = \int_4^6 g(x)\,dx - \int_4^6 f(x)\,dx = \left(\int_1^6 g(x)\,dx - \int_1^4 g(x)\,dx\right) - 5 = (5-2)-5 = -2.$

e. $\int_4^6 8g(x)\,dx = 8\left(\int_1^6 g(x)\,dx - \int_1^4 g(x)\,dx\right) = 8(5-2) = 24.$

f. $\int_4^1 2f(x)\,dx = -2\int_1^4 f(x)\,dx = -2\cdot\left(\int_1^6 f(x)\,dx - \int_4^6 f(x)\,dx\right) = -2(10-5) = -10.$

5.2.56 Because f is bounded above by the function $g(x)=3$ on the interval, we must have

$$\int_1^5 f(x)\,dx \le \int_1^5 3\,dx = 12,$$

so our integral is bounded above by 12. On the the other hand, because $2 \le f(x)$ on the interval, we must have

$$\int_1^5 f(x)\,dx \ge \int_1^5 2\,dx = 8,$$

so our integral is bounded below by 8. Therefore

$$8 \le \int_1^5 f(x)\,dx \le 12.$$

5.2.57

a. $\displaystyle\int_0^1 (4x - 2x^3)\,dx = -2\int_0^1 (x^3 - 2x)\,dx = -2 \cdot \left(-\frac{3}{4}\right) = \frac{3}{2}.$

b. $\displaystyle\int_1^0 (2x - x^3)\,dx = \int_0^1 (x^3 - 2x)\,dx = -\frac{3}{4}.$

5.2.58

a. $\displaystyle\int_0^{\pi/2} (2\sin\theta - \cos\theta)\,d\theta = -\int_0^{\pi/2} (\cos\theta - 2\sin\theta)\,d\theta = -(-1) = 1.$

b. $\displaystyle\int_{\pi/2}^0 (4\cos\theta - 8\sin\theta)\,d\theta = -4\int_0^{\pi/2} (\cos\theta - 2\sin\theta)\,d\theta = -4(-1) = 4.$

5.2.59 $\displaystyle\int_0^a f(x)\,dx = 16.$

5.2.60 $\displaystyle\int_0^b f(x)\,dx = 16 - 5 = 11.$

5.2.61 $\displaystyle\int_a^c f(x)\,dx = 11 - 5 = 6.$

5.2.62 $\displaystyle\int_0^c f(x)\,dx = 16 - 5 + 11 = 22.$

5.2.63 $\displaystyle\int_0^c |f(x)|\,dx = 16 + 5 + 11 = 32.$

5.2.64 $\displaystyle\int_0^c (2|f(x)| + 3f(x))\,dx = 2(32) + 3(22) = 130.$

5.2.65 $\displaystyle\int_a^0 f(x)\,dx = -\int_0^a f(x)\,dx = -16.$

5.2.66 $\displaystyle\int_c^0 |f(x)|\,dx = -\int_0^c |f(x)|\,dx = -32.$

5.2.67 $\displaystyle\int_0^1 (2x + \sqrt{1 - x^2} + 1)\,dx = \int_0^1 2x\,dx + \int_0^1 \sqrt{1 - x^2}\,dx + \int_0^1 1\,dx$. The first integral in this sum represents the area of a triangle with base 1 and height 2 (which has value 1), the second represents the area of a quarter of a circle of radius 1 (which has value $\frac{\pi}{4}$, and the third represents the area of a 1 × 1 square (which has value 1). So the integral's value is $1 + \frac{\pi}{4} + 1 = 2 + \frac{\pi}{4}$.

5.2.68 $\displaystyle\int_1^5 \left(|x - 2| + \sqrt{-x^2 + 6x - 5}\right)\,dx = \int_1^5 |x - 2|\,dx + \int_1^5 \sqrt{-x^2 + 6x + 5}\,dx$. The first integral in this last sum represents the areas of two triangles, one with base 1 and height 1 (over the interval $[1, 2]$) and one with base 3 and height 3 (over the interval $[2, 5]$). So the value of the first integral is $\frac{1}{2} + \frac{9}{2} = 5$. For the second integral, note that under the radical we have $-x^2 + 6x - 5 = -(x^2 - 6x + 5) = -(x^2 - 6x + 9 - 4) = -((x - 3)^2 - 2^2) = 2^2 - (x - 3)^2$. So the function $y = \sqrt{2^2 - (x - 3)^2}$ represents a circle of radius 2 centered at $(3, 0)$, so the value of the second integral is $\frac{1}{2}(4\pi) = 2\pi$. So the sum of the two integrals is $5 + 2\pi$.

5.2.69

a. True. See problem 78 in the previous section for a proof.

b. True. See problem 79 in the previous section for a proof.

c. True. Because both of those function are periodic with period $\frac{2\pi}{a}$, and both have the same amount of area above the axis as below for one period, the net area of each between 0 and $\frac{2\pi}{a}$ is zero.

d. False. For example $\displaystyle\int_0^{2\pi} \sin x\,dx = 0 = \int_{2\pi}^0 \sin x\,dx$, but $\sin x$ is not a constant function.

e. False. Because x is not a constant, it can not be factored outside of the integral. For example $\displaystyle\int_0^1 x \cdot 1\,dx \neq x\int_0^1 1\,dx$.

5.2.70

a. $\displaystyle\sum_{k=1}^{20} 3\sqrt{4+\frac{5(k-1)}{20}}\cdot\frac{5}{20} \approx 37.624.$ $\displaystyle\sum_{k=1}^{20} 3\sqrt{4+\frac{5k}{20}}\cdot\frac{5}{20} \approx 38.374.$

$\displaystyle\sum_{k=1}^{50} 3\sqrt{4+\frac{5(k-1)}{50}}\cdot\frac{5}{50} \approx 37.850.$ $\displaystyle\sum_{k=1}^{50} 3\sqrt{4+\frac{5k}{50}}\cdot\frac{5}{50} \approx 38.150.$

$\displaystyle\sum_{k=1}^{100} 3\sqrt{4+\frac{5(k-1)}{100}}\cdot\frac{5}{100} \approx 37.925.$ $\displaystyle\sum_{k=1}^{100} 3\sqrt{4+\frac{5k}{100}}\cdot\frac{5}{100} \approx 38.075.$

b. It appears that the integral's value is 38.

5.2.71

a. $\displaystyle\sum_{k=1}^{20} \left(\left(\frac{k-1}{20}\right)^2+1\right)\cdot\frac{1}{20} \approx 1.309.$ $\displaystyle\sum_{k=1}^{20} \left(\left(\frac{k}{20}\right)^2+1\right)\cdot\frac{1}{20} \approx 1.359.$

$\displaystyle\sum_{k=1}^{50} \left(\left(\frac{k-1}{50}\right)^2+1\right)\cdot\frac{1}{50} \approx 1.323.$ $\displaystyle\sum_{k=1}^{50} \left(\left(\frac{k}{50}\right)^2+1\right)\cdot\frac{1}{50} \approx 1.343.$

$\displaystyle\sum_{k=1}^{100} \left(\left(\frac{k-1}{100}\right)^2+1\right)\cdot\frac{1}{100} \approx 1.328.$ $\displaystyle\sum_{k=1}^{100} \left(\left(\frac{k}{100}\right)^2+1\right)\cdot\frac{1}{100} \approx 1.338.$

b. It appears that the integral's value is about $\frac{4}{3}$.

5.2.72

a. $\displaystyle\sum_{k=1}^{20} \ln\left(1+(k-1)\frac{(e-1)}{20}\right)\cdot\frac{e-1}{20} \approx 0.957.$ $\displaystyle\sum_{k=1}^{20} \ln\left(1+k\cdot\frac{(e-1)}{20}\right)\cdot\frac{e-1}{20} \approx 1.043.$

$\displaystyle\sum_{k=1}^{50} \ln\left(1+(k-1)\frac{(e-1)}{50}\right)\cdot\frac{e-1}{50} \approx 0.983.$ $\displaystyle\sum_{k=1}^{50} \ln\left(1+k\cdot\frac{(e-1)}{50}\right)\cdot\frac{e-1}{50} \approx 1.017.$

$\displaystyle\sum_{k=1}^{100} \ln\left(1+(k-1)\frac{(e-1)}{100}\right)\cdot\frac{e-1}{100} \approx 0.991.$ $\displaystyle\sum_{k=1}^{100} \ln\left(1+k\cdot\frac{(e-1)}{100}\right)\cdot\frac{e-1}{100} \approx 1.009.$

b. It appears that the integral's value is 1.

5.2.73

a. $\displaystyle\sum_{k=1}^{20} \cos^{-1}\left((k-1)/20\right)\cdot\frac{1}{20} \approx 1.036.$ $\displaystyle\sum_{k=1}^{20} \cos^{-1}\left(k/20\right)\cdot\frac{1}{20} \approx 0.958.$

$\displaystyle\sum_{k=1}^{50} \cos^{-1}\left((k-1)/50\right)\cdot\frac{1}{50} \approx 1.015.$ $\displaystyle\sum_{k=1}^{50} \cos^{-1}\left(k/50\right)\cdot\frac{1}{50} \approx 0.983.$

$\displaystyle\sum_{k=1}^{100} \cos^{-1}\left((k-1)/100\right)\cdot\frac{1}{100} \approx 1.008.$ $\displaystyle\sum_{k=1}^{100} \cos^{-1}\left(k/100\right)\cdot\frac{1}{100} \approx 0.992.$

b. It appears that the integral's value is 1.

5.2.74

a. $\sum_{k=1}^{20} \pi \cos\left(\frac{\pi}{2}\left(-1+\frac{2(k-1)}{20}\right)\right)\cdot\frac{2}{20} \approx 3.992.$ $\sum_{k=1}^{20} \pi \cos\left(\frac{\pi}{2}\left(-1+\frac{2k}{20}\right)\right)\cdot\frac{2}{20} \approx 3.992.$

$\sum_{k=1}^{50} \pi \cos\left(\frac{\pi}{2}\left(-1+\frac{2(k-1)}{50}\right)\right)\cdot\frac{2}{50} \approx 3.999.$ $\sum_{k=1}^{50} \pi \cos\left(\frac{\pi}{2}\left(-1+\frac{2k}{50}\right)\right)\cdot\frac{2}{50} \approx 3.999.$

$\sum_{k=1}^{100} \pi \cos\left(\frac{\pi}{2}\left(-1+\frac{2(k-1)}{100}\right)\right)\cdot\frac{2}{100} \approx 4.000.$ $\sum_{k=1}^{100} \pi \cos\left(\frac{\pi}{2}\left(-1+\frac{2k}{100}\right)\right)\cdot\frac{2}{100} \approx 4.000.$

b. It appears that the integral's value is 4.

5.2.75

a. $\sum_{k=1}^{n} 2\sqrt{1+\frac{3}{2n}+\frac{3(k-1)}{n}}\cdot\frac{3}{n}.$

b.

n	20	50	100
Midpoint Sum	9.3338	9.33341	9.33335

It appears that the integral's value is about $\frac{28}{3}$.

5.2.76

a. $\sum_{k=1}^{n} \sin\left(\frac{\pi}{4}\left(-1+\frac{3}{2n}+\frac{3(k-1)}{n}\right)\right)\cdot\frac{3}{n}.$

b.

n	20	50	100
Midpoint Sum	0.900837	0.9004	0.900337

It appears that the integral's value is about 0.9.

5.2.77

a. $\sum_{k=1}^{n} \left(4\left(\frac{2}{n}+\frac{4(k-1)}{n}\right)-\left(\frac{2}{n}+\frac{4(k-1)}{n}\right)^2\right)\cdot\frac{4}{n}.$

b.

n	20	50	100
Midpoint Sum	10.68	10.6688	10.6672

It appears that the integral's value is about $\frac{32}{3}$.

5.2.78

a. $\sum_{k=1}^{n} \left(\sin^{-1}\left(\frac{k-0.5}{n}\right)+1\right)\cdot\frac{1}{\pi n}.$

b.

n	20	50	100
Midpoint Sum	0.49973	0.49993	0.49997

It appears that the integral's value is about $\frac{1}{2}$.

5.2.79

$$\begin{aligned}\int_0^2 (2x+1)\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left[2\left(\frac{2k}{n}\right)+1\right]\frac{2}{n}\\ &= \lim_{n\to\infty}\left[\frac{8}{n^2}\sum_{k=1}^{n}k+\frac{2}{n}\sum_{k=1}^{n}1\right]\\ &= \lim_{n\to\infty}\left[\frac{8}{n^2}\cdot\frac{n(n+1)}{2}+\frac{2}{n}\cdot n\right]\\ &= \lim_{n\to\infty}\left[\frac{4(n+1)}{n}+2\right]=4+2=6.\end{aligned}$$

5.2.80

$$\begin{aligned}\int_1^5 (1-x)\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left[1-\left(1+\frac{4k}{n}\right)\right]\frac{4}{n}\\ &= \lim_{n\to\infty}\left[\frac{-16}{n^2}\sum_{k=1}^{n}k\right]\\ &= \lim_{n\to\infty}\left[\frac{-16}{n^2}\cdot\frac{n(n+1)}{2}\right]\\ &= \lim_{n\to\infty}\left[\frac{-8(n+1)}{n}\right]=-8.\end{aligned}$$

5.2.81

$$\begin{aligned}\int_3^7 (4x+6)\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left[4\left(3+\frac{4k}{n}\right)+6\right]\frac{4}{n}\\ &= \lim_{n\to\infty}\left[\frac{64}{n^2}\sum_{k=1}^{n}k+\frac{72}{n}\sum_{k=1}^{n}1\right]\\ &= \lim_{n\to\infty}\left[\frac{64}{n^2}\cdot\frac{n(n+1)}{2}+\frac{72}{n}\cdot n\right]\\ &= \lim_{n\to\infty}\left[\frac{32(n+1)}{n}+72\right]=104.\end{aligned}$$

5.2.82

$$\begin{aligned}\int_0^2 (x^2-1)\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left[\left(\frac{2k}{n}\right)^2-1\right]\frac{2}{n}\\ &= \lim_{n\to\infty}\left[\frac{2}{n}\left(\sum_{k=1}^{n}-1\right)+\frac{8}{n^3}\sum_{k=1}^{n}k^2\right]\\ &= \lim_{n\to\infty}\left[\frac{2}{n}\cdot(-n)+\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]\\ &= \lim_{n\to\infty}\left[-2+\frac{8n^2+12n+4}{3n^2}\right]=-2+\frac{8}{3}=\frac{2}{3}.\end{aligned}$$

5.2.83

$$\begin{aligned}\int_1^4 (x^2-1)\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^n \left[\left(1+\frac{3k}{n}\right)^2-1\right]\frac{3}{n}\\ &= \lim_{n\to\infty}\left[\frac{18}{n^2}\sum_{k=1}^n k+\frac{27}{n^3}\sum_{k=1}^n k^2\right]\\ &= \lim_{n\to\infty}\left[\frac{18}{n^2}\cdot\frac{n(n+1)}{2}+\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]\\ &= \lim_{n\to\infty}\left[\frac{9(n+1)}{n}+\frac{18n^2+27n+9}{2n^2}\right]=9+9=18.\end{aligned}$$

5.2.84

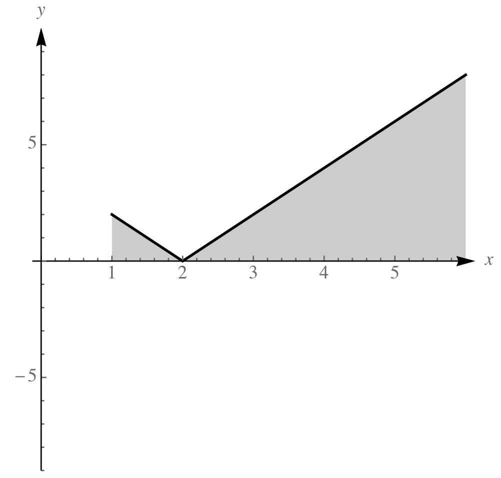
$$\begin{aligned}\int_0^2 (x^3+x+1)\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x = \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^n \left(\left(\frac{2k}{n}\right)^3+\left(\frac{2k}{n}\right)+1\right)\\ &= \lim_{n\to\infty}\left(\frac{16}{n^4}\sum_{k=1}^n k^3+\frac{4}{n^2}\sum_{k=1}^n k+2\right)\\ &= \lim_{n\to\infty}\left(\frac{16}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)+\frac{4}{n^2}\left(\frac{n(n+1)}{2}\right)+2\right)\\ &= \lim_{n\to\infty}\left(4\left(\frac{n^2(n+1)^2}{n^4}\right)+2\left(\frac{n(n+1)}{n^2}\right)+2\right)\\ &= 4+2+2=8.\end{aligned}$$

5.2.85

$$\begin{aligned}\int_0^1 (4x^3+3x^2)\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n \left(4\left(\frac{k}{n}\right)^3+3\left(\frac{k}{n}\right)^2\right)\\ &= \lim_{n\to\infty}\left(\frac{4}{n^4}\sum_{k=1}^n k^3+\frac{3}{n^3}\sum_{k=1}^n k^2\right)\\ &= \lim_{n\to\infty}\left(\frac{4}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)+\frac{3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right)\\ &= 1+1=2.\end{aligned}$$

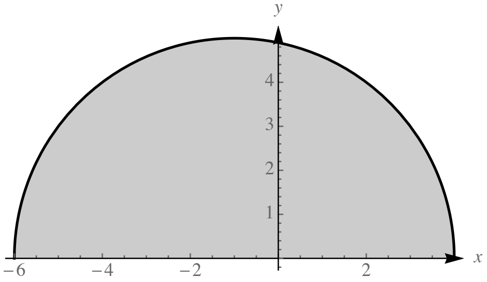
5.2.86

The region in question consists of two triangles above the axis, one with base 1 and height 2, and one with base 4 and height 8, so the net area is $\frac{1}{2}\cdot 1\cdot 2+\frac{1}{2}\cdot 4\cdot 8=17$.



5.2.87

The region in question is a semicircle above the axis with radius 5, so the area is $\frac{1}{2}\pi \cdot 5^2 = \frac{25\pi}{2}$.



5.2.88 Let the grid points for the interval $[a,p]$ be $x_i = a + i \cdot \frac{p-a}{n}$, where $1 \leq i \leq n$. Let the grid points for the interval $[p,b]$ be $x_j^* = p + j \cdot \frac{b-p}{m}$ where $1 \leq j \leq m$. Note that if we take the union of both of these sets of grid points, we get a set of grid points for $[a,b]$.

One Riemann sum for f on $[a,b]$ is $\sum_{k=1}^{n} f(x_k) \cdot \frac{p-a}{n} + \sum_{j=1}^{m} f(x_j^*) \cdot \frac{b-p}{m}$, which naturally splits into a right Riemann sum for f on $[a,p]$ plus a right Riemann sum for f on $[p,b]$.

By the definition of definite integral, taking limits as m, $n \to \infty$ shows that $\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$.

5.2.89 $\int_0^{10} f(x)\,dx = \int_0^5 2\,dx + \int_5^{10} 3\,dx = 10 + 15 = 25$.

5.2.90 $\int_1^6 f(x)\,dx = \int_1^4 2x\,dx + \int_4^6 (10-2x)\,dx = 15 + 0 = 15$.

5.2.91 $\int_1^5 x\lfloor x\rfloor\,dx = \int_1^2 x\lfloor x\rfloor\,dx + \int_2^3 x\lfloor x\rfloor\,dx + \int_3^4 x\lfloor x\rfloor\,dx + \int_4^5 x\lfloor x\rfloor\,dx = \int_1^2 x\,dx + \int_2^3 2x\,dx + \int_3^4 3x\,dx + \int_4^5 4x\,dx$. Each of these integrals represents the area of a trapezoid with base 1. The value of the integral is $\frac{1+2}{2} + \frac{4+6}{2} + \frac{9+12}{2} + \frac{16+20}{2} = 35$.

5.2.92 $\int_0^4 \frac{x}{\lceil x\rceil}\,dx = \int_0^1 \frac{x}{\lceil x\rceil}\,dx + \int_1^2 \frac{x}{\lceil x\rceil}\,dx + \int_2^3 \frac{x}{\lceil x\rceil}\,dx + \int_3^4 \frac{x}{\lceil x\rceil}\,dx = \int_0^1 x\,dx + \int_1^2 \frac{x}{2}\,dx + \int_2^3 \frac{x}{3}\,dx + \int_3^4 \frac{x}{4}\,dx$. The first of these represents the area of a triangle with base one and height 1, while the others represent the area of trapezoids with base 1. The value of the integral is $\frac{1}{2} + \frac{\frac{1}{2}+1}{2} + \frac{\frac{2}{3}+1}{2} + \frac{\frac{3}{4}+1}{2} = \frac{71}{24}$.

5.2.93

$$\begin{aligned}
\int_a^b cf(x)\,dx &= \lim_{\Delta\to 0} \sum_{k=1}^{n} cf(x_k^*)\Delta x_k \\
&= \lim_{\Delta\to 0} c \sum_{k=1}^{n} f(x_k^*)\Delta x_k \\
&= c \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = c\int_a^b f(x)\,dx.
\end{aligned}$$

5.2.94 Note that $\int_c^d x\,dx = \frac{d^2-c^2}{2}$, because it represents the area of a trapezoid with base of length $d-c$ and heights c and d. Also note that $\int_c^d b\,dx = b(d-c)$ because it represents the area of a rectangle with base $d-c$ and height b.

Therefore, $\int_c^d (x+b)\,dx = \int_c^d x\,dx + \int_c^d b\,dx = \frac{d^2-c^2}{2} + b(d-c) = (d-c)\cdot\left(\frac{d+c}{2}+b\right)$. Because $c \neq d$, this is zero exactly when $b = -\frac{c+d}{2}$.

5.2.95 Let n be a positive integer. Let $\Delta x = \frac{1}{n}$. Note that each grid point $\frac{k}{n}$ for $0 \le k \le n$ where i is an integer is a rational number. So $f(x_k) = 1$ for each grid point. So the right Riemann sum is $\sum_{k=1}^{n} f(x_k)\frac{1}{n} = \frac{1}{n}\sum_{k=1}^{n} 1 = \frac{1}{n}\cdot n = 1$. The left Riemann sum calculation is similar, as is the midpoint Riemann sum calculation (because the grid midpoints are also rational numbers – they are the average of two rational numbers and hence are rational as well).

5.2.96

a. The left Riemann sum for I(p) is $\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^p\cdot\frac{1}{n}$.

b. We have $I(p) = \int_0^1 x^p\,dx = \lim_{n\to\infty}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^p\cdot\frac{1}{n} = \frac{1}{p+1}$.

5.2.97

a. Note that for all values of $k = 1, 2, \ldots, n$, we have $x_{k-1}x_{k-1} \le x_{k-1}x_k$, so $\sqrt{x_{k-1}x_{k-1}} \le \sqrt{x_{k-1}x_k}$, and thus $x_{k-1} \le \sqrt{x_{k-1}x_k}$. Similarly, $x_{k-1}x_k \le x_kx_k$, so $\sqrt{x_{k-1}x_k} \le \sqrt{x_kx_k} = x_k$, so $\sqrt{x_{k-1}x_k} \le x_k$. Thus $x_{k-1} \le \sqrt{x_{k-1}x_k} \le x_k$ for all $k = 1, 2, \ldots, n$.

b. $\frac{1}{x_{k-1}} - \frac{1}{x_k} = \frac{x_k}{x_{k-1}x_k} - \frac{x_{k-1}}{x_{k-1}x_k} = \frac{x_k - x_{k-1}}{x_{k-1}x_k} = \frac{\Delta x_k}{x_{k-1}x_k}$, for all $k = 1, 2, \ldots, n$.

c. The Riemann sum is $\sum_{k=1}^{n}\frac{\Delta x_k}{\overline{x_k}^2}$. Using $\overline{x_k} = \sqrt{x_{k-1}x_k}$, we have

$$\sum_{k=1}^{n}\frac{\Delta x_k}{x_{k-1}x_k} = \sum_{k=1}^{n}\left(\frac{1}{x_{k-1}} - \frac{1}{x_k}\right),$$

where the last equality follows from part (b). Now note that the sum telescopes (that is, has many canceling terms).

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{1}{x_{k-1}} - \frac{1}{x_k}\right) &= \left(\frac{1}{x_0} - \frac{1}{x_1}\right) + \left(\frac{1}{x_1} - \frac{1}{x_2}\right) + \cdots + \left(\frac{1}{x_{n-2}} - \frac{1}{x_{n-1}}\right) + \left(\frac{1}{x_{n-1}} - \frac{1}{x_n}\right)\\ &= \frac{1}{x_0} + \left(-\frac{1}{x_1} + \frac{1}{x_1}\right) + \left(-\frac{1}{x_2} + \frac{1}{x_2}\right) + \cdots + \left(-\frac{1}{x_{n-1}} + \frac{1}{x_{n-1}}\right) - \frac{1}{x_n}\\ &= \frac{1}{x_0} - \frac{1}{x_n}\\ &= \frac{1}{a} - \frac{1}{b}.\end{aligned}$$

d. The integral is the limit of the Riemann sum as $n \to \infty$. Thus we have

$$\int_a^b \frac{dx}{x^2} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\Delta x_k}{\overline{x_k}^2} = \lim_{n\to\infty}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{a} - \frac{1}{b}.$$

5.2.98 Suppose $f(x) \geq g(x)$ on $[a,b]$. Then $f(x) - g(x) \geq 0$ on $[a,b]$. Then by Property 7, $\int_a^b (f(x) - g(x))\,dx \geq 0$. But by Property 3, $\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$. Therefore $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0$, so $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

5.3 Fundamental Theorem of Calculus

5.3.1 A is also an antiderivative of f.

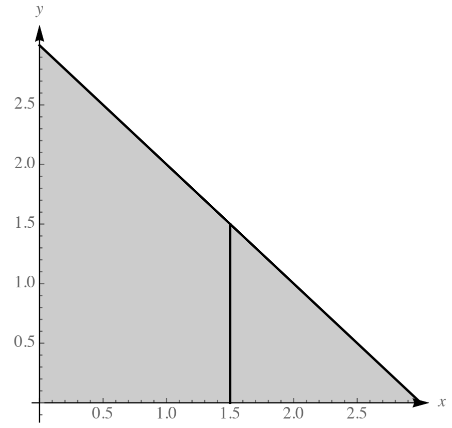
5.3.2 Because F and A are both antiderivatives of f, we have $A(x) = F(x) + C$, where C is a constant.

5.3.3 The fundamental theorem says that $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f. So to evaluate $\int_a^b f(x)\,dx$, one could find an antiderivative $F(x)$, and then evaluate this at a and b and then subtract, obtaining $F(b) - F(a)$.

5.3.4 An area function has the form $\int_a^x c\,dt$, and gives the area between a and x and under c, which is the area of a rectangle with base $x - a$ and height c. As x increases, the base $x - a$ increases while the height c remains constant, so the area increases.

5.3.5

$A(x) = \int_0^x (3 - t)\,dt$ represents the area between 0 and x and below this curve. As x increases (but remains less than 3), the trapezoidal region's area increases, so the area function increases until x is 3.



5.3.6 $\int_0^2 3x^2 = x^3 \Big|_0^2 = 8 - 0 = 8.$

$\int_{-2}^2 3x^2 = x^3 \Big|_{-2}^2 = 8 - -8 = 16.$

5.3.7 $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, and $\int f'(x)\,dx = f(x) + C$.

5.3.8 It can be omitted because it doesn't change the value of $F(b) - F(a)$. For example, suppose $F(x)$ is an antiderivative of f, and so is $G(x) = F(x) + C$. Then $G(b) - G(a) = F(b) + C - (F(a) + C) = F(b) - F(a)$.

5.3.9 $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, and $\frac{d}{dx}\int_a^b f(t)\,dt = 0$. The latter is the derivative of a constant, the former follows from the Fundamental Theorem.

5.3.10 Because f is an antiderivative of f', the Fundamental Theorem assures us that $\int_a^b f'(x)\,dx = f(b) - f(a)$.

5.3.11 $\int_3^8 f'(t)\,dt = f(8) - f(3) = 20 - 4 = 16$.

5.3.12 $\int_2^7 3\,dx = 3x\Big|_2^7 = 21 - 6 = 15$. The integral represents the area of a 5×3 rectangle, which is 15.

5.3.13

a. $A(-2) = \int_{-2}^{-2} f(t)\,dt = 0$.

b. $F(8) = \int_4^8 f(t)\,dt = -9$.

c. $A(4) = \int_{-2}^{4} f(t)\,dt = 8 + 17 = 25$.

d. $F(4) = \int_4^4 f(t)\,dt = 0$.

e. $A(8) = \int_{-2}^{8} f(t)\,dt = 25 - 9 = 16$.

5.3.14

a. $A(2) = \int_0^2 f(t)\,dt = 8$.

b. $F(5) = \int_2^5 f(t)\,dt = -5$.

c. $A(0) = \int_0^0 f(t)\,dt = 0$.

d. $F(8) = \int_2^8 f(t)\,dt = -16$.

e. $A(8) = \int_0^8 f(t)\,dt = 8 - 16 = -8$.

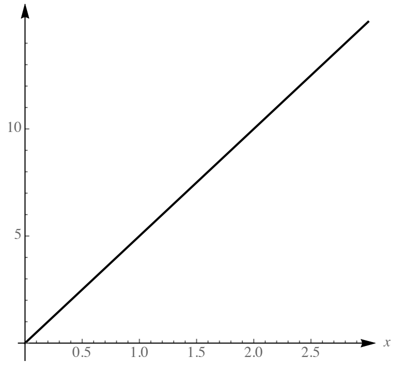
f. $A(5) = \int_0^5 f(t)\,dt = 8 - 5 = 3$.

g. $F(2) = \int_2^2 f(t)\,dt = 0$.

5.3.15

a. $A(x) = \int_0^x f(t)\,dt = \int_0^x 5\,dt = 5x$.

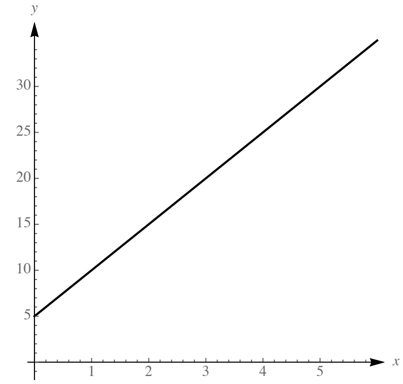
b. $A'(x) = 5 = f(x)$.



5.3.16

a. $A(x) = \int_{-5}^{x} f(t)\,dt = \int_{-5}^{x} 5\,dt = 5(x+5)$.

b. $A'(x) = 5 = f(x)$.



5.3.17

a. $A(2) = \int_0^2 t\,dt = 2$. $A(4) = \int_0^4 t\,dt = 8$. Because the region whose area is $A(x) = \int_0^x t\,dt$ is a triangle with base x and height x, its value is $\frac{1}{2}x^2$.

b. $F(4) = \int_2^4 t\,dt = 6$. $F(6) = \int_2^6 t\,dt = 16$. Because the region whose area is $A(x) = \int_2^x t\,dt$ is a trapezoid with base $x-2$ and $h_1 = 2$ and $h_2 = x$, its value is $(x-2)\frac{2+x}{2} = \frac{x^2-4}{2} = \frac{x^2}{2} - 2$.

c. We have $A(x) - F(x) = \dfrac{x^2}{2} - (\dfrac{x^2}{2} - 2) = 2$, a constant.

5.3.18

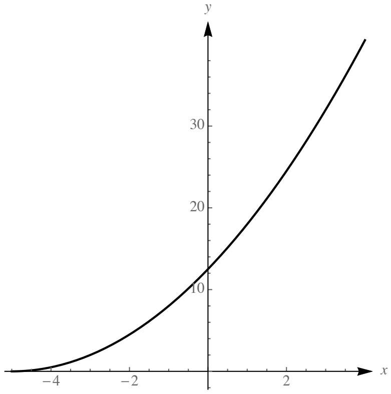
a. $A(2) = \int_1^2 (2t-2)\,dt = 1$. $A(3) = \int_1^3 (2t-2)\,dt = 4$. Because the region whose area is $A(x) = \int_1^x (2t-2)\,dt$ is a triangle with base $x-1$ and height $2x-2$, its value is $\dfrac{1}{2}\cdot(x-1)(2(x-1)) = (x-1)^2$.

b. $F(5) = \int_4^5 (2t-2)\,dt = 7$. $F(6) = \int_4^6 (2t-2)\,dt = 16$. Because the region whose area is $A(x) = \int_2^x t\,dt$ is a trapezoid with base $x-4$ and $h_1 = 6$ and $h_2 = 2x-2$, its value is $(x-4)\left(\dfrac{6+2x-2}{2}\right) = (x-4)(x+2) = x^2 - 2x - 8$.

c. We have $A(x) - F(x) = x^2 - 2x + 1 - (x^2 - 2x - 8) = 9$, a constant.

5.3.19

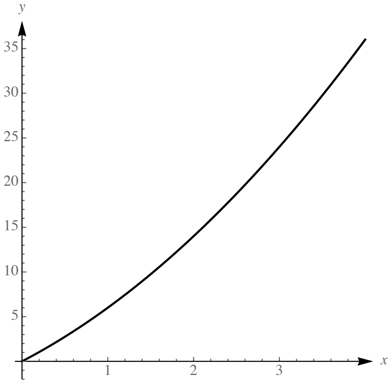
a. The region is a triangle with base $x+5$ and height $x+5$, so its area is $A(x)=\frac{1}{2}(x+5)^2$.



b. $A'(x)=x+5=f(x)$.

5.3.20

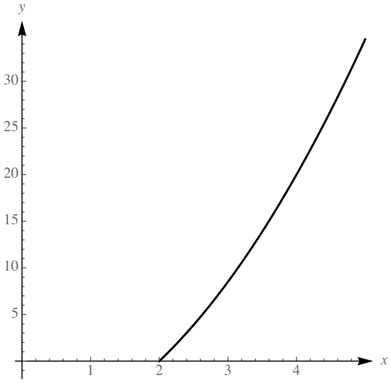
a. The region is a trapezoid with base x and heights $h_1=f(0)=5$ and $h_2=f(x)=2x+5$, so its area is $A(x)=x\cdot\dfrac{5+2x+5}{2}=x\cdot(x+5)=x^2+5x$.



b. $A'(x)=2x+5=f(x)$.

5.3.21

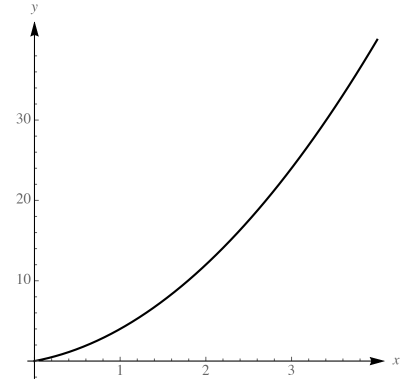
a. The region is a trapezoid with base $x-2$ and heights $h_1=f(2)=7$ and $h_2=f(x)=3x+1$, so its area is $A(x)=(x-2)\cdot\dfrac{7+3x+1}{2}=(x-2)\cdot(\frac{3}{2}x+4)=\frac{3}{2}x^2+x-8$.



b. $A'(x)=3x+1=f(x)$.

5.3.22

a. The region is a trapezoid with base x and heights $h_1 = f(0) = 2$ and $h_2 = f(x) = 4x+2$, so its area is $A(x) = (x) \cdot \dfrac{2+4x+2}{2} = (x) \cdot (2x+2) = 2x^2 + 2x$.



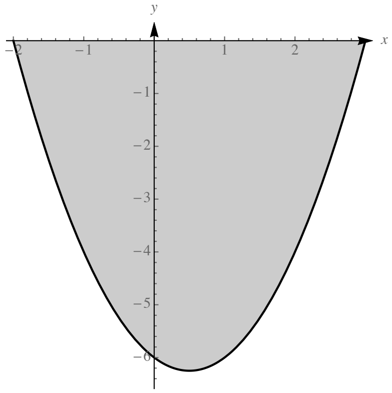
b. $A'(x) = 4x + 2 = f(x)$.

5.3.23 $\displaystyle\int_0^1 (x^2 - 2x + 3)\,dx = \left(\frac{x^3}{3} - x^2 + 3x\right)\Big|_0^1 = \frac{1}{3} - 1 + 3 - (0 - 0 + 0) = \frac{7}{3}$. It does appear that the area is between 2 and 3.

5.3.24 $\displaystyle\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)\,dx = (-\cos x + \sin x)\Big|_{-\pi/4}^{7\pi/4} = -\sqrt{2}/2 + -\sqrt{2}/2 - (-\sqrt{2}/2 + -\sqrt{2}/2) = 0$. It does appear that the area above the axis is equal to the area below, so the net area is 0.

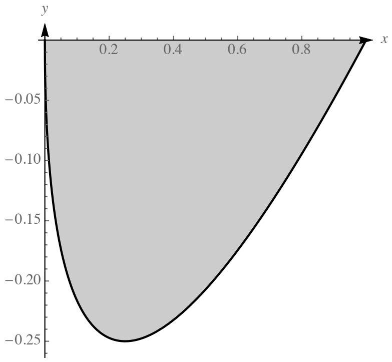
5.3.25

$\displaystyle\int_{-2}^{3} (x^2 - x - 6)\,dx = \left(\frac{x^3}{3} - \frac{x^2}{2} - 6x\right)\Big|_{-2}^{3} = -\frac{125}{6}$.



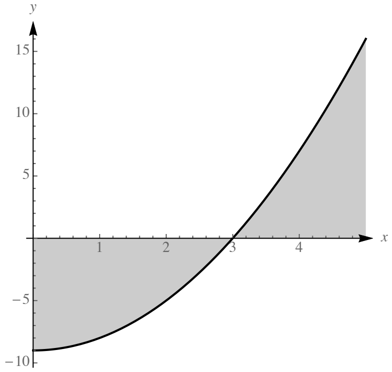
5.3.26

$\displaystyle\int_0^1 (x - \sqrt{x})\,dx = \left(\frac{x^2}{2} - \frac{2}{3}x^{3/2}\right)\Big|_0^1 = \frac{1}{2} - \frac{2}{3} - (0 - 0) = -\frac{1}{6}$.



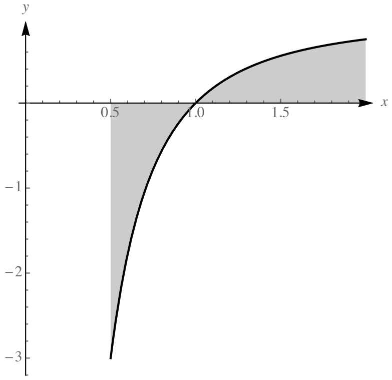
5.3.27

$$\int_0^5 (x^2-9)\,dx = \left(\frac{x^3}{3}-9x\right)\Bigg|_0^5 = \frac{125}{3}-45-(0-0) = -\frac{10}{3}.$$



5.3.28

$$\int_{1/2}^2 \left(1-\frac{1}{x^2}\right)\,dx = \left(x+\frac{1}{x}\right)\Bigg|_{1/2}^2 = 2+\frac{1}{2}-\left(\frac{1}{2}+2\right) = 0.$$



5.3.29 $\int_0^2 4x^3\,dx = x^4\Big|_0^2 = 16-0 = 16.$

5.3.30 $\int_0^2 (3x^2+2x)\,dx = \left(x^3+x^2\right)\Big|_0^2 = (8+4)-(0+0) = 12.$

5.3.31 $\int_1^8 8x^{1/3}\,dx = 6x^{4/3}\Big|_1^8 = 6(16-1) = 90.$

5.3.32 $\int_1^{16} x^{-5/4}\,dx = -4x^{-1/4}\Big|_1^{16} = -4\left(\frac{1}{2}-1\right) = 2.$

5.3.33 $\int_0^1 (x+\sqrt{x})\,dx = \left(\frac{x^2}{2}+\frac{2x^{3/2}}{3}\right)\Big|_0^1 = \frac{1}{2}+\frac{2}{3}-(0+0) = \frac{7}{6}.$

5.3.34 $\int_0^{\pi/4} 2\cos x\,dx = 2\sin x\Big|_0^{\pi/4} = \frac{2\sqrt{2}}{2}-0 = \sqrt{2}.$

5.3.35 $\int_1^9 \frac{2}{\sqrt{x}}\,dx = \int_1^9 2x^{-1/2}\,dx = 4x^{1/2}\Big|_1^9 = 12-4 = 8.$

5.3.36 $\int_4^9 \frac{2+\sqrt{t}}{\sqrt{t}}\,dt = \int_4^9 \left(2t^{-1/2}+1\right)\,dt = (4t^{1/2}+t)\Big|_4^9 = 12+9-(8+4) = 9.$

5.3.37 $\int_{-2}^{2}(x^2-4)\,dx = \left(\frac{x^3}{3}-4x\right)\Big|_{-2}^{2} = \frac{8}{3}-8-\left(-\frac{8}{3}+8\right) = \frac{16}{3}-16 = -\frac{32}{3}.$

5.3.38 $\int_0^{\ln 8} e^x\,dx = e^x\Big|_0^{\ln 8} = e^{\ln 8}-e^0 = 8-1 = 7.$

5.3.39 $\int_{1/2}^{1}(x^{-3}-8)\,dx = \left(\frac{x^{-2}}{-2}-8x\right)\Big|_{1/2}^{1} = -\frac{1}{2}-8-(-2-4) = -\frac{5}{2}.$

5.3.40 $\int_0^4 x(x-2)(x-4)\,dx = \int_0^4(x^3-6x^2+8x)\,dx = \left(\frac{x^4}{4}-2x^3+4x^2\right)\Big|_0^4 = 64-128+64-0 = 0.$

5.3.41 $\int_1^4(1-x)(x-4)\,dx = \int_1^4(-x^2+5x-4)\,dx = \left(-\frac{x^3}{3}+\frac{5x^2}{2}-4x\right)\Big|_1^4 = \frac{9}{2}.$

5.3.42 $\int_0^{1/2}\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\Big|_0^{1/2} = \sin^{-1}(1/2)-\sin^{-1}0 = \pi/6-0 = \pi/6.$

5.3.43 $\int_{-2}^{-1}x^{-3}\,dx = \frac{x^{-2}}{-2}\Big|_{-2}^{-1} = -\frac{1}{2x^2}\Big|_{-2}^{-1} = -\frac{1}{2}-\left(-\frac{1}{8}\right) = -\frac{3}{8}.$

5.3.44 $\int_0^{\pi}(1-\sin x)\,dx = (x+\cos x)\Big|_0^{\pi} = \pi-1-(0+1) = \pi-2.$

5.3.45 $\int_0^{\pi/4}\sec^2\theta\,d\theta = \tan\theta\Big|_0^{\pi/4} = 1-0 = 1.$

5.3.46 $\int_{-\pi/2}^{\pi/2}(\cos x-1)\,dx = (\sin x-x)\Big|_{-\pi/2}^{\pi/2} = 1-\frac{\pi}{2}-\left(-1-\left(-\frac{\pi}{2}\right)\right) = 2-\pi.$

5.3.47 $\int_1^2\frac{3}{t}\,dt = 3\ln|t|\Big|_1^2 = 3\ln 2-3\ln 1 = \ln 8.$

5.3.48 $\int_4^9\frac{x-\sqrt{x}}{x^2}\,dx = \int_4^9(x^{-1}-x^{-3/2})\,dx = (\ln|x|+2x^{-1/2})\Big|_4^9 = \ln 9+\frac{2}{3}-(\ln 4+1) = \ln\left(\frac{9}{4}\right)-\frac{1}{3}.$

5.3.49 $\int_1^8\sqrt[3]{y}\,dy = \frac{3}{4}y^{4/3}\Big|_1^8 = 12-\frac{3}{4} = \frac{45}{4}.$

5.3.50 $\frac{1}{2}\int_0^{\ln 2}e^x\,dx = \frac{1}{2}\left(e^x\Big|_0^{\ln 2}\right) = \frac{1}{2}(2-1) = \frac{1}{2}.$

5.3.51

$$\begin{aligned}\int_1^4\frac{x-2}{\sqrt{x}}\,dx &= \int_1^4\left(\frac{x}{\sqrt{x}}-\frac{2}{\sqrt{x}}\right)dx = \int_1^4\left(x^{1/2}-2x^{-1/2}\right)dx\\ &= \left(\frac{2}{3}x^{3/2}-4x^{1/2}\right)\Big|_1^4 = \frac{16}{3}-8-\left(\frac{2}{3}-4\right) = \frac{14}{3}-\frac{12}{3} = \frac{2}{3}.\end{aligned}$$

5.3.52 $\int_1^2\left(\frac{2}{s}-\frac{4}{s^3}\right)ds = \left(2\ln|s|+\frac{2}{s^2}\right)\Big|_1^2 = 2\ln 2+\frac{1}{2}-(0+2) = \ln 4-\frac{3}{2}.$

5.3.53 $\displaystyle\int_0^{\pi/3} \sec x \tan x\, dx = \sec x \,\Big|_0^{\pi/3} = 2 - 1 = 1.$

5.3.54 $\displaystyle\int_{\pi/4}^{\pi/2} \csc^2 \theta\, d\theta = -\cot\theta \,\Big|_{\pi/4}^{\pi/2} = 0 + 1 = 1.$

5.3.55 $\displaystyle\int_{\pi/4}^{3\pi/4} (\cot^2 x + 1)\, dx = \int_{\pi/4}^{3\pi/4} \csc^2 x\, dx = -\cot x\Big|_{\pi/4}^{3\pi/4} = -(-1-1) = 2.$

5.3.56 $\displaystyle\int_0^1 10e^{x+3}\, dx = 10e^{x+3}\Big|_0^1 = 10e^4 - 10e^3 = 10e^3(e-1).$

5.3.57 $\displaystyle\int_1^{\sqrt{3}} \frac{1}{1+x^2}\, dx = \tan^{-1}\Big|_1^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1} 1 = \pi/3 - \pi/4 = \pi/12.$

5.3.58 $\displaystyle\int_0^{\pi/4} \sec x(\sec x + \cos x)\, dx = \int_0^{\pi/4} (\sec^2 x + 1)\, dx = (\tan x + x)\Big|_0^{\pi/4} = 1 + \pi/4 - (0+0) = 1 + \pi/4.$

5.3.59 $\displaystyle\int_1^2 \frac{z^2+4}{z}\, dz = \int_1^2 \left(z + \frac{4}{z}\right)\, dz = \left(\frac{z^2}{2} + 4\ln z\right)\Bigg|_1^2 = 2 + 4\ln 2 - \left(\frac{1}{2} + 0\right) = \ln 16 + \frac{3}{2}.$

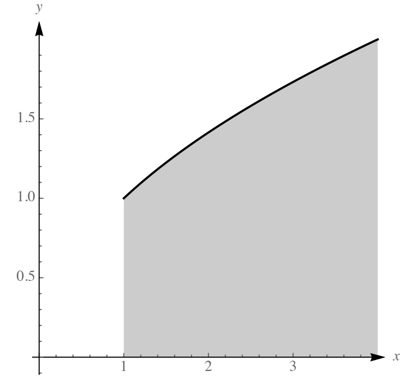
5.3.60 $\displaystyle\int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x\,\Big|_{\sqrt{2}}^2 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$

5.3.61 $\displaystyle\int_0^{\pi} f(x)\, dx = \int_0^{\pi/2} (\sin x + 1)\, dx + \int_{\pi/2}^{\pi} (2\cos x + 2)\, dx = (-\cos x + x)\Big|_0^{\pi/2} + (2\sin x + 2x)\Big|_{\pi/2}^{\pi} =$ $(0 + \pi/2) - (-1 + 0) + (0 + 2\pi) - (2 + \pi) = 3\pi/2 - 1.$

5.3.62 $\displaystyle\int_1^3 g(x)\, dx = \int_1^2 (3x^2 + 4x + 1)\, dx + \int_2^3 (2x+5)\, dx = (x^3 + 2x^2 + x)\Big|_1^2 + (x^2 + 5x)\Big|_2^3 = (8+8+2) -$ $(1+2+1) + (9+15) - (4+10) = 24.$

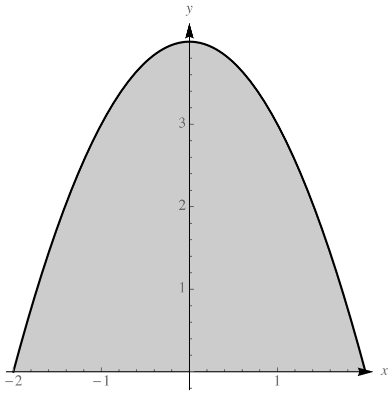
5.3.63

The area (and net area) of this region is given by $\displaystyle\int_1^4 \sqrt{x}\, dx = \frac{2}{3}x^{3/2}\,\Big|_1^4 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}.$



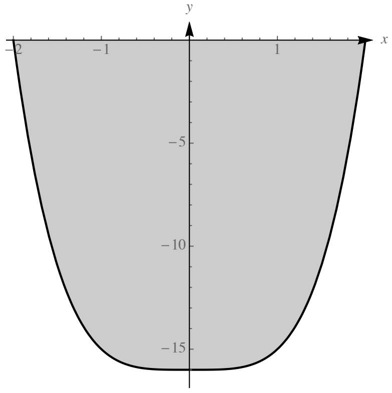
5.3.64

The area (and net area) of this region is given by $\int_{-2}^{2}(4-x^2)\,dx = \left(4x-\frac{x^3}{3}\right)\Big|_{-2}^{2} = 8-\frac{8}{3}-\left(-8+\frac{8}{3}\right) = 16-\frac{16}{3} = \frac{32}{3}$.



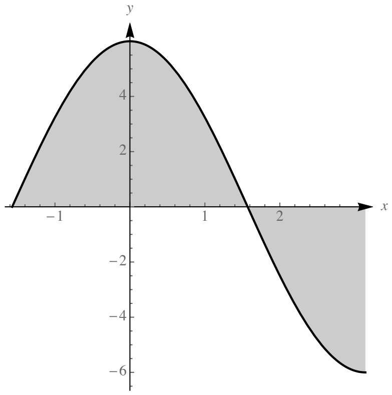
5.3.65

The net area of this region is given by $\int_{-2}^{2}(x^4-16)\,dx = \left(\frac{x^5}{5}-16x\right)\Big|_{-2}^{2} = \frac{32}{5}-32-\left(-\frac{32}{5}+32\right) = \frac{64}{5}-64 = -\frac{256}{5}$. Thus the area is $\frac{256}{5}$.



5.3.66

The net area of this region is given by $\int_{-\pi/2}^{\pi} 6\cos x\,dx = 6\sin x\Big|_{-\pi/2}^{\pi} = 0 - -6 = 6$. The area is given by $\int_{-\pi/2}^{\pi/2} 6\cos x\,dx - \int_{\pi/2}^{\pi} 6\cos x\,dx = 6\sin x\Big|_{-\pi/2}^{\pi/2} - 6\sin x\Big|_{\pi/2}^{\pi} = 6 - -6 - (0-6) = 18$.



5.3.67 Because this region is below the axis, the area of it is given by $-\int_{2}^{4}(x^2-25)\,dx = -\left(\frac{x^3}{3}-25x\right)\Big|_{2}^{4} = -\left(\frac{64}{3}-100-\left(\frac{8}{3}-50\right)\right) = 50-\frac{56}{3} = \frac{94}{3}$.

5.3.68 Because the function is below the axis between −1 and 1, and is above the axis between 1 and 2, the area of the bounded region is given by $-\int_{-1}^{1}(x^3-1)\,dx+\int_{1}^{2}(x^3-1)\,dx = -\left(\frac{x^4}{4}-x\right)\Big|_{-1}^{1}+\left(\frac{x^4}{4}-x\right)\Big|_{1}^{2} =$ $-\left(\frac{1}{4}-1-\left(\frac{1}{4}+1\right)\right)+\left(4-2-\left(\frac{1}{4}-1\right)\right) = 2+2.75 = 4.75.$

5.3.69 Because this region is below the axis, the area of it is given by $-\int_{-2}^{-1}\frac{1}{x}\,dx = -\left(\ln|x|\,\Big|_{-2}^{-1}\right) =$ $\ln 2-\ln 1 = \ln 2.$

5.3.70 Because the function is above the axis between -1 and 0 and is below the axis between 0 and 2, the area is given by $\int_{-1}^{0}(x^3-x^2-2x)\,dx-\int_{0}^{2}(x^3-x^2-2x)\,dx = \left(\frac{x^4}{4}-\frac{x^3}{3}-x^2\right)\Big|_{-1}^{0}-\left(\frac{x^4}{4}-\frac{x^3}{3}-x^2\right)\Big|_{0}^{2} =$ $\left(0-\left(\frac{1}{4}+\frac{1}{3}-1\right)\right)-\left(4-\frac{8}{3}-4-0\right) = \frac{5}{12}+\frac{8}{3} = \frac{37}{12}.$

5.3.71 The area is given by

$$-\int_{-\pi/4}^{0}\sin x\,dx+\int_{0}^{3\pi/4}\sin x\,dx = \left(\cos x\,\Big|_{-\pi/4}^{0}\right)+\left(-\cos x\,\Big|_{0}^{3\pi/4}\right) = \left(1-\frac{\sqrt{2}}{2}\right)+\left(1+\frac{\sqrt{2}}{2}\right) = 2.$$

5.3.72 Because this region is below the axis, the area is given by $-\int_{\pi/2}^{\pi}\cos x\,dx = -\left(\sin x\,\Big|_{\pi/2}^{\pi}\right) =$ $\sin(\pi/2)-\sin(\pi) = 1.$

5.3.73 By a direct application of the Fundamental Theorem, this is x^2+x+1.

5.3.74 $\frac{d}{dx}\int_{x}^{1}e^{t^2}\,dt = -\frac{d}{dx}\int_{1}^{x}e^{t^2}\,dt = -e^{x^2}.$

5.3.75 This is $-\frac{d}{dx}\int_{1}^{x}\sqrt{t^4+1}\,dt = -\sqrt{x^4+1}.$

5.3.76 This is $-\frac{d}{dx}\int_{0}^{x}\frac{dp}{p^2+1} = \frac{-1}{x^2+1}.$

5.3.77 By the Fundamental Theorem and the chain rule, this is $\frac{1}{x^6}\cdot 3x^2 = \frac{3}{x^4}.$

5.3.78 $\frac{d}{dx}\int_{0}^{x^2}\frac{1}{t^2+4}\,dt = \frac{2x}{x^4+4}.$

5.3.79 $\frac{d}{dx}\int_{0}^{\cos x}(t^4+6)\,dt = -(\cos^4 x+6)\sin x.$

5.3.80 $\frac{d}{dw}\int_{0}^{\sqrt{w}}\ln(x^2+1)\,dx = \ln(w+1)\cdot\frac{1}{2\sqrt{w}} = \frac{\ln(w+1)}{2\sqrt{w}}.$

5.3.81 $\frac{d}{dz}\int_{\sin z}^{10}\frac{dt}{t^4+1} = -\frac{d}{dz}\int_{10}^{\sin z}\frac{dt}{t^4+1} = -\frac{1}{\sin^4+1}\cdot\cos z = -\frac{\cos z}{\sin^4 z+1}.$

5.3.82 $\frac{d}{dy}\int_{y^3}^{10}\sqrt{x^6+1}\,dx = -\frac{d}{dy}\int_{10}^{y^3}\sqrt{x^6+1}\,dx = -\sqrt{y^{18}+1}\cdot 3y^2 = -3y^2\sqrt{y^{18}+1}.$

5.3.83 $\frac{d}{dt}\left(\int_{1}^{t}\frac{3}{x}\,dx-\int_{t^2}^{1}\frac{3}{x}\,dx\right) = \frac{d}{dt}\int_{1}^{t}\frac{3}{x}\,dx+\frac{d}{dt}\int_{1}^{t^2}\frac{3}{x}\,dx = \frac{3}{t}+\frac{6t}{t^2} = \frac{9}{t}.$

5.3.84 $\frac{d}{dt}\left(\int_0^t \frac{dx}{1+x^2} + \int_0^{1/t} \frac{dx}{1+x^2}\right) = \frac{1}{1+t^2} + \frac{1}{1+(1/t)^2}\left(-\frac{1}{t^2}\right) = \frac{1}{1+t^2} - \frac{1}{1+t^2} = 0.$

5.3.85 This can be written as

$$\frac{d}{dx}\left(\int_{-x}^{0} \sqrt{1+t^2}\,dt + \int_0^x \sqrt{1+t^2}\,dt\right) = \frac{d}{dx}\left(-\int_0^{-x} \sqrt{1+t^2}\,dt + \int_0^x \sqrt{1+t^2}\,dt\right)$$
$$= -\sqrt{1+(-x)^2}(-1) + \sqrt{1+x^2} = 2\sqrt{1+x^2}.$$

5.3.86 This can be written as

$$\frac{d}{dx}\left(\int_{e^x}^{0} \ln(t^2)\,dt + \int_0^{e^{2x}} \ln(t^2)\,dt\right) = \frac{d}{dx}\left(-\int_0^{e^x} \ln(t^2)\,dt + \int_0^{e^{2x}} \ln(t^2)\,dt\right)$$
$$= -\ln((e^x)^2)\cdot e^x + \ln((e^{2x})^2)\cdot 2e^{2x} = -2xe^x + 8xe^{2x} = 2xe^x(4e^x - 1).$$

5.3.87

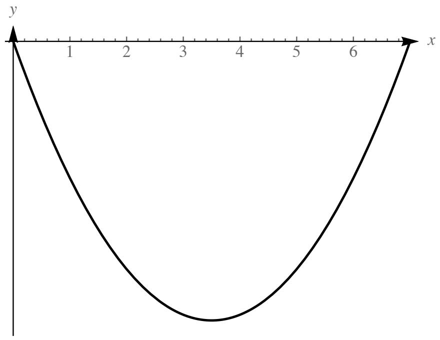
(a) matches with (C) – its area function is increasing linearly.
(b) matches with (B) – its area function increases then decreases.
(c) matches with (D) – its area function is always increasing on $[0, b]$, although not linearly.
(d) matches with (A) – its area function decreases at first and then eventually increases.

5.3.88

a. It appears that $A(x) = 0$ for $x = 0$ and $x = 10$.

b. A has a local minimum at $x = 5$ where the area function changes from decreasing to increasing.

c.

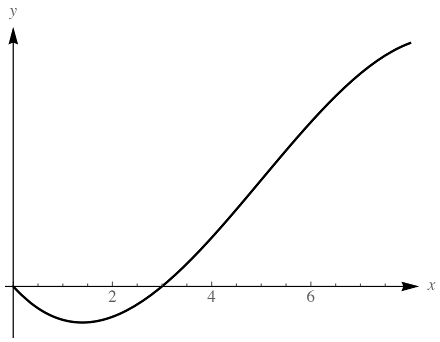


5.3.89

a. It appears that $A(x) = 0$ for $x = 0$ and at about $x = 3$.

b. A has a local minimum at about $x = 1.5$ where the area function changes from decreasing to increasing, and a local max at around $x = 8.5$ where the area function changes from increasing to decreasing.

c.

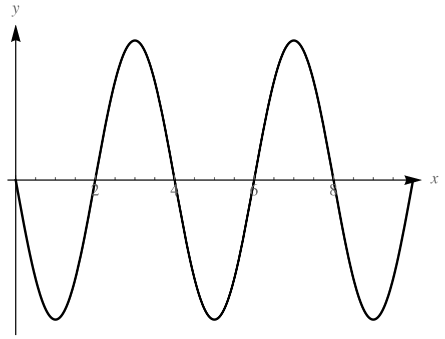


5.3.90

a. It appears that $A(x) = 0$ for $x = 0$, $x = 2$, $x = 4$, $x = 6$, $x = 8$, and $x = 10$.

b. A has a local minimum at $x = 1$, $x = 5$, and $x = 9$ where the area function changes from decreasing to increasing, and a local maximum at $x = 3$ and $x = 7$ where the area function changes from increasing to decreasing.

c.

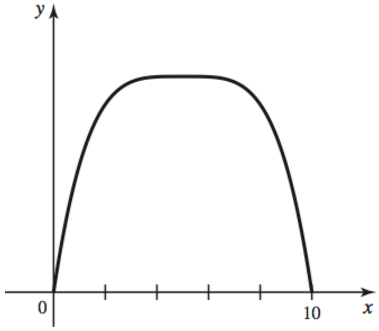


5.3.91

a. It appears that $A(x) = 0$ for $x = 0$ and $x = 10$.

b. A has a local maximum at $x = 5$ where the area function changes from increasing to decreasing.

c.



5.3.92 $A(1) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$. $A(2) = \frac{1}{2} \cdot 2 \cdot 2 = 2$. $A(4) = 2 + 2^2 = 6$. $A(6) = 6 + \frac{1}{4}\pi \cdot 2^2 = 6 + \pi$.

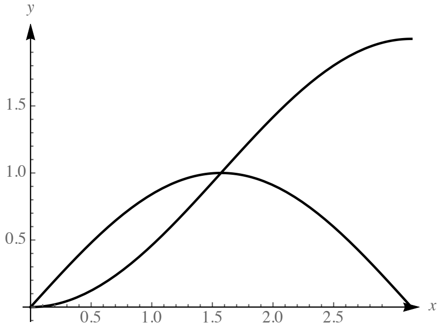
5.3.93 $A(2) = -\frac{1}{4}\pi \cdot 2^2 = -\pi$. $A(5) = -\pi + \frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2} - \pi$. $A(8) = \frac{9}{2} - \pi + \frac{1}{2} \cdot 3 \cdot 3 = 9 - \pi$. $A(12) = 9 - \pi - \frac{1}{2} \cdot 4 \cdot 2 = 5 - \pi$.

5.3.94

a. $A(x) = \int_0^x \sin t\, dt = -\cos t \Big|_0^x = -\cos x - (-1) = 1 - \cos x$.

c. $A(\pi/2) = 1 - \cos(\pi/2) = (1 - 0) = 1$ and $A(\pi) = 1 - \cos\pi = 1 - (-1) = 2$. The area under the curve between 0 and $\pi/2$ is the same as the area under the curve between $\pi/2$ and π.

b.

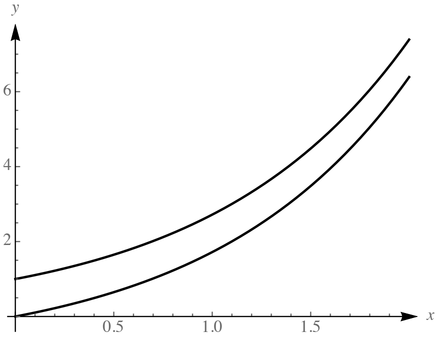


5.3.95

a. $A(x) = \int_0^x e^t\,dt = e^t \Big|_0^x = e^x - (1)$.

c. $A(\ln 2) = e^{\ln 2} - 1 = 2 - 1 = 1$. $A(\ln 4) = e^{\ln 4} - 1 = 4 - 1 = 3$. There is twice as much area under the curve between $\ln 2$ and $\ln 4$ as there is between 0 and $\ln 2$.

b.

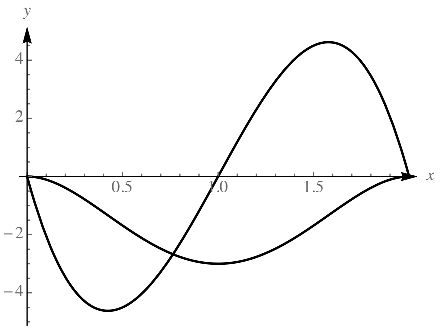


5.3.96

a. $A(x) = \int_0^x -12t(t-1)(t-2)\,dt = \int_0^x -12t^3 + 36t^2 - 24t\,dt = \left(-3t^4 + 12t^3 - 12t^2\right)\Big|_0^x = -3x^4 + 12x^3 - 12x^2$.

c. $A(1) = -3 + 12 - 12 = -3$. $A(2) = -48 + 96 - 48 = 0$. The area bounded between the x-axis and the curve on $[0,1]$ is equal to the area bounded between the x-axis and the curve on $[1,2]$.

b.

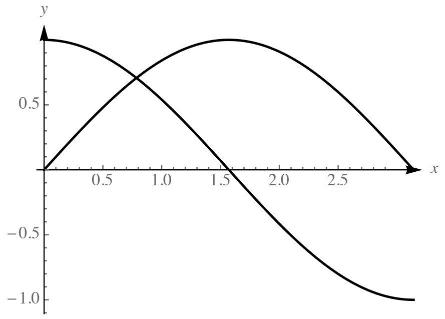


5.3.97

a. $A(x) = \int_0^x \cos t\,dt = \sin t\Big|_0^x = \sin x$.

c. $A(\pi/2) = 1$. $A(\pi) = 0$.

b.

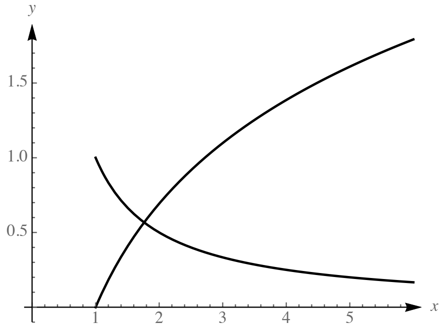


5.3.98

a. $A(x) = \int_1^x 1/t\,dt = \ln t\Big|_1^x = \ln x$.

c. $A(4) = \ln 4$ and $A(6) = \ln 6$.

b.

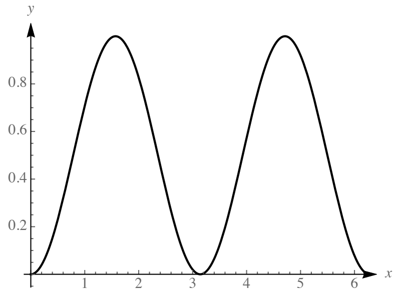


5.3.99 $f'(x) = x^2(x-3)(x-4)$, so the critical points of f are $x = 0$, $x = 3$, and $x = 4$. $f' > 0$ on $(-\infty, 0)$ and on $(0, 3)$ and on $(4, \infty)$, so f is increasing on those intervals, while $f' < 0$ on $(3, 4)$, so f is decreasing on that interval.

5.3.100 $g(x) = -\displaystyle\int_0^x \frac{t}{t^2+1}\,dt$, so $g'(x) = -\dfrac{x}{x^2+1}$ and $g''(x) = -\dfrac{x^2+1-2x^2}{(x^2+1)^2} = \dfrac{x^2-1}{(x^2+1)^2}$. $g''(x) < 0$ on $(-1, 1)$ (so g is concave down there), while $g''(x) > 0$ on $(-\infty, -1)$ and on $(1, \infty)$, so g is concave up on those intervals.

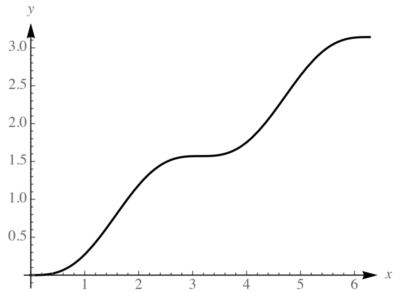
5.3.101

a.



b. $g'(x) = \sin^2 x$.

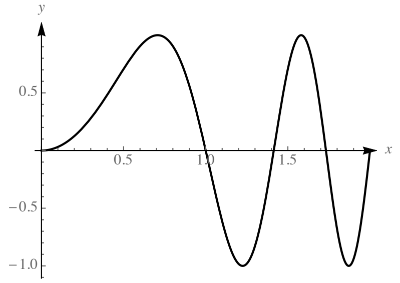
c.



Note that g' is always positive, so g is always increasing. There are inflection points where g' changes from increasing to decreasing, and vice versa.

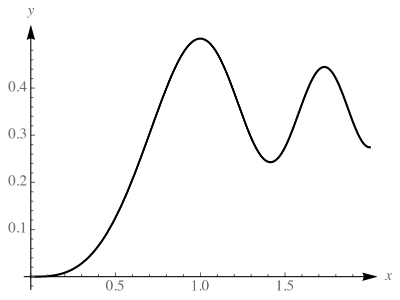
5.3.102

a.



b. $g'(x) = \sin(\pi x^2)$.

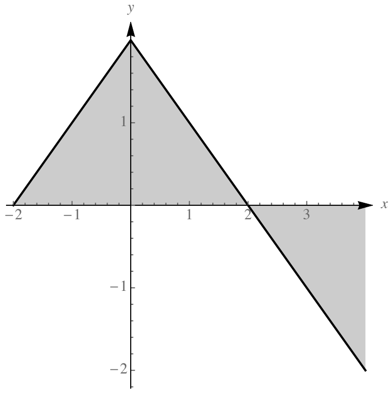
c.



Note that g is increasing where $g' > 0$ and g is decreasing when $g' < 0$. Also, where g' is increasing, g is concave up and where g' is decreasing, g is concave down.

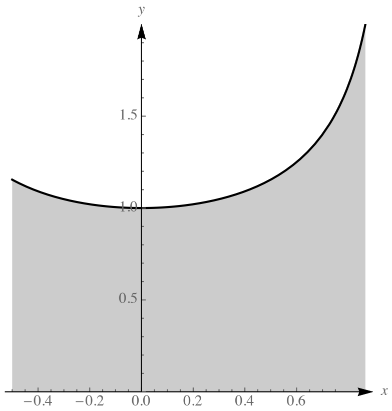
5.3.103

We can use geometry – there is a triangle with base 4 and height 2 and a triangle with base 2 and height 2, so the total area is $\frac{1}{2}\cdot 4\cdot 2+\frac{1}{2}\cdot 2\cdot 2 = 6$.



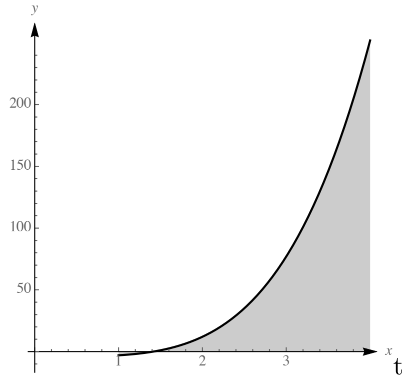
5.3.104

Because the region is above the axis, we can simply compute $\int_{-1/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_{-1/2}^{\sqrt{3}/2} = \frac{\pi}{3} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{2}$.



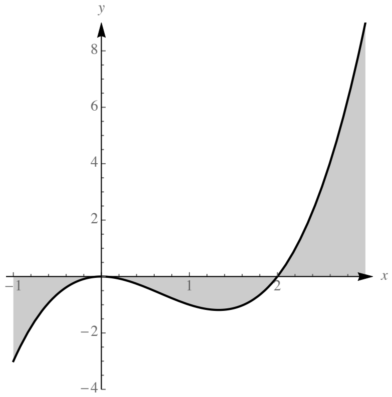
5.3.105

Because the region is below the axis on $[1,\sqrt{2}]$ and above on $[\sqrt{2},4]$ we need to compute $\int_{\sqrt{2}}^{4}(x^4 - 4)\,dx - \int_{1}^{\sqrt{2}}(x^4 - 4)\,dx = \left(\frac{x^5}{5} - 4x\right)\Big|_{\sqrt{2}}^{4} - \left(\frac{x^5}{5} - 4x\right)\Big|_{1}^{\sqrt{2}} = \frac{1024}{5} - 16-(4\sqrt{2}/5-4\sqrt{2})-(4\sqrt{2}/5-4\sqrt{2})+\frac{1}{5}-4 = 185 + \frac{32\sqrt{2}}{5}$



5.3.106

Because the function is below (or touching) the axis on $[-1,2]$ and above on $[2,3]$, the area is given by $\int_2^3 (x^3 - 2x^2)\,dx - \int_{-1}^2 (x^3 - 2x^2)\,dx = \left(\frac{x^4}{4} - \frac{2x^3}{3}\right)\Big|_2^3 - \left(\frac{x^4}{4} - \frac{2x^3}{3}\right)\Big|_{-1}^2 = \left(\frac{81}{4} - 18\right) - \left(4 - \frac{16}{3}\right) - \left(4 - \frac{16}{3}\right) + \left(\frac{1}{4} + \frac{2}{3}\right) = \frac{41}{2} - 26 + \frac{34}{3} = \frac{35}{6}$.



5.3.107

a. True. The net area under the curve increases as x increases, as long as f is above the axis.

b. True. The net area decreases as x increases, as long as f is below the axis.

c. False. These do not have the same derivative, so they are not antiderivatives of the same function.

d. True, because the two functions differ by a constant, and thus have the same derivative.

e. True, because the derivative of a constant is zero.

5.3.108

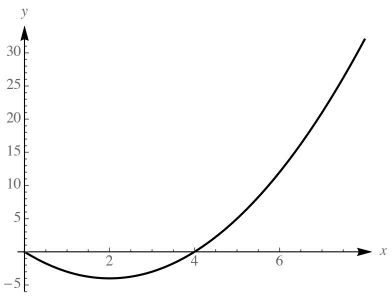
$$\lim_{x\to 2} \frac{\int_2^x \sqrt{t^2+t+3}\,dt}{x^2-4} = \lim_{x\to 2} \frac{\sqrt{x^2+x+3}}{2x} = \frac{3}{4}.$$

5.3.109 Because $\frac{d}{db}\int_{-1}^b x^2(3-x)\,dx = b^2(3-b)$ we see that this function of b has critical points at $b=0$ and $b=3$. Note also that the integrand is positive on $[0,3]$, but is negative on $[3,\infty)$. So the maximum for this area function occurs at $b=3$.

5.3.110 The function $f(x) = 8 + 2x - x^2 = (4-x)(2+x)$ is 0 for $x=4$ and $x=-2$, and is positive on $(-2,4)$ and negative on $(-\infty,-2)$ and on $(4,\infty)$. Thus, the largest possible value for the area $\int_a^b f(x)\,dx$ is when $a=-2$ and $b=4$.

5.3.111

a.



b. We seek b so that $\int_0^b (x^2-4x) = 0$ for $b>0$. We have $\left(\frac{x^3}{3} - 2x^2\right)\Big|_0^b = \frac{b^3}{3} - 2b^2 = 0$, which occurs for $\frac{b}{3} = 2$, or $b=6$.

c. We seek b so that $\int_0^b (x^2-ax) = 0$ for $b>0$. We have $\left(\frac{x^3}{3} - \frac{ax^2}{2}\right)\Big|_0^b = \frac{b^3}{3} - \frac{ab^2}{2} = 0$, which occurs for $\frac{b}{3} = \frac{a}{2}$, or $b = \frac{3a}{2}$.

5.3.112 If $0 < x < a$, then $x > 0$, $x - a < 0$, and $x - b < 0$, so the product of these three quantities is positive. If $a < x < b$, then $x > 0$, $x - a > 0$, and $x - b < 0$, so the product of these three quantities is negative. The region between $x = 0$ and $x = a$, which is above the x-axis, has area

$$\int_0^a x(x-a)(x-b)\,dx = \int_0^a (x^3 - (a+b)x^2 + abx)\,dx = \left(\frac{x^4}{4} - \frac{a+b}{3}x^3 + \frac{ab}{2}x\right)\Bigg|_0^a = \frac{a^3(2b-a)}{12},$$

while the region between $x = a$ and $x = b$, which is below the x-axis, has area

$$-\int_a^b x(x-a)(x-b)\,dx = -\int_a^b (x^3 - (a+b)x^2 + abx)\,dx = -\left(\frac{x^4}{4} - \frac{a+b}{3}x^3 + \frac{ab}{2}x\right)\Bigg|_a^b = \frac{(b-a)^3(a+b)}{12}.$$

These are equal when $a^3(2b-a) = (b-a)^3(a+b)$. Divide through by a^4 to obtain

$$2 \cdot \frac{b}{a} - 1 = \left(\frac{b}{a} - 1\right)^3 \left(1 + \frac{b}{a}\right).$$

Let $c = \frac{b}{a}$; then $2c - 1 = (c-1)^3(c+1) = c^4 - 2c^3 + 2c - 1$, so that $c^4 - 2c^3 = 0$. Because $b > 0$ we must have $c > 0$, and the only nonzero root is $c = 2$. Thus $c = \frac{b}{a} = 2$, so $b = 2a$.

5.3.113 Differentiating both sides of the given equation yields $f(x) = -2\sin x + 3$.

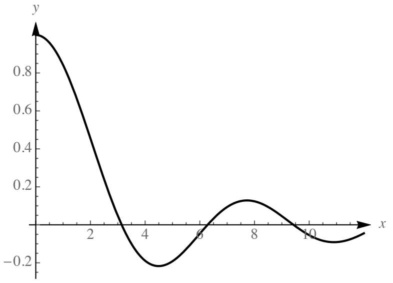
5.3.114 Suppose that a maximum of A occurs at $x = c$, and that A is not a constant function near c. Then A changes from increasing to decreasing at c. But because A is the net area from 0 to x, the only way for A to change from increasing to decreasing is for f to change from above the axis to below, so it must be the case that $f > 0$ to the left of c and $f < 0$ to the right of c, but because f is continuous, this implies that $f(c) = 0$. An analogous argument holds for the case when A has a minimum at c.

For $f(x) = x^2 - 10x$, note that $A(x) = \int_0^x (t^2 - 10t)\,dt = \frac{x^3}{3} - 5x^2$, and that this function has a minimum at $x = 10$, because $A'(x) = x^2 - 10x = f(x)$ changes from negative to positive at $x = 10$, so that A changes from decreasing to increasing there.

5.3.115 Using a computer or calculator, we obtain:

x	500	1000	1500	2000
$S(x)$	1.5726	1.57023	1.57087	1.57098

This appears to be approaching $\frac{\pi}{2}$.
Note that between 0 and π, the area is approximately half the area of a rectangle with height 1 and base π, and then from π on there is approximately as much area above the axis as below.



5.3.116 By the Fundamental Theorem, $S'(x) = \frac{\sin x}{x}$, so $S''(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x}{x} - \frac{\sin x}{x^2}$. Thus, $S'''(x) = \frac{-x\sin x - \cos x}{x^2} - \frac{x^2\cos x - 2x\sin x}{x^4} = \frac{-\sin x}{x} - \frac{2\cos x}{x^2} + \frac{2\sin x}{x^3}$.

$$xS'(x) + 2S''(x) + xS'''(x) = \sin x + \frac{2\cos x}{x} - \frac{2\sin x}{x^2} + -\sin x - \frac{2\cos x}{x} + \frac{2\sin x}{x^2} = 0.$$

5.3.117 By the Fundamental Theorem, $S'(x) = \sin x^2$, so $S''(x) = 2x\cos x^2$, so

$$S'(x)^2 + \left(\frac{S''(x)}{2x}\right)^2 = \sin^2 x^2 + \cos^2 x^2 = 1.$$

5.3.118

a. If m^* is the minimum and M^* is the maximum of f on $[a,b]$, then for every possible subinterval $[x_i, x_{i+1}]$ of $[a,x]$ of width h_i, we have $m^*h_i \leq f(x_i^*)h_i \leq M^*h_i$ where x_i^* is any value on $[x_i, x_{i+1}]$. Adding these up over any partition of $[a,x]$ and taking the limit as $n \to \infty$ gives $m^*(x-a) \leq \int_a^x f(t)\,dt \leq M^*(x-a)$, so $m^*(x-a) \leq A(x) \leq M^*(x-a)$. Now consider $\lim_{x \to a^+} m^*(x-a) = \lim_{x \to a^+} M^*(x-a) = 0$. Thus by the Squeeze Theorem we must have $\lim_{x \to a^+} A(x) = 0 = A(a)$, so A is continuous from the right at $x = a$.

b. First note that $A(b) - A(x) = \int_a^b f(t)\,dt - \int_a^x f(t)\,dt = \int_x^b f(t)\,dt$. Now if m^* is the minimum and M^* is the maximum of f on $[a,b]$, then for every possible subinterval $[x_i, x_{i+1}]$ of $[b,x]$ of width h_i, we have $m^*h_i \leq f(x_i^*)h_i \leq M^*h_i$ where x_i^* is any value on $[x_i, x_{i+1}]$. Adding these up over any partition of $[b,x]$ and taking the limit as $n \to \infty$ gives $m^*(b-x) \leq \int_x^b f(t)\,dt \leq M^*(b-x)$, so $m^*(b-x) \leq A(b) - A(x) \leq M^*(b-x)$. Now consider $\lim_{x \to b^-} m^*(b-x) = \lim_{x \to b^-} M^*(b-x) = 0$. Thus by the Squeeze Theorem we must have $\lim_{x \to b^-} (A(b) - A(x)) = 0$, so $\lim_{x \to b^-} A(x) = A(b)$, and A is continuous from the left at $x = b$.

5.3.119

a. By definition of Reimann sums, $\int_a^b f'(x)\,dx$ is approximated by $\sum_{k=1}^{n} f'(x_{k-1})\Delta x$. But $f'(x_{k-1}) = \lim_{h \to 0} \frac{f(x_{k-1}+h) - f(x_{k-1})}{h}$. If $h = \Delta x$, then we have

$$f'(x_{k-1}) \approx \frac{f(x_{k-1}+\Delta x) - f(x_{k-1})}{\Delta x} = \frac{f(x_k) - f(x_{k-1})}{\Delta x},$$

so that

$$\int_a^b f'(x)\,dx \approx \sum_{k=1}^{n} \frac{f(x_k) - f(x_{k-1})}{\Delta x} \cdot \Delta x.$$

b. Canceling the Δx factors we obtain

$$\begin{aligned}
\int_a^b f'(x)\,dx &\approx \sum_{k=1}^{n} \frac{f(x_k) - f(x_{k-1})}{\Delta x} \cdot \Delta x \\
&= \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) \\
&= (f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + \cdots + (f(x_{n-1}) - f(x_{n-2})) + (f(x_n) - f(x_{n-1})) \\
&= f(x_n) - f(x_0) = f(b) - f(a).
\end{aligned}$$

c. The analogy between the two situations is that both (a) the sum of difference quotients and (b) integral of a derivative are equal to the difference in function values at the endpoints.

5.4 Working with Integrals

5.4.1 If f is odd, it is symmetric about the origin, which guarantees that between $-a$ and a, there is as much area above the axis and under f as there is below the axis and above f, so the net area must be 0.

5.4.2 If f is even, it is symmetric about the the $y-$axis, which guarantees that the region between $-a$ and 0 has the same net area as the region between 0 and a, so $\int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$.

5.4.3

a. Because $\int_{-8}^{8} f(x)\,dx = 18 = 2\int_{0}^{8} f(x)\,dx$, we have $\int_{0}^{8} f(x)\,dx = \frac{18}{2} = 9$.

b. Because $xf(x)$ is an odd function when $f(x)$ is even, we have $\int_{-8}^{8} xf(x)\,dx = 0$.

5.4.4

a. Because $f(x)$ is odd, $\int_{-4}^{0} f(x)\,dx = -\int_{0}^{4} f(x)\,dx$. We have

$$\int_{-4}^{8} f(x)\,dx = \int_{-4}^{0} f(x)\,dx + \int_{0}^{8} f(x)\,dx = -\int_{0}^{4} f(x)\,dx + \int_{0}^{8} f(x)\,dx = -3+9 = 6.$$

b. $\int_{-8}^{4} f(x)\,dx = \int_{-8}^{0} f(x)\,dx + \int_{0}^{4} f(x)\,dx = -9+3 = -6.$

5.4.5 The integrand can be written as $(5x^4+2x^2+1)+(3x^3+x)$. The function $f(x) = 5x^4+2x^2+1$ is an even function and the function $g(x) = 3x^3+x$ is an odd function. Thus,

$$\int_{-4}^{4} (f(x)+g(x))\,dx = \int_{-4}^{4} f(x)\,dx + \int_{-4}^{4} g(x)\,dx = 2\int_{0}^{4} f(x)\,dx + 0 = 2\int_{0}^{4} (5x^4+2x^2+1)\,dx.$$

5.4.6 The number 2 should go in the first blank and the function $\cos x$ in the second.

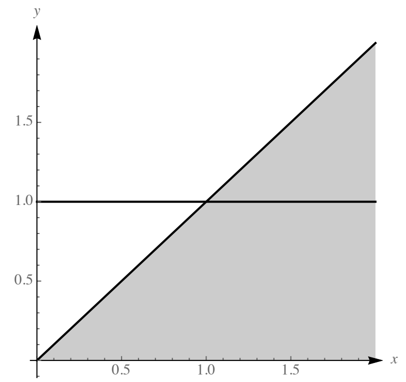
$$\int_{-\pi}^{\pi} (\sin x + \cos x)\,dx = \int_{-\pi}^{\pi} \sin x\,dx + \int_{-\pi}^{\pi} \cos x\,dx = 0 + 2\int_{0}^{\pi} \cos x\,dx = 2\int_{0}^{\pi} \cos x\,dx.$$

5.4.7 $f(x) = x^{12}$ is an even function, because $f(-x) = (-x)^{12} = x^{12} = f(x)$. $g(x) = \sin x^2$ is also even, because $g(-x) = \sin((-x)^2) = \sin x^2 = g(x)$.

5.4.8 The average value of a function f on $[a,b]$ is $\frac{1}{b-a}\int_{a}^{b} f(x)\,dx$. This is analogous to "adding up all the value of f and dividing by how many there are" – in the sense that computing the interval is like adding up all the values of the function, and dividing by $b-a$ is like dividing by how many x values there are.

5.4.9 The average value of a continuous function on a closed interval $[a,b]$ will always be between the maximum and the minimum value of f on that interval. Because the function is continuous, the Intermediate Value Theorem assures us that the function will take on each value between the maximum and the minimum somewhere on the interval.

5.4.10 Note that the area of the triangle is $\frac{1}{2}\cdot 2\cdot 2 = 2$, so the rectangle needs to have a height of 1 and a base of 2 so that its area is 2.



5.4.11 Because x^9 is an odd function, $\int_{-2}^{2} x^9\,dx = 0$.

5.4.12 Because $2x^5$ is an odd function, $\int_{-200}^{200} 2x^5\,dx = 0$.

5.4.13 $\int_{-2}^{2}(3x^8-2)\,dx = 2\int_0^2 (3x^8-2)\,dx = 2\left(\frac{x^9}{3}-2x\right)\Big|_0^2 = \left(\frac{1024}{3}\right) - 8 = \frac{1000}{3}$.

5.4.14 $\int_{-\pi/4}^{\pi/4} \cos x\,dx = 2\int_0^{\pi/4} \cos x\,dx = 2\,(\sin x)\Big|_0^{\pi/4} = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$.

5.4.15 $\int_{-2}^{2}(x^2+x^3)\,dx = 2\int_0^2 x^2\,dx = 2\left(\frac{x^3}{3}\right)\Big|_0^2 = 2\left(\frac{8}{3}-0\right) = \frac{16}{3}$.

5.4.16 Because $x^2\sin x$ is an odd function, $\int_{-\pi}^{\pi} x^2 \sin x\,dx = 0$

5.4.17 Note that the first two terms of the integrand form an odd function, and the last two terms form an even function. $\int_{-2}^{2}(x^9-3x^5+2x^2-10)\,dx = 2\int_0^2 (2x^2-10)\,dx = 2\left(\frac{2x^3}{3}-10x\right)\Big|_0^2 = \frac{32}{3} - 40 = -\frac{88}{3}$.

5.4.18 $\int_{-\pi/2}^{\pi/2} 5\sin x\,dx = 0$ because the integrand is an odd function.

5.4.19 Because the integrand is an odd function and the interval is symmetric about 0, this integral's value is 0.

5.4.20 $\int_{-1}^{1}(1-|x|)\,dx = 2\int_0^1 (1-x)\,dx = 2\left(x-\frac{x^2}{2}\right)\Big|_0^1 = 2\left(1-\frac{1}{2}\right) = 1$.

5.4.21 $\sec^2 x$ is even, so the value of this integral is $2\int_0^{\pi/4} \sec^2 x\,dx = 2\,(\tan x)\Big|_0^{\pi/4} = 2\cdot(1-0) = 2$.

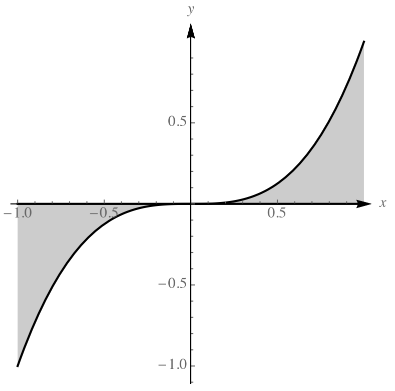
5.4.22 Recall that the tangent function is an odd function, so the value of this integral is 0.

5.4.23 The integrand is an odd function, so the value of this integral is zero.

5.4.24 The function $1-|x|^3$ is even, so the value of this integral is $2\int_0^2 (1-x^3)\,dx = 2\left(x-\frac{x^4}{4}\right)\Big|_0^2 = 2(2-4) = -4$.

5.4.25

The average value is $\frac{1}{1-(-1)}\int_{-1}^{1} x^3\,dx = \frac{1}{2}\left(x^4/4\right)\Big|_{-1}^{1} = 0$.

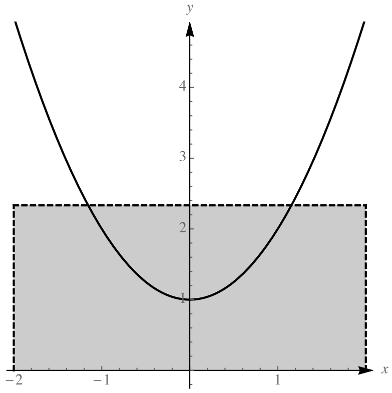


5.4.26

The average value is

$$\frac{1}{2-(-2)}\int_{-2}^{2}(x^2+1)\,dx$$

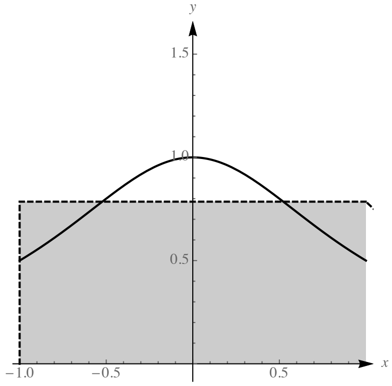
$$= \frac{1}{4}\left(x^3/3+x\right)\Big|_{-2}^{2} = \frac{8/3+2-(-8/3-2)}{4} = \frac{7}{3}.$$



5.4.27

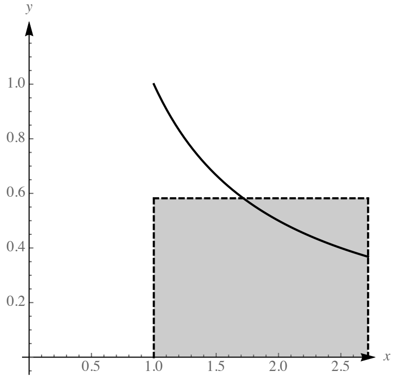
The average value is

$$\frac{1}{1-(-1)}\int_{-1}^{1}\frac{1}{x^2+1}\,dx = \frac{1}{2}\tan^{-1}x\Big|_{-1}^{1} = \frac{\pi/4-(-\pi/4)}{2} = \frac{\pi}{4}.$$



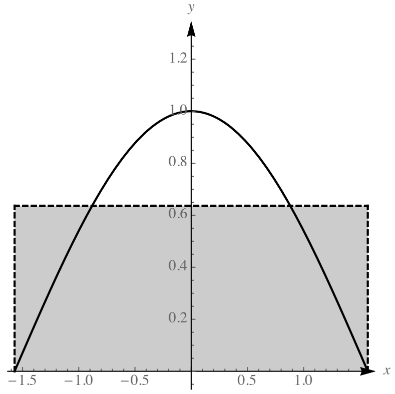
5.4.28

The average value is $\frac{1}{e-1}\int_{1}^{e}\frac{1}{x}\,dx = \frac{1}{e-1}\left(\ln|x|\right)\Big|_{1}^{e} = \frac{1}{e-1} \approx 0.582.$



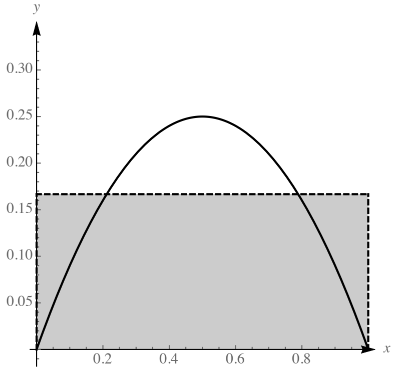
5.4.29

The average value is
$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos x\,dx = \frac{1}{\pi}\left(\sin x\right)\Big|_{-\pi/2}^{\pi/2}$$
$$= \frac{1}{\pi}\cdot(1 - -1) = \frac{2}{\pi} \approx 0.6366.$$



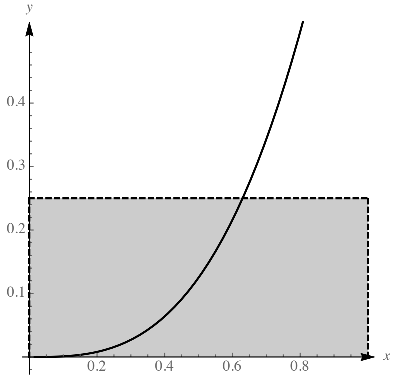
5.4.30

The average value is $\frac{1}{1}\int_0^1 (x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \approx 0.1667.$

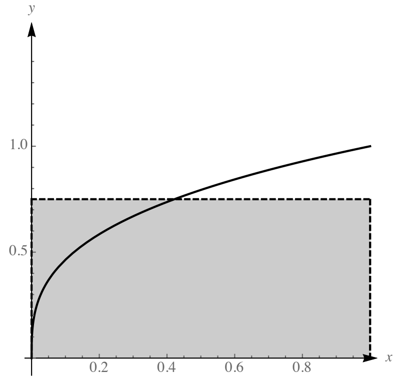


5.4.31

The average value is $\frac{1}{1}\int_0^1 x^n\,dx = \left(\frac{x^{n+1}}{n+1}\right)\Big|_0^1 = \frac{1}{n+1}$. The picture shown is for the case $n = 3$.



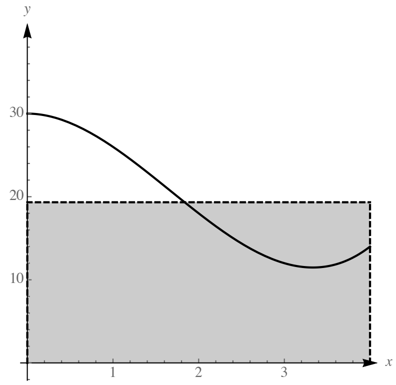
5.4.32



The average value is $\frac{1}{1}\int_0^1 x^{1/n}\,dx = \left(\frac{x^{(n+1)/n}}{(n+1)/n}\right)\Bigg|_0^1 = \frac{n}{n+1}$. The picture shown is for the case $n = 3$.

5.4.33 The average distance to the axis is given by $\frac{1}{20}\int_0^{20} 30x(20-x)\,dx$. This is equal to $\frac{1}{20}\int_0^{20}(600x - 30x^2)\,dx = \frac{1}{20}\left(300x^2 - 10x^3\right)\Big|_0^{20} = 2000$.

5.4.34



The average value is $\frac{1}{4-0}\int_0^4 (x^3 - 5x^2 + 30)\,dx = \frac{1}{4}\left(\frac{x^4}{4} - \frac{5x^3}{3} + 30x\right)\Bigg|_0^4 = \frac{1}{4}(64 - \frac{320}{3} + 120) - 0 = \frac{58}{3}$.

5.4.35 The average velocity is

$$\frac{1}{6-0}\int_0^6 (t^2+3t)\,dt = \frac{1}{6}\left(\frac{t^3}{3} + \frac{3t^2}{2}\right)\Bigg|_0^6 = \frac{1}{6}(72+54-0) = \frac{1}{6}(126) = 21\,\text{m/s}.$$

5.4.36 The average velocity is

$$\frac{1}{4-0}\int_0^4 (-32t+64)\,dt = \frac{1}{4}\left(-16t^2+64t\right)\Big|_0^4 = \frac{1}{4}(-256+256) = 0\,\text{ft/s}.$$

5.4.37 The average height is $\frac{1}{\pi}\int_0^{\pi} 10\sin x\,dx = \frac{1}{\pi}\left(-10\cos x\right)\Big|_0^{\pi} = \frac{1}{\pi}(10 - -10) = \frac{20}{\pi}$.

5.4.38 The average height is $\frac{1}{2\pi}\int_{-\pi}^{\pi}(5+5\cos x)\,dx = \frac{1}{2\pi}\left(5x+5\sin x\right)\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}(5\pi - -5\pi) = 5$.

5.4.39 The average value is $\frac{1}{4}\int_0^4 (8-2x)\,dx = \frac{1}{4}\left(8x - x^2\right)\Big|_0^4 = 4$. The function has a value of 4 when $8 - 2x = 4$, which occurs when $x = 2$.

5.4.40 The average value is $\frac{1}{2}\int_0^2 e^x\,dx = \frac{1}{2}\left(e^x\right)\Big|_0^2 = \frac{e^2-1}{2}$.The function attains this value when $\frac{e^2-1}{2} = e^x$, which is when $x = \ln\left(\frac{e^2-1}{2}\right) \approx 1.1614$.

5.4.41 The average value is $\frac{1}{a}\int_0^a \left(1-\frac{x^2}{a^2}\right)dx = \frac{1}{a}\left(x-\frac{x^3}{3a^2}\right)\Big|_0^a = \frac{2}{3}$. The function attains this value when $\frac{2}{3} = 1-\frac{x^2}{a^2}$, which is when $x^2 = \frac{a^2}{3}$, which on the given interval occurs for $x = \sqrt{3}a/3$.

5.4.42 The average value is $\frac{1}{\pi}\int_0^\pi \frac{\pi}{4}\sin x\,dx = \frac{1}{4}\left(-\cos x\right)\Big|_0^\pi = \frac{1}{4}(1--1) = \frac{1}{2}$. The function attains this value when $\frac{1}{2}\cdot\frac{4}{\pi} = \sin x$, which is when $x = \sin^{-1}\frac{2}{\pi} \approx 0.690107$ and for $x \approx 2.45149$.

5.4.43 The average value is $\frac{1}{2}\int_{-1}^1 (1-|x|)\,dx = \frac{1}{2}\int_{-1}^0 (1+x)\,dx + \frac{1}{2}\int_0^1 (1-x)\,dx = \frac{1}{2}\left(x+\frac{x^2}{2}\right)\Big|_{-1}^0 + \frac{1}{2}\left(x-\frac{x^2}{2}\right)\Big|_0^1 = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$. The function attains this value twice, once on $[-1,0]$ when $1+x = \frac{1}{2}$ which occurs when $x = -\frac{1}{2}$, and once on $[0,1]$ when $1-x = \frac{1}{2}$ which occurs when $x = \frac{1}{2}$.

5.4.44 The average value is given by $\frac{1}{3}\int_1^4 1/x\,dx = \frac{1}{3}\left(\ln x\right)\Big|_1^4 = \frac{1}{3}(\ln 4)$. The function attains this value when $x = \frac{3}{\ln 4} \approx 2.164$.

5.4.45

a. True. Because of the symmetry, the net area between 0 and 4 will be twice the net area between 0 and 2.

b. True. This follows because the symmetry implies that the net area from a to $a+2$ is the opposite of the net area from $a-2$ to a.

c. True. If $f(x) = cx+d$ on $[a,b]$ the value at the midpoint is $c\cdot\frac{a+b}{2}+d$, and the average value is $\frac{1}{b-a}\int_a^b (cx+d)\,dx = \frac{1}{b-a}\left(\frac{cx^2}{2}+dx\right)\Big|_a^b = \frac{1}{b-a}\left(\frac{cb^2}{2}+db-\left(\frac{ca^2}{2}+da\right)\right) = \frac{c}{2}\cdot(a+b)+d$.

d. False, for example, when $a=1$, we have that the maximum value of $x-x^2$ on $[0,1]$ occurs at $\frac{1}{2}$ and is equal to $\frac{1}{4}$, but the average value is $\int_0^1 (x-x^2)\,dx = \left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}$.

5.4.46

a. $d^2 = x^2+y^2 = x^2+b^2(1-(x^2/a^2))$. The average value of d^2 is $\frac{1}{2a}\int_{-a}^a \left(b^2+\left(1-\frac{b^2}{a^2}\right)x^2\right)dx = \frac{1}{2a}\left(b^2x+\frac{\left(1-(b^2/a^2)\right)x^3}{3}\right)\Big|_{-a}^a = \frac{1}{2a}\left(b^2a+\frac{a^3}{3}-\frac{b^2a}{3}-\left(-b^2a-\frac{a^3}{3}+\frac{b^2a}{3}\right)\right) = \frac{2b^2}{3}+\frac{a^2}{3}$.

b. If $a=b=R$, the above becomes $\frac{2R^2}{3}+\frac{R^2}{3} = R^2$.

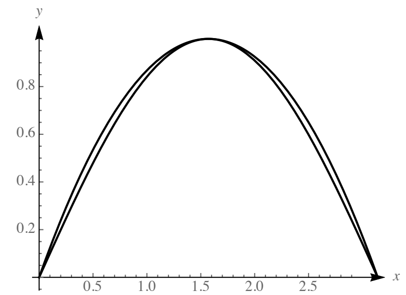
c. $D^2 = (x-\sqrt{a^2-b^2})^2+y^2 = x^2-2x\sqrt{a^2-b^2}+y^2+a^2-b^2 = \left(1-\frac{b^2}{a^2}\right)x^2-2\sqrt{a^2-b^2}x+a^2$. So the average value of D^2 is $\frac{1}{2a}\int_{-a}^a D^2\,dx = \frac{1}{2a}\int_{-a}^a \left[\left(1-\frac{b^2}{a^2}\right)x^2+a^2\right]dx - \frac{1}{a}\int_{-a}^a x\sqrt{a^2-b^2}\,dx = \frac{1}{a}\int_0^a \left[\left(1-\frac{b^2}{a^2}\right)x^2+a^2\right]dx + 0 = \frac{1}{3}(a^2-b^2)+a^2 = \frac{4a^2-b^2}{3}$.

5.4.47 The average height of the arch is given by

$$\frac{1}{630}\int_{-315}^{315}\left(630-\frac{630}{315^2}x^2\right)dx=\frac{630}{630}\left(x-\frac{x^3}{3\cdot 315^2}\right)\Bigg|_{-315}^{315}=(315-105-(-315+105))=420\text{ ft.}$$

5.4.48

a. Note that $\frac{d}{dx}\sin x = \cos x$, which is zero when $x=\pi/2$, and because the derivative is positive on $(0,\pi/2)$ and negative on $(\pi/2,0)$, there is a maximum at $x=\pi/2$. Similarly, $\frac{d}{dx}\frac{4\pi x-4x^2}{\pi^2}=\frac{4\pi-8x}{\pi^2}$, which is zero when $x=\pi/2$, and this function is increasing on $(0,\pi/2)$ and decreasing on $(\pi/2,\pi)$, so it also has a maximum at $\pi/2$. Also, both functions have the value 1 at $x=\pi/2$.



b. On $(0,\pi)$, the sine function is always less than or equal to the other function.

c. The average values are

$$\frac{1}{\pi}\int_0^{\pi}\sin x\,dx=\frac{1}{\pi}\left(-\cos x\right)\Bigg|_0^{\pi}=\frac{2}{\pi}$$

and

$$\frac{1}{\pi}\cdot\frac{4}{\pi^2}\int_0^{\pi}\pi x-x^2\,dx=\frac{4}{\pi^3}\left(\frac{\pi x^2}{2}-\frac{x^3}{3}\right)\Bigg|_0^{\pi}=\frac{4}{\pi^3}\left(\frac{\pi^3}{2}-\frac{\pi^3}{3}\right)=\frac{2}{3}.$$

5.4.49 $f(g(-x))=f(g(x))$, so $f(g(x))$ is an even function, and $\int_{-a}^{a}f(g(x))\,dx=2\int_0^a f(g(x))\,dx$.

5.4.50 $f(p(-x))=f(-p(x))=f(p(x))$, and thus $f(p(x))$ is an even function. Therefore, $\int_{-a}^{a}f(p(x))\,dx = 2\int_0^a f(p(x))\,dx$.

5.4.51 $p(g(-x))=p(g(x))$, so $p(g(x))$ is an even function, and $\int_{-a}^{a}p(g(x))\,dx=2\int_0^a p(g(x))\,dx$.

5.4.52 $p(q(-x))=p(-q(x))=-p(q(x))$, so $p(q(x))$ is an odd function, and $\int_{-a}^{a}p(q(x))\,dx=0$.

5.4.53

a. The average value is $\int_0^1(ax-ax^2)\,dx=\left(\frac{ax^2}{2}-\frac{ax^3}{3}\right)\Big|_0^1=\frac{a}{2}-\frac{a}{3}=\frac{a}{6}$.

b. The function is equal to its average value when $\frac{a}{6}=ax-ax^2$ which occurs when $6x-6x^2=1$, so when $6x^2-6x+1=0$. On the given interval, this occurs for $x=\frac{6\pm\sqrt{12}}{12}=\frac{3\pm\sqrt{3}}{6}$.

5.4.54

a. $f(0)=\frac{\int_a^b x\,dx}{\int_a^b 1\,dx}=\frac{\frac{b^2-a^2}{2}}{b-a}=\frac{a+b}{2}$.

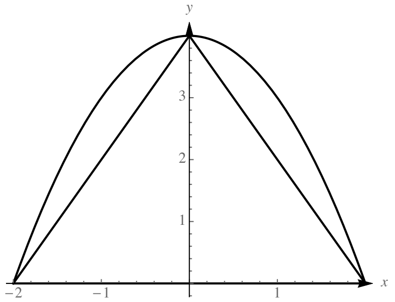
b. $f\left(-\frac{3}{2}\right)=\frac{\int_a^b x^{-1/2}\,dx}{\int_a^b x^{-3/2}\,dx}=\frac{\frac{2}{1}(b^{1/2}-a^{1/2})}{-\frac{2}{1}\cdot(b^{-1/2}-a^{-1/2})}\cdot\frac{\sqrt{ab}}{\sqrt{ab}}=\frac{b^{1/2}-a^{1/2}}{b^{1/2}-a^{1/2}}\cdot\sqrt{ab}=\sqrt{ab}$.

c. $f(-3) = \dfrac{\int_a^b x^{-2}\,dx}{\int_a^b x^{-3}\,dx} = \dfrac{-(b^{-1}-a^{-1})}{\frac{-1}{2}\cdot(b^{-2}-a^{-2})}\cdot\dfrac{a^2b^2}{a^2b^2} = \dfrac{2(a^2b-b^2a)}{a^2-b^2} = \dfrac{2ab(a-b)}{(a-b)(a+b)} = \dfrac{2ab}{a+b}.$

d. $f(-1) = \dfrac{\int_a^b 1\,dx}{\int_a^b x^{-1}\,dx} = \dfrac{b-a}{\ln b - \ln a}.$

5.4.55

a. The area of the triangle is $\frac{1}{2}\cdot 2a\cdot a^2 = a^3$. The area under the parabola is $\int_{-a}^{a}(a^2-x^2)\,dx = \left(a^2x - \frac{x^3}{3}\right)\Big|_{-a}^{a} = a^3 - \frac{a^3}{3} - \left(-a^3+\frac{a^3}{3}\right) = 2a^3 - \frac{2a^3}{3} = \frac{4a^3}{3}$, as desired. The diagram shown is for $a = 2$.



b. The area of the rectangle described is $2a\cdot a^2 = 2a^3$, and $\frac{2}{3}$ of this is $\frac{4a^3}{3}$, which is the area under the parabola derived above.

5.4.56 The average value of f' is given by $\dfrac{1}{b-a}\displaystyle\int_a^b f'(x)\,dx = \dfrac{f(b)-f(a)}{b-a}$. This result tells us that for a function with a continuous derivative, the average slope of the tangent line over an interval is the slope of the secant line through the endpoints of the interval.

5.4.57 Suppose f is even, so that $f(-x) = f(x)$. Then $f^n(x) = f^n(-x)$, so that f^n is an even function, no matter what the parity of n is.

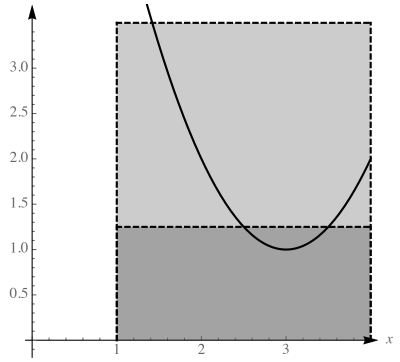
Suppose g is an odd function, so that $g(-x) = -g(x)$. Then $g^n(-x) = (-1)^n g^n(x)$, so g^n is even when n is even, and is odd when n is odd.

Summarizing, we have:

	f is even	f is odd
n is even	f^n is even	f^n is even
n is odd	f^n is even	f^n is odd

5.4.58

a. The smallest expression is the area of a rectangle on the x-axis over $[a,b]$ and height given by the value of f at the midpoint of the interval. The biggest expression is the area of a rectangle with that same base, but height equal to the average of the values of the function at the endpoints. The middle quantity represents the area under the curve.



b. After dividing, we have that

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}.$$

This says that the average value of f over the interval is greater than or equal to the value of f at the average of the endpoints, and is less than or equal to the average of the values of f at the endpoints.

5.4.59

a. Note that h is continuous and differentiable on $[a,b]$, and that $h(a) = (a-b)\int_a^a f(t)\,dt + (a-a)\int_a^b g(t)\,dt = 0+0 = 0$. Also, $h(b) = (b-b)\int_a^b f(t)\,dt + (b-a)\int_b^b g(t)\,dt = 0+0 = 0$. So by Rolle's theorem, there exists c between a and b so that $h'(c) = 0$. By the Product and Sum Rules, and the Fundamental Theorem of Calculus, we have $h'(x) = \int_a^x f(t)\,dt + (x-b)f(x) + \int_x^b g(t)\,dt - (x-a)g(x)$. So at the promised number c we have $h'(c) = \int_a^c f(t)\,dt + (c-b)f(c) + \int_c^b g(t)\,dt - (c-a)g(c) = 0$, so

$$\int_a^c f(t)\,dt + \int_c^b g(t)\,dt = (b-c)f(c) + (c-a)g(c).$$

b. Given f continuous on $[a,b]$, let g be the constant zero function, that is, $g(x) = 0$ for all x. Applying the result of part (a), we have that there exists a c between a and b with

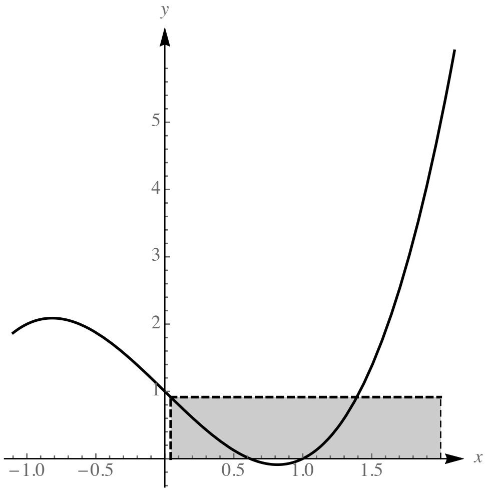
$$\int_a^c f(t)\,dt + 0 = f(c)(b-c) + 0,$$

so

$$\int_a^c f(t)\,dt = f(c)(b-c),$$

as desired.

c. There exists a rectangle with base from c to b and height $f(c)$ so that the area of the rectangle is equal to the value of the integral of f from a to c.



d. Given a function f continuous on $[a,b]$, let $g = f$. Then by part (a) there exists c between a and b so that

$$\int_a^c f(t)\,dt + \int_c^b f(t)\,dt = f(c)(b-c) + f(c)(c-a),$$

so

$$\int_a^b f(t)\,dt = f(c)(b-c+c-a) = f(c)(b-a),$$

so

$$\frac{1}{b-a}\int_a^b f(t)\,dt = f(c).$$

5.4.60

a. The left Riemann sum is given by $\dfrac{\pi}{2n}\displaystyle\sum_{k=0}^{n-1}\sin((k\pi)/(2n))$.

b.

$$\begin{aligned}\lim_{\theta\to 0}\theta\left(\frac{\cos\theta+\sin\theta-1}{2(1-\cos\theta)}\right)\left(\frac{1+\cos\theta}{1+\cos\theta}\right) &= \lim_{\theta\to 0}\frac{\theta}{2}\left(\frac{(1+\cos\theta)(\cos\theta+\sin\theta-1)}{\sin^2\theta}\right)\\ &= \left(\frac{1}{2}\lim_{\theta\to 0}\frac{\theta}{\sin\theta}\cdot\frac{1+\cos\theta}{1}\right)\left(\lim_{\theta\to 0}\frac{\cos\theta-1}{\sin\theta}+\lim_{\theta\to 0}\frac{\sin\theta}{\sin\theta}\right)\\ &= \frac{1}{2}\cdot 1\cdot 2\left(\lim_{\theta\to 0}\frac{\frac{\cos -1}{\theta}}{\frac{\sin\theta}{\theta}}+1\right) = 1(0+1) = 1.\end{aligned}$$

c. Using the previous result, the left Riemann sum is given by $\dfrac{\pi}{2n}\left(\dfrac{\cos(\pi/(2n))+\sin(\pi/(2n))-1}{2(1-\cos(\pi/(2n)))}\right)$. Let $\theta = \dfrac{\pi}{2n}$. Then as $n\to\infty$, $\theta\to 0$, and the limit of the left Riemann sum as $n\to\infty$ is 1.

5.5 Substitution Rule

5.5.1 It is based on the Chain Rule for differentiation.

5.5.2 After making a substitution, one obtains an integral in terms of a different variable, so the variable has "changed."

5.5.3 Typically u is substituted for the inner function, so $u = g(x)$.

5.5.4 One can either let $u = \tan x$, which is a good choice because the derivative is then $\sec^2 x$ which is a factor of the integrand, or one can let $u = \sec x$, because then the derivative is $\tan x \sec x$ which is also a factor of the integrand.

5.5.5 The new integral is $\displaystyle\int_{g(a)}^{g(b)} f(u)\,du$.

5.5.6 The new limits of integration are $2^2-4=0$ and $4^2-4=12$.

5.5.7 Because $u = x^2+1$, $du = 2x\,dx$. Substituting yields $\displaystyle\int u^4\,du = \frac{u^5}{5}+C = \frac{(x^2+1)^5}{5}+C$.

5.5.8 Because $u = 4x^2+3$, $du = 8x\,dx$. Substituting yields $\displaystyle\int \cos u\,du = \sin u + C = \sin(4x^2+3)+C$.

5.5.9 Because $u = \sin x$, $du = \cos x\,dx$. Substituting yields $\displaystyle\int u^3\,du = \frac{u^4}{4}+C = \frac{\sin^4(x)}{4}+C$.

5.5.10 Because $u = 3x^2+x$, $du = 6x+1\,dx$. Substituting yields $\displaystyle\int \sqrt{u}\,du = \frac{2}{3}\cdot u^{3/2}+C = \frac{2}{3}\cdot\sqrt{(3x^2+x)^3}+C$.

5.5.11 Let $u = x + 1$. Then $du = dx$, and $\int (x+1)^{12}\,dx = \int u^{12}\,du = \frac{u^{13}}{13} + C = \frac{(x+1)^{13}}{13} + C$. Check: $\frac{d}{dx}\left(\frac{(x+1)^{13}}{13} + C\right) = (x+1)^{12}$.

5.5.12 Let $u = 3x + 1$. Then $du = 3dx$, and $\int e^{3x+1}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{e^{3x+1}}{3} + C$. Check: $\frac{d}{dx}\left(\frac{e^{3x+1}}{3} + C\right) = e^{3x+1}$.

5.5.13 Let $u = 2x + 1$. Then $du = 2dx$ and $\int \sqrt{2x+1}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}u^{3/2} + C = \frac{(2x+1)^{3/2}}{3} + C$. Check: $\frac{d}{dx}\left(\frac{(2x+1)^{3/2}}{3} + C\right) = \frac{3}{2}\cdot\frac{1}{3}\cdot(2x+1)^{1/2}\cdot 2 = \sqrt{2x+1}$.

5.5.14 Let $u = 2x+5$. Then $du = 2dx$, and $\int \cos(2x+5)\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{\sin(2x+5)}{2} + C$. Check: $\frac{d}{dx}\left(\frac{\sin(2x+5)}{2} + C\right) = \cos(2x+5)$.

5.5.15

a. $\int e^{10x}\,dx = \frac{1}{10}e^{10x} + C$.

b. $\int \sec 5x \tan 5x\,dx = \frac{1}{5}\sec 5x + C$.

c. $\int \sin 7x\,dx = -\frac{1}{7}\cos 7x + C$.

d. $\int \cos\frac{x}{7}\,dx = 7\sin\frac{x}{7} + C$.

e. $\int \frac{dx}{81+9x^2} = \frac{1}{9}\int\frac{dx}{9+x^2} = \frac{1}{27}\tan^{-1}\frac{x}{3} + C$.

f. $\int \frac{dx}{\sqrt{36-x^2}} = \sin^{-1}\frac{x}{6} + C$.

5.5.16

a. $\int_0^1 10^x\,dx = \frac{1}{\ln 10}\cdot 10^x\Big|_0^1 = \frac{10}{\ln 10} - \frac{1}{\ln 10} = \frac{9}{\ln 10}$.

b. $\int_0^{\pi/40} \cos 20x\,dx = \frac{1}{20}\sin 20x\Big|_0^{\pi/40} = \frac{1}{20}(1-0) = \frac{1}{20}$.

c. $\int_{3\sqrt{2}}^6 \frac{dx}{x\sqrt{x^2-9}} = \frac{1}{3}\sec^{-1}\frac{x}{3}\Big|_{3\sqrt{2}}^6 = \frac{1}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{36}$.

d. $\int_0^{\pi/16} \sec^2 4x\,dx = \frac{1}{4}\tan 4x\Big|_0^{\pi/16} = \frac{1}{4}(1-0) = \frac{1}{4}$.

5.5.17 Let $u = x^2 - 1$. Then $du = 2x\,dx$. Substituting yields $\int u^{99}\,du = \frac{u^{100}}{100} + C = \frac{(x^2-1)^{100}}{100} + C$.

5.5.18 Let $u = x^2$. Then $du = 2x\,dx$, so $\frac{1}{2}du = x\,dx$. Substituting yields $\frac{1}{2}\int e^u\,du = \frac{1}{2}\cdot e^u + C = \frac{1}{2}\cdot e^{x^2} + C$.

5.5.19 Let $u = 1 - 4x^3$. Then $du = -12x^2\,dx$, so $-\frac{1}{6}du = 2x^2\,dx$. Substituting yields $-\frac{1}{6}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{3}\cdot\sqrt{u} + C = -\frac{1}{3}\cdot\sqrt{1-4x^3} + C$

5.5.20 Let $u = \sqrt{x}+1$. Then $du = \frac{1}{2\sqrt{x}}\,dx$. Substituting yields $\int u^4\,du = \frac{u^5}{5} + C = \frac{(\sqrt{x}+1)^5}{5} + C$.

5.5.21 Let $u = x^2 + x$. Then $du = (2x+1)\,dx$. Substituting yields $\int u^{10}\,du = \frac{u^{11}}{11} + C = \frac{(x^2+x)^{11}}{11} + C$.

5.5.22 Let $u = 10x - 3$. Then $du = 10\,dx$, so $\frac{1}{10}du = dx$. Substituting yields $\frac{1}{10}\int \frac{1}{u}\,du = \frac{1}{10}\cdot\ln|u| + C = \frac{1}{10}\ln|10x-3| + C$.

5.5.23 Let $u = x^4 + 16$. Then $du = 4x^3\,dx$, so $\frac{1}{4}du = x^3\,dx$. Substituting yields $\frac{1}{4}\int u^6\,du = \frac{1}{4}\cdot\frac{u^7}{7} + C = \frac{(x^4+16)^7}{28} + C$.

5.5.24 Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$. Substituting yields $\int u^{10}\,du = \frac{u^{11}}{11} + C = \frac{(\sin\theta)^{11}}{11} + C$.

5.5.25 $\int \frac{dx}{\sqrt{36-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{9-x^2}} = \frac{1}{2}\sin^{-1}\frac{x}{3} + C$ by equation 10 in Table 5.6.

5.5.26 $\int \frac{dx}{\sqrt{1-(3x)^2}} = \frac{1}{3}\sin^{-1}3x + C$, by equation 10 in Table 5.6.

5.5.27 Let $u = x^3$. Then $du = 3x^2\,dx$. Then $\int 6x^2 4^{x^3}\,dx = 2\int 4^u\,du = 2\cdot\frac{4^u}{\ln 4} + C = 2\cdot\frac{4^{x^3}}{\ln 4} + C = \frac{4^{x^3}}{\ln 2} + C$.

5.5.28 Let $u = x^{10}$. Then $du = 10x^9\,dx$, so $\frac{1}{10}du = x^9dx$. Substituting yields $\frac{1}{10}\int \sin u\,du = -\frac{1}{10}\cos u + C = -\frac{1}{10}\cos x^{10} + C$.

5.5.29 Let $u = x^6 - 3x^2$. Then $du = (6x^5 - 6x)\,dx$, so $\frac{1}{6}du = (x^5 - x)\,dx$. Substituting yields $\frac{1}{6}\int u^4\,du = \frac{1}{6}\cdot\frac{u^5}{5} + C = \frac{(x^6-3x^2)^5}{30} + C$.

5.5.30 Let $u = 2x$, so that $du = 2dx$. Substituting yields $\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}2x + C$.

5.5.31 Let $u = 5x$ so that $du = 5dx$. Substituting yields $\frac{3}{5}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{3}{5}\sin^{-1}u + C = \frac{3}{5}\sin^{-1}x + C$.

5.5.32 Let $u = 2x$, so that $du = 2dx$. Substituting yields $2\int \frac{1}{u\sqrt{u^2-1}}\,du = 2\sec^{-1}u + C = 2\sec^{-1}2x + C$.

5.5.33 The integral can be rewritten as $\int \frac{e^w}{36+(e^w)^2}\,dw$. Let $u = e^w$, so that $du = e^w\,dw$. Substituting yields $\int \frac{du}{36+u^2} = \frac{1}{6}\tan^{-1}\frac{u}{6} + C = \frac{1}{6}\tan^{-1}\frac{e^w}{6} + C$.

5.5.34 Let $u = 2x^2 + 3x$, so that $du = (4x + 3)\,dx = \frac{1}{2}(8x + 6)\,dx$. Substituting yields $2\int \frac{1}{u}\,du = 2\ln|u| + C = 2\ln|2x^2 + 3x| + C$.

5.5.35 Let $u = x^2$ so that $du = 2x\,dx$. Substitution yields $\frac{1}{2}\int \csc u \cot u\,du = -\frac{1}{2}\csc u + C = -\frac{1}{2}\csc x^2 + C$.

5.5.36 Let $u = 4w$. Then $du = 4\,dw$. Substituting yields $\frac{1}{4}\int \sec u \tan u\,du = \frac{1}{4}\sec u + C = \frac{1}{4}\sec 4w + C$.

5.5.37 Let $u = 10x+7$ so that $du = 10\,dx$. Substituting yields $\frac{1}{10}\int \sec^2 u\,du = \frac{1}{10}\tan u + C = \frac{1}{10}\tan(10x+7) + C$.

5.5.38 Let $u = \tan^{-1} w$ so that $du = \frac{1}{1+w^2}\,dw$. Substituting yields $\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\left(\tan^{-1} w\right)^2 + C$.

5.5.39 Let $u = 4t + 1$ so that $du = 4dt$. Substituting yields $\frac{1}{4}\int 10^u\,du = \frac{1}{4}\cdot\frac{10^u}{\ln 10} + C = \frac{10^u}{4\ln 10} + C$.

5.5.40 Let $u = \sin x$. Then $du = \cos x\,dx$. Substituting yields $\int u^5 + 3u^3 - u\,du = \frac{u^6}{6} + \frac{3u^4}{4} - \frac{u^2}{2} + C = \frac{\sin^6 x}{6} + \frac{3\sin^4 x}{4} - \frac{\sin^2 x}{2} + C$.

5.5.41 Let $u = \cot x$. Then $du = -\csc^2 x\,dx$. Substituting yields $-\int u^{-3}\,du = \frac{1}{2u^2} + C = \frac{1}{2\cot^2 x} + C$.

5.5.42 Let $u = x^{3/2} + 8$. Then $du = \frac{3}{2}\cdot\sqrt{x}\,dx$. Substituting gives $\frac{2}{3}\int u^5\,du = \frac{2}{3}\frac{u^6}{6} + C = \frac{(x^{3/2}+8)^6}{9} + C$.

5.5.43 Note that $\sin x \sec^8 x = \frac{\sin x}{\cos^8 x}$. Let $u = \cos x$, so that $du = -\sin x\,dx$. Substituting yields $-\int u^{-8}\,du = \frac{1}{7u^7} + C = \frac{1}{7\cos^7 x} + C = \frac{\sec^7 x}{7} + C$.

5.5.44 Let $u = e^{2x} + 1$. Then $du = 2e^{2x}\,dx$. Substituting yields $\frac{1}{2}\int \frac{1}{u}\,du = \frac{\ln|u|}{2} + C = \frac{\ln(e^{2x}+1)}{2} + C$.

5.5.45 $\int_0^{\pi/8} \cos 2x\,dx = \left(\frac{\sin 2x}{2}\right)\Big|_0^{\pi/8} = \frac{\sqrt{2}/2 - 0}{2} = \frac{\sqrt{2}}{4}$.

5.5.46 $\int_0^1 10e^{2x}\,dx = \left(5e^{2x}\right)\Big|_0^1 = 5(e^2 - 1)$.

5.5.47 Let $u = 4 - x^2$. Then $du = -2x\,dx$. Also, when $x = 0$ we have $u = 4$ and when $x = 1$ we have $u = 3$. Substituting yields $-\int_4^3 u\,du = \int_3^4 u\,du = \left(\frac{u^2}{2}\right)\Big|_3^4 = 8 - 4.5 = 3.5$.

5.5.48 Let $u = x^2 + 1$. Then $du = 2x\,dx$. Also, when $x = 0$ we have $u = 1$ and when $x = 2$ we have $u = 5$. Substituting yields $\int_1^5 u^{-2}\,du = \left(-\frac{1}{u}\right)\Big|_1^5 = -\frac{1}{5} + 1 = \frac{4}{5}$.

5.5.49 Let $u = 2^x + 4$ so that $du = 2^x \ln 2$. Also, when $x = 1$, $u = 6$ and when $x = 3$, $u = 12$. Substituting yields

$$\frac{1}{\ln 2}\int_6^{12} \frac{du}{u} = \frac{1}{\ln 2}\ln u\Big|_6^{12} = \frac{1}{\ln 2}\left(\ln 12 - \ln 6\right) = \frac{1}{\ln 2}(\ln 2) = 1.$$

5.5.50 Let $u = \frac{\theta}{8}$ so that $du = \frac{1}{8}\,d\theta$. Also, when $\theta = -2\pi$, $u = -\frac{\pi}{4}$, and when $\theta = 2\pi$, $u = \frac{\pi}{4}$. Substituting yields

$$8\int_{-\pi/4}^{\pi/4} \cos u\,du = 8\sin u\Big|_{-\pi/4}^{\pi/4} = 8\left(\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right) = 8\sqrt{2}.$$

5.5.51 Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$. Also, when $\theta = 0$ we have $u = 0$ and when $\theta = \pi/2$ we have $u = 1$. Substituting yields $\int_0^1 u^2\,du = \left(\frac{u^3}{3}\right)\Big|_0^1 = \frac{1}{3}$.

5.5.52 Let $u = \cos x$. Then $du = -\sin x\,dx$. Also, when $x = 0$ we have $u = 1$ and when $x = \pi/4$ we have $u = \sqrt{2}/2$. Substituting yields $-\int_1^{\sqrt{2}/2} \frac{1}{u^2}\,du = \int_{\sqrt{2}/2}^1 u^{-2}\,du = \left(-\frac{1}{u}\right)\Big|_{\sqrt{2}/2}^1 = -1 + \frac{2}{\sqrt{2}} = \sqrt{2} - 1$.

5.5.53 Let $u = e^w$. Then $du = e^w\,dw$. Also, when $w = \ln\pi/4$, $u = \pi/4$, and when $w = \ln\pi/2$, $u = \pi/2$. Substituting yields

$$\int_{\pi/4}^{\pi/2} \cos u\,du = \sin u\Big|_{\pi/4}^{\pi/2} = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}.$$

5.5.54 Let $u = 4x$ so that $du = 4\,dx$. Also, when $x = \pi/16$, $u = \pi/4$ and when $x = \pi/8$, $u = \pi/2$. Substituting yields

$$2\int_{\pi/4}^{\pi/2} \csc^2 u\,du = -2\cot u\Big|_{\pi/4}^{\pi/2} = -2(0-1) = 2.$$

5.5.55 Let $u = x^3 + 1$. Then $du = 3x^2\,dx$. Also, when $x = -1$ we have $u = 0$ and when $x = 2$ we have $u = 9$. Substituting yields $\frac{1}{3}\int_0^9 e^u\,du = \left(\frac{e^u}{3}\right)\Big|_0^9 = \frac{e^9 - 1}{3}$.

5.5.56 Let $u = 9 + p^2$. Then $du = 2p\,dp$. Also, when $p = 0$ we have $u = 9$ and when $p = 4$ we have $u = 25$. Substituting yields $\frac{1}{2}\int_9^{25} u^{-1/2}\,du = \sqrt{u}\,\Big|_9^{25} = 5 - 3 = 2$.

5.5.57 Let $u = \sin x$. Then $du = \cos x\,dx$. Also, when $x = \pi/4$ we have $u = \sqrt{2}/2$ and when $x = \pi/2$ we have $u = 1$. Substituting yields $\int_{\sqrt{2}/2}^1 \frac{1}{u^2}\,du = \left(\frac{-1}{u}\right)\Big|_{\sqrt{2}/2}^1 = \left(-1 - \left(-\frac{2}{\sqrt{2}}\right)\right) = \sqrt{2} - 1$.

5.5.58 Let $u = \cos x$. Then $du = -\sin x\,dx$. Also, when $x = 0$ we have $u = 1$ and when $x = \pi/4$ we have $u = \sqrt{2}/2$. Substituting yields $-\int_1^{\sqrt{2}/2} \frac{1}{u^3}\,du = \int_{\sqrt{2}/2}^1 u^{-3}\,du = \left(-\frac{1}{2u^2}\right)\Big|_{\sqrt{2}/2}^1 = -\frac{1}{2} + 1 = \frac{1}{2}$.

5.5.59 Let $u = 5x$, so that $du = 5\,dx$. Also, when $x = 2/(5\sqrt{3})$ we have $u = 2/\sqrt{3}$ and when $x = 2/5$ we have $u = 2$. Substituting yields $\int_{2/\sqrt{3}}^2 \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u\,\Big|_{2/\sqrt{3}}^2 = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$.

5.5.60 Let $u = v^4 + 4v + 4$, so that $du = (4v^3 + 4)\,dv$, so that $\frac{1}{4}\cdot du = (v^3 + 1)\,dv$. Also, when $v = 0$ we have $u = 4$ and when $v = 1$ we have $u = 9$. Substituting yields $\frac{1}{4}\int_4^9 u^{-1/2}\,du = \frac{1}{4}\left(2\sqrt{u}\right)\Big|_4^9 = \frac{1}{2}(3-2) = \frac{1}{2}$.

5.5.61 Let $u = x^2 + 1$, so that $du = 2x\,dx$. Substituting yields $\frac{1}{2}\int_1^{17} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\,\Big|_1^{17} = \frac{\ln 17}{2}$.

5.5.62 Let $u = 1 - 16x^2$, so that $du = -32x\,dx$. Substituting yields $-\frac{1}{32}\int_1^0 \frac{1}{\sqrt{u}}\,du = \frac{1}{16}\,\sqrt{u}\,\Big|_0^1 = \frac{1}{16}$.

5.5.63 Let $u = 3x$, so that $du = 3\,dx$. Substituting yields $\frac{4}{3}\int_1^{3/\sqrt{3}} \frac{1}{u^2+1}\,du = \frac{4}{3}\tan^{-1}u\Big|_1^{3/\sqrt{3}} = \frac{4}{3}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \frac{4}{3}\cdot\frac{\pi}{12} = \frac{\pi}{9}$.

5.5.64 Let $u = 3+2e^x$, so that $du = 2e^x\,dx$. Substituting yields $\frac{1}{2}\int_5^{11}\frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_5^{11} = \frac{\ln(11/5)}{2}$.

5.5.65 Let $u = 1-x^2$. Then $du = -2x\,dx$. Also note that when $x = 0$ we have $u = 1$, and when $x = 1$ we have $u = 0$. Substituting yields $-\frac{1}{2}\int_1^0 \sqrt{u}\,du = \frac{1}{2}\int_0^1 \sqrt{u}\,du = \left(\frac{u^{3/2}}{3}\right)\Big|_0^1 = \frac{1}{3}$.

5.5.66 Let $u = \ln p$. Then $du = \frac{1}{p}\,dp$. Also note that when $p = 1$ we have $u = 0$, and when $p = e^2$ we have $u = 2$. Substituting yields $\int_0^2 u\,du = \left(\frac{u^2}{2}\right)\Big|_0^2 = 2$.

5.5.67 Let $u = x^2-1$, so that $du = 2x\,dx$. Also note that when $x = 2$ we have $u = 3$, and when $x = 3$ we have $u = 8$. Substituting yields $\frac{1}{2}\int_3^8 u^{-1/3}\,du = \frac{1}{2}\left(\frac{3u^{2/3}}{2}\right)\Big|_3^8 = \frac{3}{4}\left(4-\sqrt[3]{9}\right)$.

5.5.68 Let $u = 5x/6$ so that $du = \frac{5}{6}\,dx$. Also note that when $x = 0$ we have $u = 0$ and when $x = 6/5$ we have $u = 1$. Substituting yields $\frac{6}{5\cdot 36}\int_0^1 \frac{1}{u^2+1}\,du = \frac{1}{30}\left(\tan^{-1}u\right)\Big|_0^1 = \frac{\pi}{120}$.

5.5.69 Let $u = 16-x^4$. Then $du = -4x^3\,dx$. Also note that when $x = 0$ we have $u = 16$, and when $x = 2$ we have $u = 0$. Substituting yields $\frac{1}{4}\int_0^{16}\sqrt{u}\,du = \frac{1}{4}\left(\frac{2u^{3/2}}{3}\right)\Big|_0^{16} = \frac{32}{3}$.

5.5.70 Let $u = x^2-2x$. Then $du = 2(x-1)\,dx$. Also note that when $x = -1$ we have $u = 3$ and when $x = 1$ we have $u = -1$. Substituting yields $\frac{1}{2}\int_3^{-1} u^7\,du = \frac{1}{16}\left(u^8\right)\Big|_3^{-1} = \frac{1}{16}\left(1-3^8\right) = -\frac{6560}{16} = -410$.

5.5.71 Let $u = 2+\cos x$ so that $du = -\sin x\,dx$. Note that when $x = -\pi$, $u = 1$ and when $x = 0$, $u = 3$. Substituting yields $\int_1^3 -\frac{1}{u}\,du = (-\ln|u|)\Big|_1^3 = -(\ln 3 - \ln 1) = -\ln 3$.

5.5.72 Let $u = 2v^3+9v^2+12v+36$, so that $du = (6v^2+18v+12)\,dv = 6(v+1)(v+2)\,dv$. Note that $u = 36$ when $v = 0$ and $u = 59$ when $v = 1$. Substituting yields $\frac{1}{6}\int_{36}^{59}\frac{1}{u}\,du = \frac{1}{6}\left(\ln|u|\right)\Big|_{36}^{59} = \frac{1}{6}(\ln 59 - \ln 36) = \frac{1}{6}\ln(59/36)$.

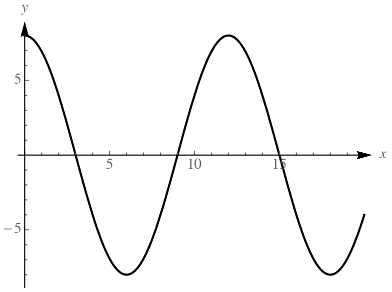
5.5.73 Let $u = 3x+1$ so that $du = 3\,dx$. Note that $9x^2+6x+1 = (3x+1)^2 = u^2$, and also that when $x = 1$, $u = 4$ and when $x = 2$, $u = 7$. Substituting yields $\frac{4}{3}\int_4^7 \frac{1}{u^2}\,du = \frac{4}{3}\left(-\frac{1}{u}\right)\Big|_4^7 = \frac{4}{3}\left(-\frac{1}{7}-\left(-\frac{1}{4}\right)\right) = \frac{4}{3}\left(\frac{3}{28}\right) = \frac{1}{7}$.

5.5.74 Let $u = \sin^2 x$, so that $du = 2\sin x\cos x\,dx = \sin 2x\,dx$. Note that when $x = 0$, $u = 0$, and when $x = \pi/4$, $u = 1/2$. Substituting yields $\int_0^{1/2} e^u\,du = e^u\Big|_0^{1/2} = \sqrt{e}-1$.

5.5.75 The average velocity is given by $\frac{1}{10-0}\int_0^{10}(8\sin\pi t + 2t)\,dt = \frac{1}{10}\left(-\frac{8}{\pi}\cos\pi t + t^2\right)\Big|_0^{10} = \frac{1}{10}\left(-\frac{8}{\pi}\cos 10\pi + 100 + \frac{8}{\pi}\cos 0 - 0\right) = 10$.

5.5.76

a.



b. $\int_0^t 8\cos(\pi y/6)\,dy = \left(\frac{48}{\pi}\sin(\pi y/6)\right)\Big|_0^t = \frac{48}{\pi}\sin(\pi t/6)$.

c. The period is $\frac{2\pi}{\pi/6} = 12$.

5.5.77

a. $\int_0^4 \frac{200}{(t+1)^2}\,dt = \left(\frac{-200}{t+1}\right)\Big|_0^4 = -40 + 200 = 160$.

b. $\int_0^6 \frac{200}{(t+1)^3}\,dt = \left(\frac{-200}{2(t+1)^2}\right)\Big|_0^6 = \frac{-100}{49} + 100 = \frac{4800}{49}$.

c. $\Delta P = \int_0^T \frac{200}{(t+1)^r}\,dt$. This decreases as r increases, because $\frac{200}{(t+1)^r} > \frac{200}{(t+1)^{r+1}}$.

d. Suppose $\int_0^{10} \frac{200}{(t+1)^r}\,dt = 350$. Then $\left(\frac{200(t+1)^{-r+1}}{1-r}\right)\Big|_0^{10} = 350$, so $11^{1-r} - 1 = \frac{350(1-r)}{200}$, and thus $\frac{11}{11^r} = \frac{7-7r}{4} + \frac{4}{4} = \frac{11-7r}{4}$, and $11^r = \frac{44}{11-7r}$. Using trial and error to find r, we arrive at $r \approx 1.278$.

e. $\int_0^T \frac{200}{(t+1)^3}\,dt = \left(-\frac{200}{2(t+1)^2}\right)\Big|_0^T = -\frac{100}{(T+1)^2} + 100$. As $T \to \infty$, this expression $\to 100$, so in the long run, the bacteria approaches a finite limit.

5.5.78 Let $u = x - 2$, so that $u + 2 = x$. Then $du = dx$. Substituting yields $\int \frac{u+2}{u}\,du = \int \left(1 + \frac{2}{u}\right)du = u + 2\ln|u| + D = x - 2 + 2\ln|x-2| + D$. The constant $-2 + D$ could be renamed as a different constant C, yielding $x + 2\ln|x-2| + C$.

5.5.79 Let $u = x - 4$, so that $u + 4 = x$. Then $du = dx$. Substituting yields $\int \frac{u+4}{\sqrt{u}}\,du = \int \left(\frac{u}{\sqrt{u}} + \frac{4}{\sqrt{u}}\right)du = \int u^{1/2} + 4u^{-1/2}\,du = \frac{2}{3}u^{3/2} + 8u^{1/2} + C = \frac{2}{3}\cdot(x-4)^{3/2} + 8\sqrt{x-4} + C$.

5.5.80 Let $u = y + 1$, so that $u - 1 = y$. Then $du = dy$. Substituting yields $\int \frac{(u-1)^2}{u^4}\,du = \int \frac{u^2 - 2u + 1}{u^4}\,du = \int \left(u^{-2} - 2u^{-3} + u^{-4}\right)du = -\frac{1}{u} + \frac{1}{u^2} - \frac{1}{3u^3} + C = -\frac{1}{y+1} + \frac{1}{(y+1)^2} - \frac{1}{3(y+1)^3} + C$.

5.5.81 Let $u = x + 4$, so that $u - 4 = x$. Then $du = dx$. Substituting yields

$$\begin{aligned}\int \frac{u-4}{\sqrt[3]{u}}\,du &= \int \left(u^{2/3} - 4u^{-1/3}\right)du = \frac{3}{5}u^{5/3} + -6u^{2/3} + C\\ &= \frac{3}{5}(x+4)^{5/3} - 6(x+4)^{2/3} + C.\end{aligned}$$

5.5.82 Let $u = e^x + e^{-x}$. Then $du = (e^x - e^{-x})\,dx$. Substituting yields $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C = \ln(e^x + e^{-x}) + C$.

5.5.83 Let $u = 2x+1$. Then $du = 2dx$ and $x = \frac{u-1}{2}$. Substituting yields $\displaystyle\frac{1}{2}\int \frac{u-1}{2}\cdot\sqrt[3]{u}\,du = \frac{1}{4}\int (u^{4/3} - u^{1/3})\,du = \frac{1}{4}\left(\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3}\right) + C = \frac{3(2x+1)^{7/3}}{28} - \frac{3(2x+1)^{4/3}}{16} + C = (2x+1)^{4/3}\left(\frac{3(2x+1)}{28} - \frac{3}{16}\right) = \frac{3}{112}(2x+1)^{4/3}(8x+4-7) = \frac{3}{112}(2x+1)^{4/3}(8x-3)$.

5.5.84 Let $u = 3z+2$. Then $du = 3dz$ and $z = \frac{u-2}{3}$. Substituting yields $\displaystyle\frac{1}{3}\int \frac{u+1}{3}\cdot\sqrt{u}\,du = \frac{1}{9}\int (u^{3/2} + u^{1/2})\,du = \frac{1}{9}\left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C = \frac{2(3z+2)^{5/2}}{45} + \frac{2(3z+2)^{3/2}}{27} + C. = \frac{2}{9}(3z+2)^{3/2}\left(\frac{3z+2}{5} + \frac{1}{3}\right) = \frac{2}{9}(3z+2)^{3/2}\left(\frac{9z+6+5}{15}\right) = \frac{2}{135}(3z+2)^{3/2}(9z+11)$.

5.5.85 Let $u = x+10$. Then $du = dx$ and $x = u - 10$. Substituting gives $\displaystyle\int (u-10)u^9\,du = \int (u^{10} - 10u^9)\,du = \frac{u^{11}}{11} - u^{10} + C = \frac{1}{11}(x+10)^{11} - (x+10)^{10} + C = (x+10)^{10}\left(\frac{x+10}{11} - 1\right) + C = \frac{(x+10)^{10}(x-1)}{11} + C$.

5.5.86 Using Table 5.6: $\displaystyle\int_0^{\sqrt{3}} \frac{3dx}{9+x^2} = \tan^{-1}(x/3)\Big|_0^{\sqrt{3}} = \tan^{-1}(\sqrt{3}/3) = \pi/6$.

5.5.87 $\displaystyle\int_{-\pi}^{\pi} \cos^2 x\,dx = 2\int_0^{\pi} \frac{1+\cos 2x}{2}\,dx = \left(x + \frac{\sin 2x}{2}\right)\Big|_0^{\pi} = \pi$.

5.5.88 $\displaystyle\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

5.5.89 $\displaystyle\int \sin^2\left(\theta + \frac{\pi}{6}\right)\,d\theta = \frac{1}{2}\int \left(1 - \cos\left(2\theta + \frac{\pi}{3}\right)\right)\,d\theta = \frac{\theta}{2} - \frac{\sin\left(2\theta + \frac{\pi}{3}\right)}{4} + C$.

5.5.90 $\displaystyle\int_0^{\pi/4} \cos^2 8\theta\,d\theta = \int_0^{\pi/4} \frac{1+\cos 16\theta}{2}\,d\theta = \left(\frac{\theta}{2} + \frac{\sin 16\theta}{32}\right)\Big|_0^{\pi/4} = \frac{\pi}{8}$.

5.5.91 $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 2\theta\,d\theta = 2\int_0^{\pi/4} \sin^2 2\theta\,d\theta = 2\int_0^{\pi/4} \frac{1-\cos 4\theta}{2}\,d\theta = \left(\theta - \frac{\sin 4\theta}{4}\right)\Big|_0^{\pi/4} = \frac{\pi}{4}$.

5.5.92 Let $u = x^2$, so that $du = 2x\,dx$. Substituting yields

$$\begin{aligned}\frac{1}{2}\int \cos^2 u\,du = \frac{1}{2}\int \frac{1+\cos 2u}{2}\,du &= \frac{1}{4}\left(u + \frac{\sin 2u}{2}\right) + C \\ &= \frac{x^2}{4} + \frac{\sin 2x^2}{8} + C.\end{aligned}$$

5.5.93 Let $u = \sin^2 y + 2$ so that $du = 2\sin y\cos y\,dy = \sin 2y\,dy$. Substituting yields $\displaystyle\int_2^{9/4} \frac{1}{u}\,du = (\ln|u|)\Big|_2^{9/4} = \ln(9/4) - \ln 2 = \ln(9/8)$.

5.5.94 Because $\sin^4\theta = (\sin^2\theta)^2 = \left(\dfrac{1-\cos 2\theta}{2}\right)^2 = \dfrac{1-2\cos 2\theta+\cos^2 2\theta}{4}$, we have

$$\int \sin^4\theta\,d\theta = \int \left(\frac{1-2\cos 2\theta+\cos^2 2\theta}{4}\right)d\theta = \frac{1}{4}\theta - \frac{\sin 2\theta}{4} + \frac{1}{4}\int \cos^2 2\theta\,d\theta.$$

Because $\dfrac{1}{4}\cos^2 2\theta = \dfrac{1+\cos 4\theta}{8}$, we have

$$\int \sin^4\theta\,d\theta = \frac{1}{4}\theta - \frac{\sin 2\theta}{4} + \frac{1}{8}\theta + \frac{\sin 4\theta}{32} = \frac{3}{8}\theta - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}.$$

Thus, $\displaystyle\int_0^{\pi/2} \sin^4\theta\,d\theta = \left(\frac{3}{8}\theta - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}\right)\Bigg|_0^{\pi/2} = \frac{3\pi}{16}$.

5.5.95

a. True. This follows by substituting $u = f(x)$ to obtain the integral $\displaystyle\int u\,du = \frac{u^2}{2} + C = \frac{f(x)^2}{2} + C$.

b. True. Again, this follows from substituting $u = f(x)$ to obtain the integral $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C = \frac{(f(x))^{n+1}}{n+1} + C$ where $n \neq -1$.

c. False. If this were true, then $\sin 2x$ and $2\sin x$ would have to differ by a constant, which they do not. In fact, $\sin 2x = 2\sin x\cos x$.

d. False. The derivative of the right hand side is $(x^2+1)^9\cdot 2x$ which is not the integrand on the left hand side.

e. False. If we let $u = f'(x)$, then $du = f''(x)\,dx$. Substituting yields $\displaystyle\int_{f'(a)}^{f'(b)} u\,du = \left(\frac{u^2}{2}\right)\Bigg|_{f'(a)}^{f'(b)} = \frac{(f'(b))^2}{2} - \frac{(f'(a))^2}{2}$.

5.5.96 $A(x) = \displaystyle\int_4^5 \frac{x}{\sqrt{x^2-9}}\,dx$. Let $u = x^2-9$, so that $du = 2x\,dx$. Also, when $x = 4$ we have $u = 7$ and when $x = 5$ we have $u = 16$. Substituting yields $\displaystyle\frac{1}{2}\int_7^{16} u^{-1/2}\,du = \sqrt{u}\,\Big|_7^{16} = 4-\sqrt{7}$.

5.5.97 $A(x) = \displaystyle\int_0^{\sqrt{\pi}} x\sin x^2\,dx$. Let $u = x^2$, so that $du = 2x\,dx$. Also, when $x = 0$ we have $u = 0$ and when $x = \sqrt{\pi}$ we have $u = \pi$. Substituting yields $\displaystyle\frac{1}{2}\int_0^{\pi} \sin u\,du = \frac{1}{2}(-\cos u)\,\Big|_0^{\pi} = 1$.

5.5.98 $A(x) = \displaystyle\int_2^6 (x-4)^4\,dx = \frac{(x-4)^5}{5}\Bigg|_2^6 = \frac{2^5}{5} - \left(-\frac{(2)^5}{5}\right) = \frac{64}{5}$.

5.5.99 $A(a) = \displaystyle\int_0^a \left(\frac{1}{a} - \frac{x^2}{a^3}\right)dx = \left(\frac{x}{a} - \frac{x^3}{3a^3}\right)\Bigg|_0^a = 1 - \frac{1}{3} = \frac{2}{3}$. This is a constant function.

5.5.100

a. Let $u = x^2$, so that $du = 2x\,dx$. Note that when $x = 1$ or $x = -1$, we have $u = 1$. Substituting gives $\displaystyle\frac{1}{2}\int_1^1 f(u)\,du = 0$. Alternatively, we could note that when f is even, $xf(x^2)$ is odd, so $\displaystyle\int_{-1}^1 xf(x^2)\,dx = 0$.

b. Let $u = x^3$ so that $du = 3x^2\,dx$. Note that when $x = -2$, $u = -8$, and when $x = 2$, $u = 8$. Substituting yields $\frac{1}{3}\int_{-8}^{8} f(u)\,du = \frac{2}{3}\int_{0}^{8} f(u)\,du = \frac{2}{3}\cdot 9 = 6$.

5.5.101

a. Let $u = \sin px$, so that $du = p\cos px\,dx$. Note that when $x = 0$, $u = 0$, and when $x = \pi/2p$, $u = 1$. Substituting yields $\frac{1}{p}\int_0^1 f(u)\,du = \frac{\pi}{p}$.

b. Let $u = \sin x$ so that $du = \cos x\,dx$. Note that when $x = -\pi/2$, $u = -1$ and when $x = \pi/2$, $u = 1$. Substituting yields $\int_{-1}^{1} f(u)\,du = 0$, because f is an odd function. Alternatively, we could note that when f is odd, $\cos x\cdot f(\sin x)$ is also odd, because $\sin x$ is odd and $\cos x$ is even. Thus the given integral must be zero because it is the definite integral of an odd function over a symmetric interval about 0.

5.5.102 The average vertical distance is given by

$$\frac{1}{a}\int_0^a \left(b - \frac{b}{a}x\right)\,dx = \frac{1}{a}\left(bx - \frac{b}{2a}x^2\right)\Big|_0^a = \frac{1}{a}\left(ba - \frac{ba^2}{2a}\right) = b - \frac{b}{2} = \frac{b}{2}.$$

5.5.103 $\frac{1}{\pi/k - 0}\int_0^{\pi/k} \sin kx\,dx = \frac{k}{\pi}\cdot\left(\frac{-\cos kx}{k}\right)\Big|_0^{\pi/k} = \frac{1}{\pi}\left(1-(-1)\right) = \frac{2}{\pi}$.

5.5.104 The area on the left is given by $\int_0^{\pi/2} 2\sin 2x\,dx$. If we let $u = 2x$ so that $du = 2\,dx$, we obtain the equivalent integral $\int_0^{\pi}\sin u\,du$ which represents the area on the right.

5.5.105 The area on the left is given by $\int_4^9 \frac{(\sqrt{x}-1)^2}{2\sqrt{x}}\,dx$. If we let $u = \sqrt{x}-1$ so that $du = \frac{1}{2\sqrt{x}}\,dx$, we obtain the equivalent integral $\int_1^2 u^2\,du$ which represents the area on the right.

5.5.106 Let $u = f(x)$, so that $du = f'(x)\,dx$. Substituting yields $\int (5u^3+7u^2+u)\,du = \frac{5u^4}{4}+\frac{7u^3}{3}+\frac{u^2}{2}+C = \frac{5f^4(x)}{4}+\frac{7f^3(x)}{3}+\frac{f^2(x)}{2}+C$.

5.5.107 Let $u = f(x)$, so that $du = f'(x)\,dx$. Substituting yields

$$\int_4^5 (5u^3+7u^2+u)\,du = \left(\frac{5u^4}{4}+\frac{7u^3}{3}+\frac{u^2}{2}\right)\Big|_4^5 = \frac{7297}{12}.$$

5.5.108 Let $u = f^{(p)}(x)$ so that $du = f^{(p+1)}(x)\,dx$. Substituting yields

$$\int u^n\,du = \frac{u^{n+1}}{n+1}+C = \frac{1}{n+1}\left(f^{(p)}(x)\right)^{n+1}+C.$$

5.5.109 If we let $u = \sqrt{x+a}$, then $u^2 = x+a$ and $2u\,du = dx$. Substituting yields $\int_{\sqrt{a}}^{\sqrt{1+a}}(u^2-a)\cdot u\cdot 2u\,du = \int_{\sqrt{a}}^{\sqrt{1+a}}(2u^4-2au^2)\,du = \left(\frac{2u^5}{5}-\frac{2au^3}{3}\right)\Big|_{\sqrt{a}}^{\sqrt{1+a}} = \frac{2(\sqrt{1+a})^5}{5}-\frac{2a(\sqrt{1+a})^3}{3}-\frac{2a^{5/2}}{5}+\frac{2a^{5/2}}{3}$.

If we let $u = x+a$, then $u-a = x$ and $du = dx$. Substituting yields $\int_a^{a+1}(u-a)\sqrt{u}\,du = \left(\frac{2u^{5/2}}{5}-\frac{2au^{3/2}}{3}\right)\Big|_a^{a+1} = \frac{2(a+1)^{5/2}}{5}-\frac{2a(a+1)^{3/2}}{3}-\frac{2a^{5/2}}{5}+\frac{2a^{5/2}}{3}$.

Note that the two results are the same.

5.5.110 If we let $u = \sqrt[p]{x+a}$, then $u^p = x + a$ and $pu^{p-1}\,du = dx$. Substituting yields

$$\begin{aligned} p\int_{\sqrt[p]{a}}^{\sqrt[p]{1+a}} (u^{2p} - au^p)\,du &= \left(\frac{pu^{2p+1}}{2p+1} - \frac{pau^{p+1}}{p+1}\right)\Bigg|_{\sqrt[p]{a}}^{\sqrt[p]{1+a}} \\ &= \frac{p(1+a)^2\sqrt[p]{1+a}}{2p+1} - \frac{ap(1+a)\sqrt[p]{1+a}}{p+1} - \frac{pa^2\sqrt[p]{a}}{2p+1} + \frac{a^2p\sqrt[p]{a}}{p+1}. \end{aligned}$$

If we let $u = x + a$, then $u - a = x$ and $du = dx$. Substituting yields

$$\begin{aligned} \int_a^{a+1} (u-a)\sqrt[p]{u}\,du = \int_a^{a+1}\left(u^{(p+1)/p} - au^{1/p}\right)du &= \left(\frac{u^{(2p+1)/p}}{(2p+1)/p} - \frac{au^{(p+1)/p}}{(p+1)/p}\right)\Bigg|_a^{a+1} \\ &= \left(\frac{pu^2u^{1/p}}{2p+1} - \frac{pau\cdot u^{1/p}}{p+1}\right)\Bigg|_a^{a+1} \\ &= \frac{p(1+a)^2\sqrt[p]{1+a}}{2p+1} - \frac{ap(1+a)\sqrt[p]{1+a}}{p+1} - \frac{pa^2\sqrt[p]{a}}{2p+1} + \frac{a^2p\sqrt[p]{a}}{p+1}. \end{aligned}$$

Note that the two results are the same.

5.5.111 If we let $u = \cos\theta$, then $du = -\sin\theta\,d\theta$. Substituting yields $\int -u^{-4}\,du = \frac{1}{3u^3} + C = \frac{1}{3\cos^3\theta} + C = \frac{\sec^3\theta}{3} + C$.

If we let $u = \sec\theta$, then $du = \sec\theta\tan\theta\,d\theta$. Substituting yields $\int u^2\,du = \frac{u^3}{3} + C = \frac{\sec^3\theta}{3} + C$. Note that the two results are the same.

5.5.112 Let $u = ax$, so that $\frac{1}{a}du = dx$. Substituting yields $\frac{1}{a}\int \sin^2 u\,du = \frac{1}{a}\int \frac{1-\cos 2u}{2}\,du = \frac{1}{2a}\int (1 - \cos 2u)\,du = \frac{1}{2a}\left(u - \frac{\sin 2u}{2}\right) + C = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$.

For the second integral, we use the same substitution to obtain $\frac{1}{a}\int \cos^2 u\,du = \frac{1}{a}\int \frac{1+\cos 2u}{2}\,du = \frac{1}{2a}\int (1+\cos 2u)\,du = \frac{1}{2a}\left(u + \frac{\sin 2u}{2}\right) + C = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$.

5.5.113

a. Because $\sin 2x = 2\sin x\cos x$, we can write $(\sin x\cos x)^2 = \left(\frac{\sin 2x}{2}\right)^2 = \frac{\sin^2 2x}{4}$. Then we have $I = \frac{1}{4}\int \sin^2 2x\,dx = \frac{1}{4}\left(\frac{x}{2} - \frac{\sin 4x}{8}\right) + C = \frac{x}{8} - \frac{\sin 4x}{32} + C$. Note that we used the result of the previous problem during this derivation.

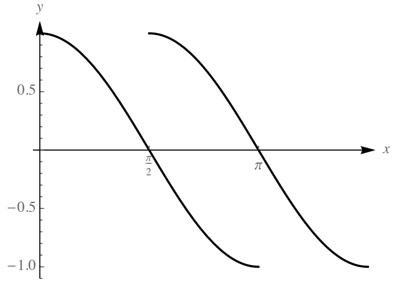
b. $I = \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)\,dx = \frac{1}{4}\int (1-\cos^2 2x)\,dx = \frac{1}{4}\int \sin^2 2x\,dx = \frac{1}{4}\left(\frac{x}{2} - \frac{\sin 4x}{8}\right) + C = \frac{x}{8} - \frac{\sin 4x}{32} + C$.

c. The results are consistent. The work involved is similar in each method.

5.5.114

a. Let $u = x + c$. Note that $du = dx$. Substitution yields $\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(u)\,du$.

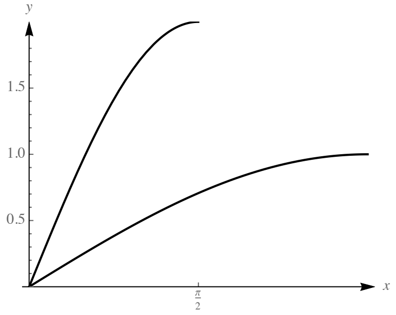
b.



5.5.115

a. Let $u = cx$. Note that $du = c \cdot dx$. Substitution yields $\int_a^b f(cx)\,dx = \frac{1}{c}\int_{ac}^{bc} f(u)\,du$.

b.



5.5.116 First let $u = x^2$, so that $du = 2x\,dx$. Substituting yields $\frac{1}{2}\int \sin^4 u \cos u\,du$. Now let $v = \sin u$, so that $dv = \cos u\,du$. This substitution yields $\frac{1}{2}\int v^4\,dv = \frac{1}{2}\cdot\frac{v^5}{5} + C = \frac{\sin^5 u}{10} + C = \frac{\sin^5 x^2}{10} + C$.

5.5.117 Let $u = \sqrt{x+1}$ so that $u^2 = x + 1$. Then $2u\,du = dx$. Substituting yields $\int 2\cdot\frac{u\,du}{\sqrt{1+u}}$. Now let $v = \sqrt{1+u}$ so that $v^2 = 1+u$ and $2v\,dv = du$. Now a substitution yields $4\int \frac{(v^2-1)v}{v}\,dv = 4\int (v^2-1)\,dv =$
$\frac{4v^3}{3} - 4v + C = \frac{4}{3}(1+u)^{3/2} - 4\sqrt{1+u} + C = \frac{4}{3}\left(1+\sqrt{x+1}\right)^{3/2} - 4\sqrt{1+\sqrt{x+1}} + C =$
$\frac{4}{3}\sqrt{1+\sqrt{x+1}}\left(1+\sqrt{x+1}-3\right) = \frac{4}{3}\sqrt{1+\sqrt{x+1}}(\sqrt{x+1}-2) + C$.

5.5.118 Let $u = 4x$, so that $du = 4\,dx$. Substituting yields $\frac{1}{4}\int \tan^{10} u \sec^2 u\,du$. Now let $v = \tan u$, so that $dv = \sec^2 u\,du$. This leads to $\frac{1}{4}\int v^{10}\,dv = \frac{v^{11}}{44} + C = \frac{\tan^{11} 4x}{44} + C$.

5.5.119 Let $u = \cos\theta$, so that $du = -\sin\theta\,d\theta$. This substitution yields $\int_0^1 \frac{u}{\sqrt{u^2+16}}\,du$. Now let $v = u^2+16$, so that $dv = 2u\,du$. Now a substitution yields $\frac{1}{2}\int_{16}^{17} v^{-1/2}\,dv = \sqrt{v}\,\Big|_{16}^{17} = \sqrt{17} - 4$.

Chapter Five Review

1

a. True. The antiderivative of a linear function is a quadratic function.

b. False. $A'(x) = f(x)$, not $F(x)$.

c. True, Note that f is an antiderivative of f', so this follows from the Fundamental Theorem.

d. True. Because $|f(x)| \geq 0$ for all x, this integral must be positive, unless f is constantly 0.

e. False. For example, the average value of $\sin x$ on $[0, 2\pi]$ is zero.

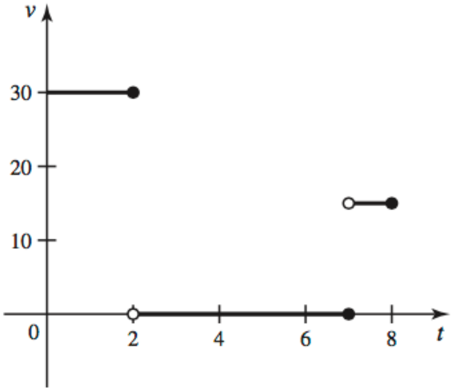
f. True. This is equal to $2\int_a^b f(x)\,dx - 3\int_a^b g(x)\,dx = 2\int_a^b f(x)\,dx + 3\int_b^a g(x)\,dx$.

g. True. The derivative of the right hand side is $f'(g(x))g'(x)$ by the Chain Rule.

2 The left Riemann sum gives $30(4)+25(4)+25(4) = (30+25+25)(4) = 80(4) = 320$ ft. The right Riemann sum gives $25(4) + 25(4) + 20(4) = (25 + 25 + 20)(4) = 70(4) = 280$ ft. The midpoint Riemann sum gives $25(4) + 20(4) + 30(4) = (25 + 20 + 30)(4) = 75(4) = 300$ ft.

3

a.



b. The area is $2 \cdot 30 + 0 + 15 \cdot 1 = 75$.

c. The area represents the distance that the diver ascends.

4 $\Delta x = \dfrac{6-0}{3} = 2$. The left Riemann sum is $f(0)(2) + f(2)(2) + f(4)(2) = 20(2) + 26(2) + 34(2) = (20 + 26 + 34)(2) = 80(2) = 160$. The right Riemann sum is $f(2)(2) + f(4)(2) + f(6)(2) = 26(2) + 34(2) + (40)(2) = (26 + 34 + 40)(2) = 100(2) = 200$. The midpoint Riemann sum is $f(1)(2) + f(3)(2) + f(5)(2) = 22(2) + 30(2) + 38(2) = (22 + 30 + 38)(2) = 90(2) = 180$.

5 $\Delta x = \dfrac{4-1}{6} = \dfrac{1}{2}$. The left Riemann sum is given by $(f(1)+f(1.5)+f(2)+f(2.5)+f(3)+f(3.5))\left(\dfrac{1}{2}\right) = (\sqrt{5} + \sqrt{7} + \sqrt{9} + \sqrt{11} + \sqrt{13} + \sqrt{15})\left(\dfrac{1}{2}\right) \approx 9.34$. The right Riemann sum is given by $(f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4))\left(\dfrac{1}{2}\right) = (\sqrt{7} + \sqrt{9} + \sqrt{11} + \sqrt{13} + \sqrt{15} + \sqrt{17})\left(\dfrac{1}{2}\right) \approx 10.28$. The midpoint Riemann sum is given by $(f(1.25) + f(1.75) + f(2.25) + f(2.75) + f(3.25) + f(3.75))\left(\dfrac{1}{2}\right) = (\sqrt{6} + \sqrt{8} + \sqrt{10} + \sqrt{12} + \sqrt{14} + \sqrt{16})\left(\dfrac{1}{2}\right) \approx 9.82$.

6 The midpoint Riemann sum is $(3\cdot3.5+4)+(3\cdot4.5+4)+(3\cdot5.5+4)+(3\cdot6.5+4)=3\cdot20+16=76$. The exact area of the region is given by $\int_3^7(3x+4)\,dx=\left(\frac{3x^2}{2}+4x\right)\Big|_3^7=\frac{147}{2}+28-\left(\frac{27}{2}+12\right)=60+16=76$.

7

n	Midpoint Riemann sum
10	114.167
30	114.022
60	114.006

From the table, it appears that $\int_1^{25}\sqrt{2x-1}\,dx=114$. This result can be verified using the Fundamental Theorem of Calculus and the substitution $u=2x-1$.

8 Suppose $f(x)=x^3+x$. Then the given expression represents $\int_3^8 f(x)\,dx$.

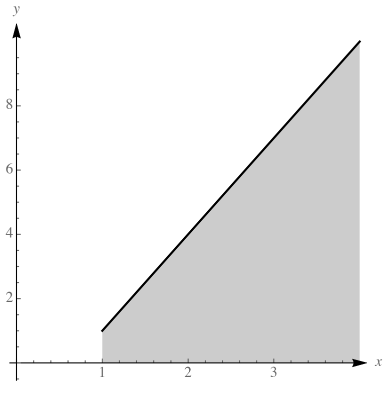
9

a. The right Riemann sum is $4\cdot1+7\cdot1+10\cdot1=21$.

b. The right Riemann sum is $\sum_{k=1}^{n}\left(3\left(1+\frac{3k}{n}\right)-2\right)\cdot\frac{3}{n}$.

c. The sum evaluates as $\sum_{k=1}^{n}\frac{3}{n}+\sum_{k=1}^{n}\frac{27k}{n^2}=3+\frac{27}{n^2}\cdot\frac{n(n+1)}{2}$. As $n\to\infty$, the limit of this expression is $3+13.5=16.5$.

d.



The area consists of a trapezoid with base 3 and heights 1 and 10, so the value is $(3)\left(\frac{1+10}{2}\right)=\frac{33}{2}=16.5$. The Fundamental Theorem assures us that $\int_1^4(3x-2)\,dx=\left(\frac{3x^2}{2}-2x\right)\Big|_1^4=(24-8)-(3/2-2)=16+1/2=16.5$.

10 Let $\Delta x=\frac{1-0}{n}=\frac{1}{n}$. Let $x_k=0+k\Delta x=\frac{k}{n}$. Then $f(x_k)=\frac{4k}{n}-2$. Thus,

$$\lim_{n\to\infty}\sum_{k=1}^{n}f(x_k)\Delta x=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{4k}{n}-2\right)\cdot\frac{1}{n}=\lim_{n\to\infty}\left(\frac{4}{n^2}\sum_{k=1}^{n}k-\frac{2}{n}\sum_{k=1}^{n}1\right)=$$

$$\lim_{n\to\infty}\left(\frac{4}{n^2}\left(\frac{n^2+n}{2}\right)-2\right)=2-2=0.$$

11 Let $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$. Let $x_k = 0 + k\Delta x = \dfrac{2k}{n}$. Then $f(x_k) = \dfrac{4k^2}{n^2} - 4$. Thus,

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{4k^2}{n^2}-4\right)\cdot\frac{2}{n} = \lim_{n\to\infty}\left(\frac{8}{n^3}\sum_{k=1}^{n}k^2 - \frac{8}{n}\sum_{k=1}^{n}1\right) =$$

$$\lim_{n\to\infty}\left(\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - 8\right) = \frac{8}{3} - 8 = \frac{-16}{3}.$$

12 Let $\Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}$. Let $x_k = 1 + k\Delta x = 1 + \dfrac{k}{n} = \dfrac{n+k}{n}$. Then $f(x_k) = 3\dfrac{(n+k)^2}{n^2} + \dfrac{n+k}{n}$. Thus,

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(3\left(\frac{(n+k)^2}{n^2}\right) + \frac{n+k}{n}\right)\cdot\frac{1}{n} = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{3}{n} + \frac{6k}{n^2} + \frac{3k^2}{n^3} + \frac{1}{n} + \frac{k}{n^2}\right) =$$

$$\lim_{n\to\infty}\left(\frac{4}{n}\sum_{k=1}^{n}1 + \frac{7}{n^2}\sum_{k=1}^{n}k + \frac{3}{n^3}\sum_{k=1}^{n}k^2\right) = \lim_{n\to\infty}\left(4 + \frac{7}{2}\cdot\frac{n^2+n}{n^2} + \frac{3}{6}\cdot\frac{n(n+1)(2n+1)}{n^3}\right) = 4 + \frac{7}{2} + 1 = 8.5$$

13 Let $\Delta x = \dfrac{4-0}{n} = \dfrac{4}{n}$. Let $x_k = 0 + k\Delta x = 0 + \dfrac{4k}{n}$. Then $f(x_k) = \dfrac{64k^3}{n^3} - \dfrac{4k}{n}$. Thus,

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{64k^3}{n^3} - \frac{4k}{n}\right)\cdot\frac{4}{n} = \lim_{n\to\infty}\left(\frac{256}{n^4}\sum_{k=1}^{n}k^3 - \frac{16}{n^2}\sum_{k=1}^{n}k\right) =$$

$$\lim_{n\to\infty}\left(\frac{256}{n^4}\cdot\frac{n^2(n+1)^2}{4} - \frac{16}{n^2}\cdot\frac{n(n+1)}{2}\right) = 64 - 8 = 56.$$

14 This sum is equal to $\displaystyle\int_0^4 (x^5+1)\,dx = \left(\frac{x^6}{6} + x\right)\Big|_0^4 = \frac{4^6}{6} + 4 = \frac{2060}{3}$.

15

a. $\displaystyle\int_{-4}^{4} f(x)\,dx = 2\int_0^4 f(x)\,dx = 2\cdot 10 = 20.$

b. $\displaystyle\int_{-4}^{4} 3g(x)\,dx = 3\cdot 0 = 0.$

c. $\displaystyle\int_{-4}^{4} 4f(x) - 3g(x)\,dx = 2\cdot 4\cdot\int_0^4 f(x)\,dx - 3\cdot 0 = 8\cdot 10 - 0 = 80.$

d. Let $u = 4x^2$, so that $du = 8x\,dx$. Substituting yields $\displaystyle\int_0^4 f(u)\,du = 10.$

e. Because f is an even function, $3xf(x)$ is an odd function. Thus, $\displaystyle\int_{-2}^{2} 3xf(x)\,dx = 0.$

16

a. $\displaystyle\int_a^c f(x)\,dx = 20 - 12 = 8.$

b. $\displaystyle\int_b^d f(x)\,dx = 15 - 12 = 3.$

c. $\displaystyle 2\int_c^b f(x)\,dx = -2\int_b^c f(x)\,dx = -2(-12) = 24.$

d. $\displaystyle 4\int_a^d f(x)\,dx = 80 - 48 + 60 = 92.$

e. $\displaystyle 3\int_a^b f(x)\,dx = 3\cdot 20 = 60.$

f. $\displaystyle 2\int_b^d f(x)\,dx = 2(15-12) = 6.$

g. $\displaystyle\int_a^c |f(x)|\,dx = 20 + 12 = 32.$

h. $\displaystyle\int_d^a |f(x)|\,dx = -(20 + 12 + 15) = -47.$

17 $\int_1^4 3f(x)\,dx = 3\int_1^4 f(x)\,dx = 3\cdot 6 = 18.$

18 $-\int_4^1 2f(x)\,dx = 2\int_1^4 f(x)\,dx = 2\cdot 6 = 12.$

19 $\int_1^4 (3f(x)-2g(x))\,dx = 3\int_1^4 f(x)\,dx - 2\int_1^4 g(x)\,dx = 3\cdot 6 - 2\cdot 4 = 18 - 8 = 10.$

20 There is not enough information to compute this integral.

21 There is not enough information to compute this integral.

22 $\int_4^1 (f(x)-g(x))\,dx = \int_1^4 (g(x)-f(x))\,dx = \int_1^4 g(x)\,dx - \int_1^4 f(x)\,dx = 4 - 6 = -2.$

23

a. This region can be divided up into a 4×2 rectangle and a right triangle with base and height equal to 1. Thus, the integral is equal to $8 + \frac{1}{2} = 8.5$.

b. $\int_6^4 f(x)\,dx = -\int_4^6 f(x)\,dx$. The region whose area is $\int_4^6 f(x)\,dx$ consists of a 1×3 rectangle, together with a right triangle with base 1 and height 3, so $\int_4^6 f(x)\,dx = 3 + \dfrac{3}{2} = 4.5$, and $\int_6^4 f(x)\,dx = -4.5$.

c. $\int_5^7 f(x)\,dx = \int_5^6 f(x)\,dx + \int_6^7 f(x)\,dx$. The region lying over $[5,6]$ is a right triangle with height 3 and base 1, so its area is $\frac{3}{2}$. The region lying under $[6,7]$ has the same area, but is below the x-axis, so $\int_6^7 f(x)\,dx = -\dfrac{3}{2}$. So $\int_5^7 f(x)\,dx = \dfrac{3}{2} + -\dfrac{3}{2} = 0$.

d. Note that $\int_4^5 f(x)\,dx = 3$, because the area represented is that of a 1×3 rectangle. Now by the work above, $\int_0^7 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^5 f(x)\,dx + \int_5^7 f(x)\,dx = 8.5 + 3 + 0 = 11.5$.

24

i. We are seeking $\int_0^5 g(t)\,dt$. Because this represents the area of a region which can be divided into a 5×1 rectangle and a right triangle with base 2 and height 2, its value is $5 + 2 = 7$.

ii. We are seeking $\int_3^7 g(t)\,dt$. Because this represents the area of a region which can be divided into a 4×1 rectangle and a right triangle with base 4 and height 2, its value is $4 + 4 = 8$.

iii. We are seeking $\int_0^8 g(t)\,dt = \int_0^3 g(t)\,dt + \int_3^7 g(t)\,dt + \int_7^8 g(t)\,dt = 3 + 8 + 1 = 12$. The first term in this sum is represented by a 1×3 rectangle, the second term is from part ii), and the third is represented by a 1×1 square.

25 $\int_0^4 \sqrt{8x - x^2}\,dx = \int_0^4 \sqrt{16 - (x^2 - 8x + 16)}\,dx = \int_0^4 \sqrt{16 - (x-4)^2}\,dx$. This represents one quarter of the area inside the circle centered at $(4,0)$ with radius 4, so its value is $\dfrac{1}{4}\cdot 16\pi = 4\pi$.

26 $\int_4^0 (2x+\sqrt{16-x^2})\,dx = -2\int_0^4 x\,dx - \int_0^4 \sqrt{16-x^2}\,dx$. The first integral in this sum is the area of a triangle with base 4 and height 4, and the second is the area of a quarter of a circle of radius 4. Therefore the given integral is equal to

$$-2\left(\frac{1}{2}\cdot 4\cdot 4\right) - \frac{1}{4}\left(\pi\cdot 4^2\right) = -16-4\pi = -4(4+\pi).$$

27 It appears that B is the derivative of A, and C is the derivative of B. Thus we must have $A = \int_0^x f(t)\,dt$, $B = f(x)$, and $C = f'(x)$. Note that A is decreasing where B is negative and increasing where B is positive, and has a minimum where B is zero. Note also that B is increasing where C is positive, and is decreasing where C is negative, and has a maximum where C is zero.

28

a. $F(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = -1+1 = 0.$
$F(-2) = \int_0^{-2} f(t)\,dt = -\int_0^{-2} f(t)\,dt = \frac{1}{4}\cdot\pi\cdot 4 = \pi.$
$F(4) = \int_0^4 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^4 f(t)\,dt = 0 + \frac{1}{4}\cdot\pi\cdot 4 = \pi.$

b. $G(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^1 f(t)\,dt = -\left(\int_{-2}^0 f(t)\,dt + \int_0^1 f(t)\,dt\right) = \pi+1.$
$G(0) = \int_1^0 f(t)\,dt = -\int_0^1 f(t)\,dt = 1.$
$G(4) = \int_1^4 f(t)\,dt = \int_1^2 f(t)\,dt + \int_2^4 f(t)\,dt = 1+\pi.$

c. One can either reason that the two functions differ by $\int_0^1 f(t)\,dt$ which is a constant, or using the Fundamental Theorem of Calculus, that the two functions have the same derivative (namely $f(x)$) and therefore differ by a constant. Note that $F(x) = G(x)-1$ for $-2 \le x \le 4$.

29 $\frac{d}{dx}\int_7^x \sqrt{1+t^4+t^6}\,dt = \sqrt{1+x^4+x^6}.$

30 $\frac{d}{dx}\int_3^{e^x} \cos t^2\,dt = \cos(e^x)^2 \cdot \frac{d}{dx}e^x = e^x\cos e^{2x}.$

31 $\frac{d}{dx}\int_x^5 \sin w^6\,dw = -\frac{d}{dx}\int_5^x \sin w^6\,dw = -\sin x^6.$

32 $\frac{d}{dx}\int_{x^2}^5 \sin w^6\,dw = -\frac{d}{dx}\int_5^{x^2} \sin w^6\,dw = -\sin(x^2)^6\cdot\frac{d}{dx}(x^2) = -\sin x^{12}\cdot 2x = -2x\sin x^{12}.$

33

$$\begin{aligned}\frac{d}{dx}\int_{-x}^x \frac{dt}{t^{10}+1} &= \frac{d}{dx}\left(\int_{-x}^0 \frac{dt}{t^{10}+1} + \int_0^x \frac{dt}{t^{10}+1}\right)\\ &= \frac{d}{dx}\left(-\int_0^{-x} \frac{dt}{t^{10}+1} + \int_0^x \frac{dt}{t^{10}+1}\right) = -\frac{1}{(-x)^{10}+1}\cdot(-1) + \frac{1}{x^{10}+1} = \frac{2}{x^{10}+1}.\end{aligned}$$

Alternatively, if we realize that the integrand is an even function, we could write

$$\frac{d}{dx}2\int_0^x \frac{dt}{t^{10}+1} = \frac{2}{x^{10}+1}.$$

34 $\dfrac{d}{dx}\displaystyle\int_{x^2}^{e^x}\sin^3 t\,dt = \dfrac{d}{dx}\left(-\int_0^{x^2}\sin^3 t\,dt + \int_0^{e^x}\sin^3 t\,dt\right) = -(\sin^3 x^2)2x + (\sin^3 e^x)\cdot e^x = e^x\sin^3 e^x - 2x\sin^3 x^2$.

35 $f(x) = -\displaystyle\int_1^x (t-3)(t-6)^{11}\,dt$, so $f'(x) = -(x-3)(x-6)^{11}$. $f' = 0$ for $x = 3$ and $x = 6$. We have $f' > 0$ on $(3,6)$, so f is increasing there, while $f' < 0$ on $(-\infty, 3)$ and on $(6, \infty)$, so f is decreasing on those intervals.

36 The area represented is a triangle with base $x-2$ and height $2x-4$, so its area is $\dfrac{(x-2)(2x-4)}{2} = x^2 - 4x + 4$. If we call this quantity $A(x)$, then $A'(x) = 2x - 4$, as desired.

37 Let $u = ax + b$. Then $du = a\,dx$, or $dx = \dfrac{du}{a}$. The substitution gives

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du = \frac{1}{a}F(u) + C = \frac{1}{a}F(ax+b) + C.$$

38 $\displaystyle\int_1^5 dx = x\Big|_1^5 = 5 - 1 = 4.$

39 $\displaystyle\int_{-2}^2 (3x^4 - 2x + 1)\,dx = \left(\frac{3x^5}{5} - x^2 + x\right)\Bigg|_{-2}^2 = \frac{96}{5} - 4 + 2 - \left(-\frac{96}{5} - 4 - 2\right) = \frac{192}{5} + 4 = \frac{212}{5}.$

40 $\displaystyle\int_0^1 (4x^{21} - 2x^{16} + 1)\,dx = \left(\frac{4x^{22}}{22} - \frac{2x^{17}}{17} + x\right)\Bigg|_0^1 = \frac{2}{11} - \frac{2}{17} + 1 = \frac{199}{187}.$

41 $\displaystyle\int (9x^8 - 7x^6)\,dx = x^9 - x^7 + C.$

42 $\displaystyle\int \frac{\sqrt{x}+1}{\sqrt{x}}\,dx = \int\left(1 + \frac{1}{\sqrt{x}}\right)dx = \int (1 + x^{-1/2})\,dx = x + 2x^{1/2} + C = x + 2\sqrt{x} + C.$

43 $\displaystyle\int_0^1 (x + \sqrt{x})\,dx = \left(\frac{x^2}{2} + \frac{2x^{3/2}}{3}\right)\Bigg|_0^1 = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}.$

44 Let $u = 3x^2 + 2x + 1$, so that $du = (6x+2)\,dx = 2(3x+1)\,dx$. Then $\displaystyle\int (3x+1)(3x^2+2x+1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{u^4}{8} + C = \frac{(3x^2+2x+1)^4}{8} + C.$

45 $\displaystyle\int_{\pi/6}^{\pi/3} (\sec^2 t + \csc^2 t)\,dt = (\tan t - \cot t)\Big|_{\pi/6}^{\pi/3} = \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) = 2\sqrt{3} - \frac{2}{\sqrt{3}} = \frac{6-2}{\sqrt{3}} = \frac{4\sqrt{3}}{3}.$

46 $\displaystyle\int_{\pi/12}^{\pi/9} (\csc 3x\cot 3x + \sec 3x\tan 3x)\,dx = \frac{1}{3}(-\csc 3x + \sec 3x)\Big|_{\pi/12}^{\pi/9} = \frac{1}{3}\left(\left(-\frac{2}{\sqrt{3}} + 2\right) - \left(-\sqrt{2} + \sqrt{2}\right)\right) = \frac{2}{3} - \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3} - 2}{3\sqrt{3}}.$

47 $\displaystyle\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x\Big|_{\sqrt{2}}^2 = \sec^{-1} 2 - \sec^{-1}\sqrt{2} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$

48 $\displaystyle\int_1^4 \left(\frac{\sqrt{v}+v}{v}\right)dv = \int_1^4 (v^{-1/2} + 1)\,dv = \left(2\sqrt{v} + v\right)\Big|_1^4 = 8 - 3 = 5.$

49 Let $u = \sin x$. Then $du = \cos x\,dx$. Then $\int \frac{\cos x}{\sin^{7/4} x}\,dx = \int u^{-7/4}\,du = -\frac{4}{3}u^{-3/4}\,du = -\frac{4}{3}(\sin x)^{-3/4} + C = -\frac{4}{3\sin^{3/4} x} + C$.

50 Let $u = 1 + \ln x$. Then $du = \frac{1}{x}\,dx$. Also, when $x = 1$, $u = 1$, and when $x = e$, $u = 2$. Substituting yields $\int_1^e \frac{dx}{x(1+\ln x)}\,dx = \int_1^2 \frac{du}{u} = \ln|u|\Big|_1^2 = \ln 2$.

51 Let $u = x^3$, so that $du = 3x^2\,dx$. Substituting gives $\int x^2 \cos x^3\,dx = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin x^3 + C$.

52 Let $u = 1+\sin 3t$, so that $du = 3\cos 3t\,dt$. Substituting yields $\int \frac{\cos 3t}{1+\sin 3t}\,dt = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|1+\sin 3t| + C$. Note that because $-1 \le \sin 3t$, we have $0 \le 1 + \sin 3t$, so the absolute value sign isn't necessary, so we can write the answer as $\frac{1}{3}\ln(1+\sin 3t) + C$.

53 Let $u = \sin 7w$. Then $du = 7\cos 7w$. Substituting yields $\frac{1}{7}\int \frac{du}{16+u^2} = \frac{1}{7}\cdot\frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right) = \frac{1}{28}\tan^{-1}\left(\frac{\sin 7w}{4}\right) + C$.

54 Let $u = 1 + \tan 2t$. Then $du = 2\sec^2 2t\,dt$. Substituting gives

$$\frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\left(\frac{2u^{3/2}}{3}\right) + C = \frac{1}{3}(1+\tan 2t)^{3/2} + C.$$

55 Let $u = x^2 + 1$. Then $du = 2x\,dx$ and when $x = 0$, $u = 1$ and when $x = 1$, $u = 2$. Substituting gives

$$\frac{1}{2}\int_1^2 2^u\,du = \frac{2^u}{2\ln 2}\Big|_1^2 = \frac{2}{\ln 2} - \frac{1}{\ln 2} = \frac{1}{\ln 2}.$$

56 $\int \cos 3x\,dx = \frac{\sin 3x}{3} + C$.

57 $\int_0^2 (2x+1)^3\,dx = \left(\frac{(2x+1)^4}{8}\right)\Big|_0^2 = \frac{625}{8} - \frac{1}{8} = 78$.

58 $\int_{-2}^2 e^{4x+8}\,dx = \left(\frac{1}{4}\cdot e^{4x+8}\right)\Big|_{-2}^2 = \frac{1}{4}\left(e^{16} - 1\right)$.

59 Let $u = 3r^2 + 2$. Then $du = 6r\,dr$. When $r = 0$, $u = 2$ and when $r = 1$, $u = 5$. Substituting gives

$$\frac{5}{6}\int_2^5 e^u\,du = \frac{5}{6}e^u\Big|_2^5 = \frac{5}{6}(e^5 - e^2) = \frac{5}{6}e^2(e^3 - 1).$$

60 Let $u = \cos z$. Then $du = -\sin z\,dz$. Substituting gives $-\int \sin u\,du = \cos u + C = \cos(\cos z) + C$.

61 Let $u = e^x$, so that $du = e^x\,dx$. We can rewrite the given integral as $\int e^{e^x} e^x\,dx$, so that substituting gives $\int e^u\,du = e^u + C = e^{e^x} + C$.

62 Let $u = y^3+27$, and note that $du = 3y^2\,dy$. Substituting yields $\frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u|+C = \frac{\ln|y^3+27|}{3}+C$.

63 Let $u = 2x$ so that $du = 2\,dx$. Substituting gives

$$\frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}2x + C.$$

64 Let $u = 3y^3+1$, and note that $du = 9y^2\,dy$. Substituting yields $\frac{1}{9}\int u^4\,du = \frac{u^5}{45}+C = \frac{(3y^3+1)^5}{45}+C$.

65 Let $u = \frac{x}{6}$, then $du = \frac{1}{6}\,dx$. When $x = 0$, $u = 0$ and when $x = 2\pi$, $u = \pi/3$. Substituting gives $6\int_0^{\pi/3}\cos^2 u\,du$. Using Example 6 of section 5.6, we have

$$6\int_0^{\pi/3}\cos^2 u\,du = 6\left(\frac{u}{2}+\frac{\sin 2u}{4}\right)\Big|_0^{\pi/3} = \pi + \frac{3\sqrt{3}}{4}.$$

66 Let $u = \cos x^2$ and note that $du = -\sin x^2 \cdot 2x\,dx$. Substituting yields $-\frac{1}{2}\int u^8\,du = -\frac{u^9}{18}+C = -\frac{\cos^9 x^2}{18}+C$.

67 $\int_0^{\pi}\sin^2 5\theta\,d\theta = \int_0^{\pi}\frac{1-\cos 10\theta}{2}\,d\theta = \left(\frac{\theta}{2}-\frac{\sin 10\theta}{20}\right)\Big|_0^{\pi} = \frac{\pi}{2}$.

68 $\int_0^{\pi}(1-\cos^2 3\theta)\,d\theta = \int_0^{\pi}\sin^2 3\theta\,d\theta = \int_0^{\pi}\frac{1-\cos 6\theta}{2}\,d\theta = \left(\frac{\theta}{2}-\frac{\sin 6\theta}{12}\right)\Big|_0^{\pi} = \frac{\pi}{2}$.

69 Let $u = x^3+3x^2-6x$, and note that $du = 3x^2+6x-6\,dx = 3(x^2+2x-2)\,dx$. Substituting yields $\frac{1}{3}\int_8^{36}\frac{1}{u}\,du = \left(\frac{1}{3}\ln u\right)\Big|_8^{36} = \frac{1}{3}(\ln 36 - \ln 8) = \frac{1}{3}\ln\left(\frac{9}{2}\right)$.

70 Let $u = e^x$ so that $du = e^x\,dx$. Substituting yields $\int_1^2 \frac{1}{1+u^2}\,du = \tan^{-1}u\,\Big|_1^2 = \tan^{-1}2 - \frac{\pi}{4}$.

71 Note that the integrand is an odd function (as it is an odd function divided by an even function). Therefore $\int_{-5}^{5}\frac{w^3}{\sqrt{w^{50}+w^{20}+1}} = 0$.

72 Because x^{17}, x^{13}, x^9, and x^3 are all odd functions, this integral reduces to $2\int_0^3 x^2\,dx = \frac{2x^3}{3}\Big|_0^3 = 18$.

73 Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}\,dx$. Substituting yields $-\int \sin u\,du = \cos u + C = \cos\left(\frac{1}{x}\right)+C$.

74 Let $u = \tan^{-1}x$. Then $du = \frac{1}{1+x^2}\,dx$. Substituting yields $\int u^5\,du = \frac{u^6}{6}+C = \frac{(\tan^{-1}x)^6}{6}+C$.

75 Let $u = \tan^{-1}x$. Then $du = \frac{1}{1+x^2}\,dx$. Substituting yields $\int \frac{1}{u}\,du = \ln|u|+C = \ln|\tan^{-1}x|+C$.

76 Let $u = \sin^{-1}x$. Then $du = \frac{1}{\sqrt{1-x^2}}\,dx$. Substituting yields $\int u\,du = \frac{u^2}{2}+C = \frac{(\sin^{-1}x)^2}{2}+C$.

77 Let $u = x+3$. Then $du = dx$ and $u - 3 = x$. Substituting gives $\int (u-3)u^{10}\,du = \int (u^{11} - 3u^{10})\,du = \frac{u^{12}}{12} - \frac{3u^{11}}{11} + C = u^{11}\left(\frac{u}{12} - \frac{3}{11}\right) = (x+3)^{11}\left(\frac{x}{12} - \frac{1}{44}\right) = (x+3)^{11}\left(\frac{11x-3}{132}\right)$.

78 Let $u = x^4 + 1$. Then $du = 4x^3\,dx$ and $u - 1 = x^4$. Substituting gives $\int x^3 x^4 \sqrt{x^4+1}\,dx =$ $\frac{1}{4}\int (u-1)\sqrt{u}\,du = \frac{1}{4}\int (u^{3/2} - u^{1/2})\,du = \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C =$ $\frac{u^{3/2}}{4}\left(\frac{2}{5}u - \frac{2}{3}\right) + C = \frac{(x^4+1)^{3/2}}{4}\left(\frac{2}{5}(x^4+1) - \frac{2}{3}\right) + C = \frac{(x^4+1)^{3/2}}{4}\left(\frac{2}{5}x^4 - \frac{4}{15}\right) + C =$ $\frac{1}{30}\left(x^4+1)\right)^{3/2}(3x^4 - 2) + C$.

79 Let $u = 25 - x^2$. Then $du = -2x\,dx$. Substituting yields $-\frac{1}{2}\int_{25}^{16} u^{-1/2}\,du = -\sqrt{u}\,\Big|_{25}^{16} = 5 - 4 = 1$.

80 Using Table 5.6, we have

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\frac{x}{2}\Big|_0^1 = \sin^{-1}\frac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6}.$$

81 Let $u = 5x$ so that $du = 5dx$. When $x = \sqrt{2}/5$, $u = \sqrt{2}$, and when $x = 2/5$, $u = 2$. Substituting then gives

$$\int_{\sqrt{2}}^2 \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u|\Big|_{\sqrt{2}}^2 = \left(\sec^{-1} 2 - \sec^{-1}\sqrt{2}\right) = \left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{12}.$$

82 Let $u = 1 + \cos^2 x$. Then $du = -2\sin x \cos x\,dx$. Substituting yields $-\int \frac{1}{u}\,du = -\ln|u| + C = -\ln(1+\cos^2 x) + C$.

83 $\int_{-10}^{10} \frac{x}{\sqrt{200-x^2}}\,dx = 0$ because the integrand is an odd function.

84 Note that the first term of the integrand is an even function, and the other two terms are odd functions. Thus, $\int_{-\pi/2}^{\pi/2} (\cos 2x + \cos x \sin x - 3\sin x^5)\,dx = 2\int_0^{\pi/2} \cos 2x\,dx = 2\left(\frac{\sin 2x}{2}\right)\Big|_0^{\pi/2} = 0$.

85 $\int_0^4 f(x)\,dx = \int_0^3 (2x+1)\,dx + \int_3^4 (3x^2+2x-8)\,dx = (x^2+x)\Big|_0^3 + (x^3+x^2-8x)\Big|_3^4 = 12 + (48-12) = 48$.

86 $\int_0^5 |2x-8|\,dx = \int_0^4 (8-2x)\,dx + \int_4^5 (2x-8)\,dx = (8x-x^2)\Big|_0^4 + (x^2-8x)\Big|_4^5 = 16 + (-15+16) = 17$.

87 The area is given by $\int_{-4}^4 (16-x^2)\,dx = \left(16x - \frac{x^3}{3}\right)\Big|_{-4}^4 = 64 - \frac{64}{3} - (-64 + \frac{64}{3}) = \frac{256}{3}$.

88 The area is given by $\int_{-1}^0 (x^3 - x)\,dx = \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^0 = 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4}$.

89 The area is given by $2\int_0^{2\pi} \sin(x/4)\,dx = 2\,(-4\cos(x/4))\Big|_0^{2\pi} = -8(0-1) = 8$.

90 The area is given by $\int_{-1}^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \tan^{-1} x\,\Big|_{-1}^{\sqrt{3}} = \frac{\pi}{3} - \left(-\frac{\pi}{4}\right) = \frac{7\pi}{12}$.

91

i. $\int_{-1}^{1}(x^4 - x^2)\,dx = \left(\frac{x^5}{5} - \frac{x^3}{3}\right)\Big|_{-1}^{1} = \left(\frac{1}{5} - \frac{1}{3}\right) - \left(-\frac{1}{5} - \left(-\frac{1}{3}\right)\right) = -\frac{4}{15}$.

ii. Because the region lies completely below the x-axis, the area bounded by the curve and the x-axis is $-\int_{-1}^{1}(x^4 - x^2)\,dx = \frac{4}{15}$.

92

i. $\int_{0}^{3}(x^2 - x)\,dx = \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_{0}^{3} = 9 - \frac{9}{2} = \frac{9}{2}$.

ii. Because the region is below the x-axis between 0 and 1 and above between 1 and 3, the area bounded by the curve and the x-axis is $-\int_{0}^{1}(x^2 - x)\,dx + \int_{1}^{3}(x^2 - x)\,dx = -\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_{0}^{1} + \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_{1}^{3} =$ $-\left(\frac{1}{3} - \frac{1}{2}\right) + \left(9 - \frac{9}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{9}{2} + 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{9}{2} + \frac{1}{3} = \frac{29}{6}$.

93 The average height of the arch is given by

$$\begin{aligned}&\frac{1}{630}\int_{-315}^{315}\left(1260 - 315\left(e^{0.00418x} + e^{-.00418x}\right)\right)\,dx\\ &= \frac{1}{630}\left(1260x - \frac{315}{0.00418}(e^{0.00418x} - e^{-.00418x})\right)\Big|_{-315}^{315} \approx 431.5 \text{ ft.}\end{aligned}$$

94 Let $T = \frac{2\pi k}{\omega}$ where k is an integer. The RMS is given by $\sqrt{\frac{\omega}{2\pi k}\int_{0}^{2\pi k/\omega} A^2\sin^2(\omega t)\,dt} =$ $A\sqrt{\frac{\omega}{2\pi k}\int_{0}^{2\pi k/\omega}\frac{1 - \cos(2\omega t)}{2}\,dt} = A\sqrt{\frac{\omega}{2\pi k}\left[\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega}\right]_{0}^{2\pi k/\omega}} = \frac{A}{\sqrt{2}}$.

95 The displacement is $\int_{0}^{2} 5\sin\pi t\,dt = -\frac{5\cos\pi t}{\pi}\Big|_{0}^{2} = -\frac{5}{\pi}(\cos 2\pi - \cos 0) = 0$. The distance traveled is

$$\begin{aligned}\int_{0}^{1} 5\sin\pi t\,dt + \int_{1}^{2} -5\sin\pi t\,dt &= -\frac{5\cos\pi t}{\pi}\Big|_{0}^{1} + \frac{5\cos\pi t}{\pi}\Big|_{1}^{2}\\ &= \frac{5}{\pi} + \frac{5}{\pi} + \frac{5}{\pi} + \frac{5}{\pi} = \frac{20}{\pi}.\end{aligned}$$

96

a. The distance traveled is given by $\int_{0}^{4}(2t + 5)\,dt = \left(t^2 + 5t\right)\Big|_{0}^{4} = 36$.

b. The average value is $\frac{1}{4}\int_{0}^{4}(2t + 5)\,dt = \frac{1}{4}\cdot 36 = 9$.

c. True. If it traveled at a rate of 9 for a time of 4, it would have gone 36 units.

97 The average value is

$$\frac{1}{\ln 2}\int_{0}^{\ln 2} e^{2x}\,dx = \frac{1}{2\ln 2}e^{2x}\Big|_{0}^{\ln 2} = \frac{4 - 1}{2\ln 2} = \frac{3}{2\ln 2}.$$

98 The baseball is in the air for x in the interval $(0, 200)$. The average height is

$$\frac{1}{200}\int_0^{200}(2x - 0.01x^2)\,dx = \frac{1}{200}\left(x^2 - \frac{0.01x^3}{3}\right)\Big|_0^{200} = 200 - \frac{400}{3} = \frac{200}{3}.$$

99

a. The average value is 2.5. This is because for a straight line, the average value occurs at the midpoint of the interval, which is at the point $(3.5, 2.5)$, so $c = 3.5$.

b. The average value is 3 over the interval $[2, 4]$ and 3 over the interval $[4, 6]$, so is 3 over the interval $[2, 6]$. The function takes on this value at $c = 3$ and $c = 5$.

100 Differentiating both sides of the given equation gives $12x^3 = f(x)$. To check, we compute $\int_2^x 12t^3\,dt = 3t^4\Big|_2^x = 3x^4 - 48$, which gives the original equation.

101 Let $u = 2x$, so that $du = 2\,dx$. We have $\frac{1}{2}\int_2^4 f'(u)\,du = \frac{1}{2}\cdot(f(4) - f(2)) = \frac{f(4)}{2} - 2$. Because we are given that this quantity is 10, we have $f(4) = 24$.

102 Note that $H'(x) = \sqrt{4 - x^2}$ by the Fundamental Theorem.

a. $H(0) = \int_0^0 \sqrt{4 - t^2}\,dt = 0$.

b. $H'(1) = \sqrt{3}$.

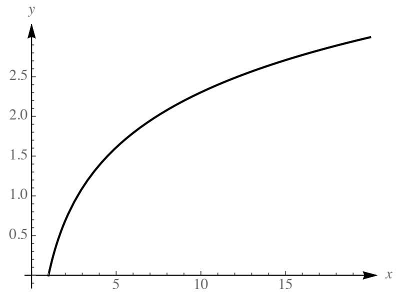
c. $H'(2) = \sqrt{4 - 4} = 0$.

d. $H(2) = \int_0^2 \sqrt{4 - t^2}\,dt = \frac{1}{4}\cdot\pi\cdot 2^2 = \pi$. This follows because the given area represents $\frac{1}{4}$ of the area inside a circle of radius 2.

e. $H(-x) = \int_0^{-x} \sqrt{4 - t^2}\,dt = -\int_{-x}^0 \sqrt{4 - t^2}\,dt = -H(x)$, because $\sqrt{4 - t^2}$ is an even function. So $s = -1$.

103

By the Fundamental Theorem, $f'(x) = \frac{1}{x}$, which is always positive for $x > 1$. Thus f is always increasing. Also, $f(1) = \int_1^1 \frac{1}{t}\,dt = 0$. Also, $f''(x) = -\frac{1}{x^2}$ which is always negative, so f is always concave down.



104 If we let $u^3 = x^2 - 1$, then $3u^2\,du = 2x\,dx$, so $x\,dx = \frac{3u^2}{2}\,du$. Also, when $x = 1$ we have $u = 0$ and when $x = 3$ we have $u = 2$. Substituting gives $\int_1^3 x\sqrt[3]{x^2 - 1}\,dx = \frac{3}{2}\int_0^2 u^2\cdot u\,du = \frac{3}{2}\left(\frac{u^4}{4}\right)\Big|_0^2 = 6$.

105

a. $F(-2) = \displaystyle\int_{-1}^{-2} f(t)\,dt = \int_{-1}^{-2} t\,dt = \left.\frac{t^2}{2}\right|_{-1}^{-2} = \frac{3}{2}$.
$F(2) = \displaystyle\int_{-1}^{2} f(t)\,dt = \int_{-1}^{0} f(t)\,dt + \int_{0}^{2} f(t)\,dt = \int_{-1}^{0} t\,dt + \int_{0}^{2} \frac{t^2}{2}\,dt = \left.\frac{t^2}{2}\right|_{-1}^{0} + \left.\frac{t^3}{6}\right|_{0}^{2} = \left(0 - \frac{1}{2}\right) + \left(\frac{8}{6} - 0\right) = \frac{5}{6}$.

b. By the Fundamental Theorem of Calculus, $F'(x) = f(x)$, so for $-2 \le x < 0$ we have $F''(x) = x$.

c. By the Fundamental Theorem of Calculus, $F'(x) = f(x)$, so for $0 \le x < 2$ we have $F''(x) = \frac{x^2}{2}$.

d. $F'(-1) = -1$ and $F'(1) = \frac{1}{2}$. These represent the rate of change of F at the given points. Because the graph in the exercise is the derivative of F, this is just the value of f at the given points.

e. Note that $F''(x) = \begin{cases} 1 & \text{if } -2 \le x < 0, \\ x & \text{if } 0 \le x \le 2. \end{cases}$ Thus we have $F''(-1) = 1$ and $F''(1) = 1$.

f. The difference $F(x) - G(x) = \displaystyle\int_{-1}^{-2} f(t)\,dt = \frac{3}{2}$, as noted in part (a).

106

a. $G(-1) = \displaystyle\int_{-2}^{-1} f(t)\,dt = \int_{-2}^{-1} t\,dt = \left.\frac{t^2}{2}\right|_{-2}^{-1} = -\frac{3}{2}$.
$G(1) = \displaystyle\int_{-2}^{1} f(t)\,dt = \int_{-2}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt = \int_{-2}^{0} t\,dt + \int_{0}^{1} \frac{t^2}{2}\,dt = \left.\frac{t^2}{2}\right|_{-2}^{0} + \left.\frac{t^3}{6}\right|_{0}^{1} = -\frac{11}{6}$.

b. By the Fundamental Theorem, $G'(x) = f(x)$, so that for $-2 \le x < 0$, we have $G'(x) = f(x) = x$.

c. By the Fundamental Theorem, $G'(x) = f(x)$, so that for $0 \le x \le 2$, we have $G'(x) = f(x) = \frac{x^2}{2}$.

d. $G'(0) = f(0) = 0$ and $G'(1) = f(1) = \frac{1}{2}$. These represent the rate of change of G at the given points. Because the graph in the exercise is the derivative of G, this is just the value of f at the given points.

e. The difference $F(x) - G(x) = \displaystyle\int_{-1}^{-2} f(t)\,dt = \frac{3}{2}$, as noted in part (f) of the previous problem.

107 By L'hôpital's rule, we have

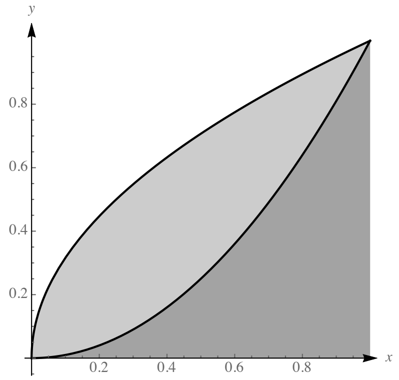
$$\lim_{x\to 2} \frac{\int_2^x e^{t^2}\,dt}{x-2} = \lim_{x\to 2} \frac{e^{x^2}}{1} = e^4.$$

108 By L'hôpital's rule, we have

$$\lim_{x\to 1} \frac{\int_1^{x^2} e^{t^3}\,dt}{x-1} = \lim_{x\to 1} \frac{2xe^{x^6}}{1} = 2e.$$

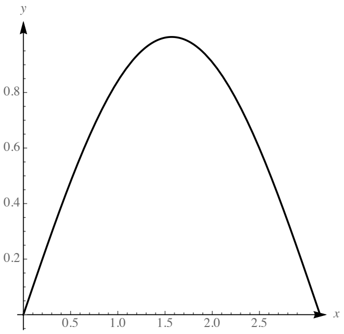
109

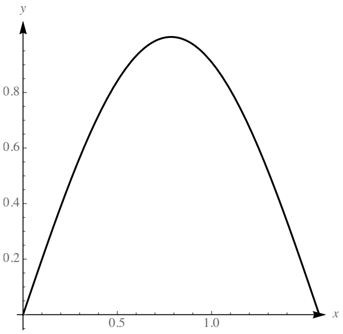
Because x^n and $\sqrt[n]{x}$ are inverse functions of each other, they are symmetric in the square $[0,1]\times[0,1]$ about the line $y=x$. Together, the two regions completely fill up the 1×1 square, so these two areas add to one.

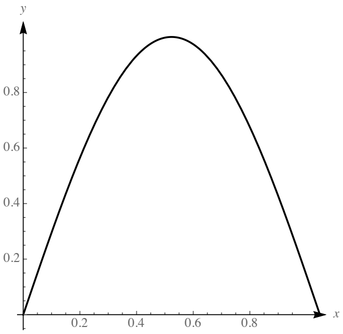


110

a.



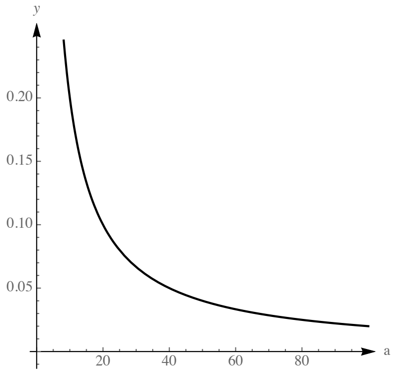




b.

$$\int_0^{\pi/a}\sin(ax)\,dx=\left(\frac{-\cos ax}{a}\right)\Big|_0^{\pi/a}=$$
$$\frac{1}{a}+\frac{1}{a}=\frac{2}{a}.$$

This is a decreasing function of a.

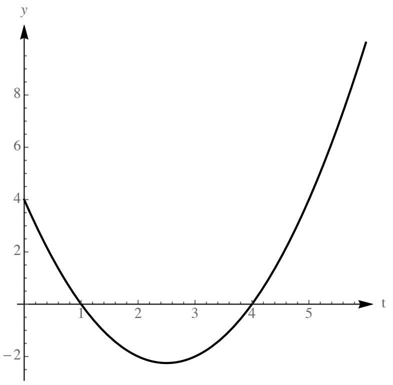


111 Factoring out $\frac{1}{b^2}$ gives $\frac{1}{b^2}\int\frac{dx}{(ax/b)^2+1}$. Now let $u=\frac{ax}{b}$, so that $du=\frac{a}{b}\,dx$.

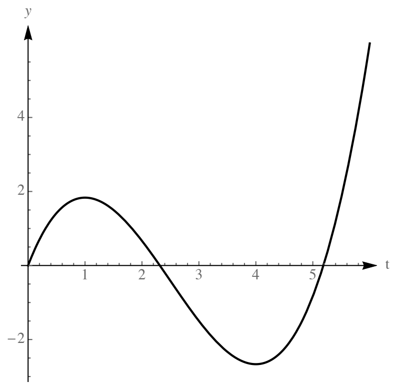
Substituting yields $\frac{1}{b^2}\cdot\frac{b}{a}\int\frac{du}{u^2+1}=\frac{1}{ab}\left(\tan^{-1}(u)\right)+C=\frac{1}{ab}\tan^{-1}\left(\frac{ax}{b}\right)+C$.

112

a.



b.



c. The zeros of f are at 1 and 4, and A has a local maximum at $x = 1$ and a local minimum at $x = 4$.

d. Geometric: Because f is above the axis from 0 to 1 and then crosses below the axis at 1, the net area from 0 to x will switch from increasing to decreasing as x moves from the left of 1 to the right of 1. A similar (but opposite) thing can be said near 4, because f switches from below the axis to above, the net area switches from decreasing to increasing at $x = 4$.

Analytic: By the Fundamental Theorem, $A'(x) = f(x)$, so the zeros of f are critical points of A, and in this example, lead to extrema.

e. Because $A(x) = \frac{x}{6} \cdot \left(2x^2 - 15x + 24\right)$, the non-zero zeros of A occur at $x = \frac{15 \pm \sqrt{225 - 4 \cdot 2 \cdot 24}}{4} = \frac{15 \pm \sqrt{33}}{4}$. So $x_1 = \frac{15 - \sqrt{33}}{4} \approx 2.31386$ and $x_2 = \frac{15+\sqrt{33}}{4} \approx 5.18614$.

f. Because $f(x) = A'(x)$, the area bounded by the graph of f and the x-axis on $[0, x_1]$ is

$$\int_0^1 f(x)\,dx - \int_1^{x_1} f(x)\,dx = A(1) - A(0) - (A(x_1) - A(1)) = 2A(1) = \frac{11}{3}.$$

Now, note that the area bounded by the graph of f on $[x_1, 4]$ is $-(A(4) - A(x_1)) = \frac{8}{3}$, so that $b > 4$. Then the area bounded by the graph of f and the x-axis on $[x_1, b]$ is

$$-\int_{x_1}^4 f(x)\,dx + \int_4^b f(x)\,dx = -(A(4) - A(x_1)) + A(b) - A(4) = A(b) - 2A(4) = A(b) + \frac{16}{3}.$$

Thus we want to solve $A(b) = -\frac{5}{3}$; using technology, we obtain $b \approx 4.756$.

g. No. For example, consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ -1 & \text{if } 1 \le x \le 2. \end{cases}$$

Then $A(x) = \int_0^x f(t)\,dt$ has a maximum at $x = 1$, even though f is never zero. An extreme point of A can occur at points of discontinuity of f.

113 Note that $f'(x) = (x-1)^{15}(x-2)^9$, and that the zeros of f' are at $x = 1$ and $x = 2$.

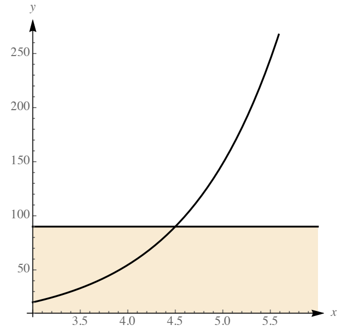
a. f' is positive and thus f is increasing on $(-\infty, 1)$ and on $(2, \infty)$, while f' is negative and f is decreasing on $(1, 2)$.

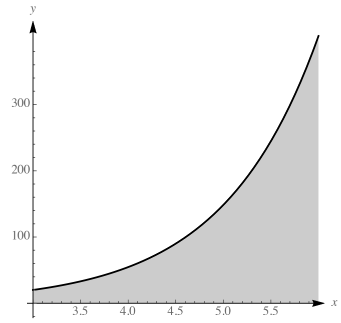
b. $f''(x) = 15(x-1)^{14}(x-2)^9 + (x-1)^{15} \cdot 9(x-2)^8 = 3(x-1)^{14}(x-2)^8(8x-13)$.
f is concave up on $\left(\frac{13}{8}, \infty\right)$ and concave down on $\left(-\infty, \frac{13}{8}\right)$.

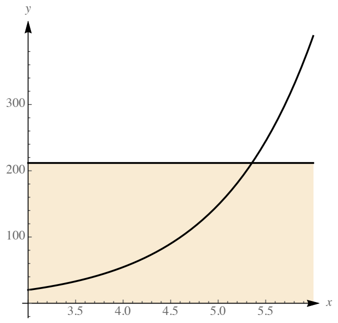
c. f has a local maximum at $x = 1$ and a local minimum at $x = 2$.

d. f has an inflection point at $x = \frac{13}{8}$.

114







The first graph on the left above shows that the area of the rectangle with height e^m where m is the midpoint of $[a, b]$ is less than the area under the curve of e^x over $[a, b]$. The last graph on the right above shows that the area of the rectangle whose height is the average of e^a and e^b is greater than the area under the curve of e^x over that interval. Note that the area under the curve is $\int_a^b e^x\,dx = (e^x)\Big|_a^b = e^b - e^a$. Putting these ideas together, we have

$$e^{(a+b)/2}(b-a) < e^b - e^a < \left(\frac{e^a + e^b}{2}\right)(b-a),$$

and dividing through by $(b-a)$ yields

$$e^{(a+b)/2} < \frac{e^b - e^a}{b-a} < \frac{e^a + e^b}{2}.$$

115 This follows by differentiating each side of the equation. $\frac{d}{dx}\left(u(x) + 2\int_0^x u(t)\,dt\right) = u'(x) + 2u(x)$, and $\frac{d}{dx}10 = 0$. The reverse is not true, because if $u(x) + 2\int_0^x u(t)\,dt = C$ for any constant C, then it would satisfy the second equation, even if $C \neq 10$.

116 The area bounded by $c \sin x$ over the stated interval is $\int_0^\pi c \sin x\,dx = (-c\cos x)\Big|_0^\pi = (c - -c) = 2c$. This is equal to 1 when $c = \frac{1}{2}$.

117 $\int_0^c x(x-c)^2\,dx = \int_0^c (x^3 - 2cx^2 + c^2x)\,dx = \left(\frac{x^4}{4} - 2c\frac{x^3}{3} + \frac{c^2x^2}{2}\right)\Big|_0^c = \frac{c^4}{4} - \frac{2c^4}{3} + \frac{c^4}{2} = \frac{c^4}{12}$. This is one when $c = \sqrt[4]{12}$.

Chapter 6

Applications of Integration

6.1 Velocity and Net Change

6.1.1 The position of an object is the coordinate of the object on the line at a given time, often denoted $s(t)$. The displacement over an interval $[a, b]$ is $s(b) - s(a)$, the difference of the object's ending position and beginning position. It can be written as $\int_a^b v(t)\, dt$ where $v(t)$ is the object's velocity at time t. The distance traveled by the object is $\int_a^b |v(t)|\, dt$, the sum of the distance traveled along the line to the right and the distance traveled along the line to the left over the given time interval.

6.1.2 If velocity is positive, then $|v(t)| = v(t)$, so the distance and displacement are equal.

6.1.3 The displacement is given by $\int_a^b v(t)\, dt$, because this quantity is equal to $s(b) - s(a)$.

6.1.4 The net change of a quantity is given by $\int_a^b f'(t)\, dt$, if $f'(t)$ is the rate of change of the quantity.

6.1.5 The value of Q at time t will be given by $Q(t) = Q(0) + \int_0^t Q'(x)\, dx$.

6.1.6 If $Q'(t)$ is the growth rate of a population Q at time t, then $\int_a^b Q'(t)\, dt = Q(b) - Q(a)$, the net change of the population over the time period $[a, b]$.

6.1.7

a. The velocity is positive for $0 \le t < 1$ and for $3 < t < 5$, so the object is moving in the positive direction on those intervals.

b. The displacement is $\int_0^3 v(t)\, dt = 12 - 16 = -4$.

c. The distance traveled is $\int_1^5 |v(t)|\, dt = 16 + 10 = 26$.

d. The displacement is $\int_0^5 v(t)\, dt = 12 - 16 + 10 = 6$.

e. After five hours the object's position is 6 miles from the original position in the positive direction.

6.1.8

a. The velocity is negative for $0 < t < 2$ and for $4 < t < 6$, so the object is moving in the negative direction on those intervals.

b. The displacement is $\displaystyle\int_2^6 v(t)\,dt = 14 - 10 = 4$.

c. The distance traveled is $\displaystyle\int_0^6 |v(t)|\,dt = 20 + 14 + 10 = 44$.

d. The displacement is $\displaystyle\int_0^8 v(t)\,dt = -20 + 14 - 10 + 6 = -10$.

e. After eight minutes the object's position is 10 meters from the original position in the negative direction.

6.1.9

a. The displacement is the net area, which is $\frac{1}{2} \cdot 2 \cdot 2 - \frac{1}{2} \cdot \frac{4}{3} \cdot 1 + \frac{1}{2} \cdot \frac{5}{3} \cdot 2 = 3$.

b. The distance traveled is $\frac{1}{2} \cdot 2 \cdot 2 + \frac{1}{2} \cdot \frac{4}{3} \cdot 1 + \frac{1}{2} \cdot \frac{5}{3} \cdot 2 = \frac{13}{3}$

c. $s(5) = s(0) + \displaystyle\int_0^5 v(t)\,dt = 0 + 3 = 3$.

d. $s(t) = \begin{cases} \displaystyle\int_0^t (-x+2)\,dx & \text{if } 0 \le t \le 3, \\ \dfrac{3}{2} + \displaystyle\int_3^t (3x-10)\,dx & \text{if } 3 < t \le 4, \\ 2 + \displaystyle\int_4^t (-2x+10)\,dx & \text{if } 4 < t \le 5 \end{cases} = \begin{cases} -\dfrac{t^2}{2} + 2t & \text{if } 0 \le t \le 3, \\ \dfrac{3t^2}{2} - 10t + 18 & \text{if } 3 < t \le 4, \\ -t^2 + 10t - 22 & \text{if } 4 < t \le 5. \end{cases}$

6.1.10

a. The displacement is the net area, which is $3 \cdot 3 + \frac{1}{2} \cdot 3 \cdot 2 = 9 + 3 = 12$.

b. Because $v(t) > 0$ on that interval, the distance traveled is the same as the displacement, so it is 12 also.

c. $s(5) = s(0) + \displaystyle\int_0^5 v(t)\,dt = 0 + 12 = 12$.

d. $s(t) = \begin{cases} 3t & \text{if } 0 \le t \le 3, \\ 9 + \displaystyle\int_3^t \left(-\frac{3}{2}x + \frac{15}{2}\right) dx & \text{if } 3 \le t \le 5 \end{cases} = \begin{cases} 3t & \text{if } 0 \le t \le 3, \\ -\dfrac{3}{4}t^2 + \dfrac{15}{2}t - \dfrac{27}{4} & \text{if } 3 \le t \le 5 \end{cases}.$

6.1.11

a. Note that $v(t) > 0$ on $[0, \pi/2]$, so the distance traveled is

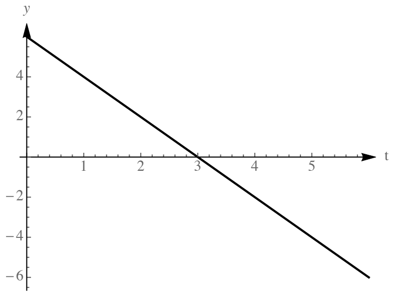
$$\int_0^{\pi/2} 3 \sin 2t\,dt = -\frac{3}{2} \cos 2t \Big|_0^{\pi/2} = -\frac{3}{2}(-1 - 1) = 3.$$

b. Note that the function is alternately positive then negative (with the same enclosed area) on each interval of length $\pi/2$. The displacement on $[0, \pi]$ is then $3 - 3 = 0$, the displacement on $[0, 3\pi/2]$ is $3 - 3 + 3 = 3$, and the displacement on $[0, 2\pi]$ is $3 - 3 + 3 - 3 = 0$.

c. The total distance traveled is $3+3+3+3=12$.

6.1.12

a.



The velocity is positive (and so the object is moving to the right) on $[0,3)$ and the velocity is negative (and so the object is moving to the left) on $(3,6]$.

b. The displacement is $\int_0^6 v(t)\,dt = \int_0^6 (6-2t)\,dt = (6t-t^2)\Big|_0^6 = 0$ m.

c. The total distance traveled is $\int_0^3 (6-2t)\,dt + \int_3^6 (2t-6)\,dt = (6t-t^2)\Big|_0^3 + (t^2-6t)\Big|_3^6 = (18-9)-(0-0)+(36-36)-(9-18) = 18$ m.

6.1.13

a. $v(t) = 3t(t-2)$ which is zero for $t=0$ and $t=2$. $v(t)>0$ on $(2,3)$ and is negative on $(0,2)$.

b. The displacement is $\int_0^3 (3t^2-6t)\,dt = (t^3-3t^2)\Big|_0^3 = 3^3-3(3)^2 = 0$ m.

c. The total distance traveled is $\int_0^2 (6t-3t^2)\,dt + \int_2^3 (3t^2-6t)\,dt = (3t^2-t^3)\Big|_0^2 + (t^3-3t^2)\Big|_2^3 = (12-8)+(27-27)-(8-12) = 8$ m.

6.1.14

a. $v(t) = 4t(t^2-6t+5) = 4t(t-5)(t-1)$, which is zero for $t=0$, $t=1$, and $t=5$. $v>0$ on $(0,1)$ and is negative on $(1,5)$.

b. The displacement is $\int_0^5 (4t^3-24t^2+20t)\,dt = (t^4-8t^3+10t^2)\Big|_0^5 = 5^4-8(5^3)+10(5^2) = -125$ m.

c. The distance traveled on the time interval $(0,1)$ is $\int_0^1 (4t^3-24t^2+20t)\,dt = (t^4-8t^3+10t^2)\Big|_0^1 = 1-8+10 = 3$ m, and the distance traveled on the time interval $(1,5)$ is $-\int_1^5 (4t^3-24t^2+20t)\,dt = -(t^4-8t^3+10t^2)\Big|_1^5 = 125+3 = 128$ m. Therefore the total distance traveled over the time interval $(0,5)$ is $3+128 = 131$ m.

6.1.15

a. $v(t) = 3(t^2-6t+8) = 3(t-4)(t-2)$, which is zero for $t=4$ and $t=2$. $v>0$ on $(0,2)$ and $(4,5)$, while v is negative on $(2,4)$.

b. The displacement is given by $\int_0^5 (3t^2-18t+24)\,dt = (t^3-9t^2+24t)\Big|_0^5 = 5^3-9(5^2)+24(5) = 20$ m.

c. On the time interval $(0, 2)$ the distance traveled is $\int_0^2 (3t^2-18t+24)\,dt = (t^3-9t^2+24t)\Big|_0^2 = 8-36+48 =$ 20 m. On the interval $(2, 4)$, the distance traveled is $-\int_2^4 (3t^2-18t+24)\,dt = -(t^3-9t^2+24t)\Big|_2^4 =$ $-(64-144+96)+(8-36+48) = 20-16 = 4$ m. On the time interval $(4, 5)$, the distance traveled is $\int_4^5 (3t^2-18t+24)\,dt = (t^3-9t^2+24t)\Big|_4^5 = 20-16 = 4$ m. Therefore the total distance traveled is $20+4+4 = 28$ m.

6.1.16

a. The motion is positive for $0 \le t \le 4$, because the exponential function is everywhere positive.

b. The displacement is $\int_0^4 50e^{-2t}\,dt = \left(-25e^{-2t}\right)\Big|_0^4 = 25(1-e^{-8}) \approx 24.992$ m.

c. Because the velocity is positive on the given interval, the distance traveled is the same as the displacement, given above.

6.1.17

a. $s(t) = \int \sin t\,dt = -\cos t + C$, and because $s(0) = 1$, we must have $C = 2$. Thus, $s(t) = 2 - \cos t$.

b. $s(t) = s(0) + \int_0^t \sin x\,dx = 1 + (-\cos x)\Big|_0^t = 2 - \cos t$.

6.1.18

a. $s(t) = \int (-t^3+3t^2-2t)\,dt = -\frac{t^4}{4} + t^3 - t^2 + C$. Because $s(0) = 4$, we have $C = 4$, and thus $s(t) = -\frac{t^4}{4} + t^3 - t^2 + 4$.

b. $s(t) = s(0) + \int_0^t (-x^3+3x^2-2x)\,dx = 4 + \left(-\frac{x^4}{4} + x^3 - x^2\right)\Big|_0^t = -\frac{t^4}{4} + t^3 - t^2 + 4$.

6.1.19

a. $s(t) = \int (6-2t)\,dt = 6t - t^2 + C$, and because $s(0) = 0$, we must have $C = 0$. Thus, $s(t) = 6t - t^2$.

b. Also, $s(t) = s(0) + \int_0^t (6-2x)\,dx = \left(6x - x^2\right)\Big|_0^t = 6t - t^2$.

6.1.20

a. $s(t) = \int 3\sin \pi t\,dt = -\frac{3}{\pi}\cos \pi t + C$, and because $s(0) = 1$, we must have $C = 1 + \frac{3}{\pi}$. Thus, $s(t) = -\frac{3}{\pi}\cos \pi t + 1 + \frac{3}{\pi}$.

b. $s(t) = s(0) + \int_0^t 3\sin \pi x\,dx = 1 + \left(-\frac{3}{\pi}\cos \pi x\right)\Big|_0^t = -\frac{3}{\pi}\cos \pi t + 1 + \frac{3}{\pi}$.

6.1.21

a. $s(t) = \int (9-t^2)\,dt = 9t - \frac{t^3}{3} + C$, and because $s(0) = -2$, we must have $C = -2$. Thus, $s(t) = 9t - \frac{t^3}{3} - 2$.

b. $s(t) = s(0) + \int_0^t (9 - x^2)\,dx = -2 + \left(9x - \frac{x^3}{3}\right)\Big|_0^t = 9t - \frac{t^3}{3} - 2.$

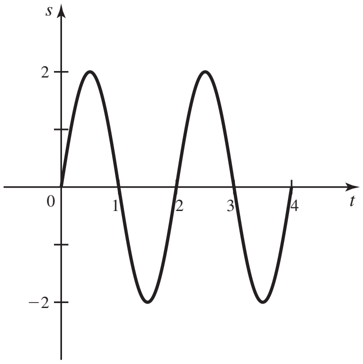
6.1.22

a. $s(t) = \int \frac{1}{t+1}\,dt = \ln(t+1) + C$, and because $s(0) = -4$, we must have $C = -4$. Thus, $s(t) = \ln(t+1) - 4$.

b. $s(t) = s(0) + \int_0^t \frac{1}{1+x}\,dx = -4 + (\ln(x+1))\Big|_0^t = \ln(t+1) - 4.$

6.1.23

a. $s(t) = s(0) + \int_0^t 2\pi\cos\pi x\,dx = 2\sin\pi x|_0^t = 2\sin\pi t.$



b.

c. The mass reaches its lowest point at $t = 1.5$, $t = 3.5$ and $t = 5.5$.

d. The mass reaches its highest point at $t = .5$, $t = 2.5$, and $t = 4.5$.

6.1.24

a. $s(5) = \int_0^5 |400 - 20t|\,dt = \int_0^5 (400 - 20t)\,dt = \left(400t - 10t^2\right)\Big|_0^5 = 1750$ m.

b. $s(10) = \int_0^{10} |400 - 20t|\,dt = \int_0^{10} (400 - 20t)\,dt = \left(400t - 10t^2\right)\Big|_0^{10} = 3000$ m.

c. Her velocity is 250 when $400 - 20t = 250$, or $t = 7.5$, and thus $s(7.5) = \int_0^{7.5} (400 - 20t)\,dt = \left(400t - 10t^2\right)\Big|_0^{7.5} = 2437.5$.

6.1.25

a. $s(t) = s(0) + \int_0^t 30(16 - x^2)\,dx = \left(480x - 10x^3\right)\Big|_0^t = 480t - 10t^3 = 10t(48 - t^2).$

b. Because the velocity is positive, this is given by $s(2) - s(0) = 960 - 80 = 880$ miles.

c. The velocity is 400 when $480 - 30t^2 = 400$, or $t = \sqrt{8/3}$. At this point the plane has traveled $s(\sqrt{8/3}) = 480\sqrt{8/3} - 10\sqrt{8/3}^3 \approx 740.290$ miles.

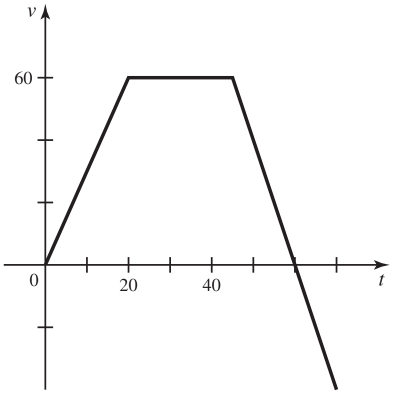
6.1.26

a. $s(t) = s(0) + \int_0^t 3\sin^2(\pi x/2)\,dx = \frac{3}{2}\int_0^t (1 - \cos(\pi x))\,dx = \frac{3}{2}\left(x - \frac{1}{\pi}\sin\pi x\right)\Big|_0^t = \frac{3}{2}\left(t - \frac{1}{\pi}\sin\pi t\right).$

b. Because the velocity is positive, this is given by $s(0.25) - s(0) = \frac{3}{2}\left(0.25 - \frac{\sin(\pi/4)}{\pi}\right) = \frac{3}{8} - \frac{3\sqrt{2}}{4\pi} \approx$ 0.0374 miles.

c. $s(3) = 4.5$ miles.

6.1.27



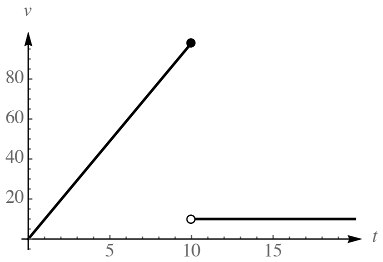
a. The velocity has a maximum of 60 for $20 \le t \le 45$. The velocity is 0 at $t = 0$ and at $t = 60$.

b. $\int_0^{20} 3t\,dt + \int_{20}^{30} 60\,dt = 1200$ m.

c. $1200 + \int_{30}^{45} 60\,dt + \int_{45}^{60} (240 - 4t)\,dt = 1200 + 900 + \left(240t - 2t^2\right)\Big|_{45}^{60} = 2550$ m.

d. At time $t = 60$ the automobile is at position 2550. In the following 15 seconds, it moves $\int_{60}^{75} (240 - 4t)\,dt = 450$ feet in the opposite direction, so it is at position $2550 - 450 = 2100$.

6.1.28



a. The velocity is given by

$$v(t) = \begin{cases} 9.8t & \text{if } 0 \le t \le 10, \\ 10 & \text{if } t > 10. \end{cases}$$

b. $s(30) = \int_0^{10} 9.8t\,dt + \int_{10}^{30} 10\,dt = 490 + 200 = 690$ m.

c. We seek t so that $490 + \int_{10}^{t} 10\,dx = 3000$, which can be written $10t - 100 = 2510$, so $t = 261$ s.

6.1.29 $v(t) = v(0) + \int_0^t a(x)\,dx = 70 + \int_0^t -32\,dx = 70 - 32x\Big|_0^t = -32t + 70.$

$s(t) = s(0) + \int_0^t (-32x + 70)\,dx = 10 + (-16x^2 + 70x)\Big|_0^t = -16t^2 + 70t + 10.$

6.1.30 $v(t) = v(0) + \int_0^t -32\,dx = 50 + -32x\Big|_0^t = -32t + 50.$

$s(t) = s(0) + \int_0^t (-32x + 50)\,dx = 0 + (-16x^2 + 50x)\Big|_0^t = -16t^2 + 50t.$

6.1.31 $v(t) = v(0) + \int_0^t -9.8\,dx = 20 + -9.8x\Big|_0^t = -9.8t + 20.$

$s(t) = s(0) + \int_0^t (-9.8x + 20)\,dx = 0 + (-4.9x^2 + 20x)\Big|_0^t = -4.9t^2 + 20t.$

6.1.32 $v(t) = v(0) + \int_0^t e^{-x}\,dx = 60 - e^{-x}\Big|_0^t = 60 - (e^{-t} - 1) = 61 - e^{-t}.$

$s(t) = s(0) + \int_0^t (61 - e^{-x})\,dx = 40 + (61x + e^{-x})\Big|_0^t = 39 + 61t + e^{-t}.$

6.1.33 $v(t) = v(0) + \int_0^t -0.01x\,dx = 10 + (-0.005x^2)\Big|_0^t = 10 - 0.005t^2.$

$s(t) = s(0) + \int_0^t (10 - .005x^2)\,dx = \left(10x - \frac{1}{600}x^3\right)\Big|_0^t = 10t - \frac{1}{600}t^3.$

6.1.34 $v(t) = v(0) + \int_0^t \frac{20}{(x+2)^2}\,dx = 20 - \frac{20}{x+2}\Big|_0^t = 30 - \frac{20}{t+2}.$

$s(t) = s(0) + \int_0^t \left(30 - \frac{20}{x+2}\right)\,dx = 10 + (30x - 20\ln|x+2|)\Big|_0^t = 10 + 20\ln 2 + 30t - 20\ln|t+2|.$

6.1.35 $v(t) = v(0) + \int_0^t \cos 2x\,dx = 5 + \left(\frac{1}{2}\sin 2x\right)\Big|_0^t = \frac{1}{2}\sin 2t + 5.$

$s(t) = s(0) + \int_0^t \left(\frac{1}{2}\sin 2x + 5\right)\,dx = 7 + \left(-\frac{1}{4}\cos 2x + 5x\right)\Big|_0^t = -\frac{1}{4}\cos 2t + 5t + \frac{29}{4}.$

6.1.36 $v(t) = v(0) + \int_0^t \frac{2x}{(x^2+1)^2}\,dx$. Let $u = x^2 + 1$ so that $du = 2x\,dx$. Then we have $0 + \int_1^{t^2+1} \frac{1}{u^2}\,du = \left(-\frac{1}{u}\right)\Big|_1^{t^2+1} = -\frac{1}{t^2+1} + 1.$

$s(t) = s(0) + \int_0^t \left(-\frac{1}{x^2+1} + 1\right)\,dx = \left(-\tan^{-1}(x) + x\right)\Big|_0^t = -\tan^{-1} t + t.$

6.1.37

a. The velocity is given by $v(0) + \int_0^t 88\,dx = 88x\Big|_0^t = 88t$ ft/s.

The position is given by $s(0) + \int_0^t 88x\,dx = 44x^2\Big|_0^t = 44t^2$ ft.

b. The car travels $s(4) = 44 \cdot 16 = 704$ feet.

c. Because a quarter mile is 1320 feet, we need $44t^2 = 1320$, so $t = \sqrt{30} \approx 5.477$ seconds.

d. We need $44t^2 = 300$, so $t \approx 2.611$ seconds.

e. It reaches that speed when $88t = 178$, or $t = 89/44$ seconds. At that time the racer has traveled $s(89/44) = 44(89/44)^2 = 89^2/44 \approx 180.023$ feet.

6.1.38

a. The velocity is given by $v(0) + \int_0^t -15\,dx = 60 + (-15x)\Big|_0^t = -15t + 60$.

The position is given by $s(0) + \int_0^t (-15x + 60)\,dx = (-7.5x^2 + 60x)\Big|_0^t = -7.5t^2 + 60t$.

b. The car comes to rest when $v(t) = 0$, which occurs for $t = 4$. At that time $s(4) = 120$ feet.

6.1.39 $v(t) = v(0) + \int_0^t -\frac{1280}{(1+8x)^3}\,dx = 80 + \left(\frac{80}{(1+8x)^2}\right)\Big|_0^t = \frac{80}{(1+8t)^2}$.

$s(t) = s(0) + \int_0^t \frac{80}{(1+8x)^2}\,dx = \left(-\frac{10}{1+8x}\right)\Big|_0^t = 10 - \frac{10}{1+8t}$.

Then in the first 0.2 seconds the train travels $s(0.2) - s(0) = 10 - \frac{10}{2.6} = 10 - \frac{50}{13} = \frac{80}{13} \approx 6.154$ miles. Between time 0.2 and 0.4 the train travels $s(0.4) - s(0.2) = 10 - \frac{10}{4.2} - \left(10 - \frac{10}{2.6}\right) = \frac{50}{13} - \frac{50}{21} = \frac{400}{273} \approx 1.465$ miles.

6.1.40

a. $55 + \int_0^6 (20 - (t/5))\,dt = 55 + \left(20t - \frac{t^2}{10}\right)\Big|_0^6 = \frac{857}{5} \approx 171.4$.

b. $P(200) = 55 + \int_0^{200} (20 - \frac{t}{5})\,dt = 55 + \left(20t - \frac{t^2}{10}\right)\Big|_0^{200} = 55$.

6.1.41

a. $P(20) = 250 + \int_0^{20} (30 + 30\sqrt{t})\,dt = 250 + \left(30t + 20t^{3/2}\right)\Big|_0^{20} = 250 + 600 + 800\sqrt{5} = 850 + 800\sqrt{5} \approx 2639$ people.

b. $P(t) = 250 + \int_0^t (30 + 30\sqrt{x})\,dx = 250 + \left(30x + 20x^{3/2}\right)\Big|_0^t = 250 + 30t + 20t^{3/2}$ people.

6.1.42

a. $P(15) = 35 + \int_0^{15} (5 + 10\sin(\pi t/5))\,dt = 35 + (5t - (50/\pi)\cos(\pi t/5))|_0^{15} = 35 + (75 + (50/\pi)) - (0 - (50/\pi)) = 110 + 100/\pi \approx 142$ foxes.

$P(35) = 35 + \int_0^{35} (5 + 10\sin(\pi t/5))\,dt = 35 + (5t - (50/\pi)\cos(\pi t/5))|_0^{35} = 35 + (175 + (50/\pi)) - (0 - (50/\pi)) = 210 + 100/\pi \approx 242$ foxes.

b. $P(t) = 35 + \int_0^t (5 + 10\sin(\pi x/5))\,dx = 35 + (5x - (50/\pi)\cos(\pi x/5))|_0^t = 35 + 5t - (50/\pi)\cos(\pi t/5) + (50/\pi)$ foxes.

6.1.43

a. $N(20) = 1500 + \int_0^{20} 100e^{-0.25t}\,dt = 1500 + \left(-400e^{-0.25t}\right)\big|_0^{20} = 1500 + (-400e^{-5} + 400) = 1900 - \frac{400}{e^5} \approx$ 1897 cells.

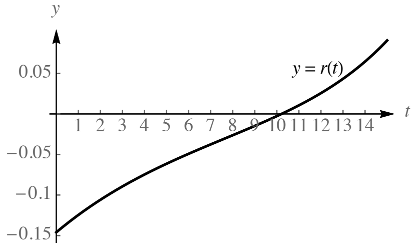
$N(40) = 1500 + \int_0^{40} 100e^{-0.25t}\,dt = 1500 + \left(-400e^{-0.25t}\right)\big|_0^{40} = 1500 + (-400e^{-10} + 400) = 1900 - \frac{400}{e^{10}} \approx$ 1900 cells.

b. $N(t) = 1500 + \int_0^t 100e^{-0.25x}\,dx = 1500 + \left(-400e^{-0.25x}\right)\Big|_0^t = 1500 + (-400e^{-0.25t} + 400) = -400e^{-0.25t} + 1900$ cells.

6.1.44

a. $r(t) = 0$ when $0.0025e^{0.25t} = 0.1485e^{-0.15t}$, which occurs when $e^{0.40t} = \frac{0.1485}{0.0025} = 59.4$. So $t_0 = \frac{\ln 59.4}{0.40} \approx 10.21$ days.

b.



The tumor decreases in size for the first 10.21 days and then starts to increase in size.

c. $\int_0^{10.21} (0.0025e^{0.25t} - 0.1485e^{-0.15t})\,dt = \left(0.01e^{0.25t} + 0.99e^{-0.15t}\right)\Big|_0^{10.21} \approx -0.658\,\text{cm}^3$. The tumor decreases in size by $0.658\,\text{cm}^3$ in the first 10.21 days.

6.1.45

a. In the first 35 days the number of barrels produced is $\int_0^{30} 800\,dt + \int_{30}^{35} (2600 - 60t)\,dt = 24000 + 3250 = 27250$.

b. In the first 50 days the number of barrels produced is $27250 + \int_{35}^{40} (2600 - 60t)\,dt + \int_{40}^{50} 200\,dt = 27250 + 1750 + 2000 = 31000$.

c. A constant 200 barrels per day times 20 days yields 4000 barrels.

6.1.46

a. $\int_0^{30} (0.25t^2 + 37.46t + 722.47)\,dt = \left(\frac{0.25}{3}t^3 + \frac{37.46}{2}t^2 + 722.47t\right)\Big|_0^{30} = 40781.1.$

b. $\int_0^{30} (0.90t^2 - 69.06t + 2053.12)\,dt = \left(\frac{0.90}{3}t^3 + \frac{69.06}{2}t^2 + 2053.12t\right)\Big|_0^{30} = 38616.6.$

c. $40781.1 + 38616.6 = 79397.7$ millions of cubic feet, or 7.93977×10^{10} cubic feet. There are $5280^3 = 147197952000$ cubic feet in a cubic mile, so the Spokane River contains $0.67 \cdot 147197952000 = 9.86226 \cdot 10^{10}$ cubic feet of water. So the percentage can be calculated by $\dfrac{7.93977 \times 10^{10}}{9.86226 \cdot 10^{10}} \approx 80.507\%$.

6.1.47

a. $Q(t) = Q(0) + \int_0^t 10^7 e^{-kx}\,dx = \left(\frac{10^7}{-k}e^{-kx}\right)\Big|_0^t = \dfrac{10^7(1 - e^{-kt})}{k}$.

b. $\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} \dfrac{10^7(1 - e^{-kt})}{k} = \dfrac{10^7}{k}$. This represents the total number of barrels extracted if the nation extracts the oil indefinitely where it is assumed that the nation has at least $\frac{10^7}{k}$ barrels of oil in reserve.

c. We seek k so that $\dfrac{10^7}{k} = 2 \times 10^9$, which gives $k = \dfrac{1}{200} = 0.005$.

d. We want T so that $(2\times 10^7)\int_0^T e^{-0.005t}\,dt = 2\times 10^9$, so $\left(-200e^{-0.005t}\right)\Big|_0^T = 100$, so $1-e^{-T/200} = 1/2$, so $T = 200\ln 2 \approx 138.629$ years.

6.1.48

a. $\int_0^{60} 3\sqrt{t}\,dt = \left(2t^{3/2}\right)\Big|_0^{60} \approx 929.52$ L.

b. $Q(t) = \int_0^t 3\sqrt{x}\,dx = \left(2x^{3/2}\right)\Big|_0^t = 2t^{3/2}$ L, where t is measured in minutes.

c. The tank will be full when $2t^{3/2} = 2000$, or $t = \sqrt[3]{1000^2} = 100$ minutes.

6.1.49

a. $\int_0^2 20(1+\cos(\pi t/12))\,dt = \left(20t + \frac{240}{\pi}\sin(\pi t/12)\right)\Big|_0^2 = 40 + \frac{240}{\pi}\cdot\frac{1}{2} = 40 + \frac{120}{\pi} \approx 78.197\,\mathrm{m}^3$.

b. $Q(t) = \int_0^t 20(1+\cos(\pi x/12))\,dx$, which is equal to

$$\left(20x + \frac{240}{\pi}\sin(\pi x/12)\right)\Big|_0^t = 20t + \frac{240}{\pi}\cdot\sin(\pi t/12)\,\mathrm{m}^3.$$

c. The reservoir is full when $20T + \frac{240}{\pi}\sin(\pi T/12) = 2500$, which occurs for $T \approx 122.6$ hours.

6.1.50

a. $\int_0^1 70(1+\sin(2\pi t))\,dt = \left(70t - \frac{35}{\pi}\cos(2\pi t)\right)\Big|_0^1 = 70 - \frac{35}{\pi} + \frac{35}{\pi} = 70$ mL.

b.

$$\begin{aligned}\int_0^t 70(1+\sin(2\pi x))\,dx = \left(70x - \frac{35}{\pi}\cos(2\pi x)\right)\Big|_0^t &= 70t - \frac{35}{\pi}\cos(2\pi t) + \frac{35}{\pi}\\ &= \frac{35}{\pi} + 70t - \frac{35}{\pi}\cos(2\pi t)\,\mathrm{mL}.\end{aligned}$$

c.

$$\begin{aligned}\int_t^{t+1} 70(1+\sin(2\pi x))\,dx = \left(70x - \frac{35}{\pi}\cos(2\pi x)\right)\Big|_t^{t+1} &= 70t + 70 - \frac{35}{\pi}\cos(2\pi(t+1)) - (70t - \frac{35}{\pi}\cos(2\pi t))\\ &= \frac{35}{\pi} + 70t - \frac{35}{\pi}\cos(2\pi t) = 70\,\mathrm{mL}.\end{aligned}$$

6.1.51

a. $V(t) = V(0) + \int_0^t -\frac{\pi}{2}\sin\frac{\pi x}{2}\,dx = 6 + \left(\cos\frac{\pi x}{2}\right)\Big|_0^t = 5 + \cos\frac{\pi t}{2}$.

b. Because $\sin\frac{\pi t}{2}$ is periodic with period 4, the breathing cycle repeats every 4 seconds, so there are $\frac{60}{4} = 15$ breaths per minute.

c. The lungs are full at $t = 0$, at which time $V(0) = 6$ L, so this is the capacity of the lungs. The tidal volume is the difference between this amount and the amount in the lungs after each exhalation, which is the minimum value of $V(t)$. This minimum occurs at the first positive zero of $V'(t)$, which is at t=2. Because $V(2) = 4$, the tidal volume is $6 - 4 = 2$L.

6.1.52 Note that the general solution for $N(t)$ is

$$\begin{aligned}N(t) &= N(0) + \int_0^t N'(x)\,dx = N(0) + \int_0^t (A\sin(2\pi x/P) + r)\,dx\\ &= N(0) + \left(-\frac{PA}{2\pi}\cos(2\pi x/P) + rx\right)\Bigg|_0^t\\ &= N(0) + rt + \frac{PA}{2\pi}(1 - \cos(2\pi t/P)).\end{aligned}$$

a. Using the general solution above, we have $N(t) = 10 + \frac{100}{\pi}(1 - \cos(\pi t/5))$. This is never 0, because it is always 10 or more.

b. Using the general solution above, we have that $N(t) = 100 + \frac{100}{\pi}(1 - \cos(\pi t/5))$. The population is never extinct, because it is always 100 or more.

c. Using the general solution above, we have $N(t) = 10 + 5t + \frac{250}{\pi}(1 - \cos(\pi t/5))$. Again, for $t \geq 0$ this is always at least 10, so the population never becomes extinct.

d. Using the general solution above, we have $N(t) = N(0) - 5t + \frac{250}{\pi}(1 - \cos(\pi t/5))$. Suppose $t = 5k$ for a positive even integer k. Then $\cos(\pi t/5) = \cos(k\pi) = 1$, so $N(t) = N(0) - 25k + 0$, which grows negatively without bound as $k \to \infty$. Thus, there is no choice of $N(0)$ which will ensure that the population won't become extinct.

6.1.53

a. $E = \int_0^{24} (300 - 200\sin(\pi t/12))\,dt = \left(300t + \frac{2400}{\pi}\cos(\pi t/12)\right)\Big|_0^{24} = 7200$ MWh. This is equivalent to $7.2 \times 10^6 \cdot 3.6 \times 10^6 = 2.592 \times 10^{13}$ Joules.

b. For one day, $\frac{7.2 \times 10^6\,\text{KWh}}{450\,\text{Kwh/kg}} = 16,000$ kg coal needed.

For one year, $16000\,\text{kg} \times 365 = 5,840,000$ kg coal needed.

c. For one day, $\frac{7.2 \times 10^6\,\text{KWh}}{1.6 \times 10^4\,\text{Kwh/g}} = 450$ g U-235 need.

For one year, $450 \times 365 = 164,250$ g needed.

d. $\frac{7.2 \times 10^6\,\text{KWh/day}}{(200\,\text{KW/turbine}) \cdot (24\,\text{hours/day})} = 1500$ turbines.

6.1.54

a. Photosynthesis occurs during daylight hours. So from approximately 5 AM to 5 PM, plants and trees take in CO_2 to produce oxygen, reducing the amount of CO_2 in the atmosphere.

b. $\int_0^{24} N(t)\,dt \approx 2(0.13) + 2(0.14) + 2(0) + 2(-0.4) + 2(-0.42) + 2(-0.39) + 2(-0.3) + 2(-0.21) + 2(0) + 2(0.24) + 2(0.17) + 2(0.16) = -1.76$.

About 1.76 grams of carbon per square meter are taken out of the coniferous forest during an average July day. (Answers will vary.)

6.1.55

a. The additional cost is $\int_{100}^{150} (2000 - .5x)\,dx = \left(2000x - \frac{x^2}{4}\right)\Big|_{100}^{150} = 96875$ dollars.

b. The additional cost is $\int_{500}^{550}(2000-.5x)\,dx=\left(2000x-\frac{x^2}{4}\right)\Big|_{500}^{550}=86875$ dollars.

6.1.56

a. The additional cost is $\int_{100}^{150}(200-.05x)\,dx=\left(200x-\frac{x^2}{40}\right)\Big|_{100}^{150}=9687.5$ dollars.

b. The additional cost is $\int_{500}^{550}(200-.05x)\,dx=\left(200x-\frac{x^2}{40}\right)\Big|_{500}^{550}=8687.5$ dollars.

6.1.57

a. The additional cost is $\int_{100}^{150}(300+10x-.01x^2)\,dx=\left(300x+5x^2-\frac{x^3}{300}\right)\Big|_{100}^{150}=69583.33$ dollars.

b. The additional cost is $\int_{500}^{550}(300+10x-.01x^2)\,dx=\left(300x+5x^2-\frac{x^3}{300}\right)\Big|_{500}^{550}=139583.33$ dollars.

6.1.58

a. The additional cost is $\int_{100}^{150}(3000-x-.001x^2)\,dx=\left(3000x-.5x^2-\frac{x^3}{3000}\right)\Big|_{100}^{150}=142958.33$ dollars.

b. The additional cost is $\int_{500}^{550}(3000-x-.001x^2)\,dx=\left(3000x-.5x^2-\frac{x^3}{3000}\right)\Big|_{500}^{550}=109958.33$ dollars.

6.1.59

a. False. This would only be the case if the motion was all in the same direction. If the object changes direction at all, then the distance traveled is greater than the displacement.

b. True. This is because $v(t)=|v(t)|$ in this case.

c. True. This is because $R(t)>0$ for $0<t<10$, but $R(t)<0$ for $t>10$.

d. True. The cost of increasing production from A to B is given by $\int_A^B C'(t)\,dt$, which is geometrically the area under the curve $y=C'(x)$ from A to B. If C' is positive and decreasing, there is more area under the curve from A to $2A$ than from $2A$ to $3A$.

6.1.60 The distance traveled is $\int_0^8(2t+6)\,dt=\left(t^2+6t\right)\Big|_0^8=112$. So the same distance could have been traveled over the given time period at a constant velocity of $\frac{112}{8}=14$.

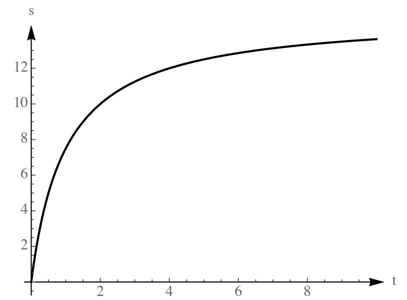
6.1.61 The distance traveled is $\int_0^4\left(1-\frac{t^2}{16}\right)\,dt=\left(t-\frac{t^3}{48}\right)\Big|_0^4=\frac{8}{3}$. So the same distance could have been traveled over the given time period at a constant velocity of $\frac{8/3}{4}=\frac{2}{3}$.

6.1.62 The distance traveled is $\int_0^\pi 2\sin t\,dt=(-2\cos t)\Big|_0^\pi=4$. So the same distance could have been traveled over the given time period at a constant velocity of $\frac{4}{\pi}$.

6.1.63 The distance traveled is $\int_0^5 t\sqrt{25-t^2}\,dt=\frac{1}{2}\int_0^{25}\sqrt{u}\,du=\left(\frac{1}{3}u^{3/2}\right)\Big|_0^{25}=\frac{125}{3}$. So the same distance could have been traveled over the given time period at a constant velocity of $\frac{125/3}{5}=\frac{25}{3}$.

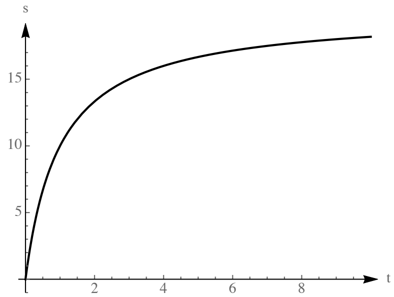
6.1.64

a.



$s_K(t) = \int_0^t \frac{15}{(x+1)^2}\,dx = 15 - \frac{15}{t+1} = \frac{15t}{t+1}$.

b.



$s_S(t) = \int_0^t \frac{20}{(x+1)^2}\,dx = 20 - \frac{20}{t+1} = \frac{20t}{t+1}$.

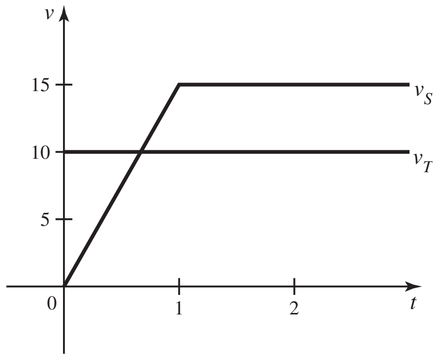
c. They meet when $\dfrac{15t}{t+1} + \dfrac{20t}{t+1} = 20$, which occurs for $t = \dfrac{4}{3}$. (Which represents 1:20 PM.) At this time, Kelly has gone $s_K(4/3) = \dfrac{20}{7/3} = \dfrac{60}{7}$ km, and Sandy has gone $s_S(4/3) = \dfrac{80}{7}$ km.

d. We would need $\dfrac{At}{t+1} + \dfrac{Bt}{t+1} = D$ to have a solution. If we solve for t, we obtain $t = \dfrac{D}{A+B-D}$, so we need $A + B > D$.

e. The maximum distances are A and B respectively, because $\lim\limits_{t\to\infty} \dfrac{At}{t+1} = A$ and $\lim\limits_{t\to\infty} \dfrac{Bt}{t+1} = B$.

6.1.65

a.



b. After 1 hour, Theo has ridden $1 \cdot 10 = 10$ miles, and Sasha has ridden $\frac{1}{2} \cdot 15 = 7.5$ miles, so Theo has ridden farther.

c. After 2 hours, Theo has ridden $2 \cdot 10 = 20$ miles, and Sasha has ridden $7.5 + 15 \cdot 1 = 22.5$ miles, so Sasha has ridden farther.

d. The times when they arrive at the various mile markers are in the following table:

	10	15	20
Theo	1	3/2	2
Sasha	7/6	3/2	11/6

Note that Theo hits the 10 mile marker first, then they are tied as they hit the 15 mile marker, and Sasha hits the 20 mile marker first. The area under v_S is the same as the area under v_T for $t = 1.5$, for $t < 1.5$ the area under v_T is greater, and for $t > 1.5$, the area under v_S is greater.

e. Theo will then hit the 20 mile mark in $18.8/10 = 1.88$ hours. Sasha hits the 20 mile mark at $t = 11/6 \approx 1.833$ hours, so Sasha will win.

f. A head start of 0.2 hours is equivalent for Theo of $10 \cdot 0.2 = 2$ miles. It will take him $18/10 = 1.8$ hours to ride the other 18 miles, while it still takes Sasha about 1.83 hours to cover 20 miles, so Theo will win.

6.1.66

a. $s_A(t) = \displaystyle\int_0^t \frac{4}{x+1}\,dx = (4\ln(x+1))\Big|_0^t = 4\ln(t+1).$

$s_B(t) = 2 + \displaystyle\int_0^t \frac{2}{x+1}\,dx = 2 + (2\ln(x+1))\Big|_0^t = 2 + 2\ln(t+1).$

b. This would occur if $4\ln(t+1) = 2 + 2\ln(t+1)$, or $\ln(t+1) = 1$, which occurs for $t = e - 1$, which is about 1 hour and 43 minutes.

6.1.67 Let $D(t)$ equal the depth of the snow t hours after it started snowing and let $t = T$ correspond to noon. This means that $D(0) = 0$ and because the snow piles up on the road at a constant rate, $D'(t) = k$ for some positive constant k. It follows that

$$D(t) = D(0) + \int_0^k = kx\Big|_0^t = kt.$$

Let $v(t)$ equal the velocity of the plow t hours after it started snowing. The plowing rate is inversely proportional to the depth of the snow which means that $v(t) = \dfrac{C}{kt}$ for some constant C and for $t \geq T$. Because the plow goes twice as far from noon to 1 PM as from 1 PM to 2 PM,

$$\int_T^{T+1} \frac{C}{kt}\,dt = 2\int_{T+1}^{T+2} \frac{C}{kt}\,dt.$$

Evaluating the integrals, we have $\dfrac{C}{k}\ln\left(\dfrac{T+1}{T}\right) = \dfrac{2C}{k}\ln\left(\dfrac{T+2}{T+1}\right)$, or $\ln\left(\dfrac{T+1}{T}\right) = \ln\left(\dfrac{T+2}{T+1}\right)^2$. This implies that $(T+1)^3 = T(T+2)^2$, Expanding both sides and collecting terms gives the quadratic equation $T^2 + T - 1 = 0$, which has a positive root of $T = \dfrac{-1+\sqrt{5}}{2} \approx 0.618\,\text{hours} \approx 37\,\text{minutes}$. So it started snowing at approximately 11:23 AM.

6.1.68

a. $y(t)$ is the position of the projectile, and the derivative of position is velocity, and the derivative of velocity is acceleration. Note that the acceleration force is due to gravity, and only depends on the position y of the projectile.

b. $\dfrac{1}{2}\dfrac{d}{dy}\left(v^2\right) = \dfrac{1}{2}\cdot 2v\dfrac{dv}{dy} = \dfrac{dy}{dt}\cdot\dfrac{dv}{dy} = \dfrac{dv}{dt}.$

c. Because $\dfrac{dv}{dt} = a(y)$ and $\dfrac{dv}{dt} = \dfrac{1}{2}\dfrac{d}{dy}(v^2)$, we must have $\dfrac{1}{2}\dfrac{d}{dy}(v^2) = a(y)$.

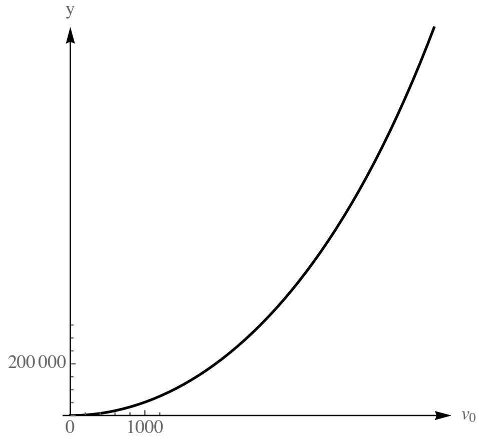
d. $\dfrac{1}{2}\displaystyle\int \frac{d}{dy}(v^2)\,dy = \int a(y)\,dy$, so $\dfrac{1}{2}(v^2) = \displaystyle\int -\frac{g}{(1+y/R)^2}\,dy = gR(\frac{1}{1+y/R}) + D$. Now when $t = 0$, we have $v = v_0$ and $y = 0$, so $\dfrac{1}{2}v_0^2 = gR + D$, so $D = -gR + \dfrac{1}{2}v_0^2$, and we can write

$$\frac{1}{2}(v^2 - v_0^2) = gR\left(\frac{1}{1+y/R} - 1\right).$$

e. When $v = 0$ we have $-\dfrac{1}{2}v_0^2\cdot\dfrac{1}{gR} + 1 = \dfrac{1}{1+y/R}$, so $\dfrac{1}{1+y/R} = \dfrac{2gR - v_0^2}{2gR}$, so $1 + y/R = \dfrac{2gR}{2gR - v_0^2}$, and

$$y = \frac{2gR^2}{2gR - v_0^2} - R = \frac{2gR^2}{2gR - v_0^2} - \frac{2gR^2 - Rv_0^2}{2gR - v_0^2} = \frac{Rv_0^2}{2gR - v_0^2}.$$

f. $y_{\max}(500) \approx 12,780$ m.
$y_{\max}(1500) \approx 116,893$ m.
$y_{\max}(5000) \approx 1,592,990$ m.



g. The denominator of the expression for $y_{\max}$ is 0 when $2gR = v_0^2$, so when $v_0 = \sqrt{2gR}$. As $v_0 \to \sqrt{2gR}$, $y_{\max} \to \infty$.

6.1.69 Using the Fundamental Theorem, we have

$$\int_a^b f'(x)\,dx = f(b) - f(a) = g(b) - g(a) = \int_a^b g'(x)\,dx.$$

6.1.70 Let $f(t)$ represent the position of the first runner at time t and let $g(t)$ represent the position of the second runner over the time interval $[a,b]$. We are given that $f(a) = g(a)$ and $f(b) = g(b)$. Then by problem 69, we have

$$\int_a^b f'(t)\,dt = \int_a^b g'(t)\,dt,$$

so the displacements of the two runners is the same, even though it might not be the case that $f'(t) = g'(t)$ for every t.

6.1.71 If $f(x)$ is the elevation of trail one at position x and $g(x)$ is the elevation of trail two at position x, then because we are given that $f(a) = g(a)$ and $f(b) = g(b)$, we must have that

$$\int_a^b f'(x)\,dx = \int_a^b g'(x)\,dx,$$

so the two trails have the same net change in elevation.

6.1.72 Let $f(x) = 12\sin(\pi x^2)$ and $g(x) = x^{10}(2-x)^2$. Note that $f(0) = 0 = g(0)$ and $f(2) = 0 = g(2)$. Thus by problem 69, $\int_0^2 f'(x)\,dx = \int_0^2 g'(x)\,dx$, which is equivalent to the statement we are trying to prove.

6.2 Regions Between Curves

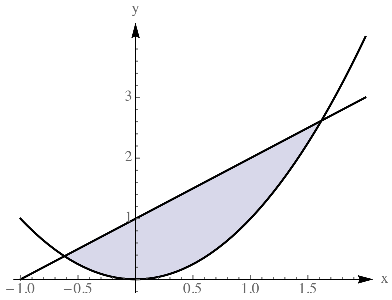
6.2.1 $\int_a^b (f(x) - g(x))\,dx + \int_b^c (g(x) - f(x))\,dx.$

6.2.2

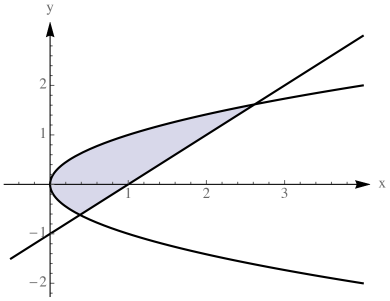
a. $\int_0^4 \left(\sqrt{x} - \frac{1}{2}x\right)\,dx.$

b. $\int_0^2 (2y - y^2)\,dy.$

6.2.3



6.2.4



6.2.5

a. The area is given by $\int_0^1 ((2-x)-x)\,dx = \int_0^1 (2-2x)\,dx = \left(2x-x^2\right)\Big|_0^1 = 1.$

b. The area is that of a triangle with base 2 and height 1, so its area is $\frac{1}{2}\cdot 2\cdot 1 = 1.$

6.2.6

a. The area is given by $\int_0^1 ((y+1)-2y)\,dy = \int_0^1 (1-y)\,dy = \left(y-\frac{y^2}{2}\right)\Big|_0^1 = \frac{1}{2}.$

b. The area is that of a triangle with base 1 and height 1, so its area is $\frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}.$

6.2.7 $\int_0^1 y\,dy + \int_1^2 (2-y)\,dy.$

6.2.8 $\int_0^1 \frac{x}{2}\,dx + \int_1^2 \left(\frac{x}{2}-x+1\right)\,dx.$

6.2.9 The curves intersect when $x = x^2-2$, or $(x+1)(x-2)=0$, so at $x=-1$ and $x=2$. The area is

$$\int_{-1}^{2} (x-(x^2-2)) = \left(2x+\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_{-1}^{2} = 4.5.$$

6.2.10 The curves intersect when $x^2-2x=-x^2+4x$, or $2x^2-6x=0$. This can be written as $2x(x-3)=0$, so $x=0$ and $x=3$ are the x-values of the points of intersection. The area is given by

$$\int_0^3 (-x^2+4x-(x^2-2x))\,dx = \int_0^3 (-2x^2+6x)\,dx = \left(\frac{-2x^3}{3}+3x^2\right)\Big|_0^3 = -18+27=9.$$

6.2.11 The curves intersect when $\frac{6}{x^2+1} = 3x^2$, or $3x^2(x^2+1)=6$. This can be written as $3x^4+3x^2-6=0$, or $(x^4+x^2-2)=0$. Factoring gives $(x^2+2)(x^2-1)=(x^2+2)(x-1)(x+1)=0$, so the curves intersect at $x=-1$ and $x=1$. Using symmetry, the area is given by

$$2\int_0^1 \left(\frac{6}{x^2+1}-3x^2\right)\,dx = 2\left(6\tan^{-1}x - x^3\right)\Big|_0^1 = 2\left(\frac{6\pi}{4}-1\right) = 3\pi-2.$$

6.2.12 The curves intersect where $\sin x = -\cos x$, which is where $\tan x = -1$, or $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$. The area is given by

$$\int_{-\pi/4}^{3\pi/4} (\sin x - (-\cos x))\,dx = (-\cos x + \sin x)\Big|_{-\pi/4}^{3\pi/4} = \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}.$$

6.2.13 By inspection, the curves intersect at $x = 1$. The area is given by

$$\int_0^1 (3 - x - 2^x)\,dx = \left(3x - x^2/2 - \frac{2^x}{\ln 2}\right)\Bigg|_0^1 = 3 - \frac{1}{2} - \frac{2}{\ln 2} - \left(-\frac{1}{\ln 2}\right) = \frac{5}{2} - \frac{1}{\ln 2}.$$

6.2.14 The given curve is 0 when $\sin x = 0$ (because $\cos x - 2$ is never zero), which occurs for $x = 0$ and $x = \pi$. The area is given by

$$\int_0^\pi (0 - (\cos x - 2)\sin x)\,dx = -\int_0^\pi (\cos x - 2)\sin x\,dx.$$

Let $u = \cos x - 2$. Then $du = -\sin x\,dx$, and substituting gives

$$\int_{-1}^{-3} u\,du = \frac{u^2}{2}\Bigg|_{-1}^{-3} = \frac{9}{2} - \frac{1}{2} = 4.$$

6.2.15 The curves intersect at $\pi/4$, so the area is given by $\displaystyle\int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx = (-\cos x)\Bigg|_0^{\pi/4} + (\sin x)\Bigg|_{\pi/4}^{\pi/2} = (1 - \sqrt{2}/2) + (1 - \sqrt{2}/2) = 2 - \sqrt{2}.$

6.2.16 The curves intersect when $x = x^3$, or $x(x-1)(x+1) = 0$, so at $x = 0$, $x = -1$ and $x = 1$. The area is given by

$$\int_{-1}^0 (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx = 2\int_0^1 (x - x^3)\,dx = 2\left(x^2/2 - x^4/4\right)\Big|_0^1 = \frac{1}{2}.$$

6.2.17 The curves intersect when $x^2 - 4x = -2x$, or $x^2 - 2x = x(x-2) = 0$, so $x = 0$ and $x = 2$. The area is given by

$$\int_0^2 (-2x - (x^2 - 4x))\,dx = \int_0^2 (2x - x^2)\,dx = \left(x^2 - \frac{x^3}{3}\right)\Bigg|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}.$$

6.2.18 The curves intersect when $\dfrac{\sec^2 x}{4} = 4\cos^2 x$, which can be written as $\cos^4 x = \dfrac{1}{16} = \dfrac{1}{2^4}$. This occurs when $\cos x = \pm\dfrac{1}{2}$, which occurs at $-\pi/3$ and $\pi/3$. Using symmetry, the area is given by

$$\begin{aligned}2\int_0^{\pi/3} (4\cos^2 x - (1/4)\sec^2 x)\,dx &= 2\int_0^{\pi/3} (2(1 + \cos 2x) - (1/4)\sec^2 x)\,dx \\ &= 2\left(2x + \sin 2x - (1/4)\tan x\right)\Bigg|_0^{\pi/3} = \frac{4\pi}{3} + \sqrt{3} - \frac{\sqrt{3}}{2} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}.\end{aligned}$$

6.2.19 The curves intersect when $2y = y^2 - 3$, or $y^2 - 2y - 3 = (y-3)(y+1) = 0$. This occurs for $y =$ and $y = -1$. The area is given by

$$\int_{-1}^3 (2y - (y^2 - 3))\,dy = \left(y^2 - \frac{y^3}{3} + 3y\right)\Bigg|_{-1}^3 = (9 - 9 + 9) - (1 + \frac{1}{3} - 3) = \frac{32}{3}.$$

6.2.20 The area is given by

$$\int_0^4 \left(\sqrt{y} - \frac{y}{4}\right)\,dy = \left(\frac{2y^{3/2}}{3} - \frac{y^2}{8}\right)\Bigg|_0^4 = \frac{16}{3} - 2 = \frac{10}{3}.$$

6.2.21 Note that the two curves intersect where $y = 2 - y^2$, so for $y^2 + y - 2 = (y+2)(y-1) = 0$. So they intersect for $y = -2$ and $y = 1$. The area is given by

$$\int_{-2}^1 (4 - 2y - 2y^2)\,dy = \left(4y - y^2 - \frac{2}{3}y^3\right)\Bigg|_{-2}^1 = \left(4 - 1 - \frac{2}{3}\right) - \left(-8 - 4 + \frac{16}{3}\right) = 9.$$

6.2.22 The curves $-\sin 2y$ and $\cos y$ intersect where $-\sin 2y = \cos y$, but $-\sin 2y = -2\sin y\cos y$, so we must solve $-2\sin y\cos y = \cos y$. So either $\cos y = 0$ or $\sin y = -\frac{1}{2}$. The first positive value of y satisfying either of these conditions is $y = \frac{\pi}{2}$, where $\cos y = 0$; the first negative value of y is $y = -\frac{\pi}{6}$ where $\sin y = -\frac{1}{2}$. So the area is

$$\begin{aligned}\int_{-\pi/6}^{\pi/2} (\cos y - (-\sin 2y))\,dy &= \left(\sin y - \frac{1}{2}\cos 2y\right)\Big|_{-\pi/6}^{\pi/2}\\ &= \sin\frac{\pi}{2} - \frac{1}{2}\cos\pi - \sin\left(-\frac{\pi}{6}\right) + \frac{1}{2}\cos\left(-\frac{\pi}{3}\right)\\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{9}{4}.\end{aligned}$$

6.2.23 The area is given by

$$\int_{-4}^{5}\left(8 - y - \frac{(y-2)^2}{3}\right)\,dy = \left(8y - \frac{y^2}{2} - \frac{(y-2)^3}{9}\right)\Big|_{-4}^{5} = 40 - \frac{25}{2} - 3 - (-32 - 8 + 24) = \frac{81}{2}.$$

6.2.24 The curves intersect where $2\sin 2y = 2\cos y$, or $4\sin y\cos y = 2\cos y$. If $\cos y \neq 0$, this reduces to $\sin y = \frac{1}{2}$, which occurs for $y = \frac{\pi}{6}$. If $\cos y = 0$, then we have $y = \pm\frac{\pi}{2}$. The area is therefore given by

$$\begin{aligned}&\int_{-\pi/2}^{\pi/6} (2\cos y - 2\sin 2y)\,dy + \int_{\pi/6}^{\pi/2} (2\sin 2y - 2\cos y)\,dy\\ &= (2\sin y + \cos 2y)\Big|_{-\pi/2}^{\pi/6} + (-\cos 2y - 2\sin y)\Big|_{\pi/6}^{\pi/2}\\ &= \left(1 + \frac{1}{2} + 2 + 1\right) + \left(1 - 2 + \frac{1}{2} + 1\right) = 5.\end{aligned}$$

6.2.25 The two curves meet at $x = 0$; at that point $y = 1$. So the integration is from $y = 0$ to $y = 1$. Solving for x gives the curves $x = 2y^2 - 2$ and $x = 1 - y^2$. Thus the area of the region is

$$\int_0^1 ((1 - y^2) - (2y^2 - 2))\,dy = \int_0^1 (-3y^2 + 3)\,dy = \left(-y^3 + 3y\right)\Big|_0^1 = 2.$$

6.2.26 Note that (by inspection) the curves intersect for $y = 1$ and $y = -1$. Using symmetry, the area is

$$2\int_0^1 ((2 - y^2) - y)\,dy = 2\left(2y - \frac{y^3}{3} - \frac{y^2}{2}\right)\Big|_0^1 = 2\left(2 - \frac{1}{3} - \frac{1}{2}\right) = 3 - \frac{2}{3} = \frac{7}{3}.$$

6.2.27 The curves intersect when $x^2 - 2 = x$, or $x^2 - x - 2 = (x-2)(x+1) = 0$, or $x = -1$ and $x = 2$. The relevant point for the shaded region is $(-1, -1)$.

The area is

$$\int_{-1}^{0} (y - (-\sqrt{y+2}))\,dy = \int_{-1}^{0} (y + \sqrt{y+2})\,dy = \left(\frac{y^2}{2} + \frac{2(y+2)^{3/2}}{3}\right)\Big|_{-1}^{0} = \frac{2\sqrt{8}}{3} - \left(\frac{1}{2} + \frac{2}{3}\right) = \frac{8\sqrt{2} - 7}{6}.$$

6.2.28 Note that $\sin 2x = 2\sin x\cos x$, so the curves intersect where $2\sin x\cos x = \sin x$, so the curves intersect at $x = 0$ and $x = \pi$, and $x = \pi/3$, because $\cos\pi/3 = \frac{1}{2}$. The area is given by

$$\begin{aligned}&\int_0^{\pi/3} (\sin 2x - \sin x)\,dx + \int_{\pi/3}^{\pi} (\sin x - \sin 2x)\,dx\\ &= (-\cos(2x)/2 + \cos x)\Big|_0^{\pi/3} + (-\cos(x) + \cos(2x)/2)\Big|_{\pi/3}^{\pi}\\ &= \left(\frac{1}{4} + \frac{1}{2} - \left(\left(-\frac{1}{2}\right) + 1\right)\right) + \left(1 + \left(\frac{1}{2}\right) - \left(\left(-\frac{1}{2}\right) + \left(-\frac{1}{4}\right)\right)\right) = \frac{1}{4} + \frac{9}{4} = \frac{5}{2}.\end{aligned}$$

6.2.29 The curves $x = y^2 - 3y + 12$ and $x = -2y^2 - 6y + 30$ intersect where $y^2 - 3y + 12 = -2y^2 - 6y + 30$, or $3y^2 + 3y - 18 = 3(y+3)(y-2) = 0$. So the points of intersection are $y = -3$ and $y = 2$. Thus the area is

$$\begin{aligned}\int_{-3}^{2} ((-2y^2 - 6y + 30) - (y^2 - 3y + 12))\,dy &= \int_{-3}^{2} (-3y^2 - 3y + 18)\,dy\\ &= \left(-y^3 - \frac{3}{2}y^2 + 18y\right)\bigg|_{-3}^{2}\\ &= -8 - 6 + 36 - 27 + \frac{27}{2} + 54 = \frac{125}{2}.\end{aligned}$$

6.2.30 These two curves intersect where $y^3 - 4y^2 + 3y = y^2 - y$, or $y^3 - 5y^2 + 4y = y(y-4)(y-1) = 0$. The three intersection points occur where $y = 0$, $y = 4$, and $y = 1$. Note that from $y = 0$ to $y = 1$ we have $y^2 - y \le y^3 - 4y^2 + 3y$; from $y = 1$ to $y = 4$ this is reversed. Thus the area between the curves is

$$\begin{aligned}&\int_0^1 (y^3 - 4y^2 + 3y - (y^2 - y))\,dy + \int_1^4 (y^2 - y - (y^3 - 4y^2 + 3y))\,dy\\ &\qquad = \int_0^1 (y^3 - 5y^2 + 4y)\,dy + \int_1^4 (-y^3 + 5y^2 - 4y)\,dy\\ &\qquad = \left(\frac{1}{4}y^4 - \frac{5}{3}y^3 + 2y^2\right)\bigg|_0^1 + \left(-\frac{1}{4}y^4 + \frac{5}{3}y^3 - 2y^2\right)\bigg|_1^4\\ &\qquad = \frac{1}{4} - \frac{5}{3} + 2 - 64 + \frac{320}{3} - 32 + \frac{1}{4} - \frac{5}{3} + 2 = \frac{71}{6}.\end{aligned}$$

6.2.31

a. The area is given by $\displaystyle\int_0^1 (\sqrt{x} - x^3)\,dx$.

b. The area can also be written $\displaystyle\int_0^1 (\sqrt[3]{y} - y^2)\,dy$.

6.2.32

a. The area is given by $\displaystyle\int_0^2 (-(x^2 - 4x))\,dx + \int_2^4 (-(2x - 8))\,dx$.

b. The area can also be written as $\displaystyle\int_{-4}^{0} (y/2 + 4 - (2 - \sqrt{y+4}))\,dy$. Note that in order to solve $y = x^2 - 4x$ for x, we needed to complete the square to obtain $y + 4 = x^2 - 4x + 4 = (x-2)^2$, so $\sqrt{y+4} + |x-2|$, and the part of this we need is $x = 2 - \sqrt{y+4}$.

6.2.33 The area is $\displaystyle\int_0^1 \left(10\sqrt{t} - 7t\right)\,dt = \left(\frac{20}{3}t^{3/2} - \frac{7t^2}{2}\right)\bigg|_0^1 = \frac{20}{3} - \frac{7}{2} \approx 3.17$ km.

The area under the curve v_1 equals the distance traveled by the first runner and the area under the curve v_2 is the distance traveled by the second runner. So the area between the curves is the difference in the distance covered by the faster runner and the slower runner, which means that the faster runner ran about 3.17 km further than the slower runner.

6.2.34

a. The area of R_1 is $\displaystyle\int_0^1 (3 - x - 2x^2)\,dx = \left(3x - \frac{x^2}{2} - \frac{2x^3}{3}\right)\bigg|_0^1 = 3 - \frac{1}{2} - \frac{2}{3} = \frac{11}{6}$.

b. The area of the R_1 and R_2 together is the area of a triangle with base 3 and height 3 so it is $\dfrac{9}{2}$. Therefore, the area of R_2 is $\dfrac{9}{2} - \dfrac{11}{6} = \dfrac{16}{6} = \dfrac{8}{3}$.

6.2.35

a. The area of R_1 is given by $\int_0^1 6x(2-x^2)^2\,dx$. Let $u = 2 - x^2$. Then $du = -2x\,dx$, and substituting yields

$$\int_2^1 -3u^2\,du = \left(-u^3\right)\Big|_2^1 = -1-(-8) = 7.$$

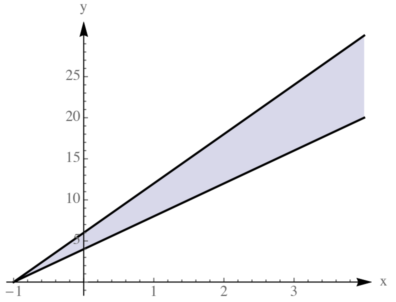
b. The area of R_1 minus the area of a triangle with base 1 and height 6 is equal to the area of R_2. Thus, the area of R_2 is $7-3=4$.

6.2.36

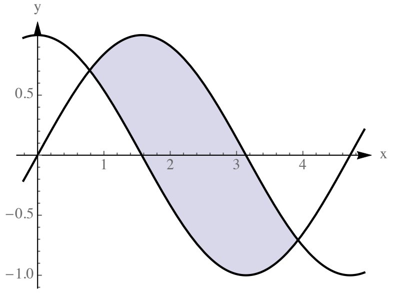
a. The area of R_1 is $\int_0^{\pi/2} \cos y\,dy = \sin y\Big|_0^{\pi/2} = 1$.

b. Together, R_1 and R_2 comprise a $1 \times \frac{\pi}{2}$ rectangle. So the area of R_2 is $\frac{\pi}{2} - 1$.

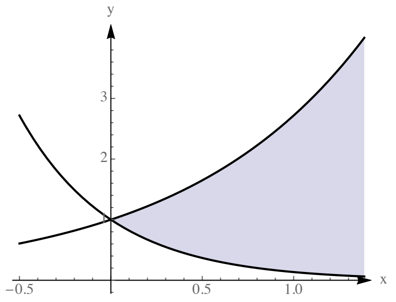
6.2.37 The nonvertical lines intersect when $4x+4 = 6x+6$, or $x=-1$. The vertical line $x=4$ intersects both nonvertical lines when $x=4$. The area is given by $\int_{-1}^4 (6x+6-(4x+4))\,dx = \int_{-1}^4 (2x+2)\,dx = \left(x^2+2x\right)\Big|_{-1}^4 = 16+8-(1-2) = 25$.



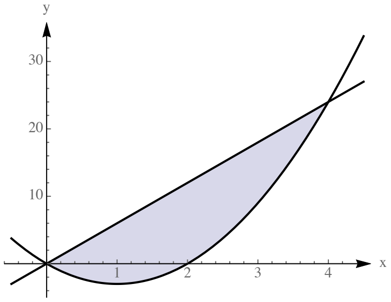
6.2.38 $\cos x = \sin x$ when $\tan x = 1$, or $x = \tan^{-1}(1) = \pi/4$. They also intersect at $\pi/4+\pi = 5\pi/4$. The area is given $\int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx = (-(\cos x+\sin x))\Big|_{\pi/4}^{5\pi/4} = -((-\sqrt{2}/2-\sqrt{2}/2) - (\sqrt{2}/2+\sqrt{2}/2)) = 2\sqrt{2}$.



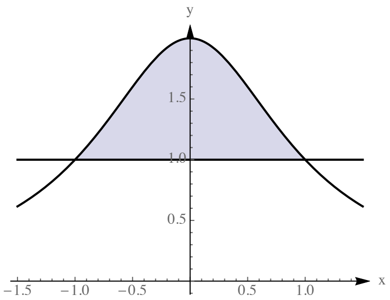
6.2.39 The curves e^x and e^{-2x} intersect when $e^x = e^{-2x}$, or $e^{3x}=1$, which occurs only for $x=0$. The vertical line $x = \ln 4$ clearly intersects the curves for $x = \ln 4$. The area is given by $\int_0^{\ln 4} (e^x - e^{-2x})\,dx = \left(e^x + e^{-2x}/2\right)\Big|_0^{\ln 4} = 4+(1/32)-(1+(1/2)) = 81/32$.



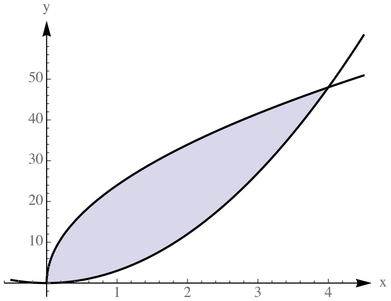
6.2.40 The curves intersect for $3x^2 - 6x = 6x$, or $3x^2 - 12x = 3x(x-4) = 0$, so for $x = 0$ and $x = 4$. The area is $\int_0^4 (6x - (3x^2 - 6x)\,dx = \int_0^4 (12x - 3x^2)\,dx = (6x^2 - x^3)\Big|_0^4 = 96 - 64 = 32$.



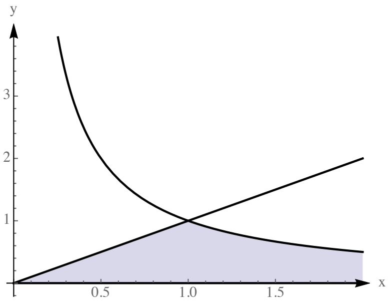
6.2.41 The curves intersect when $\dfrac{2}{1+x^2} = 1$, or $x^2 + 1 = 2$, or $x^2 - 1 = (x-1)(x+1) = 0$. So the intersections occur when $x = \pm 1$. Using symmetry, the area is $2\int_0^1 \left(\dfrac{2}{1+x^2} - 1\right)\,dx = 2\left(2\tan^{-1}x - x\right)\Big|_0^1 = \pi - 2$.



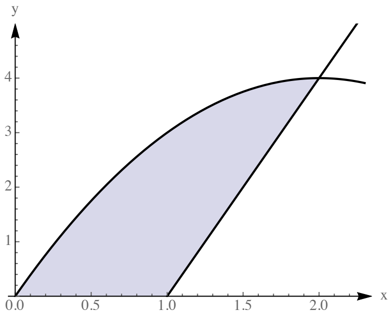
6.2.42 The curves intersect when $24\sqrt{x} = 3x^2$, so when either $x = 0$ or when $8 = x^{3/2}$. Thus happens for $x = 8^{2/3} = 4$. We have $\int_0^4 \left(24\sqrt{x} - 3x^2\right)\,dx = \left(16x^{3/2} - x^3\right)\Big|_0^4 = 128 - 64 = 64$.



6.2.43 The curves intersect when $x = 1/x$, or $x^2 = 1$. This occurs in the first quadrant when $x = 1$. The area is given by $\int_0^1 x\,dx + \int_1^2 1/x\,dx = \left(x^2/2\right)\Big|_0^1 + (\ln x)\Big|_1^2 = 1/2 + \ln 2$.

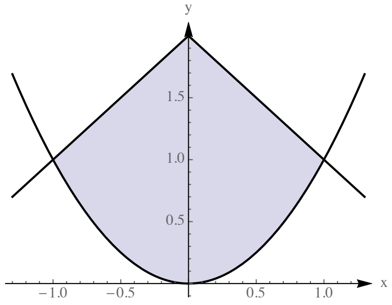


6.2.44 Note that the curves intersect when $4x - x^2 = 4x - 4$, or $x^2 - 4 = 0$. We see that the curves intersect for $x > 0$ when $x = 2$. The line $y = 4x - 4$ intersects the x-axis at $x = 1$. The area is given by $\int_0^1 (4x - x^2)\,dx + \int_1^2 (4x - x^2 - (4x - 4))\,dx = \int_0^1 (4x - x^2)\,dx + \int_1^2 (4 - x^2)\,dx = \left(2x^2 - x^3/3\right)\Big|_0^1 + \left(4x - x^3/3\right)\Big|_1^2 = (2 - 1/3) + (8 - 8/3) - (4 - 1/3) = \dfrac{10}{3}$.

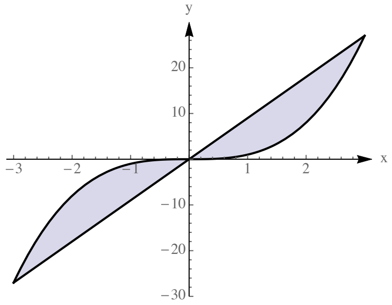


6.2.45 The curves intersect at $x = \pm 1$. Using symmetry, the area is

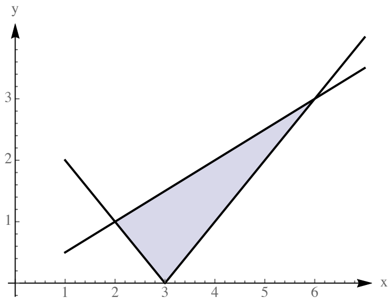
$$2\int_0^1 (2-x-x^2)\,dx = 2\left(2x - \frac{1}{2}x^2 - \frac{1}{3}x^2\right)\Big|_0^1 = \frac{7}{3}.$$



6.2.46 Using symmetry, the area is $2\int_0^3 (9x - x^3)\,dx = 2\left(9x^2/2 - x^4/4\right)\Big|_0^3 = 2(81/2 - 81/4) = 81/2$.



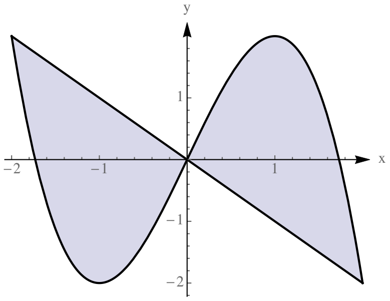
6.2.47 $\int_2^6 (x/2 - |x - 3|)\,dx = \int_2^3 (x/2 - (3 - x))\,dx + \int_3^6 (x/2-(x-3))\,dx = \int_2^3 (3x/2-3)\,dx + \int_3^6 (3-x/2)\,dx = \left(3x^2/4 - 3x\right)\Big|_2^3 + \left(3x - x^2/4\right)\Big|_3^6 = 27/4-9-(3-6)+(18-9-(9-9/4)) = 3/4+9/4 = 3$.



6.2.48

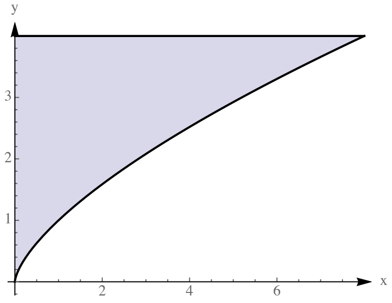
The curves intersect for $3x-x^3=-x$, or $x^3-4x=x(x^2-4)=x(x-2)(x+2)=0$, so for $x=0$ and $x=\pm2$. Using symmetry, the area is given by

$$2\int_0^2(3x-x^3+x)\,dx = 2\left(2x^2-\frac{x^4}{4}\right)\Big|_0^2 = 16-8 = 8.$$



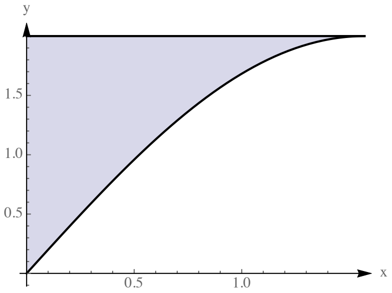
6.2.49

The area is given by $\int_0^8 (4-x^{2/3})\,dx = \left(4x-3x^{5/3}/5\right)\Big|_0^8 = 32-96/5 = 64/5.$



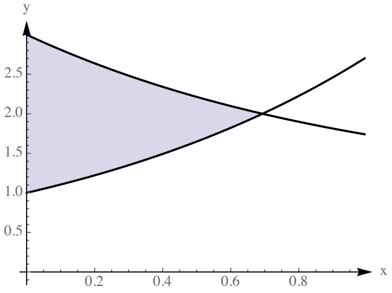
6.2.50

The area is given by $\int_0^{\pi/2}(2-2\sin x)\,dx = (2x+2\cos x)\Big|_0^{\pi/2} = \pi-2.$



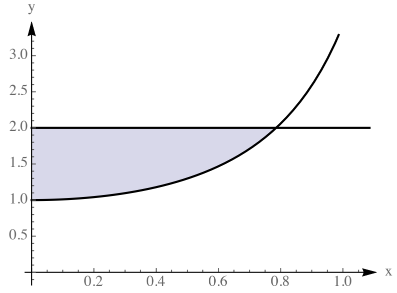
6.2.51

The area is given by $\int_0^{\ln 2}(2e^{-x}+1-e^x)\,dx = \left(-2e^{-x}+x-e^x\right)\Big|_0^{\ln 2} = -1+\ln 2-2-(-2-1) = \ln 2.$



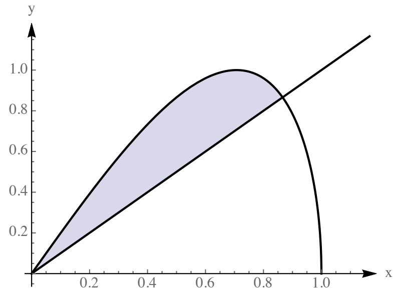
6.2.52

The area is given by $\int_0^{\pi/4}(2 - \sec^2 x)\,dx = (2x - \tan x)\Big|_0^{\pi/4} = \frac{\pi}{2} - 1.$



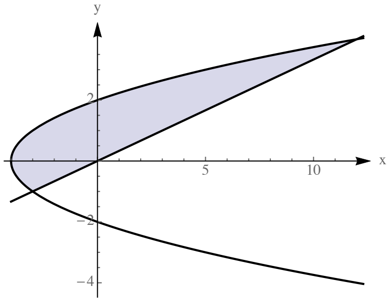
6.2.53

The area is given by $\int_0^{\sqrt{3}/2}(2x\sqrt{1-x^2} - x)\,dx = \left(-\frac{2}{3}(1-x^2)^{3/2} - \frac{x^2}{2}\right)\Big|_0^{\sqrt{3}/2} = \frac{-2}{24} - \frac{3}{8} + \frac{2}{3} = \frac{5}{24}.$



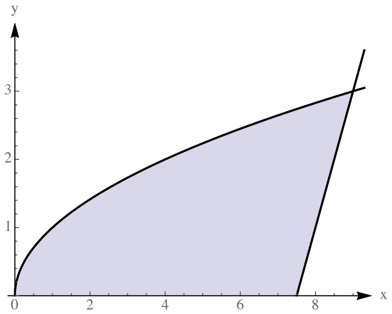
6.2.54

The area is given by $\int_{-1}^{4}(3y - (y^2 - 4))\,dy = \left(3y^2/2 - y^3/3 + 4y\right)\Big|_{-1}^{4} = 24 - 64/3 + 16 - (3/2 + 1/3 - 4) = 44 - 65/3 - 3/2 = \frac{125}{6}.$



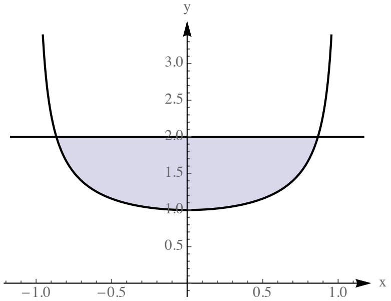
6.2.55

The curves intersect at the point $(9, 3)$. The area is given by $\int_0^3\left(\frac{y}{2} + \frac{15}{2} - y^2\right)dy = \left(\frac{y^2}{4} + \frac{15y}{2} - \frac{y^3}{3}\right)\Big|_0^3 = \frac{9}{4} + \frac{45}{2} - 9 = \frac{63}{4}.$



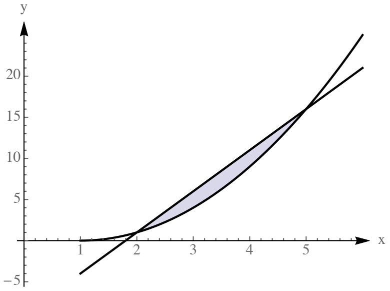
6.2.56

Note that the curves intersect for $2 = \frac{1}{\sqrt{1-x^2}}$, or $1 - x^2 = \frac{1}{4}$, or $x = \pm\frac{\sqrt{3}}{2}$. Using symmetry, the area is given by $2\int_0^{\sqrt{3}/2}\left(2 - \frac{1}{\sqrt{1-x^2}}\right)\,dx = 2\left(2x - \sin^{-1}(x)\right)\Big|_0^{\sqrt{3}/2} = 2\sqrt{3} - 2\pi/3$.



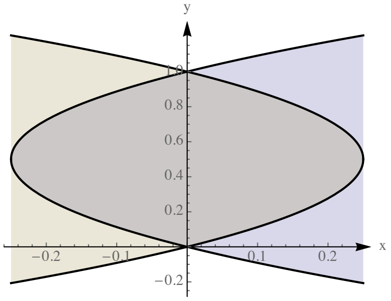
6.2.57

The curves intersect for $x^2 - 2x + 1 = 5x - 9$, or $x^2 - 7x + 10 = (x-5)(x-2) = 0$, so for $x = 2$ and $x = 5$. The area is given by $\int_2^5 (5x - 9 - (x-1)^2)\,dx = \left(5x^2/2 - 9x - (x-1)^3/3\right)\Big|_2^5 = 125/2 - 45 - 64/3 - (10 - 18 - (1/3)) = 4.5$.



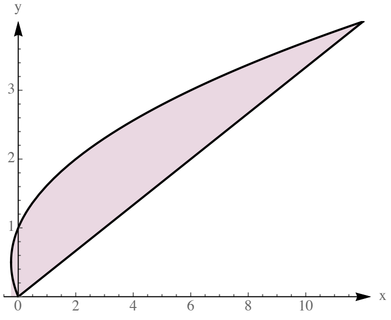
6.2.58

Using symmetry, this is $2\int_0^1 (-y(y-1))\,dy = 2\left(-y^3/3 + y^2/2\right)\Big|_0^1 = 1/3$.



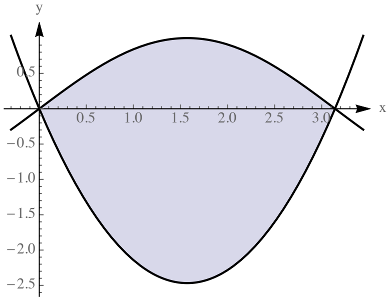
6.2.59

This is given by $\int_0^4 (3y - (y^2 - y))\,dy = \int_0^4 (4y - y^2)\,dy = \left(2y^2 - y^3/3\right)\Big|_0^4 = 32 - 64/3 = 32/3$.



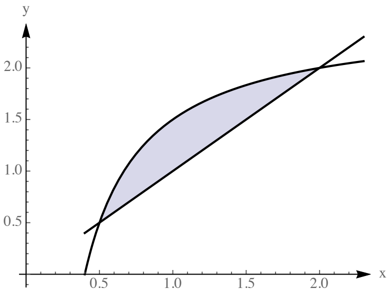
6.2.60

The area is given by $\int_0^\pi (\sin x - x^2 + \pi x)\,dx = \left(-\cos x - x^3/3 + \pi x^2/2\right)\Big|_0^\pi = 1 - \pi^3/3 + \pi^3/2 - (-1) = 2 + \pi^3/6$.



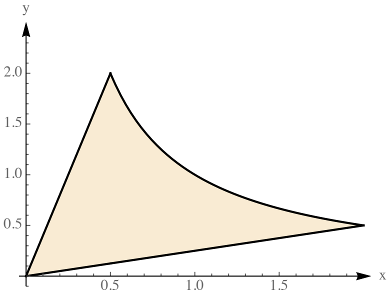
6.2.61

This area is given by $\int_{1/2}^{2} (5/2 - 1/x - x)\,dx = \left(5x/2 - \ln x - x^2/2\right)\Big|_{1/2}^{2} = 5 - \ln 2 - 2 - (5/4 - \ln(1/2) - 1/8) = 15/8 - 2\ln 2$.



6.2.62

This area is given by $\int_0^{1/2} (4x - x/4)\,dx + \int_{1/2}^{2} (1/x - x/4)\,dx = \left(15x^2/8\right)\Big|_0^{1/2} + \left(\ln x - x^2/8\right)\Big|_{1/2}^{2} = 15/32 + \ln 2 - 1/2 - (\ln 1/2 - 1/32) = 2\ln 2$.



6.2.63 $y = 8x$ and $y = 9 - x^2$ intersect when $x^2 + 8x - 9 = (x+9)(x-1) = 0$, so at $x = 1$ in the first quadrant. $y = \frac{5}{2}x$ and $y = 9 - x^2$ intersect when $x^2 + \frac{5}{2}x - 9 = 0$, or $2x^2 + 5x - 18 = (2x+9)(x-2) = 0$, so at $x = 2$ in the first quadrant. Thus the area of the region is

$$\begin{aligned} A &= \int_0^1 \left(8x - \frac{5}{2}x\right)\,dx + \int_1^2 \left(9 - x^2 - \frac{5}{2}x\right)\,dx \\ &= \int_0^1 \frac{11}{2}x\,dx + \int_1^2 \left(9 - x^2 - \frac{5}{2}x\right)\,dx \\ &= \left(\frac{11}{4}x^2\right)\Big|_0^1 + \left(9x - \frac{x^3}{3} - \frac{5}{4}x^2\right)\Big|_1^2 \\ &= \frac{11}{4} + 18 - \frac{8}{3} - 5 - 9 + \frac{1}{3} + \frac{5}{4} = \frac{17}{3}. \end{aligned}$$

6.2.64

$$\begin{aligned} A &= \int_{1/2}^{1} (4\sqrt{2x} - (-4x+6))\,dx + \int_{1}^{2} (4\sqrt{2x} - 2x^2)\,dx \\ &= \left(\frac{8\sqrt{2}}{3}x^{3/2} + 2x^2 - 6x\right)\Big|_{1/2}^{1} + \left(\frac{8\sqrt{2}}{3}x^{3/2} - 2x^3/3\right)\Big|_{1}^{2} = 19/6. \end{aligned}$$

6.2.65

a. False. This can be done either with respect to x or with respect to y. For the latter, the relevant integral is $\int_0^1 (y - y^2)\,dy$.

b. False. On the interval $(0, \pi/4)$ the cosine function is greater, but on $(\pi/4, \pi/2)$ the sine function is greater. The area is $\int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$.

c. True. They both represent the area of the region in the first quadrant under $y = x$ and above $y = x^2$.

6.2.66 Given $\int_{-a}^{a} (f(x) - g(x))\,dx = 10$, we have (by symmetry) that $\int_0^a (f(x) - g(x))\,dx = 5$. Then, $\int_0^{\sqrt{a}} x(f(x^2) - g(x^2))\,dx = \frac{1}{2}\int_0^a (f(u) - g(u))\,du = \frac{5}{2}$. (Where we used the substitution $u = x^2$.)

6.2.67 $A = 2\int_0^1 y^2\sqrt{1-y^3}\,dy = \frac{2}{3}\int_0^1 \sqrt{u}\,du$ (where $u = 1 - y^3$). So $A = \frac{2}{3}\left(\frac{2}{3}u^{3/2}\right)\Big|_0^1 = \frac{4}{9}$.

6.2.68 $A_n = \int_0^1 (x - x^n)\,dx = \left(x^2/2 - x^{n+1}/(n+1)\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{n+1} = \frac{n-1}{2n+2}$.

6.2.69 $A_n = \int_0^1 (x^{1/n} - x)\,dx = \left(\frac{nx^{(n+1)/n}}{n+1} - x^2/2\right)\Big|_0^1 = \frac{n}{n+1} - \frac{1}{2} = \frac{n-1}{2n+2}$.

6.2.70 $A_n = \int_0^1 (x^{1/n} - x^n)\,dx = \left(\frac{nx^{(n+1)/n}}{n+1} - \frac{x^{n+1}}{n+1}\right)\Big|_0^1 = \frac{n}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1}$.

6.2.71 Using the result of the previous problem, $\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{n-1}{n+1} = 1$. As $n \to \infty$, the region in question approaches the 1×1 square over the interval $[0, 1]$, which has area 1.

6.2.72 R is a right triangle with both legs equal to 1, so its area is $\frac{1}{2}$. Integrating with respect to y, we are looking for k so that $\int_0^k (1 - y)\,dy$ is half the area of the triangle, or $\frac{1}{4}$. Now,

$$\int_0^k (1-y)\,dy = \left(y - \frac{y^2}{2}\right)\Big|_0^k = k - \frac{k^2}{2}.$$

Setting $k - \frac{k^2}{2} = \frac{1}{4}$, we have $2k^2 - 4k + 1 = 0$, which has roots $k = \frac{2 \pm \sqrt{2}}{2}$. Only the negative sign choice gives a value between 0 and 1, so we want $k = \frac{2-\sqrt{2}}{2} = 1 - \frac{\sqrt{2}}{2}$.

6.2.73 This is a triangle with base 2 and height 1, so it has area 1. Note that for $0 \le x \le 1$ the line which forms the top of the triangle is $y = x$, and for $1 \le x \le 2$, the line is $y = 2 - x$. Integrating with respect to y, we are looking for k so that $\displaystyle\int_0^k (2 - y - y)\,dy = \frac{1}{2}$. Now,

$$\int_0^k (2 - 2y)\,dy = \left(2y - y^2\right)\Big|_0^k = 2k - k^2.$$

Setting $2k - k^2 = \dfrac{1}{2}$, we have $2k^2 - 4k + 1 = 0$, which has roots $k = \dfrac{2 \pm \sqrt{2}}{2}$. Only the negative sign choice gives a value between 0 and 1, so we want $k = \dfrac{2 - \sqrt{2}}{2} = 1 - \dfrac{\sqrt{2}}{2}$.

6.2.74

a. The proportion of the whole board which has the desired property is the same as the proportion of each "quarter board" with the desired property, so we can consider only the quarter board rather than the whole board.

b. Let $P(x, y)$ be a point on the curve C. Let Q be the point on line segment $\overline{AB}$ so that $\overline{QP} \perp \overline{AB}$. Then, because the distance from O to P is the same as the distance from P to Q, we must have $\sqrt{x^2 + y^2} = 1 - y$, so $y = \dfrac{1}{2}(1 - x^2)$.

c. The area of the region R is 1, and the area of $R_1 - R$ is

$$2\int_0^{\sqrt{2}-1} \left(\frac{1}{2}(1 - x^2) - x\right) dx = \left(x - x^3/3 - x^2\right)\Big|_0^{\sqrt{2}-1}$$
$$= \sqrt{2} - 1 - (\sqrt{2} - 1)^3/3 - (\sqrt{2} - 1)^2 = \sqrt{2} - 1 + 7/3 - (5/3)\sqrt{2} + 2\sqrt{2} - 34\sqrt{2}/3 - 5/3.$$

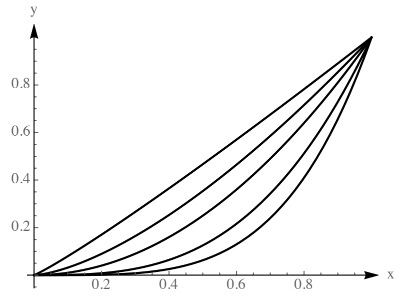
So the probability of landing in R_2 is $1 - (4\sqrt{2}/3 - 5/3) = \dfrac{8 - 4\sqrt{2}}{3} \approx .781$.

6.2.75

a. The point (n, n) on the curve $y = x$ would represent the notion that the lowest $p\%$ of the society owns $p\%$ of the wealth, which would represent a form of equality.

b. The function must be increasing and concave up because the poorest $p\%$ cannot own more than $p\%$ of the wealth.

c. $y = x^{1.1}$ is closest to $y = x$, and $y = x^4$ is furthest from $y = x$.



d. Note that $B = \displaystyle\int_0^1 L(x)\,dx$, and $A + B = 1/2$, so $A = \dfrac{1}{2} - \displaystyle\int_0^1 L(x)\,dx$. Then $G = \dfrac{A}{A + B} = \dfrac{A}{1/2} = 2A = 1 - 2\displaystyle\int_0^1 L(x)\,dx$.

e. For $L(x) = x^p$, we have $G = 1 - 2\int_0^1 x^p\,dx = 1 - 2\left(\frac{x^{p+1}}{p+1}\right)\Big|_0^1 = 1 - \frac{2}{p+1} = \frac{p-1}{p+1}$. So we have

p	1.1	1.5	2	3	4
G	1/21	1/5	1/3	1/2	3/5

f. For $p = 1$ we have $G = 0$. Because $\lim_{p\to\infty} \frac{p-1}{p+1} = 1$, the largest possible value of G approaches 1.

g. For $L(x) = 5x^2/6 + x/6$, note that $L(0) = 0$, $L(1) = 1$, $L'(x) = 5x/3 + 1/6 > 0$ on $[0, 1]$, and $L''(x) = 5/3 > 0$ as well. The Gini index is

$$G = 1-2\int_0^1 (5x^2/6+x/6)\,dx = 1-2\left(5x^3/18 + x^2/12\right)\Big|_0^1 = 1-2\,(5/18+1/12) = 1-5/9-1/6 = 5/18.$$

6.2.76

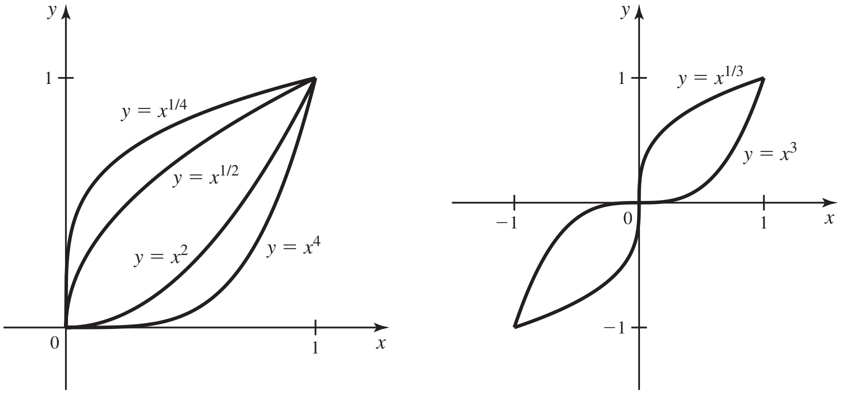
a. In this case l_Q is $y = 2ax - a^2$, l_R is $y = 0$, and l_P is the line $y = -2ax - a^2$. We have $P' = (a/2, 0)$, $R' = (0, -a^2)$, and $Q' = (-a/2, 0)$. The area of triangle PQR is $\frac{1}{2}\cdot 2a\cdot a^2 = a^3$ and the area of triangle $P'Q'R'$ is $\frac{1}{2}\cdot(2(a/2))\cdot(a^2) = a^3/2$.

b. In this case l_Q is $y = 2bx - b^2$, l_R is $y = 0$, and l_P is $y = -2ax - a^2$. We have $P' = (b/2, 0)$, $R' = ((b-a)/2, -ab)$, and $Q' = (-a/2, 0)$. The area of triangle PQR is $ab(a + b)$, and the area of $P'Q'R'$ is $ab\cdot\frac{a+b}{2}$.

c. Let the coordinates of R be (c, c^2). Assume $0 < a < c < b$ (the other cases can be handled similarly.) In this case l_Q is $y = 2bx - b^2$, l_P is $y = -2ax - a^2$, and l_R is $y = 2cx - c^2$. We have $P' = ((b+c)/2, bc)$, $Q' = ((c-a)/2, -ca)$, and $R' = ((b-a)/2, -ab)$. The area of PQR is $\frac{1}{2}(a^2b+b^2c+ab^2-ac^2-a^2c-bc^2)$. The area of triangle $P'Q'R'$ is $\frac{1}{4}(a^2b + b^2c + ab^2 - ac^2 - a^2c - bc^2)$.

6.2.77

a.

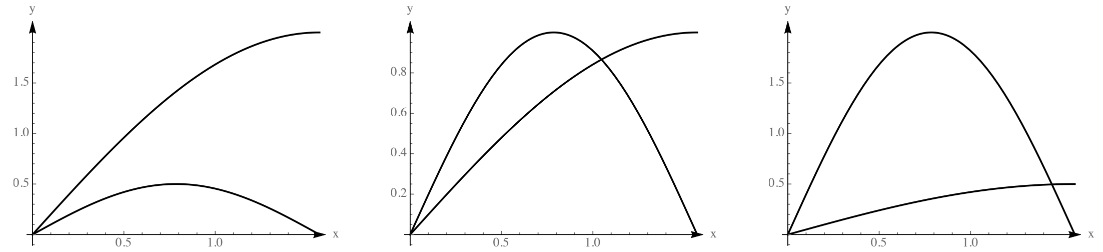


b. $A_n(x)$ is the net area of the region between the graphs of f and g from 0 to x.

c. Note that $\int_0^1 (f(s) - g(s))\,ds = \left(\frac{s^{n+1}}{n+1} - \frac{s^{(1/n)+1}}{(1/n)+1}\right)\Big|_0^1 = \frac{1}{n+1} - \frac{n}{n+1} = \frac{1-n}{n+1} < 0$. So we seek the smallest c so that $\int_1^c (f(s) - g(s))\,ds = \frac{n-1}{n+1}$. This occurs when $\left(\frac{s^{n+1}}{n+1} - \frac{s^{(1/n)+1}}{(1/n)+1}\right)\Big|_1^c = \frac{n-1}{n+1}$, or $\frac{c^{n+1}}{n+1} = \frac{c^{(n+1)/n}}{(n+1)/n}$, or $c^{n+1} = nc^{(n+1)/n}$, or $c^{n-(1/n)} = n$, so $c = n^{n/(n^2-1)}$. As n increases, this root increases as well.

6.2.78

a.



$a = 1/2.$ $a = 1.$ $a = 2.$

b. We seek a root of $a \sin 2x = (\sin x)/a$, or $2 \sin x \cos x = \frac{\sin x}{a^2}$, so $\cos x = \frac{1}{2a^2}$. This has a solution as long as $\frac{1}{2a^2} \leq 1$, which occurs when $a \geq \sqrt{1/2}$.

c.

$$\begin{aligned} A &= \int_0^{x^\star} (\sin 2x - \sin x)\,dx = \left(\cos x - \frac{1}{2}\cos 2x\right)\Bigg|_0^{x^\star} \\ &= \left(\cos x - \frac{1}{2}(2\cos^2 x - 1)\right)\Bigg|_0^{x^\star} = \left(\cos x - \cos^2 x + \frac{1}{2}\right)\Bigg|_0^{x^\star} \\ &= \cos x^\star - \cos^2 x^\star + \frac{1}{2} - \left(1 - 1 + \frac{1}{2}\right) = \frac{1}{2a^2} - \frac{1}{4a^4} = \frac{2a^2-1}{4a^4}. \end{aligned}$$

When $a = 1$ this is equal to $\frac{1}{4}$.

d. Using the result of the previous calculation, we simply note that $\frac{2a^2-1}{4a^4} \to 0$ as $a \to 1/\sqrt{2}$.

6.3 Volume by Slicing

6.3.1 $A(x)$ is the area of the cross section through the solid at the point x.

6.3.2 The general slicing method would be used.

6.3.3

a. $A(x) = (\sqrt{3-x})^2 = 3 - x$.

b. $\displaystyle\int_0^2 (3-x)\,dx$.

6.3.4 The cross sections are disks and $A(x)$ is the area of a disk.

6.3.5

a. The radius of a cross section is $\sqrt{\cos x}$.

b. $A(x) = \pi(\sqrt{\cos x})^2 = \pi \cos x$.

c. $V = \displaystyle\int_0^2 \pi \cos x\,dx$.

6.3.6

a. The radius of a cross section is $\cos y$.

b. $A(y) = \pi \cos^2 y$.

c. $V = \displaystyle\int_0^{\pi/2} \pi \cos^2 y\,dy$.

6.3.7

a. The outer radius is given by $\sqrt{x}+1$.

b. The inner radius is 1.

c. $A(x) = \pi\left((\sqrt{x}+1)^2 - 1\right)$.

d. $V = \int_0^4 \pi\left((\sqrt{x}+1)^2 - 1\right)\,dx$.

6.3.8

a. The outer radius is 4.

b. The inner radius is $(y-1)^2$.

c. $A(y) = \pi(16-(y-1)^4)$.

d. $V = \int_1^3 \pi(16-(y-1)^4)\,dy$.

6.3.9

a. The radius is $\sqrt{x}$.

b. $A(x) = \pi(\sqrt{x})^2 = \pi x$.

c. $V = \int_0^4 \pi x\,dx$.

6.3.10

a. The radius is $4-(y-1)^2$.

b. $A(y) = \pi(4-(y-1)^2)^2$.

c. $V = \int_1^3 \pi(4-(y-1)^2)^2\,dy$.

6.3.11 The curves intersect when $\sqrt{1-x^2} = 0$, or $x = \pm 1$. $A(x) = (\sqrt{1-x^2})^2 = 1-x^2$. The volume is given by

$$\int_{-1}^1 (1-x^2)\,dx = \left(x - x^3/3\right)\Big|_{-1}^1 = \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right) = \frac{4}{3}.$$

6.3.12 The curves intersect when $2-x^2 = x^2$, or $x = \pm 1$. $A(x) = ((2-x^2)-x^2)^2 = (2(1-x^2))^2 = 4(1-2x^2+x^4)$. The volume is given by

$$\int_{-1}^1 4(1-2x^2+x^4)\,dx = 4\left(x - 2x^3/3 + x^5/5\right)\Big|_{-1}^1 = 4\left(\left(1-\frac{2}{3}+\frac{1}{5}\right) - \left(-1+\frac{2}{3}-\frac{1}{5}\right)\right) = \frac{64}{15}.$$

6.3.13 The curves intersect when $\sqrt{\cos x} = 0$, or $x = \pm\pi/2$. $A(x) = \frac{1}{2}\left(\sqrt{\cos x}\right)^2 = \frac{\cos x}{2}$. The volume is given by

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x}{2}\,dx = \frac{\sin x}{2}\Big|_{-\pi/2}^{\pi/2} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1.$$

6.3.14 $A(x) = (2\sqrt{25-x^2})^2 = 100-4x^2$.

$$V = \int_0^5 A(x)\,dx = \int_0^5 (100-4x^2)\,dx = \left(100x - 4x^3/3\right)\Big|_0^5 = 500 - \frac{500}{3} = \frac{1000}{3}.$$

6.3.15 For each value of x, the height of the triangle is $2-x$, which is the diameter of the semicircle, so the area of that semicircle is

$$A(x) = \frac{\pi}{2}\left(\frac{2-x}{2}\right)^2 = \frac{\pi}{8}(4-4x+x^2).$$

Thus the volume is

$$V = \int_0^2 A(x)\,dx = \int_0^2 \frac{\pi}{8}(4-4x+x^2)\,dx = \frac{\pi}{8}\left(4x - 2x^2 + x^3/3\right)\Big|_0^2 = \frac{\pi}{3}.$$

6.3.16 The volume is $V = \int_{-1}^1 (1-x^2)^2\,dx = 2\int_0^1 (1-2x^2+x^4)\,dx = 2\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^4\right)\Big|_0^1 =$ $2\left(1-\frac{2}{3}+\frac{1}{5}\right) = \frac{16}{15}$.

6.3.17 $V = \pi \int_0^3 4x^2\,dx = 4\pi\left(x^3/3\right)\Big|_0^3 = 36\pi.$

6.3.18 $V = \pi \int_0^1 (2-2x)^2\,dx = 4\pi\left(x - x^2 + x^3/3\right)\Big|_0^1 = \frac{4\pi}{3}.$

6.3.19 $V = \pi \int_0^{\ln 4} e^{-2x}\,dx = \frac{-\pi}{2}\left(e^{-2x}\right)\Big|_0^{\ln 4} = \frac{-\pi}{2}((1/16)-1) = \frac{15\pi}{32}.$

6.3.20 $V = \pi \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \pi\,\sin^{-1} x\,\Big|_0^{1/2} = \frac{\pi^2}{6}.$

6.3.21 $V = \pi \int_{-1}^{1} \left(\frac{1}{\sqrt{1+y^2}}\right)^2 dy = \pi\tan^{-1} y\Big|_{-1}^{1} = \pi(\pi/4+\pi/4) = \pi^2/2.$

6.3.22 $V = \pi \int_0^2 e^{2y}\,dy = \pi\frac{e^{2y}}{2}\Big|_0^2 = \frac{\pi(e^4-1)}{2}.$

6.3.23 $V = \pi \int_0^4 ((2\sqrt{x})^2 - x^2)\,dx = \pi\left(2x^2 - x^3/3\right)\Big|_0^4 = \pi(32-64/3) = \frac{32\pi}{3}.$

6.3.24 The curves intersect at $x = 0$ and $x = 1$. We have $V = \pi \int_0^1 ((\sqrt[4]{x}^2) - x^2)\,dx = \pi \int_0^1 (\sqrt{x} - x^2)\,dx =$ $\pi\left(2x^{3/2}/3 - x^3/3\right)\Big|_0^1 = \pi(2/3-1/3) = \pi/3.$

6.3.25 $V = \pi \int_{\ln 2}^{\ln 3} ((e^{x/2})^2 - (e^{-x/2})^2)\,dx = \pi \int_{\ln 2}^{\ln 3} (e^x - e^{-x})\,dx = \pi\left(e^x + e^{-x}\right)\Big|_{\ln 2}^{\ln 3} = \pi((3+1/3)-(2+1/2)) = 5\pi/6.$

6.3.26 $V = \pi \int_0^4 ((x+2)^2 - x^2)\,dx = \pi \int_0^4 (4x+4)\,dx = \pi\left(2x^2+4x\right)\Big|_0^4 = 48\pi.$

6.3.27 $V = \pi \int_0^1 (1 - y^{2/3})\,dy = \pi\left(y - \frac{3}{5}y^{5/3}\right)\Big|_0^1 = \frac{2\pi}{5}.$

6.3.28 $V = \pi \int_0^6 (y^2 - y^2/4)\,dy = \frac{3\pi}{4}\left(y^3/3\right)\Big|_0^6 = 54\pi.$

6.3.29 $V = \pi \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{2}\int_0^{\pi/2}(1+\cos 2x)\,dx = \frac{\pi}{2}\left(x + \frac{\sin 2x}{2}\right)\Big|_0^{\pi/2} = \frac{\pi^2}{4}.$

6.3.30 Using symmetry, $V = 2\pi \int_0^5 (\sqrt{25-x^2})^2\,dx = 2\pi\left(25x - x^3/3\right)\Big|_0^5 = \frac{500\pi}{3}$. The volume of a sphere of radius 5 is $\frac{4}{3}\pi\cdot 5^3 = \frac{500\pi}{3}.$

6.3.31 $V = \pi \int_0^{\pi} \sin^2 x\,dx = \frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{\pi}{2}\left(x - \frac{\sin 2x}{2}\right)\Big|_0^{\pi} = \frac{\pi^2}{2}.$

6.3.32 $V = \pi \int_0^{\pi/4} \sec^2 x\,dx = \pi\,\tan x\,\Big|_0^{\pi/4} = \pi.$

6.3.33 $V = \pi \int_0^{\pi/4} \sin^2 y\, dy = \frac{\pi}{2}\int_0^{\pi/4}(1-\cos 2y)\, dy = \frac{\pi}{2}\left(y - \frac{\sin 2y}{2}\right)\Big|_0^{\pi/4} = \frac{\pi}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right) = \frac{\pi(\pi-2)}{8}$.

6.3.34 $V = \pi \int_0^{\pi/2} (1-\sin x)\, dx = \pi\left(x + \cos x\right)\Big|_0^{\pi/2} = \pi(\pi/2 - 1)$.

6.3.35 $V = \pi \int_0^{\pi/2} (\sin x - \sin^2 x)\, dx = \pi \int_0^{\pi/2} (\sin x - \frac{1}{2}(1-\cos 2x))\, dx = \pi\left(\frac{\sin 2x}{4} - \cos x - \frac{x}{2}\right)\Big|_0^{\pi/2}$ $= \pi\left(1 - \frac{\pi}{4}\right) = \frac{4\pi - \pi^2}{4}$.

6.3.36 Note that the curves intersect at $x = \pm 1$. We have $V = \pi \int_{-1}^{0}((2-x^2)^2 - (-x)^2)\, dx + \pi\int_0^1((2-x^2)^2 - x^2)\, dx = \pi\int_{-1}^{1}(4 - 4x^2 + x^4 - x^2)\, dx = \pi\int_{-1}^{1}(4 - 5x^2 + x^4)\, dx = \pi\left(4x - \frac{5x^3}{3} + \frac{x^5}{4}\right)\Big|_{-1}^{1} = \pi\left(\left(4 - \frac{5}{3} + \frac{1}{5}\right) - \left(-4 + \frac{5}{3} - \frac{1}{5}\right)\right) = \frac{76\pi}{15}$.

6.3.37 $V = \pi\int_0^2 (16 - y^4)\, dy = \pi\left(16y - \frac{y^5}{5}\right)\Big|_0^2 = \frac{128\pi}{5}$.

6.3.38 $V = \pi\int_0^4 \left(2^2 - (\sqrt{4-y})^2\right)\, dy = \pi\int_0^4 y\, dy = \pi\frac{y^2}{2}\Big|_0^4 = \pi(8-0) = 8\pi$.

6.3.39 $V = \pi\int_2^6 \left(\frac{1}{\sqrt{x}}\right)^2\, dx = \pi\left(\ln x\right)\Big|_2^6 = \pi \ln 3$.

6.3.40 $V = \pi\int_0^1 \left((2-y)^2 - (\sqrt{y})^2\right)\, dy = \pi\int_0^1 (y^2 - 5y + 4)\, dy = \pi\left(\frac{y^3}{3} - \frac{5y^2}{2} + 4y\right)\Big|_0^1 =$ $\pi\left(\frac{1}{3} - \frac{5}{2} + 4\right) = \frac{11\pi}{6}$.

6.3.41 $V = \pi\int_0^2 e^{2x}\, dx = \frac{\pi}{2}\left(e^{2x}\right)\Big|_0^2 = \frac{\pi}{2}(e^4 - 1)$.

6.3.42 $V = \pi\int_0^{\ln 4} (e^{2x} - e^{-2x})\, dx = \frac{\pi}{2}\left(e^{2x} + e^{-2x}\right)\Big|_0^{\ln 4} = \frac{\pi}{2}\left(16 + \frac{1}{16} - 2\right) = \frac{225\pi}{32}$.

6.3.43 $V = \pi\int_0^{\ln 8} (e^{2y} - e^y)\, dy = \pi\left(e^{2y}/2 - e^y\right)\Big|_0^{\ln 8} = \pi\left(32 - 8 - \left(\frac{1}{2} - 1\right)\right) = \frac{49\pi}{2}$.

6.3.44 $V(p) = \pi\int_0^p e^{-2x}\, dx = -\frac{\pi}{2}\left(e^{-2x}\right)\Big|_0^p = \frac{\pi}{2}\left(1 - e^{-2p}\right)$. The volume is bounded by $\pi/2$ as $p \to \infty$.

6.3.45 About the x-axis: $V_x = \pi\int_0^5 4x^2\, dx = \pi\left(4x^3/3\right)\Big|_0^5 = \frac{500\pi}{3}$.

About the y-axis: $V_y = \pi\int_0^{10} (25 - y^2/4)\, dy = \pi\left(25y - y^3/12\right)\Big|_0^{10} = \frac{500\pi}{3}$.
The volumes are the same.

6.3.46 About the x-axis: $V_x = \pi\int_0^2 (4-2x)^2\, dx = 4\pi\left(4x - 2x^2 + x^3/3\right)\Big|_0^2 = \frac{32\pi}{3}$.

About the y-axis: $V_y = \pi\int_0^4 (2 - (y/2))^2\, dy = \pi\left(4y - y^2 + y^3/12\right)\Big|_0^4 = \frac{16\pi}{3}$.
The volume V_x is bigger.

6.3.47 About the x-axis: $V_x = \pi \int_0^1 (1-x^3)^2\,dx = \pi \int_0^1 (1-2x^3+x^6)\,dx = \pi\left(x - x^4/2 + x^7/7\right)\Big|_0^1 = \frac{9\pi}{14}$.

About the y-axis: $V_y = \pi \int_0^1 (\sqrt[3]{1-y})^2\,dy = -\pi\left(\frac{3}{5}(1-y)^{5/3}\right)\Big|_0^1 = -\pi(0-3/5) = \frac{3\pi}{5}$. The volume V_x is bigger.

6.3.48 About the x-axis: $V_x = \pi \int_0^2 (8x - x^4)\,dx = \pi\left(4x^2 - x^5/5\right)\Big|_0^2 = \frac{48\pi}{5}$.

About the y-axis: $V_y = \pi \int_0^4 (y - y^4/64)\,dy = \pi\left(y^2/2 - y^5/320\right)\Big|_0^4 = \frac{24\pi}{5}$.

The volume V_x is bigger.

6.3.49 $V = \pi \int_0^1 \left(1 - \left(1-\sqrt{x}\right)\right)^2\,dx = \pi \int_0^1 x\,dx = \pi \frac{x^2}{2}\Big|_0^1 = \frac{\pi}{2}$.

6.3.50 $V = \pi \int_0^1 (e^{-x/2} + 2 - 2)^2\,dx = \pi \int_0^1 e^{-x}\,dx = -\pi e^{-x}\Big|_0^1 = \pi(1-e^{-1})$.

6.3.51 $V = \pi \int_0^{\pi/3} (2 - (2 - \sec y))^2\,dy = \pi \int_0^{\pi/3} \sec^2 y\,dy = \pi \tan y\Big|_0^{\pi/3} = \pi\sqrt{3}$.

6.3.52 $V = \pi \int_0^1 (1-(1-y)^2)^2\,dy = \pi \int_0^1 (y^4 - 4y^3 + 4y^2)\,dy = \pi\left(\frac{y^5}{5} - y^4 + \frac{4y^3}{3}\right)\Big|_0^1 =$ $\pi\left(\frac{1}{5} - 1 + \frac{4}{3}\right) = \frac{8\pi}{15}$.

6.3.53 Using the disk method, a disk located at x has a radius of $1-\sqrt{x}$ when revolved about the line $y=1$, so the volume is

$$V = \pi \int_0^1 (1-\sqrt{x})^2\,dx = \pi \int_0^1 (1 - 2\sqrt{x} + x)\,dx = \pi\left(x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2\right)\Big|_0^1 = \pi\left(1 - \frac{4}{3} + \frac{1}{2}\right) = \frac{\pi}{6}.$$

6.3.54 We use the washer method. Note that $x = y^2$, and for each value of y, the washer has outer radius 4 and inner radius $4 - y^2$, so that the volume is

$$V = \pi \int_0^2 (4^2 - (4-y^2)^2)\,dy = \pi \int_0^2 (8y^2 - y^4)\,dy = \pi\left(\frac{8}{3}y^3 - \frac{1}{5}y^5\right)\Big|_0^2 = \pi\left(\frac{64}{3} - \frac{32}{5}\right) = \frac{224}{15}\pi.$$

6.3.55 We use the washer method. For each x, the washer has outer radius $2 + 2\sin x$ and inner radius 2, so the volume is

$$\begin{aligned} V &= \pi \int_0^\pi \left((2+2\sin x)^2 - 2^2\right)\,dx = \pi \int_0^\pi (8\sin x + 4\sin^2 x)\,dx \\ &= \pi \int_0^\pi (8\sin x + 2(1-\cos 2x))\,dx = \pi(-8\cos x + 2x - \sin 2x)\Big|_0^\pi \\ &= \pi(8 + 2\pi + 8) = 2\pi(\pi + 8). \end{aligned}$$

6.3.56 We use the washer method. First solve for x to obtain $x = e^y$; then for each value of y, the outer radius of a washer is $e^y + 1$ and the inner radius is 1, so the volume is

$$\begin{aligned} V = \pi \int_0^1 \left((e^y+1)^2 - 1^2\right)\,dy = \pi \int_0^1 (2e^y + e^{2y})\,dy &= \pi\left(2e^y + \frac{1}{2}e^{2y}\right)\Big|_0^1 \\ &= \pi\left(2e + \frac{1}{2}e^2 - \frac{5}{2}\right) = \frac{\pi}{2}\left(e^2 + 4e - 5\right). \end{aligned}$$

6.3.57 We use the washer method. For each x, the outer radius is $1+\sin x$ and the inner radius is $1+(1-\sin x)=2-\sin x$. Thus the volume is

$$V=\pi\int_{\pi/6}^{5\pi/6}\left((1+\sin x)^2-(2-\sin x)^2\right)\,dx=\pi\int_{\pi/6}^{5\pi/6}(6\sin x-3)\,dx$$
$$=\pi(-6\cos x-3x)\Big|_{\pi/6}^{5\pi/6}=\pi\left(3\sqrt{3}-\frac{5\pi}{2}+3\sqrt{3}+\frac{\pi}{2}\right)=\pi\left(6\sqrt{3}-2\pi\right).$$

6.3.58 The lines $y=1+\frac{x}{2}$ and $y=x$ meet at the point $(2,2)$. For each x, we obtain a washer with outer radius $3-x$ and inner radius $3-\left(1+\frac{x}{2}\right)=2-\frac{x}{2}$. Hence the volume is

$$V=\pi\int_0^2\left((3-x)^2-\left(2-\frac{x}{2}\right)^2\right)\,dx=\pi\int_0^2\left(5-4x+\frac{3}{4}x^2\right)\,dx=\pi\left(5x-2x^2+\frac{1}{4}x^3\right)\Big|_0^2=4\pi.$$

6.3.59 The lines $x=2-y$ and $x=1-\frac{y}{2}$ meet at the point $(0,2)$. For each y, we obtain a washer whose outer radius is $3-(1-\frac{y}{2})=2+\frac{y}{2}$ and whose inner radius is $3-(2-y)=y+1$. Thus the volume is

$$V=\pi\int_0^2\left(\left(2+\frac{y}{2}\right)^2-(y+1)^2\right)\,dy=\pi\int_0^2\left(-\frac{3}{4}y^2+3\right)\,dy=\pi\left(-\frac{1}{4}y^3+3y\right)\Big|_0^2=4\pi.$$

6.3.60 We would expect the volume to be greater when revolved about the line $y=2$, because the wider portion of the figure then moves through a larger radius, so it traces out a larger volume. When revolving about $y=0$, we obtain a disk with radius $2x(2-x)$, so the volume is

$$V=\pi\int_0^2(2x(2-x))^2\,dx=\pi\int_0^2(16x^2-16x^3+4x^4)\,dx=\left(\frac{16}{3}x^3-4x^4+\frac{4}{5}x^5\right)\Big|_0^2=\frac{64\pi}{15}.$$

When revolving about the line $y=2$, we have washers with outer radius 2 and inner radius $2-2x(2-x)$, so the volume is

$$V=\pi\int_0^2\left(4-(2-2x(2-x))^2\right)\,dx=\pi\int_0^2(8x(2-x)-4x^2(2-x)^2)\,dx$$
$$=\pi\int_0^2(-4x^4+16x^3-24x^2+16x)\,dx=\pi\left(-\frac{4}{5}x^5+4x^4-8x^3+8x^2\right)\Big|_0^2=\frac{32}{5}\pi.$$

The second calculation yields a larger result.

6.3.61

a. False. The cross sections are not disks or washers.

b. True. It is given by $V=\pi\int_0^R(R^2-x^2)\,dx$.

c. True. This is because if we shift the sine function horizontally by $\pi/2$ units, we obtain the cosine function.

6.3.62 The relationship between the height h of tetrahedron and the edge length l is $h=l\sqrt{2/3}$, which can be deduced using triangle geometry and the Pythagorean theorem.

Let z be the distance from the top vertex of the tetrahedron down toward the base perpendicularly. The cross sections perpendicular to this axis are all equilateral triangles with height z, so their side length is $\sqrt{3/2}z$, and their area is

$$A(z)=\frac{\sqrt{3}}{4}\left(z\sqrt{\frac{3}{2}}\right)^2=\frac{3\sqrt{3}z^2}{8}.$$

The volume is thus

$$\int_0^{4\sqrt{2/3}}\frac{3\sqrt{3}}{8}z^2\,dz=\frac{3\sqrt{3}}{8}\left(z^3/3\right)\Big|_0^{4\sqrt{2/3}}=\frac{16\sqrt{2}}{3}.$$

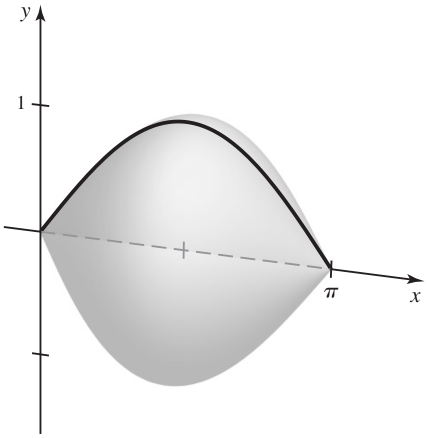
6.3.63 The volume V_S is $\pi \int_0^{\sqrt{a}} (y^2-a)^2\,dy = \pi\left(y^5/5 - 2ay^3/3 + a^2y\right)\Big|_0^{\sqrt{a}} = \frac{8\pi}{15}a^{5/2}$.

The volume V_T is $\frac{1}{3}\pi a^2 \cdot a^{1/2} = \frac{1}{3}\pi a^{5/2}$. The ratio of $V_S/V_T = \frac{8/15}{1/3} = \frac{8}{5}$.

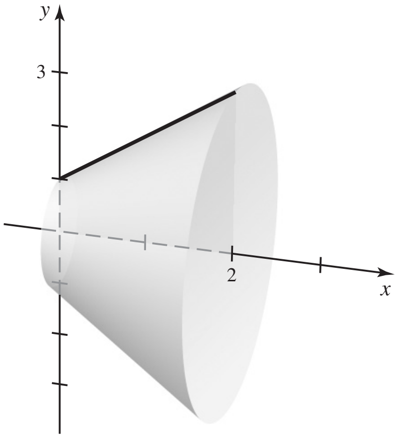
6.3.64 $V = \pi\int_0^2 x^2\,dx + \pi\int_2^5 (2x-2)^2\,dx + \pi\int_5^6 (18-2x)^2\,dx = \frac{8}{3}\pi + 84\pi + \frac{148\pi}{3} = 136\pi$.

6.3.65

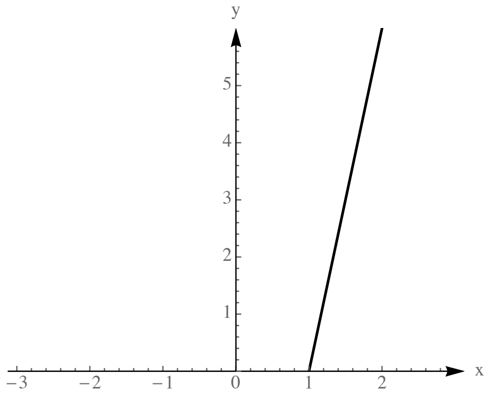
a. This comes from revolving the region in the first quadrant bounded by $\sin x$ and the line $y=0$ between $x=0$ and $x=\pi$ around the x axis.



b. This comes from revolving the region in the first quadrant bounded by $y=x+1$ and the line $y=0$ between $x=0$ and $x=2$ around the x axis.



6.3.66



By symmetry, the cup can be thought of as the result of revolving the line pictured about the y-axis. The line is given by $y = 6x - 6$ or $x = \dfrac{y}{6} + 1$. Using disks, we have

$$\pi \int_0^6 \left(\frac{y}{6}+1\right)^2 dy = \pi \int_0^6 \left(\frac{y^2}{36}+\frac{y}{3}+1\right) dy = \pi\left(\frac{y^3}{108}+\frac{y^2}{6}+y\right)\Bigg|_0^6 = \pi(2+6+6) = 14\pi.$$

6.3.67 At a given point x in $[0,4]$, let $A(x)$ equal the area of a cross section of the solid of revolution that is perpendicular to the x-axis. Use the formula $V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x$ to estimate the volume.

Left Riemann sum: $V \approx \pi(1^2+4^2+7^2+10^2) = 166\pi$.
Right Riemann sum: $V \approx \pi(4^2+7^2+10^2+12^2) = 309\pi$.
Midpoint Riemann sum: $V \approx \pi(3^2+5^2+8^2+11^2) = 219\pi$.

6.3.68 The volume is approximately

$$\pi\left((12.6/2)^2+(14.0/2)^2+(16.8/2)^2+(25.2/2)^2+(36.4/2)^2+(42.0/2)^2\right)\left(\frac{50}{6}\right) \approx 28542.7\,\text{cm}^3.$$

6.3.69

a. Think of the cone as being obtained by revolving the region under $y = x$ in the first quadrant from $x = 0$ to $x = R$ around the x-axis. The volume is $\pi \displaystyle\int_0^R x^2\,dx = \pi R^3/3 = \frac{1}{3}V_C$.

b. Think of the hemisphere as being obtained by revolving the region under $y = \sqrt{R^2 - x^2}$ between $x = 0$ and $x = R$ around the x-axis. The volume is $\pi \displaystyle\int_0^R (R^2 - x^2)\,dx = \pi\left(R^2x - x^3/3\right)\Big|_0^R = \frac{2\pi R^3}{3} = \frac{2}{3}V_C$.

6.3.70 $V(h) = \pi \displaystyle\int_{-8}^{-8+h} (\sqrt{64-y^2})^2\,dy = \pi\left(64y - y^3/3\right)\Big|_{-8}^{-8+h} = 8\pi h^2 - \frac{\pi h^3}{3}$. Note that $V(0) = 0$ and $V(8) = \dfrac{2\pi}{3}8^3$.

6.3.71 $V = \pi \displaystyle\int_{-2}^{2} ((3+\sqrt{4-y^2})^2 - (3-\sqrt{4-y^2})^2)\,dy = 24\pi \int_0^2 \sqrt{4-y^2}\,dy = 24\pi(\pi) = 24\pi^2$. Note that the last integral evaluated represents 1/4 the area of a circle of radius 2

6.3.72 Around the x-axis: $V_x = \pi \int_0^1 (x - x^4)\,dx = \pi\left(x^2/2 - x^5/5\right)\Big|_0^1 = \pi(1/2 - 1/5) = \frac{3\pi}{10}$.

Around the line $y = 1$: $V_1 = \pi \int_0^1 ((1-x^2)^2 - (1-\sqrt{x})^2)\,dx = \pi \int_0^1 (x^4 - 2x^2 - x + 2\sqrt{x})\,dx = \pi\left(x^5/5 - 2x^3/3 - x^2/2 + 4x^{3/2}/3\right)\Big|_0^1 = \pi(1/5 - 2/3 - 1/2 + 4/3) = \frac{11\pi}{30}$. Note that $V_1 > V_x$.

6.3.73

a. By the general slicing method, $V(x) = \int_a^b A(x)\,dx$. Because the two figures have the same cross sections $A(x)$, they must therefore have the same volumes.

b. According to Cavelieri's principle, the volume is $V = \pi r^2 h$ (the angle $\pi/4$ is irrelevant).

6.3.74

a.

$$\begin{aligned} V(n) &= \pi \int_0^1 (x^{2/n} - x^{2n})\,dx = \pi \left(\frac{n}{n+2} x^{(n+2)/n} - \frac{1}{2n+1} x^{2n+1} \right) \Big|_0^1 \\ &= \pi \left(\frac{n}{n+2} - \frac{1}{2n+1} \right) = \frac{2\pi(n+1)(n-1)}{(n+2)(2n+1)} = \frac{2\pi(n^2-1)}{(n+2)(2n+1)}. \end{aligned}$$

b. $\lim_{n\to\infty} V(n) = \lim_{n\to\infty} \frac{2n^2-2}{2n^2+5n+2} \cdot \pi = \pi$.

6.4 Volume by Shells

6.4.1 $V = 2\pi \int_a^b x(f(x) - g(x))\,dx$.

6.4.2 . . . revolved about the y axis. . . . using the disk/washer method and integrating with respect to $\underline{y}$ or using the shell method and integrating with respect to $\underline{x}$.

6.4.3 . . . revolved about the x axis. . . . using the disk/washer method and integrating with respect to $\underline{x}$ or using the shell method and integrating with respect to $\underline{y}$.

6.4.4 In order to use washers, one would have to find the inner and outer radii of the washers, which would involve finding inverse functions for $y = x^2 - x^3$ over different portions of the interval $[0, 1]$.

6.4.5

a. The radius is x.

b. The height is $2 - x^2 - x$.

c. The volume is $2\pi \int_0^1 x(2 - x^2 - x)\,dx$.

6.4.6

a. The radius is y.

b. The height is $4 - (2-y)^2 = 4 - (4 - 4y + y^2) = 4y - y^2$.

c. The volume is $2\pi \int_0^2 y(4y - y^2)\,dy$.

6.4.7

a. The radius is $2 - y$.

b. The height is $4 - (2-y)^2 = 4 - (4 - 4y + y^2) = 4y - y^2$.

c. The volume is $2\pi \int_0^2 (2-y)(4y - y^2)\,dy$.

6.4.8

a. The radius is $4 - x$.

b. The height is $2 - (2 - \sqrt{x}) = \sqrt{x}$.

c. The volume is $2\pi \int_0^4 (4 - x)\sqrt{x}\, dx$.

6.4.9 $V = 2\pi \int_0^1 x(x - x^2)\, dx = 2\pi \int_0^1 (x^2 - x^3)\, dx = 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{\pi}{6}$.

6.4.10 $V = 2\pi \int_0^2 \frac{x}{1 + x^2}\, dx = \pi \ln(1 + x^2) \Big|_0^2 = \pi \ln 5$.

6.4.11 $V = 2\pi \int_0^1 x(3 - 3x)\, dx = 2\pi \left(3x^2/2 - x^3 \right) \Big|_0^1 = \pi$.

The volume of a cone is $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 1^2 \cdot 3 = \pi$.

6.4.12 $V = 2\pi \int_1^4 x((-x^2 + 4x + 2) - (x^2 - 6x + 10))\, dx = 2\pi \int_1^4 (-2x^3 + 10x^2 - 8x)\, dx =$
$2\pi \left(-\frac{x^4}{2} + \frac{10x^3}{3} - 4x^2 \right) \Big|_1^4 = 2\pi \left(\left(-128 + \frac{640}{3} - 64 \right) - \left(-\frac{1}{2} + \frac{10}{3} - 4 \right) \right) = 45\pi$.

6.4.13 $V = 2\pi \int_0^2 y(4 - y^2)\, dy = 2\pi \left(2y^2 - y^4/4 \right) \Big|_0^2 = 8\pi$.

6.4.14 $V = 2\pi \int_0^1 y(2 - y - y^2)\, dy = 2\pi \int_0^1 (2y - y^2 - y^3)\, dy = 2\pi \left(y^2 - \frac{y^3}{3} - \frac{y^4}{4} \right) \Big|_0^1 =$
$2\pi \left(1 - \frac{1}{3} - \frac{1}{4} \right) = \frac{5\pi}{6}$.

6.4.15 $V = 2\pi \int_2^4 y(4 - y)\, dy = 2\pi \left(2y^2 - \frac{y^3}{3} \right) \Big|_2^4 = \frac{32\pi}{3}$.

6.4.16 $V = 2\pi \int_1^{\sqrt{3}} y \left(\frac{4}{y + y^3} - \frac{1}{\sqrt{3}} \right) dy = 2\pi \int_1^{\sqrt{3}} \left(\frac{4}{1 + y^2} - \frac{y}{\sqrt{3}} \right) dy = 2\pi \left(4 \tan^{-1} y - \frac{y^2}{2\sqrt{3}} \right) \Big|_1^{\sqrt{3}} =$
$2\pi \left((\frac{4\pi}{3} - \frac{3}{2\sqrt{3}}) - (\pi - \frac{1}{2\sqrt{3}}) \right) = 2\pi \left(\frac{\pi}{3} - \frac{1}{\sqrt{3}} \right) = \frac{2\pi}{3} \left(\pi - \sqrt{3} \right)$.

6.4.17 $V = 2\pi \int_0^{\sqrt{\pi/2}} x \cos x^2\, dx = \pi \left(\sin x^2 \right) \Big|_0^{\sqrt{\pi/2}} = \pi$.

6.4.18 $V = 2\pi \int_2^4 x(6 - x)\, dx = 2\pi \left(3x^2 - \frac{x^3}{3} \right) \Big|_2^4 = \frac{104\pi}{3}$.

6.4.19 $V = 2\pi \int_0^1 x(1 - x^2)\, dx = 2\pi \int_0^1 (x - x^3)\, dx = 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\pi}{2}$.

6.4.20 $V = 2\pi \int_0^1 x\sqrt{x}\, dx = 2\pi \left(\frac{2x^{5/2}}{5} \right) \Big|_0^1 = \frac{4\pi}{5}$.

6.4.21 $V = 2\pi \int_0^3 y(y^2)\,dy = 2\pi \int_0^3 y^3\,dy = 2\pi \left(\frac{y^4}{4}\right)\Big|_0^3 = 81\pi/2.$

6.4.22 $V = 2\pi \int_0^1 y(\sqrt[3]{y})\,dy = 2\pi \left(\frac{3}{7}y^{7/3}\right)\Big|_0^1 = \frac{6\pi}{7}.$

6.4.23 Note that the lines intersect at $(0,0)$, $(2,0)$ and $(1,1)$.
$V = 2\pi \int_0^1 y((2-y)-y)\,dy = 2\pi \int_0^1 (2y-2y^2)\,dy = 2\pi \left(y^2 - \frac{2y^3}{3}\right)\Big|_0^1 = \frac{2\pi}{3}.$

6.4.24 $V = 2\pi \int_0^{\sqrt{2}} x\sqrt{4-2x^2}\,dx$. Let $u = 4-2x^2$, so that $du = -4x\,dx$. Substituting gives

$$-\frac{\pi}{2}\int_4^0 u^{1/2}\,du = -\frac{\pi}{2}\cdot\frac{2}{3}u^{3/2}\Big|_4^0 = \frac{8\pi}{3}.$$

6.4.25 $V = 2\pi \int_1^2 \frac{x}{(x^2+1)^2}\,dx$. Let $u = x^2+1$, so that $du = 2x\,dx$. Substituting gives

$$V = \pi\int_2^5 u^{-2}\,du = -\pi\cdot\frac{1}{u}\Big|_2^5 = -\pi\left(\frac{1}{5}-\frac{1}{2}\right) = \frac{3\pi}{10}.$$

6.4.26 Note that the line $x = 4$ intersects the curve $x = y^2$ at $(4,2)$. $V = 2\pi \int_0^2 y(4-y^2)\,dy = 2\pi \int_0^2 (4y - y^3)\,dy = 2\pi \left(2y^2 - y^4/4\right)\Big|_0^2 = 8\pi.$

6.4.27 $V = 2\pi \int_2^{16} y(2/y)^{2/3}\,dy = 2^{5/3}\pi \int_2^{16} y^{1/3}\,dy = 2^{5/3}\pi\left(3y^{4/3}/4\right)\Big|_2^{16} = 2^{5/3}\pi(3\cdot 2^{10/3} - 3\cdot 2^{-2/3}) = 3\pi(2^5-2) = 90\pi.$

6.4.28 $V = 2\pi \int_0^{\sqrt{\pi/2}} y\sin y^2\,dy = -\pi\cos y^2\Big|_0^{\sqrt{\pi/2}} = -\pi(0-1) = \pi.$

6.4.29 $V = 2\pi \int_1^2 x(e^x/x)\,dx = 2\pi\, e^x\Big|_1^2 = 2\pi(e^2-e).$

6.4.30 $V = 2\pi \int_1^3 \frac{\ln x}{x}\,dx = 2\pi\left(\frac{\ln^2 x}{2}\right)\Big|_1^3 = \pi(\ln 3)^2.$

6.4.31 Note that $y = \sqrt{\cos^{-1}x}$ intersects the axes at $(0, \sqrt{\pi/2})$ and $(1,0)$. $V = 2\pi \int_0^{\sqrt{\pi/2}} y\cos y^2\,dy = \pi\sin y^2\Big|_0^{\sqrt{\pi/2}} = \pi(1-0) = \pi.$

6.4.32

$$V = 2\pi\int_0^{5\sqrt{2}} y\sqrt{(50-y^2)/2}\,dy = \sqrt{2}\pi\int_0^{5\sqrt{2}} y\sqrt{50-y^2}\,dy = \sqrt{2}\pi\cdot\frac{1}{2}\int_0^{50} u^{1/2}\,du,$$

where $u = 50-y^2$. Thus,

$$V = \frac{\sqrt{2}\pi}{2}\left(2u^{3/2}/3\right)\Big|_0^{50} = \frac{\sqrt{2}\pi}{2}\left(2\cdot 50\cdot 5\sqrt{2}/3\right) = \frac{500\pi}{3}.$$

6.4.33 Note that the line $y = 1$ intersects the curve $y = x^3 - x^8 + 1$ at $x = 0$ and $x = 1$. $V = 2\pi \int_0^1 x(x^3 - x^8 + 1 - 1)\, dx = 2\pi \int_0^1 (x^4 - x^9)\, dx = 2\pi \left(\frac{x^5}{5} - \frac{x^{10}}{10} \right) \Big|_0^1 = 2\pi \left(\frac{1}{5} - \frac{1}{10} \right) = \frac{\pi}{5}$.

6.4.34

$$V = 2\pi \int_0^2 y(e^{y^2} - e^{y^2/3})\, dy = \pi \left(e^{y^2} - 3e^{y^2/3} \right) \Big|_0^2 = \pi(e^4 - 3e^{4/3} - (1-3)) = \pi(2 + e^4 - 3e^{4/3}).$$

6.4.35 With washers we have

$$V = \pi \int_0^1 (x^{2/3} - x^2)\, dx = \pi \left(\frac{3x^{5/3}}{5} - \frac{x^3}{3} \right) \Big|_0^1 = \frac{4\pi}{15}.$$

With shells we have

$$V = 2\pi \int_0^1 y(y - y^3)\, dy = 2\pi \left(\frac{y^3}{3} - \frac{y^5}{5} \right) \Big|_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{4\pi}{15}.$$

6.4.36 Using washers we have

$$V = \pi \int_0^2 \left(\left(1 - \frac{x}{3}\right)^2 - \frac{1}{(x+1)^2} \right) dx = \pi \left(x - \frac{x^2}{3} + \frac{x^3}{27} + \frac{1}{(x+1)} \right) \Big|_0^2 = \frac{8\pi}{27}.$$

Using shells we have

$$V = 2\pi \int_{1/3}^1 y \left(3 - 3y - \left(\frac{1}{y} - 1 \right) \right) dy = 2\pi \int_{1/3}^1 (4y - 3y^2 - 1)\, dy = 2\pi \left(2y^2 - y^3 - y \right) \Big|_{1/3}^1$$
$$= 2\pi \left(0 - \left(\frac{2}{9} - \frac{1}{27} - \frac{1}{3} \right) \right) = \frac{8\pi}{27}.$$

6.4.37 Using washers we have

$$V = \pi \int_{-10}^{25} (\sqrt[3]{y+2} + 2)^2\, dy = \pi \int_{-8}^{27} (u^{1/3} + 2)^2\, du = \pi \int_{-8}^{27} (u^{2/3} + 4u^{1/3} + 4)\, du$$
$$= \pi \left(3u^{5/3}/5 + 3u^{4/3} + 4u \right) \Big|_{-8}^{27} = \pi(3^6/5 + 3^5 + 108 - (-96/5 + 48 - 32) = \pi \left(\frac{2484}{5} + \frac{16}{5} \right) = 500\pi.$$

Using shells we have

$$V = 2\pi \int_0^5 x(25 - (x-2)^3 + 2)\, dx = 2\pi \int_0^5 (27x - x(x-2)^3)\, dx = 2\pi \int_0^5 -x^4 + 6x^3 - 12x^2 + 35x\, dx$$
$$= 2\pi \left(-\frac{x^5}{5} + \frac{3x^4}{2} - 4x^3 + \frac{35x^2}{2} \right) \Big|_0^5 = 500\pi$$

6.4.38 Using shells, we have

$$V = 2\pi \int_0^2 x(8 - (2x+2))\, dx = 2\pi \int_0^2 x(6 - 2x)\, dx$$
$$= 2\pi \int_0^2 (6x - 2x^2)\, dx = 2\pi \left(3x^2 - \frac{2x^3}{3} \right) \Big|_0^2 = 2\pi \left(12 - \frac{16}{3} \right) = \frac{40\pi}{3}.$$

Using disks, we have

$$V = \pi \int_2^6 \left(\frac{1}{2}y - 1 \right)^2 dy + \pi \int_6^8 4\, dy = \pi \int_2^6 \left(\frac{y^2}{4} - y + 1 \right) dy + 8\pi$$
$$= \pi \left(\frac{y^3}{12} - \frac{y^2}{2} + y \right) \Big|_2^6 + 8\pi = \pi \left((18 - 18 + 6) - \left(\frac{8}{12} - 2 + 2 \right) \right) + 8\pi$$
$$= \frac{16\pi}{3} + 8\pi = \frac{40\pi}{3}.$$

6.4.39 $V = 2\pi \int_0^1 (x+2)x^2\,dx = 2\pi \int_0^1 (x^3+2x^2)\,dx = \pi \left(\frac{x^4}{4} + \frac{2x^3}{3} \right) \Big|_0^1 = \frac{11\pi}{6}.$

6.4.40 $V = 2\pi \int_0^1 (1-x)x^2\,dx = 2\pi \int_0^1 (x^2-x^3)\,dx = 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = \frac{\pi}{6}.$

6.4.41 $V = 2\pi \int_0^1 (2-x)x^2\,dx = 2\pi \int_0^1 (2x^2-x^3)\,dx = 2\pi \left(\frac{2x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = 2\pi \left(\frac{2}{3} - \frac{1}{4} \right) = \frac{5\pi}{6}.$

6.4.42

$$\begin{aligned} V &= 2\pi \int_0^1 (1-y)(1-\sqrt{y})\,dy = 2\pi \int_0^1 (1-\sqrt{y}-y+y^{3/2})\,dy \\ &= 2\pi \left(y - \frac{2y^{3/2}}{3} - \frac{y^2}{2} + \frac{2y^{5/2}}{5} \right) \Big|_0^1 = 2\pi \left(1 - \frac{2}{3} - \frac{1}{2} + \frac{2}{5} \right) = \frac{7\pi}{15} \end{aligned}$$

6.4.43

$$\begin{aligned} V &= 2\pi \int_0^1 (y+2)(1-\sqrt{y})\,dy = 2\pi \int_0^1 (y+2-y^{3/2}-2y^{1/2})\,dy = 2\pi \left(\frac{y^2}{2} + 2y - \frac{2y^{5/2}/}{5} - \frac{4y^{3/2}}{3} \right) \Big|_0^1 \\ &= 2\pi \left(\frac{1}{2} + 2 - \frac{2}{5} - \frac{4}{3} \right) = \frac{23\pi}{15}. \end{aligned}$$

6.4.44

$$V = 2\pi \int_0^1 (1-\sqrt{y})(2-y)\,dy = 2\pi \int_0^1 (2-y-2y^{1/2}+y^{3/2})\,dy = 2\pi \left(2y - \frac{y^2}{2} - \frac{4y^{3/2}}{3} + \frac{2y^{5/2}}{5} \right) \Big|_0^1 = \frac{17\pi}{15}.$$

6.4.45 Using washers, we have

$$\begin{aligned} V = \pi \int_0^1 (3^2-(x^2+2)^2)\,dx = \pi \int_0^1 (9-x^4-4x^2-4)\,dx &= \pi \int_0^1 (5-x^4-4x^2)\,dx \\ &= \pi \left(5x - \frac{x^5}{5} - \frac{4x^3}{3} \right) \Big|_0^1 = \pi \left(5 = \frac{1}{5} - \frac{4}{3} \right) = \frac{52\pi}{15}. \end{aligned}$$

Using shells, we have

$$\begin{aligned} V =& 2\pi \int_0^1 (y+2)\sqrt{y}\,dy = 2\pi \int_0^1 (y^{3/2}+2y^{1/2})\,dy = 2\pi \left(\frac{2y^{5/2}}{5} + \frac{4y^{3/2}}{3} \right) \Big|_0^1 \\ &= 2\pi \left(\frac{2}{5} + \frac{4}{3} \right) = \frac{52\pi}{15} \end{aligned}$$

6.4.46 Using washers, we have

$$V = \pi \int_0^1 ((\sqrt{y}+1)^2-1^2)\,dy = \pi \int_0^1 (y+2\sqrt{y})\,dy = \pi \left(\frac{y^2}{2} + \frac{4y^{3/2}}{3} \right) \Big|_0^1 = \frac{11\pi}{6}.$$

Using shells, we have

$$\begin{aligned} V &= 2\pi \int_0^1 (x+1)(1-x^2)\,dx = 2\pi \int_0^1 (x-x^3+1-x^2)\,dx = 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4} + x - \frac{x^3}{3} \right) \Big|_0^1 \\ &= 2\pi \left(\frac{1}{2} - \frac{1}{4} + 1 - \frac{1}{3} \right) = \frac{11\pi}{6}. \end{aligned}$$

6.4.47 Using washers, we have

$$V = \pi \int_0^1 ((6-x^2)^2 - 5^2)\,dx = \pi \int_0^1 (x^4 - 12x^2 + 11)\,dx = \pi \left(\frac{x^5}{5} - 4x^3 + 11x \right) \Big|_0^1 = \frac{36\pi}{5}.$$

Using shells, we have

$$V = 2\pi \int_0^2 (6-y)\sqrt{y}\,dy = 2\pi \int_0^1 (6y^{1/2} - y^{3/2})\,dy = 2\pi \left(4y^{3/2} - \frac{2y^{5/2}}{5} \right) \Big|_0^1 = 2\pi \left(4 - \frac{2}{5} \right) = \frac{36\pi}{5}.$$

6.4.48 Using washers, we have

$$V = \pi \int_0^1 (4 - (2-\sqrt{y})^2)\,dy = \pi \int_0^1 (4\sqrt{y} - y)\,dy = \pi \left(\frac{8y^{3/2}}{3} - \frac{y^2}{2} \right) \Big|_0^1 = \frac{13\pi}{6}.$$

Using shells, we have

$$\begin{aligned} V &= 2\pi \int_0^1 (2-x)(1-x^2)\,dx = 2\pi \int_0^1 (2 - 2x^2 - x + x^3)\,dx \\ &= 2\pi \left(2x - \frac{2x^3}{3} - \frac{x^2}{2} + \frac{x^4}{4} \right) \Big|_0^1 = 2\pi \left(2 - \frac{2}{3} - \frac{1}{2} + \frac{1}{4} \right) = \frac{13\pi}{6}. \end{aligned}$$

6.4.49

a. The volume is $V = 2\pi \int_0^r y(2\sqrt{r^2 - y^2})\,dy$. Let $u = r^2 - y^2$, so that $du = -2y\,dy$. Substituting gives

$$2\pi \int_{r^2}^0 -\sqrt{u}\,du = 2\pi \int_0^{r^2} u^{1/2}\,du = 2\pi \frac{2u^{3/2}}{3} \Big|_0^{r^2} = \frac{4}{3}\pi r^3.$$

b. The volume is

$$\begin{aligned} V &= \pi \int_{-r}^r (\sqrt{r^2 - x^2})^2\,dx = \pi \int_{-r}^r (r^2 - x^2)\,dx = \pi \left(r^2 x - \frac{x^3}{3} \right) \Big|_{-r}^r \\ &= \pi \left(r^3 - \frac{r^3}{3} - \left(-r^3 - \frac{(-r)^3}{3} \right) \right) = 2\pi \left(r^3 - \frac{r^3}{3} \right) = \frac{4}{3}\pi r^3. \end{aligned}$$

6.4.50

a. Using washers, the volume is

$$V = \pi \int_{-a}^a b^2 \left(1 - \frac{x^2}{a^2} \right) dx = \pi b^2 \left(x - \frac{x^3}{3a^2} \right) \Big|_{-a}^a = \pi b^2 \left(a - \frac{1}{3}a - \left(-a - \frac{(-a)^3}{3} \right) \right) = \frac{4}{3}\pi a b^2.$$

Using shells, the volume is $V = 2\pi \int_0^b y \left(2a\sqrt{1 - \frac{y^2}{b^2}} \right) dy$. Let $u = 1 - \frac{y^2}{b^2}$. Then $du = \frac{-2y}{b^2} dy$. Substituting gives

$$V = -2\pi a b^2 \int_1^0 \sqrt{u}\,du = 2\pi a b^2 \int_0^1 u^{1/2}\,du = 2\pi a b^2 \left(\frac{2u^{3/2}}{3} \right) \Big|_0^1 = \frac{4}{3}\pi a b^2.$$

b. For the rugby union football, we have $\frac{4}{3}\pi \cdot 6 \cdot (3.82)^2 \approx 366.75\,\text{in}^3$. For the American football, we have $\frac{4}{3}\pi \cdot 5.62 \cdot (3.38)^2 \approx 268.94\,\text{in}^3$.

c. $\dfrac{366.74}{269.94} \approx 1.36.$

6.4.51

a. $V = 2\pi \displaystyle\int_1^5 2x\sqrt{4-(x-3)^2}\,dx.$

b. $V = 2\pi \displaystyle\int_0^2 ((3+\sqrt{4-y^2})^2 - (3-\sqrt{4-y^2})^2)\,dy = 24\pi \int_0^2 \sqrt{4-y^2}\,dy.$

c. Note that the last integral in part b represents $\dfrac{1}{4}$ of the area of the circle of radius 2 centered at the origin, so its value is $\dfrac{1}{4}\pi \cdot 4 = \pi$. So the volume is $24\pi \cdot \pi = 24\pi^2$.

6.4.52

a. $V = \pi \displaystyle\int_0^h (rx/h)^2\,dx = \frac{\pi r^2}{h^2}\left(x^3/3\right)\Big|_0^h = \frac{\pi r^2 h}{3}.$

b. $V = 2\pi \displaystyle\int_0^r y(h-(h/r)y)\,dy = 2\pi h\left(y^2/2 - y^3/(3r)\right)\Big|_0^r = \frac{\pi r^2 h}{3}.$

6.4.53 First, note that the curve and the line intersect at $(0,0)$ and $(1,0)$.

We choose the washer method:

$$\begin{aligned} V &= \pi \int_0^1 (x-x^4)^2\,dx = \pi \int_0^1 (x^2-2x^5+x^8)\,dx \\ &= \pi\left(x^3/3 - x^6/3 + x^9/9\right)\Big|_0^1 = \pi/9 \end{aligned}$$

6.4.54 First, note that the curve and the line intersect at $(0,0)$ and $(1,0)$.
We choose the shell method:

$$\begin{aligned} V &= 2\pi \int_0^1 x(x-x^4)\,dx = 2\pi \int_0^1 (x^2-x^5)\,dx \\ &= 2\pi\left(x^3/3 - x^6/6\right)\Big|_0^1 = 2\pi(1/3-1/6) = \pi/3. \end{aligned}$$

6.4.55 Using washers:
$V = 2\pi \displaystyle\int_0^1 ((2-x^2)^2 - (x^2)^2)\,dx = 2\pi \int_0^1 (4-4x^2)\,dx = 2\pi\left(4x - 4x^3/3\right)\Big|_0^1 = \frac{16\pi}{3}.$

6.4.56 Using washers:

$$\begin{aligned} V &= \pi \int_{\pi/6}^{5\pi/6} (\sin^2 x - (1-\sin x)^2)\,dx = \pi \int_{\pi/6}^{5\pi/6} (2\sin x - 1)\,dx \\ &= -\pi\left(2\cos x + x\right)\Big|_{\pi/6}^{5\pi/6} = 2\sqrt{3}\pi - \frac{2\pi^2}{3}. \end{aligned}$$

6.4.57 Using shells:
$V = 2\pi \displaystyle\int_2^6 x(2x+2-x)\,dx = 2\pi \int_2^6 (x^2+2x)\,dx = 2\pi\left(x^3/3 + x^2\right)\Big|_2^6 = \frac{608\pi}{3}.$

6.4.58 Using disks:
$V = \pi \int_0^2 (x^3)^2\,dx = \pi\left(x^7/7\right)\Big|_0^2 = \frac{128\pi}{7}.$

6.4.59 Using shells:

$$V = 2\pi \int_0^1 y(e^{y^2} - e^{y^2/2})\,dy = 2\pi\left(e^{y^2}/2 - e^{y^2/2}\right)\Big|_0^1 = \pi(e - 2\sqrt{e} - (1-2)) = \pi(\sqrt{e}-1)^2.$$

6.4.60 Using shells:
$V = 2\pi \int_0^6 x(2x+2-2)\,dx = 2\pi\left(2x^3/3\right)\Big|_0^6 = 288\pi.$

6.4.61 The curves intersect when $x^2 = 2 - x$, or $x^2 + x - 2 = (x-1)(x+2) = 0$, so for $x = 1$ and $x = -2$. Only $x = 1$ is in the first quadrant; the corresponding y coordinate is $y = 1$. Also, $2 - x \geq x^2$ on $[0, 1]$.

Using shells, the height of each shell is $2 - x - x^2$, so the volume is

$$V = 2\pi \int_0^1 x(2 - x - x^2)\,dx = 2\pi\left(x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{5\pi}{6}.$$

6.4.62 Using disks:
$V = \pi \int_0^4 (\sqrt{x})^2\,dx = \pi\left(x^2/2\right)\Big|_0^4 = 8\pi.$

6.4.63

a. True. Otherwise, we wouldn't have shells!

b. False. Either method can be used when revolving around either axis.

c. True.

6.4.64 Consider the rectangle in the first quadrant bounded by $x = 2$, $x = 4$, $y = 6$ and $y = 0$. The solid in question is formed by revolving this region around the y-axis. The volume is

$$V = 2\pi \int_2^4 6x\,dx = 2\pi\left(3x^2\right)\Big|_2^4 = 72\pi.$$

6.4.65 Consider the region in the first quadrant bounded by the coordinate axes and the line $y = 8 - (8/3)x$. We can generate the desired cone by revolving this region around the y-axis. We then have

$$V = 2\pi \int_0^3 x(8 - (8/3)x)\,dx = \frac{16\pi}{3}\int_0^3 (3x - x^2)\,dx = \frac{16\pi}{3}\left(3x^2/2 - x^3/3\right)\Big|_0^3 = 24\pi.$$

6.4.66 Consider the region in the first and second quadrants bounded by $x = 3$, the circle $x^2 + y^2 = 36$. The solid in question is formed by revolving this region around the y-axis. The volume is $V = 4\pi \int_3^6 x\sqrt{36 - x^2}\,dx$. This integral can be computing using the substitution $u = 36 - x^2$. We then have

$$V = 2\pi \int_0^{27} u^{1/2}\,du = 2\pi\left(2u^{3/2}/3\right)\Big|_0^{27} = 108\sqrt{3}\pi.$$

6.4.67 Consider the triangle in the first quadrant bounded by $y = 0$, $x = 3$, and $y = 9 - (3/2)x$. The solid in question is formed by revolving this region around the y-axis. The volume is

$$V = 2\pi \int_3^6 x(9 - (3/2)x)\,dx = 2\pi\left(9x^2/2 - x^3/2\right)\Big|_3^6 = 54\pi.$$

6.4.68

$$V = 2\pi \int_r^R x \cdot 6(1 - x^2/R^2)\,dx = 12\pi \left(x^2/2 - x^4/4R^2\right)\Big|_r^R = \frac{3\pi(R^2 - r^2)^2}{R^2}.$$

6.4.69

a. $V_1 = \pi \int_0^1 (ax^2 + 1)^2\,dx = \pi \int_0^1 (a^2x^4 + 2ax^2 + 1)\,dx = \pi(a^2/5 + 2a/3 + 1).$

$V_2 = 2\pi \int_0^1 x(ax^2 + 1)\,dx = 2\pi(a/4 + 1/2) = \pi a/2 + \pi.$

b. These are equal when $\pi a/2 + \pi = \pi a^2/5 + 2a\pi/3 + \pi$, or when $\pi a^2/5 + a\pi/6 = 0$, which occurs when $a = 0$ and when $a = -5/6$.

6.4.70

a. $V = \pi \int_{r-h}^r (r^2 - y^2)\,dy = \pi\left(r^2y - y^3/3\right)\Big|_{r-h}^r = \pi(2r^3/3 - (r^2(r-h) - (r-h)^3/3)) = \frac{\pi h^2}{3}(3r - h).$

b.

$$\begin{aligned} V &= 2\pi \int_0^{\sqrt{2rh-h^2}} x(\sqrt{r^2 - x^2} - (r-h))\,dx = 2\pi\left(\frac{-1}{3}(\sqrt{r^2 - x^2})^3 - (r-h)x^2/2\right)\Bigg|_0^{\sqrt{2rh-h^2}} \\ &= 2\pi\left(\frac{-1}{3}(\sqrt{r^2 - 2rh + h^2})^3 - (r-h)(2rh - h^2)\frac{1}{2} + \frac{1}{3}r^3\right) \\ &= 2\pi\left(-(r-h)^3/3 - r^2h + rh^2/2 + rh^2 - h^3/2 + r^3/3\right) = \pi h^2 r - \frac{\pi h^3}{3}. \end{aligned}$$

c. If we take slices perpendicular to the y-axis, we have circles whose area $A(y) = \pi(r^2 - y^2)$. So $V = \pi \int_{r-h}^r (r^2 - y^2)\,dy$, which is exactly the integral computed in part (a) above.

Note that all three approaches led to the same result, and the result is consistent with other formulas. For example, when $h = 0$ we have no figure, so our volume is 0. When $h = r$, we have $V = \frac{2\pi r^3}{3}$, the volume of a hemisphere.

6.4.71 $V = 2\pi \int_0^2 xf(x^2)\,dx = \pi \int_0^4 f(u)\,du$, where $u = x^2$. Thus, $V = 10\pi$.

6.4.72

a. Consider the region in the first quadrant bounded by the coordinate axes and $y = 8 - 2x$. The integral on the left represents the volume of the solid obtained when this region is revolved around the x-axis, using the disk/washer method, and the integral on the right represents the same volume calculated using the shell method. Hence, they are equal.

b. Consider the region in the first quadrant bounded by the line $x = 0$, the line $y = 5$, and the curve $y = x^2 + 1$. The integral on the left represents $\frac{1}{\pi}$ times the volume of the solid obtained when this region is revolved around the x-axis, using the disk/washer method, and the integral on the right represents $\frac{1}{\pi}$ times the same volume calculated using the shell method. Hence, they are equal.

6.4.73

a. The longest diagonal of the cube is equal to the diameter of the sphere, which is $\sqrt{r^2 + r^2 + r^2} = \sqrt{3}r$. Thus, if R is the radius of the sphere, we must have $R = \frac{\sqrt{3}}{2}r$.

Now analyzing the cone which contains the sphere, we see that the height h of the cone is $3R$, so $h = \frac{3\sqrt{3}}{2}r$.

Now the volume of the cylinder is $V = \pi\left(\frac{3r}{2}\right)^2\left(\frac{3\sqrt{3}r}{2}\right) = \frac{27\sqrt{3}\pi r^3}{8}$.

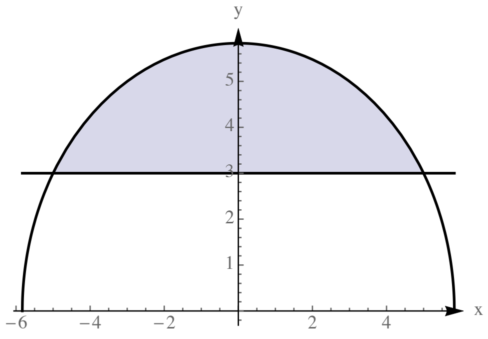
b. Imagine the cone with its vertex up. Consider the plane which contains the vertex of the cone and two non-adjacent vertices of the cube's bottom face. The cross section of this plane with the cone and cube consists of an $r \times \sqrt{2}r$ rectangle (where r is the side length of the cube) and an isosceles triangle of base 2 and height 3, with the rectangle inscribed in the triangle, and the longer side of the rectangle lying on the base of the triangle. Using similar triangles, we have $\frac{r}{3} = \frac{1-(r/\sqrt{2})}{1}$, so $r = \frac{3\sqrt{2}}{3+\sqrt{2}}$, and the volume of the cube is $\frac{54\sqrt{2}}{\left(3+\sqrt{2}\right)^3}$.

c. Imagine the sphere with the hole as being obtained by revolving the region pictured around the x-axis, where the relevant curves are $y = r$ and $y = \sqrt{R^2 - x^2}$ where r is the radius of the hole and R is the radius of the sphere. Note that if you draw the triangle with vertices $(0,0)$, $(5,0)$, and $(5,r)$, you have a right triangle with legs of length 5 and r, and hypotenuse of length R, so $25 + r^2 = R^2$.

The volume we are interested in is

$$V = 2\pi\int_0^5 ((R^2 - x^2) - r^2)\,dx = 2\pi\int_0^5 (25 - x^2)\,dx = 2\pi\left(25x - x^3/3\right)\Big|_0^5 = \frac{500\pi}{3}.$$

Note that (surprisingly!), the result doesn't depend on the radius of the original sphere.



6.4.74 Vertical slices of the wedge are triangles with area $\frac{1}{2}xh$, where x is the base and h is the height. Note that $h = x\tan\theta$, so the triangles have area $\frac{1}{2}x^2\tan\theta$. Now if we think of the curved part of the base of the wedge as having equation $x^2 + y^2 = a^2$, then we have $x^2 = a^2 - y^2$. So the volume of the wedge is given by

$$\int_{-a}^{a} \frac{1}{2}\tan\theta(a^2 - y^2)\,dy = \int_0^a \tan\theta(a^2 - y^2)\,dy = \tan\theta\left(a^2y - y^3/3\right)\Big|_0^a = \frac{2a^3}{3}\tan\theta.$$

6.4.75

a. The cross sections of such an object are washers, with inner radius given b $g(x) - y_0$ and outer radius given by $f(x) - y_0$, so the volume is given by $\pi\int_a^b ((f(x) - y_0)^2 - (g(x) - y_0)^2)\,dx$.

b. The cross section of this object are also washers, and this time the inner radius is $y_0 - f(x)$ and the outer radius is $y_0 - g(x)$, so the volume is given by $\pi\int_a^b ((y_0 - g(x))^2 - (y_0 - f(x))^2)\,dx$.

6.4.76

a. The result of revolving a slice around the line $x = x_0$ is a shell with the same height as before, but with the radius being $\overline{x}_k - x_0$ rather than $\overline{x}_k$. Thus, when the volumes of the shells are added and the limit is taken, the resulting integral is $\int_a^b 2\pi(x - x_0)(f(x) - g(x))\,dx$ instead of $\int_a^b 2\pi(x)(f(x) - g(x))\,dx$.

b. When $x_0 > b$, the radius of a typical shell is given by $x_0 - \overline{x}_k$, so the volume is given by

$$\int_a^b 2\pi(x_0 - x)(f(x) - g(x))\,dx.$$

6.5 Length of Curves

6.5.1 Given $f(x)$ and a and b, compute $f'(x)$ and then compute $\int_a^b \sqrt{1 + f'(x)^2}\,dx$.

6.5.2 Given $g(y)$ and c and d, compute $g'(y)$ and then compute $\int_c^d \sqrt{1 + g'(y)^2}\,dy$.

6.5.3 Because $f'(x) = 3x^2$, the arc length is $\int_{-2}^5 \sqrt{1 + f'(x)^2}\,dx = \int_{-2}^5 \sqrt{1 + 9x^4}\,dx$.

6.5.4 Because $f'(x) = -6\sin 3x$, the arc length is $\int_{-\pi}^{\pi} \sqrt{1 + f'(x)^2}\,dx = \int_{-\pi}^{\pi} \sqrt{1 + 36\sin^2 3x}\,dx$.

6.5.5 Because $f'(x) = -2e^{-2x}$, the arc length is $\int_0^2 \sqrt{1 + f'(x)^2}\,dx = \int_0^2 \sqrt{1 + 4e^{-4x}}\,dx$.

6.5.6 Because $f'(x) = \frac{1}{x}$, the arc length is $\int_1^{10} \sqrt{1 + f'(x)^2}\,dx = \int_1^{10} \sqrt{1 + \frac{1}{x^2}}\,dx = \int_1^{10} \frac{\sqrt{x^2+1}}{x}\,dx$.

6.5.7 $L = \int_1^5 \sqrt{1 + 2^2}\,dx = 4\sqrt{5}$. Using the fact that this is a straight line segment between the points $(1, 3)$ and $(5, 11)$, we can calculate the arc length using the distance formula in the plane:

$$L = \sqrt{(5-1)^2 + (11-3)^2} = \sqrt{16 + 64} = \sqrt{80} = 4\sqrt{5}.$$

6.5.8 $L = \int_{-3}^2 \sqrt{1 + (y')^2} = \int_{-3}^2 \sqrt{1 + (-3)^2}\,dx = \int_{-3}^2 \sqrt{10}\,dx = 5\sqrt{10}$. Using the fact that this is a straight line segment between the points $(-3, 13)$ and $(2, -2)$, we can calculate the arc length using the distance formula in the plane:

$$L = \sqrt{(2-(-3))^2 + (-2-13)^2} = \sqrt{25 + 225} = \sqrt{250} = 5\sqrt{10}.$$

6.5.9 $L = \int_{-2}^6 \sqrt{1 + (y')^2}\,dx = \int_{-2}^6 \sqrt{1 + (-8)^2}\,dx = \int_{-2}^6 \sqrt{65}\,dx = 8\sqrt{65}$.

6.5.10 $y' = \frac{1}{2}(e^x - e^{-x})$, so $1 + (y')^2 = 1 + \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) = \left(\frac{1}{2}(e^x + e^{-x})\right)^2$. Thus,

$$L = \int_{-\ln 2}^{\ln 2} \frac{1}{2}(e^x + e^{-x})\,dx = \frac{1}{2}\left(e^x - e^{-x}\right)\Big|_{-\ln 2}^{\ln 2} = \frac{1}{2}(2 - 1/2 - (1/2 - 2)) = \frac{3}{2}.$$

6.5.11 $y' = \sqrt{x}/2$, so $1 + (y')^2 = 1 + x/4$. Thus,

$$L = \int_0^{60} \sqrt{1 + (x/4)}\, dx = \int_0^{60} \frac{1}{2}\sqrt{x+4}\, dx = \left(\frac{1}{3}(x+4)^{3/2}\right)\Big|_0^{60} = \frac{504}{3} = 168.$$

6.5.12 $y' = 3/x - x/(12)$, so $1 + (y')^2 = 1 + \dfrac{9}{x^2} - \dfrac{1}{2} + \dfrac{x^2}{144} = \left(\dfrac{3}{x} + \dfrac{x}{12}\right)^2$. Thus,

$$L = \int_1^6 \left(\frac{3}{x} + \frac{x}{12}\right) dx = \left(3\ln x + x^2/24\right)\Big|_1^6 = 3\ln 6 + \frac{35}{24}.$$

6.5.13 $y' = x(x^2+2)^{1/2}$, so $1 + (y')^2 = 1 + x^2(x^2+2) = x^4 + 2x^2 + 1 = (x^2+1)^2$. Thus,

$$L = \int_0^1 (x^2+1)\, dx = \left(x^3/3 + x\right)\Big|_0^1 = \frac{4}{3}.$$

6.5.14 $y' = \sqrt{x}/2 - 1/(2\sqrt{x})$, so $1 + (y')^2 = 1 + \dfrac{x}{4} - \dfrac{1}{2} + \dfrac{1}{4x} = \left(\dfrac{\sqrt{x}}{2} + \dfrac{1}{2\sqrt{x}}\right)^2$. Thus,

$$L = \int_4^{16} (x^{1/2}/2 + (1/2)x^{-1/2})\, dx = \left(x^{3/2}/3 + x^{1/2}\right)\Big|_4^{16} = 64/3 + 4 - (8/3 + 2) = 2 + 56/3 = \frac{62}{3}.$$

6.5.15 $y' = x^3 - \frac{1}{4x^3}$, so $1 + (y')^2 = 1 + x^6 - \dfrac{1}{2} + \dfrac{1}{16x^6} = \left(x^3 + \dfrac{1}{4x^3}\right)^2$. Thus,

$$L = \int_1^2 (x^3 + \frac{1}{4x^3})\, dx = \left(x^4/4 + \frac{-1}{8}x^{-2}\right)\Big|_1^2 = 4 - (1/32) - (1/4 - (1/8)) = \frac{123}{32}.$$

6.5.16 $y' = x^{1/2} - (1/(4x^{1/2}))$, so $1 + (y')^2 = 1 + x - 1/2 + \dfrac{1}{16x} = \left(x^{1/2} + \dfrac{1}{4x^{1/2}}\right)^2$. Thus,

$$L = \int_1^9 \left(x^{1/2} + \frac{1}{4x^{1/2}}\right) dx = \left(2x^{3/2}/3 + \sqrt{x}/2\right)\Big|_1^9 = 18 + 3/2 - (2/3 + 1/2) = \frac{55}{3}.$$

6.5.17 $\dfrac{dx}{dy} = y^3 - 1/(4y^3)$, so $1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + y^6 - 1/2 + 1/(16y^6) = \left(y^3 + \dfrac{1}{4y^3}\right)^2$. So

$$L = \int_1^2 (y^3 + \frac{1}{4}\cdot y^{-3})\, dy = \left(y^4/4 - \frac{1}{8y^2}\right)\Big|_1^2 = 4 - (1/32) - (1/4 - 1/8) = \frac{123}{32}.$$

6.5.18 $\dfrac{dx}{dy} = 2\sqrt{2}e^{\sqrt{2}y} - \dfrac{\sqrt{2}}{16}e^{-\sqrt{2}y}$, so

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + 8e^{2\sqrt{2}y} - (1/2) + \frac{2}{16^2}e^{-2\sqrt{2}y} = \left(2\sqrt{2}e^{\sqrt{2}y} + \frac{\sqrt{2}}{16}e^{-\sqrt{2}y}\right)^2.$$

Then

$$L = \int_0^{(\ln 2)/\sqrt{2}} (2\sqrt{2}e^{\sqrt{2}y} + \frac{\sqrt{2}}{16}e^{-\sqrt{2}y})\, dy = \left(2e^{\sqrt{2}y} - \frac{1}{16}e^{-\sqrt{2}y}\right)\Big|_0^{(\ln 2)/\sqrt{2}} = (4 - (1/32) - (2 - (1/16))) = \frac{65}{32}.$$

6.5.19 $\dfrac{dx}{dy} = 2$, so $1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + 4 = 5$, so $L = \displaystyle\int_{-3}^4 \sqrt{5}dy = 7\sqrt{5}$.

6.5.20 $y = \ln(x - \sqrt{x^2-1})$, so $e^y = x - \sqrt{x^2-1}$, and $e^{-y} = \dfrac{1}{x-\sqrt{x^2-1}} = \dfrac{x+\sqrt{x^2-1}}{x^2-(x^2-1)} = x + \sqrt{x^2-1}$. Thus $\dfrac{e^y+e^{-y}}{2} = x$.

We have $\dfrac{dx}{dy} = \dfrac{e^y - e^{-y}}{2}$, so $1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + e^{2y}/4 - 1/2 + e^{-2y}/4 = \left(\dfrac{e^y+e^{-y}}{2}\right)^2$. Thus,

$$L = \int_{\ln(\sqrt{2}-1)}^{0} \frac{e^y+e^{-y}}{2}\,dy = \left(\frac{e^y-e^{-y}}{2}\right)\Bigg|_{\ln(\sqrt{2}-1)}^{0} = 0 - \frac{\sqrt{2}-1-\frac{1}{\sqrt{2}-1}}{2} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{-2+2\sqrt{2}-1+1}{2(\sqrt{2}-1)} = 1.$$

6.5.21

a. $y' = 2x$, so $1 + (y')^2 = 1 + 4x^2$, so $L = \displaystyle\int_{-1}^{1} \sqrt{1+4x^2}\,dx$.

b. $L = \displaystyle\int_{-1}^{1} \sqrt{1+4x^2}\,dx \approx 2.958$.

6.5.22

a. $y' = \cos x$, so $1 + (y')^2 = 1 + \cos^2 x$, so $L = \displaystyle\int_{0}^{\pi} \sqrt{1+\cos^2 x}\,dx$.

b. $L = \displaystyle\int_{0}^{\pi} \sqrt{1+\cos^2 x}\,dx \approx 3.820$.

6.5.23

a. $y' = 1/x$, so $1 + (y')^2 = 1 + (1/x)^2 = \dfrac{x^2+1}{x^2}$, so $L = \displaystyle\int_{1}^{4} \frac{\sqrt{x^2+1}}{x}\,dx$.

b. $L = \displaystyle\int_{1}^{4} \frac{\sqrt{x^2+1}}{x}\,dx \approx 3.343$.

6.5.24

a. $y' = x^2$, so $1 + (y')^2 = 1 + x^4$, so $L = \displaystyle\int_{-1}^{1} \sqrt{1+x^4}\,dx$.

b. $L = \displaystyle\int_{-1}^{1} \sqrt{1+x^4}\,dx \approx 2.179$.

6.5.25

a. $x' = \dfrac{1}{2\sqrt{y-2}}$, so $1 + (x')^2 = 1 + \dfrac{1}{4(y-2)} = \dfrac{4y-7}{4y-8}$, so $L = \displaystyle\int_{3}^{4} \sqrt{\frac{4y-7}{4y-8}}\,dy$.

b. $L = \displaystyle\int_{3}^{4} \sqrt{\frac{4y-7}{4y-8}}\,dy \approx 1.083$.

6.5.26

a. $y' = -\frac{16}{x^3}$, so $1 + (y')^2 = 1 + \dfrac{16^2}{x^6} = \dfrac{x^6+16^2}{x^6}$, so $L = \displaystyle\int_{1}^{4} \sqrt{\frac{x^6+16^2}{x^6}}\,dx$.

b. $L = \displaystyle\int_{1}^{4} \frac{\sqrt{x^6+16^2}}{x^3}\,dx \approx 8.708$.

6.5.27

a. $y' = -2\sin 2x$, so $1 + (y')^2 = 1 + 4\sin^2 2x$, so $L = \displaystyle\int_0^\pi \sqrt{1 + 4\sin^2 2x}\, dx$.

b. $L = \displaystyle\int_0^\pi \sqrt{1 + 4\sin^2 2x}\, dx \approx 5.270$.

6.5.28

a. $y' = 4 - 2x$, so $1 + (y')^2 = 1 + 16 - 16x + 4x^2 = 4x^2 - 16x + 17$, so $L = \displaystyle\int_0^4 \sqrt{4x^2 - 16x + 17}\, dx$.

b. $L = \displaystyle\int_0^4 \sqrt{4x^2 - 16x + 17}\, dx \approx 9.294$.

6.5.29

a. $y' = -1/x^2$, so $1 + (y')^2 = 1 + \dfrac{1}{x^4} = \dfrac{x^4+1}{x^4}$. Thus, $L = \displaystyle\int_1^{10} \frac{\sqrt{x^4+1}}{x^2}\, dx$.

b. $L = \displaystyle\int_1^{10} \frac{\sqrt{x^4+1}}{x^2}\, dx \approx 9.153$.

6.5.30

a. $x' = \dfrac{-2y}{(y^2+1)^2}$, so $1 + (x')^2 = 1 + \dfrac{4y^2}{(y^2+1)^4} = \dfrac{(y^2+1)^4 + 4y^2}{(y^2+1)^4}$, so $L = \displaystyle\int_{-5}^5 \frac{\sqrt{(y^2+1)^4 + 4y^2}}{(y^2+1)^2}\, dy$.

b. $L = \displaystyle\int_{-5}^5 \frac{\sqrt{(y^2+1)^4 + 4y^2}}{(y^2+1)^2}\, dy \approx 10.369$.

6.5.31 $f'(x) = 0.00074x$, so $L = \displaystyle\int_{-640}^{640} \sqrt{1 + (0.00074x)^2}\, dx \approx 1326.4$ meters.

6.5.32 $f'(x) = \dfrac{-630}{239.2}\sinh(x/239.2)$, where $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$.

So

$$L = \int_{-315}^{315} \sqrt{1 + \left(\frac{630}{239.2}\sinh(x/239.2)\right)^2}\, dx \approx 1472.17\,\text{ft}.$$

6.5.33

a. False. For example, if $f(x) = x^2$, the first integrand is $\sqrt{1+4x^2}$ and the second is $1+2x$, which clearly yield different values for (for example) $a = 0$ and $b = 1$.

b. True. They are both equal to $\displaystyle\int_a^b \sqrt{1 + f'(x)^2}\, dx$.

c. False. Because $\sqrt{1 + f'(x)^2} > 0$, arc length can't be negative.

6.5.34 $y' = m$, so $1 + (y')^2 = 1 + m^2$, and $L = \displaystyle\int_a^b \sqrt{1+m^2}\, dx = (b-a)\sqrt{1+m^2}$.

To see that this is the same result as the one given by the distance formula, consider the points $(a, ma+c)$ and $(b, mb+c)$. The distance between them is

$$\sqrt{(b-a)^2 + (mb + c - (ma+c))^2} = \sqrt{(b-a)^2 + m^2(b-a)^2} = \sqrt{(b-a)^2}\sqrt{1+m^2} = (b-a)\sqrt{1+m^2}.$$

6.5.35

a. We are seeking functions $f(x)$ so that $f'(x) = \pm 4x^2$, so any function of the form $f(x) = \pm 4x^3/3 + C$ will work.

b. We are seeking functions $f(x)$ so that $f'(x) = \pm 6\cos(2x)$, so any function of the form $f(x) = \pm 3\sin(2x) + C$ will work.

6.5.36 Because $f'(x) = \pm 4x^{-3}$, we have $f(x) = \dfrac{\pm 2}{x^2} + C$. Because $f(1) = 5$, we must have either $f(x) = 7 - \dfrac{2}{x^2}$ or $f(x) = 3 + \dfrac{2}{x^2}$.

6.5.37 The length of the parabola is $\displaystyle\int_{-1}^{1} \sqrt{1+4x^2}\,dx \approx 2.9597$.

The length of the given cosine function is $\displaystyle\int_{-1}^{1} \sqrt{1 + \frac{\pi^2}{4}\sin^2(\pi x/2)}\,dx \approx 2.924$, so the parabola is longer.

6.5.38 $f'(x) = \sin x$, so the arc length is given by $\displaystyle\int_0^{\pi} \sqrt{1+\sin^2 x}\,dx$.

6.5.39

a. Let $u = 2x$. Then the given integral is equal to $\displaystyle\frac{1}{2}\int_a^b \sqrt{1+f'(u)^2}\,du = \frac{L}{2}$.

b. Let $u = cx$. Then the given integral is equal to $\displaystyle\frac{1}{c}\int_a^b \sqrt{1+f'(u)^2}\,du = \frac{L}{c}$.

6.5.40 If f is odd, then the symmetry of f assures that the portion of the curve from 0 to b matches exactly the portion of the curve from $-b$ to 0, as can be seen by rotating the original curve about the y-axis and then about the x-axis. The same is true for f even, although the two portions match up after just rotating about the y-axis.

Suppose f is even, and recall that this means that f' is odd. Consider

$$L_- = \int_{-b}^{0} \sqrt{1+f'(x)^2}\,dx.$$

Let $u = -x$. The integral becomes

$$L_- = \int_0^b \sqrt{1+f'(-x)^2}\,dx = \int_0^b \sqrt{1+(-f'(x))^2}\,dx = \int_0^b \sqrt{1+f'(x)^2}\,dx = L_+.$$

If f is odd, so that f' is even, we have $L_- = \displaystyle\int_{-b}^{0} \sqrt{1+f'(x)^2}\,dx$. Let $u = -x$. Then the integral becomes

$$L_- = \int_0^b \sqrt{1+f'(-x)^2}\,dx = \int_0^b \sqrt{1+f'(x)^2}\,dx = L_+.$$

6.5.41

a. $f'(x) = Aae^{ax} - \dfrac{1}{4Aa}e^{-ax}$, so $1 + f'(x)^2 = 1 + (Aae^{ax})^2 - \dfrac{1}{2} + \left(\dfrac{1}{4Aa}e^{-ax}\right)^2 = (Aae^{ax})^2 + \dfrac{1}{2} + \left(\dfrac{1}{4Aa}e^{-ax}\right)^2 = \left(Aae^{ax} + \dfrac{1}{4Aa}e^{-ax}\right)^2$. So the arc length for $c \le x \le d$ is

$$L = \int_c^d (Aae^{ax} + \frac{1}{4Aa}e^{-ax})\,dx = \left(Ae^{ax} - \frac{1}{4Aa^2}e^{-ax}\right)\Bigg|_c^d.$$

b. Applying the previous result with $c = 0$ and $d = \ln 2$, we have

$$L = \left(Ae^{ax} - \frac{1}{4Aa^2} e^{-ax} \right) \Big|_0^{\ln 2} = A2^a - \frac{1}{4Aa^2} 2^{-a} - A + \frac{1}{4Aa^2} = A(2^a - 1) - \frac{1}{4a^2 A}(2^{-a} - 1).$$

6.5.42

a. $y' = \dfrac{2n+1}{2n} x^{1/(2n)}$, so $1 + (y')^2 = 1 + \left(\dfrac{2n+1}{2n} \right)^2 x^{1/n}$.

So

$$L = \int_0^a \sqrt{1 + \left(\frac{2n+1}{2n} \right)^2 x^{1/n}}\, dx.$$

b. For the sake of simplicity, let $r = \dfrac{2n+1}{2n}$, and let $d = \sqrt{1 + r^2 a^{1/n}}$. If we let $u^2 = 1 + r^2 x^{1/n}$, then $x = \left(\dfrac{u^2 - 1}{r^2} \right)^n$, and $dx = \dfrac{n}{r^{2n}} (u^2 - 1)^{n-1} \cdot 2u\, du$, and our integral becomes

$$\int_1^d u \cdot \frac{n}{r^{2n}} (u^2 - 1)^{n-1} \cdot 2u\, du = \frac{2n}{r^{2n}} \int_1^d u^2 (u^2 - 1)^{n-1}\, du.$$

c. The binomial theorem assures us that

$$(u^2 - 1)^{n-1} = u^{2n-2} - \binom{n-1}{1} u^{2n-4} + \binom{n-1}{2} u^{2n-6} - \cdots + (-1)^{n-1}.$$

Thus

$$\begin{aligned} u^2 (u^2 - 1)^{n-1} &= u^2 \left(u^{2n-2} - \binom{n-1}{1} u^{2n-4} + \binom{n-1}{2} u^{2n-6} - \cdots + (-1)^{n-1} \right) \\ &= u^{2n} - \binom{n-1}{1} u^{2n-2} + \binom{n-1}{2} u^{2n-4} - \cdots + (-1)^{n-1} u^2. \end{aligned}$$

Then

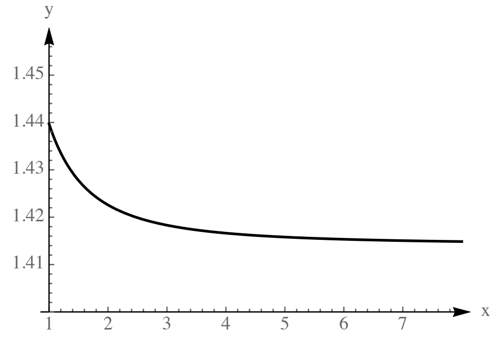
$$\begin{aligned} L = \frac{2n}{r^{2n}} \Bigg(& u^{2n+1}/(2n+1) - \binom{n-1}{1} u^{2n-1}/(2n-1) \\ & + \binom{n-1}{2} u^{2n-3}/(2n-3) - \cdots + (-1)^{n+1} u^3/3 \Bigg) \Bigg|_1^d. \end{aligned}$$

d. For $n = 2$ and $a = 1$ we have $r = (5/4)$ and $d = \sqrt{41}/4$. Using the previous result, we have $L = \dfrac{4^5}{5^4} \left(u^5/5 - u^3/3 \right) \Big|_1^{\sqrt{41}/4} = \dfrac{2048}{9375} + \dfrac{1763\sqrt{41}}{9375} \approx 1.423$.

For $n = 3$ and $a = 1$ we have $r = 7/6$ and $d = \sqrt{85}/6$. Using the previous results, we have

$$L = \frac{6^7}{7^6} \left(u^7/7 - 2u^5/5 + u^3/3 \right) \Big|_1^{\sqrt{85}/6} = \frac{142885\sqrt{85}}{823543} - \frac{746496}{4117715} \approx 1.418.$$

e.



6.6 Surface Area

6.6.1 $S = \pi r\sqrt{r^2+h^2} = 3\pi\sqrt{9+16} = 15\pi$.

6.6.2 By the formula developed in the section: $S = \pi(4)(6^2-2^2)\sqrt{1+4^2} = 128\sqrt{17}\pi$. Or it can be computed via integration:

$$S = 2\pi\int_2^6 4x\sqrt{1+4^2}\,dx = 8\sqrt{17}\pi\,\left(x^2/2\right)\Big|_2^6 = 8\sqrt{17}\pi(18-2) = 128\sqrt{17}\pi.$$

6.6.3 Evaluate $\displaystyle\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$.

6.6.4 Evaluate $\displaystyle\int_c^d 2\pi g(y)\sqrt{1+g'(y)^2}\,dy$.

6.6.5

a. $\displaystyle S = \int_0^2 2\pi(2-x)\sqrt{1+(-1)^2}\,dx = 2\sqrt{2}\pi\int_0^2(2-x)\,dx = 2\sqrt{2}\pi\left(2x-\frac{x^2}{2}\right)\Big|_0^2 = 4\sqrt{2}\pi$.

b. The surface is a cone with $\ell = 2\sqrt{2}$ and $r=2$; using the formula for the lateral surface area of a cone, we have $s = \pi r\ell = 4\sqrt{2}\pi$.

6.6.6

a. $\displaystyle S = \int_c^d 2\pi g(y)\sqrt{1+g'(y)^2}\,dy = \int_0^8 2\pi\cdot 3\sqrt{1+0}\,dy = 6\pi\int_0^8 dy = 48\pi$.

b. This surface is right circular cylinder of radius $r=3$ and height $h=8$; using the formula for the lateral surface area of a cylinder we have $S = 2\pi rh = 2\pi\cdot 3\cdot 8 = 48\pi$.

6.6.7 $\displaystyle S = 2\pi\int_0^6(3x+4)\sqrt{1+9}\,dx = 2\sqrt{10}\pi\,\left(3x^2/2+4x\right)\Big|_0^6 = 2\sqrt{10}\pi(54+24) = 156\sqrt{10}\pi$.

6.6.8 $\displaystyle S = 2\pi\int_1^3(12-3x)\sqrt{1+9}\,dx = 2\sqrt{10}\pi\,\left(12x-3x^2/2\right)\Big|_1^3 = 2\sqrt{10}\pi(36-27/2-(12-3/2)) = 24\sqrt{10}\pi$.

6.6.9

$$\begin{aligned}S &= 2\pi\int_9^{20} 8\sqrt{x}\sqrt{1+(16/x)}\,dx = 16\pi\int_9^{20}\sqrt{x+16}\,dx\\ &= 16\pi\left(\frac{2(x+16)^{3/2}}{3}\right)\Big|_9^{20} = \frac{32\pi}{3}(216-125) = \frac{2912\pi}{3}.\end{aligned}$$

6.6.10 $S = 2\pi \int_0^1 x^3\sqrt{1+9x^4}\,dx$. Let $u = 1+9x^4$ so that $du = 36x^3\,dx$. Substituting gives $S = \frac{\pi}{18}\int_1^{10} \sqrt{u}\,du = \frac{\pi}{18}\left(\frac{2u^{3/2}}{3}\right)\Big|_1^{10} = \frac{\pi}{27}(10\sqrt{10}-1)$.

6.6.11 Note that $x = y^3/3$ for $0 \le y \le 2$. The surface area is $S = 2\pi\int_0^2 (y^3/3)\sqrt{1+y^4}\,dy$. Let $u = 1+y^4$ so that $du = 4y^3\,dy$. Substituting yields $S = 2\pi\int_1^{17} \frac{1}{12}u^{1/2}\,du = \frac{\pi}{9}\left(u^{3/2}\right)\Big|_1^{17} = \frac{\pi}{9}(17\sqrt{17}-1)$.

6.6.12 Note that $x = 2\sqrt{y}$ for $1 \le y \le 4$. The surface area is $S = 2\pi\int_1^4 2\sqrt{y}\sqrt{1+1/y}\,dy =$ $4\pi\int_1^4 \sqrt{1+y}\,dy = \frac{8\pi}{3}\left((1+y)^{3/2}\right)\Big|_1^4 = \frac{8\pi}{3}(5\sqrt{5}-2\sqrt{2})$.

6.6.13

$$\begin{aligned} S &= 2\pi\int_{-1/2}^{1/2} \sqrt{1-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx \\ &= 4\pi\int_0^{1/2} \sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx = 4\pi\int_0^{1/2} \sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\,dx = 4\pi\int_0^{1/2} dx = 2\pi. \end{aligned}$$

6.6.14

$$\begin{aligned} S &= 2\pi\int_3^{3.5} \sqrt{-y^2+6y-8}\sqrt{1+\left(\frac{-2y+6}{2\sqrt{-y^2+6y-8}}\right)^2}\,dy \\ &2\pi\int_3^{3.5} \sqrt{-y^2+6y-8}\sqrt{1+\left(\frac{-y+3}{\sqrt{-y^2+6y-8}}\right)^2}\,dy \\ &= 2\pi\int_3^{3.5} \sqrt{-y^2+6y-8+y^2-6y+9}\,dy = 2\pi\int_3^{3.5} 1\,dy = \pi. \end{aligned}$$

6.6.15 Note that $x = \frac{y+1}{4}$ for $3 \le y \le 15$. The surface area is

$$\begin{aligned} S &= 2\pi\int_3^{15} \frac{y+1}{4}\sqrt{1+\frac{1}{16}}\,dy = \frac{\pi\sqrt{17}}{8}\int_3^{15}(y+1)\,dy \\ &= \frac{\pi\sqrt{17}}{8}\left(y^2/2+y\right)\Big|_3^{15} = \frac{\pi\sqrt{17}}{8}(225/2+15-(9/2+3) = \frac{\pi\sqrt{17}}{8}(120) = 15\sqrt{17}\pi. \end{aligned}$$

6.6.16

$$\begin{aligned} S &= 2\pi\int_0^5 \sqrt{4x+6}(\sqrt{1+(2/\sqrt{4x+6})^2})\,dx = 2\pi\int_0^5 \sqrt{4x+6}\sqrt{\left(\frac{4x+6+4}{4x+6}\right)}\,dx = 2\pi\int_0^5 \sqrt{4x+10}\,dx \\ &= \frac{4\pi}{3}\left((4x+10)^{3/2}/4\right)\Big|_0^5 = \frac{\pi}{3}(30\sqrt{30}-10\sqrt{10}) = \frac{10\sqrt{10}\pi}{3}(3\sqrt{3}-1). \end{aligned}$$

6.6.17

$$S = 2\pi \int_{-2}^{2} (1/4)(e^{2x} + e^{-2x}) \sqrt{1 + \frac{(2e^{2x} - 2e^{-2x})^2}{16}}\, dx = \frac{\pi}{2} \int_{-2}^{2} (e^{2x} + e^{-2x}) \sqrt{\frac{16 + 4e^{4x} - 8 + 4e^{-4x}}{16}}\, dx$$
$$= \frac{\pi}{8} \int_{-2}^{2} (e^{2x} + e^{-2x}) \sqrt{(2e^{2x} + 2e^{-2x})^2}\, dx = \frac{\pi}{8} \int_{-2}^{2} (e^{2x} + e^{-2x})(2e^{2x} + 2e^{-2x})\, dx$$
$$= \frac{\pi}{4} \int_{-2}^{2} (e^{4x} + 2 + e^{-4x})\, dx = \frac{\pi}{4} \left(\frac{e^{4x}}{4} + 2x - \frac{e^{-4x}}{4} \right) \Big|_{-2}^{2}$$
$$= \frac{\pi}{4} \left(\frac{e^8}{4} + 4 - \frac{e^{-8}}{4} - \left(\frac{e^{-8}}{4} - 4 - \frac{e^8}{4} \right) \right) = \frac{\pi}{4} \left(\frac{e^8}{2} + 8 - \frac{e^{-8}}{2} \right).$$

6.6.18

$$S = 2\pi \int_{1}^{4} \sqrt{5x - x^2} \sqrt{1 + \frac{(5 - 2x)^2}{4(5x - x^2)}}\, dx = 2\pi \int_{1}^{4} \sqrt{5x - x^2} \sqrt{\frac{20x - 4x^2 + 25 - 20x + 4x^2}{4(5x - x^2)}}\, dx$$
$$= 2\pi \int_{1}^{4} \sqrt{5x - x^2} \sqrt{\frac{25}{4(5x - x^2)}}\, dx = 2\pi \int_{1}^{4} 5/2\, dx = 5\pi\ (x) \Big|_{1}^{4} = 15\pi$$

6.6.19

$$S = 2\pi \int_{2}^{10} \sqrt{12y - y^2} \sqrt{1 + \frac{(12 - 2y)^2}{4(12y - y^2)}}\, dy = 2\pi \int_{2}^{10} \sqrt{12y - y^2} \sqrt{\frac{48y - 4y^2 + 144 - 48y + 4y^2}{4(12y - y^2)}}\, dy$$
$$= 2\pi \int_{2}^{10} 6\, dy = 12\pi(10 - 2) = 96\pi$$

6.6.20 Note that $x = \sqrt{1 - (y - 1)^2} = \sqrt{1 - (y^2 - 2y + 1)} = \sqrt{2y - y^2}$.

$$S = 2\pi \int_{1}^{3/2} \sqrt{2y - y^2} \sqrt{1 + \frac{(2 - 2y)^2}{4(2y - y^2)}}\, dy = 2\pi \int_{1}^{3/2} \sqrt{2y - y^2} \sqrt{\frac{8y - 4y^2 + 4 - 8y + 4y^2}{4(2y - y^2)}}\, dy$$
$$= 2\pi \int_{1}^{3/2} 1\, dy = 2\pi(3/2 - 1) = \pi.$$

6.6.21 $S = 2\pi \int_{1}^{7} \sqrt{8x - x^2} \sqrt{1 + \frac{(8 - 2x)^2}{4(8x - x^2)}}\, dx = 2\pi \int_{1}^{7} \sqrt{8x - x^2} \sqrt{\frac{32x - 4x^2 + 64 - 32x + 4x^2}{4(8x - x^2)}}\, dx =$ $2\pi \int_{1}^{7} \sqrt{8x - x^2} \sqrt{\frac{64}{4(8x - x^2)}}\, dx = 2\pi \int_{1}^{7} 4\, dx = 8\pi(7 - 1) = 48\pi$. Because the surface area is 48π square meters, the volume of paint required to cover the surface to a thickness of 0.0015 meters is $48\pi(.0015) \approx .226$ cubic meters. This is about 59.75 gallons.

6.6.22

$$S = 2\pi \int_{-8}^{8} \sqrt{100 - x^2} \sqrt{1 + \frac{(-2x)^2}{4(100 - x^2)}}\, dx$$
$$= 2\pi \int_{-8}^{8} \sqrt{100 - x^2} \sqrt{\frac{400 - 4x^2 + 4x^2}{4(100 - x^2)}}\, dx = 2\pi \int_{-8}^{8} 10\, dx = 20\pi(8 - (-8)) = 320\pi.$$

Because the surface area is 320π square meters, the volume of paint required to cover the surface to a thickness of 0.0015 meters is $320\pi(0.0015) \approx 1.5$ cubic meters. This is about 398.36 gallons.

6.6.23

a. False. One would need to find $x = f^{-1}(y)$ and compute the corresponding integral using f^{-1}.

b. False. For example, the curve given in number 14 above isn't one-to-one on the given interval, but the surface is still defined.

c. True. Because the curve is symmetric about the y-axis, the surface generated by revolving half the curve is half the surface generated by revolving the whole curve.

d. False. This curve is symmetric about the y-axis, so the surface generated by revolving the whole curve is the same as the surface generated by revolving the portion over $[0, 4]$.

6.6.24

a. $S = 2\pi \int_0^1 x^5 \sqrt{1 + 25x^8}\, dx$.

b. $S =\approx 3.362$.

6.6.25

a. $S = 2\pi \int_0^{\pi/2} \cos x \sqrt{1 + \sin^2 x}\, dx$.

b. $S \approx 7.21$.

6.6.26

a. $S = 2\pi \int_1^e \ln y \sqrt{1 + \frac{1}{y^2}}\, dy$.

b. $S \approx 7.05$.

6.6.27

a. $S = 2\pi \int_0^{\pi/4} \tan x \sqrt{1 + \sec^4 x}\, dx$.

b. $S \approx 3.84$.

6.6.28

a. $S = 2\pi \int_1^{\sqrt{e}} \ln x^2 \sqrt{1 + 4/x^2}\, dx$.

b. $S \approx 3.845$.

6.6.29 Let $y = f(x) = (a^{2/3} - x^{2/3})^{3/2}$. Note that $f'(x) = -x^{-1/3}\sqrt{a^{2/3} - x^{2/3}}$, so

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{a^{2/3} - x^{2/3}}{x^{2/3}}} = \sqrt{\frac{x^{2/3} + a^{2/3} - x^{2/3}}{x^{2/3}}} = \frac{a^{1/3}}{x^{1/3}}.$$

Thus (using symmetry) $S = 4\pi \int_0^a (a^{2/3} - x^{2/3})^{3/2}(a^{1/3}x^{-1/3})\, dx$. Let $u = (a^{2/3} - x^{2/3})$, so that $du = (-2x^{-1/3}/3)\, dx$. Substituting gives

$$S = -6\pi a^{1/3} \int_{a^{2/3}}^0 u^{3/2}\, du = 6\pi a^{1/3} \int_0^{a^{2/3}} u^{3/2}\, du = \frac{12\pi}{5} a^{1/3} \left(u^{5/2}\right)\Big|_0^{a^{2/3}} = \frac{12\pi a^2}{5}.$$

6.6.30 The surface area of the described cylinder is $2\pi rh$. The surface area of the described cone is $\pi r\sqrt{h^2 + r^2}$. The cone's area is $\frac{2h}{\sqrt{h^2 + r^2}}$ of the cylinder's.

6.6.31

$$\begin{aligned}
S &= 2\pi\int_1^2 \left(x^{3/2} - \frac{\sqrt{x}}{3}\right)\sqrt{1+(3\sqrt{x}/2 - x^{-1/2}/6)^2}\,dx = 2\pi\int_1^2 (x^{3/2}-\sqrt{x}/3)\sqrt{1+\left(\frac{9x-1}{6\sqrt{x}}\right)^2}\,dx \\
&= 2\pi\int_1^2 (x^{3/2}-\sqrt{x}/3)\sqrt{\frac{36x+81x^2-18x+1}{36x}}\,dx = 2\pi\int_1^2 (x^{3/2}-\sqrt{x}/3)\sqrt{\left(\frac{(1+9x)^2}{36x}\right)}\,dx \\
&= 2\pi\int_1^2 \left(\frac{(1+9x)x}{6} - \frac{1+9x}{18}\right)\,dx = \frac{2\pi}{18}\int_1^2 (3x+27x^2-1-9x)\,dx \\
&= \frac{2\pi}{18}\int_1^2 (27x^2-6x-1)\,dx = \frac{\pi}{9}\,\left(9x^3-3x^2-x\right)\,\Big|_1^2 = \frac{\pi}{9}(72-12-2-(9-3-1)) = \frac{53\pi}{9}
\end{aligned}$$

6.6.32

$$\begin{aligned}
S &= 2\pi\int_1^2 (x^4/8+x^{-2}/4)\sqrt{1+(x^3/2-x^{-3}/2)^2}\,dx = 2\pi\int_1^2 (x^4/8+x^{-2}/4)\sqrt{1+x^6/4-1/2+x^{-6}/4}\,dx \\
&= 2\pi\int_1^2 (x^4/8+x^{-2}/4)\sqrt{x^6/4+1/2+x^{-6}/4}\,dx = 2\pi\int_1^2 (x^4/8+x^{-2}/4)\sqrt{(x^3/2+x^{-3}/2)^2}\,dx \\
&= 2\pi\int_1^2 (x^4/8+x^{-2}/4)(x^3/2+x^{-3}/2)\,dx = 2\pi\int_1^2 (x^7/16+x/16+x/8+x^{-5}/8)\,dx \\
&= \frac{\pi}{8}\int_1^2 (x^7+3x+2x^{-5})\,dx = \frac{\pi}{8}\,\left(x^8/8+3x^2/2-x^{-4}/2\right)\,\Big|_1^2 \\
&= \frac{\pi}{8}(32+6-(1/32)-(1/8+3/2-1/2)) = \frac{1179\pi}{256}
\end{aligned}$$

6.6.33

$$\begin{aligned}
S &= 2\pi\int_{1/2}^2 (x^3/3+x^{-1}/4)\sqrt{1+(x^2-x^{-2}/4)^2}\,dx \\
&= 2\pi\int_{1/2}^2 (x^3/3+x^{-1}/4)\sqrt{1+x^4-1/2+x^{-4}/16}\,dx = 2\pi\int_{1/2}^2 (x^3/3+x^{-1}/4)\sqrt{x^4+1/2+x^{-4}/16}\,dx = \\
&= 2\pi\int_{1/2}^2 (x^3/3+x^{-1}/4)\sqrt{(x^2+x^{-2}/4)^2}\,dx = 2\pi\int_{1/2}^2 (x^3/3+x^{-1}/4)(x^2+x^{-2}/4)\,dx \\
&= 2\pi\int_{1/2}^2 (x^5/3+x/12+x/4+x^{-3}/16)\,dx = 2\pi\int_{1/2}^2 (x^5/3+x/3+x^{-3}/16)\,dx \\
&= 2\pi\,\left(x^6/18+x^2/6-x^{-2}/32\right)\,\Big|_{1/2}^2 = 2\pi(32/9+2/3-1/128-(1/1152+1/24-1/8)) = \frac{275\pi}{32}
\end{aligned}$$

6.6.34 We can rewrite the given curve as $2x+\sqrt{4x^2-1} = e^{2y}$. Squaring yields $4x^2+4x\sqrt{4x^2-1}+4x^2-1 = e^{4y}$. So $e^{4y}+1 = 8x^2+4x\sqrt{4x^2-1} = 4x(2x+\sqrt{4x^2-1}) = 4x(e^{2y})$. Thus

$$x = \frac{e^{4y}+1}{4e^{2y}} = \frac{e^{2y}+e^{-2y}}{4}.$$

The portion of the curve in question is for $0 \le y \le \ln 2$.

The surface area is

$$\begin{aligned}
S &= 2\pi\int_0^{\ln 2} \frac{e^{2y}+e^{-2y}}{4}\sqrt{1+(e^{2y}/2-e^{-2y}/2)^2}\,dy = 2\pi\int_0^{\ln 2} \frac{e^{2y}+e^{-2y}}{4}\sqrt{e^{4y}/4+1/2+e^{-2y}/4}\,dy \\
&= 2\pi\int_0^{\ln 2} \frac{e^{2y}+e^{-2y}}{4}\,\frac{\sqrt{(e^{2y}+e^{-2y})^2}}{2}\,dy = \frac{\pi}{4}\int_0^{\ln 2} (e^{4y}+2+e^{-4y})\,dy = \frac{\pi}{4}\,\left(e^{4y}/4+2y-e^{-4y}/4\right)\,\Big|_0^{\ln 2} \\
&= \frac{\pi}{4}(4+2\ln 2-1/64-(1/4+0-1/4)) = \frac{\pi}{4}\left(\frac{255}{64}+2\ln 2\right)
\end{aligned}$$

6.6.35

$$\begin{aligned}
S &= 2\pi\int_1^4 (4y^{3/2}-y^{1/2}/12)\sqrt{1+(6y^{1/2}-y^{-1/2}/24)^2}\,dy\\
&= 2\pi\int_1^4 (4y^{3/2}-y^{1/2}/12)\sqrt{1+36y-1/2+1/(576y)}\,dy\\
&= 2\pi\int_1^4 (4y^{3/2}-y^{1/2}/12)\sqrt{36y+1/2+1/(576y)}\,dy\\
&= 2\pi\int_1^4 (4y^{3/2}-y^{1/2}/12)\sqrt{(6y^{1/2}+y^{-1/2}/24)^2}\,dy\\
&= 2\pi\int_1^4 (4y^{3/2}-y^{1/2}/12)(6y^{1/2}+y^{-1/2}/24)\,dy\\
&= 2\pi\int_1^4 (24y^2-y/3-1/288)\,dy\\
&= 2\pi\left(8y^3-y^2/6-y/288\right)\Big|_1^4 = 2\pi\left(512-\frac{8}{3}-\frac{1}{72}\left(8-\frac{1}{6}-\frac{1}{288}\right)\right)=\frac{48143\pi}{48}.
\end{aligned}$$

6.6.36 Let $y=f(x)=a+\sqrt{r^2-x^2}$. Revolving this curve around the x-axis will give a portion of the surface of the torus. The other comes from revolving $y=g(x)=a-\sqrt{r^2-x^2}$ around the x-axis. We will compute the surface area for the first portion first.

Note that

$$\sqrt{1+f'(x)^2}=\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}=\sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}=\frac{r}{\sqrt{r^2-x^2}}.$$

So (using symmetry) the surface area for this portion of the torus is

$$\begin{aligned}
&4\pi\int_0^r (a+\sqrt{r^2-x^2})\left(\frac{r}{\sqrt{r^2-x^2}}\right)dx = 4\pi ar\int_0^r\frac{1}{\sqrt{r^2-x^2}}\,dx+4\pi r\int_0^r 1\,dx\\
&= 4\pi ar\left(\sin^{-1}(x/r)\right)\Big|_0^r+4\pi r^2=4\pi ar\,(\pi/2-0)+4\pi r^2=2\pi^2ar+4\pi r^2.
\end{aligned}$$

If we revolve $y=g(x)=a-\sqrt{r^2-x^2}$ around the x-axis, the surface area will be $2\pi^2ar-4\pi r^2$ by a very similar calculation. Thus, the total surface area of the torus is

$$S=2\pi^2ar+4\pi r^2+2\pi^2ar-4\pi r^2=4\pi^2ar.$$

6.6.37 By symmetry, we can let $y=f(x)=\sqrt{r^2-x^2}$, and imagine the surface obtained by revolving this curve around the x-axis for $a\le x\le a+h$. The surface area is

$$S=2\pi\int_a^{a+h}\sqrt{r^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx.$$

This can be written as

$$\begin{aligned}
&2\pi\int_a^{a+h}\sqrt{r^2-x^2}\sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\,dx\\
&=2\pi\int_a^{a+h}\sqrt{r^2-x^2}\frac{\sqrt{r^2}}{\sqrt{r^2-x^2}}\,dx=2\pi\int_a^{a+h}r\,dx=2\pi r(a+h-a)=2\pi rh.
\end{aligned}$$

6.6.38

a. Let $y=f(x)=\frac{b}{a}\sqrt{a^2-x^2}$. Then $f'(x)=\frac{b}{a}\cdot\frac{-x}{\sqrt{a^2-x^2}}$, so

$$\sqrt{1+f'(x)^2}=\sqrt{1+\frac{b^2}{a^2}\cdot\frac{x^2}{a^2-x^2}}=\sqrt{\frac{a^4-a^2x^2+b^2x^2}{a^2}}\cdot\frac{1}{\sqrt{a^2-x^2}}.$$

Thus $f(x)\sqrt{1+f'(x)^2}$ is

$$\frac{b}{a}\sqrt{a^2-x^2}\sqrt{\frac{a^4-a^2x^2+b^2x^2}{a^2}}\cdot\frac{1}{\sqrt{a^2-x^2}}=\frac{b}{a}\sqrt{a^2-\frac{(a^2-b^2)x^2}{a^2}}=\frac{b}{a}\sqrt{a^2-\left(1-\frac{b^2}{a^2}\right)x^2}.$$

Letting $c^2=1-\dfrac{b^2}{a^2}$ and using symmetry gives

$$S=\frac{4\pi b}{a}\int_0^a\sqrt{a^2-c^2x^2}\,dx.$$

b. Let $u=cx$, so that $du=c\,dx$, Note that when $x=0$, u is also 0, and when $x=a$, we have $u=ca=\sqrt{1-\dfrac{b^2}{a^2}}\sqrt{a^2}=\sqrt{a^2-b^2}$. Thus,

$$S=\frac{4\pi b}{ac}\int_0^{\sqrt{a^2-b^2}}\sqrt{a^2-u^2}\,du=\frac{4\pi b}{\sqrt{a^2-b^2}}\int_0^{\sqrt{a^2-b^2}}\sqrt{a^2-u^2}\,du.$$

c. S is

$$\frac{2\pi b}{\sqrt{a^2-b^2}}\left(u\sqrt{a^2-u^2}+a^2\sin^{-1}(u/a)\right)\Big|_0^{\sqrt{a^2-b^2}}=\frac{2\pi b}{\sqrt{a^2-b^2}}\left(b\sqrt{a^2-b^2}+a^2\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right).$$

Multiplying through by the factor of $\sqrt{a^2-b^2}$ in the denominator gives

$$S=2\pi b\left(b+\frac{a^2}{\sqrt{a^2-b^2}}\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right).$$

d. S would be measured in square meters.

e. Returning to our work in the first part of this problem, if $a=b$ then $f(x)\sqrt{1+f'(x)^2}=\frac{b}{a}\sqrt{a^2-0}=b$. So the surface area is $S=4\pi\displaystyle\int_0^a b\,dx=4\pi b(a-0)=4\pi ab=4\pi a^2$.

6.6.39

a. Using R for the radius of the sphere, The ratio is $\dfrac{6R^2}{R^3}=\dfrac{6}{R}$.

b. The ratio is $\dfrac{4\pi R^2}{(4/3)\pi R^3}=\dfrac{3}{R}$.

c. We have an ellipsoid whose axes have lengths $2R$, R, and R, where $R=\ell\sqrt[3]{4}$. The volume of the ellipsoid is

$$V=\frac{4\pi}{3}\cdot R\cdot\frac{R}{2}\cdot\frac{R}{2}=\frac{\pi R^3}{3}=\frac{4\pi\ell^3}{3}.$$

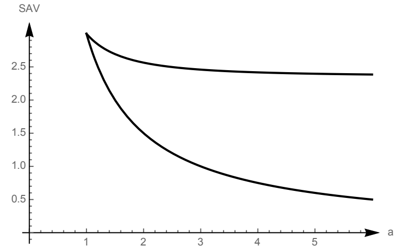
The surface area is

$$\begin{aligned}S&=2\pi\cdot\frac{R}{2}\left(\frac{R}{2}+\frac{R^2}{\sqrt{R^2-(R^2/4)}}\sin^{-1}\frac{\sqrt{R^2-(R^2/4)}}{R}\right)\\&=\pi R\left(\frac{R}{2}+\frac{2R^2}{\sqrt{3}R}\sin^{-1}\left(\frac{\sqrt{3}R}{2R}\right)\right)\\&=\pi\ell\sqrt[3]{4}\left(\frac{\ell\sqrt[3]{4}}{2}+\frac{2\ell\sqrt[3]{4}}{\sqrt{3}}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\\&=\pi\ell\sqrt[3]{4}\left(\frac{\ell\sqrt[3]{4}}{2}+\frac{2\ell\sqrt[3]{4}}{\sqrt{3}}\cdot\frac{\pi}{3}\right)\\&=\pi\ell^2 4^{2/3}\left(\frac{1}{2}+\frac{2\pi}{3\sqrt{3}}\right).\end{aligned}$$

The SAV ratio is therefore

$$\frac{3\pi\ell^2 4^{2/3}\left(\frac{1}{2}+\frac{2\pi}{3\sqrt{3}}\right)}{4\pi\ell^3}=\frac{3(3\sqrt{3}+4\pi)}{4^{1/3}\ell 6\sqrt{3}}=\frac{9+4\sqrt{3}\pi}{6\cdot 4^{1/3}\ell}.$$

d.



The lower curve is for the sphere and the upper for the ellipsoid.

e. More spherical is better for minimizing heat loss.

6.6.40 Let $c=\dfrac{g(b)-g(a)}{b-a}$.

The slant height ℓ can be written as $\sqrt{(g(b)-g(a))^2+(b-a)^2}$, so the quantity $\pi(g(b)+g(a))l$ can be written as

$$\frac{b-a}{\sqrt{(b-a)^2}}\pi(g(b)+g(a))\sqrt{(g(b)-g(a))^2+(b-a)^2}=(b-a)\pi(g(b)+g(a))\sqrt{c^2+1}.$$

We will show that the surface area of the frustrum is this quantity.

The line between $(a,g(a))$ and $(b,g(b))$ has slope $\dfrac{g(b)-g(a)}{b-a}=c$. Then the equation of the line is $y=f(x)=g(a)+c(x-a)$. The surface area is

$$\begin{aligned}S&=2\pi\int_a^b(g(a)+c(x-a))\sqrt{1+c^2}\,dx=2\pi\sqrt{1+c^2}\int_a^b(g(a)+c(x-a))\,dx\\ &2\pi\sqrt{1+c^2}\left(g(a)x+\frac{c(x-a)^2}{2}\right)\Big|_a^b=2\pi\sqrt{1+c^2}\left(g(a)(b-a)+\frac{c(b-a)^2}{2}\right)\\ &=\pi\sqrt{1+c^2}\,(2g(a)(b-a)+(g(b)-g(a))(b-a))=\pi\sqrt{1+c^2}(b-a)(2g(a)+g(b)-g(a))\\ &=\pi\sqrt{1+c^2}(b-a)(g(a)+g(b))\end{aligned}$$

as desired.

6.6.41

a. Note that $g(x)=cf(x)$, so that $g'(x)=cf'(x)$.

$$2\pi\int_a^b g(x)\sqrt{c^2+g'(x)^2}\,dx=2\pi\int_a^b cf(x)\sqrt{c^2+c^2f'(x)^2}\,dx=2c^2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx=c^2A.$$

b. Note that $h'(x)=cf'(cx)$. Consider

$$2\pi\int_{a/c}^{b/c}f(cx)\sqrt{c^2+c^2f'(cx)^2}\,dx=2c\pi\int_{a/c}^{b/c}f(cx)\sqrt{1+f'(cx)^2}\,dx.$$

Let $u=cx$ so that $du=c\,dx$. Then our integral is equal to

$$2\pi\int_a^b f(u)\sqrt{1+f'(u)^2}\,du=A.$$

6.6.42 The area obtained by revolving $f(x)+C$ around the x-axis is

$$2\pi\int_a^b (f(x)+C)\sqrt{1+f'(x)^2}\,dx = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx + 2\pi C\int_a^b \sqrt{1+f'(x)^2}\,dx = S + 2\pi CL,$$

as desired.

6.7 Physical Applications

6.7.1 $m = \rho_1 \cdot l_1 + \rho_2 \cdot l_2 = 1\,\text{g/cm}\cdot 50\,\text{cm} + 2\,\text{g/cm}\cdot 50\,\text{cm} = 150$ g.

6.7.2 The mass is given by $m = \displaystyle\int_a^b \rho(x)\,dx$.

6.7.3 The work is the product of 5 Newtons and 5 meters, which is 25 J.

6.7.4 If the force is not constant, the interval must be divided up into pieces, and on each small piece the work can be approximated by assuming a constant force. These approximations are then added up and then the sum is refined through a limiting process, which leads to a definite integral.

6.7.5 Different volumes of water are moved different distances.

6.7.6 Different parts of the dam have different depths and thus different amount of pressure.

6.7.7 $F = \rho g h = 1000 \cdot 9.8 \cdot 4 = 39,200\ \text{N/m}^2$.

6.7.8 Along a thin horizontal strip, the pressure is the same, because the depth is constant.

6.7.9 $W = \displaystyle\int_5^{10} 25\pi\rho g(15-y)\,dy$.

6.7.10 $W = \displaystyle\int_0^{5} 25\pi\rho g(15-y)\,dy$.

6.7.11 $W = \displaystyle\int_0^{10} 25\pi\rho g(10-y)\,dy$.

6.7.12 $W = \displaystyle\int_0^{3} 25\pi\rho g(10-y)\,dy$.

6.7.13 $m = \displaystyle\int_0^{\pi} (1+\sin x)\,dx = (x-\cos x)\Big|_0^{\pi} = \pi-(-1)-(0-1) = \pi+2$.

6.7.14 $m = \displaystyle\int_0^{1} (1+x^3)\,dx = \left(x+x^4/4\right)\Big|_0^{1} = \frac{5}{4}$.

6.7.15 $m = \displaystyle\int_0^{2} (2-x/2)\,dx = \left(2x-x^2/4\right)\Big|_0^{2} = 3$.

6.7.16 $m = \displaystyle\int_0^{4} 5e^{-2x}\,dx = \left((-5/2)e^{-2x}\right)\Big|_0^{4} = \frac{5}{2}(1-e^{-8})$.

6.7.17 $m = \displaystyle\int_0^{1} x\sqrt{2-x^2}\,dx = \left(\frac{-1}{3}(2-x^2)^{3/2}\right)\Big|_0^{1} = (-1/3)+(2\sqrt{2}/3) = \frac{2\sqrt{2}-1}{3}$.

6.7.18 $m = \int_0^2 1\,dx + \int_2^3 2\,dx = 2 + 2 = 4.$

6.7.19 $m = \int_0^2 1\,dx + \int_2^4 (1+x)\,dx = 2 + \left(x + x^2/2\right)\Big|_2^4 = 2 + (4+8) - (2+2) = 10.$

6.7.20 $m = \int_0^1 x^2\,dx + \int_1^2 x(2-x)\,dx = \left(x^3/3\right)\Big|_0^1 + \left(x^2 - x^3/3\right)\Big|_1^2 = \frac{1}{3} + 4 - \frac{8}{3} - \left(1 - \frac{1}{3}\right) = 1.$

6.7.21 $W = \int_0^3 2x\,dx = x^2\Big|_0^3 = 9$ J.

6.7.22 $W = \int_1^3 2/x^2\,dx = (-2/x)\Big|_1^3 = -\frac{2}{3} - (-2) = \frac{4}{3}$ J.

6.7.23

a. Because $f(0.2) = 0.2k = 30$, we have $k = 150$.

b. $W = \int_0^{-0.4} 150x\,dx = 75x^2\Big|_0^{-0.4} = 75 \cdot 0.16 = 12$ J.

c. $W = \int_0^{0.3} 150x\,dx = 75x^2\Big|_0^{0.3} = 75 \cdot 0.09 = 6.75$ J.

d. $W = \int_{0.2}^{0.4} 150x\,dx = 75x^2\Big|_{0.2}^{0.4} = 75(0.16 - 0.04) = 9$ J.

6.7.24

a. $f(0.25) = 0.25k = 15$, so $k = 60$.

b. $W = \int_0^{-0.2} 60x\,dx = 30x^2\Big|_0^{-0.2} = 30 \cdot 0.04 = 1.2$ J.

c. $W = \int_{0.25}^{0.55} 60x\,dx = 30x^2\Big|_{0.25}^{0.55} = 30(0.3025 - 0.0625) = 30 \cdot 0.24 = 7.2$ J.

6.7.25

a. $f(x) = kx$, and $f(0.5) = 50$, so $k(0.5) = 50$, so $k = 100$.

Therefore $W = \int_0^{1.5} 100x\,dx = \left(50x^2\right)\Big|_0^{1.5} = 112.5$ J.

b. $\int_0^{-0.5} 100x\,dx = \left(50x^2\right)\Big|_0^{-0.5} = 12.5$ J.

6.7.26 $f(x) = kx$, and $f(0.02) = 0.02k = 500 \cdot 9.8 = 4900$, so $k = 245000$.
$W = \int_0^{0.04} 245000x\,dx = \left(122500x^2\right)\Big|_0^{0.04} = 196$ J.

6.7.27

a. $f(0.2) = 0.2k = 50$, so $k = 250$. $W = \int_0^{0.5} 250x\,dx = 125x^2\Big|_0^{0.5} = 125 \cdot 0.25 = 31.25$ J.

b. $\int_0^{0.2} kx\,dx = kx^2/2 \Big|_0^{0.2} = 0.02k = 50$, so $k = 2500$. $W = \displaystyle\int_0^{0.5} 2500x\,dx = 1250x^2 \Big|_0^{0.5} = 1250 \cdot 0.25 =$ 312.5 J.

6.7.28

a. $f(0.1) = 0.1k = 50$, so $k = 500$. $W = \displaystyle\int_0^{0.4} 500x\,dx = 250x^2 \Big|_0^{0.4} = 250 \cdot 0.16 = 40$ J.

b. $\int_0^{0.1} kx\,dx = kx^2/2 \Big|_0^{0.1} = 0.005k = 2$, so $k = 400$. $W = \displaystyle\int_0^{0.4} 400x\,dx = 200x^2 \Big|_0^{0.4} = 200 \cdot 0.16 = 32$ J.

6.7.29

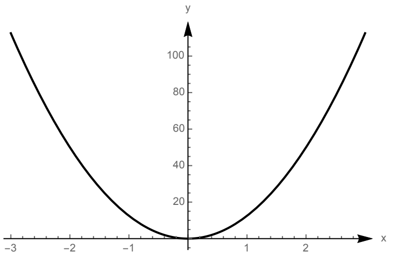
a. We have $\displaystyle\int_0^{0.5} kx\,dx = \frac{k}{2}x^2\Big|_0^{0.5} = 0.125k = 100$, so that $k = 800$. Then

$$W = \int_0^{1.25} 800x\,dx = 400x^2\Big|_0^{1.25} = 625\,\text{J}.$$

b. $f(0.5) = 0.5k = 250$, so $k = 500$ and thus

$$W = \int_0^{1.25} 500x\,dx = 250x^2\Big|_0^{1.25} = 390.625\,\text{J}.$$

6.7.30 $W(x) = \displaystyle\int_0^x 25t\,dt = \left(25t^2/2\right)\Big|_0^x = 25x^2/2$. Note that W is an even function, so that $W(-x) = W(x)$, and thus the work is the same to compress or stretch the spring a given distance from its equilibrium position.



6.7.31

a. $W_1 = \displaystyle\int_0^{30} \rho g(30-y)\,dy = 5g\left(30y - y^2/2\right)\Big|_0^{30} = 2250g = 2250 \cdot 9.8 = 22050$ J.

b. $W = W_1 + W_2$, where W_2 is the work to just lift the block. $W_2 = 50g \cdot 30 = 1500g$, so $W = 2250g + 1500g = 3750g = 3750 \cdot 9.8 = 36750$ J.

6.7.32 $W = \displaystyle\int_0^{60} \frac{55}{1000}g(60-y)\,dy = \frac{55g}{1000}\left(60y - y^2/2\right)\Big|_0^{60} = 99g = 99 \cdot 9.8 = 970.2$ J.

6.7.33 The bottom half of the chain weighs 25 kg, so this problem is equivalent to determining the sum of the work done by the winch in winding up a 25-kg, 10-m chain and lifting a 25-kg mass a distance of 10 m. So

$$W = \int_{10}^{20} 2.5(9.8)(20-y)\,dy + 25(9.8)(10) = 2450 + 24.5\int_{10}^{20}(20-y)\,dy = 2450 + 24.5\left(20y - \frac{y^2}{2}\right)\Big|_{10}^{20}$$
$$= 2450 + 24.5\,(400 - 200 - (200 - 50)) = 2450 + 1225 = 3675\,\text{J}.$$

6.7.34 $F(x) = 9.8\left(4 - \frac{1}{5}y\right)$, so

$$W = \int_0^{10} 9.8\left(4 - \frac{1}{5}y\right)\,dy = 9.8\left(4y - \frac{y^2}{10}\right)\Big|_0^{10} = 9.8(40-10) = 294\,\text{J}.$$

6.7.35 $W = \int_0^{2.5} \rho g A(y)(2.5-y)\,dy = 1000\cdot 9.8\cdot 25\cdot 15\int_0^{2.5}(2.5-y)\,dy = 3675000\left(2.5y - \frac{y^2}{2}\right)\Big|_0^{2.5} = 3675000\cdot\frac{(2.5)^2}{2} = 11,484,375$ J.

6.7.36

a. $W = \int_0^8 \rho g\pi\cdot 2^2(8-y)\,dy = 4\pi\rho g\left(8y - y^2/2\right)\Big|_0^8 \approx 3.941\times 10^6$ J.

b. Not true. For pumping half the water from a full tank the work is

$$\int_4^8 4\rho\pi g(8-y)\,dy = 4\pi\rho g\left(8y - y^2/2\right)\Big|_4^8 = 32\pi\rho g \approx 985203\,\text{J}.$$

To empty a half-full tank, the work is

$$\int_0^4 4\rho\pi g(8-y)\,dy = 4\pi\rho g\left(8y - y^2/2\right)\Big|_0^4 = 96\pi\rho g \approx 2.95561\times 10^6\,\text{J}.$$

6.7.37 $W = \int_0^4 \rho\pi g 2^2(10-y)\,dy = 4\pi\rho g\left(10y - \frac{y^2}{2}\right)\Big|_0^4 = 4\pi\rho g(40-8) = 128\pi\rho g \approx 3.941\times 10^6$ J.

6.7.38 $W = \int_0^2 \rho g A(y)(3-y)\,dy = 1000\cdot 9.8\cdot 25\cdot 15\int_0^2(3-y)\,dy = 3675000\left(3y - \frac{y^2}{2}\right)\Big|_0^2 = (3675000)\cdot 4 = 1.47\times 10^7$ J.

6.7.39

a. Let the vertex of the cone be at $(0,0)$, with the y-axis vertically oriented. Note that the area of a horizontal slice at height y is $\pi y^2/16$, and it must move $6-y$ meters to get to the top.

$$W = \int_0^6 \rho g\pi\frac{y^2}{16}(6-y)\,dy = \frac{\pi\rho g}{16}\int_0^6(6y^2 - y^3)\,dy = \frac{\pi\rho g}{16}\left(2y^3 - \frac{y^4}{4}\right)\Big|_0^6 = \frac{\pi\rho g}{16}(108) = 66,150\pi\,\text{J}.$$

b. Not true.

$$\int_0^3 \rho g\pi\frac{y^2}{16}(6-y)\,dy = \frac{\pi\rho g}{16}\int_0^3(6y^2 - y^3)\,dy = \frac{\pi\rho g}{16}\left(2y^3 - \frac{y^4}{4}\right)\Big|_0^3 = \frac{\pi\rho g}{16}\cdot 33.75 \approx 20672\pi\,\text{J},$$

less than half the amount from part (a). Note that while the water must be raised further than water in the top half, due to the shape of the tank, there is far less water in the bottom half than in the top.

6.7.40

a. $W = \pi\rho g\int_4^8 9(10-y)\,dy = \pi\rho g\left(90y - 9y^2/2\right)\Big|_4^8 = (720-288-360+72)\pi\rho g = 144\pi\rho g \approx 4.433\times 10^6$ J.

b. $W = \pi\rho g\int_0^4 9(10-y)\,dy = \pi\rho g\left(90y - 9y^2/2\right)\Big|_0^4 = (360-72)\pi\rho g = 288\pi\rho g \approx 4.43342\times 10^6 \approx 8.867\times 10^6$ J.

c. The water in the lower part of the tank has to travel farther, so it requires more work to pump it out.

6.7.41 A vertical cross section of the tank that passes through the center of the tank is a circle of radius 8. If we place the origin of the xy-plane at the center of this circle, then we have

$$\begin{aligned}
W &= \int_{-8}^{8} 1000(9.8)(64-y^2)\pi(10+y)\,dy = 9800\pi\int_{-8}^{8}(640+64y-10y^2-y^3)\,dy\\
&= 9800\pi\left(640y+32y^2-\frac{10y^3}{3}-\frac{y^4}{4}\right)\Big|_{-8}^{8}\\
&= 9800\pi\left(5120+2048-\frac{5120}{3}-1024-\left(-5120+2048+\frac{5120}{3}-1024\right)\right)\\
&= 9800\pi\left(10240-\frac{10240}{3}\right) = 9800\pi\left(\frac{20480}{3}\right)\approx 210,176,737\,\text{J}.
\end{aligned}$$

6.7.42

a. Orient the axes so that $(0,0)$ is the south pole of the semicircle at one end of the trough. The equation of the semicircle is $x^2+(y-(1/4))^2=(1/4)^2$. At a height of y, a slice has area $2x\cdot 3 = 6\sqrt{(1/2)y-y^2}$. The distance it must travel to the top is $1/4-y$.

$$W = \int_0^{1/4}\rho g 6\sqrt{(1/2)y-y^2}((1/4)-y)\,dy.$$

Let $u=(1/2)y-y^2$, so that $du=((1/2)-2y)\,dy = 2((1/4)-y)\,dy$. Then we have

$$W = 3\rho g\int_0^{1/16}u^{1/2}\,du = 2\rho g\left(u^{3/2}\right)\Big|_0^{1/16} = \rho g/32 = 306.25\,\text{J}.$$

b. Yes. If we double the length of the trough, the area of a slice is doubled, and the work integral is doubled.

c. No. If the radius is doubled, the work is more than doubled. (There are "more slices," and each must travel farther to get to the top of the tank.)

6.7.43

a. Orient the axes so that the lower corners of the trough are at $(-0.25,0)$ and at $(0.25,0)$. Then the upper corners are at $(-0.5,1)$ and at $(0.5,1)$. Note that the line between $(0.25,0)$ and $(0.5,1)$ is given by $y=4x-1$. The area of a slice at height y is $2x\cdot 10 = 20\cdot\frac{1}{4}(y+1)=5(y+1)$. Thus,

$$W=\rho g\int_0^1 5(y+1)(1-y)\,dy = 5\rho g\int_0^1(1-y^2)\,dy = 5\rho g\left(y-y^3/3\right)\Big|_0^1 = \frac{10\rho g}{3}\approx 32,667\,\text{J}.$$

b. Yes. If the length is doubled, the area of each slice is doubled, so the work integral is doubled as well.

6.7.44 Note that this tank is full of water, rather than gasoline.

$$W = 1000\cdot 9.8\int_{-5}^{5}20\sqrt{25-y^2}(10-y)\,dy = 196,000\int_{-5}^{5}\sqrt{25-y^2}(10-y)\,dy.$$

The integral can be divided into two integrals as in the example. We have

$$10\int_{-5}^{5}\sqrt{25-y^2}\,dy - \int_{-5}^{5}y\sqrt{25-y^2}\,dy.$$

The first represents 10 times the area of a half-circle of radius 5, so the value is $(10)\frac{25\pi}{2} = 125\pi$. The second integral can be computed with the substitution $u = 25 - y^2$, $du = -2y\,dy$. The resulting integral is $\frac{1}{2}\int_0^0 \sqrt{u}\,du = 0$. So the total work is

$$W = 196000 \cdot 125\pi \approx 7.697 \times 10^7 \text{ J}.$$

6.7.45 Let the vertex of the cone be at $(0, 0)$, with the y-axis vertically oriented. Note that the area of a horizontal slice at height y is $\pi y^2/16$, and it must move $3 - y$ meters to get to the point 1 meter above the top.

$$\begin{aligned} W &= \int_0^2 \rho g \pi \frac{y^2}{16}(3-y)\,dy = (\pi\rho g/16)\int_0^2 (3y^2 - y^3)\,dy \\ &= (\pi\rho g/16)\left(y^3 - y^4/4\right)\Big|_0^2 = (\pi\rho g/16)(4) = \frac{\pi\rho g}{4} \approx 7696.9\,\text{J}. \end{aligned}$$

6.7.46 $F = \int_0^{10} \rho g(10-y)\cdot 40\,dy = 40\rho g\left(10y - y^2/2\right)\Big|_0^{10} = 200\rho g = 1.960 \times 10^6$ N.

6.7.47 Orient the axes so that the lower corners of the trapezoid are at $(5, 0)$ and $(-5, 0)$, and the upper corners are at $(10, 15)$ and $(-10, 15)$. Note that the line between the corners for $x > 0$ is given by $y = 3(x-5)$, so at level y, we have a width of $2x = \frac{2y}{3} + 10$.

$$\begin{aligned} F &= \rho g \int_0^{15} (15-y)\left(\frac{2y+30}{3}\right) dy = \frac{2\rho g}{3}\int_0^{15} (225 - y^2)\,dy \\ &= \frac{2\rho g}{3}\left(225y - y^3/3\right)\Big|_0^{15} = \frac{2\rho g}{3}\cdot 2250 = 1500\rho g = 1.470 \times 10^7 \text{ N}. \end{aligned}$$

6.7.48 Orient the axes so that $(0, 0)$ is at the "south pole" of the semicircle. The center of the circle is $(0, 20)$ and the radius is 20, so the equation is $x^2 + (y - 20)^2 = 20^2$, so $x = \sqrt{40y - y^2}$. The width is $2x = 2\sqrt{40y - y^2}$.

$$F = 2\rho g \int_0^{20} (20 - y)\sqrt{40y - y^2}\,dy.$$

Let $u = 40y - y^2$, so that $du = 2(20 - y)\,dy$. Then

$$F = \rho g \int_0^{400} \sqrt{u}\,du = \rho g\left(2u^{3/2}/3\right)\Big|_0^{400} = \frac{16000}{3}\rho g = 5.227 \times 10^7 \text{ N}.$$

6.7.49 Orient the axes so that the bottom vertex is at $(0, 0)$. The other vertices are at $(\pm 10, 30)$, and the line between $(0, 0)$ and $(10, 30)$ is given by $y = 3x$. Thus, a slice at height y has width $2x = 2y/3$.

$$F = \int_0^{30} \rho g(30-y)\frac{2y}{3}\,dy = \frac{2\rho g}{3}\left(15y^2 - y^3/3\right)\Big|_0^{30} = \frac{2\rho g}{3}(4500) = 3000\rho g = 2.940 \times 10^7 \text{ N}.$$

6.7.50 At a height of y, the width of a slice is $2x = 2(4\sqrt{y}) = 8y^{1/2}$. This gives

$$F = 8\rho g \int_0^{25} (25-y)y^{1/2}\,dy = 8\rho g\left(50y^{3/2}/3 - 2y^{5/2}/5\right)\Big|_0^{25} = 8\rho g\left(\frac{2500}{3}\right) = 6.533 \times 10^7 \text{ N}.$$

6.7.51 The width of the plate at depth y is $2 - y$, so the force on the plate is

$$F = \int_1^2 \rho g(2-y)y\,dy = \rho g\left(y^2 - \frac{y^3}{3}\right)\Big|_1^2 = \rho g \cdot \frac{2}{3} \approx 6533\,\text{N}.$$

6.7.52 $F = \int_0^{1/2} \rho g(4-y)\cdot 0.5\,dy = \rho g\left(2y - y^2/4\right)\Big|_0^{1/2} = \dfrac{15\rho g}{16} = 9187.5$ N.

6.7.53 $F = \int_1^{1.5} \rho g(4-y)\cdot 0.5\,dy = \rho g\left(2y - y^2/4\right)\Big|_1^{1.5} = \dfrac{11\rho g}{16} = 6737.5$ N.

6.7.54 Orient the axes so that $(0,0)$ is at the bottom of the circle at the bottom of the pool. Then the equation of the circle is $x^2 + (y-1/2)^2 = (1/2)^2$, so the width of a slice at a height of y is $2x = 2\sqrt{y-y^2}$. Thus,

$$F = 2\rho g\int_0^1 (4-y)(\sqrt{y-y^2})\,dy = 2\rho g\int_0^1 \frac{7}{2}\sqrt{y-y^2}\,dy + 2\rho g\int_0^1 (\frac{1}{2}-y)\sqrt{y-y^2}\,dy$$

$$= 7\rho g\left(\frac{\pi}{8}\right) + 2\rho g\left(\frac{1}{3}(y-y^2)^{3/2}\right)\Big|_0^1 = \frac{7\pi\rho g}{8} + 0 \approx 2.694\times 10^4\,\text{N}.$$

6.7.55 $F = \int_0^{50} (150+2y)\cdot 80\,dy = 80\left(150y + y^2\right)\Big|_0^{50} = 8\times 10^5$ N.

6.7.56 Arrange the axes so that the origin is in the center of the circular end of the tank. The equation of the circle is $x^2 + y^2 = 25$, so the width of the tank at height y is $2\sqrt{25-y^2}$, and the depth is $5-y$. Thus the force is given by

$$\rho g\int_{-5}^5 (4-y)2\sqrt{25-y^2}\,dy = 14445.2\int_{-5}^5 (5-y)\sqrt{25-y^2}\,dy$$

$$= 14445.\left(\int_{-5}^5 5\sqrt{25-y^2}\,dy + \int_{-5}^5 y\sqrt{25-y^2}\,dy\right).$$

The first integral is five times the area of a semicircle of radius 5, while the second integral is zero since it is a symmetric integral of an odd function. So the total force is $14445.2\cdot\dfrac{125\pi}{2} \approx 2.836\times 10^6$ N.

6.7.57

a. True. $m = \int_a^b \rho(x)\,dx = \dfrac{1}{b-a}\int_a^b \rho(x)\,dx\cdot(b-a) = \overline{\rho}\cdot L.$

b. True. $\int_0^L kx\,dx = \dfrac{kL^2}{2} = \int_0^{-L} kx\,dx.$

c. True. This follows because work is force times distance.

d. False. Although they have the same geometry, they are placed at different depths of the water, so the force is different.

6.7.58

a. $m_1 = \int_0^L 4e^{-x}\,dx = \left(-4e^{-x}\right)\Big|_0^L = 4(1-e^{-L}).$

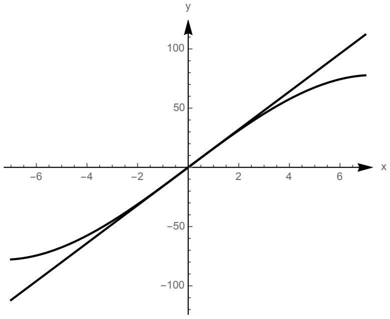
$m_2 = \int_0^L 6e^{-2x}\,dx = \left(-3e^{-2x}\right)\Big|_0^L = 3(1-e^{-2L}).$

These are the same when $3(e^{-L})^2 - 4e^{-L} + 1 = 0$, or $(e^{-L}-1)(3e^{-L}-1) = 0$, so $L = \ln 3$. m_2 is bigger on $(0,\ln 3)$ and m_1 is bigger for $L > \ln 3$.

b. No. $\lim_{L\to\infty} m_1 = 4$ and $\lim_{L\to\infty} m_2 = 3$.

6.7.59

a. Compared to the linear spring $F(x) = 16x$, the restoring force is less for large displacements.



b. $W = \int_0^{1.5} (16x - 0.1x^3)\,dx = \left(8x^2 - .025x^4\right)\Big|_0^{1.5} = 17.87$ J.

c. $W = \int_0^{-2} (16x - 0.1x^3)\,dx = \left(8x^2 - .025x^4\right)\Big|_0^{-2} = 31.6$ J.

6.7.60

a. $f(x) = kx$, and $f(2) = 2k = 10g$, so $k = 5g$. To compress the spring:

$$W = g\int_{1.5}^{2} 5x\,dx = g\left(5x^2/2\right)\Big|_{1.5}^{2} = g(10 - 45/8) = 35g/8 = 42.875\,\text{J}.$$

To move the mass: $10 \cdot 0.5 \cdot 9.8 = 49$ J. So the total work is $42.875 + 49 = 91.875$ J.

b. To stretch the spring:

$$W = g\int_{2}^{2.5} 5x\,dx = g\left(5x^2/2\right)\Big|_{2}^{2.5} = g(125/8 - 10) = 45g/8 = 55.125\,\text{J}.$$

To move the mass: $10 \cdot 0.5 \cdot 9.8 = 49$ J. So the total work is $55.125 + 49 = 104.125$ J.

6.7.61 Dividing the leak rate by the lifting rate, we find that the bucket is leaking at 20 kg/m. So the force exerted by the bucket after it has been raised y meters is $F(y) = 9.8(5000 - 20y)$. Therefore, the work required to lift the bucket 30 m is

$$W = \int_0^{30} 9.8(5000 - 20y)\,dy = 9.8\left(5000y - 10y^2\right)\Big|_0^{30} = 9.8\,(150,000 - 9000) = 1,381,800\,\text{J}.$$

6.7.62 Orient the y-axis vertically, with the point $(10, 0)$ representing a point on the bottom of the pool in one corner of the deep end, and $(-10, 1)$ representing a point on the bottom of the pool in the shallow end on the same side of the pool. Note that the straight line between these two points is given by $y = \frac{1}{20}(10-x)$. So in the first 1 meter of depth, at a height of y the length of the pool is $10 - x = 20y$, so the area of a slice is $20y \cdot 10$. In the 2nd meter of depth, the slices are uniformly of area 200 square meters.

$$W = \int_0^1 \rho g 200y(2.2 - y)\,dy + \int_1^2 \rho g 200(2.2 - y)\,dy = 200\rho g\left(\left(1.1y^2 - y^3/3\right)\Big|_0^1 + \left(2.2y - y^2/2\right)\Big|_1^2\right)$$

$$= 200\rho g\,(1.1 - (1/3) + (4.4 - 2 - (2.2 - 1/2))) = \frac{880}{3}\rho g \approx 2.87467 \times 10^6\ \text{J}.$$

6.7.63 Orient the axes so that $(0,0)$ is in the middle of the bottom of the cup. Note that the line between $(0.02,0)$ and $(0.025,0.15)$ is given by $y=30x-3/5$, so $x=\frac{y}{30}+\frac{1}{50}$. The area of a cross section at height y is given by $\pi x^2=\pi\left(\frac{5y+3}{150}\right)^2$. Note that the distance the slice must travel is $0.2-y$, because it must go 0.05 above the top of the glass. Thus,

$$\begin{aligned}W&=\pi\rho g\int_0^{0.15}\left(\frac{5y+3}{150}\right)^2(.2-y)\,dy=\frac{\pi\rho g}{5\cdot150^2}\int_0^{0.15}(5y+3)^2(1-5y)\,dy\\&=\frac{\pi\rho g}{5\cdot150^2}\int_0^{0.15}(-125y^3-125y^2-15y+9)\,dy\\&=\frac{\pi\rho g}{5\cdot150^2}\left(-125y^4/4-125y^3/3-15y^2/2+9y\right)\Big|_0^{0.15}\\&=\frac{\pi\rho g}{5\cdot150^2}\cdot1.0248\approx0.280\,\text{J}.\end{aligned}$$

6.7.64

a. The acceleration due to gravity is $F=mg$, and is in the vertical direction. The tangent direction to the curve is perpendicular to the normal, which makes an angle of θ with the the vertical. If we form a right triangle with hypotenuse of length mg, and angle θ, then the two legs must have lengths $mg\sin\theta$ (parallel to the curve) and $mg\cos\theta$ (normal to the curve).

b. Note that the angle t is $t=S/L$, where S is arc length. Then $S=Lt$ and $dS=L\,dt$. So

$$W=\int_0^\theta F\,ds=\int_0^\theta mg\sin\theta\cdot L\,d\theta=mgL\left(-\cos\theta)\right)\Big|_0^\theta=mg(L-L\cos\theta)=mgh.$$

6.7.65

a. $F=\int_1^3\rho g(4-y)\cdot2\,dy=\rho g\left(8y-y^2\right)\Big|_1^3=8\rho g=78,400$ N. This is less than 90,000 N., so the window can withstand the force.

b. $F(h)=\int_1^3\rho g(h-y)\cdot2\,dy=\rho g\left(2hy-y^2\right)\Big|_1^3=\rho g(6h-9-(2h-1))=4\rho g(h-2)$. This is less than or equal to 90,000 Newtons when $h\le4.296$ meters.

6.7.66 The plate pictured on the left should have more force, because it has its wider part lower in the pool.

For each plate, orient the axes so that the vertical line through the vertex pointing up (or down) is the y-axis, and the x-axis lies along the horizontal base of the triangle. For the plate on the left, the line through $(1/2,0)$ and $(0,\sqrt{3}/2)$ is given by $y=\sqrt{3}((1/2)-x)$, so $x=\frac{1}{2}-(\sqrt{3}/3)y$, and the width at height y is given by $1-\frac{2\sqrt{3}}{3}y$.

For the plate on the left,

$$\begin{aligned}F&=\int_0^{\sqrt{3}/2}\rho g(1+(\sqrt{3}/2)-y)(1-\frac{2}{\sqrt{3}}y)\,dy\\&=\rho g\int_0^{\sqrt{3}/2}\left(\frac{2y^2}{\sqrt{3}}-\frac{2y}{\sqrt{3}}-2y+\frac{\sqrt{3}}{2}+1\right)dy\\&=\rho g\left(\frac{1}{6}\left(\frac{4y^3}{\sqrt{3}}-2(3+\sqrt{3})y^2+3\left(2+\sqrt{3}\right)y\right)\right)\Big|_0^{\sqrt{3}/2}=\frac{1}{4}+\frac{\sqrt{3}}{4}\approx0.683\,\text{N}.\end{aligned}$$

For the plate on the right, the line through $(0,0)$ and $(1/2,\sqrt{3}/2)$ is given by $y=\sqrt{3}x$, so $x=\dfrac{y}{\sqrt{3}}$, and the width at height y is $\dfrac{2y}{\sqrt{3}}$.

Thus,

$$\begin{aligned}F &= \int_0^{\sqrt{3}/2} \rho g(1+\sqrt{3}/2-y)\frac{2y}{\sqrt{3}}\,dy = \rho g\int_0^{\sqrt{3}/2}\left(-\frac{2y^2}{\sqrt{3}}+\frac{2y}{\sqrt{3}}+y\right)dy\\ &= \rho g\left(-\frac{2y^3}{3\sqrt{3}}+\frac{y^2}{\sqrt{3}}+\frac{y^2}{2}\right)\Bigg|_0^{\sqrt{3}/2} = \frac{1+2\sqrt{3}}{8}\approx 0.558\,\text{N}.\end{aligned}$$

6.7.67 The plate on the left has more than half of its area below the horizontal line which is 3/2 below the surface, while the plate on the right has exactly half its area below that line, so the plate on the left should have more force than the plate on the right.

a. Note that the line in the first quadrant forom $(0,0)$ to $(\sqrt{2}/2,\sqrt{2}/2)$ is given by $y=x$,while the line from $(\sqrt{2}/2,\sqrt{2}/2)$ to $(0,\sqrt{2})$ is given by $y=-x+\sqrt{2}$. So the width of a slice at height y in the lower part of the region is $2y$, and in the upper part of the region is $2(\sqrt{2}-y)$.

$$\begin{aligned}F &= \rho g\int_0^{\sqrt{2}/2}(\sqrt{2}+1-y)(2y)\,dy+\rho g\int_{\sqrt{2}/2}^{\sqrt{2}}(\sqrt{2}+1-y)2(\sqrt{2}-y)\,dy\\ &= \rho g\int_0^{\sqrt{2}/2}(-2y^2+2\sqrt{2}y+2y)\,dy+\rho g\int_{\sqrt{2}/2}^{\sqrt{2}}(2y^2-4\sqrt{2}y-2y+2\sqrt{2}+4)\,dy\\ &= \rho g\left(-\frac{2y^3}{3}+\sqrt{2}y^2+y^2\right)\Bigg|_0^{\sqrt{2}/2}+\rho g\left(2\left(\frac{y^3}{3}-\frac{1}{2}\left(1+2\sqrt{2}\right)y^2+\sqrt{2}y+2y\right)\right)\Bigg|_{\sqrt{2}/2}^{\sqrt{2}}\\ &= \rho g\left(\frac{1}{2}+\frac{\sqrt{2}}{3}+\frac{1}{2}+\frac{1}{3\sqrt{2}}\right)=\rho g\left(1+\frac{\sqrt{2}}{2}\right)\text{ N.}\end{aligned}$$

This is 16,730 N.

b. $F=\displaystyle\int_0^1 \rho g(2-y)\cdot 1\,dy=\rho g\left(2y-y^2/2\right)\Big|_0^1=\frac{3\rho g}{2}$ N. This is 14,700 N.

6.7.68

a. $W=\displaystyle\int_{x_0}^{x_f} mx''\,dx=\int_0^{x(5)} 2\cdot 8\,dx=16(x(5))=1600$ J.

b. $W=\displaystyle\int_{x_0}^{x_f} mx''\,dx=\int_0^5 2\cdot 8\frac{dx}{dt}\,dt=\int_0^5 16\cdot 8t\,dt=\left(64t^2\right)\Big|_0^5=64\cdot 25=1600$ J.

6.7.69

a.

$$\begin{aligned}W &= \int_0^{2500000}\frac{GMm}{(x+R)^2}\,dx=GMm\left(\frac{-1}{x+R}\right)\Bigg|_0^{2500000}\\ &= GMm\left(\frac{1}{R}-\frac{1}{R+2500000}\right)=\frac{GMm2500000}{R(R+2500000)}\approx 8.87435\times 10^9\text{ J.}\end{aligned}$$

b. $W(x)=\displaystyle\int_0^x\frac{GMm}{(t+R)^2}\,dt=GMm\left(\frac{-1}{t+R}\right)\Bigg|_0^x=GMm\left(\frac{1}{R}-\frac{1}{R+x}\right)=\frac{GMmx}{R(R+x)}=\frac{500GMx}{R(R+x)}$.

c. $\lim_{x\to\infty} \frac{GMmx}{R(R+x)} = \lim_{x\to\infty} \frac{GMm}{R\left(\left(\frac{R}{x}\right)+1\right)} = \frac{GMm}{R}$.

d. Suppose $\frac{GMmx}{R(R+x)} = \frac{1}{2}mv^2$, then $v^2 = \frac{2GMx}{R(R+x)}$, and as $x \to \infty$ we have $v^2 = \frac{2GM}{R}$, so $v = \sqrt{\frac{2GM}{R}}$.

6.7.70 When the box is fully submerged in the water, the buoyant force is $8g\rho_w$. The force required is $F = 8g\rho_w - 8g\rho = 8g\rho_w - 8g(\rho_w/2) = 4g\rho_w = 39,200$ N.

Chapter Six Review

1

a. True. A vertical slice would lead to shells, while a horizontal slice would lead to either disks or washers.

b. True. In order to find position, you would also need to know either its initial position, or at least its position at some time.

c. True. If dV/dt is constant, then V is a linear function of time.

2

a. $v(t) = 0$ when $\frac{1}{2} = \cos\left(\frac{\pi t}{3}\right)$, which occurs on the given interval for $t = 1$. For $0 \le t < 1$, we have $v(t) < 0$, so the motion is in the negative direction, and for $1 < t \le 3$, $v(t) > 0$, so the motion is in the positive direction.

b. The displacement is given by $\int_0^3 \left(1 - 2\cos\left(\frac{\pi t}{3}\right)\right) dt = \left(t - \frac{6\sin(\pi t/3)}{\pi}\right)\Big|_0^3 = 3$ m.

c. The distance traveled is

$$\begin{aligned}
&\int_1^3 \left(1 - 2\cos\left(\frac{\pi t}{3}\right)\right) dt - \int_0^1 \left(1 - 2\cos\left(\frac{\pi t}{3}\right)\right) dt \\
&= \left(t - \frac{6\sin(\pi t/3)}{\pi}\right)\Big|_1^3 - \left(t - \frac{6\sin(\pi t/3)}{\pi}\right)\Big|_0^1 \\
&= (3 - 0 - (1 - 3\sqrt{3}/\pi)) - (1 - 3\sqrt{3}/\pi - (0 - 0)) \\
&= 1 + \frac{6\sqrt{3}}{\pi} = \frac{6\sqrt{3}+\pi}{\pi}.
\end{aligned}$$

d. $s(t) = s(0) + \int_0^t (1 - 2\cos \pi x/3)\, dx = \left(x - \frac{6\sin(\pi x/3)}{\pi}\right)\Big|_0^t = t - \frac{6}{\pi}\sin\frac{\pi t}{3}$.

With the antiderivative method, we would have

$$s(t) = \int v(t)\, dt = \int \left(1 - 2\cos\left(\frac{\pi t}{3}\right)\right) dt = \left(t - \frac{6\sin(\pi t/3)}{\pi}\right) + C,$$

and because $s(0) = 0$, we would have $C = 0$. Therefore $s(t) = t - \frac{6}{\pi}\sin\frac{\pi t}{3}$.

3

a. $12t^2 - 30t + 12 = 6(2t^2 - 5t + 2) = 6(2t-1)(t-2)$, which is zero for $t = 2$ and $t = \frac{1}{2}$. $v(t) > 0$ for $0 \le t < \frac{1}{2}$ and for $2 < t \le 3$, so the motion is in the positive direction on those intervals, while $v < 0$ for $\frac{1}{2} < t < 2$, so the motion is in the negative direction on that interval.

b. The displacement is given by $\int_0^3 (12t^2 - 30t + 12)\,dt = \left(4t^3 - 15t^2 + 12t\right)\Big|_0^3 = 108 - 135 + 36 = 9\,\text{m}$.

c. The distance traveled is

$$\begin{aligned}&\int_0^{1/2} (12t^2 - 30t + 12)\,dt + \int_2^3 (12t^2 - 30t + 12)\,dt - \int_{1/2}^2 (12t^2 - 30t + 12)\,dt\\ &= \left(4t^3 - 15t^2 + 12t\right)\Big|_0^{1/2} + \left(4t^3 - 15t^2 + 12t\right)\Big|_2^3 - \left(4t^3 - 15t^2 + 12t\right)\Big|_{1/2}^2\\ &= (1/2 - 15/4 + 6) + (108 - 135 + 36) - (32 - 60 + 24) - ((32 - 60 + 24) - (1/2 - 15/4 + 6))\\ &= 22.5\,\text{m}.\end{aligned}$$

d. $s(t) = s(0) + \int_0^t (12x^2 - 30x + 12)\,dx = 1 + \left(4x^3 - 15x^2 + 12x\right)\Big|_0^t = 4t^3 - 15t^2 + 12t + 1$.

With the antiderivative method, we would have

$$s(t) = \int v(t)\,dt = \int (12t^2 - 30t + 12)\,dt = 4t^3 - 15t^2 + 12t + C,$$

and because $s(0) = 1$, we would have $C = 1$. Therefore, $s(t) = 4t^3 - 15t^2 + 12t + 1$.

4 The displacement is $\int_0^{1.5} 20\cos \pi t\,dt = \left(\frac{20}{\pi}\sin \pi t\right)\Big|_0^{1.5} = -\frac{20}{\pi}$.

5 The position $s(t)$ and the displacement are the same, because the projective started on the ground at position 0.

$$s(t) = \int_0^t v(x)\,dx = \int_0^t (20 - 10x)\,dx = \left(20x - 5x^2\right)\Big|_0^t = 20t - 5t^2.$$

Note that the projectile is moving up for $0 < t < 2$ and down for $2 < t < 4$. Thus the distance traveled is equal to the position for $0 < t < 2$, but for $2 < t < 4$ the distance traveled is $20 + (20 - (20t - 5t^2)) = 40 - 20t + 5t^2$.

6 $a(t) = -5$, so $v(t) = -5t + C$, and because $v(0) = 80$, we have $v(t) = 80 - 5t$. The position function $s(t) = \int v(t)\,dt = \int (80 - 5t)\,dt = 80t - 5t^2/2 + D$, and because $s(0) = 0$ we have $D = 0$. Thus, $s(t) = 80t - 5t^2/2$.

7

a. $v(t) = \int a(t)\,dt = \int 2\sin(\pi t/4)\,dt = -\frac{8}{\pi}\cos(\pi t/4) + C$, and because $v(0) = -\frac{8}{\pi}$, we have $C = 0$. Thus, $v(t) = -\frac{8}{\pi}\cos(\pi t/4)$.

$s(t) = \int v(t)\,dt = \int -\frac{8}{\pi}\cos(\pi t/4)\,dt = -\frac{32}{\pi^2}\sin(\pi t/4) + D$, and because $s(0) = 0$ we have $D = 0$. Thus, $s(t) = -\frac{32}{\pi^2}\sin(\pi t/4)$.

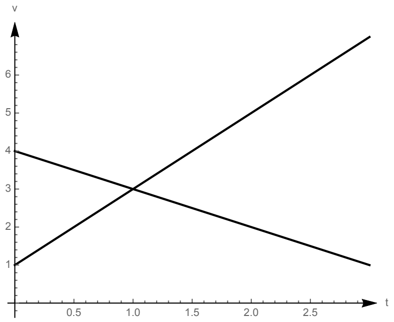
b. s is periodic with period 8, so we only consider $0 \le t \le 8$. There are critical numbers for s at $t = 2$ and $t = 6$. There is a maximum for s of $\frac{32}{\pi^2}$ at $t = 6$ and a minimum of $-\frac{32}{\pi^2}$ at $t = 2$.

c. The average velocity is $\frac{1}{8}\int_0^8 \frac{8}{\pi}\cos(\pi t/4)\,dt = -\frac{4}{\pi^2}\left(\sin(\pi t/4)\right)\Big|_0^8 = 0$.

The average position is $\frac{1}{8}\int_0^8 -\frac{32}{\pi^2}\sin(\pi t/4)\,dt = \frac{16}{\pi^3}\left(\cos(\pi t/4)\right)\Big|_0^8 = 0$.

8

a.



b. Anna runs $\int_0^1 (2t+1)\,dt = \left(t^2+t\right)\Big|_0^1 = 2$ miles, while Benny runs $\int_0^1 (4-t)\,dt = \left(4t - t^2/2\right)\Big|_0^1 = (4-1/2) = 3.5$ miles. Note that the area under Anna's graph from 0 to 1 is smaller than the corresponding area under Benny's graph.

c. $s_A(t) = t^2 + t$, which is 6 for $t = 2$. $s_B(t) = 4t - t^2/2$ which is 6 for $t = 2$, so they both take exactly two hours to run 6 miles. At $t = 2$, both velocity function have the same area under them, namely 6.

9

a. For $0 \le t \le 8$ we have $R'(t) = 4t^{1/3}$, so $R(t) = 3t^{4/3} + C$, but $C = 0$, so $R(t) = 3t^{4/3}$.

b. Note that $R(8) = 48$, so for $t > 8$, $R(t) = 48 + \int_8^t 2\,dx = 48 + 2(t-8)$.

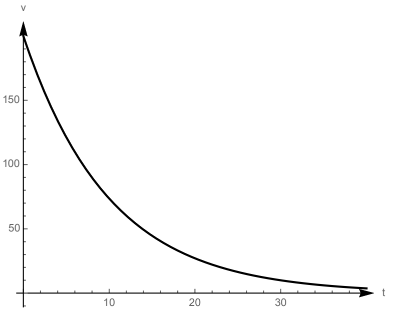
$$\text{We have } R(t) = \begin{cases} 3t^{4/3} & \text{if } 0 \le t \le 8; \\ 2t + 32 & \text{if } t > 8. \end{cases}$$

c. The fuel runs out when $150 = 48 + 2(t-8)$, which occurs for $t = 59$.

10 Let $V'(t) = -\dfrac{15}{t+1}$. Then $V(t) = -15\ln(t+1) + C$, and because $V(0) = 75$, we have $C = 75$. Thus, $V(t) = 75 - 15\ln(t+1)$. This is 0 when $\ln(t+1) = 5$, which occurs when $t = e^5 - 1 \approx 147.413$ hours, so don't try to hold your breath while the tank is emptying.

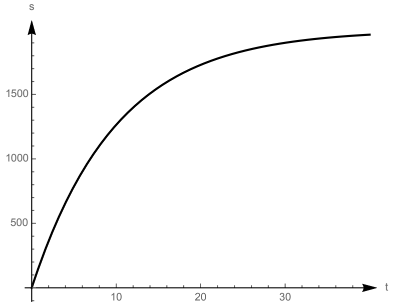
11

a.



b. The velocity is 50 when $200e^{-t/10} = 50$, which occurs when $e^{t/10} = 4$, so when $t = 10\ln 4$.

c. .

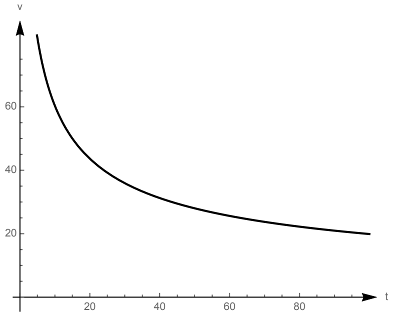


The position is given by $\int_0^t 200e^{-x/10}\,dx = -2000e^{-x/10}\Big|_0^t = 2000(1-e^{-t/10})$.

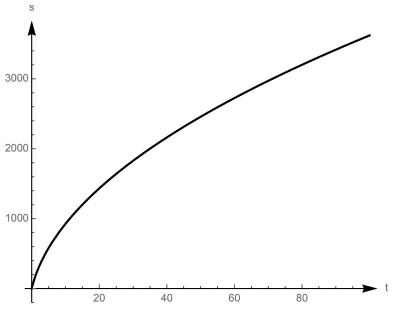
d. No. $\lim_{t\to\infty} s(t) = 2000 < 2500$.

12

a.



b.

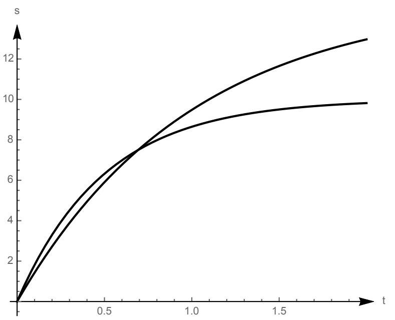


The position is given by $\int_0^t 200/\sqrt{x+1}\,dx = \left(400\sqrt{x+1}\right)\Big|_0^t = 400(\sqrt{t+1}-1)$.

c. Yes, the position is 2500 when $400\sqrt{t+1} = 2900$, which occurs when $t+1 = \left(\frac{29}{4}\right)^2$, so $t = \left(\frac{29}{4}\right)^2 - 1 = 51.5625$.

13

a. Tom's position function is given by $\int_0^t 20e^{-2x}\,dx = \left(-10e^{-2x}\right)\Big|_0^t = 10(1-e^{-2t})$. Sue's position is given by $\int_0^t 15e^{-x}\,dx = \left(-15e^{-x}\right)\Big|_0^t = 15(1-e^{-t})$.



b. We are looking for t so that $10(1-e^{-2t}) = 15(1-e^{-t})$, which occurs when $10(e^{-t})^2 - 15e^{-t} + 5 = 0$, or $2u^2 - 3u + 1 = 0$ where $u = e^{-t}$. This quadratic factors as $(2u-1)(u-1)$, so if $e^{-t} = 1$ or $e^{-t} = 1/2$, so $t = 0$ or $t = \ln 2$.

c. Sue takes the lead and doesn't relinquish it at $t = \ln 2$.

14 The area is given by

$$\int_0^1 ((x+1) - (2^x + x - 1))\,dx = \int_0^1 (2 - 2^x)\,dx = \left(2x - \frac{2^x}{\ln 2}\right)\Big|_0^1 = \left(2 - \frac{2}{\ln 2}\right) - \left(0 - \frac{1}{\ln 2}\right) = 2 - \frac{1}{\ln 2}.$$

15 The area is given by

$$\int_0^{\pi/4} (\sin 2x - 1 - (-\cos 2x))\,dx = \left(-\frac{\cos 2x}{2} - x + \frac{\sin 2x}{2}\right)\Big|_0^{\pi/4} = \left(0 - \frac{\pi}{4} + \frac{1}{2}\right) - \left(-\frac{1}{2} - 0 + 0\right) = 1 - \frac{\pi}{4}.$$

16 The area is given by

$$\int_0^{\pi/3} \cos y\,dy = \sin y\Big|_0^{\pi/3} = \frac{\sqrt{3}}{2}.$$

17 The area is given by

$$\int_0^1 (e^y - 1)\,dy = (e^y - y)\Big|_0^1 = (e-1) - (1-0) = e - 2.$$

18 Because $\sin\frac{\pi}{4} = \cos\frac{\pi}{4}$, the curves intersect at $x = \frac{\pi}{2}$. By symmetry, the area is given by

$$2\int_{\pi/2}^{5\pi/2} \left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)\,dx = 2\left(-2\cos\frac{x}{2} - 2\sin\frac{x}{2}\right)\Big|_{\pi/2}^{5\pi/2} = -4\left(\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\right) = 8\sqrt{2}.$$

19 These two curves meet when $4x = x\sqrt{25-x^2}$, so at $x = 0$ and when $4 = \sqrt{25-x^2}$, which occurs for $x = 3$. On $[0,3]$, we have $x\sqrt{25-x^2} > 4x$ (for example check at $x = 1$), so the area is

$$A = \int_0^3 \left(x\sqrt{25-x^2} - 4x\right)\,dx.$$

Note that $\int_0^3 -4x\,dx = -2x^2\Big|_0^3 = -18$. For the first term, use the substitution $u = 25 - x^2$, so that $du = -2x\,dx$ and the integration bounds become $u = 25$ to $u = 16$. Then we have

$$A = -\frac{1}{2}\int_{25}^{16} u^{1/2}\,du - 18 = -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Big|_{25}^{16} - 18 = -\frac{64}{3} + \frac{125}{3} - 18 = \frac{7}{3}.$$

20 The curves meet where $6-3x^2=6x-3$, or $3x^2+6x-9=3(x^2+2x-3)=3(x+3)(x-1)=0$, which occurs at $x=1$ and $x=-3$. The area is given by

$$\int_{-3}^{1}(6-3x^2-(6x-3))\,dx=\int_{-3}^{1}(-3x^2-6x+9)\,dx=\left(-x^3-3x^2+9x\right)\Big|_{-3}^{1}=(-1-3+9)-(27-27-27)=32.$$

21 The curves intersect for $x^2=2x^2-4x$, or $x^2-4x=x(x-4)=0$, so for $x=0$ and $x=4$. For $0\le x\le 2$ the region is bounded above by x^2 and below by the x-axis, and for $2\le x\le 4$ the region is bounded above by x^2 and below by $2x^2-4x$. The area is given by

$$\int_0^2 x^2\,dx+\int_2^4(x^2-2x^2+4x)\,dx=\left(x^3/3\right)\Big|_0^2+\left(-x^3/3+2x^2\right)\Big|_2^4=(8/3)+-64/3+32-(-8/3+8)=8.$$

22 The area is given by

$$\int_0^1(1-\sqrt{x})^2\,dx=\int_0^1(1-2\sqrt{x}+x)\,dx=\left(x-4x^{3/2}/3+x^2/2\right)\Big|_0^1=\frac{1}{6}.$$

23 Note that we can write $1-\left|\frac{x}{2}-1\right|$ as $\begin{cases}\frac{x}{2} & \text{if } x\le 2;\\ 2-\frac{x}{2} & \text{if } x\ge 2.\end{cases}$

For $0\le x\le 2$, the region is bounded above by $\frac{x}{2}$ and below by $\frac{x}{6}$. For $2\le x\le 3$, the region is bounded above by $2-\frac{x}{2}$ and below by $\frac{x}{6}$. The area is then given by

$$\int_0^2(x/2-x/6)\,dx+\int_2^3(2-x/2-x/6)\,dx=\left(x^2/6\right)\Big|_0^2+\left(2x-x^2/3\right)\Big|_2^3=\frac{2}{3}+(6-3)-(4-4/3)=1.$$

24 $\int_0^1(x^{1/p}-x^p)\,dx$ is evaluated as

$$\left(\frac{p}{p+1}x^{(p+1)/p}-\frac{1}{p+1}x^{p+1}\right)\Big|_0^1=\frac{p}{p+1}-\frac{1}{p+1}=\frac{p-1}{p+1}.$$

For $p=100$ the area is $99/101$, and for $p=1000$ the area is $999/1001$.

25 The curve and the line intersect where $2x^2-6x+5=1$, or $2x^2-6x+4=2(x^2-3x+2)=2(x-1)(x-2)=0$, which is for $x=1$ and $x=2$. The area is given by

$$\int_1^2(1-(2x^2-6x+5))\,dx=\int_1^2(-4+6x-2x^2)\,dx=\left(-4x+3x^2-\frac{2x^3}{3}\right)\Big|_1^2$$
$$=-8+12-\frac{16}{3}-\left(-4+3-\frac{2}{3}\right)=\frac{1}{3}.$$

26 Substituting x for y^2 in the equation $x=(2-y^2)^2$, we obtain $x=(2-x)^2$, which can be written as $x=4-4x+x^2$ or $x^2-5x+4=(x-4)(x-1)=0$, so the curves intersect at $x=4$ and $x=1$. The points of intersection are $(1,\pm1)$ and $(4,\pm2)$. It will be easier to integrate with respect to y. Exploiting symmetry, we can integrate with respect to y from 0 to 1 and then again from 1 to 2 and double our result.

Note that between $y=1$ and $y=2$ we have $y^2>(2-y^2)^2$, while between $y=0$ and $y=1$ the reverse is true. Thus the area is given by

$$\begin{aligned} A &= 2\int_0^1 ((2-y^2)^2) - y^2)\,dy + 2\int_1^2 (y^2 - (2-y^2)^2)\,dy \\ &= 2\int_0^1 (y^4 - 5y^2 + 4)\,dy + 2\int_1^2 (-y^4 + 5y^2 - 4)\,dy \\ &= 2\left(\frac{y^5}{5} - \frac{5y^3}{3} + 4y\right)\Big|_0^1 + 2\left(-\frac{y^5}{5} + \frac{5y^3}{3} - 4y\right)\Big|_1^2 \\ &= 2\left(\frac{38}{15}\right) + 2\left(-\frac{16}{15} + \frac{38}{15}\right) = \frac{120}{15} = 8. \end{aligned}$$

27 The area of R_1 is

$$\int_0^1 (3 - x - 2\sqrt{x})\,dx = \left(3x - \frac{x^2}{2} - \frac{4x^{3/2}}{3}\right)\Big|_0^1 = 3 - \frac{1}{2} - \frac{4}{3} = \frac{7}{6}.$$

The area of R_2 can be computed by subtracting the area of R_1 from the area of a triangle with base 3 and height 3, so it is

$$\frac{9}{2} - \frac{7}{6} = \frac{20}{6} = \frac{10}{3}.$$

The area of R_3 is the difference of $\int_1^3 2\sqrt{x}\,dx$ and the area of a triangle with base 2 and height 2, so it is

$$\int_1^3 2\sqrt{x}\,dx - 2 = \left(\frac{4x^{3/2}}{3}\right)\Big|_1^3 - 2 = \frac{12\sqrt{3}}{3} - \frac{4}{3} - 2 = 4\sqrt{3} - \frac{10}{3}.$$

28 $V = \pi\int_0^1 ((3-x)^2 - 4x)\,dx = \pi\int_0^1 (x^2 - 10x + 9)\,dx = \pi\left(\frac{x^3}{3} - 5x^2 + 9x\right)\Big|_0^1 = \pi\left(\frac{1}{3} - 5 + 9\right) = \frac{13\pi}{3}.$

29

$$\begin{aligned} V &= 2\pi\int_0^1 x(3 - x - 2\sqrt{x})\,dx = 2\pi\int_0^1 (3x - x^2 - 2x^{3/2})\,dx \\ &= 2\pi\left(\frac{3x^2}{2} - \frac{x^3}{3} - \frac{4x^{5/2}}{5}\right)\Big|_0^1 = 2\pi\left(\frac{3}{2} - \frac{1}{3} - \frac{4}{5}\right) = \frac{11\pi}{15}. \end{aligned}$$

30

$$\begin{aligned} V &= \pi\int_0^2 \left((3-y)^2 - \frac{y^4}{16}\right)\,dy = \pi\int_0^2 \left(9 - 6y + y^2 - \frac{y^4}{16}\right)\,dy \\ &= \pi\left(9y - 3y^2 + \frac{y^3}{3} - \frac{y^5}{80}\right)\Big|_0^2 = \pi\left(18 - 12 + \frac{8}{3} - \frac{32}{80}\right) = \frac{124\pi}{15}. \end{aligned}$$

31

$$\begin{aligned} V &= 2\pi\int_0^2 y\left(3 - y - \frac{y^2}{4}\right)\,dy = 2\pi\left(3y - y^2 - \frac{y^3}{4}\right)\,dy \\ &= 2\pi\left(\frac{3y^2}{2} - \frac{y^3}{3} - \frac{y^4}{16}\right)\Big|_0^2 = 2\pi\left(6 - \frac{8}{3} - 1\right) = \frac{14\pi}{3}. \end{aligned}$$

32 $V = \pi\int_0^2 (3-y)^2\,dy + \pi\int_2^{2\sqrt{3}} \left(3 - \frac{y^2}{4}\right)^2\,dy.$

33 $V = 2\pi \int_1^3 (3-x)\left(2\sqrt{x} - 3 + x\right)\, dx.$

34

a. $A = \int_0^2 ((4+y) - (y^2+2))\, dy.$

b. $V = 2\pi \int_0^2 y\left[(4+y) - (y^2+2)\right]\, dy.$

c. $V = \pi \int_0^2 \left[(4+y)^2 - (y^2+2)^2\right]\, dy.$

d. $V = \int_0^2 A(y)\, dy = \dfrac{\pi}{2}\int_0^2 \left(\dfrac{(4+y)-(y^2+2)}{2}\right)^2\, dy.$

35 $A = \int_0^1 1\, dx + \int_1^4 (2-\sqrt{x})\, dx = 1 + \left(2x - \dfrac{2x^{3/2}}{3}\right)\Big|_1^4 = 1 + \left(8 - \dfrac{16}{3} - \left(2 - \dfrac{2}{3}\right)\right) = \dfrac{7}{3}.$

36 $V = 2\pi \int_0^1 y(2-y)^2\, dy = 2\pi \int_0^1 (4y - 4y^2 + y^3)\, dy = 2\pi\left(2y^2 - \dfrac{4y^3}{3} + \dfrac{y^4}{4}\right)\Big|_0^1 = 2\pi\left(2 - \dfrac{4}{3} + \dfrac{1}{4}\right) = \dfrac{11\pi}{6}.$

37 $V = \pi \int_0^1 (2-y)^4\, dy = \pi\left(-\dfrac{(2-y)^5}{5}\right)\Big|_0^1 = \pi\left(-\dfrac{1}{5} + \dfrac{32}{5}\right) = \dfrac{31\pi}{5}.$

38 For the portion generated over $0 \le x \le 1$, we have

$$S_1 = 2\pi \int_0^1 1\sqrt{1+0^2}\, dx = 2\pi.$$

For the portion over $1 \le x \le 4$, we have

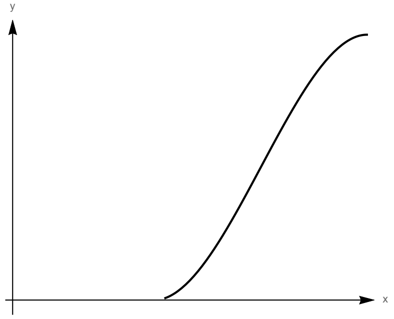
$$S_2 = 2\pi \int_1^4 (2-\sqrt{x})\sqrt{1 + \frac{1}{4x}}\, dx \approx 8.96177.$$

This calculation was done using a computer algebra system. The surface area for the whole surface is approximately $S_1 + S_2 = 2\pi + 8.96177 \approx 15.24$.

39 The area of region R_1 is $\int_0^{\pi/3} \sec^2 x\, dx = \tan x\Big|_0^{\pi/3} = \sqrt{3}$. The area of region R_2 can be calculated by subtracting the area of R_1 from the area of a rectangle with base $\dfrac{\pi}{3}$ and height 4. Thus the area of R_2 is $\dfrac{4\pi}{3} - \sqrt{3}$.

40

a.



b. $R(x) = \int_a^x (f(t) - g(t))\,dt$, and $R'(x) = f(x) - g(x)$.

41 $A(a) = \int_0^{\sqrt[3]{a}} \left(\sqrt{x/a} - x^2/a\right)\,dx = \left(\frac{2}{3}\frac{x^{3/2}}{\sqrt{a}} - \frac{x^3}{3a}\right)\Big|_0^{\sqrt[3]{a}} = \frac{1}{3}$.

42

a. $V = \pi\int_0^4 (\sqrt{4-y})^2\,dy = \pi\int_0^4 (4-y)\,dy = \pi\left(4y - y^2/2\right)\Big|_0^4 = \pi(16 - 8 - (0-0)) = 8\pi$.

b. $V = 2\pi\int_0^2 x(4-x^2)\,dx = 2\pi\int_0^2 (4x - x^3)\,dx = 2\pi\left(2x^2 - x^4/4\right)\Big|_0^2 = 2\pi\left(8 - 4 - (0-0)\right) = 8\pi$.

43 From $x = 0$ to $x = 1$ the area function is $A(x) = (\sqrt{x})^2 = x$, while from $x = 1$ to $x = 2$ it is $A(x) = (2-x)^2$.

$$\int_0^1 x\,dx + \int_1^2 (2-x)^2\,dx = \frac{1}{2}x^2\Big|_0^1 - \frac{1}{3}(2-x)^3\Big|_1^2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

44 The region is the same as in the previous exercise. From $x = 0$ to $x = 1$ the slices are semicircles with diameter $\sqrt{x}$, so their area is $\frac{\pi(\sqrt{x})^2}{8} = \frac{\pi}{8}x$. From $x = 1$ to $x = 2$, the slices are semicircles with diameter $2 - x$, so their area is $\frac{\pi}{8}(2-x)^2$. Thus by the general slicing method, the volume is

$$\frac{\pi}{8}\left(\int_0^1 x\,dx + \int_1^2 (2-x)^2\,dx\right) = \frac{\pi}{8}\left(\frac{1}{2}x^2\Big|_0^1 - \frac{1}{3}(2-x)^3\Big|_1^2\right) = \frac{\pi}{8}\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{5\pi}{48}.$$

45 Because the cross sections are parallel to the x-axis, solve for x to get $x = y^2$ and $x = 2 - y$. Then from $y = 0$ to $y = 1$ the area function is $A(y) = (y^2)^2 = y^4$ and from $y = 1$ to $y = 2$ it is $A(y) = (2-y)^2$. The by the general slicing method, the volume is

$$\int_0^1 y^4\,dy + \int_1^2 (2-y)^2\,dy = \frac{y^5}{5}\Big|_0^1 - \frac{1}{3}(2-y)^3\Big|_1^2 = \frac{1}{5} + \frac{1}{3} = \frac{8}{15}.$$

46

$$V = \pi\int_0^2 ((-x^2 + 2x + 2)^2 - (2x^2 - 4x + 2)^2)\,dx = \pi\int_0^2 (24x - 24x^2 + 12x^3 - 3x^4)\,dx$$
$$= \pi\left(12x^2 - 8x^3 + 3x^4 - 3x^5/5\right)\Big|_0^2 = \frac{64\pi}{5}.$$

47

a. $V = 2\pi\int_0^1 x((1+\sqrt{x}) - (1-\sqrt{x}))\,dx = 4\pi\int_0^1 x^{3/2}\,dx = 4\pi\cdot\frac{2x^{5/2}}{5}\Big|_0^1 = \frac{8\pi}{5}$.

b. $V = \pi\int_0^2 (1^2 - ((1-y)^2)^2)\,dy = \pi\int_0^2 (1 - (y^4 - 4y^3 + 6y^2 - 4y + 1))\,dy = \pi\int_0^2 (-y^4 + 4y^3 - 6y^2 + 4y)\,dy = \pi\left(-\frac{y^5}{5} + y^4 - 2y^3 + 2y^2\right)\Big|_0^2 = \frac{8\pi}{5}$.

48 $V = \pi\int_0^{(\ln 2)/2} ((2e^{-x})^2 - (e^x)^2)\,dx = \pi\left(-2e^{-2x} - \frac{1}{2}e^{2x}\right)\Big|_0^{(\ln 2)/2} = \frac{\pi}{2}$.

49 It appears to be easiest to use shells, because then we won't have to split the integral up, and also the functions will integrate more easily. Solving for y we have $y = e^{x^2}$ and $y = e^{2-x^2}$. The height of each shell is then $e^{2-x^2} - e^{x^2}$ and the volume is

$$2\pi \int_0^1 x(e^{2-x^2} - e^{x^2})\, dx = 2\pi \left(-\frac{1}{2} e^{2-x^2} - \frac{1}{2} e^{x^2} \right) \Bigg|_0^1 = \pi(-e - e + e^2 + 1) = \pi(e^2 - 2e + 1) = \pi(e-1)^2.$$

50 Using washers, we have

$$\pi \int_0^{\pi/3} (4 - \sec^2 x)\, dx = \pi\ (4x - \tan x)\ \Bigg|_0^{\pi/3} = \pi \left(4\pi/3 - \sqrt{3} - (0 - 0)\right) = 4\pi^2/3 - \pi\sqrt{3}.$$

51 We use the shell method. $V = 2\pi \displaystyle\int_0^{\sqrt{3}/2} \frac{x}{\sqrt{1-x^2}}\, dx$. Let $u = 1 - x^2$ so that $du = -2x\, dx$. Substituting gives $V = 2\pi \displaystyle\int_1^{1/4} \frac{-1}{2\sqrt{u}}\, du = \pi \int_{1/4}^1 u^{-1/2}\, du = 2\pi\ \sqrt{u}\ \Bigg|_{1/4}^1 = 2\pi \left(1 - \frac{1}{2}\right) = \pi$.

52 Use the shell method. Then each shell has height $4 - x^2$ and the radius of the shell at x is $x - (-2) = x + 2$. The volume is then

$$\begin{aligned} V &= 2\pi \int_{-2}^2 (x+2)(4-x^2)\, dx = 2\pi \int_{-2}^2 (-x^3 - 2x^2 + 4x + 8)\, dx \\ &= 2\pi \left(-\frac{x^4}{4} - \frac{2}{3}x^3 + 2x^2 + 8x \right) \Bigg|_{-2}^2 = 2\pi \left(-\frac{32}{3} + 32 \right) = \frac{128\pi}{3}. \end{aligned}$$

53 We use the disk method. The radius of each disk is $4 - (x-2)^2 = 4x - x^2$, so the volume is

$$\begin{aligned} V &= \pi \int_0^4 (4x - x^2)^2\, dx = \pi \int_0^4 (x^4 - 8x^3 + 16x^2)\, dx \\ &= \pi \left(\frac{x^5}{5} - 2x^4 + \frac{16}{3}x^3 \right) \Bigg|_0^4 = \pi \left(\frac{1024}{5} - 512 + \frac{1024}{3} \right) = \frac{512\pi}{15}. \end{aligned}$$

54 The curves intersect when $6x = x^2 + 5$, or when $x^2 - 6x + 5 = (x-1)(x-5) = 0$, thus at $x = 1$ and $x = 5$. To revolve about the line $y = -1$, we use the washer method; the outer radius of each washer is $6x - (-1) = 6x + 1$ and the inner radius is $x^2 + 5 - (-1) = x^2 + 6$. The volume is thus

$$\begin{aligned} V &= \pi \int_1^5 ((6x+1)^2 - (x^2+6)^2)\, dx = \pi \int_1^5 (-x^4 + 24x^2 + 12x - 35)\, dx \\ &= \pi \left(-\frac{x^5}{5} + 8x^3 + 6x^2 - 35x \right) \Bigg|_1^5 = \pi \left(-625 + 1000 + 150 - 175 + \frac{1}{5} - 8 - 6 + 35 \right) \\ &= \pi \left(371 + \frac{1}{5} \right) = \frac{1856\pi}{5}. \end{aligned}$$

To revolve about the line $x = -1$, we use the shell method. The height of each shell is $6x - (x^2 + 5) = 6x - x^2 - 5$, and the radius at x is $x - (-1) = x + 1$. The volume is thus

$$\begin{aligned} V &= 2\pi \int_1^5 (x+1)(6x - x^2 - 5)\, dx = 2\pi \int_1^5 (-x^3 + 5x^2 + x - 5)\, dx = 2\pi \left(-\frac{x^4}{4} + \frac{5}{3}x^2 + \frac{1}{2}x^2 - 5x \right) \Bigg|_1^5 \\ &= 2\pi \left(-\frac{625}{4} + \frac{625}{3} + \frac{25}{2} - 25 + \frac{1}{4} - \frac{5}{3} - \frac{1}{2} + 5 \right) = \frac{256\pi}{3}. \end{aligned}$$

55 The two non-horizontal lines intersect at $(2, 4)$. To revolve about $y = -2$, we use the shell method. First solve for x to obtain $x = \frac{y}{2}$ and $x = 6 - y$; then the height of each shell is $6 - y - \frac{y}{2} = 6 - \frac{3}{2}y$, and the radius at y is $y - (-2) = y + 2$. The volume is

$$2\pi \int_0^4 (y + 2) \left(6 - \frac{3}{2}y\right) dy = 2\pi \int_0^4 \left(12 + 3y - \frac{3}{2}y^2\right) dy = 2\pi \left(12y + \frac{3}{2}y^3 - \frac{1}{2}y^3\right) \bigg|_0^4 = 80\pi.$$

To revolve about $x = -2$, we use the washer method. With the equations above, the outer radius of each asher is $6 - y - (-2) = 8 - y$ and the inner radius is $\frac{y}{2} - (-2) = 2 + \frac{y}{2}$. The volume is

$$V = \pi \int_0^4 \left((8 - y)^2 - \left(2 + \frac{y}{2}\right)^2\right) dy = \pi \int_0^4 \left(\frac{3}{4}y^2 - 18y + 60\right) dy$$
$$= \pi \left(\frac{1}{4}y^3 - 9y^2 + 60y\right) \bigg|_0^4 = \pi(16 - 144 + 240) = 112\pi.$$

56

a. $V_x = \pi \int_1^2 \frac{1}{x^2}\, dx = \pi \left(-1/x\right) \Big|_1^2 = \frac{\pi}{2}.$

$V_y = 2\pi \int_1^2 x \cdot x^{-1}\, dx = 2\pi$, so $V_y > V_x$.

b. $V_x = \pi \int_1^4 \frac{1}{x^6}\, dx = \pi \left(-x^{-5}/5\right) \Big|_1^4 = \frac{\pi}{5}(1 - \frac{1}{1024}).$

$V_y = 2\pi \int_1^4 x \cdot x^{-3}\, dx = 2\pi \left(-1/x\right) \Big|_1^4 = \frac{3\pi}{2}$, so $V_y > V_x$.

c. $V_x = \pi \int_1^a \frac{1}{x^{2p}}\, dx = \begin{cases} \frac{\pi}{1 - 2p}(-1 + a^{1-2p}) & \text{if } p \neq 1/2, \\ \pi \ln a & \text{if } p = 1/2. \end{cases}$

d. $V_y = 2\pi \int_1^a \frac{1}{x^{p-1}}\, dx = \begin{cases} \frac{2\pi(a^{2-p} - 1)}{2 - p} & \text{if } p \neq 2, \\ 2\pi \ln a & \text{if } p = 2. \end{cases}$

e. Using part d, let $h = 2 - p$, and note that V_y is continuous. So $\lim_{h\to 0} V_y = \lim_{p\to 2} \frac{2\pi(a^{2-p} - 1)}{2 - p} = \lim_{h\to 0} \frac{2\pi(a^h - 1)}{h} = 2\pi \ln a$, so $\lim_{h\to 0} \frac{(a^h - 1)}{h} = \ln a$. A similar calculation can be done with the result of part c.

f. No, $V_y > V_x$ for all values of a and p, because $\frac{1}{x^{2p}} < \frac{1}{x^{p-1}}$ for $x > 1$ and $p > 0$.

57 We use disks when revolving about the x-axis. The volume is

$$\pi \int_0^1 c^2x^2(1 - x)^2\, dx = \pi c^2 \int_0^1 (x^2 - 2x^3 + x^4)\, dx = \pi c^2 \left(\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right) \bigg|_0^1 = \pi c^2 \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{\pi c^2}{30}.$$

We use shells when revolving about the y-axis. The volume is

$$2\pi \int_0^1 x \cdot cx(1 - x)\, dx = 2\pi c \int_0^1 (x^2 - x^3)\, dx = 2\pi c \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right) \bigg|_0^1 = \frac{\pi c}{6}.$$

These are equal when $\frac{\pi c^2}{30} = \frac{\pi c}{6}$, which for $c \neq 0$ occurs when $c = 5$.

58 $L = \int_{-2}^{2} \sqrt{1+4}\,dx = \sqrt{5}(2-(-2)) = 4\sqrt{5}$.

59 $L = \int_{\sqrt{2}}^{\sqrt{5}} \sqrt{1+\frac{1}{x^2-1}}\,dx = \int_{\sqrt{2}}^{\sqrt{5}} \frac{x}{\sqrt{x^2-1}}\,dx$. Let $u = x^2-1$ so that $du = 2x\,dx$. Substituting gives $\frac{1}{2}\int_1^4 u^{-1/2}\,du = \left(\sqrt{u}\right)\Big|_1^4 = 2-1 = 1$.

60 Note that $y' = x^2/2 - 1/(2x^2)$, so $1+y'^2 = (x^2/2+1/(2x^2))^2$. $L = \int_1^2 \sqrt{1+y'^2}\,dy = \int_1^2 (x^2/2 + 1/(2x^2))\,dx = \left(x^3/6 - 1/(2x)\right)\Big|_1^2 = \frac{17}{12}$.

61 Note that $y' = 1/(2\sqrt{x}) - \sqrt{x}/2$, so $1+y'^2 = (1/(2\sqrt{x})+\sqrt{x}/2)^2$. $L = \int_1^3 \sqrt{1+y'^2}\,dy = \int_1^3 1/(2\sqrt{x}) + \sqrt{x}/2\,dx = \left(\sqrt{x}+x^{3/2}/3\right)\Big|_1^3 = 2\sqrt{3}-4/3$.

62 $f'(x) = \pi\cos\pi x$. Therefore the arc length is $L = \int_0^1 \sqrt{1+\pi^2\cos^2\pi x}\,dx \approx 2.305$.

63 $y' = 2(x+1)$, so $\sqrt{1+f'(x)^2} = \sqrt{1+(2x+2)^2} = \sqrt{4x^2+8x+5}$.
The arc length is $\int_2^4 \sqrt{4x^2+8x+5}\,dx \approx 16.127$.

64 $y' = x^2+x$, so $\sqrt{1+f'(x)^2} = \sqrt{1+(x^2+x)^2} = \sqrt{x^4+2x^3+x^2+1}$. So the arc length is $L = \int_0^2 \sqrt{x^4+2x^3+x^2+1}\,dx \approx 5.351$.

65 Note that $y' = 1/x$, so $1+y'^2 = \frac{x^2+1}{x^2}$, and $\sqrt{1+y'^2} = \frac{\sqrt{x^2+1}}{x}$.
So

$$L = \int_1^b \frac{\sqrt{x^2+1}}{x}\,dx = \left(\sqrt{x^2+1} - \ln\left(\frac{1+\sqrt{x^2+1}}{x}\right)\right)\Big|_1^b = \sqrt{b^2+1}-\sqrt{2}+\ln\left(\frac{(\sqrt{b^2+1}-1)(1+\sqrt{2})}{b}\right).$$

Using a computer algebra system, we see that this has value 2 for $b \approx 2.715$.

66

a. $S = 2\pi\int_0^2 (x^3/3)\sqrt{1+x^4}\,dx$. Let $u = 1+x^4$, so that $du = 4x^3\,dx$. Substituting gives $\frac{\pi}{6}\int_1^{17} u^{1/2}\,du = \frac{\pi}{9}\left(u^{3/2}\right)\Big|_1^{17} = \frac{\pi}{9}\left(17^{3/2}-1\right)$.

b. $V = 2\pi\int_0^2 x(x^3/3)\,dx = 2\pi\int_0^2 x^4/3\,dx = 2\pi\left(x^5/15\right)\Big|_0^2 = 2\pi(32/15-0) = 64\pi/15$.

c. $V = \pi\int_0^2 x^6/9\,dx = \pi\left(x^7/63\right)\Big|_0^2 = \pi(128/63) = 128\pi/63$.

67

a. $S = 2\pi\int_0^3 \sqrt{3x-x^2}\sqrt{1+\frac{(3-2x)^2}{4(3x-x^2)}}\,dx = 2\pi\int_0^3 \sqrt{3x-x^2}\sqrt{\frac{12x-4x^2+9-12x+4x^2}{4(3x-x^2)}}\,dx =$ $\pi\int_0^3 3\,dx = 3\pi(3-0) = 9\pi.$

b. $V = \pi\int_0^3 (3x-x^2)\,dx = \pi\left(3x^2/2 - x^3/3\right)\Big|_0^3 = \pi\,(27/2-9) = 9\pi/2.$

68 Consider the line $y = x/2$ over the interval $[0,8]$. If we revolve this around the x-axis, we generate a cone with radius 4 and height 8. The surface of this cone is $2\pi\int_0^8 (x/2)\sqrt{1+1/4}\,dx = \sqrt{5}\pi/2\left(x^2/2\right)\Big|_0^8 = 16\sqrt{5}\pi.$

69

a. $S = 2\pi\int_1^2 \left(\frac{x^4}{2}+\frac{1}{16x^2}\right)\sqrt{1+(2x^3-(1/8x^3))^2}\,dx =$ $2\pi\int_1^2 \left(\frac{x^4}{2}+\frac{1}{16x^2}\right)\sqrt{1+(4x^6-1/2+(64/x^6)}\,dx = 2\pi\int_1^2 \left(\frac{x^4}{2}+\frac{1}{16x^2}\right)\sqrt{4x^6+1/2+64/x^6}\,dx =$ $2\pi\int_1^2 \left(\frac{x^4}{2}+\frac{1}{16x^2}\right)\sqrt{(2x^3+(1/8x^3))^2}\,dx = 2\pi\int_1^2 \left(\frac{x^4}{2}+\frac{1}{16x^2}\right)\left(2x^3+\frac{1}{8x^3}\right)dx = 2\pi\int_1^2 (x^7 +$ $x/16 + x/8 + x^{-5}/128)\,dx = 2\pi\int_1^2 (x^7+3x/16+x^{-5}/128)\,dx = 2\pi\left(x^8/8+3x^2/32-x^{-4}/512\right)\Big|_1^2 =$ $2\pi((32+3/8-1/8192)-(1/8+3/32-1/512)) = \frac{263439\pi}{4096}.$

b. Using the fact that $\sqrt{1+f'(x)^2} = 2x^3+1/(8x^3)$ which was discovered during the previous calculation, we have $L = \int_1^2 (2x^3+x^{-3}/8)\,dx = \left(x^4/2 - x^{-2}/16\right)\Big|_1^2 = (8-1/64)-(1/2-1/16) = \frac{483}{64}.$

c. $V = 2\pi\int_1^2 (x^5/2+x^{-1}/16)\,dx = 2\pi\left(x^6/12+(1/16)\ln x\right)\Big|_1^2 = 2\pi(16/3+(\ln 2)/16-(1/12+0)) =$ $\frac{21\pi}{2}+\frac{\pi\ln 2}{8}.$

d.

$$V = \pi\int_1^2 (x^4/2+x^{-2}/16)^2\,dx = \pi\int_1^2 (x^8/4+x^2/16+x^{-4}/256)\,dx = \pi\left(x^9/36+x^3/48-x^{-3}/768\right)\Big|_1^2$$
$$= \pi\,(512/36+8/48-1/6144-(1/36+1/48-1/768)) = \frac{264341\pi}{18432}.$$

70 $m = \int_0^9 (3+2\sqrt{x})\,dx = \left(3x+4x^{3/2}/3\right)\Big|_0^9 = 63$ gm.

71 $m = \int_0^3 150e^{-x/3}\,dx = \left(-450e^{-x/3}\right)\Big|_0^3 = 450(1-e^{-1})$ gm.

72 $m = \int_0^2 dx + \int_2^4 2\,dx + \int_4^6 4\,dx = 2+4+8 = 14.$

73

a. Because $50 = \int_0^{.2} kx\,dx = \frac{kx^2}{2}\Big|_0^{0.2} = \frac{k}{50}$, we must have $k = 2500$.

$$W = \int_{0.2}^{0.7} 2500x\,dx = 1250x^2\Big|_{0.2}^{0.7} = 562.5\,\text{J}.$$

b. $f(0.2) = 0.2k = 50$, so $k = 250$, and thus

$$W = \int_{0.2}^{0.7} 250x\,dx = 125x^2\Big|_{0.2}^{0.7} = 56.25\,\mathrm{J}.$$

74 The force function is given by $f(y) = 9.8\left(7 - \frac{y}{4}\right)$. So the work required is

$$W = 9.8\int_0^8 \left(7 - \frac{y}{4}\right)\,dy = 9.8\left(7y - \frac{y^2}{8}\right)\Big|_0^8 = 9.8(56 - 8) = 9.8(48) = 470.4\,\mathrm{J}.$$

75

a. Note that the density of the chain is 2 kg/m.

$$W = \int_0^{10} 2(9.8)(10-y)\,dy = 19.6\left(10y - \frac{y^2}{2}\right)\Big|_0^{10} = 19.6\,(100-50) = 980\,\text{J}.$$

b. The lower 6 meters of the chain are lifted 4 meters. The work required for this is $6(2)(9.8)(4) = 470.4\,\text{J}$ The work required to lift the upper four meters of the chain is

$$\int_6^{10} 2(9.8)(10-y)\,dy = 19.6\left(10y - \frac{y^2}{2}\right)\Big|_6^{10} = 19.6(100-50-(60-18) = 156.8.$$

So the total work required is $470.4 + 156.8 = 672.2\,\text{J}$.

76 The density of the chain is 3 kg/m. The work to lift the mass 5 meters is $4(9.8)(5) = 196\,\text{J}$. The work to lift the chain is $W = \int_0^5 3(9.8)(5-y)\,dy = 29.4\left(5y - \frac{y^2}{2}\right)\Big|_0^5 = 29.4(25-12.5) = 29.4(12.5) = 367.5\,\text{J}$. Therefore the total work is $196 + 367.5 = 563.5\,\text{J}$.

77

a. $W = \int_0^6 1000(9.8)(8)(6-y)\,dy = 78,400\left(6y - \frac{y^2}{2}\right)\Big|_0^6 = 78,400(36-18) = 1,411,200\,\text{J}$.

b. $W = \int_0^2 1000(9.8)(8)(7-y)\,dy = 78,400\left(7y - \frac{y^2}{2}\right)\Big|_0^2 = 78,400(14-2) = 940,800\,\text{J}$.

78 $W = \int_0^6 \pi 1000(9.8)\cdot 16(6-y)\,dy = 16\pi(9800)\left(6y - \frac{y^2}{2}\right)\Big|_0^6 = 288\pi(9800) \approx 8,866,830$ J.

79

a.

$$\begin{aligned} W &= \int_0^6 1000(9.8)\frac{4}{9}\pi y^2(6-y)\,dy = \frac{39200}{9}\pi\int_0^6 (6y^2 - y^3)\,dy \\ &= \frac{39200}{9}\pi\left(2y^3 - \frac{y^4}{4}\right)\Big|_0^6 = \frac{39200}{9}\pi\,(432-324) \approx 1,477,805\,\text{J}. \end{aligned}$$

b. The work to pump out the top 3 feet is

$$\int_3^6 \frac{39200}{9}\pi(6y^2 - y^3)\,dy = \frac{39200}{9}\pi\left(2y^3 - \frac{y^4}{4}\right)\Big|_3^6 = \frac{39200}{9}\pi(432-324-(54-20.25)) \approx 1,015,592\,\text{J}.$$

Therefore, the work to pump out the bottom 3 feet is about $1,477,805 - 1,015,592 = 461,814\,\text{J}$.

80

$$\begin{aligned} W &= \int_{-2}^0 9.8(1000)(4-y^2)\pi(2-y)\,dy = 9800\pi\int_{-2}^0 (8-4y-2y^2+y^3)\,dy \\ &= 9800\pi\left(8y - 2y^2 - \frac{2y^3}{3} + \frac{y^4}{4}\right)\Big|_{-2}^0 = 9800\pi\left(16+8-\frac{16}{3}-4\right) = 9800\pi\cdot\frac{44}{3} \approx 451,552\,\text{J}. \end{aligned}$$

81 $W = \int_0^9 1000(9.8)\pi y(10-y)\,dy = 9800\pi\int_0^9 (10y - y^2)\,dy = 9800\pi\left(5y^2 - \frac{y^3}{3}\right)\Big|_0^9 = 9800\pi(405-243) \approx 4,987,592\,\text{J}$.

82 $F = \int_0^1 1000(9.8)(3-y)\,dy = 9800\left(3y - \frac{y^2}{2}\right)\Big|_0^1 = 9800(2.5) = 24,500\,\text{N}.$

83 $F = \int_0^1 1000(9.8)(3-y)\frac{y}{2}\,dy = 4900\int_0^1 (3y-y^2)\,dy = 4900\left(\frac{3y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 = 4900\left(\frac{3}{2} - \frac{1}{3}\right) \approx 5716.7\,\text{N}.$

84 $F = \int_{-2}^2 1000(9.8)(4-y)2\sqrt{4-y^2}\,dy = 19600\int_{-2}^2 4\sqrt{4-y^2}\,dy - 19600\int_{-2}^2 y\sqrt{4-y^2}\,dy.$ The first term in this sum is $19,600\cdot 4$ times the area of the upper half of a circle of radius 2 so it is $19600\cdot 4\cdot 2\pi = 156,800\pi$. The second term is zero because it is the integral of an odd function over a symmetric interval. Therefore $F = 156,800\pi \approx 492,602\,\text{N}$.

85 Orient the semicircle so that the center is at the point $(0,20)$. Then the force is given by

$$\int_0^{20} \rho g(20-y)(2)\sqrt{40y-y^2}\,dy.$$

. Let $u = 40y - y^2$ so that $du = (40-2y)\,dy$. Then we have

$$\rho g\int_0^{400} u^{1/2}\,du = \rho g\left(\frac{2}{3}u^{3/2}\right)\Big|_0^{400} \approx 5.2\times 10^7.$$

86 The equation of L_1 is $y = (2ap+b)(x-p) + f(p)$, which can be written $y = (2ap+b)x - ap^2 + c$. The equation of L_2 is $y = (2aq+b)(x-q) + f(q)$, which can be written $y = (2aq+b)x - aq^2 + c$. If we set these equal to each other and solve to find the point of intersection, we find $s = \frac{p+q}{2}$. Thus, the area

$$\begin{aligned} R_1 &= \int_p^s f(x) - ((2ap+b)x - ap^2 + c)\,dx = \int_p^s ax^2 - 2apx + ap^2\,dx \\ &= \left(ax^3/3 - apx^2 + ap^2x\right)\Big|_p^s = \frac{as^3}{3} - aps^2 + ap^2s - \frac{ap^3}{3} = \frac{a}{24}(q-p)^3. \end{aligned}$$

Similarly, the area

$$\begin{aligned} R_2 &= \int_s^q f(x) - ((2aq+b)x - aq^2 + c)\,dx = \int_s^q ax^2 - 2aqx + aq^2\,dx \\ &= \left(ax^3/3 - aqx^2 + aq^2x\right)\Big|_s^q = \frac{aq^3}{3} - \frac{as^3}{3} + aqs^2 - aq^2s = \frac{aq^3}{3} - \frac{as}{3}(s^2 - 3qs + 3q^2) = \frac{a}{24}(q-p)^3. \end{aligned}$$

Chapter 7

Logarithmic, Exponential, and Hyperbolic Functions

7.1 Logarithmic and Exponential Functions Revisited

7.1.1 The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

7.1.2 This represents the area under the function $\frac{1}{t}$ and above the t axis between the vertical line $t = 1$ and the vertical line $t = x$.

7.1.3 $\int 4^x \, dx = \int e^{x \ln 4} \, dx = \frac{1}{\ln 4} e^{x \ln 4} + C = \frac{4^x}{\ln 4} + C.$

7.1.4 The inverse is $y = e^x$, whose domain is $(-\infty, \infty)$ and whose range is $(0, \infty)$.

7.1.5 $3^x = e^{x \ln 3}$. $x^\pi = e^{\pi \ln x}$. $x^{\sin x} = e^{sinx \cdot \ln x}$.

7.1.6 $\frac{d}{dx} 3^x = \frac{d}{dx} e^{x \ln 3} = e^{x \ln 3} \cdot \ln 3 = \ln 3 \cdot 3^x$.

7.1.7 $\frac{d}{dx}(x \ln x^3) = x \cdot \frac{1}{x^3} \cdot 3x^2 + \ln x^3 = 3 + \ln x^3 = 3 + 3 \ln x = 3(1 + \ln x)$.

7.1.8 $\frac{d}{dx}(\ln(\ln(x))) = \frac{1}{\ln x} \cdot \frac{1}{x}$.

7.1.9 $\frac{d}{dx} \sin(\ln x) = \cos(\ln x) \cdot \frac{1}{x} = \frac{\cos(\ln x)}{x}$.

7.1.10 $\frac{d}{dx} \ln(\cos^2 x) = \frac{1}{\cos^2 x} \cdot (2 \cos x)(-\sin x) = -\frac{2 \sin x}{\cos x} = -2 \tan x$.

7.1.11 $\frac{d}{dx}((\ln 2x)^{-5}) = -5(\ln 2x)^{-6}(1/x) = -\frac{5}{x \ln^6 2x}$.

7.1.12 $\frac{d}{dx}(\ln^3(3x^2 + 2)) = 3 \ln^2(3x^2 + 2) \left(\frac{1}{3x^2 + 2} \right) (6x) = \frac{18x \ln^2(3x^2 + 2)}{3x^2 + 2}$.

7.1.13 $2x^{4x} = e^{4x \ln 2x}$, so

$$\frac{d}{dx} 2x^{4x} = \frac{d}{dx} e^{4x \ln 2x} = e^{4x \ln(2x)} \left(4x \cdot \frac{1}{x} + \ln 2x \cdot 4 \right) = (2x)^{4x}(4 + 4 \ln 2x) = 4^{2x+1} x^{4x}(1 + \ln 2x).$$

7.1.14 $\frac{d}{dx}x^\pi = \pi x^{\pi-1}$.

7.1.15 $\frac{d}{dx}2^{(x^2)} = 2^{(x^2)} \cdot \ln 2 \cdot (2x) = 2^{(x^2+1)}x\ln 2$.

7.1.16 $\frac{d}{dt}(\sin t)^{\sqrt{t}} = \frac{d}{dt}e^{\sqrt{t}\ln(\sin t)} = e^{\sqrt{t}\ln(\sin t)} \cdot \left(\sqrt{t}\cot t + \ln(\sin t) \cdot \frac{1}{2\sqrt{t}}\right)$, which can be written as

$$(\sin t)^{\sqrt{t}}\left(\sqrt{t}\cot t + \ln(\sin t) \cdot \frac{1}{2\sqrt{t}}\right) = (\sin t)^{\sqrt{t}}\left(\frac{2t\cot t + \ln(\sin t)}{2\sqrt{t}}\right).$$

7.1.17 $\frac{d}{dx}e^{2x\ln(x+1)} = e^{2x\ln(x+1)} \cdot \left(\frac{2x}{x+1} + 2\ln(x+1)\right)$, which can be written as

$$(x+1)^{2x}\left(\frac{2x}{x+1} + 2\ln(x+1)\right).$$

7.1.18 $\frac{d}{dx}e^{-(\ln x)^2} = e^{-(\ln x)^2} \cdot \left(-\frac{2\ln x}{x}\right)$, which can be written as $-\frac{x^{-\ln x} \cdot 2\ln x}{x}$.

7.1.19 $\frac{d}{dy}e^{\sin y\ln y} = e^{\sin y \ln y}\left(\cos y\ln y + \frac{\sin y}{y}\right)$, which can be written as $y^{\sin y}\left(\cos y\ln y + \frac{\sin y}{y}\right)$.

7.1.20 $\frac{d}{dt}e^{(\ln t)/t} = e^{(\ln t)/t}\left(\frac{t(1/t) - \ln t}{t^2}\right)$, which can be witten as $t^{1/t}\left(\frac{1-\ln t}{t^2}\right)$.

7.1.21 $\frac{d}{dx}e^{-10x^2} = e^{-10x^2} \cdot -20x = -20xe^{-10x^2}$.

7.1.22 $\frac{d}{dx}(x^e + e^x) = ex^{e-1} + e^x$.

7.1.23 $\frac{d}{dx}x^{2x} = \frac{d}{dx}e^{2x\ln x} = e^{2x\ln x}(2 + 2\ln x) = 2x^{2x}(1 + \ln x)$.

7.1.24 $\frac{d}{dx}x^{\tan x} = \frac{d}{dx}e^{\tan x\ln x} = e^{\tan x\ln x}((\tan x)/x + \sec^2 x\ln x) = x^{\tan x}\left(\frac{\tan x}{x} + \sec^2 x\ln x\right)$.

7.1.25 $\frac{d}{dx}(1/x)^x = \frac{d}{dx}e^{x\ln(1/x)} = \frac{d}{dx}e^{-x\ln x} = e^{-x\ln x}(-1 - \ln x) = \left(\frac{1}{x}\right)^x(-1-\ln x)$.

7.1.26 $\frac{d}{dx}(x^{(x^{10})}) = \frac{d}{dx}e^{(x^{10}\ln x)} = e^{(x^{10}\ln x)}(x^9 + 10x^9\ln x) = x^{(x^{10})}(x^9)(1 + 10\ln x)$.

7.1.27

$$\begin{aligned}\frac{d}{dx}(1+(4/x))^x = \frac{d}{dx}e^{x\ln(1+(4/x))} &= e^{x\ln(1+(4/x))}\left(\ln(1+(4/x)) + \frac{x}{1+(4/x)} \cdot -\frac{4}{x^2}\right)\\ &= (1+(4/x))^x(\ln(1+(4/x)) - \frac{4}{x+4})\end{aligned}$$

7.1.28

$$\begin{aligned}\frac{d}{dx}\cos(x^{2\sin x}) = \frac{d}{dx}\cos(e^{2\sin x\ln x}) &= -\sin(x^{2\sin x}) \cdot e^{2\sin x\ln x} \cdot \left(\frac{2\sin x}{x} + 2\cos x\ln x\right)\\ &= -\sin(x^{2\sin x}) \cdot x^{2\sin x} \cdot \left(\frac{2\sin x}{x} + 2\cos x\ln x\right).\end{aligned}$$

7.1.29 $\int_0^3 \frac{2x-1}{x+1}\,dx = \int_0^3 \left(2 - \frac{3}{x+1}\right)\,dx = (\,2x - 3\ln(x+1))\,\Big|_0^3 = 6(1-\ln 2).$

7.1.30 Let $u = 4x^3+7$. Then $du = 12x^2\,dx$. Substituting gives

$$\frac{1}{12}\int \frac{1}{u}\,du = \frac{1}{12}\ln|u| + C = \frac{1}{12}\ln|4x^3+7| + C.$$

7.1.31 Let $u = \ln x$ so that $du = \frac{1}{x}\,dx$. Then

$$\int_e^{e^2} \frac{dx}{x\ln^3 x}\,dx = \int_1^2 u^{-3}\,du = \frac{-1}{2}\left(u^{-2}\right)\Big|_1^2 = \frac{-1}{2}(1/4-1) = \frac{3}{8}.$$

7.1.32 Let $u = 1+\cos x$ so that $du = -\sin x\,dx$. Then

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,dx = \int_1^2 \frac{1}{u}\,du = (\,\ln u)\,\Big|_1^2 = \ln 2.$$

7.1.33 Let $u = 4+e^{2x}$ so that $du = 2e^{2x}\,dx$. Substituting gives

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(4+e^{2x}) + C.$$

7.1.34 Let $u = \ln(\ln x)$ so that $du = \dfrac{1}{x\ln x}\,dx$. Substituting yields

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln(\ln x)| + C.$$

7.1.35 Let $u = \ln(\ln x)$ so that $du = \dfrac{1}{x\ln x}\,dx$. Substituting yields

$$\int_{\ln 2}^{\ln 3} \frac{1}{u^2}\,du = \left(\frac{-1}{u}\right)\Big|_{\ln 2}^{\ln 3} = \frac{1}{\ln 2} - \frac{1}{\ln 3}.$$

7.1.36 Let $u = \ln(y^2+1)$ so that $du = \dfrac{2y}{y^2+1}\,dy$. Substituting yields

$$\frac{1}{2}\int_0^{\ln 2} u^4\,du = \frac{1}{2}\left(u^5/5\right)\Big|_0^{\ln 2} = (\ln 2)^5/10.$$

7.1.37 Let $u = -x^2/2$ so that $du = -x\,dx$. Substituting yields

$$-4\int_0^{-2} e^u\,du = 4\int_{-2}^0 e^u\,du = 4\left(e^u\right)\Big|_{-2}^0 = 4(1-e^{-2}) = 4 - \frac{4}{e^2}.$$

7.1.38 Let $u = \sin x$ so that $du = \cos x\,dx$. Then

$$\int \frac{e^{\sin x}}{\sec x}\,dx = \int e^{\sin x}\cos x\,dx = \int e^u\,du = e^u + C = e^{\sin x} + C.$$

7.1.39 Let $u = \sqrt{x}$ so that $du = \dfrac{1}{2\sqrt{x}}\,dx$. Then

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C.$$

7.1.40 Let $u = e^{z/2} + 1$, so that $du = \frac{1}{2}e^{z/2}\,dz$.
Then

$$\int_{-2}^{2} \frac{e^{z/2}}{e^{z/2}+1}\,dz = \int_{(1/e)+1}^{e+1} \frac{2}{u}\,du = (2\ln u)\Big|_{(1/e)+1}^{e+1} = 2\ln\left(\frac{e+1}{e^{-1}+1}\right) = 2\ln(e) = 2.$$

7.1.41 Let $u = e^x - e^{-x}$ so that $du = e^x + e^{-x}\,dx$. Then we have

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|e^x - e^{-x}| + C.$$

7.1.42 Note that the denominator of the integrand is $(e^x - e^{-x})^2$. Let $u = e^x - e^{-x}$ so that $du = (e^x + e^{-x})\,dx$. Substituting gives

$$\int_{3/2}^{8/3} u^{-2}\,du = (-1/u)\Big|_{3/2}^{8/3} = -3/8 + 2/3 = \frac{-9+16}{24} = \frac{7}{24}.$$

7.1.43 $\displaystyle\int_{-1}^{1} 10^x\,dx = \left(\frac{10^x}{\ln 10}\right)\Big|_{-1}^{1} = \frac{10 - 10^{-1}}{\ln 10} = \frac{99}{10\ln 10}.$

7.1.44 Let $u = \sin x$ so that $du = \cos x\,dx$. Then

$$\int_0^{\pi/2} 4^{\sin x}\cos x\,dx = \int_0^1 4^u\,du = \left(\frac{4^u}{\ln 4}\right)\Big|_0^1 = \frac{3}{\ln 4}.$$

7.1.45 $\displaystyle\int_1^2 (1 + \ln x)x^x\,dx = (x^x)\Big|_1^2 = 4 - 1 = 3.$

7.1.46 Let $u = 1/p$, so that $du = -\frac{1}{p^2}\,dp$. Then

$$\int_{1/3}^{1/2} \frac{10^{1/p}}{p^2}\,dp = \int_2^3 10^u\,du = \left(\frac{10^u}{\ln 10}\right)\Big|_2^3 = \frac{900}{\ln 10}.$$

7.1.47 Let $u = x^3 + 8$ so that $du = 3x^2\,dx$. Substituting gives

$$\frac{1}{3}\int 6^u\,du = \frac{6^u}{3\ln 6} + C = \frac{6^{x^3+8}}{3\ln 6} + C.$$

7.1.48 Let $u = \cot x$ so that $du = -\csc^2 x\,dx = \frac{-1}{\sin^2 x}\,dx$. Substituting yields

$$-\int 4^u\,du = \frac{-4^u}{\ln 4} + C = \frac{-4^{\cot x}}{\ln 4} + C.$$

7.1.49 Let $u = 3x^2 + 1$ so that $du = 6x\,dx$. Substituting yields

$$\frac{1}{6}\int e^u\,du = \frac{1}{6}e^u + C = \frac{1}{6}e^{3x^2+1} + C.$$

7.1.50 $\displaystyle\int 7^{2x}\,dx = \int e^{2x\ln 7}\,dx = e^{2x\ln 7}\cdot\frac{1}{2\ln 7} + C = \frac{7^{2x}}{2\ln 7} + C.$

7.1.51 $\displaystyle\int 3^{-2x}\,dx = \int e^{-2x\ln 3}\,dx = e^{-2x\ln 3}\cdot\frac{1}{-2\ln 3} + C = -\frac{3^{-2x}}{2\ln 3} + C = \frac{-1}{9^x\ln 9} + C.$

7.1.52 $\displaystyle\int_0^5 5^{5x}\,dx = \left(\frac{5^{5x}}{5\ln 5}\right)\Big|_0^5 = \frac{5^{25} - 1}{5\ln 5}.$

7.1.53 Let $u = x^3$, so that $du = 3x^2\,dx$. Then we have

$$\int \frac{1}{3}10^u\,du = \frac{10^u}{3\ln 10} + C = \frac{10^{x^3}}{3\ln 10} + C.$$

7.1.54 Let $u = \sin x$ so that $du = \cos x\,dx$. Then we have

$$\int_0^0 2^u\,du = \left(\frac{2^u}{\ln 2}\right)\Big|_0^0 = 0.$$

7.1.55 Let $u = \ln x$ so that $du = \frac{1}{x}\,dx$. Then we have

$$\int_0^{\ln(2)+1} 3^u\,du = \left(\frac{3^u}{\ln 3}\right)\Big|_0^{\ln(2)+1} = \frac{3\cdot 3^{\ln 2} - 1}{\ln 3}.$$

7.1.56 Let $u = \ln x$ so that $du = \frac{1}{x}\,dx$. Then we have

$$\int \frac{1}{4}\sin u\,du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos(\ln x) + C.$$

7.1.57 Let $u = \ln x$ so that $du = \frac{1}{x}\,dx$. Then we have

$$\int_0^2 u^5\,du = \left(\frac{u^6}{6}\right)\Big|_0^2 = \frac{32}{3}.$$

7.1.58 Let $u = \ln x$ so that $du = \frac{1}{x}\,dx$. Then we have

$$\int (u^2 + 2u - 1)\,du = u^3/3 + u^2 - u + C = (\ln x)^3/3 + (\ln x)^2 - \ln x + C.$$

7.1.59 Let $u = e^{3x} + e^{-3x}$ so that $du = 3(e^{3x} - e^{-3x})\,dx$. Substituting gives

$$\frac{1}{3}\int_2^{65/8} \frac{1}{u}\,du = \frac{1}{3}\,(\ln u)\Big|_2^{65/8} = \frac{1}{3}\,(\ln(65/8) - \ln 2) = \frac{1}{3}\ln(65/16).$$

7.1.60 Let $u = e^{2x} + 1$. Then $du = 2e^{2x}\,dx$. Substituting yields

$$\frac{1}{2}\int u^{-2}\,du = \frac{1}{2}\left(-\frac{1}{u}\right) = -\frac{1}{2(e^{2x}+1)}.$$

7.1.61 Let $u = 5 + \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}$. Substituting yields

$$\int 2e^u\,du = 2e^u + C = 2e^{5+\sqrt{x}} + C.$$

7.1.62 Note that $4^{2x} = (4^2)^x = 16^x$. So we are computing $\displaystyle\int_0^1 1\,dx = 1$.

7.1.63

h	$(1+2h)^{1/h}$	h	$(1+2h)^{1/h}$
10^{-1}	6.1917	-10^{-1}	9.3132
10^{-2}	7.2446	-10^{-2}	7.5404
10^{-3}	7.3743	-10^{-3}	7.4039
10^{-4}	7.3876	-10^{-4}	7.3905
10^{-5}	7.3889	-10^{-5}	7.3892
10^{-6}	7.3890	-10^{-6}	7.3891

Let $y = (1+2h)^{1/h}$. Then $\ln y = \dfrac{\ln(1+2h)}{h}$. $\displaystyle\lim_{h\to 0}\ln y = \lim_{h\to 0}\frac{\ln(1+2h)}{h} = \lim_{h\to 0}\frac{2}{1+2h} = 2$, so the limit of y as $h \to 0$ is e^2.

7.1.64

h	$(1+3h)^{2/h}$	h	$(1+3h)^{2/h}$
10^{-1}	190.0496	-10^{-1}	1253.25
10^{-2}	369.3558	-10^{-2}	442.2350
10^{-3}	399.8214	-10^{-3}	407.0834
10^{-4}	403.0659	-10^{-4}	403.7921
10^{-5}	403.3925	-10^{-5}	403.4651
10^{-6}	403.4252	-10^{-6}	403.4324
10^{-7}	403.4284	-10^{-7}	403.4288
10^{-8}	403.4288	-10^{-8}	403.4388
10^{-9}	403.4290	-10^{-9}	403.4290

Let $y = (1+3h)^{2/h}$. Then $\ln y = \dfrac{2\ln(1+3h)}{h}$. $\displaystyle\lim_{h\to 0} \ln y = \lim_{h\to 0} \frac{2\ln(1+3h)}{h} = \lim_{h\to 0} \frac{6}{1+3h} = 6$, so the limit of y as $h \to 0$ is e^6.

7.1.65

x	$\dfrac{2^x-1}{x}$	x	$\dfrac{2^x-1}{x}$
10^{-1}	.71773	-10^{-1}	.66967
10^{-2}	.69556	-10^{-2}	.69075
10^{-3}	.69339	-10^{-3}	.69291
10^{-4}	.69317	-10^{-4}	.69312
10^{-5}	.69315	-10^{-5}	.69314
10^{-6}	.69315	-10^{-6}	.69315

$$\lim_{x\to 0} \frac{2^x-1}{x} = \lim_{x\to 0} \frac{2^x \ln 2}{1} = \ln 2.$$

7.1.66

x	$\dfrac{\ln(1+x)}{x}$	x	$\dfrac{\ln(1+x)}{x}$
10^{-1}	.9531	-10^{-1}	1.0536
10^{-2}	.9950	-10^{-2}	1.0050
10^{-3}	.9995	-10^{-3}	1.0005
10^{-4}	1.0000	-10^{-4}	1.0001
10^{-5}	1.0000	-10^{-5}	1.0000
10^{-6}	1.0000	-10^{-6}	1.0000

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1/(1+x)}{1} = 1.$$

7.1.67

a. True. This follows because $e^{\ln xy} = xy = e^{\ln x} e^{\ln y} = e^{\ln x + \ln y}$, and because the exponential function is one-to-one.

b. False. Zero is not in the domain of the natural logarithm function.

c. False. For example, $\ln(1+1) = \ln 2 \neq \ln(1) + \ln(1) = 0$.

d. False. $e^{2\ln x} = e^{\ln x^2} = x^2 \neq 2^x$.

e. False. $\displaystyle\int_0^e \frac{1}{x}\,dx = \lim_{b\to 0+} \int_b^e \frac{1}{x}\,dx = \lim_{b\to 0^+} (\ln x)\Big|_b^e = 1 - \lim_{b\to 0^+} \ln b$, which does not exist.

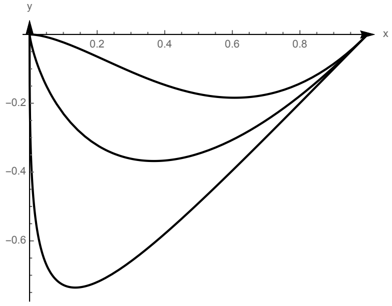
7.1.68 $\ln(x/y) = \int_1^{x/y} \frac{1}{t}\,dt = \int_1^{x} \frac{1}{t}\,dt + \int_x^{x/y} \frac{1}{t}\,dt$. For the second integral, let $u = x/t$ so that $du = \frac{1}{t}dx$. Then we have $\ln(x) + \int_1^{1/y} \frac{1}{u}\,du = \ln(x) + \ln(1/y)$. All that remains is to prove that $\ln(1/y) = -\ln(y)$.

In the text it was proved that $\ln xy = \ln x + \ln y$, so if we apply this rule to $0 = \ln(1) = \ln(y(1/y)) = \ln(y) + \ln(1/y)$, we see that $\ln(1/y) = -\ln(y)$.

7.1.69 The average value is given by $\frac{1}{p-1}\int_1^p \frac{1}{x}\,dx = \frac{1}{p-1}\left(\ln x\right)\Big|_1^p = \frac{\ln p}{p-1}$. Note that

$$\lim_{p\to\infty} \frac{\ln p}{p-1} = \lim_{p\to\infty} \frac{1/p}{1} = 0.$$

7.1.70 Each has limit 0 as $x \to 0^+$ (by L'Hôpital's rule), and each has value 0 at $x = 1$. Each is negative on $(0,1)$. Note that $\frac{d}{dx}x^p \ln x = x^{p-1} + px^{p-1}\ln x = x^{p-1}(1 + p\ln x)$, which is 0 on $(0,1)$ for $x = e^{-1/p}$. This critical point leads to a minimum in each case, but the location of the minimums increases as p increases, and the value of the minimum increases as well.



7.1.71

a. No. Let

$$h(a) = \int_{1-a}^{1+a} \frac{1-x}{x}\,dx = \int_{1-a}^{1} \frac{1-x}{x}\,dx + \int_1^{1+a} \frac{1-x}{x}\,dx = -\int_1^{1-a} \frac{1-x}{x}\,dx + \int_1^{1+a} \frac{1-x}{x}\,dx.$$

Then

$$h'(a) = \frac{1-(1-a)}{1-a} + \frac{1-(1+a)}{1+a} = \frac{a}{1-a} - \frac{a}{1+a} = \frac{a+a^2+(-a)+a^2}{1-a^2} = \frac{2a^2}{1-a^2} > 0.$$

Also, $h(a) = 0$. Because h is increasing on $(0,1)$, there are no other numbers on that interval where h is zero.

b. No. Let

$$g(a) = \int_{1/a}^{a} f(x)\,dx = \int_{1/a}^{1/2} f(x)\,dx + \int_{1/2}^{a} f(x)\,dx = -\int_{1/2}^{1/a} f(x)\,dx + \int_{1/2}^{a} f(x)\,dx.$$

Then

$$g'(a) = -f(1/a)\cdot -\frac{1}{a^2} + f(a) = \frac{1-(1/a)}{a} + \frac{1-a}{a} = \frac{2-(a+1/a)}{a} < 0,$$

because $a + 1/a > 2$ for $a > 1$. Because g has value 0 at $a = 1$ and is decreasing on $(1,\infty)$ it is never 0 on that interval.

7.1.72 For $x > 0$, $\frac{d}{dx}\ln x = \frac{1}{x}$ For $x < 0$, $\frac{d}{dx}\ln(-x) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$. So $\frac{d}{dx}\ln|x| = \frac{1}{x}$.

7.1.73

a. Let $\exp 0 = z$. Then $\ln \exp 0 = \ln z$, so $\ln z = 0$. Then because $\ln z = \ln 1$, we have $z = 1$, so $\exp 0 = 1$.

b. $\ln\left(\dfrac{\exp x}{\exp y}\right) = \ln(\exp x) - \ln(\exp y) = x - y$. Thus, $\exp(x-y) = \dfrac{\exp x}{\exp y}$.

c. $\ln((\exp x)^p) = p\ln(\exp x) = px$, so $\exp px = (\exp x)^p$.

7.1.74

a. The region under $1/x$ and above the x-axis between $x = 1$ and $x = 2$ has area $\displaystyle\int_1^2 1/x\,dx = (\ln x)\Big|_1^2 = \ln 2$.

b. Because the region bounded by the x-axis and the lines $y = 1/2$ and $x = 1$ and $x = 2$ has area $1/2$, and because this region is encompassed by the region under $y = 1/x$ and above the x-axis between $x = 1$ and $x = 2$, we must have $\ln 2 > 1/2$.

c. Let $n > 0$. Because of the result of part b), we must have $n\ln 2 > n/2$, so $\ln 2^n > n/2$. Also, $-n\ln 2 < -n/2$, so $\ln(2^{-n}) < -n/2$.

d. Because $n/2 \to \infty$ as $n \to \infty$, and because $2^n > 2/n$, we must have $\ln(2^n) \to \infty$ as $n \to \infty$. So $\ln x \to \infty$ as $x \to \infty$.

Also because $\ln(2^{-n}) \to -\infty$ as $n \to \infty$ (because $\ln(2^{-n}) < -n/2$), we must have that $\ln x \to -\infty$ as $x \to 0^+$.

7.1.75 Because $1/x$ is decreasing, we know that the left Riemann sum is an overestimate. Using $n = 2$ subintervals, we have $\displaystyle\int_1^2 1/x\,dx = \ln 2 < \frac{1}{1}\cdot\frac{1}{2} + \frac{1}{3/2}\cdot\frac{1}{2} = \frac{5}{6} < 1$.

Because $1/x$ is decreasing, we know that the right Riemann sum is an underestimate. We will use $n = 8$ subintervals. We have $\displaystyle\int_1^3 1/x\,dx = \ln 3 > 2\left(\frac{1}{9} + \frac{1}{11} + \cdots + \frac{1}{21}\right) > 1$.

7.1.76 With y fixed, we have $\dfrac{d}{dx}\ln(xy) = \dfrac{1}{xy}\cdot y = \dfrac{1}{x} = \dfrac{d}{dx}\ln(x)$.

Thus, $\ln(xy) = \ln(x) + C$. If we let $x = 1$, we see that $\ln(y) = 0 + C$, so $C = \ln(y)$, and $\ln(xy) = \ln(x) + \ln(y)$.

7.1.77 Because $1/x$ is decreasing, the left Riemann sum is an overestimate to the integral. Over the interval $[1, n+1]$ with n subdivisions, we must have $\displaystyle\int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1) < \frac{n+1-1}{n}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}) = \sum_{i=1}^{n}\frac{1}{i}$. Because $\ln(n+1) \to \infty$ as $n \to \infty$, it must follow that the harmonic partial sum $\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \to \infty$ as well.

7.1.78 Note that the area of the triangle described is $\dfrac{1}{2}xy$, so if $xy \ge \frac{1}{2}$, then $\dfrac{1}{2}xy \ge \dfrac{1}{4}$, which is not what we are seeking. However, if $xy < 1/2$, then the triangle formed will give us what we want.

Note that for any choice of x with $0 < x < 1/2$ and any choice of possible y, we have $xy < 1/2$ because $0 < y < 1$. Thus because there is a probability of $\dfrac{1}{2}$ of choosing $0 < x < 1/2$, the probability we seek is at least $1/2$. In addition, for $1/2 < x < 1$, if $y < \dfrac{1}{2x}$, then $xy < \dfrac{1}{2}$. So we should add $\displaystyle\int_{1/2}^1 \frac{1}{2x}\,dx = \left(\frac{1}{2}\ln x\right)\Big|_{1/2}^1 = (\ln 2)/2$. Thus the probability we seek is $\dfrac{1}{2}(1 + \ln 2)$.

7.2 Exponential Models

7.2.1 Exponential growth occurs for a constant relative growth rate.

7.2.2 The initial value, and the relative growth rate.

7.2.3 It is the time it takes the population to double in size.

7.2.4 It is the time it takes for the population to become half its size.

7.2.5 For exponential growth modeled by $y = y_0e^{kt}$, the doubling time T_2 is $T_2 = \frac{\ln 2}{k}$, where $k > 0$ is the growth constant.

7.2.6 For exponential decay modeled by $y = y_0e^{kt}$, the half-life $T_{1/2}$ is $T_{1/2} = \frac{\ln(1/2)}{k}$, where $k < 0$ is the decay constant.

7.2.7 The doubling time is $\frac{\ln 2}{k}$, so we solve $\frac{\ln 2}{k} = 20$, or $k = \frac{\ln 2}{20} \approx 0.03466$.

7.2.8 $\frac{\ln(1/2)}{k} = \frac{\ln(1/2)}{-0.05} \approx 13.9$ years.

7.2.9 Compound interest and world population growth.

7.2.10 Radioactive decay and depreciation of durable goods.

7.2.11 We have $y(t) = y_0e^{kt}$ and $y(1) = y_0e^k = 1.11y_0$. Therefore $e^k = 1.11$, and $k = \ln 1.11 \approx 0.10436$.

7.2.12 We have $y(t) = y_0e^{kt}$ and $y(1) = y_0e^k = 0.955y_0$. Therefore $e^k = 0.955$, and $k = \ln 0.955 \approx -0.046$.

7.2.13 For $f(t)$, $\frac{df}{dt} = 10.5$, so the absolute growth rate is constant. For $g(t)$, $\frac{dg}{dt} = 100e^{t/10} \cdot \frac{1}{10} = 10e^{t/10}$, so the growth rate is not constant but the relative growth rate $\frac{1}{g(t)}g'(t) = \frac{1}{10}$ is constant.

7.2.14 For $f(t)$, $\frac{df}{dt} = 400$, so the absolute growth rate is constant. For $g(t)$, $\frac{dg}{dt} = 400 \cdot 2^{t/20} \cdot \frac{\ln 2}{20}$, so the growth rate is not constant but the relative growth rate $\frac{1}{g(t)}g'(t) = \frac{\ln 2}{20}$ is constant.

7.2.15

a. The growth is modeled by $p(t) = 90,000e^{kt}$, with $t = 0$ corresponding to 2016, and time measured in years. Because $(1.024)(90,000) = 90,000e^k$, we have $k = \ln(1.024) \approx 0.02373$.

b. $120,000 = 90,000e^{t\ln 1.024}$, so $\frac{4}{3} = e^{t\ln 1.024}$, and $\ln\frac{4}{3} = t\ln 1.024$, so $t = \frac{\ln(4/3)}{\ln 1.024} \approx 12.1$ years. The population will reach 120,000 in 2028.

7.2.16

a. The growth is modeled by $y(t) = 2.115e^{kt}$ where $t = 0$ corresponds to 2015 and time is measured in years. Because the growth rate is 1.5%, $k = \ln(1.015) \approx 0.01489$.

b. In 2025 we would have $t = 10$. We have $y(10) = 2.115e^{10\ln 1.015} \approx 2.455$ million people.

7.2.17

a. The growth is modeled by $y(t) = 50000e^{kt}$. When $t = 10$, we have $p(10) = 50000e^{10k} = 55000$, so $k = \frac{\ln 1.1}{10} \approx 0.00953$.

b. In 20 years, the population should be about $50000e^{2\ln 1.1} = 50000(1.1)^2 = 60,500$.

7.2.18

a. Our model is given by $y(t) = 1500e^{kt}$, and $y(1) = 1500 \cdot 1.031 = 1500e^k$, so $k = \ln(1.031) \approx 0.03053$.

b. We seek t so that $2500 = 1500e^{\ln(1.031)t}$. Thus $\ln(5/3) = \ln(1.031)t$, so $t = \dfrac{\ln(5/3)}{\ln(1.031)} \approx 16.7$ years.

7.2.19

a. The price of a cart of groceries is modeled by $y(t) = 100e^{kt}$ where $t = 0$ corresponds to 2010, and t is measured in years and $k = \ln(1.016) \approx 0.01587$.

b. The price of the groceries in 2025 when $t = 15$ should be about $y(15) = 100e^{15\ln(1.016)} \approx \$126.88$.

7.2.20

a. Because the doubling time is 6 weeks, $k = \dfrac{\ln 2}{6}$. The number of cells is modeled by $y(t) = 8e^{(\ln 2/6)t}$ where t is measured in weeks.

b. The population will reach 1500 cells when $\ln(1500/8) = \dfrac{\ln 2}{6}t$, so when $t = \dfrac{6\ln(1500/8)}{\ln 2} \approx 45.3$ weeks.

7.2.21 $1200 = 1000e^{5k}$, so $k = \dfrac{\ln 1.2}{5}$, and therefore $y(t) = 1000e^{t(\ln 1.2)/5}$. In one year, the account grows by a factor of $e^{(\ln 1.2)/5} \approx 1.0371$, so the APY is about 3.71%.

7.2.22 Because $73 = 64e^{33k}$, we have $\dfrac{73}{64} = e^{33k}$, so $33k = \ln\left(\dfrac{73}{64}\right)$, and $k = \dfrac{\ln(73/64)}{33} \approx 0.004$. We are looking for t so that $64e^{0.004t} = 100$. We have $e^{0.004t} = \dfrac{100}{64}$, so $t = \dfrac{\ln(100/64)}{0.004} \approx 111.572$ days, so about 112 days.

7.2.23

a. The population is modeled by $P(t) = 334.4e^{t\ln(1.0079)}$. The doubling time is $\dfrac{\ln 2}{\ln 1.0079} \approx 88.1$ years. The population in 2050 will be about $P(30) = 334.4e^{30\ln(1.0079)} \approx 423.44$ million.

b. If the growth rate is 0.7%, the doubling time is $\dfrac{\ln 2}{\ln 1.007} \approx 99.4$ years and the population in 2050 will be about $334.4e^{30\ln(1.007)} \approx 412.2$ million

7.2.24

a. $P(t)$ has the form $P(t) = 2000e^{kt}$, and because $P(1) = 1.013 \cdot 2000$, we have $k = \ln(1.013)$. Thus $P(t) = 2000e^{t\ln(1.013)}$.

b. This is given by $\displaystyle\int_0^4 P(t)\,dt = \frac{2000}{\ln(1.013)}e^{t\ln(1.013)}\Big|_0^4 \approx 8210.3$.

c. This is given by $\displaystyle\int_0^t P(s)\,ds = \frac{2000}{\ln(1.013)}e^{s\ln(1.013)}\Big|_0^t = -154844 + 154844e^{t\ln(1.013)}$.

7.2.25 The growth is modeled by $p(t) = 25.1e^{kt}$. When $t = 6$, we have $p(6) = 25.1e^{6k} = 26.47$, so $k = \dfrac{\ln(26.47/25.1)}{6} \approx 0.00886$. Thus $p(15) \approx 25.1e^{0.00886(15)} \approx 28.67$ million people.

7.2.26

a. Note that $k = \ln(1.015)$. The amount consumed over the year would be

$$\int_0^1 1.2e^{\ln(1.015)t}\,dt = \left(\frac{1.2}{\ln(1.015)}e^{\ln(1.015)t}\right)\Bigg|_0^1 \approx 1.20898$$

million barrels of oil.

b. $\displaystyle\int_0^t 1.2e^{\ln(1.015)x}\,dx = \left(\frac{1.2}{\ln(1.015)}e^{\ln(1.015)x}\right)\Bigg|_0^t = \frac{1.2}{\ln(1.015)}(e^{\ln(1.015)t} - 1)$ million barrels.

c. This will occur when $\dfrac{1.2}{\ln(1.015)}(e^{\ln(1.015)t} - 1) = 10$, or $e^{\ln(1.015)t} = \dfrac{10\ln(1.015)}{1.2} + 1$, or

$$t = \ln\left(\frac{10\ln(1.015)}{1.2} + 1\right)/(\ln(1.015)) \approx 7.85551\,\text{years}.$$

7.2.27 The homicide rate is modeled by $H(t) = 800e^{kt}$, where $t = 0$ corresponds to 2018, and time is measured in years, and $k = \ln 0.97$. The rate will reach 600 when $\dfrac{600}{800} = e^{(\ln 0.97)t}$, or $t = \dfrac{\ln(6/8)}{\ln 0.97} \approx 9.44$ years. So it should achieve this rate in 2027.

7.2.28 The amount of drug in the body is modeled by $y(t) = y_0e^{kt}$, where y_0 is the initial dose and time is measured in hours and $k = \ln(0.85)$. The amount in the body should be 10 percent of the initial dose when $0.1 = e^{\ln(0.85)t}$, or $t = \dfrac{\ln(0.1)}{\ln(0.85)} \approx 14.1681$ hours.

7.2.29 The amount of Valium in the bloodstream is modeled by $a(t) = 20e^{kt}$. If the half-life is 36 hours, then $36 = \dfrac{\ln 2}{k}$, so $k = \dfrac{\ln 1/2}{36}$. So $a(12) = 20e^{-12\ln(2)/36} = 20e^{-\ln(2)/3} \approx 15.87$ mg.

The concentration of Valium will reach 2 mg when $0.1 = e^{-\ln(2)t/36}$, which is when $t = \dfrac{-36\ln(0.1)}{\ln 2} \approx$ 119.59 hours.

7.2.30 The population is modeled by $p(t) = 1.2e^{\ln(.995)t}$. After 50 years, the population should be about $p(50) = 1.2e^{\ln(.995)\cdot 50} \approx 0.933975$ billion people (or about 934 million people). This rate is not sufficient to meet the goal of 700 million people.

7.2.31 The population is modeled by $p(t) = 1.853e^{kt}$. Because $p(6) = 1.853e^{6k} = 1.831$, we have $e^{6k} = \dfrac{1.831}{1.853}$, so $k = \dfrac{\ln(1.831/1.853)}{6}$. The population in 2025 is predicated to be $1.853e^{\frac{1}{6}\ln(1.831/1.853)15} \approx 1.798$ million. This may not be an appropriate long-term model because the downturn in the population of West Virginia may be temporary.

7.2.32

a. The value after t years is given by $V(t) = 2.5e^{kt}$. Because $V(1) = 2.5 \cdot 0.932$, we have $k = \ln(0.932)$. Thus, after 10 years, we have $V(10) = 2.5e^{10\ln(0.932)} \approx 1.2$ million dollars.

b. The value is 10 percent of the original when $0.10 = e^{t\ln(0.932)}$, which occurs when $t = \dfrac{\ln(0.10)}{\ln(0.932)} \approx 32.7$ years.

7.2.33 Let $y = 1000e^{kx}$ model the pressure at x feet above sea level. We know that $\frac{1}{3} = e^{30000k}$, so $k = \dfrac{\ln(1/3)}{30000}$. We want to know for what x does $\dfrac{1}{2} = e^{kx}$, so we are seeking $x = \dfrac{\ln 2}{k} = \dfrac{30000\ln(2)}{\ln 3} \approx 18,928$ feet above sea level.

The pressure is 1/100th of the sea-level pressure when $x = \dfrac{30000\ln(100)}{\ln 3} \approx 125,754$ feet.

7.2.34

a. The decay is modeled by $a(t) = a_0e^{kt}$ with $k = \dfrac{\ln(0.5)}{5730}$. We seek t so that $0.77a_0 = a_0e^{kt}$, so $t = \dfrac{\ln(0.77)}{k} = \dfrac{\ln(0.77)(5730)}{\ln(0.5)} \approx 2160.6$. So the cloth was painted about 2160.6 year ago.

b. In a similar manner to part (a) we have $t = \dfrac{\ln(0.062)(5730)}{\ln(0.5)} \approx 22,986.4$, so the wood was cut about 23,000 year ago.

7.2.35 The amount of U-238 in the rock is modeled by $a(t) = a_0e^{kt}$ with $k = \dfrac{\ln(0.5)}{4.5}$, where time is measured in billions of years. We seek t so that $a_0 = a_0e^{kt}$, so $t = \dfrac{\ln(0.85)}{k} = \dfrac{\ln(0.85)(4.5)}{\ln(0.5)} \approx 1.055$. So the cloth was painted about 1.055 billion years ago.

7.2.36

a. After t days there would be $y = 100e^{(-t\ln 2)/8}$ millicuries after t days.

b. We seek t so that $10 = 100e^{(-t\ln 2)/8}$, so $t = \dfrac{-8\ln(0.1)}{\ln 2} \approx 26.58$ days.

c. We seek t so that $10 = 105e^{(-t\ln 2)/8}$, so $t = \dfrac{-8\ln(2/21)}{\ln 2} \approx 27.1385$ days.

7.2.37 The amount of caffeine is modeled by $y(t) = e^{kt}$, and $0.8 = e^{2k}$, so $k = \dfrac{\ln 0.8}{2}$. The half-life is $\dfrac{\ln(0.5)}{k} = \dfrac{2\ln(0.5)}{\ln(0.8)} \approx 6.2$ hours.

7.2.38

a. Because the half-life is $\dfrac{\ln(0.5)}{k} = 5.7$, we have $k = \dfrac{\ln(0.5)}{5.7}$. The amount of caffeine in the bloodstream after 1 hour is $90e^{\ln(0.5)/5.7} \approx 79.7$ mg.

b. Right before the individual drinks the second cup of coffee (which is two hours after drinking the first cup) the amount of caffeine in the bloodstream is $90e^{2\ln(0.5)/5.7}$. As soon as he chugs the second cup, he has this amount plus 90 in his bloodstream, so $90e^{2\ln(0.5)/5.7} + 90$. Therefore the amount of caffeine t hours after the second cup of coffee is $y(t) = \left(90e^{2\ln(0.5)/5.7} + 90\right)e^{t\ln(0.5)/5.7}$, and $y(1)$ is

$$y(1) = \left(90e^{2\ln(0.5)/5.7} + 90\right)e^{\ln(0.5)/5.7} \approx 142.2\,\text{mg}.$$

7.2.39 Let $y(t)$ equal the cost of the light t years after 2018. Then $y(t) = 4e^{kt}$. Ten years from now, it is predicted that the light will cost 1/10 of its current cost, which means that $y(10) = 4/10 = 0.4$. Replace y with 0.4 and t with 10 in the equation $y(t) = 4e^{kt}$, we have $0.4 = 4e^{10k}$, or $e^{10k} = 0.1$. Solving for k we find that $k = (\ln 0.1)/10 \approx -0.2303$. This means that $y(t) = 4e^{-0.2302t}$ and the cost of the light in 2021 is predicted to be $y(3) = 4e^{-0.2302(3)} \approx \$2.01$.

7.2.40 Let y_0 equal the current light output of an LED package. In the equation $y(t) = y_0e^{kt}$, if we replace $y(t)$ with $20y_0$ and t with 10, we have $20y_0 = y_0e^{10k}$, or $20 = e^{10k}$. Solving for k, we find that $k = \dfrac{\ln 20}{10} \approx 0.2996$. So three years from now, $y(3) \approx 2.46y_0$. So three years from now, an LED package will produce almost two and a half times as much light output as the current LED package.

7.2.41 The initial volume is $V_0 = \dfrac{4\pi}{3}\left(\dfrac{5}{10000}\right)^3 = \dfrac{\pi}{6\times 10^9}$ cubic cm. $k = \dfrac{\ln 2}{35}$. Suppose $0.5 = V_0e^{kt} = \dfrac{\pi}{6\times 10^9}e^{(t\ln 2)/35}$. Then $\dfrac{t\ln 2}{35} = \ln\left(\dfrac{3\times 10^9}{\pi}\right)$, and $t = \dfrac{35}{\ln 2}\ln\left(\dfrac{3\times 10^9}{\pi}\right) \approx 1044$ days.

7.2.42 The time required to increase p-fold occurs when $py_0 = y_0e^{kt}$, or when $t = \dfrac{\ln p}{k}$. Thus, the tripling time is $\dfrac{\ln 3}{k}$.

7.2.43

a. False. If that was the correct formula, then $y(1) = y_0e^{.06} = (1.06184)y_0 \neq (1.06)y_0$.

b. False. If it increases by ten precent per year, then after 3 years it increases by a factor of $(1.1)^3 = 1.331$ which corresponds to 33.1 percent.

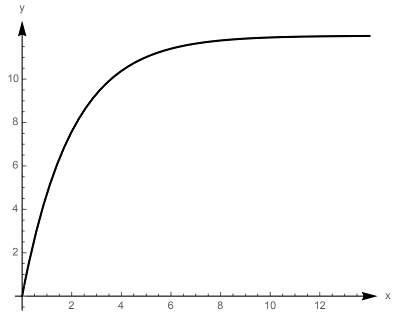
c. True. The relative decay rate is constant, so the decay is exponential.

d. True. This follows because the doubling time is related to k by the equation $T_2 = \dfrac{\ln 2}{k}$.

e. True. This time would be the constant $\dfrac{\ln 10}{k}$.

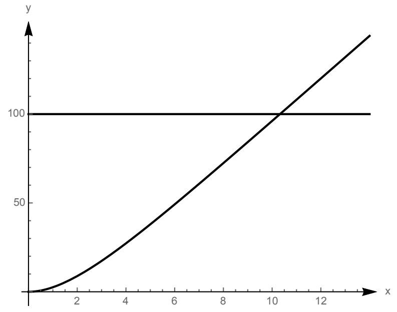
7.2.44

a. The runner's velocity approaches but does not attain 12 meters per second.



b. The position function is $s(t) = \displaystyle\int_0^t 12(1 - e^{-x/2})\,dx = \left(12(x + 2(e^{-x/2}))\right)\Big|_0^t = 12t + 24e^{-t/2} - 24$.

c. $s = 100$ when $12t+24e^{-t/2}-24 = 100$, which occurs when $t \approx 10.325$ seconds.



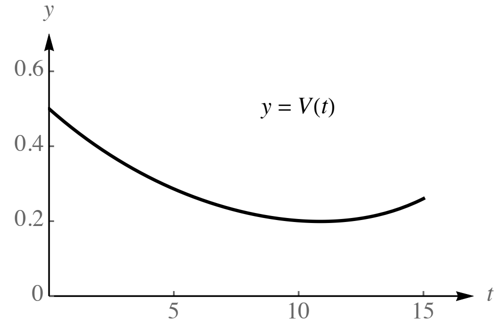
7.2.45

a. $V_1(t) = 0.99(0.5)e^{kt}$, where $k = \dfrac{\ln(0.5)}{5.7} \approx -0.1216$. So $V_1(t) = 0.495e^{-0.1216t}$.

b. $V_2(t) = 0.01(0.5)e^{kt}$, where $k = \dfrac{\ln 2}{2.9} \approx 0.239$. So $V_2(t) = 0.005e^{0.239t}$.

c. $V(t) = V_1(t) + V_2(t) = 0.495e^{-0.1216t} + 0.005e^{0.239t}$.

d. The tumor initially shrinks significantly in size but eventually starts growing again.



d. Solving the equation $V'(t) = 0$ for t, we find that $t \approx 10.9$ days. So it's best to treat the mouse again just before the end of the 10th day after the first treatment.

7.2.46

a. We are seeking t so that $500,000e^{\ln(1.03)t} = 300,000e^{\ln(1.05)t}$. This occurs when $\frac{5}{3} = e^{(\ln(1.05)-\ln(1.03))t}$, so $\ln(5/3) = (\ln(1.05) - \ln(1.03))t$, so $t = \dfrac{\ln(5/3)}{(\ln(1.05) - \ln(1.03))} \approx 26.5621$ years.

b. We are seeking y_0 and p so that $500,000e^{\ln(1.03)10} = y_0e^{\ln(1+p)10}$. This occurs when

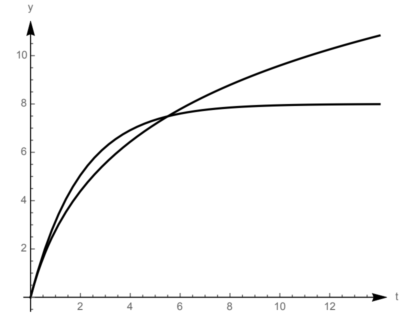
$$y_0 = 500,000e^{(10(\ln(1.03)-\ln(1+p)))} = 500,000\left(\frac{1.03}{1+p}\right)^{10}.$$

7.2.47

a. After 5 hours, Abe has run $\int_0^5 \frac{4}{t+1}\,dt = (4\ln(t+1))\big|_0^5 = 4\ln 6 \approx 7.17$ miles. After 5 hours, Bob has run $\int_0^5 4e^{-t/2}\,dt = \left(8e^{-t/2}\right)\Big|_5^0 = 8(1 - e^{-5/2}) \approx 7.34$. So after 5 hours, Bob is ahead.

After 10 hours, Abe has run $\int_0^{10} \frac{4}{t+1}\,dt = (4\ln(t+1))\big|_0^{10} = 4\ln 11 \approx 9.59$ miles. After 10 hours, Bob has run $\int_0^{10} 4e^{-t/2}\,dt = \left(8e^{-t/2}\right)\Big|_{10}^0 = 8(1 - e^{-5}) \approx 7.95$. So after 10 hours, Abe is ahead.

b. Bob's distance function is given by $8(1-e^{-t/2})$ and is bounded above by 8. Abe's distance function is given by $4\ln(t+1)$ and is unbounded.



7.2.48 $T_2 = \dfrac{\ln 2}{k} = \dfrac{\ln 2}{\ln(1+.0p)} \approx \dfrac{0.693}{\ln(1+.0p)} \approx \dfrac{70}{p}$. The approximation is best when $\ln(1 + .0p) \approx .0p$, which occurs when p is small.

7.2.49 $(1 + 0.008)^{12} - 1 \approx 10.034\%$, which is more than $12 \cdot 0.008 \approx 9.6\%$.

7.2.50

a. If $\frac{dv}{dt} = -kv$, then $\int \frac{1}{v}\frac{dv}{dt}\,dt = \int -k\,dt$, so $\ln(v(t)) = -kt + C$, so $v(t) = v_0 e^{-kt} = 10e^{-kt}$.

b. $s(t) = \int 10e^{-kt}\,dt = \frac{10}{-k}e^{-kt} + C = \frac{10}{-k}e^{-kt} + \frac{10}{k}$. (Because $s(0) = 0$.)

c. $dv/dt = \frac{dv}{ds}\frac{ds}{dt}$, so $-kv = \frac{dv}{ds}\cdot v$, so $\frac{dv}{ds} = -k$. Thus $v = -ks + C$ for some constant C, and because $v(0) = 10$, we have $v = 10 - ks$.

7.2.51 As in the previous problem, $s(t) = \frac{v_0}{k}(1 - e^{-kt})$. Consider the equations $s(t_2) - s(t_1) = 1200 = \frac{v_0}{k}(e^{-kt_1} - e^{-kt_2})$ and $v(t_2) - v(t_1) = -100 = v_0(e^{-kt_2} - e^{-kt_1})$. Dividing these two equations yields $k = \frac{1}{12}$.

Now $v(t_1) = 1000 = v_0 e^{-t_1/12}$ and $v(t_2) = 900 = v_0 e^{-t_2/12}$, so $\frac{1000}{900} = e^{(-1/12)(t_1 - t_2)}$, so $t_1 - t_2 = -12\ln(10/9) \approx -1.2643$ seconds. The deceleration takes about 1.2643 seconds to occur.

7.2.52 $R_T = \frac{y(t+T) - y(t)}{y(t)} = \frac{y_0 e^{k(t+T)} - y_0 e^{kt}}{y_0 e^{kt}} = \frac{y_0 e^{kt}(e^{kT} - 1)}{y_0 e^{kt}} = e^{kT} - 1$. Hence, R_T is constant for all t.

7.2.53 If $y_0 e^{kt} = y_0(1+r)^t$, then $(e^k)^t = (1+r)^t$, so $e^k = 1 + r$, and $k = \ln(1+r)$. If $y_0 e^{kt} = y_0 2^{t/T_2}$, then $(e^k)^t = (2^{1/T_2})^t$, so $e^k = 2^{1/T_2}$, and $k = \frac{\ln 2}{T_2}$, or $T_2 = \frac{\ln 2}{k}$. Also, we have $T_2 = \frac{\ln 2}{k} = \frac{\ln 2}{\ln(1+r)}$, so $T_2 \ln(1+r) = \ln 2$. Then $(1+r)^{T_2} = 2$, so $r = 2^{1/T_2} - 1$.

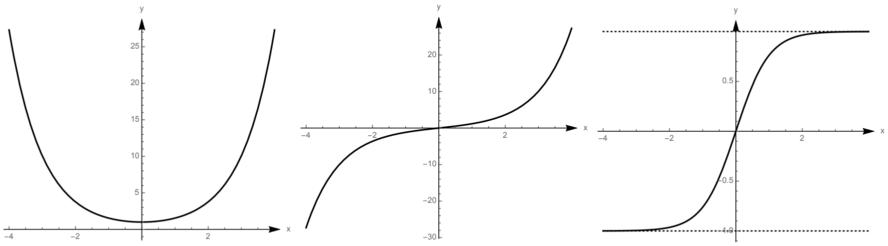
7.2.54 If $y(p) = \sqrt{y(m)y(n)}$, then $y_0 e^{kp} = \sqrt{y_0^2 e^{mk} e^{nk}}$, so $e^{kp} = e^{(m+n)k/2}$, so $p = \frac{m+n}{2}$.

7.2.55 The time required to double occurs when $2y_0 = y_0 e^{kt}$, or when $t = \frac{\ln 2}{k}$. Therefore, the doubling time is $\frac{\ln 2}{k}$ which is constant as a function of t.

7.3 Hyperbolic Functions

7.3.1 $\cosh x = \frac{e^x + e^{-x}}{2}$; $\sinh x = \frac{e^x - e^{-x}}{2}$.

7.3.2 From left to right, $y = \cosh x$ (an even function), $y = \sinh x$ (an odd function) , and $y = \tanh x$ (an odd function). Only $y = \tanh x$ has asymptotes: they are $y = 1$ and $y = -1$.



7.3.3 $\cosh^2 x - \sinh^2 x = 1$.

7.3.4 They have a very similar basic form, but the plus and minus signs appear in spots that aren't completely analogous.

7.3.5 $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$.

7.3.6 The domain is $\{x : 0 < x \leq 1\}$. $\operatorname{sech}^{-1} x = \cosh^{-1}(1/x)$.

7.3.7 Evaluate $\sinh^{-1}(1/5)$, which has the same value.

7.3.8 This is valid on the domain of the inverse hyperbolic tangent, which is the interval $(-1, 1)$.

7.3.9 $\displaystyle\int \frac{dx}{16-x^2} = \frac{1}{4}\coth^{-1}(x/4)+C$ for $|x| > 4$; in this case, the values in the interval of integration satisfy $|x| > 4$.

7.3.10 As a increases, the curves become wider and less steep. The value of the minimum at $x = 0$ increases.

7.3.11 $\displaystyle \tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \frac{e^{2x}-1}{e^{2x}+1}.$

7.3.12 $\displaystyle \tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = \frac{\frac{e^{-x}-e^{x}}{2}}{\frac{e^{-x}+e^{x}}{2}} = \frac{-\sinh x}{\cosh x} = -\tanh x.$

7.3.13

$$\begin{aligned}\cosh^2 x + \sinh^2 x &= \left(\frac{e^x+e^{-x}}{2}\right)^2 + \left(\frac{e^x-e^{-x}}{2}\right)^2 = \frac{e^{2x}+2+e^{-2x}}{4} + \frac{e^{2x}-2+e^{-2x}}{4} \\ &= \frac{2e^{2x}+2e^{-2x}}{4} = \frac{e^{2x}+e^{-2x}}{2} = \cosh(2x).\end{aligned}$$

7.3.14 $\displaystyle 2\sinh \ln \sec x = 2\left(\frac{e^{\ln \sec x} - e^{-\ln \sec x}}{2}\right) = \sec x - \cos x = \frac{1-\cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x} = \sin x \tan x.$

7.3.15 $\displaystyle \cosh x + \sinh x = \frac{e^x+e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \frac{2e^x}{2} = e^x.$

7.3.16 $\displaystyle \coth^2 x - 1 = \frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} = \operatorname{csch}^2 x.$

7.3.17 $\displaystyle \frac{1+\cosh 2x}{2} = \frac{1+\cosh^2 x + \sinh^2 x}{2} = \frac{\cosh^2 x + (\sinh^2 x + 1)}{2} = \frac{\cosh^2 x + \cosh^2 x}{2} = \cosh^2 x.$
$\displaystyle \frac{\cosh 2x - 1}{2} = \frac{\cosh^2 x + \sinh^2 x - 1}{2} = \frac{\sinh^2 x + (\cosh^2 x - 1)}{2} = \frac{\sinh^2 x + \sinh^2 x}{2} = \sinh^2 x.$

7.3.18 $\cosh(2x) = \cosh(x+x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x.$

7.3.19 $\displaystyle \frac{d}{dx}(\coth x) = \frac{d}{dx}\frac{\cosh x}{\sinh x} = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{csch}^2 x.$

7.3.20 $\displaystyle \frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}\frac{1}{\cosh x} = \frac{\cosh x \cdot 0 - \sinh x}{\cosh^2 x} = -\operatorname{sech} x \tanh x.$

7.3.21 $\displaystyle \frac{d}{dx}(\operatorname{csch} x) = \frac{d}{dx}\frac{1}{\sinh x} = \frac{\sinh x \cdot 0 - \cosh x}{\sinh^2 x} = -\operatorname{csch} x \coth x.$

7.3.22 $f'(x) = 4\cosh 4x$.

7.3.23 $f'(x) = 2(\cosh x)(\sinh x)$.

7.3.24 $f'(x) = -3(\sinh^2 4x)(\cosh 4x)(4) = -12(\sinh^2 4x)(\cosh 4x)$.

7.3.25 $f'(x) = 2(\tanh x)(\operatorname{sech}^2 x)$.

7.3.26 $f'(x) = \dfrac{1}{2\sqrt{\coth 3x}}\left(-\mathrm{csch}^2(3x)\right)(3) = \dfrac{-3\mathrm{csch}^2(3x)}{2\sqrt{\coth(3x)}}$.

7.3.27 $f'(x) = \dfrac{1}{\mathrm{sech}\, 2x}(-\mathrm{sech}\, 2x \tanh 2x)(2) = -2\tanh 2x$.

7.3.28 $f'(x) = \tanh x + x\,\mathrm{sech}^2 x$.

7.3.29 $f'(x) = 2x\cosh^2 3x + x^2(2\cosh(3x)\sinh(3x)(3) = 2x\cosh(3x)(\cosh(3x) + 3x\sinh(3x))$.

7.3.30 $f'(x) = \dfrac{\mathrm{csch}\, x + x\mathrm{csch}\, x \coth x}{\mathrm{csch}^2 x} = \dfrac{1}{\mathrm{csch}\, x} + \dfrac{x\coth x}{\mathrm{csch}\, x} = \sinh x + x\cosh x$.

7.3.31 $f'(x) = \dfrac{1}{\sqrt{16x^2-1}} \cdot 4 = \dfrac{4}{\sqrt{16x^2-1}}$.

7.3.32 $f'(t) = \dfrac{2}{1-t}\left(\dfrac{1}{2\sqrt{t}}\right) = \dfrac{1}{\sqrt{t}(1-t)}$.

7.3.33 $f'(v) = \dfrac{1}{\sqrt{v^4+1}} \cdot (2v) = \dfrac{2v}{\sqrt{v^4+1}}$.

7.3.34 $f'(x) = \dfrac{-1}{|2/x|\sqrt{1+4/x^2}}\left(\dfrac{-2}{x^2}\right) = \dfrac{1}{|x|\sqrt{1+4/x^2}} = \dfrac{1}{\sqrt{x^2+4}}$.

7.3.35 $f'(x) = \sinh^{-1}(x) + \dfrac{x}{\sqrt{x^2+1}} - \dfrac{x}{\sqrt{x^2+1}} = \sinh^{-1}(x)$.

7.3.36 $f'(u) = \dfrac{1}{\sqrt{\tan^2 u+1}} \cdot \sec^2 u = \dfrac{1}{\sqrt{\sec^2 u}} \cdot \sec^2 u = \dfrac{\sec^2 u}{|\sec u|} = |\sec u|$.

7.3.37 $\displaystyle\int \cosh 2x\, dx = \frac{1}{2}\sinh 2x + C$.

7.3.38 Let $u = \tanh x$ so that $du = \mathrm{sech}^x\, dx$. Then we have $\displaystyle\int \mathrm{sech}^2 x \tanh x\, dx = \int u\, du = u^2/2 + C = \frac{\tanh^2 x}{2} + C$. Note: It is also possible to let $u = \mathrm{sech}\, x$ and obtain a different-looking but equivalent answer.

7.3.39 Let $u = 1 + \cosh x$ so that $du = \sinh x\, dx$. Substituting gives $\displaystyle\int \frac{1}{u}\, du = \ln|u| + C = \ln|1+\cosh x| + C$.

7.3.40 Let $u = \coth x$ so that $du = -\mathrm{csch}^2 x\, dx$. Substituting gives $\displaystyle\int -u^2\, du = -u^3/3 + C = -(\coth x)^3/3 + C$.

7.3.41 Recall that $\tanh^2 x = 1 - \mathrm{sech}^2 x$. So we have $\displaystyle\int (1 - \mathrm{sech}^2 x)\, dx = x - \tanh x + C$.

7.3.42 Recall that $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$. Thus we have $\displaystyle\frac{1}{2}\int (\cosh 2x - 1)\, dx = \frac{1}{2}\left(\frac{\sinh 2x}{2} - x\right) + C = \frac{\sinh 2x}{4} - \frac{x}{2} + C$.

7.3.43 Let $u = \cosh 3x$ so that $du = 3\sinh 3x\, dx$. Substituting gives $\displaystyle\frac{1}{3}\int_1^{\cosh 3} u^3\, du = \left(u^4/12\right)\Big|_1^{\cosh 3} = \cosh^4 3/12 - 1/12 \approx 856.034$.

7.3.44 Let $u = \sqrt{x}$ so that $du = \dfrac{1}{2\sqrt{x}}\,dx$. Substituting yields $2\displaystyle\int_0^2 \operatorname{sech}^2 u\,du = 2\,(\tanh u)\Big|_0^2 = 2(\tanh(2) - 0) = 2\tanh(2) \approx 1.93$.

7.3.45 $\displaystyle\int_0^{\ln 2} \tanh x\,dx = \int_0^{\ln 2} \frac{\sinh x}{\cosh x}\,dx$. Let $u = \cosh x$ so that $du = \sinh x\,dx$. Substituting gives $\displaystyle\int_1^{5/4} \frac{1}{u}\,du = (\ln u)\Big|_1^{5/4} = \ln(5/4)$.

7.3.46 We use the formula in Theorem 6.9: $\displaystyle\int \operatorname{csch} y\,dy = \ln|\tanh(y/2)| + C$. We have $\displaystyle\int_{\ln 2}^{\ln 3} \operatorname{csch} y\,dy = (\ln|\tanh(y/2)|)\Big|_{\ln 2}^{\ln 3} = \ln(\tanh((\ln 3)/2)) - \ln(\tanh((\ln 2)/2)) \approx 0.405$.

7.3.47 We have $\displaystyle\frac{1}{8}\int \frac{dx}{1-(x/\sqrt{8})^2}\,dx$. Let $u = x/\sqrt{8}$ so that $du = \dfrac{dx}{\sqrt{8}}$. Substituting gives $\displaystyle\frac{1}{\sqrt{8}}\int \frac{du}{1-u^2} = \frac{1}{\sqrt{8}}\coth^{-1}(u) + C = \frac{1}{\sqrt{8}}\coth^{-1}(x/\sqrt{8}) + C$.

7.3.48 $\displaystyle\frac{1}{4}\int \frac{dx}{\sqrt{(x/4)^2-1}}$. Let $u = x/4$ so that $du = dx/4$. Substituting gives $\displaystyle\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1}(u) + C = \cosh^{-1}(x/4) + C$.

7.3.49 We have $\displaystyle\frac{1}{36}\int \frac{e^x}{1-(e^x/6)^2}\,dx$.

Let $u = e^x/6$ so that $du = e^x/6\,dx$. Substituting gives

$$\frac{1}{6}\int \frac{du}{1-u^2} = \frac{1}{6}\tanh^{-1}(u) + C = \frac{1}{6}\tanh^{-1}(e^x/6) + C.$$

7.3.50 This can be computed using part 5 of Theorem 7.9. We have

$$\int \frac{1}{x\sqrt{4^2+x^2}}\,dx = -\frac{1}{4}\operatorname{csch}^{-1}\left(\frac{|x|}{4}\right) + C.$$

7.3.51 We have $\displaystyle\frac{1}{2}\int \frac{dx}{x\sqrt{1-(x^4/2)^2}}$. Let $u = x^4/2$ so that $du = 2x^3\,dx$. Substituting gives

$$\frac{1}{2}\int \frac{du}{4u\sqrt{1-u^2}} = -\frac{1}{8}\operatorname{sech}^{-1} u + C = -\frac{1}{8}\operatorname{sech}^{-1}(x^4/2) + C.$$

7.3.52 Let $u = x^2$ so that $du = 2x\,dx$. Substituting gives

$$\int \frac{du}{2u\sqrt{1+u^2}} = -\frac{1}{2}\operatorname{csch}^{-1}|u| + C = -\frac{1}{2}\operatorname{csch}^{-1} x^2 + C.$$

7.3.53 $\displaystyle\int \frac{\cosh z}{\sinh^2 z}\,dz = \int \coth z \operatorname{csch} z\,dx = -\operatorname{csch} z + C$.

7.3.54 Let $u = \sin\theta$ so that $du = \cos\theta\,d\theta$. Substituting gives

$$\int \frac{du}{3^2-u^2} = \frac{1}{3}\tanh^{-1}(u/3) + C = \frac{1}{3}\tanh^{-1}((\sin\theta)/3) + C.$$

7.3.55 Let $u = \sinh^{-1} x$ so that $du = \frac{dx}{\sqrt{x^2+1}}$. Substituting gives

$$\int_{\sinh^{-1}(5/12)}^{\sinh^{-1}(3/4)} u\,du = \left(u^2/2\right)]\Big|_{\sinh^{-1}(5/12)}^{\sinh^{-1}(3/4)} = \frac{1}{2}\left(\left(\sinh^{-1}(3/4)\right)^2 - \left(\sinh^{-1}(5/12)\right)^2\right) \approx 0.158.$$

7.3.56 Consider $\int_{25}^{225} \frac{dx}{\sqrt{x}\sqrt{\sqrt{x}^2+25}}$, and let $u = \sqrt{x}$, so that $du = \frac{dx}{2\sqrt{x}}$. Substituting gives

$$2\int_5^{15} \frac{du}{\sqrt{u^2+5^2}} = 2\left(\sinh^{-1}(u/5)\right)\Big|_5^{15} = 2(\sinh^{-1}(3) - \sinh^{-1}(1)) = 2(\ln(3+\sqrt{10}) - \ln(1+\sqrt{2})) \approx 1.874.$$

7.3.57

a. Note that $\sinh \ln x = \frac{e^{\ln x} - e^{-\ln x}}{2} = \frac{x - 1/x}{2} = x/2 - 1/(2x)$. So $\int \frac{\sinh \ln x}{x} = \int (1/2 - x^{-2}/2)\,dx = x/2 + x^{-1}/2 + C = \frac{x^2+1}{2x} + C$.

b. Let $u = \ln x$, so that $du = \frac{1}{x}\,dx$. Substituting gives $\int \sinh u\,du = \cosh u + C = \cosh \ln x + C = \frac{e^{\ln x} + e^{-\ln x}}{2} + C = \frac{x + 1/x}{2} + C = \frac{x^2+1}{2x} + C$.

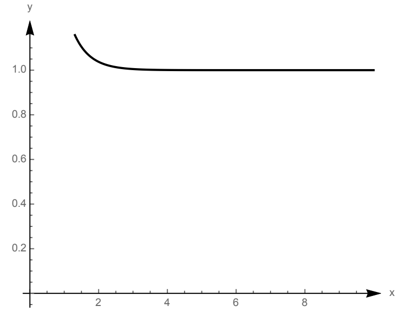
7.3.58

a. Note that $\cosh \ln x = \frac{e^{\ln x} + e^{-\ln x}}{2} = \frac{x + 1/x}{2} = \frac{x^2+1}{2x}$. Thus $\operatorname{sech} \ln x = \frac{1}{\cosh \ln x} = \frac{2x}{x^2+1}$. We have $\int_1^{\sqrt{3}} \frac{\sinh \ln x}{x}\,dx = \int_1^{\sqrt{3}} \frac{2}{x^2+1}\,dx = \left(2\tan^{-1} x\right)\Big|_1^{\sqrt{3}} = 2(\pi/3 - \pi/4) = \pi/6$.

b. Let $u = \ln x$, so that $du = \frac{1}{x}\,dx$. Substituting gives $\int_0^{\ln\sqrt{3}} \operatorname{sech} u\,du = \left(\tan^{-1}|\sinh(u)|\right)\Big|_0^{\ln\sqrt{3}} = \tan^{-1}(\frac{\sqrt{3} - (1/\sqrt{3})}{2}) - 0 = \tan^{-1}\left(1/\sqrt{3}\right) = \pi/6$.

7.3.59

a. The values of $\coth x$ on the interval $[5, 10]$ are very close to 1. So the area under the curve is very close to that of a 5×1 rectangle.

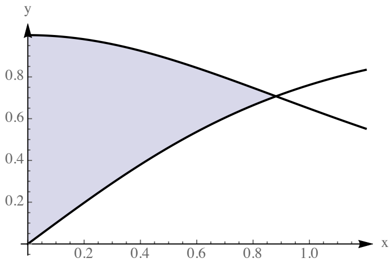


b. $\int_5^{10} \coth x\,dx = \int_5^{10} \frac{\cosh x}{\sinh x}\,dx = (\ln|\sinh x|)\Big|_5^{10} = \ln \sinh 10 - \ln \sinh 5 \approx 5.0000454$. The absolute value of the error is about 0.0000454.

7.3.60 Let $u = \sinh x$ so that $du = \cosh x\,dx$. Then $\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sinh x| + C$.

7.3.61

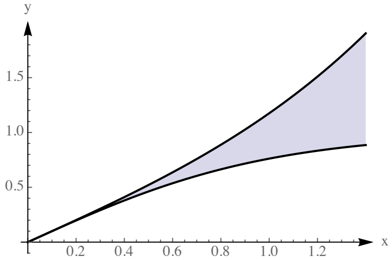
a. The curves intersect when $\frac{1}{\cosh x} = \frac{\sinh x}{\cosh x}$, or when $\sinh x = 1$, so at $x = \sinh^{-1}(1) = \ln(1+\sqrt{2})$.



b. We need to compute $A = \int_0^{\ln(1+\sqrt{2}} \operatorname{sech} x - \tanh x\, dx$. Recall that $\int \operatorname{sech} x\, dx = \tan^{-1}|\sinh x| + C$ and $\int \tanh x\, dx = \ln\cosh x + C$. Thus, $A = \left(\tan^{-1}\sinh x - \ln\cosh x\right)\Big|_0^{\ln(1+\sqrt{2})} = \tan^{-1}\sinh(\ln(1+\sqrt{2})) - \ln\cosh(\ln(1+\sqrt{2})) - (0-0) \approx 0.44$.

7.3.62

a. The curves intersect when $\sinh x = \dfrac{\sinh x}{\cosh x}$, or when $\sinh x = 0$, (or $\cosh x = 1$), so at $x = 0$.



b. We need to compute $\int_0^{\ln 3} \sinh x - \tanh x\, dx = (\cosh x - \ln\cosh x)\Big|_0^{\ln 3} = \cosh(\ln 3) - \ln\cosh\ln(3) - 1 = 5/3 - \ln(5/3) - 1 = 2/3 - \ln(5/3) \approx 0.156$.

7.3.63 Let $u = \ln x$ so that $du = \frac{dx}{x}$. Substituting gives

$$\int_0^2 \frac{du}{\sqrt{u^2+1}} = \left(\sinh^{-1} u\right)\Big|_0^2 = \sinh^{-1}(2) - 0 = \ln(2+\sqrt{5}).$$

7.3.64 $\displaystyle\int_5^{3\sqrt{5}} \frac{dx}{\sqrt{x^2-3^2}} = \left(\cosh^{-1}(x/3)\right)\Big|_5^{3\sqrt{5}} = \cosh^{-1}(\sqrt{5}) - \cosh^{-1}(5/3) = \ln(\sqrt{5}+2) - \ln(5/3+4/3) = \ln\left(\frac{\sqrt{5}+2}{3}\right)$.

7.3.65 $\displaystyle\int_{-2}^2 \frac{dt}{t^2-3^2} = -\frac{1}{3}\left(\tanh^{-1}(x/3)\right)\Big|_{-2}^2 = -\frac{1}{3}\left(\tanh^{-1}(2/3) - \tanh^{-1}(-2/3)\right) = -\frac{1}{3}\left(\frac{1}{2}\ln((5/3)/(1/3)) - \frac{1}{2}\ln((1/3)/(5/3))\right) = -\frac{1}{3}\ln 5 \approx -0.54$.

7.3.66 Let $u = 2t$ so that $du = 2\,dt$. Substituting gives $\displaystyle\int_{1/3}^{1/2} \frac{du}{u\sqrt{1-u^2}} = \left(-\operatorname{sech}^{-1} u\right)\Big|_{1/3}^{1/2} = \operatorname{sech}^{-1}(1/3) - \operatorname{sech}^{-1}(1/2) = \cosh^{-1}(3) - \cosh^{-1}(2) = \ln(3+\sqrt{8}) - \ln(2+\sqrt{3}) = \ln((3+\sqrt{8})/(2+\sqrt{3})) \approx 0.45$.

7.3.67 Let $u = x^{1/3}$ so that $du = \dfrac{1}{3}x^{-2/3}\,dx$. Substituting gives $\displaystyle\int_{1/2}^1 \frac{3u^2du}{u^3\sqrt{1+u^2}} = 3\int_{1/2}^1 \frac{du}{u\sqrt{1+u^2}} = -3\left(\operatorname{csch}^{-1} x\right)\Big|_{1/2}^1 = 3(\operatorname{csch}^{-1}(1/2) - \operatorname{csch}^{-1}(1)) = 3(\sinh^{-1}(2) - \sinh^{-1}(1)) = 3(\ln(2+\sqrt{5}) - \ln(1+\sqrt{2}) \approx 1.69$.

7.3.68 Let $u = \sinh x$ so that $du = \cosh x\,dx$. Substituting gives $\displaystyle\int_{12/5}^{40/9} \frac{du}{4-u^2} = \frac{1}{2}\left(\coth^{-1}(u/2)\right)\Big|_{12/5}^{40/9} =$ $\dfrac{1}{2}\left(\coth^{-1}(20/9) - \coth^{-1}(6/5)\right) = \dfrac{1}{2}\left(\tanh^{-1}(9/20) - \tanh^{-1}(5/6)\right) =$ $\dfrac{1}{4}\left(\ln((29/20)/(11/20)) - \ln((11/6)/(1/6))\right) = \dfrac{1}{4}\left(\ln(29/121)\right) \approx -0.36$.

7.3.69 By symmetry, we compute $I = 2\displaystyle\int_0^{\cosh^{-1}(17/15)} (17/15 - \cosh x)\,dx$ and divide by $2\cosh^{-1}(17/15)$. The value of I is $2\left((17/15)x - \sinh x\right)\Big|_0^{\cosh^{-1}(17/15)} = 2\left((17/15)\cosh^{-1}(17/15) - \sinh(\cosh^{-1}(17/15))\right) =$ $(34/15)\cosh^{-1}(17/15) - 16/15$. When we divide by $2\cosh^{-1}(17/15)$ we obtain $\dfrac{17}{15} - \dfrac{8}{15\cosh^{-1}(17/15)} \approx 0.09$.

7.3.70 $L = \displaystyle\int_0^a \sqrt{1+\sinh^2 x}\,dx = \int_0^a \cosh x\,dx = (\sinh x)\Big|_0^a = \sinh a$.

7.3.71

a. Let a be the value that produces a sag of 10 feet. Note that $f(0)+\text{sag} = f(50)$, so $a+\text{sag} = a\cosh(50/a)$, and dividing by a yields $1 + \dfrac{\text{sag}}{a} = \cosh(50/a)$, which can be written as $\cosh(50/a) - 1 = 10/a$.

b. If $t = 10/a$, we have $\cosh(5t) - 1 = t$. Solving for t yields $t \approx 0.08$.

c. If $0.08 = 10/a$, then $a = 10/0.08 = 125$. The length of the power line is $2\displaystyle\int_0^{50} \sqrt{1+\sinh^2(x/125)}\,dx =$ $2\displaystyle\int_0^{50} \cosh(x/125)\,dx = 250\left(\sinh(x/125)\right)\Big|_0^{50} = 250\sinh(50/125) = 250\sinh(2/5) \approx 102.7$ feet.

7.3.72 We have a right triangle with hypotenuse 50.5 and side adjacent to θ of length 50, so $\cos\theta = 50/50.5$, so $\theta = \cos^{-1}(50/50.5) \approx 0.14$ radians.

7.3.73 We are seeking λ so that $7 = \sqrt{\dfrac{9.8\lambda}{2\pi}\tanh\left(\dfrac{20\pi}{\lambda}\right)}$. A computer algebra system finds the approximate root to be 32.81 meters.

7.3.74

a. If $\lambda = 50$ and $d = 20$ then $v = \sqrt{\dfrac{(9.8)(50)}{2\pi}\tanh\left(\dfrac{40\pi}{50}\right)} \approx 8.77$ meters per second.

b. If $4.5 = \sqrt{\dfrac{(9.8)(15)}{2\pi}\tanh\left(\dfrac{2\pi d}{15}\right)}$, then $d \approx 3.14$ meters.

7.3.75

a. $f'(x) = \operatorname{sech}^2(x)$, so $f'(0) = 1$. The point of tangency is $(0, \tanh 0) = (0,0)$, so the linearization is $L(x) = 0 + 1(x-0) = x$.

b. When $d/\lambda < 0.05$, $2\pi d/\lambda$ is small. Because $\tanh x \approx x$ for small values of x, $\tanh(2\pi d/\lambda) \approx 2\pi d/\lambda$. Thus,

$$v = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi d}{\lambda}\right)} \approx \sqrt{\frac{g\lambda}{2\pi}\cdot\frac{2\pi d}{\lambda}} = \sqrt{gd}.$$

c. $v = \sqrt{gd}$ is a function of depth only. As $d \to 0$, $v \to 0$ as well.

7.3.76 With $\lambda = 150$ km which is 150,000 meters, we still have $\dfrac{d}{\lambda} = \dfrac{4000}{150000} \approx 0.027$, which is less than 0.05. The conclusions from the previous solution (part b) still apply.

7.3.77

a. False. $\sinh(\ln 3)$ is a constant, so its derivative is zero.

b. False. $\dfrac{d}{dx}\cosh x = \sinh x$, not $-\sinh x$.

c. True. Note that $-\ln(\sqrt{2}-1) = \ln\left(\dfrac{1}{\sqrt{2}-1}\right) = \ln\left(\dfrac{1}{\sqrt{2}-1}\cdot\dfrac{\sqrt{2}+1}{\sqrt{2}+1}\right) = \ln(\sqrt{2}+1)$.

d. False. $\displaystyle\int_0^1 \frac{dx}{2^2-x^2} = \frac{1}{2}\tanh^{-1}(x/2)\Big|_0^1 = \frac{1}{2}\tanh^{-1}\left(\frac{1}{2}\right) \neq \frac{1}{2}\left(\coth^{-1}\frac{1}{2} - \coth^{-1} 0\right)$.

7.3.78

a. $\coth(4) \approx 1.00067$

b. Does not exist

c. $\operatorname{csch}^{-1}(5) \approx 0.19869$

d. $\operatorname{csch}\Big|_{1/2}^{2} \approx -1.64331$

e. $\ln|\tanh(x/2)|\Big|_1^{10} \approx 0.771846$

f. $\tan^{-1}(\sinh x)\Big|_{-3}^{3} \approx 2.94261$

g. $\frac{1}{4}\coth^{-1}(x/4)\Big|_{20}^{36} \approx -0.0227902$

7.3.79

a. $\cosh(0) = 1$

b. $\tanh(0) = 0$

c. Does not exist

d. $\operatorname{sech}(0) = 1$

e. $\coth(\ln 5) = \dfrac{5+1/5}{5-1/5} = \dfrac{13}{12}$

f. $\sinh(\ln 9) = \dfrac{9-1/9}{2} = \dfrac{40}{9}$

g. $\left(\dfrac{e+(1/e)}{2}\right)^2 = \left(\dfrac{e^2+1}{2e}\right)^2$

h. Does not exist

i. $\ln\left(17/8 + \sqrt{\dfrac{17^2-8^2}{8^2}}\right) = \ln 4$

j. $\sinh^{-1}\left(\dfrac{e^1-e^{-1}}{2}\right) = 1$

7.3.80 $f'(x) = \cosh x > 0$ for all x, so f is increasing on $(-\infty,\infty)$. $f''(x) = \sinh x$, which is zero only for $x = 0$. Note that f'' is positive on $(0,\infty)$ and negative on $(-\infty,0)$, so f is concave up on $(0,\infty)$ and concave down on $(-\infty,0)$. There are no extrema, but there is an inflection point at $(0,0)$.

7.3.81 $f'(x) = 2\sinh x\cosh x\cosh x + \sinh^2 x\sinh x = \sinh x(2\cosh^2 x + \sinh^2 x)$. Note that $\sinh x = 0$ for $x = 0$, while $2\cosh^2 x + \sinh^2 x > 0$ for all x. So the only critical point is $x = 0$.

7.3.82

a. $f'(x) = \dfrac{x\sinh x - \cosh x}{x^2}$. This is zero when $x\sinh x = \cosh x$, or $x = \coth x$.

b. Using a computer algebra system, we find that $x = \pm 1.2$ are critical points of f.

7.3.83 $f'(x) = 2\tanh x \operatorname{sech}^2 x = \dfrac{2\sinh x}{\cosh^3 x}$.

$$f''(x) = \frac{2\cosh^4 x - 2(\sinh x)(3\cosh^2 x)(\sinh x)}{\cosh^6 x} = \frac{2\cosh^2 x - 6\sinh^2 x}{\cosh^4 x}.$$

This is zero for $\tanh^2 x = 1/3$, or $x = \tanh^{-1}(\pm\sqrt{1/3}) = \pm\tanh^{-1}(\sqrt{1/3})$.

7.3.84 $f'(x) = -\operatorname{sech} x \tanh x$. $f''(x) = (\operatorname{sech} x \tanh x)\tanh x - \operatorname{sech} x \operatorname{sech}^2 x = \operatorname{sech} x(\tanh^2 x - \operatorname{sech}^2 x)$. Because $\operatorname{sech} x > 0$ for all x, this quantity is zero when $\tanh x = \pm\operatorname{sech} x$, which occurs for $\sinh x = \pm 1$, or $x = \sinh^{-1}(\pm 1)$. The potential inflection points occur at $\ln(\sqrt{2}+1)$ and $\ln(\sqrt{2}-1)$. A check of the sign of f'' on the relevant intervals confirms that there are inflection points at these x-values.

7.3.85 This area would be the area under $\operatorname{sech} x$ between 0 and 1, minus the area of a quarter circle. Thus we have $\displaystyle\int_0^1 \operatorname{sech} x\,dx - \frac{\pi}{4} = \left(\tan^{-1}|\sinh x|\right)\Big|_0^1 - \frac{\pi}{4} = \tan^{-1}(\sinh(1)) - \frac{\pi}{4} \approx 0.08$.

7.3.86 $\displaystyle V = \pi\int_0^1 (\operatorname{sech}^2 x - (1-x^2))\,dx = \pi\left(\tanh x - x + x^3/3\right)\Big|_0^1 = \tanh(1) - 1 + 1/3 \approx 0.095$ cubic units.

7.3.87 Applying L'Hôpital's rule twice brings you back to the initial limit. If we write $\dfrac{\sinh x}{\cosh x} = \dfrac{(\sinh x)e^x}{(\cosh x)e^x} = \dfrac{e^{2x}-1}{e^{2x}+1}$, then applying L'Hôpital's rule once shows that the limit is 1.

7.3.88 $\displaystyle\lim_{x\to\infty}\frac{1-\coth x}{1-\tanh x} = \lim_{x\to\infty}\frac{\operatorname{csch}^2 x}{-\operatorname{sech}^2 x} = -\lim_{x\to\infty}\frac{\cosh^2 x}{\sinh^2 x} = -1.$

7.3.89 $\displaystyle\lim_{x\to 0}\frac{\tanh^{-1} x}{\tan(\pi x/2)} = \lim_{x\to 0}\frac{\frac{1}{1-x^2}}{(\pi/2)\sec^2(\pi x/2)} = \frac{2}{\pi}\lim_{x\to 0}\frac{\cos^2(\pi x/2)}{1-x^2} = \frac{2}{\pi}.$

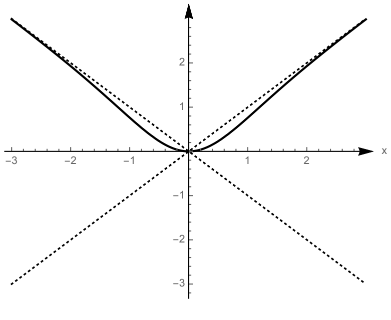
7.3.90

$$\begin{aligned}\lim_{x\to 1}\frac{\tanh^{-1} x}{\tan(\pi x/2)} &= \lim_{x\to 1}\frac{\frac{1}{1-x^2}}{(\pi/2)\sec^2(\pi x/2)} = \frac{2}{\pi}\lim_{x\to 1}\frac{\cos^2(\pi x/2)}{1-x^2}\\ &= \frac{2}{\pi}\lim_{x\to 1}\frac{2\cos(\pi x/2)(-\sin(\pi x/2))(\pi/2)}{-2x} = \frac{2}{\pi}\cdot\frac{0}{-2} = 0.\end{aligned}$$

7.3.91 Let $y = (\tanh x)^x$. Then $\ln y = x\ln(\tanh x) = \dfrac{\ln\tanh x}{1/x}$. Note that $\displaystyle\lim_{x\to 0}\frac{\ln\tanh x}{1/x} = \lim_{x\to 0}\frac{1/(\sinh x\cosh x)}{-1/x^2} = \lim_{x\to 0}\frac{-x^2}{\sinh x\cosh x} = \lim_{x\to 0}\frac{-2x}{\sinh^2 x + \cosh^2 x} = 0.$
Thus, $\displaystyle\lim_{x\to 0} y = \lim_{x\to 0}(\tanh x)^x = e^0 = 1.$

7.3.92

a. Note that $y = -x$ is also a slant asymptote, by symmetry.



b. Because $\lim_{x\to\infty} \tanh x = 1$, we expect $x \tanh x$ to behave like $x \cdot 1$ as x gets large.

c. Note that $x \tanh x - x = x\left(\frac{e^{2x}-1}{e^{2x}+1} - 1\right) = x\left(\frac{e^{2x}-1-(e^{2x}+1)}{e^{2x}+1}\right) = \frac{-2x}{e^{2x}+1}$. So $\lim_{x\to\infty} x\tanh x - x = \lim_{x\to\infty} \frac{-2x}{e^{2x}+1} = \lim_{x\to\infty}\frac{-2}{2e^{2x}} = 0$.

7.3.93 Let A be the area under $3 - \cosh x$ between $\cosh^{-1}(-3)$ and $\cosh^{-1}(3)$. Then the volume of the kiln will be $6A$. By symmetry, $A = 2\int_0^{\cosh^{-1}(3)} (3 - \cosh x)\,dx = 2\,(3x - \sinh x)\Big|_0^{\cosh^{-1}(3)} = 6\cosh^{-1}(3) - 2\sinh(\cosh^{-1}(3)) \approx 4.92$. So the volume of the kiln is about $6 \cdot 4.92 \approx 29.5$

7.3.94 Note that f has odd symmetry, so nonzero critical points should come in positive, negative pairs. $f'(x) = \operatorname{sech} x - x\operatorname{sech} x \tanh x = \operatorname{sech} x(1 - x\tanh x)$. Because $\operatorname{sech} x > 0$ for all x, any roots must come from solutions to the equation $1 - x\tanh x = 0$. Let $g(x) = 1 - x\tanh x$. Applying Newton's method with the recursion $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = x_n - \frac{1 - x_n \tanh x_n}{-\tanh x_n - x_n \operatorname{sech}^2 x_n}$ starting with $x_0 = 1$ yields $x_1 = 1.20177$, $x_2 = 1.9968$, $x_3 = 1.9968$ and so on, so the local extreme values for f occur at ± 1.9968.

7.3.95

a. $d(10) = \frac{75}{0.2}\ln\left(\cosh\left(\sqrt{\frac{(0.2)(9.8)}{75}} \cdot 10\right)\right) \approx 360.8$.

b. We need to solve $100 = \frac{75}{0.2}\ln\left(\cosh\left(\sqrt{\frac{(0.2)(9.8)}{75}} \cdot t\right)\right)$ for t. We have $t = \cosh^{-1}(e^{\frac{100\cdot 0.2}{75}}) \cdot \sqrt{\frac{75}{(0.2)(9.8)}} \approx 4.72$ seconds. The first 200 feet take $t = \cosh^{-1}(e^{\frac{200\cdot 0.2}{75}}) \cdot \sqrt{\frac{75}{(0.2)(9.8)}} \approx 6.97$ seconds, which means that the second 100 feet took about $6.97 - 4.72 = 2.25$ seconds. The average velocity over the first 100 feet is $\frac{100}{4.72} \approx 21.2$ meters per second, while over the second 100 feet is $\frac{100}{2.25} \approx 44.5$ meters per second.

7.3.96

a. $d'(t) = \frac{m}{k}\frac{1}{\cosh\left(\sqrt{\frac{kg}{m}}t\right)}\sinh\left(\sqrt{\frac{kg}{m}}t\right)\sqrt{\frac{kg}{m}} = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}t\right)$.

b. $d'(10) = \sqrt{\frac{(75)(9.8)}{0.2}}\tanh\left(\sqrt{\frac{(75)(9.8)}{0.2}}(10)\right) \approx 56.02$ meters per second.

c. We seek t so that $d'(t) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}t\right) = 45$. This can be written as

$60.6218\tanh(0.161658t) = 45$, so $t \approx \tanh^{-1}\left(\frac{45}{60.618}\right) \cdot \frac{1}{0.161658} \approx 5.911$ seconds.

7.3.97

a. $\sqrt{\frac{mg}{k}}\lim_{t\to\infty}\tanh\left(\sqrt{\frac{kg}{m}}t\right) = \sqrt{\frac{mg}{k}} \cdot 1 = \sqrt{\frac{mg}{k}}$.

b. $\sqrt{\frac{(75)(9.8)}{0.2}} \approx 60.6$ meters per second.

c. We seek t so that $v(t) = \sqrt{\dfrac{mg}{k}} \tanh\left(\sqrt{\dfrac{kg}{m}}t\right) = 0.95\sqrt{\dfrac{mg}{k}}$.

So $\tanh\left(\sqrt{\frac{kg}{m}}t\right) = 0.95$, and $t = \tanh^{-1}(0.95)\sqrt{\dfrac{m}{kg}}$.

d. It takes $\tanh^{-1}(0.95)\sqrt{\dfrac{75}{(0.2)(9.8)}} \approx 11.33$ seconds to achieve 95 percent of terminal velocity. It takes $d(11.33) \approx 436.5$ feet to achieve this. So the cliff should be at least $436.5 + 300 = 736.5$ feet high.

7.3.98

a. $a(t) = \dfrac{d}{dt}\sqrt{\dfrac{mg}{k}} \tanh\left(\sqrt{\dfrac{kg}{m}}t\right) = \sqrt{\dfrac{mg}{k}}\operatorname{sech}^2\left(\sqrt{\dfrac{kg}{m}}t\right)\sqrt{\dfrac{kg}{m}} = g\operatorname{sech}^2\left(\sqrt{\dfrac{kg}{m}}t\right)$.

b. $\lim\limits_{t\to\infty} g\operatorname{sech}^2\left(\sqrt{\dfrac{kg}{m}}t\right) = g\left(\lim\limits_{t\to\infty}\operatorname{sech}\left(\sqrt{\dfrac{kg}{m}}t\right)\right)^2 = g(0)^2 = 0$. Once terminal velocity is achieved, the velocity is constant, so the acceleration is zero.

7.3.99 For $y = A\sinh kx$, we have $y' = kA\cosh kx$ and $y'' = k^2A\sinh kx$. So $y'' - k^2y = k^2A\sinh kx - k^2A\sinh kx = 0$.

For $y = B\cosh kx$, we have $y' = kB\sinh kx$ and $y'' = k^2B\cosh kx$. So $y'' - k^2y = k^2B\cosh kx - k^2B\cosh kx = 0$.

7.3.100 Using symmetry,

$$\begin{aligned} A &= 4\pi\int_0^{\ln 2}\cosh x\sqrt{1+\sinh^2 x}\,dx = 4\pi\int_0^{\ln 2}\cosh^2 x\,dx = 4\pi\int_0^{\ln 2}\frac{1+\cosh 2x}{2}\,dx \\ &= 2\pi\left(x + (1/2)\sinh 2x\right)\Big|_0^{\ln 2} = 2\pi(\ln 2 + 15/16) = \pi(\ln 4 + 15/8) \approx 10.25. \end{aligned}$$

7.3.101

$$\begin{aligned} \sinh(\cosh^{-1}(x)) = \sinh(\ln(x+\sqrt{x^2-1})) &= \frac{e^{\ln(x+\sqrt{x^2-1})} - \frac{1}{e^{\ln(x+\sqrt{x^2-1})}}}{2} \\ &= \frac{1}{2}\left(x+\sqrt{x^2-1} - \frac{1}{x+\sqrt{x^2-1}}\right). \end{aligned}$$

Multiplying the last term by $\dfrac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}}$ gives

$$\frac{1}{2}\left((x+\sqrt{x^2-1}) - (x-\sqrt{x^2-1})\right) = \sqrt{x^2-1}.$$

7.3.102

$$\begin{aligned} \cosh(\sinh^{-1}(x)) = \cosh(\ln(x+\sqrt{x^2+1})) &= \frac{e^{\ln(x+\sqrt{x^2+1})} + \frac{1}{e^{\ln(x+\sqrt{x^2+1})}}}{2} \\ &= \frac{1}{2}\left(x+\sqrt{x^2+1} + \frac{1}{x+\sqrt{x^2+1}}\right). \end{aligned}$$

Multiplying the last term by $\dfrac{x-\sqrt{x^2+1}}{x-\sqrt{x^2+1}}$ gives

$$\frac{1}{2}\left((x+\sqrt{x^2+1}) - (x-\sqrt{x^2+1})\right) = \sqrt{x^2+1}.$$

7.3.103 $\cosh(x+y) = \dfrac{e^{x+y}+e^{-(x+y)}}{2} = \dfrac{2e^{x+y}+2e^{-(x+y)}}{4}$. This can be written as

$$\begin{aligned}
&\frac{1}{4}(e^{x+y}+e^{x-y}+e^{y-x}+e^{-(x+y)}+e^{x+y}-e^{x-y}-e^{y-x}+e^{-(x+y)})\\
&=\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^y+e^{-y}}{2}\right)+\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^y-e^{-y}}{2}\right)\\
&=\cosh x\cosh y+\sinh x\sinh y
\end{aligned}$$

7.3.104 $\sinh(x+y) = \dfrac{e^{x+y}-e^{-(x+y)}}{2} = \dfrac{2e^{x+y}-2e^{-(x+y)}}{4}$. This can be written as

$$\begin{aligned}
&\frac{1}{4}(e^{x+y}+e^{x-y}-e^{y-x}-e^{-(x+y)}+e^{x+y}-e^{x-y}+e^{y-x}-e^{-(x+y)})\\
&=\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^y+e^{-y}}{2}\right)+\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^y-e^{-y}}{2}\right)\\
&=\sinh x\cosh y+\cosh x\sinh y
\end{aligned}$$

7.3.105 First suppose $x \ge 0$. Then $\cosh^{-1}(\cosh x) = \ln(\cosh x + \sqrt{1+\cosh^2 x}) = \ln(\cosh x + \sqrt{\sinh^2 x}) = \ln(\cosh x + \sinh x) = \ln e^x = x$.

Now suppose $x < 0$. Then $\cosh^{-1}(\cosh x) = \ln(\cosh x + \sqrt{1+\cosh^2 x}) = \ln(\cosh x + \sqrt{\sinh^2 x}) = \ln(\cosh x - \sinh x) = \ln e^{-x} = -x$.

Thus $\cosh^{-1}(\cosh x) = |x|$.

7.3.106

a. Let $y = \cosh^{-1}(x)$. Then $x = \cosh y$, so $1 = \sinh y \cdot \frac{dy}{dx}$. Thus $\dfrac{dy}{dx} = \dfrac{1}{\sinh y} = \dfrac{1}{\sinh(\cosh^{-1}(x))} = \dfrac{1}{\sqrt{x^2-1}}$.

b. $\dfrac{d}{dx}\ln(x+\sqrt{x^2+1}) = \dfrac{1}{x+\sqrt{x^2+1}}\left(1+\dfrac{2x}{2\sqrt{x^2+1}}\right) = \dfrac{1}{x+\sqrt{x^2+1}}\left(\dfrac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right) = \dfrac{1}{\sqrt{x^2+1}}$.

7.3.107

a. $\displaystyle\int \operatorname{sech} x\,dx = \int \frac{\cosh x}{1+\sinh^2 x}\,dx$. Let $u = \sinh x$ so that $du = \cosh x\,dx$. Then we have $\displaystyle\int \frac{1}{1+u^2}\,du = \tan^{-1}(u) + C = \tan^{-1}(\sinh x) + C$.

b. Note that $\operatorname{sech} x = \dfrac{\operatorname{sech}^2 x}{\sqrt{\operatorname{sech}^2 x}} = \dfrac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}}$. So $\displaystyle\int \operatorname{sech} x\,dx = \int \frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}}\,dx$. Let $u = \tanh x$ so that $du = \operatorname{sech}^2 x\,dx$. Substituting gives $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C = \sin^{-1}(\tanh x) + C$.

c. $\dfrac{d}{dx}(2\tan^{-1}(e^x)+C) = 2\left(\dfrac{1}{1+e^{2x}}\right)e^x + 0 = \dfrac{2e^x}{1+e^{2x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \dfrac{2}{e^{-x}+e^x} = \dfrac{1}{\cosh x} = \operatorname{sech} x$.

7.3.108

a. Consider $I = \displaystyle\int \operatorname{csch} x\,dx$ and let $u = x/2$, so that $du = dx/2$. Substituting gives $I = 2\displaystyle\int \operatorname{csch} 2u\,du = \int \frac{2\,du}{\sinh 2u}$.

b. Because $\sinh 2x = 2\sinh x\cosh x$, we have $I = \displaystyle\int \frac{2\,du}{2\sinh u\cosh u} = \int \frac{1}{\sinh u\cosh u} \cdot \frac{\frac{1}{\cosh^2 u}}{\frac{1}{\cosh^2 u}}\,du = \int \frac{\operatorname{sech}^2 u}{\tanh u}\,du$.

c. Let $w = \tanh u$ so that $dw = \operatorname{sech}^2 u\,du$. Substituting gives $I = \displaystyle\int \frac{1}{w}\,dw = \ln|w| + C = \ln|\tanh u| + C = \ln|\tanh(x/2)| + C$.

7.3.109 $L = \displaystyle\int_{\ln 2}^{\ln 8} \sqrt{1+\operatorname{csch}^x}\,dx = \int_{\ln 2}^{\ln 8} \sqrt{\coth^2 x}\,dx = \int_{\ln 2}^{\ln 8} \coth x\,dx = \int_{\ln 2}^{\ln 8} \frac{\cosh x}{\sinh x}\,dx$. Let $u = \sinh x$ so that $du = \cosh x\,dx$. Substituting gives $\displaystyle\int_{3/4}^{63/16} \frac{1}{u}\,du = (\ln u)\Big|_{3/4}^{63/16} = \ln(63/16) - \ln(3/4) = \ln(21/4) \approx 1.66$.

7.3.110 Let $x = \tanh y$ so that $x = \dfrac{e^y - e^{-y}}{e^y + e^{-y}} \cdot \dfrac{e^y}{e^y} = \dfrac{e^{2y}-1}{e^{2y}+1}$. Then $xe^{2y} + x = e^{2y} - 1$, so $x + 1 = e^{2y} - xe^{2y}$, and $x + 1 = e^{2y}(1-x)$. Dividing through by $1 - x$ gives $\dfrac{x+1}{1-x} = e^{2y}$, so $\ln\left(\dfrac{x+1}{1-x}\right) = 2y$, and $y = \dfrac{1}{2}\ln\left(\dfrac{x+1}{1-x}\right)$.

7.3.111 Suppose $r > 0$ and $0 < x < 1$. If $u = x^r$, then $du = rx^{r-1}\,dx$. Substituting gives $\displaystyle\int \frac{du}{ru\sqrt{1-u^2}} = -\frac{1}{r}(\operatorname{sech}^{-1}(u)) + C = -\frac{1}{r}(\operatorname{sech}^{-1}(x^r)) + C$.

7.3.112

a. Consider the triangle with vertices $(0,0)$, $(x,0)$ and (x,y). Twice the area of the shaded region is twice the area of the triangle less twice the area under the curve $y = \sqrt{z^2-1}$ between $z = 1$ and $z = x$. The area of the triangle is $\frac{1}{2}xy$ and the area under the curve is given by $\displaystyle\int_1^x \sqrt{z^2-1}\,dz$. Thus, twice the area is given by

$$t = 2\left(\frac{1}{2}xy - \int_1^x \sqrt{z^2-1}\,dz\right) = x\sqrt{x^2-1} - 2\int_1^x \sqrt{z^2-1}\,dz.$$

b. Given the formula, we have $\displaystyle\int_1^x \sqrt{z^2-1}\,dz = \left(\frac{z}{2}\sqrt{z^2-1} - \frac{1}{2}\ln|z+\sqrt{z^2-1}|\right)\Big|_1^x = \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln|x+\sqrt{x^2-1}| - (0-0) = \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln(x+\sqrt{x^2-1})$. The absolute values can be dropped because $x > 1$ so $x + \sqrt{x^2-1} > 0$.

Now using the previous part of this problem, we have

$$t = x\sqrt{x^2-1} - 2\left(\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln(x+\sqrt{x^2-1})\right) = \ln(x+\sqrt{x^2-1}).$$

c. Because $t = \ln(x+\sqrt{x^2-1})$, we have $e^t = x + \sqrt{x^2-1}$, so $e^{-t} = \dfrac{1}{x+\sqrt{x^2-1}} = \dfrac{1}{x+\sqrt{x^2-1}} \cdot \dfrac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}} = x - \sqrt{x^2-1}$. Thus $e^t + e^{-t} = 2x$, so $x = \dfrac{e^t+e^{-t}}{2}$.

d. We have $y = \sqrt{x^2-1} = \sqrt{\dfrac{e^{2t}+2+e^{-2t}}{4} - \dfrac{4}{4}} = \sqrt{\dfrac{e^{2t}-2+e^{-2t}}{4}} = \sqrt{\left(\dfrac{e^t-e^{-t}}{2}\right)^2} = \dfrac{e^t-e^{-t}}{2}$.

Chapter Seven Review

1

a. False. For example, when t goes from 2 to 3, the value of y goes from 3 to 4, which is not doubling.

b. False. It grows from A to $A \cdot e^{.1} \approx 1.10517A$, so the growth is about 10.517%.

c. False. For example, if $x = e$ and $y = 1$, this equation would imply that $\ln(e \cdot 1) = (\ln e)(\ln 1)$, which is not true because $1 \neq 0$.

d. True. $\sinh(\ln x) = \dfrac{e^{\ln x} - e^{-\ln x}}{2} = \dfrac{x - 1/x}{2} \cdot \dfrac{x}{x} = \dfrac{x^2 - 1}{2x}$.

2 Let $u = 4e^x + 6$ so that $du = 4e^x\,dx$. Then we have $\dfrac{1}{4}\displaystyle\int \frac{1}{u}\,du = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln(4e^x + 6) + C$.

3 Let $u = \ln x$ so that $du = \dfrac{1}{x}\,dx$. Then we have $\displaystyle\int_2^8 \frac{1}{u}\,du = (\ln u)\Big|_2^8 = \ln 4$.

4 Let $u = \sqrt{x}$ so that $du = \dfrac{1}{2\sqrt{x}}\,dx$. Then we have $2\displaystyle\int_1^2 10^u\,du = 2\left(10^u/(\ln 10)\right)\Big|_1^2 = \frac{2}{\ln 10} \cdot 90 = \frac{180}{\ln 10}$.

5 Let $u = x^2 + 8x + 25$ so that $du = (2x + 8)\,dx = 2(x + 4)\,dx$. Then we have $\dfrac{1}{2}\displaystyle\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2 + 8x + 25) + C$.

6 $\displaystyle\int_{\ln 2}^{\ln 3} \frac{\cosh s}{\sinh s}\,ds = (\ln(\sinh s))\Big|_{\ln 2}^{\ln 3} = \ln(4/3) - \ln(3/4) = \ln(16/9)$.

7 $\displaystyle\int \frac{dx}{\sqrt{x^2 - 9}} = \cosh^{-1}(x/3) + C = \ln(x + \sqrt{x^2 - 9}) + C$.

8 Let $u = e^x$ so that $du = e^x\,dx$. Then we have $\displaystyle\int \frac{du}{\sqrt{u^2 + 4}} = \sinh^{-1}(u/2) + C = \sinh^{-1}(e^x/2) + C$.

9 Let $u = x^3$ so that $du = 3x^2\,dx$. Substituting gives

$$\frac{1}{3}\int_0^1 \frac{du}{3^2 - u^2} = \frac{1}{9}\left(\tanh^{-1}(u/3)\right)\Big|_0^1 = \frac{1}{9}\tanh^{-1}(1/3) \approx 0.0385.$$

10 We can write $f(x)$ as $f(x) = \ln 3 + 2\ln(\sin 4x)$, so $f'(x) = 2 \cdot \dfrac{1}{\sin 4x} \cdot (\cos 4x) \cdot 4 = \dfrac{8\cos 4x}{\sin 4x} = 8\cot 4x$.

11 We can write $g(x)$ as $g(x) = e^{(3x^2+1)\ln x}$, so

$$g'(x) = e^{(3x^2+1)\ln x}\left(6x\ln x + (3x^2 + 1)\frac{1}{x}\right) = x^{3x^2+1}\left(6x\ln x + 3x + \frac{1}{x}\right).$$

12 $f'(x) = \dfrac{(1 + \sinh x)\cosh x - (\sinh x)(\cosh x)}{(1 + \sinh x)^2} = \dfrac{\cosh x}{(1 + \sinh x)^2}$.

13 $f'(t) = \sinh t \sinh t + \cosh t \cosh t = \sinh^2 t + \cosh^2 t = \cosh 2t$.

14 $g'(w) = \operatorname{sech}^2(4w^2 + w)(8w + 1) = (8w + 1)\operatorname{sech}^2(4w^2 + w)$.

15 $f'(x) = 2\cosh(3x - 1) \cdot \sinh(3x - 1) \cdot 3 = 6\cosh(3x - 1)\sinh(3x - 1) = 3\sinh(2(3x - 1)) = 3\sinh(6x - 2)$.

16 $h'(z) = \dfrac{\cosh z}{\sinh z} - \coth z + z \operatorname{csch}^2 z = z \operatorname{csch}^2 z.$

17 $f'(x) = \dfrac{-\sin x}{1-\cos^2 x} = -\dfrac{\sin x}{\sin^2 x} = -\dfrac{1}{\sin x} = -\csc x.$

18 $g'(t) = \dfrac{1}{\sqrt{(\sqrt{t})^2+1}} \cdot \dfrac{1}{2\sqrt{t}} = \dfrac{1}{2\sqrt{t^2+t}}.$

19 $f'(x) = -\dfrac{1}{\frac{1}{x^2}\sqrt{1-\frac{1}{x^4}}} \cdot \dfrac{-2}{x^3} = \dfrac{2}{x\sqrt{1-\frac{1}{x^4}}} = \dfrac{2x}{\sqrt{x^4-1}}.$

20

a. We can model the population size by $y(t) = 1.3e^{kt}$. Because $1.45 = 1.3e^{10k}$, we have $\dfrac{1.45}{1.3} = e^{10k}$, so $k = \dfrac{\ln(1.45/1.3)}{10} \approx 0.01092$. So $y(20) = 1.3e^{20(0.01092)} \approx 1.62$ million.

b. The doubling time is approximately $\dfrac{\ln 2}{0.01092} \approx 63.5$ years.

21 Because the half-life is 5.5, $k = \dfrac{\ln(1/2)}{5.5}$, and $y(t) = 75e^{\frac{\ln(1/2)}{5.5}t}$. We are seeking t so that $30 = 75e^{\frac{\ln(1/2)}{5.5}t}$, or $\dfrac{30}{75} = e^{\frac{\ln(1/2)}{5.5}t}$, so $\ln\left(\dfrac{30}{75}\right) = \dfrac{-\ln 2}{5.5}t$, so $t = \dfrac{5.5\ln(30/75)}{-\ln 2} \approx 7.3$ hours.

22 $65 = 80e^{1.5k}$, so $1.5k = \ln\dfrac{65}{80}$, and $k = \dfrac{2}{3}\ln\dfrac{65}{80}$.

The amount of caffeine in his system t hours after his second cup of coffee is $y(t) = 145e^{\frac{2}{3}t\ln\frac{65}{80}}$. The amount of caffeine in his system 7 hours after his first cup of coffee is $y(5.5) = 145e^{\frac{2}{3}\cdot 5.5\cdot\ln\frac{65}{80}} \approx 67.7$ mg.

23

a. Because the doubling time is 2, $k = \dfrac{\ln 2}{2}$, so $y(t) = 29{,}000e^{\frac{\ln 2}{2}t}$.

b. $y(21) = 29{,}000e^{\frac{21\ln 2}{2}} \approx 41{,}996{,}486$ transistors. This closely approximates the actual number of transistors.

24 Because the half-life is 1500 years, we know that $k = \dfrac{\ln(1/2)}{1500}$. We are seeking t so that $0.7y_0 = y_0e^{kt}$, so $\ln(0.7) = kt$, and thus $t = \dfrac{\ln(0.7)}{k} = \dfrac{\ln(0.7)\cdot 1500}{-\ln 2} \approx 771.86$ years ago.

25 Growth is modeled by $p(t) = 150{,}000e^{kt}$ where $k = \ln(1.04)$. The population reaches 1,000,000 when $1{,}000{,}000 = 150{,}000e^{\ln(1.04)t}$, or when $\ln(20/3) = \ln(1.04)t$, so when $t = \dfrac{\ln(20/3)}{\ln(1.04)} \approx 48.37$ years.

26

a. The balance at time t years is given by $1500(1.054)^t$.

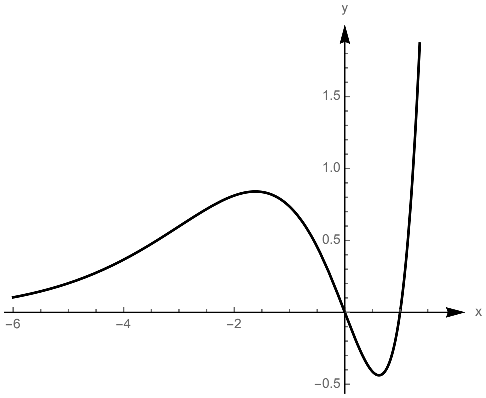
b. The balance doubles when $(1.054)^t = 2$, which occurs for $t = \dfrac{\ln 2}{\ln 1.054} \approx 13.18$ years.

c. The balance reaches \$5000 when $5000 = 1500(1.054)^t$, which occurs when $t = \dfrac{\ln(10/3)}{\ln(1.054)} \approx 22.89$ years.

27 The domain of f is the set of all real numbers. Note that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = 0$.

$f'(x) = e^x(2x-1)+(x^2-x)e^x = e^x(x^2+x-1)$, which is 0 for $x = \dfrac{-1\pm\sqrt{5}}{2}$. Note that $f'(x) > 0$ on the interval $(-\infty, (-1-\sqrt{5})/2)$ and on $((-1+\sqrt{5})/2, \infty)$, so f is increasing on those intervals, while $f'(x) < 0$ (and so f is decreasing) on $((-1-\sqrt{5})/2, (-1+\sqrt{5})/2)$. There is a local maximum at $(-1-\sqrt{5})/2$ and a local minimum at $(-1+\sqrt{5})/2$.

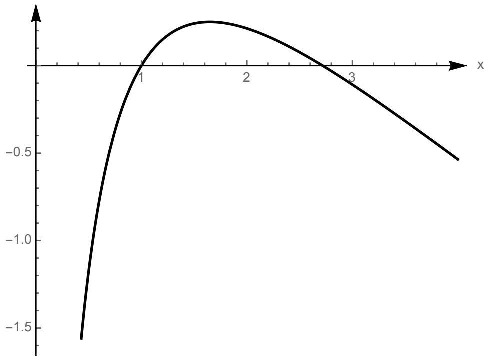
Note that $f''(x) = e^x(2x+1)+(x^2+x-1)e^x = e^x(x^2+3x)$, which is 0 for $x = 0$ and $x = -3$. $f''(x) > 0$ on $(-\infty, -3)$ and on $(0, \infty)$, so f is concave up on those intervals, while $f''(x) < 0$ on $(-3, 0)$, so f is concave down there. There are inflection points at $x = -3$ and at $x = 0$.



28 The domain of f is $(0, \infty)$, and has x-intercepts where $\ln x(1-\ln x) = 0$, so at $x = 1$ and $x = e$. Note that $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = -\infty$.

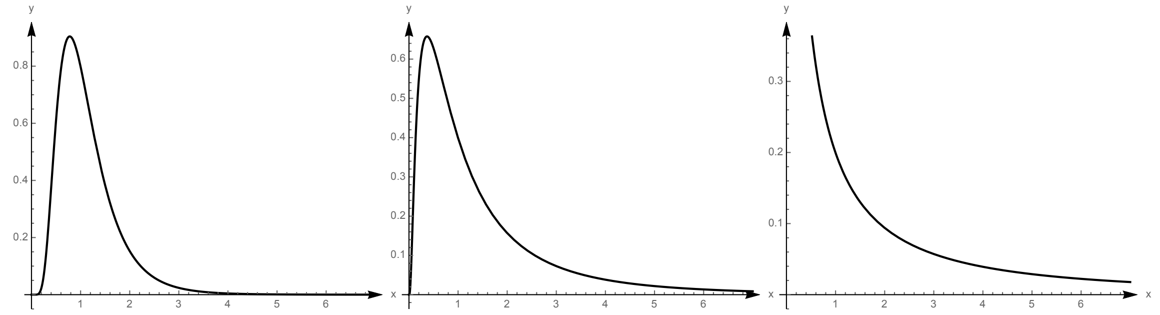
$f'(x) = \dfrac{1-2\ln x}{x}$, which is 0 for $x = \sqrt{e}$. Note that $f'(x) > 0$ on the interval $(0, \sqrt{e})$, so f is increasing on that interval, while $f'(x) < 0$ (and so f is decreasing) on $(\sqrt{e}, \infty)$. There is a local maximum at $x = \sqrt{e}$.

Note that $f''(x) = \dfrac{-2-(1-2\ln x)}{x^2} = \dfrac{2\ln x - 3}{x^2}$, which is 0 for $x = e^{3/2}$. Note that $f''(x) < 0$ on $(0, e^{3/2})$, so f is concave down on that interval, while $f''(x) > 0$ on $(e^{3/2}, \infty)$ so f is concave up there. The point at $x = e^{3/2}$ is an inflection point.



29

a.



$\sigma = 1/2$ $\sigma = 1$ $\sigma = 2$

It appears that $\lim\limits_{x\to 0} f(x) = 0$.

b. Let $x = e^y$, so that $y = \ln x$ and as $x \to 0$ we have $y \to -\infty$. Then $\lim\limits_{x\to 0} f(x) = \lim\limits_{y\to -\infty} \dfrac{e^{-y^2/2\sigma^2}}{\sigma\sqrt{2\pi}e^y} = \dfrac{1}{\sigma\sqrt{2\pi}} \lim\limits_{y\to-\infty} \dfrac{1}{e^y \cdot e^{y^2/2\sigma^2}} = 0$.

c. Write f as $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\left(\dfrac{1}{xe^{(\ln^2 x)/2\sigma^2}}\right)$. Then

$$f'(x) = \frac{-1}{\sigma\sqrt{2\pi}}\left(\frac{1}{x^2 e^{(\ln^2 x)/\sigma^2}}\right)\left(e^{(\ln^2 x)/2\sigma^2} + xe^{(\ln^2 x)/2\sigma^2} \cdot \frac{1}{\sigma^2}\ln x \cdot \frac{1}{x}\right) =$$

$$\frac{-1}{\sigma\sqrt{2\pi}}\left(\frac{1}{x^2}\right)\left(1 + \frac{\ln x}{\sigma^2}\right).$$

This quantity is 0 only when $1 + \dfrac{\ln x}{\sigma^2} = 0$, which occurs for $x^* = e^{-\sigma^2}$, and this critical number yields a maximum.

d. $f(e^{-\sigma^2}) = \dfrac{1}{\sigma\sqrt{2\pi}}\left(\dfrac{1}{(e^{-\sigma^2}e^{(\ln^2(e^{-\sigma^2}))/2\sigma^2}}\right) = \dfrac{1}{\sigma\sqrt{2\pi}}\left(\dfrac{1}{e^{-\sigma^2}e^{\sigma^2/2}}\right) = \dfrac{e^{\sigma^2/2}}{\sigma\sqrt{2\pi}}$.

e. Let $g(\sigma) = \dfrac{e^{\sigma^2/2}}{\sigma\sqrt{2\pi}}$. Then $g'(\sigma) = \dfrac{e^{\frac{\sigma^2}{2}}\left(\sigma^2 - 1\right)}{\sqrt{2\pi}\sigma^2}$, so there is a critical point at $\sigma = 1$. Note that g is decreasing on $(0,1)$ and increasing on $(1,\infty)$, so g has a minimum at $\sigma = 1$.

30 Note that the curves intersect when $\dfrac{8}{\cosh^2 x} = \cosh x$, which occurs when $\cosh^3 x = 8$ or $\cosh x = 2$, which is $x = \cosh^{-1}(2)$. by symmetry, we are seeking $2\displaystyle\int_0^{\cosh^{-1}(2)} (8\,\mathrm{sech}^2 x - \cosh x)\, dx =$
$2\left(8\tanh x - \sinh x\right)\Big|_0^{\cosh^{-1}(2)} = 2(4\sqrt{3} - \sqrt{3}) = 6\sqrt{3}$.

31 $f'(\ln 3) = \sinh(\ln 3) = 4/3$. So the linearization is $L(x) = 5/3 + (4/3)(x - \ln 3)$. Using this approximation, we have $\cosh(1) \approx L(1) = (5/3) + (4/3)(1 - \ln 3) \approx 1.535$.

32 Let $y = (\tanh x)^x$. Then we have $\ln y = x\ln(\tanh x)$. So

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\tanh x}{1/x} = \lim_{x\to\infty} \frac{\mathrm{sech}^2 x/\tanh x}{-1/x^2} = \lim_{x\to\infty} \frac{-x^2}{\sinh x\cosh x}$$
$$= \lim_{x\to\infty} \frac{-2x}{\sinh^2 x + \cosh^2 x} = \lim_{x\to\infty} \frac{-2x}{\cosh 2x} = \lim_{x\to\infty} \frac{-2}{2\sinh 2x} = 0.$$

Because $\lim_{x\to\infty} \ln y = 0$, we have $\lim_{x\to\infty} y = e^0 = 1$.

33

a. Because $\frac{d}{dx}\cosh x = \sinh x$ and $\frac{d}{dx}\sinh x = \cosh x$, we see that every nth derivative of $\cosh x$ where n is even is equal to $\cosh x$. So $\frac{d^6}{dx^6}\cosh x = \cosh x$.

b. $\frac{d}{dx} x \operatorname{sech} x = \operatorname{sech} x - x \operatorname{sech} x \tanh x = \operatorname{sech} x(1 - x\tanh x)$.

34 Note that $y' = 1/x$, so $1 + y'^2 = \frac{x^2+1}{x^2}$, and $\sqrt{1+y'^2} = \frac{\sqrt{x^2+1}}{x}$.
So

$$L = \int_1^b \frac{\sqrt{x^2+1}}{x}\, dx = \left(\sqrt{x^2+1} - \ln\left(\frac{1+\sqrt{x^2+1}}{x}\right)\right)\Bigg|_1^b = \sqrt{b^2+1} - \sqrt{2} + \ln\left(\frac{(\sqrt{b^2+1}-1)(1+\sqrt{2})}{b}\right).$$

Using a computer algebra system, we see that this has value 2 for $b \approx 2.715$.

Chapter 8

Integration Techniques

8.1 Basic Approaches

8.1.1 Let $u = 4 - 7x$. Then $du = -7\,dx$ and we obtain $-\frac{1}{7}\int u^{-6}\,du$.

8.1.2 $\int (\sec x + 1)^2\,dx = \int (\sec^2 x + 2\sec x + 1)\,dx = \tan x + 2\ln|\sec x + \tan x| + x + C$.

8.1.3 $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.

8.1.4 $\displaystyle\int \frac{4x^3 + x^2 + 4x + 2}{x^2 + 1}\,dx = \int \left(4x + 1 + \frac{1}{x^2 + 1}\right)dx = 2x^2 + x + \tan^{-1} x + C$.

8.1.5 Complete the square in the denominator to get $\displaystyle\int \frac{10}{(x-2)^2 + 1}\,dx$.

8.1.6

a. $\displaystyle\int \frac{2x+1}{x^2+1}\,dx = \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$.

b. $\ln(x^2 + 1) + \tan^{-1} x + C$.

8.1.7 Let $u = 3 - 5x$ so that $du = -5\,dx$. Substituting gives

$$-\frac{1}{5}\int u^{-4}\,du = \frac{1}{15}u^{-3} + C = \frac{1}{15(3-5x)^3} + C.$$

8.1.8 Let $u = 9x - 2$ so that $du = 9\,dx$. Substituting gives

$$\frac{1}{9}\int u^{-3}\,du = \frac{-1}{18}u^{-2} + C = \frac{-1}{18(9x-2)^2} + C.$$

8.1.9 Let $u = 2x - \pi/4$ so that $du = 2\,dx$. Substituting gives

$$\frac{1}{2}\int_{-\pi/4}^{\pi/2} \sin u\,du = \frac{1}{2}\left(-\cos u\right)\Big|_{-\pi/4}^{\pi/2} = \frac{1}{2}\left(0 + \sqrt{2}/2\right) = \frac{\sqrt{2}}{4}.$$

8.1.10 Let $u = 3 - 4x$ so that $du = -4\,dx$. Substituting gives

$$-\frac{1}{4}\int e^u\,du = -\frac{1}{4}e^u + C = -\frac{1}{4}e^{3-4x} + C.$$

8.1.11 Let $u = \ln(2x)$ so that $du = \dfrac{dx}{x}$. Substituting gives

$$\int u\,du = u^2/2 + C = \frac{1}{2}\ln^2 2x + C.$$

8.1.12 Let $u = 4 - x$ so that $du = -dx$. Substituting gives

$$-\int_9^4 u^{-1/2}\,du = \int_4^9 u^{-1/2}\,du = 2\sqrt{u}\Big|_4^9 = 6 - 4 = 2.$$

8.1.13 Let $u = e^x + 1$ so that $du = e^x\,dx$ Substituting gives

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln(e^x + 1) + C.$$

8.1.14 Let $u = x^2 + 1$ so that $du = 2x\,dx$. Note that $0^2 + 1 = 1$ and $1^2 + 1 = 2$. Substituting gives

$$\frac{1}{2}\int_1^2 3^u\,du = \frac{1}{2\ln 3}3^u\Big|_1^2 = \frac{1}{2\ln 3}(9 - 3) = \frac{3}{\ln 3}.$$

8.1.15 Let $u = \ln x^2 = 2\ln x$. Then $du = \dfrac{2}{x}\,dx$. Substituting gives

$$\frac{1}{2}\int_0^4 u^2\,du = \left(u^3/6\right)\Big|_0^4 = 32/3.$$

8.1.16 Let $u = t^3$ so that $du = 3t^2\,dt$. Note that $0^3 = 0$ and $1^3 = 1$. Substituting gives

$$\frac{1}{3}\int_0^1 \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}u\Big|_0^1 = \frac{1}{3}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{12}.$$

8.1.17 Let $u = s - 1$, so that $du = ds$ and $s = u + 1$. Note that $1 - 1 = 0$ and $2 - 1 = 1$, so the new limits of integration are 0 and 1. Substituting gives

$$\int_0^1 u^9(u+1)\,du = \int_0^1 (u^{10} + u^9)\,du = \left(\frac{u^{11}}{11} + \frac{u^{10}}{10}\right)\Big|_0^1 = \frac{1}{11} + \frac{1}{10} = \frac{21}{110}.$$

8.1.18 Let $u = t - 3$ so that $du = dt$ and $t = u + 3$. Note that $3 - 3 = 0$ and $7 - 3 = 4$, so the new limits of integration are 0 and 4. Substituting gives

$$\int_0^4 (u-3)\sqrt{u}\,du = \int_0^4 (u^{3/2} - 3u^{1/2})\,du = \left(\frac{2}{5}u^{5/2} - 2u^{3/2}\right)\Big|_0^4 = \frac{64}{5} - 16 = -\frac{16}{5}.$$

8.1.19 Let $u = \ln w - 1$. Then $du = \dfrac{dw}{w}$ and $\ln w = u + 1$. Substituting gives

$$\int u^7(u+1)\,du = \int (u^8 + u^7)\,du = \frac{u^9}{9} + \frac{u^8}{8} + C = \frac{(\ln w - 1)^9}{9} + \frac{(\ln w - 1)^8}{8} + C.$$

8.1.20 Let $u = 1 + e^x$ so that $du = e^x\,dx$ and $e^x = u - 1$. Substituting gives

$$\int u^9(2-u)\,du = \int (2u^9 - u^{10})\,du = \frac{u^{10}}{5} - \frac{u^{11}}{11} + C = \frac{(1+e^x)^{10}}{5} - \frac{(1+e^x)^{11}}{11} + C$$
$$= (1+e^x)^{10}\left(\frac{11}{55} - \frac{5(1+e^x)}{55}\right) + C = \frac{(1+e^x)^{10}(6-5e^x)}{55} + C.$$

8.1.21 $\displaystyle\int \frac{x}{x^2+4}\,dx + 2\int \frac{1}{x^2+4}\,dx = \frac{1}{2}\ln(x^2+4) + \tan^{-1}(x/2) + C.$

8.1.22 $\displaystyle\int \frac{\sin x+1}{\cos x}\,dx = \int(\tan x+\sec x)\,dx = \ln|\sec x| + \ln|\sec x+\tan x| + C.$

8.1.23 Let $u = 3e^x+4$ so that $du = 3e^x\,dx$. Substituting gives

$$\frac{1}{3}\int \csc u\,du = -\frac{1}{3}\ln|\csc u+\cot u| + C = -\frac{1}{3}\ln|\csc(3e^x+4)+\cot(3e^x+4)| + C.$$

8.1.24 $\displaystyle\int_4^9 x\,dx - \int_4^9 x^{-1}\,dx = \left.\frac{x^2}{2}\right|_4^9 - \ln x\Big|_4^9 = \frac{81}{2} - \frac{16}{2} - (\ln 9 - \ln 4) = \frac{65}{2} - \ln\left(\frac{9}{4}\right).$

8.1.25 We may rewrite the integrand as $\displaystyle\int_0^{\pi/4}\left(\frac{\sec\theta}{\sec\theta\csc\theta} + \frac{\csc\theta}{\sec\theta\csc\theta}\right)d\theta$. This can then be simplified as

$$\int_0^{\pi/4}(\sin\theta+\cos\theta)\,d\theta = (-\cos\theta+\sin\theta)\Big|_0^{\pi/4} = \left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right) - (-1-0) = 1.$$

8.1.26 $\displaystyle\int 4e^{-3x}\,dx + \int e^{-5x}\,dx = (-4/3)e^{-3x} + (-1/5)e^{-5x} + C.$

8.1.27

$$\int \frac{2}{\sqrt{1-x^2}}\,dx - 3\int\frac{x}{\sqrt{1-x^2}}\,dx = 2\sin^{-1}x - 3\int\frac{x}{\sqrt{1-x^2}}\,dx.$$

Let $u = 1-x^2$ so that $du = -2x\,dx$. Substituting gives

$$2\sin^{-1}x + \frac{3}{2}\int u^{-1/2}\,du = 2\sin^{-1}x + 3\sqrt{u} + C = 2\sin^{-1}x + 3\sqrt{1-x^2} + C.$$

8.1.28

$$\int\frac{3x}{\sqrt{4-x^2}}\,dx + \int\frac{1}{\sqrt{4-x^2}}\,dx = \int\frac{3x}{\sqrt{4-x^2}}\,dx + \sin^{-1}(x/2).$$

Let $u = 4-x^2$ so that $du = -2x\,dx$. Substituting gives

$$-\frac{3}{2}\int u^{-1/2}\,du + \sin^{-1}(x/2) = -3\sqrt{u} + \sin^{-1}(x/2) + C = -3\sqrt{4-x^2} + \sin^{-1}(x/2) + C.$$

8.1.29 $\displaystyle\int_{\pi/4}^{\pi/2}\sqrt{1+\cot^2 x}\,dx = \int_{\pi/4}^{\pi/2}\csc x\,dx = -\ln|\csc x+\cot x|\Big|_{\pi/4}^{\pi/2} = -\ln|1+0| + \ln|\sqrt{2}+1| = \ln(\sqrt{2}+1).$

8.1.30 Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$. Substituting gives

$$\int_{\pi/4}^{\pi/3}\cot u\,du = \ln|\sin u|\Big|_{\pi/4}^{\pi/3} = \ln\frac{\sqrt{3}}{2} - \ln\frac{\sqrt{2}}{2} = \frac{\ln 3 - \ln 2}{2} = \frac{1}{2}\ln\frac{3}{2}.$$

8.1.31 Note that by completing the square, we have $x^2-2x+10 = (x^2-2x+1)+9 = (x-1)^2+9$. So

$$\int\frac{dx}{(x-1)^2+3^2} = \frac{1}{3}\tan^{-1}\left(\frac{x-1}{3}\right) + C.$$

8.1.32 Note that by completing the square, we have $x^2+4x+8 = (x+2)^2+4$. Thus we have

$$\int_0^2\frac{(x+2)-2}{(x+2)^2+4}\,dx = \int_2^4\frac{u}{u^2+4}\,du - \int_2^4\frac{2}{u^2+4}\,du$$

where $u = x+2$. This is equal to

$$\left(\frac{1}{2}\ln|u^2+4| - \tan^{-1}(u/2)\right)\Big|_2^4 = \frac{1}{2}\ln(5/2) + \frac{\pi}{4} - \tan^{-1}2.$$

8.1.33 We reduce the integrand by long division to $x+1+\dfrac{2x+1}{x^2+x+2}$. Then we have

$$\int\left(x+1+\frac{2x+1}{x^2+x+2}\right)dx=\frac{x^2}{2}+x+\int\frac{2x+1}{x^2+x+1}\,dx.$$

For the remaining integral, we let $u=x^2+x+1$ so that $du=(2x+1)\,dx$. The last integral is therefore equal to $\ln|u|+C=\ln|x^2+x+1|+C$. So our final result is

$$\frac{x^2}{2}+x+\ln|x^2+x+1|+C.=\frac{x^2}{2}+x+\ln(x^2+x+1)+C.$$

8.1.34 Note that long division gives $\dfrac{x^2+2}{x-1}=x+1+\dfrac{3}{x-1}$. Thus we have

$$\int_2^4\left(x+1+\frac{3}{x-1}\right)dx=\left(x^2/2+x+3\ln|x-1|\right)\Big|_2^4=8+4+3\ln 3-(2+2+0)=8+3\ln 3.$$

8.1.35 By long division, we can write the integrand as $t^2+t+\dfrac{1}{t^2+1}$. Then we have

$$\int_0^1\left(t^2+t+\frac{1}{t^2+1}\right)dt=\left(\frac{t^3}{3}+\frac{t^2}{2}+\tan^{-1}t\right)\Big|_0^1=\left(\frac{1}{3}+\frac{1}{2}+\frac{\pi}{4}\right)-0=\frac{3\pi+10}{12}.$$

8.1.36 Note that long division gives $\dfrac{t^3-2}{t+1}=t^2-t+1-\dfrac{3}{t+1}$. Our integral is therefore

$$\int\left(t^2-t+1-\frac{3}{t+1}\right)dt=t^3/3-t^2/2+t-3\ln|t+1|+C.$$

8.1.37 Note that $27-6\theta-\theta^2=-(\theta^2+6\theta+9-36)=-((\theta+3)^2-36)=36-(\theta+3)^2$. Thus our integral is

$$\int\frac{d\theta}{\sqrt{36-(\theta+3)^2}}=\sin^{-1}((\theta+3)/6)+C.$$

8.1.38 The integral can be written as $\displaystyle\int\frac{x}{(x^2+1)^2}\,dx$. Let $u=x^2+1$ so that $du=2x\,dx$. Substitution gives

$$\frac{1}{2}\int u^{-2}\,du=\frac{-1}{2u}+C=\frac{-1}{2(x^2+1)}+C.$$

8.1.39

$$\begin{aligned}\int\frac{1}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta}\,d\theta&=\int\frac{1-\sin\theta}{1-\sin^2\theta}\,d\theta=\int\frac{1-\sin\theta}{\cos^2\theta}\,d\theta\\&=\int\sec^2\theta\,d\theta-\int\frac{\sin\theta}{\cos^2\theta}\,d\theta=\tan\theta-\int\frac{\sin\theta}{\cos^2}\theta\,d\theta.\end{aligned}$$

Let $u=\cos\theta$ so that $du=-\sin\theta\,d\theta$. Substituting gives

$$\tan\theta+\int u^{-2}\,du=\tan\theta-\frac{1}{u}+C=\tan\theta-\sec\theta+C.$$

8.1.40 $\displaystyle\int\frac{1-x}{1-\sqrt{x}}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}}\,dx=\int\frac{(1-x)(1+\sqrt{x})}{1-x}\,dx=\int(1+\sqrt{x})\,dx=x+2x^{3/2}/3+C.$

8.1.41 $\displaystyle\int\frac{1}{\sec x-1}\cdot\frac{\sec x+1}{\sec x+1}\,dx=\int\frac{\sec x+1}{\sec^2 x-1}\,dx=\int\frac{\sec x+1}{\tan^2 x}\,dx=\int\frac{\sec x}{\tan^2 x}\,dx+\int\cot^2 x\,dx=$ $\displaystyle\int\cot x\csc x\,dx+\int\cot^2 x\,dx=-\csc x+\int(\csc^2 x-1)\,dx=-\csc x-\cot x-x+C.$

8.1.42 $\int \frac{1}{1-\csc\theta}\cdot\frac{1+\csc\theta}{1+\csc\theta}\,d\theta = \int \frac{1+\csc\theta}{1-\csc^2\theta}\,d\theta = \int \frac{1+\csc\theta}{-\cot^2\theta}\,d\theta = \int \frac{-\csc\theta}{\cot^2\theta}\,d\theta - \int \frac{1}{\cot^2\theta}\,d\theta = \int -\tan\theta\sec\theta\,d\theta - \int \tan^2\theta\,d\theta = -\sec\theta - \int(\sec^2\theta - 1)\,d\theta = -\sec\theta - \tan\theta + \theta + C.$

8.1.43 Let $u = 1+\sinh 3x$. Then $du = 3\cosh 3x\,dx$. Substituting gives

$$\frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|1+\sinh x| + C.$$

8.1.44 Let $u = x^2+1$. Then $du = 2x\,dx$. Note that $0^2+1 = 1$ and $\sqrt{3}^2+1 = 4$, and $6x^3\,dx = 3x^2 2x\,dx = 3(u-1)\,du$. Substituting gives

$$3\int_1^4 \frac{u-1}{\sqrt{u}}\,du = 3\int_1^4 (u^{1/2}-u^{-1/2})\,du = 3\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right)\Bigg|_1^4 = 3\left(\frac{16}{3} - 4 - \frac{2}{3} + 2\right) = 8.$$

8.1.45 We rewrite the integral by multiplying the numerator and denominator of the integrand by e^x. We have

$$\int \frac{e^x}{e^x - 2e^{-x}}\,dx = \int \frac{e^x}{e^x-2e^{-x}}\cdot\frac{e^x}{e^x}\,dx = \int \frac{e^{2x}}{e^{2x}-2}\,dx.$$

Now let $u = e^{2x}-2$ so that $du = 2e^{2x}\,dx$. Substituting gives

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|e^{2x}-2| + C.$$

8.1.46 $\int \frac{e^{2z}}{e^{2z}-4e^{-z}}\cdot\frac{e^z}{e^z}\,dz = \int \frac{e^{3z}}{e^{3z}-4}\,dz$. Let $u = e^{3z}-4$ so that $du = 3e^{3z}\,dz$. Substituting gives

$$\frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|e^{3z}-4| + C.$$

8.1.47 $\int \frac{dx}{x^{-1}+1} = \int \frac{1}{x^{-1}+1}\cdot\frac{x}{x}\,dx = \int \frac{x}{x+1}\,dx$. Using long division, we have $\frac{x}{x+1} = 1 - \frac{1}{x+1}$. Then

$$\int\left(1 - \frac{1}{x+1}\right)dx = x - \ln|x+1| + C.$$

8.1.48 $\int \frac{dy}{y^{-1}+y^{-3}} = \int \frac{1}{y^{-1}+y^{-3}}\cdot\frac{y^3}{y^3}\,dy = \int \frac{y^3}{y^2+1}\,dy$. Using long division, we have

$$\frac{y^3}{y^2+1} = y - \frac{y}{y^2+1}.$$

Thus our integral is equal to

$$\int\left(y - \frac{y}{y^2+1}\right)dy = \frac{y^2}{2} - \int \frac{y}{y^2+1}\,dy.$$

To compute this last integral, let $u = y^2+1$ so that $du = 2y\,dy$. We have

$$\frac{y^2}{2} - \frac{1}{2}\int \frac{1}{u}\,du = \frac{y^2}{2} - \frac{1}{2}\ln|u| + C = \frac{y^2}{2} - \frac{1}{2}\ln|y^2+1| + C.$$

8.1.49 Let $u = 9+\sqrt{t+1}$. Then $du = \frac{dt}{2\sqrt{t+1}}$, or $dt = 2(u-9)\,du$. Our integral is

$$\begin{aligned}\int \sqrt{9+\sqrt{t+1}}\,dt &= \int \sqrt{u}(2(u-9))\,du = 2\int(u^{3/2}-9u^{1/2})\,du = 2\left(\frac{2}{5}u^{5/2} - 6u^{3/2}\right) + C\\ &= \frac{4}{5}u^{3/2}(u-15) + C = \frac{4}{5}(9+\sqrt{t+1})^{3/2}(9+\sqrt{t+1}-15) + C\\ &= \frac{4}{5}(9+\sqrt{t+1})^{3/2}(\sqrt{t+1}-6) + C.\end{aligned}$$

8.1.50 Let $u = \sqrt{x}$ so that $du = \frac{1}{2\sqrt{x}}\,dx$, or $dx = 2u\,du$. Substituting gives

$$\int_2^3 \frac{2u}{1-u}\,du = -\int_2^3 \frac{2u}{u-1}\,du.$$

By long division, $\frac{2u}{u-1} = 2 + \frac{2}{u-1}$. Thus we have

$$-\int_2^3 \left(2 + \frac{2}{u-1}\right) du = (-2u - 2\ln|u-1|)\bigg|_2^3 = -6 - 2\ln 2 - (-4-0) = -2 - 2\ln 2.$$

8.1.51 $\int_{-1}^0 \frac{x}{x^2+2x+2}\,dx = \int_{-1}^0 \frac{x}{(x+1)^2+1}\,dx$. Let $u = x+1$ so that $du = dx$. Substituting gives

$$\int_0^1 \frac{u-1}{u^2+1}\,du = \int_0^1 \frac{u}{u^2+1}\,du - \int_0^1 \frac{1}{u^2+1}\,du$$
$$= \left((1/2)\ln(u^2+1) - \tan^{-1}(u)\right)\bigg|_0^1 = (1/2)\ln 2 - \frac{\pi}{4} = \frac{1}{4}(\ln 4 - \pi).$$

8.1.52 $\int_{\pi/6}^{\pi/2} (\csc y)\,dy = -\ln|\csc y + \cot y|\bigg|_{\pi/6}^{\pi/2} = -\ln|1+0| + \ln|2+\sqrt{3}| = \ln(2+\sqrt{3})$.

8.1.53 Let $u = e^x + 1$ so that $du = e^x\,dx$. Substituting gives

$$\int \sec u\,du = \ln|\sec u + \tan u| + C = \ln|\sec(e^x+1) + \tan(e^x+1)| + C.$$

8.1.54 Let $u = 1 + \sqrt{x}$ so that $du = \frac{1}{2\sqrt{x}}\,dx$, or $dx = 2\sqrt{x}\,du = 2(u-1)\,du$. Substituting gives

$$2\int_1^2 \sqrt{u}(u-1)\,du = 2\int_1^2 (u^{3/2} - u^{1/2})\,du = 2\left(2u^{5/2}/5 - 2u^{3/2}/3\right)\bigg|_1^2$$
$$= 2(8\sqrt{2}/5 - 4\sqrt{2}/3 - (2/5 - 2/3)) = \frac{8}{15}\left(1+\sqrt{2}\right).$$

8.1.55 Using the identity $\sin 2x = 2\sin x\cos x$, we have $2\int \sin^2 x\cos x\,dx$. Let $u = \sin x$ so that $du = \cos x\,dx$. We have $2\int u^2\,du = 2u^3/3 + C = 2(\sin^3 x)/3 + C$.

8.1.56 Using the double angle identity $\cos 2x = 2\cos^2 x - 1$, we have

$$\int_0^{\pi/2} \sqrt{2\cos^2 x}\,dx = \int_0^{\pi/2} \sqrt{2}\cos x\,dx = \sqrt{2}\sin x\bigg|_0^{\pi/2} = \sqrt{2}(1-0) = \sqrt{2}.$$

8.1.57 Rewrite the integral as $\int \frac{1}{\sqrt{x}} \cdot \frac{1}{1+(\sqrt{x})^2}\,dx$ and let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$, and substituting gives

$$2\int \frac{1}{1+u^2}\,du = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C.$$

8.1.58 Let $u = \sqrt{p}$ so that $u^2 = p$ and $2u\,du = dp$. Substituting gives

$$\int_0^1 \frac{2u}{4-u}\,du = 2\int_0^1 \left(-1 - \frac{4}{u-4}\right) du$$
$$= 2\left(-u - 4\ln|u-4|\right)\bigg|_0^1 = 2(-1 - 4\ln 3 - (0 - 4\ln 4)) = 2(\ln(256/81) - 1).$$

8.1.59 Note that $x^2+6x+13=(x^2+6x+9)+4=(x+3)^2+4$. Also note that we can write the numerator $x-2=x+3-5=\frac{1}{2}(2x+6)-5$. We have

$$\int \frac{\frac{1}{2}(2x+6)}{x^2+6x+13}\,dx - \int \frac{5}{(x+3)^2+4}\,dx.$$

For the first integral, let $u=x^2+6x+13$ so that $du=(2x+6)\,dx$. We have (for just the first integral)

$$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|+C=\frac{1}{2}\ln(x^2+6x+13)+C.$$

The second integrand has antiderivative equal to $\frac{5}{2}\tan^{-1}((x+3)/2)$, so the original integral is equal to

$$\frac{1}{2}\ln(x^2+6x+13)-\frac{5}{2}\tan^{-1}((x+3)/2)+C.$$

8.1.60 $3\int_0^{\pi/4}\sqrt{1+\sin 2x}\cdot\frac{\sqrt{1-\sin 2x}}{\sqrt{1-\sin 2x}}\,dx = 3\int_0^{\pi/4}\frac{\cos 2x}{\sqrt{1-\sin 2x}}\,dx$. Let $u=1-\sin 2x$ so that $du=-2\cos 2x\,dx$. Substituting gives

$$\frac{-3}{2}\int_1^0 u^{-1/2}\,du = 3\sqrt{u}\Big|_0^1 = 3.$$

8.1.61 Let $u=e^x$ so that $du=e^x\,dx$. Substituting gives

$$\int \frac{1}{u^2+2u+1}\,du=\int (u+1)^{-2}\,du = -\frac{1}{u+1}+C=-\frac{1}{e^x+1}+C.$$

8.1.62 By long division, the integrand can be written as $-x^3-x+1+\frac{4x+2}{x^2+x+1}$. Then

$$\int\left(-x^3-x+1+\frac{2(2x+1)}{x^2+x+1}\right)=-\frac{x^4}{4}-\frac{x^2}{2}+x+2\ln|x^2+x+1|+C.$$

8.1.63 The denominator factors as $(x+1)^2$.

$$\int_1^3\frac{2}{(x+1)^2}\,dx=-2\left(\frac{1}{x+1}\right)\Big|_1^3=-2\left(\frac{1}{4}-\frac{1}{2}\right)=\frac{1}{2}.$$

8.1.64 The denominator factors as $(s+1)^3$.

$$\int_0^2\frac{2}{(s+1)^3}\,ds=-\left(s+1\right)^{-2}\Big|_0^2=-\left(\frac{1}{9}-1\right)=\frac{8}{9}.$$

8.1.65

a. False. This seem to use the untrue "identity" that $\frac{a}{b+c}=\frac{a}{b}+\frac{a}{c}$.

b. False. The degree of the numerator is already less than the degree of the denominator, so long division won't help.

c. False. This is false because $\frac{d}{dx}\ln|\sin x+1|+C\neq\frac{1}{\sin x+1}$. The substitution $u=\sin x+1$ can't be carried out because $du=\cos x\,dx$ can't be accounted for.

d. False. In fact, $\int e^{-x}\,dx=-e^{-x}+C\neq\ln e^x+C$.

8.1.66 $\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx$. Let $u = \sin x$ so that $du = \cos x\, dx$. We then have

$$\int \frac{1}{u}\, du = \ln|u| + C = \ln|\sin x| + C.$$

8.1.67

$$\int \csc x\, dx = \int (\csc x)\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right)\, dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\, dx.$$

Let $u = \csc x + \cot x$ so that $du = -\csc^2 x - \csc x \cot x\, dx$. Substituting then yields

$$-\int \frac{1}{u}\, du = -\ln|u| + C = -\ln|\csc x + \cot x| + C.$$

8.1.68

a. If $u = \cot x$, then $du = -\csc^2 x\, dx$. Substituting gives $-\int u\, du = -\frac{u^2}{2} + C = -\frac{\cot^2 x}{2} + C$.

b. If $u = \csc x$, then $du = -\csc x \cot x\, dx$. Substituting gives $-\int u\, du = -\frac{u^2}{2} + C = -\frac{\csc^2 x}{2} + C$.

c. The seemingly different answers are the same, since $-(\cot^2 x)/2$ and $-(\csc^2 x)/2$ differ by a constant. In fact, $-\frac{\cot^2 x}{2} - \left(-\left(\frac{\csc^2 x}{2}\right)\right) = \frac{1}{2}$.

8.1.69

a. If $u = \tan x$ then $du = \sec^2 x\, dx$. Substituting gives $\int u\, du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C$.

b. If $u = \sec x$, then $du = \sec x \tan x\, dx$. Substituting gives $\int u\, du = \frac{u^2}{2} + C = \frac{\sec^2 x}{2} + C$.

c. The seemingly different answers are the same, since $\frac{\tan^2 x}{2}$ and $\frac{\sec^2 x}{2}$ differ by a constant. In fact, $\frac{\tan^2 x}{2} - \frac{\sec^2 x}{2} = -\frac{1}{2}$.

8.1.70

a. Note that long division gives $\frac{x+2}{x+4} = 1 - \frac{2}{x+4}$. Thus our integral is equal to

$$\int \left(1 - \frac{2}{x+4}\right)\, dx = x - 2\ln|x+4| + C.$$

b. Let $u = x + 4$. Then $du = dx$ and $x + 2 = u - 2$. Substituting gives

$$\int \frac{u-2}{u}\, du = \int \left(1 - \frac{2}{u}\right)\, du = u - 2\ln|u| + C = x + 2 - 2\ln|x+2| + C = x - 2\ln|x+2| + 2 + C.$$

c. In part b, $2 + C$ is a constant, so we can replace $2 + C$ by C to obtain the answer to part a.

8.1.71

a. Let $u = x + 1$ so that $du = dx$. Note that $x = u - 1$, so that $x^2 = (u-1)^2$. Substituting gives

$$\int \frac{u^2 - 2u + 1}{u}\, du = \int (u - 2 + (1/u))\, du = u^2/2 - 2u + \ln|u| + C = (x+1)^2/2 - 2(x+1) + \ln|x+1| + C.$$

b. By long division, $\dfrac{x^2}{x+1} = x - 1 + \dfrac{1}{x+1}$. Thus,

$$\int \frac{x^2}{x+1}\,dx = \int \left(x - 1 + \frac{1}{x+1}\right)\,dx = x^2/2 - x + \ln|x+1| + C.$$

c. The seemingly different answers are the same, because they differ by a constant. In fact,

$$\frac{(x+1)^2}{2} - 2(x+1) + \ln|x+1| - \left(\frac{x^2}{2} - x + \ln|x+1|\right) = -\frac{3}{2}.$$

8.1.72 The curves intersect when $x^3 = 8x$, so at $x = 0$ and $x = \pm\sqrt{8}$. By symmetry, we have $A = 2\displaystyle\int_0^{\sqrt{8}} \frac{8x - x^3}{x^2+1}\,dx$. Using long division, we can write $\dfrac{8x - x^3}{x^2+1} = -x + \dfrac{9x}{x^2+1}$. Thus,

$$\begin{aligned} A &= 2\int_0^{\sqrt{8}} (-x + \frac{9x}{x^2+1})\,dx = 2\int_0^{\sqrt{8}} (-x)\,dx + 2\int_0^{\sqrt{8}} \frac{9x}{x^2+1}\,dx \\ &= -2\,x^2/2\,\Big|_0^{\sqrt{8}} + 2\int_0^{\sqrt{8}} \frac{9x}{x^2+1}\,dx = -8 + 2\int_0^{\sqrt{8}} \frac{9x}{x^2+1}\,dx. \end{aligned}$$

To compute this last integral, let $u = x^2 + 1$ so that $du = 2x\,dx$. Then we have

$$A = -8 + 9\int_1^9 \frac{1}{u}\,du = -8 + 9\,\ln u\,\Big|_1^9 = -8 + 9\ln 9 \approx 11.775.$$

8.1.73

$$A = \int_2^4 \frac{x^2 - 1}{x^3 - 3x}\,dx.$$

Let $u = x^3 - 3x$ so that $du = 3x^2 - 3\,dx$. Substituting gives

$$A = \frac{1}{3}\int_2^{52} \frac{1}{u}\,du = \frac{1}{3}\,\ln u\,\Big|_2^{52} = \frac{1}{3}(\ln 52 - \ln 2) = \frac{\ln 26}{3}.$$

8.1.74

$$\begin{aligned} V &= 2\pi\int_0^3 \frac{x}{x+2}\,dx = 2\pi\int_0^3 (1 - \frac{2}{x+2})\,dx \\ &= 2\pi\,\left(x - 2\ln(x+2)\right)\Big|_0^3 = 2\pi(3 - 2\ln 5 - (0 - 2\ln 2)) = 2\pi(3 - 2\ln(2/5)). \end{aligned}$$

8.1.75

a. $V = \pi\displaystyle\int_0^2 (x^2+1)\,dx = \pi\,\left(x^3/3 + x\right)\Big|_0^2 = \pi(8/3 + 2) = \frac{14\pi}{3}$.

b. $V = 2\pi\displaystyle\int_0^2 x\sqrt{x^2+1}\,dx$. Let $u = x^2 + 1$ so that $du = 2x\,dx$. Substituting gives

$$\pi\int_1^5 u^{1/2}\,du = \frac{2\pi}{3}\left(u^{3/2}\right)\Big|_1^5 = \frac{2\pi}{3}(5\sqrt{5} - 1).$$

8.1.76

a. Note that $x - x^2 = -(x^2 - x + 1/4 - 1/4) = -((x-1/2)^2 - 1/4) = 1/4 - (x-1/2)^2$. We can write our integral as
$$\int \frac{dx}{\sqrt{1/4-(x-1/2)^2}} = 2\int \frac{dx}{\sqrt{1-(2x-1)^2}}.$$
Let $u = 2x-1$. Then $du = 2\,dx$. Substituting gives
$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(2x-1) + C.$$

b. We can write our integral as $\displaystyle\int \frac{dx}{\sqrt{x}\sqrt{1-x}}$. Let $u = \sqrt{x}$ so that $du = \dfrac{1}{2\sqrt{x}}\,dx$. Substituting gives $\displaystyle 2\int \frac{du}{\sqrt{1-u^2}} = 2\sin^{-1} u + C = 2\sin^{-1}\sqrt{x} + C$.

c. By parts a and b, it follows that both $\sin^{-1}(2x-1)$ and $2\sin^{-1}(\sqrt{x})$ are antiderivatives of $\dfrac{1}{\sqrt{x-x^2}}$. Therefore, $2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1) = C$ for some constant C. To determine C, we let $x = 0$, giving $2\sin^{-1}(0) - \sin^{-1}(-1) = C$. Thus $0 - \left(-\dfrac{\pi}{2}\right) = C$, so $C = \dfrac{\pi}{2}$.

8.1.77

$$\begin{aligned} A &= 2\pi\int_0^1 \sqrt{x+1}\sqrt{1+\frac{1}{4(x+1)}}\,dx = 2\pi\int_0^1 \sqrt{x+5/4}\,dx \\ &= 2\pi\left((2/3)(x+5/4)^{3/2}\right)\Big|_0^1 = \frac{4\pi}{3}(27/8 - 5\sqrt{5}/8) = \frac{9\pi}{2} - \frac{5\sqrt{5}\pi}{6}. \end{aligned}$$

8.1.78 $A = 2\pi\displaystyle\int_0^{\ln 2}(e^x + e^{-x}/4)\sqrt{1+(e^x - e^{-x}/4)^2}\,dx$. Note that
$$1 + (e^x - e^{-x}/4)^2 = 1 + e^{2x} - 1/2 + e^{-2x}/16 = e^{2x} + 1/2 + e^{-2x}/16 = (e^x + e^{-x}/4)^2.$$

Thus we have
$$\begin{aligned} 2\pi\int_0^{\ln 2}(e^x+e^{-x}/4)^2\,dx &= 2\pi\int_0^{\ln 2}(e^{2x}+1/2+e^{-2x}/16)\,dx = 2\pi\left(e^{2x}/2 + x/2 - e^{-2x}/32\right)\Big|_0^{\ln 2} \\ &= 2\pi(2 + (\ln 2)/2 - 1/128 - (1/2 - 1/32)) = \pi\left(\frac{195}{64} + \ln 2\right). \end{aligned}$$

8.1.79 $L = \displaystyle\int_0^1 \sqrt{1+\frac{25x^{1/2}}{16}}\,dx$. Let $u^2 = 1 + \dfrac{25x^{1/2}}{16}$. Then $2u\,du = \dfrac{25}{32\sqrt{x}}\,dx$. Note that $\sqrt{x} = \dfrac{16}{25}(u^2-1)$, and that $dx = \dfrac{64\sqrt{x}}{25}u\,du = \dfrac{1024}{625}(u^3-u)\,du$. Substituting gives
$$\begin{aligned} L &= \int_1^{\sqrt{41/16}} \frac{1024}{625}(u^4-u^2)\,du = \frac{1024}{625}\left(u^5/5 - u^3/3\right)\Big|_1^{\sqrt{41/16}} \\ &= \frac{1024}{625}((\sqrt{41/16})^5/5 - (\sqrt{41/16})^3/3 - (1/5-1/3)) = \frac{1024}{625}\left(\frac{2}{15} + \frac{1763\sqrt{41}}{15360}\right) \\ &= \frac{2048 + 1763\sqrt{41}}{9375} \approx 1.423. \end{aligned}$$

8.1.80

$$d(t) = \int_0^t v(y)\,dy = \int_0^t v_T\left(\frac{e^{ay}-1}{e^{ay}+1}\right)dy = v_T\int_0^t \frac{e^{ay}}{e^{ay}+1}\,dy + -v_T\int_0^t \frac{1}{e^{ay}+1}\,dy.$$

The first integral can be computed by letting $u = e^{ay}+1$ so that $du = ae^{ay}\,dy$. The first integral is then equal to $(v_T/a)\int_2^{e^{at}+1}\frac{1}{u}\,du = (v_T/a)(\ln(e^{at}+1)-\ln(2))$. The second integral is $-v_T\int_0^t \frac{1}{e^{ay}+1}\cdot\frac{e^{ay}}{e^{ay}}\,dy$. Again, let $u = e^{ay}+1$ and note that $du = ae^{ay}\,dy$. Substitution gives

$$-(v_T/a)\int_2^{e^{at}+1}\frac{1}{u(u-1)}\,du = -(v_T/a)\int_2^{e^{at}+1}\left(\frac{1}{u-1}-\frac{1}{u}\right)dy$$

$$= -(v_T/a)\left(\ln(u-1)-\ln(u)\right)\Big|_2^{e^{at}+1} = -(v_T/a)(at-\ln(e^{at}+1)-(0-\ln 2)).$$

Adding the results of the two integrations gives

$d(t) = (v_T/a)(\ln(e^{at}+1)-\ln(2)) + -(v_T/a)(at-\ln(e^{at}+1)-(0-\ln 2)) = (v_T/a)(2\ln(e^{at}+1)-at-\ln 4)$

8.2 Integration by Parts

8.2.1 It is based on the product rule. In fact, the rule can be obtained by writing down the product rule, then integrating both sides and rearranging the terms in the result.

8.2.2 $u = x$ so $du = dx$, and $dv = \cos x\,dx$ so $v = \sin x$. Then the integration by parts formula gives

$$x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

8.2.3 $u = \ln x$ so $du = \frac{dx}{x}$, and $dv = x\,dx$ so $v = \frac{x^2}{2}$. Then the integration by parts formula gives

$$\frac{x^2\ln x}{2} - \int \frac{x^2}{2x}\,dx = \frac{x^2\ln x}{2} - \int \frac{x}{2}\,dx = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C = \frac{x^2(2\ln x - 1)}{4} + C.$$

8.2.4 One can use integration by parts for definite integrals via the formula

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx.$$

8.2.5 Those for which the choice for dv is easily integrated and when the resulting new integral is no more difficult than the original.

8.2.6 It is generally a good idea to let dv be something easy to integrate. In this case, we would let $dv = e^{ax}\,dx$, leaving $u = x^n$. Note that differentiating x^n results in something simpler (lower degree,) while integrating it make it more complicated (higher degree). However, differentiating or integrating e^{ax} yields essentially the same thing (a constant times the function e^{ax}).

8.2.7 Let $u = \tan x + 2$, so that $du = \sec^2 x\,dx$. Then

$$\int \sec^2 x \ln(\tan x + 2)\,dx = \int \ln u\,du = u\ln u - u + C = (\tan x + 2)\ln(\tan x + 2) - \tan x + C.$$

8.2.8 Let $u = \sin x$ so that $du = \cos x\,dx$. Then

$$\int \cos x \ln(\sin x)\,dx = \int \ln u\,du = u\ln u - u + C = \sin x\ln(\sin x) - \sin x + C.$$

8.2.9 Let $u = x$ and $dv = \cos 5x\,dx$. Then $du = dx$ and $v = \dfrac{1}{5}\sin 5x$. Then

$$\int x\cos 5x\,dx = \frac{1}{5}x\sin 5x - \frac{1}{5}\int \sin 5x\,dx = \frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x + C.$$

8.2.10 Let $u = x$ and $dv = \sin 2x\,dx$. Then $du = dx$ and $v = -\frac{1}{2}\cos(2x)$. So

$$\int x\sin 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C.$$

8.2.11 Let $u = t$ and $dv = e^{6t}\,dt$. Then $du = dt$ and $v = \dfrac{1}{6}\cdot e^{6t}$. Then

$$\int te^{6t}\,dt = \frac{1}{6}te^{6t} - \frac{1}{6}\int e^{6t}\,dt = \frac{1}{6}te^{6t} - \frac{1}{36}e^{6t} + C.$$

8.2.12 Let $u = 2x$ and $dv = e^{3x}\,dx$. Then $du = 2\,dx$ and $v = \frac{e^{3x}}{3}$. Then

$$\int 2xe^{3x}\,dx = \frac{2xe^{3x}}{3} - \frac{2}{3}\int e^{3x}\,dx = \frac{2xe^{3x}}{3} - \frac{2e^{3x}}{9} + C.$$

8.2.13 Let $u = \ln 10x$ and $dv = x\,dx$. Then $du = \dfrac{1}{x}\,dx$ and $v = \dfrac{x^2}{2}$. Then

$$\int x\ln 10x\,dx = \frac{x^2}{2}\ln 10x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln 10x - \frac{x^2}{4} + C = \frac{x^2}{4}\left(2\ln 10x - 1\right) + C.$$

8.2.14 Let $u = s$ and $dv = e^{-2s}\,ds$. Then $du = ds$ and $v = -\dfrac{1}{2}e^{-2s}$. Then

$$\int se^{-2s}\,ds = -\frac{1}{2}se^{-2s} + \frac{1}{2}\int e^{-2s}\,ds = -\frac{1}{2}se^{-2s} - \frac{1}{4}e^{-2s} + C.$$

8.2.15 Let $u = 2w + 4$ and $dv = \cos 2w\,dw$. Then $du = 2\,dw$ and $v = \dfrac{1}{2}\sin 2w$. The integration by parts formula gives

$$(w+2)\sin 2w - \int \sin 2w\,dw = (w+2)\sin 2w + \frac{1}{2}\cos 2w + C.$$

8.2.16 Let $u = \theta$ and $dv = \sec^2\theta\,d\theta$. Then $du = d\theta$ and $v = \tan\theta$. Then

$$\int \theta\sec^2\theta\,d\theta = \theta\tan\theta - \int \tan\theta\,d\theta = \theta\tan\theta + \ln|\cos\theta| + C.$$

8.2.17 Let $u = x$ and $dv = 3^x\,dx$. Then $du = dx$ and $v = \dfrac{3^x}{\ln 3}$. Then we have

$$\frac{x3^x}{\ln 3} - \frac{1}{\ln 3}\int 3^x\,dx = \frac{x3^x}{\ln 3} - \frac{3^x}{\ln^2 3} + C = \frac{3^x}{\ln 3}\left(x - \frac{1}{\ln 3}\right) + C.$$

8.2.18 Let $u = \ln x$ and $dv = x^9\,dx$. Then $du = \dfrac{1}{x}\,dx$ and $v = \dfrac{1}{10}x^{10}$. Then we have

$$\frac{1}{10}x^{10}\ln x - \frac{1}{10}\int x^9\,dx = \frac{1}{10}x^{10}\ln x - \frac{1}{100}x^{10} + C = \frac{1}{100}x^{10}(10\ln x - 1) + C.$$

8.2.19 Let $u = \ln x$ and $dv = x^{-10}\,dx$. Then $du = \dfrac{1}{x}\,dx$ and $v = -\dfrac{1}{9}x^{-9}$. Then

$$\int \frac{\ln x}{x^{10}}\,dx = -\frac{1}{9x^9}\ln x + \frac{1}{9}\int x^{-10}\,dx = -\frac{1}{9x^9}\ln x + -\frac{1}{81x^9} + C.$$

8.2.20 Let $u = \sin^{-1} x$ and $dv = dx$. Then $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$ and $v = x$. Then

$$\int \sin^{-1} x\,dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C.$$

The fact that $-\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx = \sqrt{1-x^2} + C$ follows from the ordinary substitution $u = 1-x^2$.

8.2.21

$$\int x\sin x\cos x\,dx = \frac{1}{2}\int x\cdot(2\sin x\cos x)\,dx = \frac{1}{2}\int x\sin 2x\,dx.$$

Now using the result of problem 10, we have

$$\int x\sin x\cos x\,dx = \frac{1}{2}\cdot\left(-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right) + C = -\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C.$$

8.2.22 Let $u = e^x$ and $dv = e^x \sin e^x\,dx$. Then $du = e^x\,dx$ and $v = -\cos e^x$. Then we have

$$-e^x\cos e^x + \int e^x\cos e^x\,dx = -e^x\cos e^x + \sin e^x + C = \sin e^x - e^x\cos e^x + C.$$

8.2.23 Let $u = x^2$ and $dv = \sin 2x\,dx$. Then $du = 2x\,dx$ and $v = -\frac{1}{2}\cos 2x$. Then we have

$$\int x^2\sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx.$$

Now we consider computing this last term $\displaystyle\int x\cos 2x\,dx$ as a new problem. Let $u = x$ and $dv = \cos 2x\,dx$. Then $du = dx$ and $v = \dfrac{1}{2}\sin 2x$. So

$$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C.$$

Combining these results we have

$$\int x^2\sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C.$$

8.2.24 Let $u = x^2$ and $dv = e^{4x}\,dx$. Then $du = 2x\,dx$ and $v = \dfrac{e^{4x}}{4}$. Then we have

$$\int x^2e^{4x}\,dx = \frac{1}{4}x^2e^{4x} - \frac{1}{2}\int xe^{4x}\,dx.$$

Now we consider computing this last integral $\displaystyle\int xe^{4x}\,dx$ as a new problem. Let $u = x$ and $dv = e^{4x}\,dx$. Then $du = dx$ and $v = \dfrac{e^{4x}}{4}$. Then we have

$$\int xe^{4x}\,dx = \frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x}\,dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C.$$

Combining these results gives

$$\int x^2e^{4x}\,dx = \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C = e^{4x}\left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right) + C.$$

8.2.25 Let $u = t^2$ and $dv = e^{-t}\,dt$. Then $du = 2t\,dt$ and $v = -e^{-t}$. We have

$$\int t^2 e^{-t}\,dt = -t^2e^{-t} + 2\int te^{-t}\,dt.$$

To compute this last integral, we let $u = t$ and $dv = e^{-t}\,dt$. Then

$$\int te^{-t}\,dt = -te^{-t} + \int e^{-t}\,dt = -te^{-t} - e^{-t} + C.$$

Putting these results together, we obtain

$$\int t^2e^{-t}\,dt = -t^2e^{-t} + 2(-te^{-t} - e^{-t}) + C = -e^{-t}(t^2 + 2t + 2) + C.$$

8.2.26 Let $u = t^3$ and $dv = \sin t\,dt$. Then $du = 3t^2\,dt$ and $v = -\cos t$. Then

$$\int t^3 \sin t\,dt = -t^3\cos t + 3\int t^2\cos t.$$

We then consider the last integral. We now let $u = t^2$ and $dv = \cos t\,dt$. Then $du = 2t\,dt$ and $v = \sin t$. We have

$$3\int t^2\cos t\,dt = 3t^2\sin t - 6\int t\sin t\,dt.$$

For the last integral, we let $u = t$ and $dv = \sin t\,dt$. Then $du = dt$ and $v = \cos t$. We have

$$-6\int t\sin t\,dt = 6t\cos t - 6\int \cos t = 6t\cos t - 6\sin t + C.$$

Putting these results together, we have

$$\int t^3\sin t\,dt = -t^3\cos t + 3t^2\sin t + 6t\cos t - 6\sin t + C.$$

8.2.27 Let $u = \cos x$ and $dv = e^x\,dx$. Then $du = -\sin x\,dx$ and $v = e^x$. We have

$$\int e^x\cos x\,dx = e^x\cos x + \int e^x\sin x\,dx.$$

Now in order to compute the integral which comprises this last term, we let $u = \sin x$ and $dv = e^x\,dx$. Then $du = \cos x\,dx$ and $v = e^x$. Thus,

$$\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx.$$

Putting these results together gives

$$\begin{aligned}\int e^x\cos x\,dx &= e^x\cos x + e^x\sin x - \int e^x\cos x\,dx\\ 2\int e^x\cos x\,dx &= e^x(\cos x + \sin x) + C\\ \int e^x\cos x\,dx &= \frac{e^x}{2}(\cos x + \sin x) + C.\end{aligned}$$

8.2.28 Let $u = \cos 2x$ and $dv = e^{3x}\,dx$. Then $du = -2\sin 2x\,dx$ and $v = \frac{1}{3}e^{3x}$. We have

$$\int e^{3x}\cos 2x\,dx = \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\int e^{3x}\sin 2x\,dx.$$

Now in order to compute the integral which comprises this last term, we let $u = \sin 2x$ and $dv = e^{3x}\,dx$. Then $du = 2\cos 2x\,dx$ and $v = \frac{1}{3}e^{3x}$. Thus,

$$\int e^{3x}\sin 2x\,dx = \frac{1}{3}e^{3x}\sin 2x - \frac{2}{3}\int e^{3x}\cos 2x\,dx.$$

Putting these results together gives

$$\begin{aligned}\int e^{3x}\cos 2x\,dx &= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{9}e^{3x}\sin 2x - \frac{4}{9}\int e^{3x}\cos 2x\,dx\\ \frac{13}{9}\int e^{3x}\cos 2x\,dx &= \frac{1}{3}e^{3x}\left(\cos 2x + \frac{2}{3}\sin 2x\right) + C\\ \int e^{3x}\cos 2x\,dx &= \frac{3}{13}e^{3x}\left(\cos 2x + \frac{2}{3}\sin 2x\right) + C.\end{aligned}$$

8.2.29 Let $u = \sin 4x$ and $dv = e^{-x}\,dx$. Then $du = 4\cos 4x\,dx$ and $v = -e^{-x}$. We have

$$\int e^{-x}\sin 4x\,dx = -e^{-x}\sin 4x + 4\int e^{-x}\cos 4x\,dx.$$

Now in order to compute the integral which comprises this last term, we let $u = \cos 4x$ and $dv = e^{-x}\,dx$. Then $du = -4\sin 4x\,dx$ and $v = -e^{-x}$. Thus,

$$\int e^{-x}\cos 4x\,dx = -e^{-x}\cos 4x - 4\int e^{-x}\sin 4x\,dx.$$

Putting these results together gives

$$\begin{aligned}\int e^{-x}\sin 4x\,dx &= -e^{-x}\sin 4x - 4e^{-x}\cos 4x - 16\int e^{-x}\sin 4x\,dx\\ 17\int e^{-x}\sin 4x\,dx &= -e^{-x}\sin 4x - 4e^{-x}\cos 4x + C\\ \int e^{-x}\sin 4x\,dx &= -\frac{e^{-x}}{17}(\sin 4x + 4\cos 4x) + C.\end{aligned}$$

8.2.30 Let $u = \sin 6\theta$ and $dv = e^{-2\theta}\,d\theta$. Then $du = 6\cos 6\theta\,d\theta$ and $v = -\frac{1}{2}e^{-2\theta}$. We have

$$\int e^{-2\theta}\sin 6\theta\,d\theta = -\frac{1}{2}e^{-2\theta}\sin 6\theta + 3\int e^{-2\theta}\cos 6\theta\,d\theta.$$

Now in order to compute the integral which comprises this last term, we let $u = \cos 6\theta$ and $dv = e^{-2\theta}\,d\theta$. Then $du = -6\sin 6\theta\,d\theta$ and $v = -\frac{1}{2}e^{-2\theta}$. Thus,

$$\int e^{-2\theta}\cos 6\theta\,d\theta = -\frac{1}{2}e^{-2\theta}\cos 6\theta - 3\int e^{-2\theta}\sin 6\theta\,d\theta.$$

Putting these results together gives

$$\begin{aligned}\int e^{-2\theta}\sin 6\theta\,d\theta &= -\frac{1}{2}e^{-2\theta}\sin 6\theta - \frac{3}{2}e^{-2\theta}\cos 6\theta - 9\int e^{-2\theta}\sin 6\theta\,d\theta\\ 10\int e^{-2\theta}\sin 6\theta\,d\theta &= -\frac{1}{2}e^{-2\theta}(\sin 6\theta + 3\cos 6\theta) + C\\ \int e^{-2\theta}\sin 6\theta\,d\theta &= -\frac{e^{-2\theta}}{20}(\sin 6\theta + 3\cos 6\theta) + C.\end{aligned}$$

8.2.31 Let $u = e^{2x}$ and $dv = e^x \sin e^x\,dx$. Then $du = 2e^{2x}\,dx$ and $v = -\cos e^x$. Then we have

$$\int e^{3x}\sin e^x\,dx = -e^{2x}\cos e^x + 2\int e^{2x}\cos e^x\,dx.$$

To compute the last integral, we let $u = e^x$ and $dv = e^x \cos e^x\,dx$. Then $du = e^x\,dx$ and $v = \sin e^x$. Then the last integral is

$$2\int e^{2x}\cos e^x\,dx = 2e^x\sin e^x - 2\int e^x \sin e^x\,dx = 2e^x\sin e^x + 2\cos e^x + C.$$

Combining these results gives a final answer of

$$\int e^{3x}\sin e^x\,dx = -e^{2x}\cos e^x + 2e^x\sin e^x + 2\cos e^x + C.$$

8.2.32 Let $u = x^2$ and $dv = 2^x\,dx$, so that $du = 2x\,dx$ and $v = \dfrac{2^x}{\ln 2}$. Then we have

$$\left.\frac{x^2 2^x}{\ln 2}\right|_0^1 - \frac{2}{\ln 2}\int_0^1 x2^x\,dx = \frac{2}{\ln 2} - \frac{2}{\ln 2}\int_0^1 x2^x\,dx.$$

Now we let $u = x$ and $dv = 2^x\,dx$, so that $du = dx$ and $v = \dfrac{2^x}{\ln 2}$. The given integral is then equal to

$$\begin{aligned}\frac{2}{\ln 2} - \frac{2}{\ln 2}\left(\left.\frac{x2^x}{\ln 2}\right|_0^1 - \frac{1}{\ln 2}\int_0^1 2^x\,dx\right) &= \frac{2}{\ln 2} - \frac{2}{\ln 2}\left(\frac{2}{\ln 2} - \frac{1}{\ln 2}\left(\left.\frac{2^x}{\ln 2}\right|_0^1\right)\right)\\ &= \frac{2}{\ln 2} - \frac{2}{\ln 2}\left(\frac{2}{\ln 2} - \left(\frac{2}{\ln^2 2} - \frac{1}{\ln^2 2}\right)\right).\end{aligned}$$

This can be written as

$$\frac{2}{\ln 2} - \frac{2}{\ln 2}\left(\frac{2}{\ln 2} - \frac{1}{\ln^2 2}\right) = \frac{2\ln^2 2}{\ln^3 2} - \frac{2(2\ln 2 - 1)}{\ln^3 2} = \frac{2(\ln 2 - 1)^2}{\ln^3 2}.$$

8.2.33 Let $u = x$ and $dv = \sin x\,dx$. Then $du = dx$ and $v = -\cos x$. Then

$$\int_0^\pi x\sin x\,dx = -x\cos x\,\Big|_0^\pi + \int_0^\pi \cos x\,dx = \pi - 0 + \sin x\,\Big|_0^\pi = \pi - 0 + 0 - 0 = \pi.$$

8.2.34 First note that $\displaystyle\int_1^e \ln 2x\,dx = \int_1^e \ln 2\,dx + \int_1^e \ln x\,dx = \ln 2(e-1) + \int_1^e \ln x\,dx.$

Let $u = \ln x$ and $dv = dx$. Then $du = \dfrac{1}{x}\,dx$ and $v = x$. Then

$$\int_1^e \ln x\,dx = x\ln x\,\Big|_1^e - \int_1^e dx = e - (e-1) = 1.$$

Thus $\displaystyle\int_1^e \ln 2x\,dx = \ln 2(e-1) + 1.$

8.2.35 Let $u = x$ and $dv = \cos 2x\,dx$. Then $du = dx$ and $v = \dfrac{1}{2}\sin 2x$. Then

$$\int_0^{\pi/2} x\cos 2x\,dx = \frac{1}{2}x\sin 2x\,\Big|_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2}\sin 2x\,dx = 0 - \left(\frac{1}{2}\cdot\frac{(-\cos 2x)}{2}\right)\Big|_0^{\pi/2} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

8.2.36 Let $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$. Then

$$\int_0^{\ln 2} xe^x\,dx = xe^x\,\Big|_0^{\ln 2} - \int_0^{\ln 2} e^x\,dx = 2\ln 2 - (e^x)\,\Big|_0^{\ln 2} = 2\ln 2 - (2-1) = 2\ln 2 - 1.$$

8.2.37 Let $u = \ln x$ and $dv = x^2\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{x^3}{3}$. Then

$$\int_1^{e^2} x^2\ln x\,dx = \frac{1}{3}x^3\ln x\,\Big|_1^{e^2} - \frac{1}{3}\int_1^{e^2} x^2\,dx = \frac{2}{3}e^6 - \frac{1}{9}x^3\,\Big|_1^{e^2} = \frac{2}{3}e^6 - \frac{1}{9}\left(e^6 - 1\right) = \frac{5}{9}e^6 + \frac{1}{9}.$$

8.2.38 Let $u = \ln^2 x$ and $dv = x^2\,dx$. Then $du = \frac{2\ln x}{x}$ and $v = \frac{x^3}{3}$. We have

$$\int x^2\ln^2 x\,dx = \frac{1}{3}x^3\ln^2 x - \frac{2}{3}\int x^2\ln x\,dx.$$

To compute this last integral, let $u = \ln x$ and $dv = x^2\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{x^3}{3}$. Then

$$\int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C = \frac{x^3}{9}(3\ln x - 1) + C.$$

Thus, $\displaystyle\int x^2\ln^2 x\,dx = \frac{x^3}{3}\ln^2 x - \frac{2x^3}{27}(3\ln x - 1) + C = \frac{x^3}{27}(9\ln^2 x - 6\ln x + 2) + C.$

8.2.39 By problem 20, $\int \sin^{-1} y\,dy = y\sin^{-1} y + \sqrt{1-y^2}$. Thus,

$$\int_0^1 \sin^{-1} y\,dy = \left(y\sin^{-1} y + \sqrt{1-y^2}\right)\Big|_0^1 = \left(\frac{\pi}{2} + 0\right) - (0+1) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}.$$

8.2.40 Let $u = \sqrt{x}$ and $dv = \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. Then $du = \frac{dx}{2\sqrt{x}}$ and $v = 2e^{\sqrt{x}}$. Then

$$\int e^{\sqrt{x}}\,dx = \int \frac{\sqrt{x}e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\sqrt{x}e^{\sqrt{x}} - \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C.$$

8.2.41

a. Let $u = \tan^{-1} x$ and $dv = dx$. Then $du = \frac{dx}{1+x^2}$ and $v = x$. Then

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C.$$

b. Let $u = x^2$. Then $du = 2x\,dx$. Then

$$\int x\tan^{-1} x^2\,dx = \int \frac{1}{2}\tan^{-1} u\,du = \frac{1}{2}u\tan^{-1} u - \frac{1}{4}\ln(1+u^2) + C = \frac{1}{2}x^2\tan^{-1} x^2 - \frac{1}{4}\ln(1+x^4) + C.$$

8.2.42 The volume is given by $V = \int_0^1 2\pi y e^y\,dy$. Let $u = y$ and $dv = e^y\,dy$. Then $du = dy$ and $v = e^y$. Then

$$V = 2\pi\,(ye^y)\Big|_0^1 - 2\pi\int_0^1 e^y\,dy = 2\pi\,(e - 0) - 2\pi(e^y)\Big|_0^1 = 2\pi e - 2\pi(e-1) = 2\pi.$$

8.2.43 Using shells, we have $\frac{V}{2\pi} = \int_0^{\ln 2} xe^{-x}\,dx$. Let $u = x$ and $dv = e^{-x}\,dx$, so that $du = dx$ and $v = -e^{-x}$. Then

$$\frac{V}{2\pi} = -xe^{-x}\Big|_0^{\ln 2} + \int_0^{\ln 2} e^{-x}\,dx = -\frac{1}{2}\ln 2 - e^{-x}\Big|_0^{\ln 2} = -\frac{\ln 2}{2} - \left(\frac{1}{2} - 1\right) = \frac{1}{2}\left(1 - \ln 2\right).$$

Thus $V = \pi(1 - \ln 2)$.

8.2.44 Using shells, we have $\frac{V}{2\pi} = \int_0^{\pi} x\sin x\,dx$. Let $u = x$ and $dv = \sin x\,dx$, so that $du = dx$ and $v = -\cos x$. Then

$$\frac{V}{2\pi} = -x\cos x\Big|_0^{\pi} + \int_0^{\pi}\cos x\,dx = \pi + \sin x\Big|_0^{\pi} = \pi.$$

Thus $V = 2\pi^2$.

8.2.45 We have $V = \int_1^e \pi\ln x\,dx = \pi(x\ln x - x)\Big|_1^e = \pi(e - e) - \pi(0 - 1) = \pi$.

8.2.46 Using shells, we have $\frac{V}{2\pi} = \int_0^{\ln 2}(\ln 2 - x)e^{-x}\,dx = \ln 2\int_0^{\ln 2} e^{-x}\,dx - \int_0^{\ln 2} xe^{-x}\,dx$.

In the course of solving problem 43, we deduced that $\int_0^{\ln 2} xe^{-x}\,dx = \frac{1 - \ln 2}{2}$. Thus,

$$\frac{V}{2\pi} = \ln 2\left(-e^{-x}\right)\Big|_0^{\ln 2} - \frac{1 - \ln 2}{2} = \ln 2\left(-\frac{1}{2} + 1\right) - \frac{1 - \ln 2}{2} = \ln 2 - \frac{1}{2}.$$

Thus, $V = 2\pi\left(\ln 2 - \frac{1}{2}\right) = \pi(\ln 4 - 1)$.

8.2.47 Using disks, we have $\frac{V}{\pi} = \int_1^{e^2} x^2\ln^2 x\,dx$. By problem 38, we have $\int x^2\ln^2 x\,dx = \frac{1}{3}x^3\ln^2 x - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3 + C$. Thus,

$$\frac{V}{\pi} = \left(\frac{1}{3}x^3\ln^2 x - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3\right)\Big|_1^{e^2} = \left(\frac{4}{3}e^6 - \frac{4}{9}e^6 + \frac{2}{27}e^6\right) - \left(\frac{2}{27}\right) = \frac{26}{27}e^6 - \frac{2}{27}.$$

Thus, $V = \frac{\pi}{27}\left(26e^6 - 2\right)$.

8.2.48 Let $u = \sec x$ and $dv = \sec^2 x\,dx$, so that $du = \sec x\tan x\,dx$ and $v = \tan x$. Then $\int \sec^3 x\,dx = \sec x\tan x - \int \sec x\tan^2 x\,dx = \sec x\tan x - \int \sec x\left(\sec^2 x - 1\right)\,dx = \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$.

Thus $2\int \sec^3 x\,dx = \sec x\tan x + \int \sec x\,dx$, so

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\int \sec x\,dx.$$

8.2.49

a. False. For example, suppose $u = x$ and $dv = x\,dx$. Then $\int uv'\,dx = \int x^2\,dx = \frac{x^3}{3} + C$, but $\int u\,dx\int v'\,dx = \left(\int x\,dx\right)^2 = \left(\frac{x^2}{2} + C\right)^2$.

b. True. This is one way to write the integration by parts formula.

c. True. This is the integration by parts formula with the roles of u and v reversed.

8.2.50 Let $u = x^n$ and $dv = e^{ax}\,dx$. Then $du = nx^{n-1}\,dx$ and $v = \dfrac{e^{ax}}{a}$. Then

$$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx.$$

8.2.51 Let $u = x^n$ and $dv = \cos ax\,dx$. Then $du = nx^{n-1}\,dx$ and $v = \dfrac{\sin ax}{a}$. Then

$$\int x^n \cos ax\,dx = \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin ax\,dx.$$

8.2.52 Let $u = x^n$ and $dv = \sin ax\,dx$. Then $du = nx^{n-1}\,dx$ and $v = -\dfrac{\cos ax}{a}$. Then

$$\int x^n \sin ax\,dx = -\frac{x^n \cos ax}{a} + \frac{n}{a}\int x^{n-1}\cos ax\,dx.$$

8.2.53 Let $u = \ln^n x$ and $dv = dx$. Then $du = \dfrac{n\ln^{n-1}(x)}{x}\,dx$ and $v = x$. Then

$$\int \ln^n(x)\,dx = x\ln^n x - n\int \ln^{n-1}(x)\,dx.$$

8.2.54

$$\begin{aligned}\int x^2e^{3x}\,dx &= \frac{x^2e^{3x}}{3} - \frac{2}{3}\int xe^{3x}\,dx\\ &= \frac{x^2e^{3x}}{3} - \frac{2}{3}\left(\frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}\,dx\right)\\ &= \frac{1}{3}\left(x^2e^{3x} - \frac{2}{3}xe^{3x} + \frac{2}{9}e^{3x}\right) + C\\ &= \frac{e^{3x}}{3}\left(x^2 - \frac{2}{3}x + \frac{2}{9}\right) + C.\end{aligned}$$

8.2.55

$$\begin{aligned}\int x^2\cos 5x\,dx &= \frac{x^2\sin 5x}{5} - \frac{2}{5}\int x\sin 5x\,dx\\ &= \frac{x^2\sin 5x}{5} - \frac{2}{5}\left(-\frac{x\cos 5x}{5} + \frac{1}{5}\int \cos 5x\,dx\right)\\ &= \frac{1}{5}\left(x^2\sin 5x + \frac{2}{5}x\cos 5x - \frac{2}{25}\sin 5x\right) + C.\end{aligned}$$

8.2.56

$$\begin{aligned}\int x^3\sin x\,dx &= -x^3\cos x + 3\int x^2\cos x\,dx\\ &= -x^3\cos x + 3\left(x^2\sin x - 2\int x\sin x\,dx\right)\\ &= -x^3\cos x + 3x^2\sin x - 6\left(-x\cos x + \int \cos x\,dx\right)\\ &= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C.\end{aligned}$$

8.2.57

$$\begin{aligned}\int_1^e \ln^3 x\,dx &= x\ln^3 x\Big|_1^e - 3\int_1^e \ln^2 x\,dx\\ &= e - 3\left(x\ln^2 x\right)\Big|_1^e + 6\int_1^e \ln x\,dx\\ &= e - 3e + 6(x\ln x - x)\Big|_1^e\\ &= e - 3e + 6((e-e)-(0-1)) = 6-2e.\end{aligned}$$

8.2.58

a. Let $u = \ln\cos x$ and $dv = \sin x\,dx$. Then $du = -\tan x\,dx$ and $v = -\cos x$. We have

$$\begin{aligned}\int_0^{\pi/3} \sin x \ln(\cos x)\,dx = -\cos x\ln\cos x\Big|_0^{\pi/3} - \int_0^{\pi/3}\sin x\,dx = -\frac{1}{2}\ln\frac{1}{2} + \left(\cos x\Big|_0^{\pi/3}\right)\\ = -\frac{1}{2}\ln\frac{1}{2} + \frac{1}{2} - 1 = \frac{1}{2}\ln 2 - \frac{1}{2} - \frac{\ln 2 - 1}{2}.\end{aligned}$$

b. Let $u = \cos x$. Then $du = -\sin x\,dx$. Substituting gives

$$-\int_1^{1/2}\ln u\,du = (u\ln u - u)\Big|_{1/2}^1 = (0-1) - \left(\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\right) = \frac{\ln 2 - 1}{2}.$$

8.2.59

a. Let $u = x$ and $dv = \dfrac{dx}{\sqrt{x+1}}$. Then $du = dx$ and $v = 2\sqrt{x+1}$. Then

$$\int \frac{x}{\sqrt{x+1}}\,dx = 2x\sqrt{x+1} - \int 2\sqrt{x+1}\,dx. = 2x\sqrt{x+1} - \frac{4}{3}(x+1)^{3/2} + C = \frac{2}{3}\sqrt{x+1}(x-2) + C.$$

b. Let $u = x+1$. Then $du = dx$ and $x = u-1$. Substituting gives

$$\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du = \int (u^{1/2} - u^{-1/2})\,du = \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C.$$

c. The answer to b can be written as the answer to a:

$$\frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C = \frac{2}{3}(x+1)^{1/2}(x+1-3) + C = \frac{2}{3}\sqrt{x+1}(x-2) + C.$$

8.2.60

a. Let $u = x^2$, so that $du = 2x\,dx$. Then

$$\int x\ln x^2\,dx = \frac{1}{2}\int \ln u\,du = \frac{1}{2}(u\ln u - u) + C = \frac{1}{2}\left(x^2\ln(x^2) - x^2\right) + C.$$

b. Let $u = \ln x$ and $dv = x\,dx$. Then $du = \dfrac{1}{x}\,dx$ and $v = \dfrac{x^2}{2}$. Then

$$\int x\ln x^2\,dx = 2\int x\ln x\,dx = 2\left(\frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx\right) = x^2\ln x - \frac{x^2}{2} + C.$$

c. The answer to the first part is $\dfrac{1}{2}\left(x^2\ln(x^2) - x^2\right) + C = x^2\ln(x) - \dfrac{x^2}{2} + C.$ which is the answer to the second part.

8.2.61 Using the change of base formula, we have $\int \log_b x\,dx = \int \frac{\ln x}{\ln b}\,dx = \frac{1}{\ln b}\,(x\ln x - x) + C$.

8.2.62 By parts: Let $u = \sin x$ and $dv = \cos x\,dx$, so that $du = \cos x\,dx$ and $v = \sin x$. Then

$$\int \sin x \cos x\,dx = \sin^2 x - \int \sin x \cos x\,dx,$$

so $\int \sin x \cos x\,dx = \frac{\sin^2 x}{2} + C$.

By substitution: Let $u = \sin x$, so that $du = \cos x\,dx$. Then we have $\int \sin x \cos x\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C$. The two answers are the same.

8.2.63 Let $z = \sqrt{x}$, so that $dz = \frac{1}{2\sqrt{x}}\,dx$. Substituting yields $2\int \frac{\sqrt{x}\cos\sqrt{x}}{2\sqrt{x}}\,dx = 2\int z\cos z\,dz$. Now let $u = z$ and $dv = \cos z\,dz$. then $du = dz$ and $v = \sin z$. Then by Integration by Parts, we have

$$2\int z\cos z\,dz = 2\left(z\sin z - \int \sin z\,dz\right) = 2z\sin z + 2\cos z + C.$$

Thus, the original given integral is equal to $2(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}) + C$.

8.2.64 Let $z = \sqrt{x}$, so that $dz = \frac{1}{2\sqrt{x}}\,dx$. Substituting yields $\int_0^{\pi^2/4} \sin\sqrt{x}\,dx = 2\int_0^{\pi/2} z\sin z\,dz$. Now let $u = z$ and $dv = \sin z\,dz$. Then $du = dz$ and $v = -\cos z$. Then by Integration by Parts, we have

$$2\int_0^{\pi/2} z\sin z\,dz = 2(-z\cos z)\Big|_0^{\pi/2} + 2\int_0^{\pi/2} \cos z = 0 + 2\sin z\Big|_0^{\pi/2} = 2.$$

8.2.65 Let $u = x$ and $dv = f''(x)\,dx$. Then $du = dx$ and $v = f'(x)$. We have

$$\int_a^b xf''(x)\,dx = xf'(x)\Big|_a^b - \int_a^b f'(x)\,dx = (0-0) - f(x)\Big|_a^b = -(f(b) - f(a)) = f(a) - f(b).$$

8.2.66 Let $u = f(x)$ and $dv = f'(x)\,dx$. Then $du = f'(x)\,dx$ and $v = f(x)$. We have

$$\int_a^b f(x)f'(x)\,dx = f(x)^2\Big|_a^b - \int_a^b f(x)f'(x)\,dx.$$

Thus, $2\int_a^b f(x)f'(x)\,dx = f(x)^2\Big|_a^b$, so $\int_a^b f(x)f'(x)\,dx = \frac{1}{2}\,f(x)^2\Big|_a^b = \frac{1}{2}\left(f(b)^2 - f(a)^2\right)$.

8.2.67 By the Fundamental Theorem, $f'(x) = \sqrt{\ln^2 x - 1}$. So the arc length is $\int_e^{e^3} \sqrt{1 + (f'(x))^2}\,dx = \int_e^{e^3} \ln x\,dx = (x\ln x - x)\Big|_e^{e^3} = 3e^3 - e^3 - (e - e) = 2e^3$.

8.2.68 Suppose $m \neq -1$ and let $u = \ln x$ and $dv = x^m\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{x^{m+1}}{m+1}$. Then $\int x^m \ln x\,dx = \frac{x^{m+1}}{m+1}\ln x - \frac{1}{m+1}\int x^m\,dx = \frac{x^{m+1}}{m+1}\left(\ln x - \frac{1}{m+1}\right) + C$.

For the case $m = -1$ we are computing $\int \frac{1}{x}\ln x\,dx$, so letting $u = \ln x$ so that $du = \frac{1}{x}\,dx$ yields $\int u\,du = \frac{u^2}{2} + C = \frac{\ln^2 x}{2} + C$.

8.2.69 Let V_1 be the volume generated when R is revolved about the x-axis, and V_2 the volume generated when R is revolved about the y-axis.

Using disks, we have

$$\frac{V_1}{\pi} = \int_0^\pi \sin^2 x\,dx = \frac{1}{2}\int_0^\pi (1-\cos 2x)\,dx = \frac{1}{2}\left(\pi - \frac{1}{2}\int_0^\pi 2\cos 2x\,dx\right) = \frac{1}{2}\left(\pi - \frac{1}{2}\int_0^{2\pi} \cos u\,du\right) = \frac{\pi}{2},$$

where the ordinary substitution $u = 2x$ was made. So $V_1 = \frac{\pi^2}{2}$.

Using shells to compute V_2, we have $\frac{V_2}{2\pi} = \int_0^\pi x\sin x\,dx$. Letting $u = x$ and $dv = \sin x\,dx$, we have $du = dx$ and $v = -\cos x$. Thus, $\frac{V_2}{2\pi} = -x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx = \pi$. Thus, $V_2 = 2\pi^2$, and $V_2 > V_1$.

8.2.70

a. Let $u = x$ and $dv = f'(x)\,dx$. Then $du = dx$ and $v = f(x)$. So $\int xf'(x)\,dx = xf(x) - \int f(x)\,dx$.

b. Letting $f'(x) = e^{3x}$ we have $\int xe^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$.

8.2.71 Using shells, we have $\frac{V}{2\pi} = \int_0^{\pi/2} x\cos x\,dx$. Let $u = x$ and $dv = \cos x\,dx$, so that $du = dx$ and $v = \sin x$. We have $\frac{V}{2\pi} = x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x\,dx = \frac{\pi}{2} - 1$.

Thus, $V = \pi(\pi - 2)$.

8.2.72 We are looking to compute $\int_0^{1/2} (\sin^{-1} x - \sin x)\,dx = \int_0^{1/2} \sin^{-1} x\,dx + \cos 1 - \cos(1/2)$.

Consider $\int \sin^{-1} x$. Let $u = \sin^{-1} x$ and $dv = dx$. Then $du = \frac{1}{\sqrt{1-x^2}}\,dx$ and $v = x$. Then

$$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx.$$

Now we perform the ordinary substitution $u = 1 - x^2$, so that $du = -2x\,dx$. We have

$$x\sin^{-1} x + \frac{1}{2}\int u^{-1/2}\,du = x\sin^{-1} x + \sqrt{1-x^2} + C.$$

Thus $\int_0^{1/2} \sin^{-1} x\,dx = \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right) - 1$, and

$$\int_{1/2}^1 (\sin^{-1} x - \sin x)\,dx = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 - (\cos(1/2) - 1) = \frac{\pi}{12} + \frac{\sqrt{3}}{2} + \cos(1/2) - 2.$$

8.2.73 Let $u = \sin bx$ and $dv = e^{ax}\,dx$ so that $du = b\cos bx\,dx$ and $v = \frac{e^{ax}}{a}$. Then $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx$. Now let $u = \cos bx$ and $dv = e^{ax}$, so that $du = -b\sin bx$ and $v = \frac{e^{ax}}{a}$. Then $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx$. Putting these together yields

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\left(\frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx\right).$$

Multiplying through by a^2 and combining like terms yields

$$(a^2+b^2)\int e^{ax}\sin bx\,dx = e^{ax}\,(a\sin bx - b\cos bx) + C,$$

so $\displaystyle\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\,(a\sin bx - b\cos bx) + C$ as desired.

Now, returning to the integral mentioned above, and incorporating this last result, we have

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\cdot e^{ax}\cdot\frac{a\sin bx - b\cos bx}{a^2+b^2} + C = \frac{e^{ax}}{a}\cdot\frac{(a^2+b^2)\cos bx + b(a\sin bx - b\cos bx)}{a^2+b^2} + C$$

$$= e^{ax}\cdot\frac{a\cos bx + b\sin bx}{a^2+b^2} + C.$$

8.2.74

a. Let $y = f^{-1}(x)$, so that $dy = \frac{1}{f'(y)}\,dx$, $f'(y) \neq 0$. Then $\displaystyle\int f^{-1}(x)\,dx = \int f'(y)f^{-1}(x)\frac{1}{f'(y)}\,dx = \int yf'(y)\,dy$.

b. Using the result of exercise 73, $\displaystyle\int f^{-1}(x)\,dx = \int yf'(y)\,dy = yf(y) - \int f(y)\,dy$.

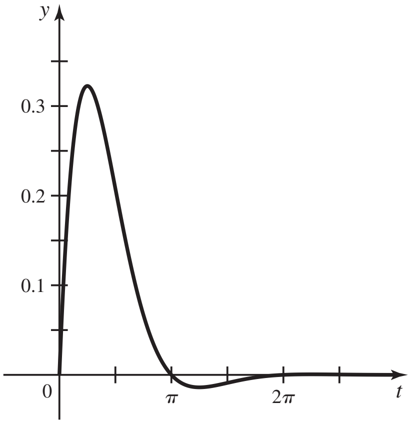
c. $\displaystyle\int \ln x\,dx = x\ln x - \int e^y\,dy = x\ln x - e^y + C = x\ln x - x + C$.

d. $\displaystyle\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \sin y\,dy = x\sin^{-1} x + \cos y + C = x\sin^{-1} x + \sqrt{1-x^2} + C$.

e. $\displaystyle\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \tan y\,dy = x\tan^{-1} x + \ln|\cos y| + C = x\tan^{-1} x + \ln\left(1/\sqrt{1+x^2}\right) + C = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$.

8.2.75

a. We have $s(t) = 0$ when $\sin t = 0$, which occurs for $t = k\pi$, where k is an integer.



b. This is given by $\displaystyle\frac{1}{\pi}\int_0^{\pi} e^{-t}\sin t\,dt$. Using the previous problem, this is equal to

$$\frac{1}{\pi}\,e^{-t}\cdot\frac{-\sin t - \cos t}{2}\Bigg|_0^{\pi} = \frac{1}{2\pi}(e^{-\pi}+1).$$

c. This is given by $\frac{1}{\pi}\int_{n\pi}^{(n+1)\pi} e^{-t}\sin t\,dt$. Using the previous problem, this is equal to

$$\frac{1}{\pi}\,e^{-t}\cdot\frac{-\sin t-\cos t}{2}\bigg|_{n\pi}^{(n+1)\pi} = \frac{1}{2\pi}\left(e^{-(n+1)\pi}\left(-\sin(n+1)\pi-\cos(n+1)\pi\right)-e^{-n\pi}\left(-\sin n\pi-\cos n\pi\right)\right)$$
$$=\frac{1}{2\pi}\left(-e^{-(n+1)\pi}\cos(n+1)\pi+e^{-n\pi}\cos n\pi\right)=\frac{e^{-n\pi}}{2\pi}\left(\cos n\pi-e^{-\pi}\cos(n+1)\pi\right)$$
$$=\frac{e^{-n\pi}}{2\pi}\left((-1)^n-e^{-\pi}(-1)^{n+1}\right)=(-1)^n\frac{e^{-n\pi}}{2\pi}\left(1+e^{-\pi}\right).$$

d. Each a_n is $e^{-\pi}$ times a_{n-1}.

8.2.76 The fallacy comes from thinking that $\int \frac{dx}{x}$ represents a unique quantity, as opposed to a family of functions which differ by a constant. When we subtract this quantity from both sides, we should write the arbitrary constant, giving $0+C_1=1+C_2$, which is a valid statement.

8.2.77

a. If $u=f(x)$ then $du=f'(x)\,dx$, and if $dv=g(x)\,dx$, then $v=G_1(x)$. The integration by parts formula gives

$$\int f(x)g(x)\,dx=f(x)G_1(x)-\int f'(x)G_1(x)\,dx.$$

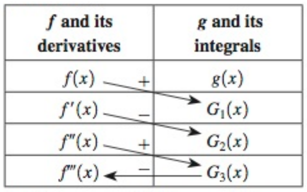
b. Letting $u=f'(x)$ yields $du=f''(x)\,dx$ and letting $dv=G_1(x)\,dx$ yields $v=G_2(x)$. Then continuing from part (a) we have

$$\int f(x)g(x)\,dx=f(x)G_1(x)-\left(f'(x)G_2(x)-\int f''(x)G_2(x)\,dx\right)$$
$$=f(x)G_1(x)-f'(x)G_2(x)+\int f''(x)G_2(x)\,dx.$$

Note that we end up with three terms which consist of products as indicated by the arrows, with the last product inside an integral sign. The signs alternate as indicated on the chart.

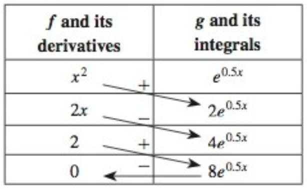
c. For the trailing integral from part (b), we let $u=f''(x)$ so that $du=f'''(x)\,dx$, and let $dv=G_2(x)\,dx$ so that $v=G_3(x)$. Then we have

$$\int f(x)g(x)\,dx=f(x)G_1(x)-f'(x)G_2(x)+\int f''(x)G_2(x)\,dx$$
$$=f(x)G_1(x)-f'(x)G_2(x)+f''(x)G_3(x)-\int f'''(x)G_3(x)\,dx.$$



f **and its derivatives**	***g*** **and its integrals**
$f(x)$ +	$g(x)$
$f'(x)$ −	$G_1(x)$
$f''(x)$ +	$G_2(x)$
$f'''(x)$ −	$G_3(x)$

d. The table is as follows:



f **and its derivatives**	***g*** **and its integrals**
x^2 +	$e^{0.5x}$
$2x$ −	$2e^{0.5x}$
2 +	$4e^{0.5x}$
0 −	$8e^{0.5x}$

So
$$\int x^2e^{0.5x}\,dx = 2x^2e^{0.5x} - 8xe^{0.5x} + 16e^{0.5x} + C.$$
Note that the last integral is $\int 0\cdot 8e^{0.5x}\,dx = \int 0\,dx = C$.

e. We have:

f and its derivatives	g and its integrals
x^3 $+$	$\cos x$
$3x^2$ $-$	$\sin x$
$6x$ $+$	$-\cos x$
6 $-$	$-\sin x$
0 $+$	$\cos x$

So $\int x^3\cos x\,dx = x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x + C$.

f. $\dfrac{d^k}{dx^k}(p_n(x)) = 0$ for $k \geq n+1$, so the process will end.

8.2.78

a. The chart looks like this:

f and its derivatives	g and its integrals
x^4 $+$	e^x
$4x^3$ $-$	e^x
$12x^2$ $+$	e^x
$24x$ $-$	e^x
24 $+$	e^x

Thus $\int x^4e^x\,dx = x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C$.

b. The chart looks like this:

f and its derivatives	g and its integrals
$7x$ $+$	e^{3x}
7 $-$	$\frac{1}{3}e^{3x}$
0 $+$	$\frac{1}{9}e^{3x}$

Thus $\int 7xe^{3x}\,dx = \dfrac{7}{3}xe^{3x} - \dfrac{7}{9}e^{3x} + C$.

c. The chart looks like this:

f and its derivatives	g and its integrals
$2x^2$ $+$	$\sqrt{x+1}$
$4x$ $-$	$\frac{2}{3}(x+1)^{3/2}$
4 $+$	$\frac{4}{15}(x+1)^{5/2}$
0 $-$	$\frac{8}{105}(x+1)^{7/2}$

Thus $\int 2x^2\sqrt{x+1}\,dx = \frac{4}{3}x^2(x+1)^{3/2} - \frac{16}{15}x(x+1)^{5/2} + \frac{32}{105}(x+1)^{7/2} + C$. Evaluating at -1 and 0 gives

$$\int_{-1}^{0} 2x^2\sqrt{x+1} = \frac{32}{105}.$$

d. The chart looks like this:

f and its derivatives	g and its integrals
$x^3 - 2x$ $+$	$\sin 2x$
$3x^2 - 2$ $-$	$-\frac{1}{2}\cos 2x$
$6x$ $+$	$-\frac{1}{4}\sin 2x$
6 $-$	$\frac{1}{8}\cos 2x$
0 $+$	$\frac{1}{16}\sin 2x$

Thus $\int (x^3 - 2x)\sin 2x\,dx = -\frac{1}{2}(x^3 - 2x)\cos 2x + \frac{1}{4}(3x^2 - 2)\sin 2x + \frac{3}{4}x\sin 2x - \frac{3}{8}\sin 2x + C$.

e. The chart looks like this:

f and its derivatives	g and its integrals
$2x^2 - 3x$ $+$	$(x-1)^{-3}$
$4x - 3$ $-$	$-\frac{1}{2}(x-1)^{-2}$
4 $+$	$\frac{1}{2}(x-1)^{-1}$
0 $-$	$\frac{1}{2}\ln\lvert x-1\rvert$

Thus $\int \frac{2x^2 - 3x}{(x-1)^3}\,dx = -\frac{1}{2}(2x^2 - 3x)(x-1)^{-2} - \frac{4x-3}{2}(x-1)^{-1} + 2\ln\lvert x-1\rvert + C$.

f. The chart looks like this:

f and its derivatives	g and its integrals
$x^2 + 3x + 4$ $+$	$(2x+1)^{-1/3}$
$2x + 3$ $-$	$\frac{3}{4}(2x+1)^{2/3}$
2 $+$	$\frac{9}{40}(2x+1)^{5/3}$
0 $-$	$\frac{27}{640}(2x+1)^{8/3}$

Thus $\int \frac{x^2 + 3x + 4}{\sqrt[3]{2x+1}}\,dx = \frac{3}{4}(x^2 + 3x + 4)(2x+1)^{2/3} - \frac{9}{40}(2x+3)(2x+1)^{5/3} + \frac{27}{320}(2x+1)^{8/3} + C$.

g. The chart looks like this:

f and its derivatives	g and its integrals
x $+$	$\frac{1}{\sqrt{1-x^2}}$
1 $-$	$\sin^{-1} x$
0 $+$	$x\sin^{-1} x + \sqrt{1-x^2}$

So $\int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}x - (x\sin^{-1}x + \sqrt{1-x^2}) + C = -\sqrt{1-x^2} + C$. However, this is a little ridiculous because finding the integrals in the 2nd column of the table is more difficult than just integrating the given function using the substitution $u = 1 - x^2$. This gives $du = -2x\,dx$, so we have $-\frac{1}{2}\int u^{-1/2}\,du = -\sqrt{u} + C = -\sqrt{1-x^2} + C$.

8.2.79

a. $\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$

b. Adding $\int e^x \cos x\,dx$ to both sides of the previous equation gives $2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$, so

$$\int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

c. The chart looks like this:

f and its derivatives	g and its integrals
e^{-2x} $+$	$\sin 3x$
$-2e^{-2x}$ $-$	$-\frac{1}{3}\cos 3x$
$4e^{-2x}$ $+$	$-\frac{1}{9}\sin 3x$

So we have

$$\int e^{-2x} \sin 3x = -\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{9}e^{-2x}\sin 3x - \frac{4}{9}e^{-2x}\sin 3x.$$

Adding $\frac{4}{9}\int e^{-2x}\sin 3x\,dx$ to both sides gives

$$\frac{13}{9}\int e^{-2x}\sin 3x\,dx = -\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{9}e^{-2x}\sin 3x + D,$$

where D is an arbitrary constant so

$$\int e^{-2x}\sin 3x\,dx = -\frac{3}{13}e^{-2x}\cos 3x - \frac{2}{13}e^{-2x}\sin 3x + C.$$

8.2.80 Let $u = g(x)$ and $dv = f''(x)\,dx$. Then $du = g'(x)\,dx$ and $v = f'(x)$. Then

$$\int_0^1 f''(x)g(x)\,dx = f'(x)g(x)\,\Big|_0^1 - \int_0^1 f'(x)g'(x)\,dx = -\int_0^1 f'(x)g'(x)\,dx.$$

Now let $u = g'(x)$ and $dv = f'(x)\,dx$. Then $du = g''(x)\,dx$ and $v = f(x)$. Then

$$\int_0^1 f'(x)g'(x)\,dx = f(x)g'(x)\,\Big|_0^1 - \int_0^1 f(x)g''(x)\,dx = -\int_0^1 f(x)g''(x)\,dx.$$

Thus, $\int_0^1 f''(x)g(x)\,dx = \int_0^1 f(x)g''(x)\,dx.$

8.2.81

a. To compute I_1, we let $u = -x^2$, so that $du = -2x\,dx$. Then an ordinary substitution yields

$$-\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C.$$

b. To compute I_3, we let $u = x^2$ and $dv = xe^{-x^2}\,dx$, so that $du = 2x\,dx$ and $v = -\frac{1}{2}e^{-x^2}$ (by part (a)). Then $I_3 = -\frac{1}{2}x^2e^{-x^2} + \int xe^{-x^2}\,dx = -\frac{1}{2}x^2e^{-x^2} - \frac{1}{2}e^{-x^2} + C = -\frac{1}{2}e^{-x^2}(x^2+1) + C.$

c. To compute I_5, we let $u = x^4$ and $dv = xe^{-x^2}\,dx$. Then $du = 4x^3\,dx$ and $v = -\frac{1}{2}e^{-x^2}$. Then $I_5 = -\frac{1}{2}x^4e^{-x^2} + 2\int x^3e^{-x^2}\,dx = -\frac{1}{2}x^4e^{-x^2} + 2I_3 = -\frac{1}{2}e^{-x^2}(x^4 + 2x^2 + 2)$.

d. Suppose that n is odd and that it is true that $I_n = -\frac{1}{2}e^{-x^2}p_{n-1}(x)$ where $p_{n-1}(x)$ is an even polynomial of degree $n-1$. We will show that I_{n+2} also has this property, so the result will follow by induction. Let $u = x^{n+1}$ and let $dv = xe^{-x^2}\,dx$, so that $du = (n+1)x^n\,dx$ and $v = -\frac{1}{2}e^{-x^2}$. Then $I_{n+2} = -\frac{1}{2}x^{n+1}e^{-x^2} + \frac{n+1}{2}I_n = -\frac{1}{2}e^{-x^2}(x^{n+1} + \frac{n+1}{2}p_{n-1}(x)) + C = -\frac{1}{2}e^{-x^2}p_{n+1}(x) + C$. Note that if $p_{n-1}(x)$ is an even polynomial of degree $n-1$ then $\frac{n+1}{2}p_{n-1}(x) + x^{n+1}$ is an even polynomial of degree $n+1$.

e. Note that $I_2 = -\frac{1}{2}xe^{-x^2} + \frac{1}{2}I_0$. (Using a similar technique to that above). Now if I_2 were expressible in terms of elementary functions, then I_0 would be as well, but we are given that it isn't. Similarly, we can express I_{2k} in terms of I_{2k-2} using Integration by Parts, and if any of these were expressible in terms of elementary functions, then the even numbered one below it would be. So the inability to express I_0 that way implies the inability to express I_2 that way, which implies the inability to express I_4 that way, and so on.

8.2.82

a. This is given by $\int_0^4 xe^{-x}\,dx$. Let $u = x$ and $dv = e^{-x}\,dx$. Then $du = dx$ and $v = -e^{-x}$. Then $\int_0^4 xe^{-x}\,dx = -xe^{-x}\Big|_0^4 + \int_0^4 e^{-x}\,dx = -\frac{4}{e^4} + \left(-e^{-x}\right)\Big|_0^4 = -\frac{4}{e^4} - \frac{1}{e^4} + 1 = 1 - \frac{5}{e^4}$.

b. This is given by $\int_0^4 xe^{-ax}\,dx$. Let $u = x$ and $dv = e^{-ax}\,dx$. Then $du = dx$ and $v = -\frac{1}{a}e^{-ax}$. Then $\int_0^4 xe^{-ax}\,dx = -\frac{1}{a}xe^{-ax}\Big|_0^4 + \frac{1}{a}\int_0^4 e^{-ax}\,dx = -\frac{4}{ae^{4a}} + \left(-\frac{1}{a^2}e^{-ax}\right)\Big|_0^4 = -\frac{4}{ae^{4a}} - \frac{1}{a^2}\cdot\left(\frac{1}{e^{4a}} - 1\right) = \frac{1}{a^2}\left(1 - \frac{4a+1}{e^{4a}}\right)$.

c. This is given by $\int_0^b xe^{-ax}\,dx$. Let $u = x$ and $dv = e^{-ax}\,dx$. Then $du = dx$ and $v = -\frac{1}{a}e^{-ax}$. Then $\int_0^b xe^{-ax}\,dx = -\frac{1}{a}xe^{-ax}\Big|_0^b + \frac{1}{a}\int_0^b e^{-ax}\,dx = -\frac{b}{ae^{ba}} + \left(-\frac{1}{a^2}e^{-ax}\right)\Big|_0^b = -\frac{b}{ae^{ba}} - \frac{1}{a^2}\cdot\left(\frac{1}{e^{ba}} - 1\right) = \frac{1}{a^2}\left(1 - \frac{ba+1}{e^{ba}}\right)$.

d. $A(1, \ln b) = 1 - \frac{\ln b + 1}{e^{\ln b}} = 1 - \frac{\ln b + 1}{b}$.

$A(2, (\ln b)/2) = \frac{1}{4}\left(1 - \frac{\ln b + 1}{e^{\ln b}}\right) = \frac{1}{4}\left(1 - \frac{\ln b + 1}{b}\right) = \frac{1}{4}A(1, \ln b)$. So $A(1, \ln b) = 4A(2, (\ln b)/2)$.

e. Yes. $A(a, (\ln b)/a) = \frac{1}{a^2}\left(1 - \frac{\ln b + 1}{e^{\ln b}}\right) = \frac{1}{a^2}\left(1 - \frac{\ln b + 1}{b}\right) = \frac{1}{a^2}A(1, \ln b)$.

So $A(1, \ln b) = a^2A(a, (\ln b)/a)$.

8.3 Trigonometric Integrals

8.3.1 The half-angle identities for sine and cosine:

$$\sin^2 x = \frac{1-\cos 2x}{2} \text{ and } \cos^2 x = \frac{1+\cos 2x}{2},$$

which are conjugates.

8.3.2 The Pythagorean Identities:

$$\cos^2 x + \sin^2 x = 1 \text{ and } 1+\tan^2 x = \sec^2 x \text{ and } \cot^2 x + 1 = \csc^2 x,$$

where the second follows from the first by dividing through by $\cos^2 x$, and the third follows from the first by dividing through by $\sin^2 x$.

8.3.3 To integrate $\sin^3 x$, write $\sin^3 x = \sin x \sin^2 x = \sin x(1-\cos^2 x)$, and let $u = \cos x$ so that $du = -\sin x\, dx$.

8.3.4 To integrate $\sin^m x \cos^n x$ where m is even and n is odd, write $\sin^m x \cos^n x = \sin^m x \cos^{n-1} x \cos x = \sin^m x(1-\sin^2 x)^{(n-1)/2} \cos x$ and let $u = \sin x$ so that $du = \cos x\, dx$.

8.3.5 A reduction formula is a recursive formula involving integrals. Using it, one can rewrite an integral of a certain type in a simpler form – which can then perhaps be evaluated or further reduced.

8.3.6 One would compute this integral by writing $\cos^2 x \sin^3 x$ as $\cos^2 x(\sin^2 x)\sin x = \cos^2 x(1-\cos^2 x)\sin x$ and then performing the ordinary substitution $u = \cos x$.

8.3.7 One would compute this integral by letting $u = \tan x$, so that $du = \sec^2 x\, dx$. This substitution leads to the integral $\displaystyle\int u^{10}\, du$, which can easily be evaluated.

8.3.8 One would compute this integral by letting $u = \sec x$, so that $du = \sec x \tan x\, dx$. This substitution leads to the integral $\displaystyle\int u^{11}\, du$, which can easily be evaluated.

8.3.9 $\displaystyle\int \cos^3 x\, dx = \int \cos x(1-\sin^2 x)\, dx$Let $u = \sin x$. Then $du = \cos x\, dx$. Substituting gives

$$\int (1-u^2)\, du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C.$$

8.3.10 $\displaystyle\int \sin^3 x\, dx = \int \sin x(1-\cos^2 x)\, dx.$. Let $u = \cos x$. Then $du = -\sin x\, dx$. Substituting gives

$$\int (u^2-1)\, du = \frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C.$$

8.3.11 $\displaystyle\int \sin^2 3x\, dx = \frac{1}{2}\int (1-\cos 6x)\, dx = \frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right) + C = \frac{x}{2} - \frac{1}{12}\sin 6x + C.$

8.3.12

$$\begin{aligned}\int \cos^4 2\theta\, d\theta &= \int \left(\frac{1+\cos 4\theta}{2}\right)^2 d\theta = \int \frac{1}{4}(1+2\cos 4\theta + \cos^2 4\theta)\, d\theta \\ &= \frac{1}{4}\int \left(1 + 2\cos 4\theta + \frac{1+\cos 8\theta}{2}\right) d\theta = \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 4\theta + \frac{1}{2}\cos 8\theta\right) d\theta \\ &= \frac{1}{4}\left(\frac{3\theta}{2} + \frac{\sin 4\theta}{2} + \frac{\sin 8\theta}{16}\right) + C = \frac{1}{8}\left(3\theta + \sin 4\theta + \frac{1}{8}\sin 8\theta\right) + C.\end{aligned}$$

8.3.13 $\int \sin^5 x\,dx = \int (\sin^2 x)^2 \sin x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx$. Let $u = \cos x$ so that $du = -\sin x\,dx$. Substituting yields

$$-\int (1-u^2)^2\,du = \int (-u^4+2u^2-1)\,du = \frac{-u^5}{5} + \frac{2u^3}{3} - u + C = \frac{-\cos^5 x}{5} + \frac{2\cos^3 x}{3} - \cos x + C.$$

8.3.14 $\int \cos^3 20x\,dx = \int \cos^2 20x \cos 20x\,dx = \int (1-\sin^2 20x)\cos 20x\,dx$. Let $u = \sin 20x$, so that $du = 20\cos 20x\,dx$. Substituting yields

$$\frac{1}{20}\int (1-u^2)\,du = \frac{1}{20}\left(u - \frac{u^3}{3}\right) + C = \frac{1}{20}\left(\sin 20x - \frac{\sin^3 20x}{3}\right) + C.$$

8.3.15 $\int \sin^3 x \cos^2 x\,dx = \int \sin x(1-\cos^2 x)(\cos^2 x)\,dx$. Let $u = \cos x$ so that $du = -\sin x\,dx$. Substituting gives

$$\int (u^4-u^2)\,du = u^5/5 - u^3/3 + C = (\cos^5 x)/5 - (\cos^3 x)/3 + C.$$

8.3.16 $\int \sin^2\theta \cos^5\theta\,d\theta = \int (\sin^2\theta)(1-\sin^2\theta)^2 \cos\theta\,d\theta$. Let $u = \sin\theta$ so that $du = \cos\theta\,d\theta$. Substituting gives

$$\int u^2(1-u^2)^2\,du = \int (u^2-2u^4+u^6)\,du = u^3/3 - 2u^5/5 + u^7/7 + C = (\sin^3\theta)/3 - 2(\sin^5\theta)/5 + (\sin^7\theta)/7 + C.$$

8.3.17 $\int \cos^3 x\sqrt{\sin x}\,dx = \int \cos x(1-\sin^2 x)\sqrt{\sin x}\,dx$. Let $u = \sin x$ so that $du = \cos x\,dx$. Substituting gives

$$\int (1-u^2)u^{1/2}\,du = \int (u^{1/2}-u^{5/2})\,du = 2u^{3/2}/3 - 2u^{7/2}/7 + C = 2(\sin^{3/2} x)/3 - 2(\sin^{7/2} x)/7 + C.$$

8.3.18 $\int \frac{\sin^3\theta}{\cos^2\theta}\,d\theta = \int (\sin\theta)\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)\,d\theta$. Let $u = \cos\theta$ so that $du = -\sin\theta\,d\theta$. Substituting gives

$$\int \frac{u^2-1}{u^2}\,du = \int (1-u^{-2})\,du = u + \frac{1}{u} + C = \cos\theta + \sec\theta + C.$$

8.3.19 $\int_0^{\pi/3} \sin^5 x \cos^{-2} x\,dx = \int_0^{\pi/3} (\sin x)\left(\frac{(1-\cos^2 x)^2}{\cos^2 x}\right)\,dx$. Let $u = \cos x$ so that $du = -\sin x\,dx$. Substituting yields

$$\begin{aligned}-\int_1^{1/2} \frac{(1-u^2)^2}{u^2}\,du &= \int_1^{1/2} (-u^{-2}+2-u^2)\,du = \left(\frac{1}{u} + 2u - \frac{u^3}{3}\right)\Bigg|_1^{1/2} \\ &= \left(2+1-\frac{1}{24} - \left(1+2-\frac{1}{3}\right)\right) = \frac{7}{24}.\end{aligned}$$

8.3.20 $\int \sin^{-3/2} x \cos^3 x\,dx = \int \sin^{-3/2} \cos^2 x \cos x\,dx = \int (\sin^{-3/2} x)(1-\sin^2 x)\cos x\,dx$. Let $u = \sin x$ so that $du = \cos x\,dx$. Then a substitution yields

$$\int u^{-3/2}(1-u^2)\,du = \int u^{-3/2} - u^{1/2}\,du = \frac{-2}{u^{1/2}} - \frac{2u^{3/2}}{3} + C = \frac{-2}{\sqrt{\sin x}} - \frac{2\sin^{3/2} x}{3} + C.$$

8.3.21 $\int_0^{\pi/2} \cos^3 x\sqrt{\sin^3 x}\,dx = \int_0^{\pi/2}(1-\sin^2 x)\sin^{3/2} x\cos x\,dx$. Let $u = \sin x$ so that $du = \cos x\,dx$. Substituting gives

$$\int_0^1 (1-u^2)u^{3/2}\,du = \int_0^1 (u^{3/2}-u^{7/2})\,du = \left(\frac{2}{5}u^{5/2}-\frac{2}{9}u^{9/2}\right)\Big|_0^1 = \frac{2}{5}-\frac{2}{9} = \frac{8}{45}.$$

8.3.22 $\int_{\pi/4}^{\pi/2} \sin^2 2x\cos^3 2x\,dx = \int_{\pi/4}^{\pi/2} \sin^2 2x\cos^2 2x\cos 2x\,dx = \int_{\pi/4}^{\pi/2} \sin^2 2x(1-\sin^2 2x)\cos 2x\,dx$. Let $u = \sin 2x$ so that $du = 2\cos 2x\,dx$. Substituting gives

$$\frac{1}{2}\int_1^0 (u^2-u^4)\,du = \frac{1}{2}\left(\frac{u^3}{3}-\frac{u^5}{5}\right)\Big|_1^0 = \frac{1}{2}\left(\frac{1}{5}-\frac{1}{3}\right) = -\frac{1}{15}.$$

8.3.23

$$\begin{aligned}\int \sin^2 x\cos^2 x\,dx &= \int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{4}\int 1-\cos^2 2x\,dx\\ &= \frac{1}{4}\int\left(1-\frac{1+\cos 4x}{2}\right)dx = \frac{1}{4}\int\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right)dx = \frac{1}{4}\left(\frac{x}{2}-\frac{\sin 4x}{8}\right)+C.\end{aligned}$$

8.3.24 $\int \sin^3 x\cos^5 x\,dx = \int \sin^2 x\cos^5 x\sin x\,dx = \int (1-\cos^2 x)\cos^5 x\sin x\,dx$. Let $u = \cos x$ so that $du = -\sin x\,dx$. Substituting gives

$$-\int (u^5-u^7)\,du = -\frac{\cos^6 x}{6}+\frac{\cos^8 x}{8}+C.$$

8.3.25 $\int \sin^2 x\cos^4 x\,dx = \int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{8}\int (1-\cos^2 2x)(1+\cos 2x)\,dx = \frac{1}{8}\int 1+\cos 2x-\cos^2 2x-\cos^3 2x\,dx = \frac{1}{8}\int 1+\cos 2x-\frac{1+\cos 4x}{2}-\cos^3 2x\,dx = \frac{1}{8}\int \frac{1}{2}+\cos 2x-\frac{1}{2}\cos 4x\,dx - \frac{1}{8}\int (1-\sin^2 2x)\cos 2x\,dx = \frac{x}{16}+\frac{\sin 2x}{16}-\frac{\sin 4x}{64}-\frac{1}{8}\int (1-\sin^2 2x)\cos 2x\,dx$. To compute this last integral, we let $u = \sin 2x$, so that $du = 2\cos 2x\,dx$. Then

$$\int (1-\sin^2 2x)\cos 2x\,dx = \frac{1}{2}\int (1-u^2)\,du = \frac{1}{2}\left(u-\frac{u^3}{3}\right)+C = \frac{1}{2}\left(\sin 2x-\frac{\sin^3 2x}{3}\right)+C.$$

Thus, our original given integral is equal to

$$\frac{x}{16}+\frac{\sin 2x}{16}-\frac{\sin 4x}{64}-\frac{1}{8}\left(\frac{1}{2}\left(\sin 2x-\frac{\sin^3 2x}{3}\right)\right)+C = \frac{x}{16}-\frac{\sin 4x}{64}+\frac{\sin^3 2x}{48}+C.$$

8.3.26 $\int \sin^3 x\cos^{3/2} x\,dx = \int (\sin x)(1-\cos^2 x)\cos^{3/2} x\,dx$. Let $u = \cos x$ so that $du = -\sin x\,dx$. Then substituting yields

$$-\int (1-u^2)u^{3/2}\,du = \int u^{7/2}-u^{3/2}\,du = \frac{2u^{9/2}}{9}-\frac{2u^{5/2}}{5}+C = \frac{2\cos^{9/2} x}{9}-\frac{2\cos^{5/2} x}{5}+C.$$

8.3.27 $\int \tan^2 x\,dx = \int (\sec^2 x-1)\,dx = \tan x - x + C.$

8.3.28 $\int 6\sec^4 x\,dx = \int 6\sec^2 x(\tan^2 x+1)\,dx$. Let $u = \tan x$, so that $du = \sec^2 x\,dx$. Then we have

$$6\int (u^2+1)\,du = 2u^3+6u+C = 2\tan^3 x+6\tan x+C.$$

8.3.29 $\int \cot^4 x\,dx = \int \cot^2 x(\csc^2 x - 1)\,dx = \int(\cot^2 x\csc^2 x - (\csc^2 x - 1))\,dx = \int \cot^2 x\csc^2 x\,dx + \cot x + x$. Let $u = \cot x$ so that $du = -\csc^2 x\,dx$. Substituting gives

$$-\int u^2\,du + \cot x + x = -u^3/3 + \cot x + x + C = -(\cot^3 x)/3 + \cot x + x + C.$$

8.3.30 $\int \tan^3\theta\,d\theta = \int \tan\theta(\sec^2\theta - 1)\,d\theta = \int(\tan\theta\sec^2\theta - \tan\theta)\,d\theta = \int \tan\theta\sec^2\theta\,d\theta + \ln|\cos\theta|$. Let $u = \tan\theta$ so that $du = \sec^2\theta\,d\theta$. Substituting gives

$$\int u\,du + \ln|\cos\theta| = u^2/2 + \ln|\cos\theta| + C = (\tan^2\theta)/2 + \ln|\cos\theta| + C.$$

Note that this can also be written as $\sec^2\theta/2 + \ln|\cos\theta| + C$, because $\sec^2\theta$ and $\tan^2\theta$ differ by a constant.

8.3.31

$$\begin{aligned}\int 20\tan^6 x\,dx &= 20\int(\tan^4 x)(\sec^2 x - 1)\,dx = 20\int((\tan^4 x)\sec^2 x - (\tan^2 x)(\sec^2 x - 1))\,dx\\ &= 20\int(\tan^4 x\sec^2 x - \tan^2 x\sec^2 x + \sec^2 x - 1)\,dx.\end{aligned}$$

Let $u = \tan x$ so that $du = \sec^2 x\,dx$. We have

$$\begin{aligned}20\left(\int(u^4 - u^2)\,du + \tan x - x\right) + C &= 4u^5 - \frac{20u^3}{3} + 20\tan x - 20x + C\\ &= 4\tan^5 x - \frac{20\tan^3 x}{3} + 20\tan x - 20x + C.\end{aligned}$$

8.3.32

$$\begin{aligned}\int \cot^5 3x\,dx &= \int(\cot^3 3x)(\csc^2 3x - 1)\,dx = \int(\cot^3 3x\csc^2 3x - (\cot 3x)(\csc^2 3x - 1))\,dx\\ &= \cot^3 3x\csc^2 3x - \cot 3x\csc^2 3x\,dx + \frac{\ln|\sin 3x|}{3} + C.\end{aligned}$$

Now let $u = \cot 3x$, so that $du = -3\csc^2 3x\,dx$. Substituting gives

$$\begin{aligned}-\frac{1}{3}\int(u^3 - u)\,du + \frac{\ln|\sin 3x|}{3} + C &= -\frac{u^4}{12} + \frac{u^2}{6} + \frac{\ln|\sin 3x|}{3} + C\\ &= -\frac{\cot^4 3x}{12} + \frac{\cot^2 3x}{6} + \frac{\ln|\sin 3x|}{3} + C.\end{aligned}$$

8.3.33 Let $u = \tan x$ so that $du = \sec^2 x\,dx$. Substituting gives $\int 10u^9\,du = u^{10} + C = \tan^{10} x + C$.

8.3.34 $\int \tan^9 x(\tan^2 x + 1)\sec^2 x\,dx$. Let $u = \tan x$ so that $du = \sec^2 x\,dx$. Substituting gives

$$\int u^9(u^2 + 1)\,du = \int(u^{11} + u^9)\,du = u^{12}/12 + u^{10}/10 + C = (\tan^{12} x)/12 + (\tan^{10} x)/10 + C.$$

8.3.35 Let $u = \sec x$ so that $du = \sec x\tan x\,dx$. Substituting gives

$$\int u^2\,du = u^3/3 + C = (\sec^3 x)/3 + C.$$

8.3.36 $\int \tan 4x \sec^{3/2} 4x\,dx = \int \sec^{1/2} 4x \sec 4x \tan 4x\,dx$. Let $u = \sec 4x$. Then $du = 4\sec 4x \tan 4x$. Substituting gives

$$\frac{1}{4}\int u^{1/2}\,du = \frac{1}{4}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{6}\sec^{3/2} 4x + C.$$

8.3.37 Let $u = \ln\theta$ so that $du = \frac{1}{\theta}\,d\theta$. Substituting yields $\int \sec^4 u\,du = \int (\sec^2 u)(1+\tan^2 u)\,du$. Let $w = \tan u$ so that $dw = \sec^2 u\,du$. Substituting again gives

$$\int (1+w^2)\,dw = w + \frac{w^3}{3} + C = \tan(\ln(\theta)) + \frac{\tan^3(\ln(\theta))}{3} + C.$$

8.3.38 $\int \tan^5\theta \sec^4\theta\,d\theta = \int (\tan^5\theta)(\tan^2\theta+1)(\sec^2\theta)\,d\theta$. Let $u = \tan\theta$ so that $du = \sec^2\theta\,d\theta$. Substituting yields $\int u^7 + u^5\,du = \frac{u^8}{8} + \frac{u^6}{6} + C = \frac{\tan^8\theta}{8} + \frac{\tan^6\theta}{6} + C.$

8.3.39

$$\begin{aligned}\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2\theta - 1}\,d\theta &= 2\int_0^{\pi/3} \sqrt{\sec^2\theta - 1}\,d\theta = 2\int_0^{\pi/3} \tan\theta\,d\theta \\ &= -2\,\ln|\cos\theta|\,\Big|_0^{\pi/3} = -2\ln(1/2) + 2\ln(1) = 2\ln 2.\end{aligned}$$

8.3.40 $\int_0^{\pi/6} \tan^5 2x \sec 2x\,dx = \int_0^{\pi/6} (\tan^2 2x)^2 \tan 2x \sec 2x\,dx = \int_0^{\pi/6} (\sec^2 2x - 1)^2 \tan 2x \sec 2x\,dx$. Let $u = \sec 2x$. Then $du = 2\sec 2x \tan 2x\,dx$. Substituting gives

$$\begin{aligned}\frac{1}{2}\int_1^2 (u^2-1)^2\,du &= \frac{1}{2}\int_1^2 (u^4 - 2u^2 + 1)\,du = \frac{1}{2}\left(\frac{u^5}{5} - \frac{2u^3}{3} + u\right)\Big|_1^2 \\ &= \frac{1}{2}\left(\frac{32}{5} - \frac{16}{3} + 2 - \left(\frac{1}{5} - \frac{2}{3} + 1\right)\right) = \frac{19}{15}.\end{aligned}$$

8.3.41 $\int_0^{\pi/4} \sec^7 x \sin x\,dx = \int_0^{\pi/4} \sec^6 x \tan x\,dx = \int_0^{\pi/4} \sec^5 x \sec x \tan x\,dx$. Let $u = \sec x$ so that $du = \sec x \tan x\,dx$. Substituting gives

$$\int_1^{\sqrt{2}} u^5\,du = \frac{u^6}{6}\Big|_1^{\sqrt{2}} = \frac{8}{6} - \frac{1}{6} = \frac{7}{6}.$$

8.3.42 $\int \sqrt{\tan x}\sec^4 x\,dx = \int \sqrt{\tan x}(\tan^2 x + 1)(\sec^2 x)\,dx$. Let $u = \tan x$ so that $du = \sec^2 x\,dx$. Substituting gives

$$\int \sqrt{u}(u^2+1)\,du = \int (u^{5/2} + u^{1/2})\,du = 2u^{7/2}/7 + 2u^{3/2}/3 + C = 2(\tan^{7/2} x)/7 + 2(\tan^{3/2} x)/3 + C.$$

8.3.43

$$\begin{aligned}\int \tan^3 4x\,dx &= \int (\tan 4x)(\sec^2 4x - 1)\,dx = \int (\tan 4x)\sec^2 4x\,dx - \int \tan 4x\,dx \\ &= \int (\tan 4x)\sec^2 4x\,dx + \frac{\ln|\cos 4x|}{4} + C.\end{aligned}$$

Let $u = \tan 4x$ so that $du = 4\sec^2 4x\,dx$. Substituting gives

$$\frac{1}{4}\int u\,du + \frac{\ln|\cos 4x|}{4} + C = \frac{u^2}{8} + \frac{\ln|\cos 4x|}{4} + C = \frac{\tan^2 4x}{8} + \frac{\ln|\cos 4x|}{4} + C.$$

8.3.44 Let $u = \tan x$ so that $du = \sec^2 x\,dx$. Substituting gives

$$\int u^{-5}\,du = -\frac{u^{-4}}{4} + C = -\frac{1}{4\tan^4 x} + C.$$

8.3.45 Let $u = \tan x$ so that $du = \sec^2 x\,dx$. Then

$$\int \sec^2 x \tan^{1/2} x\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\tan^{3/2} x + C.$$

8.3.46

$$\begin{aligned}\int \sec^{-2} x \tan^3 x\,dx &= \int \sec^{-2}(\sec^2 x - 1)\tan x\,dx = \int (\tan x - \sec^{-2} x \tan x)\,dx \\ &= \int \left(\frac{\sin x}{\cos x} - \cos x \sin x\right)\,dx.\end{aligned}$$

Let $u = \cos x$ so that $du = -\sin x\,dx$. Then we have

$$\int \left(-\frac{1}{u} + u\right)\,du = -\ln|u| + \frac{u^2}{2} + C = -\ln|\cos x| + \frac{1}{2}\cos^2 x + C.$$

8.3.47 $\displaystyle\int \frac{\csc^4 x}{\cot^2 x}\,dx = \int (\csc^2 x)\left(\frac{\cot^2 x + 1}{\cot^2 x}\right)\,dx$. Let $u = \cot x$ so that $du = -\csc^2 x\,dx$. Substituting gives

$$-\int \frac{u^2+1}{u^2}\,du = \int -1 - u^{-2}\,du = -u + \frac{1}{u} + C = -\cot x + \tan x + C.$$

8.3.48 Let $u = \csc x$ so that $du = -\csc x \cot x\,dx$. Substituting gives

$$-\int u^9\,du = -u^{10}/10 + C = -(\csc^{10} x)/10 + C.$$

8.3.49 Let $u = \cot 5w$. Then $du = -5\csc^2 5w$. Substituting gives

$$-\frac{1}{5}\int_1^0 u^4\,du = -\frac{1}{25}u^5\Big|_1^0 = -\frac{1}{25}(0-1) = \frac{1}{25}.$$

8.3.50 $\displaystyle\int \csc^{10} x \cot^3 x\,dx = \int \csc^9 x \cot^2 x \csc x \cot x\,dx = \int \csc^9 x(\csc^2 x - 1)\csc x \cot x\,dx$. Let $u = \csc x$ so that $du = -\csc x \cot x\,dx$. Substituting gives

$$-\int u^9(u^2-1)\,du = -\int (u^{11} - u^9)\,du = -(u^{12}/12 - u^{10}/10) + C = \frac{-\csc^{12} x}{12} + \frac{\csc^{10} x}{10} + C.$$

8.3.51 $\displaystyle\int (\csc^2 x + \csc^4 x)\,dx = \int (1 + \csc^2 x)\csc^2 x\,dx$. Using the identity $\csc^2 x = 1 + \cot^2 x$, we can write this integral as $\displaystyle\int (2 + \cot^2 x)\csc^2 x\,dx$. Substituting $u = \cot x$ so that $du = -\csc^2 x\,dx$ gives

$$-\int (2 + u^2)\,du = -2u - \frac{u^3}{3} + C = -2\cot x - \frac{\cot^3 x}{3} + C.$$

8.3.52

$$\begin{aligned}\int_0^{\pi/8} (\tan 2x + \tan^3 2x)\,dx &= \int_0^{\pi/8} (1 + \tan^2 2x)\tan 2x\,dx = \int_0^{\pi/8} \sec^2 2x \tan 2x\,dx \\ &= \int_0^{\pi/8} \sec 2x \sec 2x \tan 2x\,dx.\end{aligned}$$

Let $u = \sec 2x$. Then $du = 2\sec 2x \tan 2x\,dx$. Substituting gives

$$\frac{1}{2}\int_1^{\sqrt{2}} u\,du = \frac{1}{4}u^2\Big|_1^{\sqrt{2}} = \frac{1}{4}(2-1) = \frac{1}{4}.$$

8.3.53 $\int_0^{\pi/4} \sec^4\theta\,d\theta = \int_0^{\pi/4}(\sec^2\theta)(1+\tan^2\theta)\,d\theta$. Let $u = \tan\theta$ so that $du = \sec^2\theta\,d\theta$. Note that when $\theta = 0$ we have $u = 0$ and when $\theta = \frac{\pi}{4}$ we have $u = 1$. So the original integral is equal to

$$\int_0^1 (1+u^2)\,du = \left(u + \frac{u^3}{3}\right)\Big|_0^1 = 1 + \frac{1}{3} = \frac{4}{3}.$$

8.3.54 $\int_0^{\sqrt{\pi/2}} x\sin^3(x^2)\,dx = \frac{1}{2}\int_0^{\pi/2}\sin^3 u\,du$, where $u = x^2$ and $du = 2x\,dx$. Now $\frac{1}{2}\int_0^{\pi/2}\sin^3 u\,du = \frac{1}{2}\int_0^{\pi/2}(\sin u)(1-\cos^2 u)\,du$. Now let $w = \cos u$ so that $dw = -\sin u\,du$. We then have

$$-\frac{1}{2}\int_1^0 (1-w^2)\,dw = \frac{1}{2}\left(w - \frac{w^3}{3}\right)\Big|_0^1 = \frac{1}{3}.$$

8.3.55 $\int_{\pi/6}^{\pi/3}\cot^3\theta\,d\theta = \int_{\pi/6}^{\pi/3}(\cot\theta)(\csc^2\theta - 1)\,d\theta = \int_{\pi/6}^{\pi/3}\cot\theta\csc^2\theta\,d\theta - \int_{\pi/6}^{\pi/3}\frac{\cos\theta}{\sin\theta}\,d\theta$. For the first integral, let $u = \cot\theta$ so that $du = -\csc^2\theta\,d\theta$. For the second integral, let $w = \sin\theta$ so that $dw = \cos\theta\,d\theta$. Substituting gives

$$\begin{aligned}-\int_{\sqrt{3}}^{1/\sqrt{3}} u\,du - \int_{1/2}^{\sqrt{3}/2}\frac{1}{w}\,dw &= -\frac{u^2}{2}\Big|_{\sqrt{3}}^{1/\sqrt{3}} - \ln w\,\Big|_{1/2}^{\sqrt{3}/2}\\ &= -\frac{1}{2}\left(\frac{1}{3} - 3\right) - \left(\ln\sqrt{3} - \ln 2 - \ln 1 + \ln 2\right) = \frac{4}{3} - \frac{\ln 3}{2}.\end{aligned}$$

8.3.56 $\int_0^{\pi/4}\tan^3\theta\sec^2\theta\,d\theta$. Let $u = \tan\theta$ so that $du = \sec^2\theta\,d\theta$. Substituting yields $\int_0^1 u^3\,du = \frac{u^4}{4}\Big|_0^1 = \frac{1}{4}$.

8.3.57 $\int_0^{\pi}(1-\cos 2x)^{3/2}\,dx = \int_0^{\pi}(2\sin^2 x)^{3/2}\,dx = 2\sqrt{2}\int_0^{\pi}\sin^3 x\,dx = 2\sqrt{2}\int_0^{\pi}(\sin x)(1-\cos^2 x)\,dx$. Let $u = \cos x$ so that $du = -\sin x\,dx$. Substituting yields

$$-2\sqrt{2}\int_1^{-1}(1-u^2)\,du = 2\sqrt{2}\int_{-1}^{1}(1-u^2)\,du = 4\sqrt{2}\int_0^1(1-u^2)\,du = 4\sqrt{2}\left(u - \frac{u^3}{3}\right)\Big|_0^1 = \frac{8\sqrt{2}}{3}.$$

8.3.58 $\int_{-\pi/4}^{\pi/4}\sqrt{1+\cos 4x}\,dx = 2\int_0^{\pi/4}\sqrt{1+\cos 4x}\,dx = 2\sqrt{2}\int_0^{\pi/4}\cos 2x\,dx$. Let $u = 2x$, so that $du = 2\,dx$. Substituting yields

$$\sqrt{2}\int_0^{\pi/2}\cos u\,du = \sqrt{2}\sin u\,\Big|_0^{\pi/2} = \sqrt{2}.$$

8.3.59 $\int_0^{\pi/2}\sqrt{1-\cos 2x}\,dx = \sqrt{2}\int_0^{\pi/2}\sin x\,dx = -\sqrt{2}\cos x\,\Big|_0^{\pi/2} = \sqrt{2}$.

8.3.60 $\int_0^{\pi/8}\sqrt{1-\cos 8x}\,dx = \sqrt{2}\int_0^{\pi/8}\sin 4x\,dx = -\sqrt{2}\cdot\frac{\cos 4x}{4}\Big|_0^{\pi/8} = \frac{\sqrt{2}}{4}$.

8.3.61 $\int_0^{\pi/4}(1+\cos 4x)^{3/2}\,dx = \int_0^{\pi/4}(2\cos^2 2x)^{3/2}\,dx = 2\sqrt{2}\int_0^{\pi/4}\cos^3 2x\,dx = 2\sqrt{2}\int_0^{\pi/4}(\cos 2x)(1-\sin^2 2x)\,dx$. Let $u = \sin 2x$ so that $du = 2\cos 2x\,dx$. Substituting gives

$$\sqrt{2}\int_0^1(1-u^2)\,du = \sqrt{2}\left(u - \frac{u^3}{3}\right)\Big|_0^1 = \frac{2\sqrt{2}}{3}.$$

8.3.62 If $y = \ln(\sec x)$ then $\dfrac{dy}{dx} = \tan x$. Thus,

$$L = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4} \sec x\,dx = \ln|\sec x + \tan x|\,\Big|_0^{\pi/4} = \ln(\sqrt{2}+1).$$

8.3.63

a. True. We have $\int_0^{\pi} \cos^{2m+1} x\,dx = \int_0^{\pi} (\cos^2 x)^m \cos x\,dx = \int_0^{\pi} (1-\sin^2 x)^m \cos x\,dx$. Let $u = \sin x$ so that $du = \cos x\,dx$. Substituting yields $\int_0^0 (1-u^2)^m\,du = 0$.

b. False. For example, suppose $m = 1$. Then $\int_0^{\pi} \sin x\,dx = -\cos x\,\Big|_0^{\pi} = -(-1-1) = 2 \neq 0$.

8.3.64 Using the disk method, we have

$$\frac{V}{\pi} = \int_0^{\pi} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi} (1-\cos 2x)\,dx = \frac{\pi}{2} - \frac{1}{2}\int_0^{\pi} \cos 2x\,dx = \frac{\pi}{2} - \left(\frac{\sin 2x}{4}\right)\Big|_0^{\pi} = \frac{\pi}{2}.$$

Thus, $V = \dfrac{\pi^2}{2}$.

8.3.65 $V = \int_0^{\pi/2} \pi \sin^4 x \cos^3 x\,dx = \pi \int_0^{\pi/2} \sin^4 x(1-\sin^2 x)\cos x\,dx = \int_0^{\pi/2} (\sin^4 x - \sin^6 x)\cos x\,dx$. Let $u = \sin x$. Then $du = \cos x\,dx$ and substitution gives

$$\pi \int_0^1 (u^4 - u^6)\,du = \pi\left(\frac{u^5}{5} - \frac{u^7}{7}\right)\Big|_0^1 = \pi\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{2\pi}{35}.$$

8.3.66 $s(t) = s(0) + \int_0^t \sec^4\left(\frac{\pi x}{12}\right)\,dx = 0 + \int_0^t \left(\tan^2\left(\frac{\pi x}{12}\right)+1\right)\sec^2\left(\frac{\pi x}{12}\right)\,dx$. Let $u = \tan\left(\frac{\pi x}{12}\right)$. Then $du = \frac{\pi}{12}\sec^2\left(\frac{\pi x}{12}\right)$. Substituting gives

$$\frac{12}{\pi}\int_0^{\tan(\pi t/12)} (u^2+1)\,du = \frac{12}{\pi}\left(\frac{\tan^3\left(\frac{\pi t}{12}\right)}{3} + \tan\left(\frac{\pi t}{12}\right)\right) = \frac{4}{\pi}\left(\tan^3\left(\frac{\pi t}{12}\right) + 3\tan\left(\frac{\pi t}{12}\right)\right).$$

8.3.67 $\int \sin 3x \cos 7x\,dx = \frac{1}{2}\left(\int \sin(-4x)\,dx + \int \sin 10x\,dx\right) = \frac{1}{2}\left(\frac{\cos(-4x)}{4} - \frac{\cos 10x}{10}\right) + C = \frac{\cos 4x}{8} - \frac{\cos 10x}{20} + C$.

8.3.68 $\int \sin 5x \sin 7x\,dx = \frac{1}{2}\left(\int \cos(-2x)\,dx - \int \cos 12x\,dx\right) = \frac{1}{2}\left(\int \cos 2x\,dx - \int \cos 12x\,dx\right) = \frac{1}{2}\left(\frac{\sin 2x}{2} - \frac{\sin 12x}{12}\right) + C = \frac{\sin 2x}{4} - \frac{\sin 12x}{24} + C$.

8.3.69 $\int \sin 3x \sin 2x\,dx = \frac{1}{2}\left(\int \cos x\,dx - \int \cos 5x\,dx\right) = \frac{\sin x}{2} - \frac{\sin 5x}{10} + C$.

8.3.70 $\int \cos x \cos 2x\,dx = \frac{1}{2}\left(\int \cos(-x)\,dx + \int \cos 3x\,dx\right) = \frac{\sin x}{2} + \frac{\sin 3x}{6} + C$.

8.3.71

a. $\displaystyle\int_0^{\pi} \sin mx \sin nx\, dx = \frac{1}{2}\left(\int_0^{\pi} \cos(m-n)x\, dx - \int_0^{\pi} \cos(m+n)x\, dx\right) =$
$\displaystyle\frac{1}{2}\left(\frac{1}{m-n}\int_0^{(m-n)\pi} \cos u\, du - \frac{1}{m+n}\int_0^{(m+n)\pi} \cos v\, dv\right)$ where $u = (m-n)x$ and $v = (m+n)x$. But this yields $\displaystyle\frac{1}{2}\left(\frac{1}{m-n}\sin u\,\Bigg|_0^{(m-n)\pi} - \frac{1}{m+n}\sin v\,\Bigg|_0^{(m+n)\pi}\right) = \frac{1}{2}(0-0) = 0.$

b. $\displaystyle\int_0^{\pi} \cos mx \cos nx\, dx = \frac{1}{2}\left(\int_0^{\pi} \cos(m-n)x\, dx + \int_0^{\pi} \cos(m+n)x\, dx\right) = 0$ by the previous part of this problem,

c. $\displaystyle\int_0^{\pi} \sin mx \cos nx\, dx = \frac{1}{2}\left(\int_0^{\pi} \sin(m-n)x\, dx + \int_0^{\pi} \sin(m+n)x\, dx\right) =$
$\displaystyle\frac{1}{2}\left(\frac{1}{m-n}\int_0^{(m-n)\pi} \sin u\, du + \frac{1}{m+n}\int_0^{(m+n)\pi} \sin v\, dv\right)$ where $u = (m-n)x$ and $v = (m+n)x$. This quantity is equal to $\displaystyle\frac{-1}{2}\left(\frac{1}{m-n}\cos u\,\Bigg|_0^{(m-n)\pi} + \frac{1}{m+n}\cos v\,\Bigg|_0^{(m+n)\pi}\right) =$
$\displaystyle\frac{-1}{2}\left(\frac{1}{m-n}(\cos(m-n)\pi - 1) + \frac{1}{m+n}(\cos(m+n)\pi - 1)\right) =$
$$\begin{cases} 0 & \text{if } m \text{ and } n \text{ are both even or both odd;} \\ \dfrac{1}{m-n} + \dfrac{1}{m+n} = \dfrac{2m}{m^2-n^2} & \text{otherwise.} \end{cases}$$

8.3.72 $\displaystyle\int \sin^n x\, dx = \int \sin^{n-1} x \sin x\, dx$. Let $u = \sin^{n-1} x$ and $dv = \sin x\, dx$. Then we have $du = (n-1)\sin^{n-2} x \cos x\, dx$ and $v = -\cos x$. We have $\displaystyle\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int (\sin^{n-2} x)(1-\sin^2 x)\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$. Adding the appropriate quantity to both sides of this last equation gives

$$n\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx,$$

so $\displaystyle\int \sin^n x\, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx.$

Thus,

$$\int \sin^6 x\, dx = \frac{-\sin^5 x \cos x}{6} + \frac{5}{6}\left(\frac{-\sin^3 x \cos x}{4} + \frac{3}{4}\left(\frac{x}{2} - \frac{\sin 2x}{4}\right)\right) + C.$$

8.3.73 For $n \neq 1$, $\displaystyle\int \tan^n x\, dx = \int (\tan^{n-2} x)(\sec^2 x - 1)\, dx = \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx$. Let $u = \tan x$ so that $du = \sec^2 x\, dx$. Then substituting in the first of these last two integrals yields

$$\int u^{n-2}\, du - \int \tan^{n-2} x\, dx = \frac{u^{n-1}}{n-1} - \int \tan^{n-2} x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx.$$

Thus $\displaystyle\int_0^{\pi/4} \tan^3 x\, dx = \frac{\tan^2 x}{2}\,\Bigg|_0^{\pi/4} - \int_0^{\pi/4} \tan x\, dx = \frac{1}{2} + \ln|\cos x|\,\Bigg|_0^{\pi/4} = \frac{1}{2} - \frac{\ln 2}{2}.$

8.3.74 $\displaystyle\int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx$. Let $u = \sec^{n-2} x$ and $dv = \sec^2 x\, dx$. Then $du = (n-2)\sec^{n-2} x \tan x\, dx$ and $v = \tan x$. Integration by Parts gives us $\displaystyle\sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\, dx =$

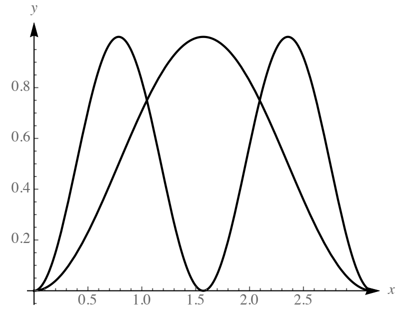
$$\sec^{n-2}x\tan x-(n-2)\int(\sec^{n-2}x)(\sec^2 x-1)\,dx=\sec^{n-2}x\tan x-(n-2)\int\sec^n x\,dx+(n-2)\int\sec^{n-2}x\,dx.$$

Combining like terms then gives $(n-1)\int\sec^n x\,dx=\sec^{n-2}x\tan x+(n-2)\int\sec^{n-2}x\,dx$, so as long as $n\neq 1$ we have

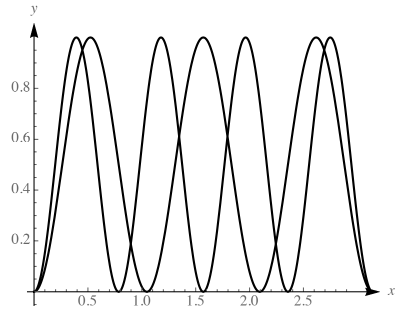
$$\int\sec^n x\,dx=\frac{\sec^{n-2}x\tan x}{n-1}+\frac{n-2}{n-1}\int\sec^{n-2}x\,dx.$$

8.3.75

a. $\int_0^\pi \sin^2 x\,dx=\frac{1}{2}\int_0^\pi(1-\cos 2x)\,dx=\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)\Big|_0^\pi=\frac{\pi}{2}.$
$\int_0^\pi \sin^2 2x\,dx=\frac{1}{2}\int_0^\pi(1-\cos 4x)\,dx=\frac{1}{2}\left(x-\frac{\sin 4x}{4}\right)\Big|_0^\pi=\frac{\pi}{2}.$



b. $\int_0^\pi \sin^2 3x\,dx=\frac{1}{2}\int_0^\pi(1-\cos 6x)\,dx=\frac{1}{2}\left(x-\frac{\sin 6x}{6}\right)\Big|_0^\pi=\frac{\pi}{2}.$
$\int_0^\pi \sin^2 4x\,dx=\frac{1}{2}\int_0^\pi(1-\cos 8x)\,dx=\frac{1}{2}\left(x-\frac{\sin 8x}{8}\right)\Big|_0^\pi=\frac{\pi}{2}.$



c. $\int_0^\pi \sin^2 nx\,dx=\frac{1}{2}\int_0^\pi(1-\cos 2nx)\,dx=\frac{1}{2}\left(x-\frac{\sin 2nx}{2n}\right)\Big|_0^\pi=\frac{\pi}{2}.$

d. Yes. $\int_0^\pi \cos^2 nx\,dx=\frac{1}{2}\int_0^\pi(1+\cos 2nx)\,dx=\frac{1}{2}\left(x+\frac{\sin 2nx}{2n}\right)\Big|_0^\pi=\frac{\pi}{2}.$

e. Claim: The corresponding integrals are all equal to $\frac{3\pi}{8}$. Proof:

$$\begin{aligned}\int_0^\pi \sin^4 nx\,dx&=\int_0^\pi\left(\frac{1-\cos 2nx}{2}\right)^2dx\\&=\int_0^\pi\frac{1-2\cos 2nx+\cos^2 2nx}{4}\,dx=\int_0^\pi\frac{1}{4}\,dx-\frac{1}{2}\int_0^\pi\cos 2nx\,dx+\frac{1}{4}\int_0^\pi\cos^2 2nx\,dx\\&=\frac{\pi}{4}-\frac{1}{2}\left(\frac{\sin 2nx}{2n}\right)\Big|_0^\pi+\frac{1}{4}\cdot\frac{\pi}{2}=\frac{\pi}{4}-0+\frac{\pi}{8}=\frac{3\pi}{8}.\end{aligned}$$

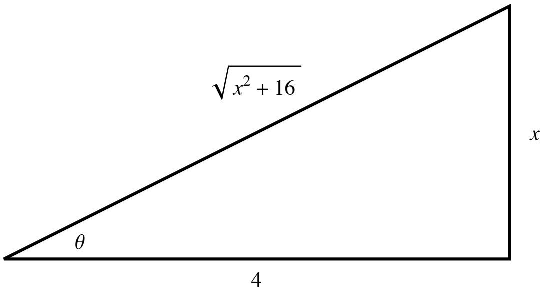
8.4 Trigonometric Substitutions

8.4.1 This would suggest $x=3\sec\theta$, because then $\sqrt{x^2-9}=3\sqrt{\sec^2\theta-1}=3\sqrt{\tan^2\theta}=3\tan\theta$, for $\theta\in[0,\pi/2)$.

8.4.2 This would suggest $x = 6\tan\theta$, because then $\sqrt{x^2+36} = 6\sqrt{\tan^2\theta+1} = 6\sqrt{\sec^2\theta} = 6\sec\theta$, for $|\theta| < \frac{\pi}{2}$.

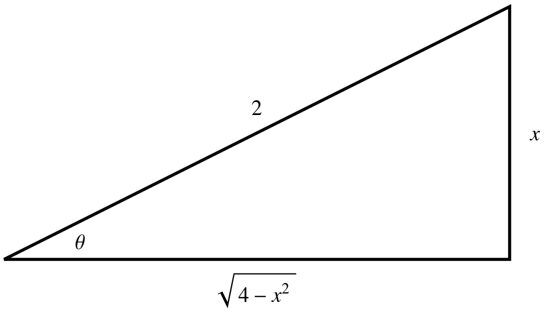
8.4.3 This would suggest $x = 10\sin\theta$, because then $\sqrt{100-x^2} = 10\sqrt{1-\sin^2\theta} = 10\sqrt{\cos^2\theta} = 10\cos\theta$, for $|\theta| \le \frac{\pi}{2}$.

8.4.4



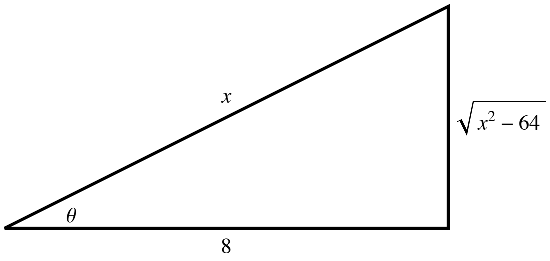
If $\tan\theta = \dfrac{x}{4}$, then $16\tan^2\theta = x^2$, so $16(\sec^2\theta - 1) = x^2$. Thus $\sec^2\theta = \dfrac{x^2+16}{16}$ and $\cos^2\theta = 1-\sin^2\theta = \dfrac{16}{x^2+16}$. Thus $\sin^2\theta = \dfrac{x^2}{16+x^2}$ and we have $\sin\theta = \dfrac{x}{\sqrt{16+x^2}}$, for $|\theta| < \dfrac{\pi}{2}$.

8.4.5



If $x = 2\sin\theta$ then $\dfrac{x^2}{4} = \sin^2\theta = \dfrac{1}{\csc^2\theta}$. Then $\cot^2\theta = \csc^2\theta - 1 = \dfrac{4}{x^2} - 1 = \dfrac{4-x^2}{x^2}$. So $\cot\theta = \dfrac{\sqrt{4-x^2}}{x}$ for $0 < |\theta| \le \dfrac{\pi}{2}$.

8.4.6



If $x = 8\sec\theta$, the $\tan^2\theta = \sec^2\theta - 1 = \dfrac{x^2}{64} - 1 = \dfrac{x^2-64}{64}$. Thus $\tan\theta = \dfrac{\sqrt{x^2-64}}{8}$.

8.4.7 Let $x = 5\sin\theta$, so that $dx = 5\cos\theta\,d\theta$. Note that $\sqrt{25-x^2} = 5\cos\theta$. Then $\displaystyle\int_0^{5/2} \frac{1}{\sqrt{25-x^2}}\,dx = \int_0^{\pi/6} \frac{5\cos\theta}{5\cos\theta}\,d\theta = \frac{\pi}{6}$.

Checking without using a trigonometric substitution:

$$\int_0^{5/2} \frac{dx}{\sqrt{25-x^2}} = \arcsin\left(\frac{x}{5}\right)\Big|_0^{5/2} = \left(\frac{\pi}{6}-0\right) = \frac{\pi}{6}.$$

8.4.8 Let $x = 3\sin\theta$ so that $dx = 3\cos\theta\,d\theta$. Note that $\sqrt{9-x^2} = 3\cos\theta$. Then

$$\int_0^{3/2} \frac{1}{(9-x^2)^{3/2}}\,dx = \int_0^{\pi/6} \frac{3\cos\theta}{27\cos^3\theta}\,d\theta = \frac{1}{9}\int_0^{\pi/6} \sec^2\theta\,d\theta = \frac{1}{9}\tan\theta\Big|_0^{\pi/6} = \frac{1}{9}\left(\frac{1}{\sqrt{3}}-0\right) = \frac{\sqrt{3}}{27}.$$

8.4.9 Let $x = 10\sin\theta$ so that $dx = 10\cos\theta\,d\theta$. Note that $\sqrt{100-x^2} = 10\cos\theta$. Then $\int_5^{5\sqrt{3}} \sqrt{100-x^2}\,dx =$ $100\int_{\pi/6}^{\pi/3} \cos^2\theta\,d\theta = 50\int_{\pi/6}^{\pi/3} (1+\cos 2\theta)\,d\theta = 50\left(\theta+\frac{\sin 2\theta}{2}\right)\Big|_{\pi/6}^{\pi/3} = 50\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}-\left(\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right)\right) = \frac{25\pi}{3}.$

8.4.10 Let $x = 2\sin\theta$, so that $dx = 2\cos\theta\,d\theta$. Note that $\sqrt{4-x^2} = 2\cos\theta$. Thus, $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx =$ $\int_0^{\pi/4} \frac{4\sin^2\theta\cdot 2\cos\theta}{2\cos\theta}\,d\theta = 4\int_0^{\pi/4} \sin^2\theta\,d\theta = 2\left(\frac{\pi}{4}-\int_0^{\pi/4}\cos 2\theta\,d\theta\right) = \frac{\pi}{2}-2\left(\frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/4} = \frac{\pi}{2}-1.$

8.4.11 Let $x = \sin\theta$ so that $dx = \cos\theta\,d\theta$. Note that $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$. Substituting gives

$$\int_{\pi/6}^{\pi/3} \sin^2\theta\,d\theta = \int_{\pi/6}^{\pi/3} \frac{1-\cos 2\theta}{2}\,d\theta = \frac{\theta}{2}\Big|_{\pi/6}^{\pi/3} - \frac{\sin 2\theta}{4}\Big|_{\pi/6}^{\pi/3} = \frac{\pi}{12}.$$

8.4.12 Let $x = \sin\theta$ so that $dx = \cos\theta\,d\theta$. Note that $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$. Substituting gives

$$\int_{\pi/6}^{\pi/2} \cot^2\theta\,d\theta = \int_{\pi/6}^{\pi/2} (\csc^2\theta-1)\,d\theta = (-\cot\theta-\theta)\Big|_{\pi/6}^{\pi/2} = 0-\pi/2-(-\sqrt{3}-\pi/6) = \sqrt{3}-\frac{\pi}{3}.$$

8.4.13 Let $x = 4\sin\theta$ so that $dx = 4\cos\theta\,d\theta$. Note that $\sqrt{16-x^2} = 4\cos\theta$. Thus,

$$\int \frac{1}{\sqrt{16-x^2}}\,dx = \int \frac{4\cos\theta}{4\cos\theta}\,d\theta = \theta + C = \sin^{-1}\left(\frac{x}{4}\right) + C.$$

8.4.14 Let $t = 6\sin\theta$ so that $dt = 6\cos\theta\,d\theta$ and $\sqrt{36-t^2} = 6\cos\theta$. Then

$$\int \sqrt{36-t^2}\,dt = \int 36\cos^2\theta\,d\theta = 18\int (1+\cos 2\theta)\,d\theta = 18\left(\theta+\frac{\sin 2\theta}{2}\right) + C = 18\left(\theta+\sin\theta\cos\theta\right)$$
$$= 18\left(\sin^{-1}\left(\frac{t}{6}\right)+\frac{t}{6}\cdot\frac{\sqrt{36-t^2}}{6}\right) + C = 18\sin^{-1}\left(\frac{t}{6}\right) + \frac{t\sqrt{36-t^2}}{2} + C.$$

8.4.15 Let $x = 3\tan\theta$ so that $dx = 3\sec^2\theta\,d\theta$. Note that $\sqrt{x^2+9} = \sqrt{9(\tan^2\theta+1)} = 3\sec\theta$. Substituting gives

$$\int \frac{1}{9}\cot\theta\csc\theta\,d\theta = -\frac{1}{9}\csc\theta + C = -\frac{1}{9}\csc(\tan^{-1}(x/3)) + C = -\frac{\sqrt{x^2+9}}{9x} + C.$$

8.4.16 Let $x = 5\tan\theta$ so that $dx = 5\sec^2\theta\,d\theta$. Note that $25+x^2 = 25\sec^2\theta$. Thus, $\int \frac{x^2}{(25+x^2)^2}\,dx =$ $\int \frac{25\tan^2\theta\cdot 5\sec^2\theta}{25^2\sec^4\theta}\,d\theta = \frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{5}\int \sin^2\theta\,d\theta = \frac{1}{10}\int (1-\cos 2\theta)\,d\theta = \frac{1}{10}\left(\theta-\frac{1}{2}\sin 2\theta\right) + C =$ $\frac{1}{10}(\tan\theta-\sin\theta\cos\theta) + C = \frac{1}{10}\left(\tan^{-1}\left(\frac{x}{5}\right)-\frac{5x}{25+x^2}\right) + C.$

8.4.17 Let $x = 2\tan\theta$ so that $dx = 2\sec^2\theta\,d\theta$. Note that $x^2+4 = 4\tan^2\theta + 4 = 4(\tan^2\theta+1) = 4\sec^2\theta$. Then

$$\begin{aligned}\int_0^1 \frac{x^2}{x^2+4}\,dx &= \int_0^{\pi/4} \frac{4\tan^2\theta}{4\sec^2\theta} 2\sec^2\theta\,d\theta = 2\int_0^{\pi/4} \tan^2\theta\,d\theta \\ &= 2\int_0^{\pi/4} (\sec^2\theta - 1)\,d\theta = 2(\tan\theta - \theta)\Big|_0^{\pi/4} = 2\left(1-\frac{\pi}{4}\right) = 2 - \frac{\pi}{2}.\end{aligned}$$

8.4.18 Let $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$. Note that $\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sec\theta$. Substituting gives

$$\int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C = \frac{x}{\sqrt{x^2+1}} + C.$$

8.4.19 Let $x = 9\sec\theta$ with $\theta \in (0, \pi/2)$. Then $dx = 9\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-81} = 9\tan\theta$. Then $\displaystyle\int \frac{1}{\sqrt{x^2-81}}\,dx = \int \frac{9\sec\theta\tan\theta}{9\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{9} + \frac{\sqrt{x^2-81}}{9}\right| + C.$ Note that because $x > 9$, the absolute value signs are unnecessary, and the final result can be written as $\ln(\sqrt{x^2-81}+x)+C$.

8.4.20 Let $x = 7\sec\theta$ where $\theta \in (0, \pi/2)$. Then $dx = 7\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-49} = 7\tan\theta$. Then $\displaystyle\int \frac{1}{\sqrt{x^2-49}}\,dx = \int \frac{7\sec\theta\tan\theta}{7\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C = \ln\left|\frac{x}{7} + \frac{\sqrt{x^2-49}}{7}\right| + C$. (Note also that the absolute value signs can be omitted because $x > 7$, and if we replace $-\ln(7)+C$ by a different arbitrary constant D, we can write the result as $\ln(x+\sqrt{x^2-49})+D$.

8.4.21 Let $x = 8\sin\theta$ so that $dx = 8\cos\theta\,d\theta$ and $\sqrt{64-x^2} = 8\cos\theta$. Then,

$$\begin{aligned}\int \sqrt{64-x^2}\,dx &= \int 64\cos^2\theta\,d\theta = 32\int(1+\cos 2\theta)\,d\theta = 32\theta + 16\sin 2\theta + C \\ &= 32\theta + 32\sin\theta\cos\theta + C = 32\sin^{-1}\left(\frac{x}{8}\right) + \frac{x\sqrt{64-x^2}}{2} + C.\end{aligned}$$

8.4.22 Let $t = 3\sin\theta$, so that $dt = 3\cos\theta\,d\theta$. Note that $\sqrt{9-t^2} = \sqrt{9(\cos^2\theta)} = 3\cos\theta$. Substituting gives

$$\int \frac{1}{9}\csc^2\theta\,d\theta = -\frac{1}{9}\cot\theta + C = -\frac{\sqrt{9-t^2}}{9t} + C.$$

8.4.23 Let $x = 5\sin\theta$ so that $dx = 5\cos\theta\,d\theta$. Note that $\sqrt{25-x^2} = \sqrt{25-25\sin^2\theta} = 5\cos\theta$. Substituting gives

$$\int \frac{5\cos\theta}{125\cos^3\theta}\,d\theta = \frac{1}{25}\int \sec^2\theta\,d\theta = \frac{1}{25}\tan\theta + C = \frac{x}{25\sqrt{25-x^2}} + C.$$

8.4.24 Let $x = 3\sin\theta$ so that $dx = 3\cos\theta\,d\theta$. Note that $\sqrt{9-x^2} = 3\cos\theta$. Thus, $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta\cdot 3\cos\theta}{9\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta = \int \csc^2 - 1\,d\theta = -\cot\theta - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}(x/3) + C.$

8.4.25 Let $x = 3\sin\theta$ so that $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$. Then

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x}\,dx &= \int \frac{3\cos\theta\cdot 3\cos\theta}{3\sin\theta}\,d\theta = 3\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta \\ &= 3\left(\int \csc\theta\,d\theta - \int \sin\theta\,d\theta\right)\,d\theta = 3\left(-\ln|\csc\theta + \cot\theta| + \cos\theta\right) \\ &= -3\ln\left|\frac{3}{x} + \frac{\sqrt{9-x^2}}{x}\right| + \sqrt{9-x^2} + C.\end{aligned}$$

8.4.26 Let $x = \sec\theta$ so that $dx = \sec\theta\tan\theta\,d\theta$. Note that $\sqrt{x^2-1} = \sqrt{\tan^2\theta} = \tan\theta$. Substituting gives

$$\int_{\pi/4}^{\pi/3} \tan^2\theta\,d\theta = \int_{\pi/4}^{\pi/3} (\sec^2\theta - 1)\,d\theta = (\tan\theta - \theta)\Big|_{\pi/4}^{\pi/3} = \sqrt{3} - \pi/3 - (1-\pi/4) = \sqrt{3} - 1 - \frac{\pi}{12}.$$

8.4.27 Let $x = \frac{1}{3}\tan\theta$ so that $dx = \frac{1}{3}\sec^2\theta\,d\theta$. Note that $\sqrt{9x^2+1} = \sec\theta$. Thus $\int_0^{1/3} \frac{1}{(9x^2+1)^{3/2}}\,dx = \int_0^{\pi/4} \frac{\frac{1}{3}\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{3}\int_0^{\pi/4} \cos\theta\,d\theta = \frac{1}{3}\sin\theta\Big|_0^{\pi/4} = \frac{\sqrt{2}}{6}$.

8.4.28 Let $z = 6\tan\theta$ so that $dz = 6\sec^2\theta\,d\theta$. Note that $z^2+36 = 36\sec^2\theta$. Thus

$$\begin{aligned}\int_0^6 \frac{z^2}{(z^2+36)^2}\,dz &= \int_0^{\pi/4} \frac{36\tan^2\theta\cdot 6\sec^2\theta}{36^2\sec^4\theta}\,d\theta = \frac{1}{6}\int_0^{\pi/4} \frac{\tan^2\theta}{\sec^2\theta}\,d\theta \\ &= \frac{1}{6}\int_0^{\pi/4} \sin^2\theta\,d\theta = \frac{1}{12}\int_0^{\pi/4} (1-\cos 2\theta)\,d\theta \\ &= \frac{1}{12}\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/4} = \frac{1}{12}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi-2}{48}\end{aligned}$$

8.4.29 Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$ and $4+x^2 = 4(\sec^2\theta)$. Then

$$\begin{aligned}\int \frac{dx}{(4+x^2)^2} &= \int \frac{2\sec^2\theta}{2^4\sec^4\theta}\,d\theta = \frac{1}{8}\int \cos^2\theta\,d\theta = \frac{1}{16}\int (1+\cos 2\theta)\,d\theta \\ &= \frac{1}{16}\left(\theta + \frac{\sin 2\theta}{2}\right) + C = \frac{1}{16}(\theta + \sin\theta\cos\theta) + C = \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right) + C.\end{aligned}$$

8.4.30 Let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$ and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$. Substituting gives

$$\begin{aligned}\int x^3\sqrt{1-x^2}\,dx \int \sin^3\theta\sqrt{1-\sin^2\theta}\cos\theta\,d\theta &= \int \sin^3\theta\cos^2\theta\,d\theta \\ &= \int (1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta = -\int (u^2-u^4)\,du \\ &= -\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C = -\left(\frac{1}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta\right) + C \\ &= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C = -\frac{1}{15}(1-x^2)^{3/2}(3x^2+2) + C.\end{aligned}$$

Along the way we made the substitution $u = \cos x$.

8.4.31 Let $x = 4\sin\theta$ so that $dx = 4\cos\theta\,d\theta$. Note that $\sqrt{16-x^2} = 4\cos\theta$. Then $\int \frac{x^2}{\sqrt{16-x^2}}\,dx = \int \frac{16\sin^2\theta\cdot 4\cos\theta}{4\cos\theta}\,d\theta = 16\int \sin^2\theta\,d\theta = 8\int (1-\cos 2\theta)\,d\theta = 8\left(\theta - \frac{\sin 2\theta}{2}\right) + C = 8\theta - 8\sin\theta\cos\theta + C = 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C$.

8.4.32 Let $x = 6\sec\theta$ with $\theta \in (0,\pi/2)$. Then $dx = 6\sec\theta\tan\theta\,d\theta$, and $\sqrt{x^2-36} = 6\tan\theta$. Then

$$\int \frac{1}{(x^2-36)^{3/2}}\,dx = \int \frac{6\sec\theta\tan\theta}{6^3\tan^3\theta}\,d\theta = \frac{1}{36}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{36}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta.$$

Let $u = \sin\theta$ so that $du = \cos\theta\,d\theta$. Then we have

$$\frac{1}{36}\int u^{-2}\,du = -\frac{1}{36u} + C = -\frac{1}{36\sin\theta} + C = -\frac{x}{36\sqrt{x^2-36}} + C.$$

8.4.33 Let $x = 3\sec\theta$ where $\theta \in (0, \pi/2)$. Then $dx = 3\sec\theta\tan\theta$ and $\sqrt{x^2-9} = 3\tan\theta$. Thus we have $\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{3\sec\theta\tan\theta\cdot 3\tan\theta}{3\sec\theta}\,d\theta = 3\int \tan^2\theta\,d\theta = 3\int \sec^2\theta - 1\,d\theta = 3\,(\tan\theta - \theta) + C = \sqrt{x^2-9} - 3\tan^{-1}\left(\frac{\sqrt{x^2-9}}{3}\right) + C = \sqrt{x^2-9} - 3\sec^{-1}(x/3) + C$.

8.4.34 Let $x = \sec\theta$ where $\theta \in (0, \pi/2)$. Then $dx = \sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-1} = \tan\theta$. Thus,

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \int \frac{\sec\theta\tan\theta}{\sec^3\theta\tan\theta}\,d\theta = \int \cos^2\theta\,d\theta = \frac{1}{2}\int 1 + \cos 2\theta\,d\theta = \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$
$$= \frac{1}{2}\,(\theta + \sin\theta\cos\theta) + C = \frac{1}{2}\left(\tan^{-1}\sqrt{x^2-1} + \frac{\sqrt{x^2-1}}{x^2}\right) + C.$$

8.4.35 Let $x = \sec\theta$ where $\theta \in (0, \pi/2)$. Then $dx = \sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-1} = \tan\theta$. Then

$$\int \frac{1}{x(x^2-1)^{3/2}}\,dx = \int \frac{\sec\theta\tan\theta}{\sec\theta\tan^3\theta}\,d\theta = \int \cot^2\theta\,d\theta$$
$$= \int \csc^2\theta - 1\,d\theta = -\cot\theta - \theta + C = -\frac{1}{\sqrt{x^2-1}} - \sec^{-1} x + C.$$

8.4.36 Let $x = 8\sec\theta$ so that $dx = 8\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-64} = 8\tan\theta$. Then $\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2-64}} = \int_{\pi/4}^{\pi/3} \frac{8\sec\theta\tan\theta}{8\tan\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|\Big|_{\pi/4}^{\pi/3} = \ln(2+\sqrt{3}) - \ln(\sqrt{2}+1) = \ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right)$.

8.4.37 Let $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$ and $\sqrt{1+x^2} = \sec\theta$. Substituting gives

$$\int_{\pi/6}^{\pi/4} \cot\theta\csc\theta\,d\theta = (-\csc\theta)\Big|_{\pi/6}^{\pi/4} = -(\sqrt{2}-2) = 2-\sqrt{2}.$$

8.4.38 Let $x = 2\sin\theta$, so that $dx = 2\cos\theta\,d\theta$. Note that when $x = 1$ we have $\theta = \frac{\pi}{6}$ and when $x = \sqrt{2}$ we have $\theta = \frac{\pi}{4}$. Also $\sqrt{4-x^2} = 2\sqrt{\cos^2\theta} = 2\cos\theta$. Substituting gives

$$\frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2\theta\,d\theta = -\frac{1}{4}\,(\cot\theta)\Big|_{\pi/6}^{\pi/4} = -\frac{1}{4}\left(1-\sqrt{3}\right) = \frac{\sqrt{3}-1}{4}.$$

8.4.39 Let $x = 10\sin\theta$ so that $dx = 10\cos\theta\,d\theta$. Note that $\sqrt{100-x^2} = 10\cos\theta$. Thus,

$$\int \frac{x^2}{(100-x^2)^{3/2}}\,dx = \int \frac{100\sin^2\theta\cdot 10\cos\theta}{1000\cos^3\theta}\,d\theta = \int \tan^2\theta\,d\theta$$
$$= \int (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta + C = \frac{x}{\sqrt{100-x^2}} - \sin^{-1}(x/10) + C.$$

8.4.40 Let $y = 5\sec\theta$ so that $dy = 5\sec\theta\tan\theta\,d\theta$. Then $\sqrt{y^2-25} = 5\tan\theta$. Then, $\int_{10/\sqrt{3}}^{10} \frac{1}{\sqrt{y^2-25}}\,dy = \int_{\pi/6}^{\pi/3} \frac{5\sec\theta\tan\theta}{5\tan\theta}\,d\theta = \int_{\pi/6}^{\pi/3} \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|\Big|_{\pi/6}^{\pi/3} = \ln(2+\sqrt{3}) - \ln(\sqrt{3}) = \ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)$.

8.4.41 Let $x = \frac{\tan\theta}{2}$ so that $dx = \frac{\sec^2\theta}{2}\,d\theta$ and $\sqrt{1+4x^2} = \sec\theta$. Then

$$\int \frac{1}{(1+4x^2)^{3/2}}\,dx = \frac{1}{2}\int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{2}\int \cos\theta\,d\theta = \frac{\sin\theta}{2} + C = \frac{x}{\sqrt{1+4x^2}} + C.$$

8.4.42 Let $x = \dfrac{1}{3}\sec\theta$, where $\theta \in (0, \pi/2)$. Then $dx = \dfrac{1}{3}\sec\theta\tan\theta\; d\theta$. Note that $\sqrt{9x^2-1} = \tan\theta$. Then

$$\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx = \int \frac{\frac{1}{3}\sec\theta\tan\theta}{\frac{1}{9}\sec^2\theta\tan\theta}\,d\theta = 3\int \cos\theta\,d\theta = 3\sin\theta + C = \frac{\sqrt{9x^2-1}}{x} + C.$$

8.4.43 Let $x = 4\tan\theta$ so that $dx = 4\sec^2\theta\,d\theta$. Note that $\sqrt{x^2+16} = 4\sec\theta$. Thus, $\displaystyle\int_0^{4/\sqrt{3}} \frac{1}{\sqrt{x^2+16}}\,dx =$ $\displaystyle\int_0^{\pi/6} \frac{4\sec^2\theta}{4\sec\theta}\,d\theta = \int_0^{\pi/6} \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta|\,\Big|_0^{\pi/6} = \ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) - \ln 1 = \ln\frac{3}{\sqrt{3}} = \ln 3 - \frac{1}{2}\ln 3 =$ $\dfrac{1}{2}\ln 3$.

8.4.44 Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$ and $\sqrt{16+4x^2} = 4\sec\theta$. Then

$$\int \frac{1}{\sqrt{16+4x^2}}\,dx = \int \frac{2\sec^2\theta}{4\sec\theta}\,d\theta = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta| + C = \frac{1}{2}\ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C.$$

8.4.45 Let $x = 9\sin\theta$ so that $dx = 9\cos\theta\,d\theta$. Note that $81 - x^2 = 81\cos^2\theta$. Thus,

$$\begin{aligned}\int \frac{x^3}{(81-x^2)^2}\,dx &= \int \frac{9^3\sin^3\theta\cdot 9\cos\theta}{9^4\cos^4\theta}\,d\theta = \int \tan^3\theta\,d\theta \\ &= \int (\tan\theta)(\sec^2\theta - 1)\,d\theta = \int \sec^2\theta\tan\theta\,d\theta - \int \tan\theta\,d\theta \\ &= \frac{\sec^2\theta}{2} + \ln|\cos\theta| + C = \frac{81}{2(81-x^2)} + \ln\left|\frac{\sqrt{81-x^2}}{9}\right| + C.\end{aligned}$$

This can be written as $\dfrac{81}{2(81-x^2)} + \ln\sqrt{81-x^2} + C$.

8.4.46 Let $x = (1/\sqrt{2})\sin\theta$ so that $dx = (1/\sqrt{2})\cos\theta\,d\theta$ and $\sqrt{1-2x^2} = \cos\theta$. Then

$$\int \frac{1}{\sqrt{1-2x^2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{\cos\theta}{\cos\theta}\,d\theta = \frac{1}{\sqrt{2}}\cdot\theta + C = \frac{1}{\sqrt{2}}\sin^{-1}(\sqrt{2}x) + C.$$

8.4.47 Let $x = 2\sec\theta$ so that $dx = 2\sec\theta\tan\theta\,d\theta$ and $x^2 - 4 = 4\tan^2\theta$. Thus,
$\displaystyle\int_{4/\sqrt{3}}^{4} \frac{1}{x^2(x^2-4)}\,dx = \int_{\pi/6}^{\pi/3} \frac{2\sec\theta\tan\theta}{4\sec^2\theta\cdot 4\tan^2\theta}\,d\theta = \frac{1}{8}\int_{\pi/6}^{\pi/3} \frac{\cos^2\theta}{\sin\theta}\,d\theta = \frac{1}{8}\int_{\pi/6}^{\pi/3} \frac{1-\sin^2\theta}{\sin\theta}\,d\theta =$ $\displaystyle\frac{1}{8}\int_{\pi/6}^{\pi/3} \csc\theta - \sin\theta\,d\theta = \frac{1}{8}\left(-\ln|\csc\theta+\cot\theta| + \cos\theta\right)\Big|_{\pi/6}^{\pi/3} = \frac{1}{8}\left(-\ln(\sqrt{3}(2-\sqrt{3})) + \frac{1-\sqrt{3}}{2}\right) =$ $\dfrac{1}{16}\left(1 - \sqrt{3} - \ln(21 - 12\sqrt{3})\right)$.

8.4.48 Let $x = \dfrac{3}{2}\sin\theta$, so that $dx = \dfrac{3}{2}\cos\theta\,d\theta$. Note that $\sqrt{9-4x^2} = 3\cos\theta$. Thus $\displaystyle\int \sqrt{9-4x^2}\,dx =$ $\displaystyle\frac{3}{2}\int \cos\theta\cdot 3\cos\theta\,d\theta = \frac{9}{2}\int \cos^2\theta\,d\theta = \frac{9}{4}\int 1 + \cos 2\theta\,d\theta = \frac{9}{4}\left(\theta + \frac{\sin 2\theta}{2}\right) + C = \frac{9\theta}{4} + \frac{9\sin\theta\cos\theta}{4} + C =$ $\dfrac{9\sin^{-1}(2x/3)}{4} + \dfrac{x\sqrt{9-4x^2}}{2} + C$.

8.4.49 Let $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$. Note that $\sqrt{x^2+1} = \sqrt{\tan^2\theta+1} = \sqrt{\sec^2\theta} = \sec\theta$. Substituting gives $\displaystyle\int_0^{\pi/6} \sec^3\theta\,d\theta$. Recall from section 8.2 number 48 that $\displaystyle\int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta +$ $\tan\theta|$. Thus the original integral is equal to $\displaystyle\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/6} = \frac{1\cdot 2\cdot 1}{2\cdot\sqrt{3}\cdot\sqrt{3}} +$ $\displaystyle\frac{1}{2}\ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) = \frac{1}{3} + \frac{\ln 3}{4}$.

8.4.50 Let $x = 2\sin\theta$ so that $dx = 2\cos\theta\,d\theta$ and $\sqrt{4-x^2} = 2\cos\theta$. Then $\int (36-9x^2)^{-3/2}\,dx = \frac{1}{27}\int \frac{1}{(4-x^2)^{3/2}}\,dx = \frac{1}{27}\int \frac{2\cos\theta}{8\cos^3\theta}\,d\theta = \frac{1}{108}\int \sec^2\theta\,d\theta = \frac{1}{108}\tan\theta + C = \frac{x}{108\sqrt{4-x^2}} + C.$

8.4.51 Let $x = 2\tan\theta$ so that $dx = 2\sec^2\theta\,d\theta$. Note that $\sqrt{4+x^2} = 2\sec\theta$. Then $\int \frac{x^2}{\sqrt{4+x^2}}\,dx = \int \frac{4\tan^2\theta\cdot 2\sec^2\theta}{2\sec\theta}\,d\theta = 4\int \tan^2\theta\sec\theta\,d\theta = 4\int(\sec^2\theta - 1)\sec\theta\,d\theta = 4\left(\int \sec^3\theta\;d\theta - \int \sec\theta\,d\theta\right) = 4\left(\frac{1}{2}\left(\sec\theta\tan\theta + \int\sec\theta\,d\theta\right) - \int\sec\theta\,d\theta\right) = 2\sec\theta\tan\theta - 2\int\sec\theta\,d\theta = 2\sec\theta\tan\theta - 2\ln|\sec\theta+\tan\theta| + C = \frac{x\sqrt{4+x^2}}{2} - 2\ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$. This can be written as $\frac{x\sqrt{4+x^2}}{2} - 2\ln(x+\sqrt{4+x^2}) + C$.

8.4.52 Let $x = \frac{1}{2}\sec\theta$ where $\theta\in[0,\pi/2)$. Then $dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$ and $\sqrt{4x^2-1} = \tan\theta$. Thus, $\int \frac{\sqrt{4x^2-1}}{x^2}\,dx = \int \frac{\tan\theta\cdot\frac{1}{2}\sec\theta\tan\theta}{\frac{1}{4}\cdot\sec^2\theta}\,d\theta = 2\int\frac{\tan^2\theta}{\sec\theta}\,d\theta = 2\int\frac{\sec^2\theta-1}{\sec\theta}\,d\theta = 2\int\sec\theta - \cos\theta\,d\theta = 2\left(\ln|\sec\theta+\tan\theta| - \sin\theta\right) + C = 2\ln(2x+\sqrt{4x^2-1}) - \frac{\sqrt{4x^2-1}}{x} + C.$

8.4.53 Let $x = \frac{5}{3}\sec\theta$ where $\theta\in[0,\pi/2)$. Then $dx = \frac{5}{3}\sec\theta\tan\theta\,d\theta$ and $\sqrt{9x^2-25} = 5\tan\theta$. Thus, $\int\frac{\sqrt{9x^2-25}}{x^3}\,dx = \int\frac{5\tan\theta\cdot\frac{5}{3}\cdot\sec\theta\tan\theta}{\frac{125}{27}\sec^3\theta}\,d\theta = \frac{9}{5}\int\frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \frac{9}{5}\int\frac{\sec^2\theta-1}{\sec^2\theta}\,d\theta = \frac{9}{5}\int 1-\cos^2\theta\,d\theta = \frac{9}{5}\int\sin^2\theta\,d\theta = \frac{9}{10}\int(1-\cos2\theta)\,d\theta = \frac{9\theta}{10} - \frac{9\sin2\theta}{20} + C = \frac{9\theta}{10} - \frac{9\sin\theta\cos\theta}{10} = \frac{9\cos^{-1}(5/3x)}{10} - \frac{\sqrt{9x^2-25}}{2x^2} + C.$

8.4.54 Let $y = \tan\theta$ so that $dy = \sec^2\theta\,d\theta$. Note that $1+y^2 = \sec^2\theta$. Thus,

$$\begin{aligned}\int\frac{y^4}{1+y^2}\,dy &= \int\frac{\tan^4\theta\sec^2\theta}{\sec^2\theta}\,d\theta = \int\tan^4\theta\,d\theta\\ &= \int(\tan^2\theta)(\sec^2\theta-1)\,d\theta = \int\tan^2\theta\sec^2\theta - \tan^2\theta\,d\theta\\ &= \int\tan^2\theta\sec^2\theta + 1 - \sec^2\theta\,d\theta = \int\tan^2\theta\sec^2\theta\,d\theta + \theta - \tan\theta.\end{aligned}$$

Now recall that $y = \tan\theta$, so we have $\int y^2\,dy + \theta - \tan\theta = \frac{y^3}{3} + \theta - \tan\theta + C = \frac{y^3}{3} + \tan^{-1}(y) - y + C.$

8.4.55 Let $x = 10\sec\theta$ where $\theta\in(0,\pi/2)$. Then $dx = 10\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-100} = 10\tan\theta$. Thus,

$$\begin{aligned}\int\frac{1}{x^3\sqrt{x^2-100}}\,dx &= \int\frac{10\sec\theta\tan\theta}{10^3\sec^3\theta\cdot10\tan\theta}\,d\theta = \frac{1}{1000}\int\cos^2\theta\,d\theta\\ &= \frac{1}{2000}\int(1+\cos2\theta)\,d\theta = \frac{\theta}{2000} + \frac{\sin2\theta}{4000} + C = \frac{\theta}{2000} + \frac{\sin\theta\cos\theta}{2000} + C\\ &= \frac{\sec^{-1}(x/10)}{2000} + \frac{\sqrt{x^2-100}}{200x^2} + C.\end{aligned}$$

8.4.56 Let $x = 4\sec\theta$ where $\theta\in(0,\pi/2)$. Then $dx = 4\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-16} = 4\tan\theta$. Then

$$\begin{aligned}\int\frac{x^3}{(x^2-16)^{3/2}}\,dx &= \int\frac{4^3\sec^3\theta\cdot4\sec\theta\tan\theta}{64\tan^3\theta}\,d\theta = 4\int\frac{\sec^4\theta}{\tan^2\theta}\,d\theta\\ &= 4\int\frac{(\sec^2\theta)(1+\tan^2\theta)}{\tan^2\theta}\,d\theta.\end{aligned}$$

Let $u = \tan\theta$ so that $du = \sec^2\theta\, d\theta$. Then we have

$$4\int \frac{1+u^2}{u^2}\, du = 4\int u^{-2} + 1\, du = -\frac{4}{u} + 4u + C$$
$$= -4\cot\theta + 4\tan\theta + C = -\frac{16}{\sqrt{x^2-16}} + \sqrt{x^2-16} + C.$$

8.4.57

a. False. In fact, we would have $\csc\theta = \dfrac{\sqrt{x^2+16}}{x}$.

b. True. Almost every number in the interval $[1, 2]$ is not in the domain of $\sqrt{1-x^2}$, so this integral isn't defined.

c. False. It does represent a finite real number, because $\sqrt{x^2-1}$ is continuous on the interval $[1, 2]$.

d. False. It can be so evaluated. The integral is equivalent to $\displaystyle\int \frac{1}{(x+2)^2+5}\, dx$, and this can be evaluated by the substitution $x + 2 = \sqrt{5}\tan\theta$.

8.4.58 Let A be the area of the ellipse. Using symmetry, we have $\dfrac{A}{4} = \displaystyle\int_0^b \frac{b}{a}\sqrt{a^2-x^2}\, dx$. Let $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$. Substituting yields

$$\frac{b}{a}\int_0^{\pi/2} a\cos\theta\cdot a\cos\theta\, d\theta = ab\int_0^{\pi/2}\cos^2\theta\, d\theta = \frac{ab}{2}\int_0^{\pi/2}(1+\cos 2\theta)\, d\theta = \frac{ab}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_0^{\pi/2} = \frac{\pi ab}{4}.$$

So the total area of the ellipse is $A = \pi ab$.

8.4.59

a. Recall that the area of a circular sector subtended by an angle θ is given by $\dfrac{\theta r^2}{2}$. So the area of the cap is this area minus the area of the isosceles triangle with two sides of length r and angle between them θ. So

$$A_{\text{cap}} = A_{\text{sector}} - A_{\text{triangle}} = \frac{\theta r^2}{2} - \frac{r^2\sin\theta}{2} = \frac{r^2}{2}\left(\theta - \sin\theta\right).$$

b. For a cap we have $0 \le \theta \le \pi$ so $0 \le \theta/2 \le \pi/2$. By symmetry, $\dfrac{A_{\text{cap}}}{2} = \displaystyle\int_{r\cos\theta/2}^{r}\sqrt{r^2-x^2}\, dx$. Let $x = r\cos\alpha/2$ so that $dx = -\frac{r}{2}\sin\alpha/2\, d\alpha$. Then we have

$$\frac{A_{\text{cap}}}{2} = \int_\theta^0 r\sin(\alpha/2)\cdot -\frac{r}{2}\sin(\alpha/2)\, d\alpha$$
$$= \frac{r^2}{2}\int_0^\theta \sin^2(\alpha/2)\, d\alpha = \frac{r^2}{4}\int_0^\theta (1-\cos\alpha)\, d\alpha = \frac{r^2}{4}\left(\alpha - \sin\alpha\right)\Bigg|_0^\theta = \frac{r^2}{4}\left(\theta - \sin\theta\right).$$

Thus $A_{\text{cap}} = \dfrac{r^2}{2}(\theta - \sin\theta)$.

8.4.60 Note that the given integral can be written $\displaystyle\int \frac{1}{(x-3)^2+25}\, dx = \int \frac{1}{u^2+25}\, du$ with $u = x-3$. Now let $u = 5\tan\theta$ so that $du = 5\sec^2\theta\, d\theta$ and $u^2 + 25 = 25\sec^2\theta$. Thus we have

$$\int \frac{5\sec^2\theta}{25\sec^2\theta}\, d\theta = \frac{\theta}{5} + C = \frac{\tan^{-1}((x-3)/5)}{5} + C.$$

8.4.61 $\int \frac{1}{\sqrt{3-2x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x+1)^2}}\,dx = \int \frac{1}{\sqrt{4-u^2}}\,du$ where $u = x+1$. Then let $u = 2\sin\theta$ so that $du = 2\cos\theta\,d\theta$. We have $\int \frac{2\cos\theta}{2\cos\theta}\,d\theta = \theta + C = \sin^{-1}\left(\frac{x+1}{2}\right) + C$.

8.4.62 Note that the given integral can be written $\frac{1}{2}\int \frac{1}{(u-3)^2+9}\,du = \frac{1}{2}\int \frac{1}{w^2+9}\,dw$ where $w = u-3$. Now let $w = 3\tan\theta$ so that $dw = 3\sec^2\theta\,d\theta$ and $w^2+9 = 9\sec^2\theta$. Thus we have

$$\frac{1}{2}\int \frac{3\sec^2\theta}{9\sec^2\theta}\,d\theta = \frac{\theta}{6} + C = \frac{\tan^{-1}((u-3)/3)}{6} + C.$$

8.4.63 Note that the given integral can be written $\int \frac{1}{(x+3)^2+9}\,dx = \int \frac{1}{u^2+9}\,du$ where $u = x+3$. Now let $u = 3\tan\theta$ so that $du = 3\sec^2\theta\,d\theta$ and $u^2+9 = 9\sec^2\theta$. Thus we have

$$\int \frac{3\sec^2\theta}{9\sec^2\theta}\,d\theta = \frac{\theta}{3} + C = \frac{\tan^{-1}((x+3)/3)}{3} + C.$$

8.4.64 Note that the given integral can be written as $\int \frac{(x-1)^2}{\sqrt{(x-1)^2+9}}\,dx = \int \frac{u^2}{\sqrt{u^2+9}}\,du$ where $u = x-1$. Now let $u = 3\tan\theta$ so that $du = 3\sec^2\theta\,d\theta$ and $u^2+9 = 9\sec^2\theta$. Thus we have

$$\begin{aligned}
\int \frac{9\tan^2\theta\cdot 3\sec^2\theta}{3\sec\theta}\,d\theta &= 9\int \tan^2\theta\sec\theta\,d\theta = 9\int(\sec^2\theta - 1)(\sec\theta)\,d\theta \\
&= 9\int(\sec^3\theta - \sec\theta)\,d\theta = 9\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\,d\theta - \int\sec\theta\,d\theta\right) \\
&= \frac{9}{2}\left(\sec\theta\tan\theta - \int\sec\theta\,d\theta\right) \\
&= \frac{9}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\right) + C \\
&= \frac{9}{2}\left(\frac{(x-1)\sqrt{(x-1)^2+9}}{9} - \ln\left|\frac{\sqrt{(x-1)^2+9}}{3} + \frac{x-1}{3}\right|\right) + C.
\end{aligned}$$

This can be written as $\frac{x-1}{2}\sqrt{x^2-2x+10} - \frac{9}{2}\ln(x-1+\sqrt{x^2-2x+10}) + C$. Note that in the middle of this derivation we used the reduction formula for $\int \sec^3\theta\,d\theta$ given in the previous section.

8.4.65 $\int_{1/2}^{(\sqrt{2}+3)/2\sqrt{2}} \frac{1}{8x^2-8x+11}\,dx = \int_{1/2}^{(\sqrt{2}+3)/2\sqrt{2}} \frac{1}{8(x-1/2)^2+9}\,dx$. Let $u = x - 1/2$, so that our integral becomes $\int_0^{3/2\sqrt{2}} \frac{1}{8u^2+9}\,du$. Now let $u = \frac{3}{\sqrt{8}}\tan\theta$ so that $du = \frac{3}{\sqrt{8}}\sec^2\theta\,d\theta$. Substituting gives

$$\int_0^{\pi/4} \frac{\frac{3}{\sqrt{8}}\sec^2\theta}{9\sec^2\theta}\,d\theta = \frac{1}{6\sqrt{2}}\int d\theta = \frac{1}{6\sqrt{2}}\theta\,\bigg|_0^{\pi/4} = \frac{\pi\sqrt{2}}{48}.$$

8.4.66 $\int_1^4 \frac{1}{t^2-2t+10}\,dt = \int_1^4 \frac{1}{(t-1)^2+9}\,dt$. Let $3\tan\theta = t-1$, so that $dt = 3\sec^2\theta\,d\theta$. Note that $(t-1)^2+9 = 9\sec^2\theta$. Then we have $\int_0^{\pi/4} \frac{3\sec^2\theta}{9\sec^2\theta}\,d\theta = \frac{1}{3}\theta\,\bigg|_0^{\pi/4} = \frac{\pi}{12}$.

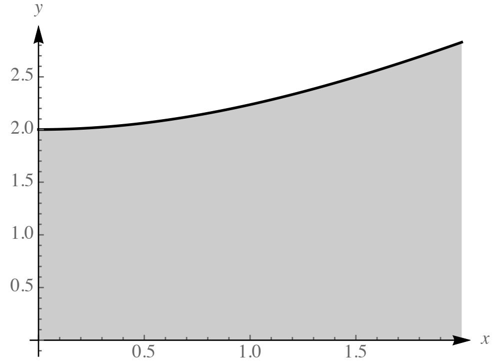
8.4.67 Note that the given integral can be written as $\int \frac{(x-4)^2}{(25-(x-4)^2)^{3/2}}\,dx$. Let $u = x-4$, and note that we have $\int \frac{u^2}{(25-u^2)^{3/2}}\,du$. Now let $u = 5\sin\theta$ so that $du = 5\cos\theta\,d\theta$, and note that $\sqrt{25-u^2} = 5\cos\theta$. Thus we have

$$\int \frac{25\sin^2\theta\cdot 5\cos\theta}{5^3\cos^3\theta}\,d\theta = \int \tan^2\theta\,d\theta = \int \sec^2\theta - 1\,d\theta = \tan\theta - \theta + C = \frac{x-4}{\sqrt{25-(x-4)^2}} - \sin^{-1}\left(\frac{x-4}{5}\right) + C.$$

8.4.68 $\int \frac{1}{\sqrt{(x-1)(3-x)}}\,dx = \int \frac{1}{\sqrt{1-(x-2)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}u + C = \sin^{-1}(x-2) + C.$

8.4.69 $\int_{2+\sqrt{2}}^{4} \frac{1}{\sqrt{(x-1)(x-3)}}\,dx = \int_{2+\sqrt{2}}^{4} \frac{1}{\sqrt{(x-2)^2-1}}\,dx = \int_{\sqrt{2}}^{2} \frac{1}{\sqrt{u^2-1}}\,du$, where $u = x-2$. Now let $u = \sec\theta$, so that $du = \sec\theta\tan\theta\,d\theta$. Then $\int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta}{\tan\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \sec\theta\,d\theta = \ln(\sec\theta + \tan\theta)\Big|_{\pi/4}^{\pi/3} = \ln(2+\sqrt{3}) - \ln(\sqrt{2}+1) = \ln\left(\frac{2+\sqrt{3}}{\sqrt{2}+1}\right) = \ln((2+\sqrt{3})(\sqrt{2}-1)).$

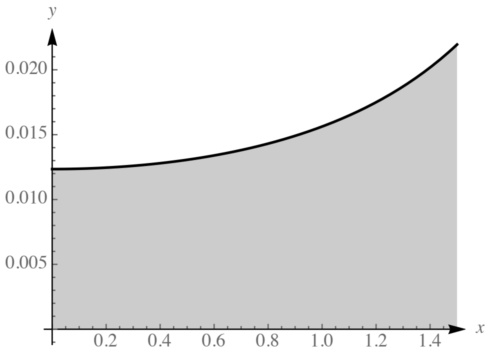
8.4.70



The area is given by $\int_0^2 \sqrt{4+x^2}\,dx$. Let $x = 2\tan\theta$ so that $dx = 2\sec^2\theta\,d\theta$. Then we have

$$\int_0^{\pi/4} 2\sec^2\theta\cdot 2\sec\theta\,d\theta = 4\int_0^{\pi/4} \sec^3\theta\,d\theta = 2\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\pi/4} = 2\left(\sqrt{2} + \ln(\sqrt{2}+1)\right).$$

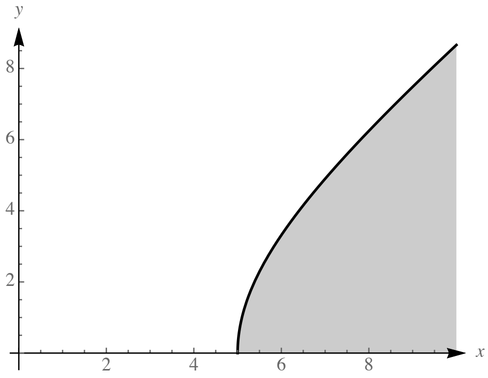
8.4.71



The area is given by $\displaystyle\int_0^{3/2} \frac{1}{(9-x^2)^2}\,dx$. Let $x = 3\sin\theta$ so that $dx = 3\cos\theta\,d\theta$. Then we have

$$\int_0^{\pi/6} \frac{3\cos\theta}{9^2\cos^4\theta}\,d\theta = \frac{1}{27}\int_0^{\pi/6} \sec^3\theta\,d\theta = \frac{1}{54}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/6}$$
$$= \frac{1}{54}\left(\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \ln\left(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)\right) = \frac{1}{81} + \frac{\ln(\sqrt{3})}{54}.$$

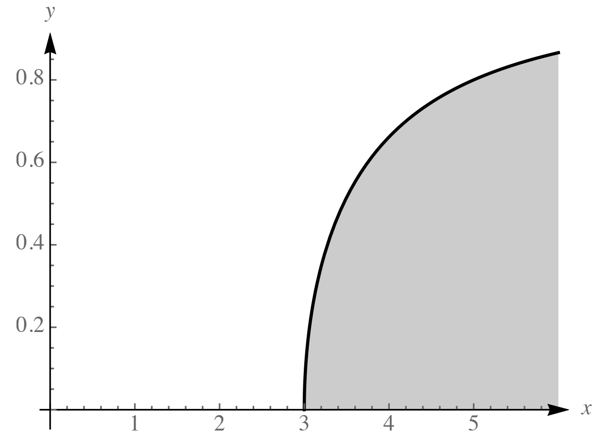
8.4.72



The area is given by $\displaystyle\int_5^{10} \sqrt{x^2-25}\,dx$. Let $x = 5\sec\theta$ so that $dx = 5\sec\theta\tan\theta\,d\theta$. Then we have

$$\int_0^{\pi/3} 5\sec\theta\tan\theta\cdot 5\tan\theta\,d\theta = 25\int_0^{\pi/3}\sec\theta\tan^2\theta\,d\theta = 25\int_0^{\pi/3}\sec\theta(\sec^2\theta-1)\,d\theta$$
$$= 25\int_0^{\pi/3}(\sec^3\theta - \sec\theta)\,d\theta = \frac{25}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/3}$$
$$= \frac{25}{2}\left(2\sqrt{3} - \ln(2+\sqrt{3})\right) = 25\sqrt{3} - \frac{25}{2}\cdot\ln(2+\sqrt{3}).$$

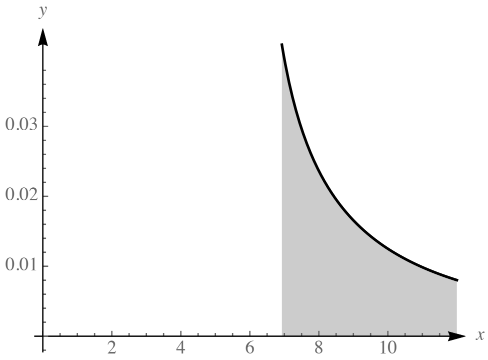
8.4.73



Let $x = 3\sec\theta$ for $x \in (0, \pi/2)$. Then $dx = 3\sec\theta\tan\theta\, d\theta$, and note that $|\tan\theta| = \tan\theta$. We have

$$\int_3^6 \frac{\sqrt{x^2-9}}{x}\, dx = \int_0^{\pi/3} \frac{3\sec\theta\tan\theta(3\tan\theta)}{3\sec\theta}\, d\theta =$$

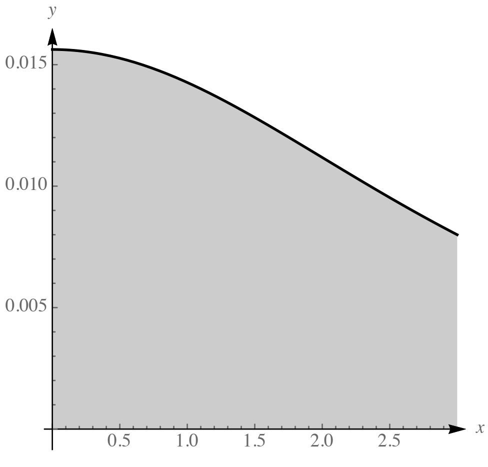
8.4.74



Let $x = 6\sec\theta$ so that $dx = 6\sec\theta\tan\theta\, d\theta$. Then

$$\int_{12/\sqrt{3}}^{12} \frac{1}{x\sqrt{x^2-36}}\, dx = \int_{\pi/6}^{\pi/3} \frac{6\sec\theta\tan\theta}{6\sec\theta\cdot 6\tan\theta}\, d\theta = \frac{1}{6}\int_{\pi/6}^{\pi/3} d\theta = \frac{\pi}{36}.$$

8.4.75



The area bounded by the curve and the axis on $[0,3]$ is given by $\displaystyle\int_0^3 \frac{1}{(16+x^2)^{3/2}}\,dx$. Let $x = 4\tan\theta$ so that $dx = 4\sec^2\theta\,d\theta$. Substituting yields

$$\int_0^{\tan^{-1}(3/4)} \frac{4\sec^2\theta}{4^3\sec^3\theta}\,d\theta = \frac{1}{16}\int_0^{\tan^{-1}(3/4)} \cos\theta\,d\theta = \frac{\sin\theta}{16}\bigg|_0^{\tan^{-1}(3/4)} = \frac{3}{80}.$$

8.4.76

a. The area is given by $\displaystyle\int_0^4 \frac{1}{\sqrt{9+x^2}}\,dx$. Let $x = 3\tan\theta$, so that $dx = 3\sec^2\theta\,d\theta$. Substituting yields $\displaystyle\int_0^{\tan^{-1}(4/3)} \frac{3\sec^2\theta}{3\sec\theta}\,d\theta = \int_0^{\tan^{-1}(4/3)} \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta|\,\bigg|_0^{\tan^{-1}(4/3)} = \ln\left(\frac{5}{3}+\frac{4}{3}\right) = \ln 3.$

b. Using disks, we have $\displaystyle\frac{V}{\pi} = \int_0^4 \frac{1}{9+x^2}\,dx$. Let $x = 3\tan\theta$ so that $dx = 3\sec^2\theta\,d\theta$. Substituting yields $\displaystyle\int_0^{\tan^{-1}(4/3)} \frac{3\sec^2\theta}{9\sec^2\theta}\,d\theta = \frac{1}{3}\int_0^{\tan^{-1}(4/3)} 1\,d\theta = \frac{1}{3}\theta\,\bigg|_0^{\tan^{-1}(4/3)} = \frac{1}{3}\tan^{-1}(4/3)$. Thus $V = \dfrac{\pi}{3}\tan^{-1}(4/3)$.

c. Using shells, we have $\displaystyle\frac{V}{2\pi} = \int_0^4 \frac{x}{(9+x^2)^{1/2}}\,dx$. Let $u = 9+x^2$, so that $du = 2x\,dx$. Then we have $\displaystyle\frac{1}{2}\int_9^{25} u^{-1/2}\,du = \sqrt{u}\,\bigg|_9^{25} = 5-3 = 2$. So $V = 4\pi$.

8.4.77 Because $y = ax^2$, we have $1+\left(\dfrac{dy}{dx}\right)^2 = 1+4a^2x^2$, so the arc length is given by $\displaystyle\int_0^{10}\sqrt{1+4a^2x^2}\,dx$. Let $x = \dfrac{1}{2a}\tan\theta$ so that $dx = \dfrac{1}{2a}\sec^2\theta\,d\theta$. Then we have

$$\begin{aligned}\frac{1}{2a}\int_0^{\tan^{-1}(20a)} \sec^2\theta\sec\theta\,d\theta &= \frac{1}{2a}\int_0^{\tan^{-1}(20a)} \sec^3\theta\,d\theta = \frac{1}{4a}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)\bigg|_0^{\tan^{-1}(20a)}\\ &= \frac{1}{4a}\left(\sqrt{1+400a^2}(20a) + \ln(\sqrt{1+400a^2}+20a)\right).\end{aligned}$$

8.4.78

a. $\displaystyle\int_0^2 \frac{x^2}{3}\,dx = \frac{x^3}{9}\bigg|_0^2 = \frac{8}{9}.$

b. $\displaystyle\int_0^2 \frac{x^2}{\sqrt{9-x^2}}\,dx$. Let $x = 3\sin\theta$ so that $dx = 3\cos\theta\,d\theta$. Then

$$\begin{aligned}\int_0^{\sin^{-1}(2/3)} \frac{9\sin^2\theta\cdot 3\cos\theta}{3\cos\theta}\,d\theta &= 9\int_0^{\sin^{-1}(2/3)} \sin^2\theta\,d\theta = \frac{9}{2}\int_0^{\sin^{-1}(2/3)} 1-\cos 2\theta\,d\theta\\ &= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\bigg|_0^{\sin^{-1}(2/3)}\\ &= \frac{9\sin^{-1}(2/3)}{2} - \frac{9}{2}\left(\frac{2\sqrt{5}}{3\cdot 3}\right) = \frac{9}{2}\sin^{-1}(2/3) - \sqrt{5} \approx 1.05.\end{aligned}$$

c. The area under g is larger than the area under f.

8.4.79 If $1 < x = \sec\theta$ then we need $\theta \in (0,\pi/2)$. Alternatively, if $-1 > x = \sec\theta$, then we need $\theta \in (\pi/2,\pi)$. So, in the former, $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\,d\theta = \theta + C = \sec^{-1}x + C$. In the latter case, $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{\sec\theta}{\sec\theta(-\tan\theta)}\,d\theta = -\theta + C = -\sec^{-1}x + C.$

8.4.80 For $1 \le x = \sec\theta$ (where $\theta \in [0, \pi/2)$) we have $dx = \sec\theta\tan\theta\,d\theta$. Note that $|\tan\theta| = \tan\theta$ in this case. We have $\displaystyle\int \frac{\sqrt{x^2-1}}{x^3}\,dx = \int \frac{\sec\theta\tan\theta|\tan\theta|}{\sec^3\theta}\,d\theta = \int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \int \sin^2\theta\,d\theta = \frac{1}{2}\int 1 - \cos 2\theta\,d\theta = \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C = \frac{1}{2}\left(\sec^{-1}x - \frac{\sqrt{x^2-1}}{x^2}\right) + C.$

For the case $-1 \ge x = \sec\theta$, we have that $|\tan\theta| = -\tan\theta$. So in this case, $\displaystyle\int \frac{\sqrt{x^2-1}}{x^3}\,dx = \int \frac{\sec\theta\tan\theta|\tan\theta|}{\sec^3\theta}\,d\theta = -\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = -\int \sin^2\theta\,d\theta = -\frac{1}{2}\int 1 - \cos 2\theta\,d\theta = -\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C = -\frac{1}{2}(\theta - \sin\theta\cos\theta) + C$. Now note that because $\theta \in (\pi/2, \pi]$, we must have that $\cos\theta < 0$. Thus we have $\displaystyle -\frac{1}{2}\left(\sec^{-1}x + \frac{\sqrt{x^2-1}}{x^2}\right) + C.$

8.4.81

a. $\displaystyle E_x(a) = \frac{kQa}{2L}\int_{-L}^{L} \frac{dy}{(a^2+y^2)^{3/2}}$. Let $y = a\tan\theta$ so that $dy = a\sec^2\theta\,d\theta$. Then note that

$$\int_{-L}^{L} \frac{dy}{(a^2+y^2)^{3/2}} = 2\int_0^{L} \frac{dy}{(a^2+y^2)^{3/2}} = 2\int_0^{\tan^{-1}(L/a)} \frac{a\sec^2\theta}{a^3\sec^3\theta}\,d\theta = \frac{2}{a^2}\int_0^{\tan^{-1}(L/a)} \cos\theta\,d\theta$$
$$= \frac{2}{a^2}\sin\theta\,\Big|_0^{\tan^{-1}(L/a)} = \frac{2}{a^2}\cdot\frac{L}{\sqrt{a^2+L^2}} = \frac{2L}{a^2\sqrt{a^2+L^2}}.$$

Thus, $\displaystyle E_x(a) = \frac{kQ}{a\sqrt{a^2+L^2}}$.

b. Set $\rho = Q/(2L)$. Then because $\displaystyle\lim_{L\to\infty} \frac{2L}{a^2\sqrt{a^2+L^2}} = \frac{2}{a^2}$, we have

$$E_x(a,0) \approx \frac{kQa}{2L}\lim_{L\to\infty}\int_{-L}^{L} \frac{dy}{(a^2+y^2)^{3/2}} = \frac{kQa}{2L}\lim_{L\to\infty}\frac{2L}{a^2\sqrt{a^2+L^2}} = \frac{kQa}{2L}\left(\frac{2}{a^2}\right) = \frac{2kQ}{2aL} = \frac{2k\rho}{a}.$$

8.4.82

a. Let β be the angle which forms a linear pair with θ, and let α be the angle in the pictured triangle with vertex at $(a, 0)$. Note that $\theta + \beta = \pi$ and $\alpha + \beta = \frac{\pi}{2}$, so $\theta = \pi - \beta = \pi/2 + \alpha$. Thus, $\sin(\theta) = \sin(\pi/2 + \alpha) = \cos\alpha$. Now, because $r^2 = a^2 + y^2$ and $\cos\alpha = \frac{a}{r}$, we have that

$$\frac{\sin\theta}{r^2} = \frac{a}{r^3} = \frac{a}{(a^2+y^2)^{3/2}}.$$

Thus

$$\int_{-L}^{L} \frac{\sin\theta}{r^2}\,dy = \int_{-L}^{L} \frac{a}{(a^2+y^2)^{3/2}}\,dy = 2\int_0^{L} \frac{a}{(a^2+y^2)^{3/2}}\,dy.$$

Let $y = a\tan u$ so that $dy = a\sec^2 u\,du$. Substituting, we have

$$2\int_0^{\tan^{-2}(L/a)} \frac{a^2\sec^2 u}{a^3\sec^3 u}\,du = \frac{2}{a}\int_0^{\tan^{-1}(L/a)} \cos u\,du = \frac{2}{a}\sin u\,\Big|_0^{\pi/2-\theta_0} = \frac{2}{a}\sin(\pi/2-\theta_0) = \frac{2}{a}\cos\theta_0$$

where $\displaystyle\tan\theta_0 = \frac{a}{L}$. Thus,

$$B(a) = \frac{\mu_0 I}{4\pi}\cdot\frac{2}{a}\cos\theta_0 = \frac{\mu_0 I}{2\pi a}\cos\theta_0.$$

b. From part (a), $\cos\theta_0 = \dfrac{L}{\sqrt{a^2+L^2}}$, and thus $\lim\limits_{L\to\infty}\cos\theta_0 = \lim\limits_{L\to\infty}\dfrac{L}{\sqrt{a^2+L^2}} = \lim\limits_{L\to\infty}\dfrac{1}{\sqrt{(a^2/L^2)+1}} = 1$. Therefore, $\lim\limits_{L\to\infty} B(a) = \dfrac{\mu_0 I}{2\pi a}$.

8.4.83

a. The area of sector OAB is given by the formula $\dfrac{\theta}{2}a^2$ where $\sin\theta = x/a$. The area of triangle OBC is $\dfrac{x}{2}\sqrt{a^2-x^2}$. Thus,

$$F(x) = \frac{a^2\sin^{-1}(x/a)}{2} + \frac{x\sqrt{a^2-x^2}}{2}.$$

b. By the first fundamental theorem of calculus, $F(x)$ is an antiderivative of $\sqrt{a^2-x^2}$, because $F(x) = \int_0^x \sqrt{a^2-t^2}\,dt$. Thus, any other antiderivative differs from this by a constant, so

$$\int \sqrt{a^2-x^2}\,dx = \frac{a^2\sin^{-1}(x/a)}{2} + \frac{x\sqrt{a^2-x^2}}{2} + C.$$

8.4.84

a. $0 = -\dfrac{1}{2}ka^2 + y_{\max}$ when $a = \sqrt{\dfrac{2y_{\max}}{k}}$.

b. $y'(x) = -kx$, so $1+(y'(x))^2 = 1+k^2x^2$, so the arc length is given by

$$L = \int_{-a}^{a}\sqrt{1+(y'(x))^2}\,dx = \int_{-a}^{a}\sqrt{1+k^2x^2}\,dx = 2\int_0^a \sqrt{1+k^2x^2}\,dx$$

because the function is an even function.

c. Let $x = \dfrac{1}{k}\tan\theta$ so that $dx = \dfrac{1}{k}\sec^2\theta\,d\theta$. Then the arc length integral is equal to

$$\begin{aligned}2\int_0^{\tan^{-1}(ak)} \frac{1}{k}\sec^2\theta\cdot\sec\theta\,d\theta &= \frac{1}{k}\left(\sec\theta\tan\theta\,\Big|_0^{\tan^{-1}(ak)} + \ln|\sec\theta+\tan\theta|)\,\Big|_0^{\tan^{-1}(ak)}\right.\\ &= \frac{1}{k}\left(ak\sqrt{1+a^2k^2} + \ln(\sqrt{1+a^2k^2}+ak)\right).\end{aligned}$$

Noting that $\tan(\theta) = ak$, $a = (V^2/g)\sin\theta\cos\theta$, and $k = \dfrac{g}{V^2\cos^2\theta}$, we can write L as a function of θ as $L(\theta) = (V^2/g)\sin\theta + (V^2/g)\cos^2\theta(\ln(\sqrt{1+\tan^2\theta}+\tan\theta))$, and using an identity for the inverse hyperbolic sine function, this can be written as $L(\theta) = \dfrac{V^2}{g}\left(\sin\theta + \cos^2\theta\sinh^{-1}(\tan\theta)\right)$.

d. Using the expression for $L(\theta)$ above, we have $L'(\theta) = (V^2/g)\left(\cos\theta - 2\cos\theta\sin\theta(\sinh^{-1}(\tan\theta)) - \cos^2\theta(1+\tan^2\theta)^{-1/2}\sec^2\theta\right) = \dfrac{2V^2\cos\theta}{g}\left(1-\sin\theta\sinh^{-1}(\tan\theta)\right)$. This expression is 0 when $\sin\theta(\sinh^{-1}\tan\theta) = 1$, which is equivalent to $\sin\theta(\ln(\tan\theta+\sec\theta)) = 1$. Either the first or second derivative tests can be used to see that this critical number yields the maximum.

e. Some experimenting reveals that the maximum occurs for $\theta \approx 56.5$ degrees.

8.4.85

a. Because $t \in [0, \pi]$ so that $sint \geq 0$, we have

$$\int_a^b \sqrt{\frac{1-\cos t}{g(\cos a - \cos t)}}\,dt = \int_a^b \sqrt{\frac{(1-\cos t)(1+\cos t)}{g(1+\cos t)(\cos a - \cos t)}}\,dt = \int_a^b \sin t\sqrt{\frac{1}{g(1+\cos t)(\cos a - \cos t)}}\,dt.$$

Let $u = \cos t$ so that $du = -\sin t\,dt$. Then the given integral is equal to

$$-\frac{1}{\sqrt{g}}\int_{\cos a}^{\cos b}\sqrt{\frac{1}{(1+u)(\cos a - u)}}\,du.$$

Now we complete the square:

$$\begin{aligned}(1+u)(\cos a - u) &= \cos a + (\cos a - 1)u - u^2\\ &= -\left(u^2 - (\cos a - 1)u + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\frac{\cos a - 1}{2}\right)^2\right) + \cos a\\ &= \cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(u - \frac{\cos a - 1}{2}\right)^2 = \left(\frac{\cos a + 1}{2}\right)^2 - \left(u - \frac{\cos a - 1}{2}\right)^2.\end{aligned}$$

Thus, setting $v = u - \dfrac{\cos a - 1}{2}$ we have that the original integral is equal to $-\dfrac{1}{\sqrt{g}}\displaystyle\int_{(\cos a+1)/2}^{\cos b - \frac{\cos a - 1}{2}} \frac{1}{\sqrt{k^2 - v^2}}\,dv$ where $k = \dfrac{(\cos a + 1)}{2}$.

Now, $\displaystyle\int \frac{1}{\sqrt{k^2 - v^2}}\,dv = \int \frac{k\cos\theta}{k\cos\theta}\,d\theta = \theta + C = \sin^{-1}(v/k) + C$ where $v = k\sin\theta$.

Therefore, the original integral is equal to

$$\begin{aligned}-\frac{1}{\sqrt{g}}\sin^{-1}\left(\frac{2v}{\cos a + 1}\right)\Bigg|_{(\cos a+1)/2}^{\cos b - (\cos a - 1)/2} &= \frac{1}{\sqrt{g}}\left(\sin^{-1}\left(\frac{\cos a + 1}{\cos a + 1}\right) - \sin^{-1}\left(\frac{2\cos b - \cos a + 1}{\cos a + 1}\right)\right)\\ &= \frac{1}{\sqrt{g}}\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{2\cos b - \cos a + 1}{\cos a + 1}\right)\right).\end{aligned}$$

b. Letting $b = \pi$, we have that the integral is equal to

$$\begin{aligned}&\frac{1}{\sqrt{g}}\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{-2 - \cos a + 1}{\cos a + 1}\right)\right)\\ &= \frac{1}{\sqrt{g}}\left(\frac{\pi}{2} - \sin^{-1}(-1)\right) = \frac{1}{\sqrt{g}}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{\sqrt{g}}.\end{aligned}$$

8.4.86 Let $x = 2\tan^{-1} u$ so that $u = \tan(x/2)$ and $\sec^2(x/2) = 1 + \tan^2(x/2) = 1 + u^2$. Also, $\cos^2(x/2) = 1/(1+u^2)$. By the double angle identity,

$$\cos x = \cos^2(x/2) - \sin^2(x/2) = \cos^2(x/2) - u^2\cos^2(x/2) = (1-u^2)\cos^2(x/2) = \frac{1-u^2}{1+u^2}.$$

Also, $\sin x = 2\sin(x/2)\cos(x/2) = 2\tan(x/2)\cos^2(x/2) = \dfrac{2u}{1+u^2}$. Now $\displaystyle\int \frac{1}{1+\sin x + \cos x}\,dx = 2\int \frac{1}{1+u^2}\cdot\frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\,du = 2\int \frac{1}{1+u^2+2u+1-u^2}\,du = 2\int \frac{1}{2+2u}\,du = \int \frac{1}{1+u}\,du = \ln|1+u| + C = \ln|1+\tan(x/2)| + C.$

8.5 Partial Fractions

8.5.1 Proper rational functions can be integrated using partial fraction decomposition.

8.5.2 Your answers may vary.

a. $x - 1$.

b. $(x-1)^3$.

c. $x^2 + x + 1$.

d. $(x^2 + x + 1)^2$

8.5.3

a. $\dfrac{A}{x-3}$.

b. $\dfrac{A_1}{x-4}, \dfrac{A_2}{(x-4)^2}, \dfrac{A_3}{(x-4)^3}$.

c $\dfrac{Ax+B}{x^2+2x+6}$.

8.5.4 The first step is to divide the numerator by the denominator via long division in order to write the quotient as the sum of a polynomial and a proper rational function. Thus we would write

$$\frac{x^2+2x-3}{x+1} = x+1-\frac{4}{x+1}.$$

8.5.5 $\dfrac{4x}{(x-4)(x-5)} = \dfrac{A}{x-4} + \dfrac{B}{x-5}$.

8.5.6 $\dfrac{4x+1}{(2x-1)(2x+1)} = \dfrac{A}{2x-1} + \dfrac{B}{2x+1}$.

8.5.7 $\dfrac{x+3}{(x-5)^2} = \dfrac{A}{x-5} + \dfrac{B}{(x-5)^2}$.

8.5.8 $\dfrac{2}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$.

8.5.9 $\dfrac{4}{x(x+1)(x-1)(x+2)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1} + \dfrac{D}{x+2} + \dfrac{E}{x-2}$.

8.5.10 $\dfrac{20x}{(x-1)^2(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{Cx+D}{x^2+1}$.

8.5.11 $\dfrac{1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$.

8.5.12 $\dfrac{2x^2+3}{(x^2-8x+16)(x^2+3x+4)} = \dfrac{2x^2+3}{(x-4)^2(x^2+3x+4)} = \dfrac{A}{x-4} + \dfrac{B}{(x-4)^2} + \dfrac{Cx+D}{x^2+3x+4}$.

8.5.13 $\dfrac{x^4+12x^2}{(x-2)^2(x^2+x+2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{Cx+D}{x^2+x+2} + \dfrac{Ex+F}{(x^2+x+2)^2}$.

8.5.14 $\dfrac{6x^4-4x^3+15x^2-5x+7}{(x-2)(2x^2+3)^2} = \dfrac{A}{x-2} + \dfrac{Bx+C}{2x^2+3} + \dfrac{Dx+E}{(2x^2+3)^2}$.

8.5.15 $\dfrac{x}{(x-2)^2(x+2)^2(x^2+4)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2} + \dfrac{Ex+F}{x^2+4} + \dfrac{Gx+H}{(x^2+4)^2}$.

8.5.16 $\dfrac{x^2+2x+6}{x^3(x^2+x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+x+1} + \dfrac{Fx+G}{(x^2+x+1)^2}$.

8.5.17 $\dfrac{5x-7}{x^2-3x+2} = \dfrac{5x-7}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$. Thus, $A(x-2)+B(x-1) = 5x-7$. Equating coefficients gives $A+B=5$ and $-2A-B=-7$. Solving this system yields $A=2$, $B=3$. Thus,

$$\frac{5x-7}{x^2-3x+2} = \frac{2}{x-1} + \frac{3}{x-2}.$$

8.5.18 $\dfrac{11x-10}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$. Thus, $A(x-1)+Bx = 11x-10$. Equating coefficients gives $A+B=11$, $-A=-10$. Solving this system yields $A=10$, $B=1$. Thus,

$$\frac{11x-10}{x^2-x} = \frac{10}{x} + \frac{1}{x-1}.$$

8.5.19 $\dfrac{6}{x^2-2x-8} = \dfrac{6}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$. Thus, $6 = A(x+2)+B(x-4)$. Equating coefficients gives $A+B=0$ and $2A-4B=6$. Solving this system yields $A=1$ and $B=-1$. Thus,

$$\frac{6}{x^2-2x-8} = \frac{1}{x-4} - \frac{1}{x+2}.$$

8.5.20 $\dfrac{x^2-4x+11}{(x-3)(x-1)(x+1)} = \dfrac{A}{x-3} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$. Thus, $x^2-4x+11 = A(x-1)(x+1)+B(x-3)(x+1)+C(x-3)(x-1)$. Letting $x=1$ gives $B=-2$, letting $x=-1$ gives $C=2$, and letting $x=3$ gives $A=1$. Thus,

$$\frac{x^2-4x+11}{(x-3)(x-1)(x+1)} = \frac{1}{x-3} + \frac{-2}{x-1} + \frac{2}{x+1}.$$

8.5.21 By long division, $\dfrac{2x^2+5x+6}{x^2+3x+2} = 2 + \dfrac{-x+2}{(x+1)(x+2)}$. We use a partial fraction decomposition on $\dfrac{-x+2}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$. Therefore $A(x+2)+B(x+1) = -x+2$, so $A+B=-1$ and $2A+B=2$. So $A=3$ and $B=-4$. Thus

$$\frac{2x^2+5x+6}{x^2+3x+2} = 2 + \frac{3}{x+1} - \frac{4}{x+2}.$$

8.5.22 By long division, we can write $\dfrac{x^4+2x^3+x}{x^2-1}$ as $x^2+2x+1+\dfrac{3x+1}{x^2-1} = (x+1)^2 + \dfrac{3x+1}{(x-1)(x+1)}$. We use a partial fraction decomposition on $\dfrac{3x+1}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$. We have $A(x+1)+B(x-1) = 3x+1$, so $A+B=3$ and $A-B=1$. Therefore, $A=2$ and $B=1$. We have

$$\frac{x^4+2x^3+x}{x^2-1} = (x+1)^2 + \frac{2}{x-1} + \frac{1}{x+1}.$$

8.5.23 If we write $\dfrac{3}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$, we have $3 = A(x+2)+B(x-1)$. Letting $x=-2$ yields $B=-1$ and letting $x=1$ yields $A=1$. Thus, the original integral is equal to

$$\int \left(\frac{1}{x-1} - \frac{1}{x+2} \right) dx = \ln|x-1| - \ln|x+2| + C = \ln\left|\frac{x-1}{x+2}\right| + C.$$

8.5.24 If we write $\frac{8}{(x-2)(x+6)} = \frac{A}{x-2} + \frac{B}{x+6}$, we have $8 = A(x+6) + B(x-2)$. Letting $x = -6$ yields $B = -1$ and letting $x = 2$ yields $A = 1$. Thus the original integral is equal to

$$\int \left(\frac{1}{x-2} - \frac{1}{x+6} \right) dx = \ln|x-2| - \ln|x+6| + C.$$

8.5.25 If we write $\frac{6}{x^2-1} = \frac{6}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$, then we have $6 = A(x+1) + B(x-1)$. Letting $x = -1$ yields $B = -3$ and letting $x = 1$ yields $A = 3$. Thus, the original integral is equal to

$$\int \left(\frac{3}{x-1} - \frac{3}{x+1} \right) dx = 3\left(\ln|x-1| - \ln|x+1|\right) + C = 3\ln\left|\frac{x-1}{x+1}\right| + C.$$

8.5.26 If we write $\frac{1}{t^2-9} = \frac{A}{t-3} + \frac{B}{t+3}$, then we have $1 = A(t+3) + B(t-3)$. Letting $t = -3$ yields $B = -1/6$ and letting $t = 3$ yields $A = 1/6$. Thus the original integral is equal to

$$\int_0^1 \left(\frac{1/6}{t-3} - \frac{1/6}{t+3} \right) dt = \frac{1}{6}\left(\ln|t-3| - \ln|t+3|\right)\Big|_0^1 = -\frac{\ln 2}{6}.$$

8.5.27 $\frac{8x-5}{(x-1)(3x-2)} = \frac{A}{x-1} + \frac{B}{3x-2}$. Thus $8x - 5 = A(3x-2) + B(x-1) = (3A+B)x - 2A - B$. So $3A + B = 8$ and $-2A - B = -5$. Solving gives $A = 3$ and $B = -1$. We have

$$\int \frac{8x-5}{3x^2-5x+2}\, dx = \int \left(\frac{3}{x-1} - \frac{1}{3x-2} \right) dx = 3\ln|x-1| - \frac{1}{3}\ln|3x-2| + C.$$

8.5.28 $\frac{7x-2}{3x^2-2x} = \frac{7x-2}{x(3x-2)} = \frac{A}{x} + \frac{B}{3x-2}$. Then $7x - 2 = A(3x-2) + Bx = (3A+B)x - 2A$. So $A = 1$ and $B = 4$. We have

$$\int_1^2 \frac{7x-2}{3x^2-2x}\, dx = \int_1^2 \left(\frac{1}{x} + \frac{4}{3x-2} \right) dx = \left(\ln x + \frac{4}{3}\ln(3x-2) \right)\Big|_1^2 = \ln 2 + \frac{4}{3}\ln 4 = \ln 2 + \frac{8}{3}\ln 2 = \frac{11\ln 2}{3}.$$

8.5.29 If we write $\frac{5x}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$, then we have $5x = A(x+2) + B(x-3)$. Letting $x = -2$ yields $B = 2$ and letting $x = 3$ yields $A = 3$. Thus the original integral is equal to

$$\int_{-1}^2 \left(\frac{3}{x-3} + \frac{2}{x+2} \right) dx = \left(3\ln|x-3| + 2\ln|x+2|\right)\Big|_{-1}^2 = \ln(16) - \ln(64) = -\ln 4.$$

8.5.30 If we write $\frac{21x^2}{x^3-x^2-12x} = \frac{21x^2}{x(x-4)(x+3)} = \frac{21x}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3}$, then we have $21x = A(x+3) + B(x-4)$. Letting $x = 4$ yields $A = 12$. Letting $x = -3$ yields $B = 9$. Thus, the original integral is equal to

$$\int \left(\frac{12}{x-4} + \frac{9}{x+3} \right) dx = 12\ln|x-4| + 9\ln|x+3| + C.$$

8.5.31 Let $\frac{6x^2}{x^4-5x^2+4} = \frac{6x^2}{(x-2)(x+2)(x-1)(x+1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x-1} + \frac{D}{x+1}$. Then

$$6x^2 = A(x+2)(x-1)(x+1) + B(x-2)(x-1)(x+1) + C(x-2)(x+2)(x+1) + D(x-2)(x+2)(x-1).$$

Letting $x = 2$ gives $A = 2$, letting $x = -2$ gives $B = -2$, letting $x = 1$ gives $C = -1$, and letting $x = -1$ gives $D = 1$. Thus, the original integral is equal to

$$\int \left(\frac{2}{x-2} - \frac{2}{x+2} - \frac{1}{x-1} + \frac{1}{x+1} \right) dx = \ln\left|\frac{(x-2)^2(x+1)}{(x+2)^2(x-1)}\right| + C.$$

8.5.32 Let $\frac{4x-2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$. Thus,

$$4x-2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1).$$

Letting $x = 0$ gives $A = 2$, letting $x = 1$ gives $B = 1$, and letting $x = -1$ gives $C = -3$. Thus, the original integral is equal to

$$\int \left(\frac{2}{x} + \frac{1}{x-1} - \frac{3}{x+1} \right) dx = \ln \left| \frac{x^2(x-1)}{(x+1)^3} \right| + C.$$

8.5.33 After performing long division, we have that the original integrand is equal to $3 + \frac{13x-12}{(x-1)(x-2)}$, and if we write $\frac{13x-12}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$, then $13x - 12 = A(x-2) + B(x-1)$. Letting $x = 1$ yields $A = -1$ and letting $x = 2$ yields $B = 14$. Thus the original integral is equal to

$$3x - \int \frac{1}{x-1}\, dx + 14 \int \frac{1}{x-2}\, dx = 3x - \ln|x-1| + 14 \ln|x-2| + C.$$

8.5.34 After performing long division, we have that the original integrand is equal to $2z - 1 + \frac{7z+1}{(z+3)(z-2)}$. If we write $\frac{7z+1}{(z+3)(z-2)} = \frac{A}{z+3} + \frac{B}{z-2}$, then $7z + 1 = A(z-2) + B(z+3)$. Letting $z = 2$ yields $B = 3$ and letting $z = -3$ yields $A = 4$. Thus, the original integral is equal to

$$\int \left(2z - 1 + \frac{4}{z+3} + \frac{3}{z-2} \right) dz = z^2 - z + 4\ln|z+3| + 3\ln|z-2| + C.$$

8.5.35 Let $\frac{x^2+12x-4}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$. Then $x^2 + 12x - 4 = A(x-2)(x+2) + Bx(x+2) + Cx(x-2)$. Letting $x = 0$ gives $A = 1$, letting $x = 2$ gives $B = 3$, and letting $x = -2$ gives $C = -3$. Thus, the original integral is equal to $\int \left(\frac{1}{x} + \frac{3}{x-2} - \frac{3}{x+2} \right) dx = \ln \left| \frac{x(x-2)^3}{(x+2)^3} \right| + C.$

8.5.36 Let $\frac{z^2+20z-15}{z(z+5)(z-1)} = \frac{A}{z} + \frac{B}{z+5} + \frac{C}{z-1}$. Then $z^2 + 20z - 15 = A(z+5)(z-1) + Bz(z-1) + Cz(z+5)$. Letting $z = 0$ gives $A = 3$, letting $z = -5$ gives $B = -3$, and letting $z = 1$ gives $C = 1$. Thus, the original integral is equal to $\int \left(\frac{3}{z} - \frac{3}{z+5} + \frac{1}{z-1} \right) dz = \ln \left| \frac{z^3(z-1)}{(z+5)^3} \right| + C.$

8.5.37 If we write $\frac{1}{x^4 - 10x^2 + 9} = \frac{1}{(x-1)(x+1)(x-3)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-3} + \frac{D}{x+3}$ then $1 = A(x+1)(x-3)(x+3) + B(x-1)(x-3)(x+3) + C(x-1)(x+1)(x+3) + D(x-1)(x+1)(x-3)$. Letting $x = -1$ yields $B = 1/16$. Letting $x = 3$ yields $C = 1/48$. Letting $x = -3$ yields $D = -1/48$, and letting $x = 1$ yields $A = -1/16$. Thus the original integral is equal to

$$\begin{aligned}
&\int \left(-\frac{1/16}{x-1} + \frac{1/16}{x+1} + \frac{1/48}{x-3} - \frac{1/48}{x+3} \right) dx \\
&= -\frac{1}{16} \ln|x-1| + \frac{1}{16} \ln|x+1| + \frac{1}{48} \ln|x-3| - \frac{1}{48} \ln|x+3| + C \\
&= \ln \left| \frac{(x+1)^3(x-3)}{(x-1)^3(x+3)} \right|^{1/48} + C.
\end{aligned}$$

8.5.38 If we write $\frac{2}{x^2 - 4x - 32} = \frac{A}{x-8} + \frac{B}{x+4}$, then we have $2 = A(x+4) + B(x-8)$. Letting $x = -4$ yields $B = -1/6$ and letting $x = 8$ yields $A = 1/6$. Thus, the original integral is equal to

$$\int_0^5 \left(\frac{1/6}{x-8} - \frac{1/6}{x+4} \right) dx = \frac{1}{6} \left(\ln|x-8| - \ln|x+4| \right) \Big|_0^5 = \frac{1}{6}(\ln(1/3) - \ln 2) = -\frac{\ln 6}{6}.$$

8.5.39 If we write $\frac{81}{x^3-9x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-9}$, then $81=Ax(x-9)+B(x-9)+C(x^2)$. Letting $x=0$ yields $B=-9$. Letting $x=9$ yields $C=1$. If we let $x=10$, then we have $81=10A+B+100C=10A-9+100$, so $A=-1$. Thus, the original integral is equal to $\int\left(-\frac{1}{x}-\frac{9}{x^2}+\frac{1}{x-9}\right)dx=\ln\left|\frac{(x-9)}{x}\right|+\frac{9}{x}+C$.

8.5.40 If we write $\frac{16x^2}{(x-6)(x+2)^2}=\frac{A}{x-6}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$, then we have $16x^2=A(x+2)^2+B(x-6)(x+2)+C(x-6)$. Letting $x=6$ yields $A=9$. Letting $x=-2$ yields $C=-8$. Letting $x=0$ gives $0=36-12B+48$, so $B=7$. Thus the original integral is equal to

$$\int\left(\frac{9}{x-6}+\frac{7}{x+2}-\frac{8}{(x+2)^2}\right)dx=\ln\left|(x+2)^7(x-6)^9\right|+\frac{8}{x+2}+C.$$

8.5.41 If we write $\frac{x}{(x+3)^2}=\frac{A}{x+3}+\frac{B}{(x+3)^2}$, then we have $x=A(x+3)+B$. Letting $x=-3$ yields $B=-3$, and then letting $x=-2$ yields $A=1$. Thus the original integral is equal to

$$\int_{-1}^{1}\left(\frac{1}{x+3}-\frac{3}{(x+3)^2}\right)dx=\left(\ln|x+3|+\frac{3}{x+3}\right)\Big|_{-1}^{1}=\ln 4+\frac{3}{4}-\left(\ln 2+\frac{3}{2}\right)=\ln 2-\frac{3}{4}.$$

8.5.42 If we write $\frac{1}{x^3-2x^2-4x+8}=\frac{1}{(x+2)(x-2)^2}=\frac{A}{x+2}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$, then $1=A(x-2)^2+B(x+2)(x-2)+C(x+2)$. Letting $x=2$ yields $C=1/4$. Letting $x=-2$ yields $A=1/16$. Letting $x=3$ yields $1=\frac{1}{16}+5B+5\cdot\frac{1}{4}$, so $B=-\frac{1}{16}$. Thus, the original integral is equal to

$$\int\left(\frac{1/16}{x+2}-\frac{1/16}{x-2}+\frac{1/4}{(x-2)^2}\right)dx=\frac{1}{16}\left(\ln|x+2|-\ln|x-2|\right)-\frac{1}{4(x-2)}+C.$$

8.5.43 If we write $\frac{2}{x^3+x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$, then $2=Ax(x+1)+B(x+1)+Cx^2$. Letting $x=0$ yields $B=2$, and letting $x=-1$ yields $C=2$. Then letting $x=1$ yields $A=-2$. So the original integral is equal to

$$\int\left(-\frac{2}{x}+\frac{2}{x^2}+\frac{2}{x+1}\right)dx=2\left(\ln|x+1|-\ln|x|\right)-\frac{2}{x}+C.$$

8.5.44 If we write $\frac{2}{t^3(t+1)}=\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t^3}+\frac{D}{t+1}$ then we have $2=At^2(t+1)+Bt(t+1)+C(t+1)+Dt^3$. Letting $t=0$ yields $C=2$, and letting $t=-1$ reveals that $D=-2$. Now if we let $t=1$ we have that $2=2A+2B+4-2$, so $A=-B$. Letting $t=2$ yields the equation $12=12A+6B=12A-6A=6A$, so $A=2$ and $B=-2$. So the original integral is equal to

$$\int_1^2\left(\frac{2}{t}-\frac{2}{t^2}+\frac{2}{t^3}-\frac{2}{t+1}\right)dt=\left(2\left(\ln|t|-\ln|t+1|\right)+\frac{2}{t}-\frac{1}{t^2}\right)\Big|_1^2=\ln(16/9)-1/4.$$

8.5.45 If we write $\frac{x-5}{x^2(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$, then we have $x-5=Ax(x+1)+B(x+1)+Cx^2$. Letting $x=0$ yields $B=-5$, and letting $x=-1$ yields $C=-6$. Then letting $x=1$ yields $-4=2A-10-6$, so $A=6$. The original integral is thus equal to

$$\int\left(\frac{6}{x}-\frac{5}{x^2}-\frac{6}{x+1}\right)dx=6\left(\ln|x|-\ln|x+1|\right)+\frac{5}{x}+C.$$

8.5.46 Let $\frac{x^2}{(x-2)^3}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}$. Then $x^2=A(x-2)^2+B(x-2)+C$. Letting $x=2$ gives $C=4$. Letting $x=0$ gives $0=4A-2B+4$, and letting $x=1$ gives $1=A-B+4$. Solving the system of two linear equations results in $A=1$ and $B=4$. The original integral is therefore equal to

$$\int\left(\frac{1}{x-2}+\frac{4}{(x-2)^2}+\frac{4}{(x-2)^3}\right)dx=\ln|x-2|-\frac{4}{x-2}-\frac{2}{(x-2)^2}+C.$$

8.5.47 By long division, we can write the integrand as $x + \frac{2x}{(x-5)^2}$. We write

$$\frac{2x}{(x-5)^2} = \frac{A}{x-5} + \frac{B}{(x-5)^2},$$

so that $A(x-5) + B = 2x$. Then $A = 2$ and $B = 10$. We have

$$\int \frac{x^3 - 10x^2 + 27x}{x^2 - 10x + 25}\, dx = \int \left(x + \frac{2}{x-5} + \frac{10}{(x-5)^2} \right) dx = \frac{x^2}{2} + 2\ln|x-5| - \frac{10}{x-5} + C.$$

8.5.48 By long division, we can write the integrand as $1 + \frac{2x^2 - x + 2}{x(x-1)^2}$. We write

$$\frac{2x^2 - x + 2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

So

$$2x^2 - x + 2 = A(x-1)^2 + Bx(x-1) + Cx.$$

Letting $x = 0$ gives $A = 2$. Letting $x = 1$ gives $C = 3$. Letting $x = 2$ then gives $8 = 2 + 2B + 6$, so $B = 0$. We have

$$\int \frac{x^3 + 2}{x^3 - 2x^2 + x}\, dx = \int \left(1 + \frac{2}{x} + \frac{3}{(x-1)^2} \right) dx = x + 2\ln|x| - \frac{3}{x-1} + C.$$

8.5.49 Let $\frac{x^2 - 4}{x^3 - 2x^2 + x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$. Then $x^2 - 4 = A(x-1)^2 + Bx(x-1) + Cx$. Letting $x = 0$ gives $A = -4$. Letting $x = 1$ gives $C = -3$. Letting $x = 2$ gives $0 = -4 + 2B - 6$, so $B = 5$. The given integral is thus equal to

$$\int \left(-\frac{4}{x} + \frac{5}{x-1} - \frac{3}{(x-1)^2} \right) dx = \ln\left| \frac{(x-1)^5}{x^4} \right| + \frac{3}{x-1} + C.$$

8.5.50 Let $\frac{8(x^2+4)}{x(x^2+8)} = \frac{Ax+B}{x^2+8} + \frac{C}{x}$. Then $8(x^2+4) = Ax^2 + Bx + C(x^2+8)$. Letting $x = 0$ gives $C = 4$. Letting $x = 1$ gives $40 = A + B + 36$, so $A + B = 4$. Letting $x = -1$ gives $A - B = 4$, so $A = 4$ and $B = 0$. The original integral is thus equal to

$$\int \left(\frac{4x}{x^2+8} + \frac{4}{x} \right) dx = 2\ln(x^2+8) + 4\ln|x| + C = \ln((x^2+8)^2 x^4) + C.$$

8.5.51 Let $\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$. Then $x^2 + x + 2 = (Ax+B)(x+1) + C(x^2+1)$. Letting $x = -1$ gives $C = 1$. Letting $x = 0$ gives $2 = B + 1$, so $B = 1$. Letting $x = 1$ gives $4 = 2A + 2 + 2$, so $A = 0$. The original integral is therefore equal to

$$\int \left(\frac{1}{x^2+1} + \frac{1}{x+1} \right) dx = \tan^{-1}(x) + \ln|x+1| + C.$$

8.5.52 Let $\frac{x^2+3x+2}{x(x^2+2x+2)} = \frac{Ax+B}{x^2+2x+2} + \frac{C}{x}$. Then $x^2 + 3x + 2 = Ax^2 + Bx + C(x^2+2x+2)$. Letting $x = 0$ gives $C = 1$. Letting $x = 1$ gives $6 = A + B + 5$, so $A + B = 1$. Letting $x = -1$ gives $0 = A - B + 1$. Solving the system of linear equations gives $A = 0$ and $B = 1$. The original integral is therefore equal to

$$\begin{aligned}\int \left(\frac{1}{x^2+2x+2} + \frac{1}{x} \right) dx &= \int \left(\frac{1}{(x^2+2x+1)+1} + \frac{1}{x} \right) dx \\ &= \int \left(\frac{1}{(x+1)^2+1} + \frac{1}{x} \right) dx = \tan^{-1}(x+1) + \ln|x| + C.\end{aligned}$$

8.5.53 Let $\frac{2x^2+5x+5}{(x+1)(x^2+2x+2)} = \frac{Ax+B}{x^2+2x+2}+\frac{C}{x+1}$. Then $2x^2+5x+5 = (Ax+B)(x+1)+C(x^2+2x+2)$. Letting $x=-1$ gives $C=2$. Letting $x=0$ gives $5=B+4$, so $B=1$. Letting $x=1$ gives $12=2A+2+2(5)$, so $A=0$. The original integral is therefore equal to

$$\int\left(\frac{1}{x^2+2x+2}+\frac{2}{x+1}\right)dx = \int\left(\frac{1}{(x+1)^2+1}+\frac{2}{x+1}\right)dx = \tan^{-1}(x+1)+\ln((x+1)^2)+C.$$

8.5.54 If we write $\frac{z+1}{z(z^2+4)} = \frac{A}{z}+\frac{Bz+C}{z^2+4}$, then we have that $z+1 = A(z^2+4)+(Bz+C)z$. Letting $z=0$ yields $A=1/4$, and we have $z+1=(1/4+B)z^2+Cz+1$, so equating coefficients gives $B=-1/4$ and $C=1$. So the original integral is equal to $\int\left(\frac{1}{4z}-\frac{z}{4(z^2+4)}+\frac{1}{z^2+4}\right)dz$. The middle term can be handled via the substitution $u=z^2+4$, and the last term is recognizable as the derivative of $\frac{1}{2}\tan^{-1}(z/2)$. Thus the original integral is equal to

$$\frac{1}{4}\ln|z|-\frac{1}{8}\ln(z^2+4)+\frac{1}{2}\tan^{-1}\frac{z}{2}+C.$$

8.5.55 If we write $\frac{20x}{(x-1)(x^2+4x+5)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+4x+5}$, then $20x = A(x^2+4x+5)+(Bx+C)(x-1)$. Letting $x=1$ yields $A=2$. Letting $x=0$ yields $0=10-C$, so $C=10$. Letting $x=2$ yields $40=34+2B+10$, so $B=-2$. The original integral is thus

$$\begin{aligned}\int\left(\frac{2}{x-1}-\frac{2x-10}{x^2+4x+5}\right)dx &= \int\left(\frac{2}{x-1}-\frac{(2x+4)-14}{x^2+4x+5}\right)dx\\ &= \int\left(\frac{2}{x-1}-\frac{(2x+4)}{x^2+4x+5}+\frac{14}{(x+2)^2+1}\right)dx\\ &= \ln\left|\frac{(x-1)^2}{x^2+4x+5}\right|+14\tan^{-1}(x+2)+C.\end{aligned}$$

8.5.56 Note that this rational function is already in decomposition form, so any attempt to decompose it further will be futile. Instead we write the given integral as the sum $\int\frac{2x}{x^2+4}\,dx+\int\frac{1}{x^2+4}\,dx$. For the first integral, let $u=x^2+4$ so that $du=2x\,dx$. It is then equal to $\int\frac{1}{u}\,du=\ln|x^2+4|+C$. The second integral can be written as $\frac{1}{4}\int\frac{1}{(x/2)^2+1}\,dx=\frac{1}{2}\tan^{-1}\frac{x}{2}+D$. So the original integral is equal to

$$\ln|x^2+4|+\frac{1}{2}\tan^{-1}\frac{x}{2}+E.$$

8.5.57 $\frac{x^3+5x}{(x^2+3)^2} = \frac{Ax+B}{x^2+3}+\frac{Cx+D}{(x^2+3)^2}$. Then

$$x^3+5x = (Ax+B)(x^2+3)+Cx+D = Ax^3+Bx^2+(3A+C)x+3B+D.$$

Then $A=1$, $B=0$, $3A+C=5$, and $3B+D=0$. So $C=2$ and $D=0$. We have

$$\int\frac{x^3+5x}{(x^2+3)^2}\,dx = \int\left(\frac{x}{x^2+3}+\frac{2x}{(x^2+3)^2}\right)dx = \frac{1}{2}\ln|x^2+3|-\frac{1}{x^2+3}+C.$$

8.5.58 After performing long division, we have that the original integrand is equal to $x-\frac{9x^2-1}{x(x^2+9)}$. If we write $\frac{9x^2-1}{x(x^2+9)} = \frac{A}{x}+\frac{Bx+C}{x^2+9}$, we have $9x^2-1 = A(x^2+9)+(Bx+C)x$. Letting $x=0$ yields $A=-1/9$, and letting $x = 1$ yields $8 = 10(-1/9) + B + C$. Letting $x = -1$ yields $8 = 10(-1/9) + B - C$. Thus $C = 0$ and therefore $B = 82/9$. Therefore the original integral is equal to

$$\frac{x^2}{2} + \frac{1}{9}\ln|x| - \frac{82}{9}\int \frac{x}{x^2+9}\,dx = \frac{x^2}{2} + \frac{1}{9}\ln|x| - \frac{41}{9}\ln|x^2+9| + C.$$

8.5.59 $\dfrac{x^3+6x^2+12x+6}{(x^2+6x+10)^2} = \dfrac{Ax+B}{x^2+6x+10} + \dfrac{Cx+D}{(x^2+6x+10)^2}$. Then

$$x^3+6x^2+12x+6 = (Ax+B)(x^2+6x+10)+Cx+D = Ax^3+(B+6A)x^2+(10A+6B+C)x+10B+D.$$

Then $A = 1$, $B = 0$, $C = 2$, and $D = 6$. The given integral is then

$$\int\left(\frac{x}{x^2+6x+10} + \frac{2x+6}{(x^2+6x+10)^2}\right)dx = \int \frac{x}{(x+3)^2+1}\,dx - \frac{1}{x^2+6x+10} + C.$$

Note that

$$\int \frac{x}{(x+3)^2+1}\,dx = \int\left(\frac{x+3}{(x+3)^2+1} - \frac{3}{(x+3)^2+1}\right)dx = \frac{1}{2}\ln(x^2+6x+10) - 3\tan^{-1}(x+3) + C.$$

So the given integral is equal to

$$\frac{1}{2}\ln(x^2+6x+10) - 3\tan^{-1}(x+3) - \frac{1}{x^2+6x+10} + C.$$

8.5.60 If we write $\dfrac{1}{(y^2+1)(y^2+2)} = \dfrac{Ay+B}{y^2+1} + \dfrac{Cy+D}{y^2+2}$ then

$$1 = (Ay+B)(y^2+2)+(Cy+D)(y^2+1) = (A+C)y^3+(B+D)y^2+(2A+C)y+2B+D.$$

Equating coefficients gives us the equations $A+C = 0$, $B+D = 0$, $2A+C = 0$, and $2B+D = 1$. Solving this system yields $A = C = 0$, $B = 1$ and $D = -1$. Thus the original integral is equal to

$$\int\left(\frac{1}{y^2+1} - \frac{1}{y^2+2}\right)dy = \tan^{-1}y - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + C.$$

8.5.61 If we write $\dfrac{2}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$, then

$$2 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x.$$

Letting $x = 0$ yields $A = 2$. Expanding the right-hand side yields $2 = (2+B)x^4 + Cx^3 + (4+B+D)x^2 + (C+E)x + 2$. Equating coefficients gives us the equations $2+B = 0$, $C = 0$, $4+B+D = 0$, and $C+E = 0$, from which we can deduce that $B = -2$, $C = 0$, $D = -2$ and $E = 0$. The original integral is thus equal to

$$\int\left(\frac{2}{x} - \frac{2x}{x^2+1} - \frac{2x}{(x^2+1)^2}\right)dx = 2\ln|x| - \ln|x^2+1| + \frac{1}{x^2+1} + C.$$

8.5.62 If we write $\dfrac{1}{(x+1)(x^2+2x+2)^2} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+2x+2} + \dfrac{Dx+E}{(x^2+2x+2)^2}$, then

$$1 = A(x^2+2x+2)^2 + (Bx+C)(x+1)(x^2+2x+2) + (Dx+E)(x+1).$$

Letting $x = -1$ yields $A = 1$. Then expanding the polynomial on the right-hand side gives

$$1 = (1+B)x^4 + (4+3B+C)x^3 + (8+4B+3C+D)x^2 + (8+2B+4C+D+E)x + (4+E+2C).$$

Equating coefficients and then solving for the unknowns yields $B = -1$, $C = -1$, $D = -1$, and $E = -1$. The original integral is thus equal to

$$\int\left(\frac{1}{x+1} - \frac{x+1}{x^2+2x+2} - \frac{x+1}{(x^2+2x+2)^2}\right)dx = \ln|x+1| - \frac{1}{2}\ln|x^2+2x+2| + \frac{1}{2(x^2+2x+2)} + C.$$

8.5.63 $\frac{9x^2+x+21}{(3x^2+7)^2} = \frac{Ax+B}{3x^2+7} + \frac{Cx+D}{(3x^2+7)^2}$. Then

$$9x^2+x+21 = (Ax+B)(3x^2+7)+Cx+D = 3Ax^3+3Bx^2+(7A+C)x+7B+D.$$

So $A=0$, $B=3$, $C=1$, and $D=0$. We have

$$\int \frac{9x^2+x+21}{(3x^2+7)^2}\,dx = \int\left(\frac{3}{3x^2+7}+\frac{x}{(3x^2+7)^2}\right)dx = \int\frac{1}{x^2+(7/3)}\,dx + \int\frac{x}{(3x^2+7)^2}\,dx$$
$$= \sqrt{\frac{3}{7}}\tan^{-1}\left(\sqrt{\frac{3}{7}}x\right) - \frac{1}{6(3x^2+7)} + C.$$

8.5.64 $\frac{9x^5+6x^3}{(3x^2+1)^3} = \frac{Ax+B}{3x^2+1}+\frac{Cx+D}{(3x^2+1)^2}+\frac{Ex+F}{(3x^2+1)^3}$. Then

$$9x^5+6x^3 = (Ax+B)(3x^2+1)^2+(Cx+D)(3x^2+1)+Ex+F.$$

This can be written as

$$9x^5+6x^3 = 9Ax^5+9Bx^4+(6A+3C)x^3+(6B+3D)x^2+(A+C+E)x+B+D+F.$$

Therefore, $A=1$, $B=C=D=0$, $E=-1$, and $F=0$. We have

$$\int\frac{9x^5+6x^3}{(3x^2+1)^3}\,dx = \int\left(\frac{x}{3x^2+1}-\frac{x}{(3x^2+1)^3}\right)dx = \frac{1}{6}\ln(3x^2+1)+\frac{1}{12(3x^2+1)^2}+C.$$

8.5.65

a. False. Because the given integrand is improper, the first step would be to use long division to write the integrand as the sum of a polynomial and a proper rational function.

b. False. This is easy to evaluate via the substitution $u=3x^2+x$.

c. False. The discriminant of the denominator is $b^2-4ac=169-168=1>0$, so the denominator factors into linear factors of the real numbers.

d. True. The discriminant of the denominator is $b^2-4ac=169-172=-3<0$, so the given quadratic expression is irreducible.

8.5.66 We are seeking $\int_0^1 \frac{x-x^2}{(x+1)(x^2+1)}\,dx$. We can write the integrand as $\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$, and then

$$x-x^2 = A(x^2+1)+(Bx+C)(x+1) = (A+B)x^2+(C+B)x+A+C.$$

Thus $A+B=-1$, $C+B=1$, and $A+C=0$. Solving gives $A=-1$, $B=0$, and $C=1$. Thus we have

$$\int_0^1\frac{x-x^2}{(x+1)(x^2+1)}\,dx = \int_0^1\left(-\frac{1}{x+1}+\frac{1}{x^2+1}\right)dx = \left(-\ln|x+1|+\tan^{-1}x\right)\Big|_0^1 = \frac{\pi}{4}-\ln 2.$$

8.5.67 We are seeking $\int_{-2}^2\frac{10}{x^2-2x-24}\,dx = \int_{-2}^2\frac{10}{(x-6)(x+4)}\,dx$. If we write

$$\frac{10}{x^2-2x-24} = \frac{10}{(x-6)(x+4)} = \frac{A}{x-6}+\frac{B}{x+4},$$

then $10 = A(x+4)+B(x-6)$. Letting $x=-4$ gives $B=-1$ and letting $x=6$ gives $A=1$. Thus the area in question is given by

$$-\int_{-2}^2\left(-\frac{1}{x+4}+\frac{1}{x-6}\right)dx = \left(\ln(x+4)-\ln|x-6|\right)\Big|_{-2}^2 = \ln 6-\ln 4-(\ln 2-\ln 8) = \ln 6.$$

8.5.68 The curves intersect when $\dfrac{3x^2+2x+1}{x(x^2+x+1)} = \dfrac{2}{x}$, or when $3x^2+2x+1 = 2x^2+2x+2$, or $x^2-1=0$, so $x=-1$ and $x=1$, so in the first quadrant the bounded area lies between $x=1$ and $x=2$. The area is given by

$$\int_1^2 \left(\frac{3x^2+2x+1}{x(x^2+x+1)} - \frac{2}{x}\right) dx = \int_1^2 \frac{x^2-1}{x(x^2+x+1)}\, dx.$$

We decompose this integrand via partial fractions. Let $\dfrac{x^2-1}{x(x^2+x+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+x+1}$. Then $x^2-1 = A(x^2+x+1)+(Bx+C)x = (A+B)x^2+(A+C)x+A$, so $A=-1$, $B=2$, and $C=1$. The area is then given by

$$\int_1^2 \left(-\frac{1}{x} + \frac{2x+1}{x^2+x+1}\right) dx = \big(\ln|x^2+x+1| - \ln|x|\big)\Big|_1^2 = \ln 7 - \ln 2 - \ln 3 = \ln\frac{7}{6}.$$

8.5.69 Using disks, we have $\dfrac{V}{\pi} = \displaystyle\int_0^4 \frac{x^2}{(x+1)^2}\, dx$. Now we can perform long division to rewrite this integral as $\displaystyle\int_0^4 \left(1 - \frac{2x+2}{x^2+2x+1} + \frac{1}{(x+1)^2}\right) dx$. Thus we have $V = \pi\left(x - \ln|x^2+2x+1| - \dfrac{1}{x+1}\Big|_0^4\right) = \pi\left(4 - \ln(25) - \dfrac{1}{5} - (-1)\right) = \pi\left(\dfrac{24}{5} - \ln(25)\right)$.

8.5.70 Using shells, the volume is given by $V = 2\pi\displaystyle\int_1^2 \frac{x}{x^2(x^2+2)^2}\, dx = 2\pi\int_1^2 \frac{1}{x(x^2+2)^2}\, dx$. If we write

$$\frac{1}{x(x^2+2)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2},$$

then

$$1 = A(x^2+2)^2 + (Bx+C)(x)(x^2+2) + (Dx+E)x = (A+B)x^4 + Cx^3 + (D+2B+4A)x^2 + 2Cx + 4A.$$

This leads to $A = \dfrac{1}{4}$, $B = -\dfrac{1}{4}$, $C=0$, $D = -\dfrac{1}{2}$, and $E=0$. The volume is then

$$\begin{aligned}\frac{2\pi}{4}\int_1^2 \left(\frac{1}{x} - \frac{x}{x^2+2} - \frac{2x}{(x^2+2)^2}\right) dx &= \frac{\pi}{2}\left(\ln x - \frac{1}{2}\ln(x^2+2) + \frac{1}{x^2+2}\right)\Big|_1^2 \\ &= \frac{\pi}{2}\left(\ln 2 - \frac{1}{2}\ln 6 + \frac{1}{6} - \left(0 - \frac{1}{2}\ln 3 + \frac{1}{3}\right)\right) \\ &= \frac{\pi}{2}\left(\frac{1}{2}\ln 2 - \frac{1}{6}\right) = \frac{\pi(3\ln 2 - 1)}{12}.\end{aligned}$$

8.5.71 Using disks, we have $\dfrac{V}{\pi} = \displaystyle\int_1^2 \frac{1}{x(3-x)}\, ,dx$. If we write

$$\frac{1}{x(3-x)} = \frac{A}{x} + \frac{B}{3-x},$$

then we have $1 = A(3-x)+Bx$. Letting $x=0$ yields $A=1/3$ and letting $x=3$ yields $B=1/3$. Thus we have

$$V = \frac{\pi}{3}\int_1^2 \left(\frac{1}{x} - \frac{1}{x-3}\right) dx = \frac{\pi}{3}\left(\ln x - \ln|x-3|\,\Big|_1^2\right) = \frac{\pi}{3}\cdot 2\ln 2.$$

8.5.72 Using shells, we have $V = 2\pi \int_5^{10} \frac{x(3x^2+25)}{x^2(x^2+25)}\,dx = 2\pi \int_5^{10} \frac{3x^2+25}{x(x^2+25)}\,dx$. Write

$$\frac{3x^2+25}{x(x^2+25)} = \frac{A}{x} + \frac{Bx+C}{x^2+25}.$$

Then

$$3x^2+25 = A(x^2+25) + (Bx+C)x = (A+B)x^2 + Cx + 25A,$$

so $A = 1$, $B = 2$, and $C = 0$. We have

$$V = 2\pi \int_5^{10} \left(\frac{1}{x} + \frac{2x}{x^2+25} \right) dx = 2\pi \left(\ln x + \ln(x^2+25) \right) \Big|_5^{10} = 2\pi \ln \frac{1250}{250} = 2\pi \ln 5.$$

8.5.73 If we write $\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$, then $1 = A(x+1) + B(x-1)$, so $A = 1/2$ and $B = -1/2$. Thus we have

$$\frac{1}{2} \int \frac{1}{x-1} - \frac{1}{x+1}\,dx = \frac{1}{2} \left(\ln|x-1| - \ln|x+1| \right) + C.$$

Now let $x = \sec\theta$, so that $dx = \sec\theta\tan\theta\,d\theta$. Then the original integral is equal to $\int \csc\theta\,d\theta = -\ln|\csc\theta + \cot\theta| + C = -\ln\left| \frac{x}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2-1}} \right| + C = -\ln\left(\frac{|x+1|}{\sqrt{x^2-1}} \right) + C = -\ln\left(\sqrt{\left| \frac{x+1}{x-1} \right|} \right) + C = \ln\left(\sqrt{\left| \frac{x-1}{x+1} \right|} \right) + C$. The two answers are equivalent.

8.5.74 $\frac{3x^2+2x+1}{(x+1)^3(x^2+x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+x+1} + \frac{Fx+G}{(x^2+x+1)^2}$. Using a computer algebra system, we find that $A = -3$, $B = 0$, $C = 2$, $D = 3$, $E = 0$, $F = 1$, and $G = 2$. Thus

$$\frac{3x^2+2x+1}{(x+1)^3(x^2+x+1)^2} = -\frac{3}{x+1} + \frac{2}{(x+1)^3} + \frac{3x}{x^2+x+1} + \frac{x+2}{(x^2+x+1)^2}.$$

8.5.75 $\frac{x^4+3x^2+1}{x(x^2+1)^2(x^2+x+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} + \frac{Fx+G}{x^2+x+4} + \frac{Hx+I}{(x^2+x+4)^2}$. Using a computer algebra system, we find that $A = 1/16$, $B = -1/100$, $C = -1/10$, $D = 2/25$, $E = 3/50$, $F = -21/400$, $G = 19/400$, $H = -1/5$, and $I = -1/20$. Thus

$$\frac{x^4+3x^2+1}{x(x^2+1)^2(x^2+x+4)^2} = \frac{1}{16x} - \frac{x+10}{100(x^2+1)} + \frac{4x+3}{50(x^2+1)^2} + \frac{-21x+19}{400(x^2+x+4)} - \frac{4x+1}{20(x^2+x+4)^2}.$$

8.5.76 Let $u = \sin x$ so that $du = \cos x\,dx$. Then we have $\int \frac{1}{u(u-2)(u+2)}\,du$. If we write $\frac{1}{u(u-2)(u+2)} = \frac{A}{u} + \frac{B}{u-2} + \frac{C}{u+2}$, then we have $1 = A(u-2)(u+2) + Bu(u+2) + Cu(u-2)$. Letting $u = 2$ yields $B = 1/8$ and letting $u = -2$ yields $C = 1/8$. Letting $u = 0$ yields $A = -1/4$. Thus we have

$$\begin{aligned} \int \left(-\frac{1}{4u} + \frac{1/8}{u-2} + \frac{1/8}{u+2} \right) du &= -\frac{1}{4} \ln|u| + \frac{1}{8} \left(\ln|u-2| + \ln|u+2| \right) + C \\ &= -\frac{1}{4} \ln|\sin x| + \frac{1}{8} \ln\left|(\sin x - 2)(\sin x + 2)\right| + C \\ &= -\frac{1}{4} \ln|\sin x| + \frac{1}{8} \ln(4 - \sin^2 x) + C. \end{aligned}$$

8.5.77 Let $u = e^x$ so that $du = e^x\,dx$. Then the original integral is equal to $\int \frac{1}{(u-1)(u+2)}\,du$. If we write $\frac{1}{(u-1)(u+2)} = \frac{A}{u-1} + \frac{B}{u+2}$, then $1 = A(u+2) + B(u-1)$. Letting $u = -2$ yields $B = -1/3$ and letting $u = 1$ yields $A = 1/3$. Thus we have

$$\int \left(\frac{1/3}{u-1} - \frac{1/3}{u+2} \right) du = \frac{1}{3}\left(\ln|u-1| - \ln|u+2|\right) + C = \frac{1}{3}\ln\left|\frac{e^x-1}{e^x+2}\right| + C.$$

8.5.78 Let $u = \sqrt{y}$, so that $du = \frac{1}{2\sqrt{y}}\,dy$. So $dy = 2u\,du$. Substituting gives $\int \frac{2u}{u^2(\sqrt{a}-u)}\,du = \int \frac{2}{u(\sqrt{a}-u)}\,du$. Write $\frac{2}{u(\sqrt{a}-u)} = \frac{A}{u} + \frac{B}{\sqrt{a}-u}$. Then $2 = A(\sqrt{a}-u) + Bu$. Letting $u = 0$ gives $A = \frac{2}{\sqrt{a}}$, and letting $u = \sqrt{a}$ gives $B = \frac{2}{\sqrt{a}}$. Our integral is therefore equal to

$$\int \left(\frac{2/\sqrt{a}}{u} - \frac{2/\sqrt{a}}{u-\sqrt{a}} \right) du = \frac{2}{\sqrt{a}}\ln\left|\frac{u}{u-\sqrt{a}}\right| + C = \frac{2}{\sqrt{a}}\ln\left|\frac{\sqrt{y}}{\sqrt{y}-\sqrt{a}}\right| + C.$$

8.5.79 This can be written as

$$\int \frac{dt}{\cos t(1+\sin t)} = \int \frac{\cos t}{\cos^2 t(1+\sin t)}\,dt = \int \frac{\cos t}{(1-\sin^2 t)(1+\sin t)}\,dt.$$

Now let $u = \sin t$ so that $du = \cos t\,dt$. Substituting gives $\int \frac{du}{(1-u^2)(1+u)} = \int \frac{du}{(1+u)^2(1-u)}$. If we write $\frac{1}{(1+u)^2(1-u)} = \frac{A}{1+u} + \frac{B}{(1+u)^2} + \frac{C}{1-u}$ then $1 = A(1+u)(1-u) + B(1-u) + C(1+u)^2$. Letting $u = -1$ yields $B = 1/2$. Letting $u = 1$ yields $C = 1/4$. Letting $u = 0$ then yields $A = 1/4$. Then our integral is equal to

$$\int \left(\frac{1/4}{1+u} + \frac{1/2}{(1+u)^2} + \frac{1/4}{1-u} \right) du = \frac{1}{4}\left(\frac{-2}{1+u} + \ln\left|\frac{1+u}{1-u}\right| \right) + C = \frac{1}{4}\left(\frac{-2}{1+\sin t} + \ln\left(\frac{1+\sin t}{1-\sin t}\right) \right) + C.$$

8.5.80 Let $u = \sqrt{e^x+1}$, so that $u^2 = e^x + 1$ and $2u\,du = e^x\,dx = (u^2-1)\,dx$. Then the original integral is equal to $\int \frac{2u^2}{u^2-1}\,du = \int 2 + \frac{2}{u^2-1}\,du$. If we write $\frac{2}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1}$, then $2 = A(u-1) + B(u+1)$. Letting $u = 1$ yields $B = 1$ and letting $u = -1$ yields $A = -1$. Thus we have $\int \left(2 - \frac{1}{u+1} + \frac{1}{u-1}\right) du = 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{e^x+1} + \ln|\sqrt{e^x+1}-1| - \ln|\sqrt{e^x+1}+1| + C$.

8.5.81 Let $u = e^x$. Then $du = e^x\,dx$. We have $\int \frac{e^{3x}+e^{2x}+e^x}{(e^{2x}+1)^2}\,dx = \int \frac{u^2+u+1}{(u^2+1)^2}\,du$.

Let $\frac{u^2+u+1}{(u^2+1)^2} = \frac{Au+B}{u^2+1} + \frac{Cu+D}{(u^2+1)^2}$. Then $u^2+u+1 = (Au+B)(u^2+1) + Cu + D = Au^3 + Bu^2 + (A+C)u + B + D$. Then $A = 0$, $B = 1$, $C = 1$, and $D = 0$. So we have

$$\int \frac{u^2+u+1}{(u^2+1)^2}\,du = \int \left(\frac{1}{u^2+1} + \frac{u}{(u^2+1)^2} \right) du = \tan^{-1} u - \frac{1}{2(u^2+1)} + C = \tan^{-1} e^x - \frac{1}{2(e^{2x}+1)} + C.$$

8.5.82 If we let $u^2 = 2x + 1$, then $2u\,du = 2\,dx$. Substituting yields

$$\int \frac{2u}{u(u-1)(u+1)}\,du = 2\int \frac{1}{(u-1)(u+1)}\,du = \int \frac{1}{u-1} - \frac{1}{u+1}\,du$$
$$= \ln|u-1| - \ln|u+1| + C = \ln\left|\frac{\sqrt{1+2x}-1}{\sqrt{1+2x}+1}\right| + C.$$

8.5.83 Let $u = e^x$, so that $du = e^x\,dx$. Then $\int \frac{1}{1+e^x}\cdot\frac{e^x}{e^x}\,dx = \int \frac{1}{u(1+u)}\,du$. If we write $\frac{1}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u}$, then $1 = A(1+u) + Bu$. Letting $u = 0$ yields $A = 1$ and letting $u = -1$ yields $B = -1$, so the original integral is equal to $\int \left(\frac{1}{u} - \frac{1}{1+u}\right) du = \ln|u| - \ln|1+u| + C = x - \ln(1+e^x) + C$.

8.5.84 Because $\frac{x^2}{(x-4)(x+5)}$ is not proper, it is not of the proper form to be decomposed via partial fractions.

8.5.85

a. $\sec x = \frac{1}{\cos x}\cdot\frac{\cos x}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1-\sin^2 x}$.

b. $\int \sec x\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx$. Let $u = \sin x$ so that $du = \cos x\,dx$. Then our integral becomes $\int \frac{du}{1-u^2}$. If we write $\frac{1}{1-u^2} = \frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$, we have $1 = A(1+u) + B(1-u)$, so $A = \frac{1}{2}$ and $B = \frac{1}{2}$. We have

$$\int \frac{du}{1-u^2} = \frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u}\right) du = \frac{1}{2}(\ln|1+u| - \ln|1-u|) + C = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C.$$

8.5.86 Differentiating the right-hand side, we have

$$\frac{5}{x-1} - \frac{3}{x+1} + \frac{3}{(x+1)^2} + \frac{1}{2}\cdot\frac{1}{1+(x^2/4)}\cdot\frac{1}{2} = \frac{5}{x-1} - \frac{3}{x+1} + \frac{3}{(x+1)^2} + \frac{1}{x^2+4}.$$

This is the partial fraction decomposition of the integrand.

8.5.87 If $x = 2\tan^{-1} u$, then $dx = \frac{2}{1+u^2}\,du$.

If $u = \tan(x/2)$, then $u^2 + 1 = \tan^2(x/2) + 1 = \sec^2(x/2) = \frac{1}{\cos^2(x/2)}$. Thus, $\frac{1}{u^2+1} = \cos^2(x/2) = \frac{1+\cos x}{2}$, so $\cos x = \frac{2}{u^2+1} - 1 = \frac{1-u^2}{u^2+1} = \frac{1-u^2}{1+u^2}$.

Also, $\sin x = 2\sin(x/2)\cos(x/2) = 2\tan(x/2)\cos^2(x/2) = \frac{2u}{1+u^2}$.

8.5.88 Using the substitution $x = 2\tan^{-1} u$ yields

$$\int \frac{1}{2+\cos x}\,dx = \int \frac{2}{1+u^2}\cdot\frac{1}{2+\frac{1-u^2}{1+u^2}}\,du = 2\int \frac{1}{3+u^2}\,du = \frac{2\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)}{\sqrt{3}} + C$$
$$= \frac{2\tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right)}{\sqrt{3}} + C.$$

8.5.89 Using the substitution $x = 2\tan^{-1} u$ yields

$$\int \frac{1}{1-\frac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}\,du = \int u^{-2}\,du = -\frac{1}{u} + C = -\cot(x/2) + C.$$

8.5.90 Using the substitution $x = 2\tan^{-1} u$ yields

$$\int \frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}\,du$$
$$= \int \frac{2}{2+2u}\,du = \int \frac{1}{1+u}\,du = \ln|1+u| + C = \ln|1+\tan(x/2)| + C.$$

8.5.91 Using the substitution $\theta = 2\tan^{-1} u$ yields $\displaystyle\int_0^1 \frac{1}{\frac{1-u^2}{1+u^2}+\frac{2u}{1+u^2}}\cdot\frac{2}{1+u^2}\,du = \int_0^1 \frac{2}{1+2u-u^2}\,du = -2\int_0^1 \frac{1}{u^2-2u+1-2}\,du = -2\int_0^1 \frac{1}{(u-1)^2-2}\,du$. If we factor the denominator as the difference of squares we have $(u-1)^2-2 = (u-1-\sqrt{2})(u-1+\sqrt{2})$, and using a partial fractions decomposition yields

$$\frac{1}{\sqrt{2}}\int_0^1\left(\frac{1}{u+\sqrt{2}-1}-\frac{1}{u-\sqrt{2}-1}\right)du = \frac{1}{\sqrt{2}}\ln\left|\frac{u+\sqrt{2}-1}{u-\sqrt{2}-1}\right|\Bigg|_0^1 = \frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

8.5.92 Using the substitution $\theta = 2\tan^{-1} u$ yields

$$\int_0^{1/\sqrt{3}}\frac{\frac{2u}{1+u^2}}{1-\frac{2u}{1+u^2}}\cdot\frac{2}{1+u^2}\,du = \int_0^{1/\sqrt{3}}\frac{4u}{(1+u^2)(u^2-2u+1)}\,du = \int_0^{1/\sqrt{3}}\frac{4u}{(u^2+1)(u-1)^2}\,du.$$

Decomposing this integral via partial fractions yields

$$\int_0^{1/\sqrt{3}}\left(\frac{2}{(u-1)^2}-\frac{2}{u^2+1}\right)du = \left(\frac{-2}{u-1}-2\tan^{-1}u\right)\Bigg|_0^{1/\sqrt{3}} = \frac{-2}{\frac{1}{\sqrt{3}}-1}-\frac{\pi}{3}-2 = 1+\sqrt{3}-\frac{\pi}{3}.$$

8.5.93 $\displaystyle s_A(t) = \int v_A(t)\,dt = 88\int\frac{t}{t+1}\,dt = 88\int\left(1-\frac{1}{1+t}\right)dt = 88t - 88\ln(t+1) + C$. Because $s_A(0) = 0$, we see that $C = 0$, so $s_A(t) = 88t - 88\ln(1+t)$.

$\displaystyle s_B(t) = \int v_B(t)\,dt = 88\int\frac{t^2}{(t+1)^2}\,dt = 88\int\left(1-\frac{2t+2}{t^2+2t+1}+\frac{1}{(t+1)^2}\right)dt = 88(t-\ln(t^2+2t+1)-\frac{1}{t+1}) + D$. Because $s_B(0) = 0$, we see that $D = 88$, so $s_B(t) = 88(t-\ln(t^2+2t+1)-\frac{1}{t+1}+1)$.

$\displaystyle s_C(t) = \int v_C(t)\,dt = 88\int\frac{t^2}{t^2+1}\,dt = 88\int\left(1-\frac{1}{t^2+1}\right)dt = 88(t-\tan^{-1}(t)) + E$. Because $s_C(0) = 0$, we have that $E = 0$. Thus $s_C(t) = 88t - 88\tan^{-1}(t)$.

a. $s_A(1) = 88(1-\ln(2)) \approx 27$. $s_B(1) = 88(1-\ln 4-(1/2)+1) \approx 88(3/2-\ln 4) \approx 10$. $s_C(1) = 88(1-\tan^{-1}(1)) \approx 18.9$. So car A travels farthest.

b. $s_A(5) = 88(5-\ln 6) \approx 282$. $s_B(5) = 88(5-\ln 36-(1/6)+1) \approx 198$. $s_C(5) = 88(5-\tan^{-1}(5)) \approx 319$. So car C travels farthest.

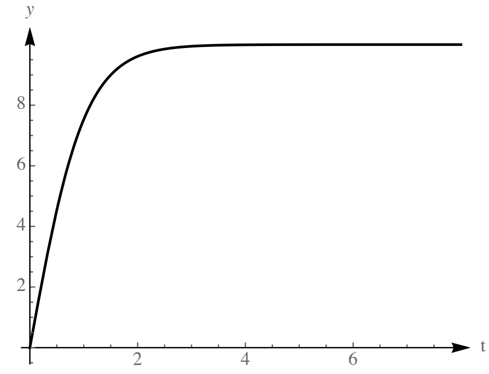
c. See the development above.

d. Ultimately car C gains the lead. This can be seen by the fact that car C's velocity function is greater than that of the other cars.

8.5.94

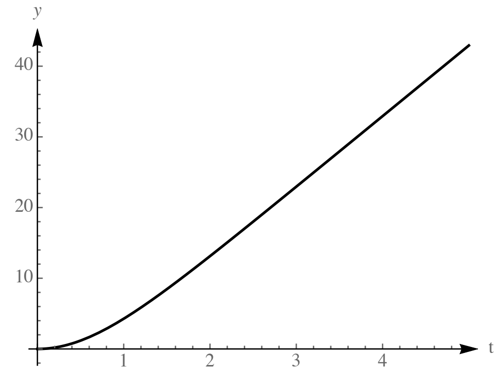
a. $v(0) = V\cdot\dfrac{1-1}{1+1} = 0$. $\displaystyle\lim_{t\to\infty}\frac{1-(1/e^{kt})}{1+(1/e^{kt})} = 1$, when $k = 2g/V > 0$ (which it is.) Thus $\displaystyle\lim_{t\to\infty} v(t) = V\cdot 1 = V$.

b.



c. $s(t) = \int v(t)\,dt = V\int \frac{e^{kt}-1}{e^{kt}+1}\,dt = V\int\left(\frac{e^{kt}}{e^{kt}+1} - \frac{1}{e^{kt}+1}\right)dt$. Let $u = e^{kt}+1$ and $w = e^{kt}$. Then we have $V\left(\frac{1}{k}\int\frac{1}{u}\,du - \frac{1}{k}\int\frac{1}{w(w+1)}\,dw\right) = \frac{V}{k}\ln|e^{kt}+1| - \frac{V}{k}\int\left(\frac{1}{w} - \frac{1}{w+1}\right)dw = \frac{V}{k}\ln|e^{kt}+1| - \frac{V}{k}\ln\left(\frac{e^{kt}}{e^{kt}+1}\right) + C = \frac{V}{k}\left(2\ln(e^{kt}+1) - kt\right) + C$. Now because $s(0) = 0$, we must have $C = -\frac{V}{k}\ln 4$. Thus, $s(t) = \frac{V}{k}(\ln(e^{kt}+1)^2 - kt - \ln 4) = \frac{V}{k}\left(\ln\left(\frac{e^{kt}+1}{2}\right)^2 - kt\right) = \frac{V}{k}2\ln\left(\frac{e^{kt}+1}{2}\right) - Vt = \frac{V^2}{g}\left(\ln\frac{1+e^{-kt}}{2} + \ln e^{kt}\right) - Vt = \frac{V^2}{g}\ln\frac{1+e^{-kt}}{2} + \frac{V^2}{g}kt - Vt = \frac{V^2}{g}\ln\frac{1+e^{-kt}}{2} + Vt.$

d.



8.5.95 First note that the numerator of the given integrand can be written as $x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$, and this quantity when divided by x^2+1 yields $x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$. Thus the given integral is equal to $\left(\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x\right)\Big|_0^1 - \left(4\tan^{-1}(x)\right)\Big|_0^1 = \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi = \frac{22}{7} - \pi$. Because the given integrand is positive on the interval $(0,1)$, we know that this integral is positive. Thus,

$$0 < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi.$$

Adding π to both sides of this inequality yields $\pi < \frac{22}{7}$.

8.5.96 Let $u^2 = \tan x$ so that $2u\,du = \sec^2 x\,dx = 1 + u^4\,dx$. Then we can write

$$\int \sqrt{\tan x}\,dx = \int \frac{2u^2}{1+u^4}\,du = \int \frac{u^2}{(u^2-\sqrt{2}u+1)(u^2+\sqrt{2}u+1)}\,du.$$

If we write $\dfrac{u^2}{(u^2-\sqrt{2}u+1)(u^2+\sqrt{2}u+1)} = \dfrac{Au+B}{u^2-\sqrt{2}u+1} + \dfrac{Cu+D}{u^2+\sqrt{2}u+1}$ then we have $u^2 = (Au + B)(u^2+\sqrt{2}u+1) + (Cu+D)(u^2-\sqrt{2}u+1)$. Multiplying out the right-hand side, equating coefficients, and solving for the unknowns yields $A = \dfrac{\sqrt{2}}{4}$, $B = 0$, $C = -\dfrac{\sqrt{2}}{4}$, and $D = 0$. Now

$$\begin{aligned}\frac{u^2}{1+u^4} &= \frac{\sqrt{2}}{4}\left(\frac{u}{u^2-\sqrt{2u}+1} - \frac{u}{u^2+\sqrt{2u}+1}\right) = \frac{\sqrt{2}}{8}\left(\frac{2u}{u^2-\sqrt{2u}+1} - \frac{2u}{u^2+\sqrt{2u}+1}\right)\\ &= \frac{\sqrt{2}}{8}\left(\frac{2u-\sqrt{2}}{u^2-\sqrt{2u}+1} + \frac{\sqrt{2}}{u^2-\sqrt{2u}+1} - \frac{2u+\sqrt{2}}{u^2+\sqrt{2u}+1} + \frac{\sqrt{2}}{u^2+\sqrt{2u}+1}\right).\end{aligned}$$

Then

$$\begin{aligned}2\int \frac{u^2}{1+u^4}\,du &= \frac{\sqrt{2}}{4}\left(\ln|u^2-\sqrt{2}u+1| - \ln|u^2+\sqrt{2}u+1|\right)\\ &+ \frac{1}{2}\int\left(\frac{1}{(u^2-\sqrt{2}u+1)} + \frac{1}{(u^2+\sqrt{2}u+1)}\right)du.\end{aligned}$$

This last integral can be written as

$$\int\left(\frac{1}{(\sqrt{2}u+1)^2+1} + \frac{1}{(1-\sqrt{2}u)^2+1}\right)du = \frac{1}{\sqrt{2}}\left(\tan^{-1}(\sqrt{2}u+1) - \tan^{-1}(1-\sqrt{2}u)\right) + C.$$

Putting this all together and replacing u by $\sqrt{\tan x}$ yields

$$\begin{aligned}&\frac{\sqrt{2}}{4}\left(\ln|\tan x - \sqrt{2\tan x}+1| - \ln|\tan x + \sqrt{2\tan x}+1|\right)\\ &+ \frac{1}{\sqrt{2}}\left(\tan^{-1}(1+\sqrt{2\tan x}) - \tan^{-1}(1-\sqrt{2\tan x})\right) + C.\end{aligned}$$

Thus,

$$\int_0^{\pi/4} \sqrt{\tan x}\,dx = \frac{\sqrt{2}}{4}\left(\ln(2-\sqrt{2}) - \ln(2+\sqrt{2}\right) + \frac{1}{\sqrt{2}}\left(\tan^{-1}(1+\sqrt{2}) - \tan^{-1}(1-\sqrt{2})\right) \approx 0.4875.$$

8.6 Integration Strategies

8.6.1 Because the integrand is a product of a polynomial and a trigonometric function, it makes sense to integrate by parts.

8.6.2 Because $\dfrac{d}{dx}(1+\tan x) = \sec^2 x$, use a u-substitution of $u = 1 + \tan x$.

8.6.3 The presence of $64 - x^2$ in the integrand suggests the trigonometric substitution $x = 8\sin\theta$.

8.6.4 One approach is to rewrite the integrand by replacing $\tan^2 x + 1$ with $\sec^2 x$ and then proceed with with a u-substitution of $u = \tan x$. Another approach is to split up the fraction and simplify the integrand by writing

$$\int \frac{\tan^2 x + 1}{\tan x}dx = \int\left(\frac{\tan^2 x}{\tan x} + \frac{1}{\tan x}\right)dx = \int(\tan x + \cot x)dx.$$

Then use the integral formulas for $\tan x$ and $\cot x$.

8.6.5 The method of partial fractions is appropriate because the integrand is a proper rational function.

8.6.6 Using the identity $1 - \sin^2 x = \cos^2 x$, rewrite the integrand as $\dfrac{\cos^5 x \sin^4 x}{\cos^2 x}$, or $\cos^3 x \sin^4 x$. Then split off a factor of $\cos x$, express the remaining powers in terms of $\sin x$, and make the substitution $u = \sin x$.

8.6.7 Let $u = \cos\theta$, which implies that $du = -\sin\theta\, d\theta$, or $-du = \sin\theta\, d\theta$. The lower limit of integration becomes $u = 1$ and the upper limit becomes $u = 0$. So we have

$$\int_0^{\pi/2} \frac{\sin\theta}{1+\cos^2\theta} d\theta = -\int_1^0 \frac{1}{1+u^2} = \int_0^1 \frac{1}{1+u^2} = \tan^{-1} u\Big|_0^1 = \frac{\pi}{4}.$$

8.6.8 Using the identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ with $\theta = 10x$, we have

$$\int \cos^2 10x\, dx = \int \frac{1+\cos 20x}{2} dx = \frac{x}{2} + \frac{\sin 20x}{40} + C.$$

8.6.9 After completing the square of $8x - x^2$, we make a substitution of $u = x - 4$. The lower limit of integration becomes $u = 0$ and the upper limit of integration becomes $u = 2$. Integrating, we have

$$\int_4^6 \frac{dx}{\sqrt{8x - x^2}} = \int_4^6 \frac{dx}{\sqrt{16 - (x-4)^2}} = \int_0^2 \frac{dx}{\sqrt{16 - u^2}} = \sin^{-1} \frac{u}{4}\Big|_0^2 = \frac{\pi}{6}.$$

8.6.10 Because $\sin x$ and $\cos x$ both have odd powers, we have a choice of either splitting off a factor $\sin x$ or a factor of $\cos x$. In this case, you can verify that it is easier to split off $\cos x$ by rewriting $\cos^3 x$ as $\cos^2 x \cos x$. Here is the integration process:

$$\begin{aligned}
\int \sin^9 x \cos^3 x\, dx &= \int \sin^9 x \cos^2 x \cos x\, dx && \text{Split off } \cos x. \\
&= \int \sin^9 x (1 - \sin^2 x) \cos x\, dx && \text{Pythagorean identity} \\
&= \int (\sin^9 x - \sin^{11} x) \cos x\, dx && \text{Expand.} \\
&= \int (u^9 - u^{11}) du && \text{Let } u = \sin x;\ du = \cos\ dx. \\
&= \frac{u^{10}}{10} - \frac{u^{12}}{12} + C && \text{Integrate.} \\
&= \frac{\sin^{10} x}{10} - \frac{\sin^{12} x}{12} + C. && \text{Replace } u \text{ with } \sin x.
\end{aligned}$$

8.6.11 The integral is evaluated after expanding the integrand.

$$\begin{aligned}
\int_0^{\pi/4} (\sec x - \cos x)^2\, dx &= \int_0^{\pi/4} (\sec^2 x - 2\underbrace{\sec x \cos x}_{1} + \cos^2 x)\, dx && \text{Expand integrand.} \\
&= \int_0^{\pi/4} \left(\sec^2 x - 2 + \frac{1+\cos 2x}{2}\right) dx && \cos^2 x = \frac{1+\cos 2x}{2} \\
&= \int_0^{\pi/4} \left(\sec^2 x - \frac{3}{2} + \frac{1}{2}\cos 2x\right) dx && \text{Simplify integrand.} \\
&= \left(\tan x - \frac{3}{2}x + \frac{1}{4}\sin 2x\right)\Bigg|_0^{\pi/4} = \frac{5}{4} - \frac{3\pi}{8} && \text{Evaluate.}
\end{aligned}$$

8.6.12 Letting $u = e^x$, it follows that $du = e^x dx$. So we have

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} e^x + C.$$

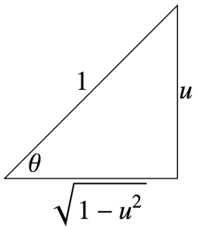
8.6.13 Let $u = e^x$, which implies that $du = e^x dx$ and $dx = \frac{du}{e^x} = \frac{du}{u}$. The integrand is then expressed in terms of u:

$$\int \frac{dx}{e^x\sqrt{1-e^{2x}}} = \int \frac{du}{u^2\sqrt{1-u^2}}.$$

The expression $1-u^2$ suggests the trigonometric substitution $u = \sin\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. It follows that $du = \cos\theta\,d\theta$ and

$$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta.$$

Using the equation $u = \sin\theta$, we create a reference triangle that helps us evaluate the integral.



Summarizing what we've done so far and then evaluating the integral, we have

$$\begin{aligned}
\int \frac{dx}{e^x\sqrt{1-e^{2x}}} &= \int \frac{du}{u^2\sqrt{1-u^2}} && u\text{-substitution}\\
&= \int \frac{\cos\theta}{\sin^2\theta\cos\theta}\,d\theta && \text{Trigonometric substitution}\\
&= \int \csc^2\theta\,d\theta && \text{Simplify.}\\
&= -\cot\theta + C && \text{Integrate.}\\
&= -\frac{\sqrt{1-u^2}}{u} + C && \text{Use the reference triangle.}\\
&= -\frac{\sqrt{1-e^{2x}}}{e^x} + C. && \text{Replace } u \text{ with } e^x.
\end{aligned}$$

8.6.14 We start by eliminating the negative powers in the integrand by multiplying the numerator and denominator by x^3. Then we split the integral into a sum of integrals:

$$\int \frac{x^{-2}+x^{-3}}{x^{-1}+16x^{-3}} \cdot \frac{x^3}{x^3}\,dx = \int \frac{x+1}{x^2+16}dx = \int \frac{x}{x^2+16}dx + \int \frac{1}{x^2+16}dx.$$

The second integral on the right equals $\frac{1}{4}\tan^{-1}\frac{x}{4} + C$. For the first integral on the right, we let $u = x^2+16$. It follows that $du = 2x\,dx$, or $\frac{1}{2}du = x\,dx$. Therefore

$$\int \frac{x}{x^2+16}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+16) + C.$$

Adding the integrals, we have

$$\int \frac{x^{-2}+x^{-3}}{x^{-1}+16x^{-3}}\,dx = \frac{1}{2}\ln(x^2+16) + \frac{1}{4}\tan^{-1}\frac{x}{4} + C.$$

8.6.15 We let $u = \sqrt{x}$, which implies that $du = \dfrac{dx}{2\sqrt{x}}$ or $2du = \dfrac{dx}{\sqrt{x}}$. The lower limit of integration is $u = 1$ and the upper limit is $u = 2$. Therefore

$$\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int_1^2 2^u du = \left.\frac{2\cdot 2^u}{\ln 2}\right|_1^2 = \frac{4}{\ln 2}.$$

8.6.16 The denominator factors as $x^4 - 1 = (x^2+1)(x^2-1) = (x^2+1)(x+1)(x-1)$. Therefore the appropriate form of the partial fraction decomposition is

$$\frac{1}{x^4-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}.$$

Multiplying both sides of the equation by $(x^2+1)(x+1)(x-1)$ leads to

$$1 = (Ax+B)(x+1)(x-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1).$$

We can use this equation to find the constants A, B, C, and D.

$$x = 1 \Rightarrow 1 = 4D \text{ or } D = \frac{1}{4}$$

$$x = -1 \Rightarrow 1 = -4C \text{ or } C = -\frac{1}{4}$$

$$x = 0 \Rightarrow 1 = -B + \frac{1}{4} + \frac{1}{4} \text{ or } B = -\frac{1}{2}$$

$$\text{Equating coefficients of } x^3 \Rightarrow 0 = A + C + D \Rightarrow A = 0$$

So the partial fraction decomposition is

$$\frac{1}{x^4-1} = -\frac{1}{2(x^2+1)} - \frac{1}{4(x+1)} + \frac{1}{4(x-1)}.$$

Using this decomposition, integration is straightforward:

$$\int \frac{dx}{x^4-1} = -\int \frac{dx}{2(x^2+1)} - \int \frac{dx}{4(x+1)} + \int \frac{dx}{4(x-1)} = -\frac{1}{2}\tan^{-1}x - \frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + C.$$

8.6.17 Let's evaluate the corresponding indefinite integral first. Let $z = w^2$ which implies that $dz = 2w\,dw$, or $\dfrac{dz}{2} = w\,dw$. We then have

$$\int w^3 e^{w^2}\,dw = \frac{1}{2}\int ze^z\,dz.$$

For the integral on the right, use integration by parts with the following choices.

$$u = z \qquad dv = e^z\,dz$$

$$du = dz \quad v = e^z$$

Integrating by parts, we have

$$\int w^3 e^{w^2} dw = \frac{1}{2}\int ze^z dz = \frac{1}{2}\left(ze^z - \int e^z\,dz\right) = \frac{e^z}{2}(z-1) + C$$

After replacing z with w^2, we evaluate the definite integral:

$$\int_1^2 w^3 e^{w^2} dx = \left.\left(\frac{1}{2}e^{w^2}(w^2-1)\right)\right|_1^2 = \frac{3e^4}{2}.$$

8.6.18 Letting $u = x - 5$, it follows that $du = dx$ and $x = u + 5$. The lower limit of integration is $u = 0$ and the upper limit is $u = 1$.

$$\int_5^6 x(x-5)^{10}\,dx = \int_0^1 (u+5)u^{10}\,du = \int_0^1 (u^{11} + 5u^{10})\,du = \left(\frac{u^{12}}{12} + \frac{5u^{11}}{11}\right)\Bigg|_0^1 = \frac{71}{132}.$$

8.6.19 The integral has an odd power of $\sin x$, so we start by splitting off a factor of $\sin x$.

$$\begin{aligned}
\int_0^{\pi/2} \sin^7 x\,dx &= \int_0^{\pi/2} (\sin^2 x)^3 \sin x\,dx && \text{Split off } \sin x. \\
&= \int_0^{\pi/2} (1-\cos^2 x)^3 \sin x\,dx && \sin^2 x = 1 - \cos^2 x \\
&= \int_1^0 -(1-u^2)^3\,du && \text{Let } u = \cos x; du = -\sin x\,dx. \\
& && x = 0 \Rightarrow u = 1; x = \frac{\pi}{2} \Rightarrow u = 0 \\
&= \int_0^1 (-u^6 + 3u^4 - 3u^2 + 1)\,du && \text{Expand.} \\
&= \left(-\frac{u^7}{7} + \frac{3}{5}u^5 - u^3 + u\right)\Bigg|_0^1 && \text{Integrate.} \\
&= \frac{16}{35}. && \text{Evaluate.}
\end{aligned}$$

8.6.20 Let $u = \sqrt{t}$ which implies that $du = \frac{dt}{2\sqrt{t}}$, $t = u^2$, and $\dfrac{dt}{\sqrt{t}} = 2du$. The lower limit of integration becomes $u = 1$ and the upper limit of integration becomes $u = \sqrt{3}$. Changing variables, we have

$$\int_1^3 \frac{dt}{\sqrt{t}(t+1)} = 2\int_1^{\sqrt{3}} \frac{du}{u^2+1} = 2\tan^{-1} u\Big|_1^{\sqrt{3}} = 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}.$$

8.6.21 We use integration by parts with the following choices.

$$u = \ln 3x \qquad dv = x^9\,dx$$

$$du = \frac{1}{x}dx \qquad v = \tfrac{x^{10}}{10}$$

Integrating by parts, we have

$$\int x^9 \ln 3x\,dx = \frac{x^{10}}{10}\ln 3x - \frac{1}{10}\int x^9\,dx = \frac{x^{10}}{10}\ln 3x - \frac{x^{10}}{100} + C.$$

8.6.22 The denominator consists of two distinct linear terms $x - a$ and $x - b$ because $a \neq b$. An appropriate form of the partial fraction decomposition is

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}.$$

Multiplying both sides of this equation by $(x-a)(x-b)$ gives us

$$1 = A(x-b) + B(x-a).$$

Choosing $x = b$, we have $1 = B(b-a)$ or $B = 1/(b-a)$ and if $x = a$, $1 = A(a-b)$ or $A = 1/(a-b)$. Substituting these values for A and B in the partial fraction decomposition, we carry out integration.

$$\begin{aligned}\int \frac{dx}{(x-a)(x-b)} &= \frac{1}{a-b}\int \frac{dx}{x-a} + \frac{1}{b-a}\int \frac{dx}{x-b} && \text{Partial fractions}\\ &= \frac{1}{a-b}\ln|x-a| + \frac{1}{b-a}\ln|x-b| + C && \text{Integrate.}\\ &= \frac{\ln|x-a| - \ln|x-b|}{a-b} + C. && \text{Simplify.}\end{aligned}$$

8.6.23 Letting $u = \cos x$ and $du = -\sin x\,dx$, or $-du = \sin x\,dx$, we have

$$\int \frac{\sin x}{\cos^2 x + \cos x}dx = -\int \frac{du}{u^2+u} = -\int \frac{du}{u(u+1)}.$$

To evaluate the integral on the right, we first find the constants A and B in the partial fraction decomposition

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}.$$

Multiplying both sides of this equation by $u(u+1)$, we have

$$1 = A(u+1) + Bu = (A+B)u + A.$$

It follows that $A = 1$ and $B = -1$. Using these constants in the partial fraction decomposition, integration is straightforward.

$$\begin{aligned}\int \frac{\sin x}{\cos^2 x + \cos x}dx &= -\int \frac{du}{u(u+1)} && u = \cos x;\ du = -\sin x\,dx\\ &= -\int \left(\frac{1}{u} - \frac{1}{u+1}\right)du && \text{Partial fractions}\\ &= -(\ln|u| - \ln|u+1|) + C && \text{Integrate.}\\ &= \ln(\cos x + 1) - \ln|\cos x| + C. && \text{Replace } u \text{ with } \cos x.\end{aligned}$$

8.6.24 The degree of the polynomial in the numerator is greater the the degree of the polynomial in the denominator, so we perform long division to obtain

$$\frac{3w^5 + 2w^4 - 12w^3 - 12w - 32}{w^3 - 4w} = 3w^2 + 2w + \frac{8w^2 - 12w - 32}{w^3 - 4w}.$$

The first part of the right side of the equation is easy to integrate, so we focus on the fractional part. Factoring the denominator and then setting up the partial fraction decomposition, we have

$$\frac{8w^2 - 12w - 32}{w(w+2)(w-2)} = \frac{A}{w} + \frac{B}{w+2} + \frac{C}{w-2}.$$

We multiply both sides by $w(w+2)(w-2)$:

$$8w^2 - 12w - 32 = A(w+2)(w-2) + Bw(w-2) + Cw(w+2).$$

Now we find the values of unknown constants:

$$w = 0 \Rightarrow -32 = -4A \text{ or } A = 8$$

$$w = -2 \Rightarrow 24 = 8B \text{ or } B = 3$$

$$w = 2 \Rightarrow -24 = 8C \text{ or } C = -3$$

Substituting these values into decomposition and evaluating the integral, we have

$$\int \frac{3w^5 + 2w^4 - 12w^3 - 12w - 32}{w^3 - 4w} dw = \int \left(3w^2 + 2w + \frac{8}{w} + \frac{3}{w+2} - \frac{3}{w-2}\right) dw$$

$$= w^3 + w^2 + 8\ln|w| + 3\ln|w+2| - 3\ln|w-2| + C.$$

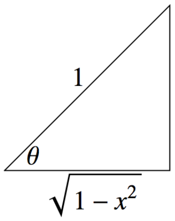
8.6.25 One approach for evaluating this integral is to use a trigonometric substitution. The expression $1-x^2$ suggests the substitution $x = \sin\theta$ with $-\pi/2 < \theta < \pi/2$. With this substitution, $dx = \cos\theta\, d\theta$ and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta.$$

Substituting these values into the integral and simplifying, we have

$$\int \frac{dx}{x\sqrt{1-x^2}} = \int \csc\theta\, d\theta = -\ln|\csc\theta + \cot\theta| + C.$$

Given that $\sin\theta = x$, a geometric description of the relationship between x and θ is given by the following reference triangle.



With the help of the reference triangle, we continue:

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\ln|\csc\theta + \cot\theta| + C = -\ln\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| + C = \ln\left|\frac{x}{1+\sqrt{1-x^2}}\right| + C.$$

8.6.26 Let $u = \ln x$ which implies that $du = \dfrac{dx}{x}$. Using this substitution, the lower limit of integration is $u = -1$ and the upper limit of integration is $u = 0$. After making these substitutions and then completing the square, we evaluate the integral.

$$\int_{1/e}^{1} \frac{dx}{x(\ln^2 x + 2\ln x + 2)} = \int_{-1}^{0} \frac{du}{u^2 + 2u + 2} = \int_{-1}^{0} \frac{du}{(u+1)^2 + 1} = \tan^{-1}(u+1)\Big|_{-1}^{0} = \frac{\pi}{4}$$

8.6.27 Because we have an even power of $\sin\frac{x}{2}$, we use the half-angle formula for $\sin^2\dfrac{x}{2}$ to rewrite the integrand:

$$\int \sin^4\frac{x}{2}\, dx = \int \Bigg(\underbrace{\frac{1-\cos x}{2}}_{\sin^2 \frac{x}{2}}\Bigg)^2 dx = \frac{1}{4}\int \left(1 - 2\cos x + \cos^2 x\right) dx.$$

Using the half-angle formula for $\cos^2 x$, the evaluation may be completed:

$$\int \sin^4 \frac{x}{2}\,dx = \frac{1}{4}\int \big(1 - 2\cos x + \underbrace{\frac{1+\cos 2x}{2}}_{\cos^2 x}\big)\,dx = \frac{1}{4}\int \left(\frac{3}{2} - 2\cos x + \frac{\cos 2x}{2}\right)dx$$

$$= \frac{3x}{8} - \frac{1}{2}\sin x + \frac{1}{16}\sin 2x + C.$$

8.6.28 The denominator factors as $(x^2+1)^2$, which is repeated irreducible quadratic factor. So the form of the partial fraction decomposition is

$$\frac{3x^2+2x+3}{x^4+2x^2+1} = \frac{3x^2+2x+3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}.$$

Multiplying by $(x^2+1)^2$, we have

$$3x^2+2x+3 = (Ax+B)(x^2+1) + Cx + D = Ax^3 + Bx^2 + (A+C)x + B + D.$$

Comparing coefficients of equal powers, starting with the highest power, we find that $A = 0$, $B = 3$, $C = 2$, and $D = 0$. Now we use the partial fraction decomposition, with these constant values, to integrate:

$$\int \frac{3x^2+2x+3}{x^4+2x^2+1}dx = \int \left(\frac{3}{x^2+1} + \frac{2x}{(x^2+1)^2}\right)dx \qquad \text{Partial fractions}$$

$$= \int \frac{3}{x^2+1}dx + \int \frac{2x}{(x^2+1)^2}dx \qquad \text{Split into 2 integrals.}$$

$$= 3\tan^{-1} x + \int \frac{du}{u^2} \qquad \text{Evaluate first integral;}$$

Second integral: $u = x^2+1$, $du = 2x\,dx$

$$= 3\tan^{-1} x - \frac{1}{x^2+1} + C. \qquad \text{Evaluate and replace } u \text{ with } x^2+1.$$

8.6.29 Before integrating, let's write $\cot x$ in terms of $\sin x$ and $\cos x$ and then simplify the integrand by multiplying the numerator and denominator by $\sin x$.

$$\frac{2\cos x + \overbrace{\frac{\cos x}{\sin x}}^{\cot x}}{1+\sin x} = \frac{2\sin x\cos x + \cos x}{\sin x + \sin^2 x}.$$

The integral is evaluated with the help of a u-substitution.

$$\int \frac{2\cos x + \cot x}{1+\sin x}\,dx = \int \frac{2\sin x\cos x + \cos x}{\sin x + \sin^2 x}dx \qquad \text{Write in terms of } \sin x \text{ and } \cos x\text{; simplify.}$$

$$= \int \frac{du}{u} \qquad \text{Let } u = \sin x + \sin^2 x;\ du = (\cos x + 2\sin x\cos x)\,dx.$$

$$= \ln|u| + C \qquad \text{Integrate.}$$

$$= \ln|\sin x + \sin^2 x| + C. \qquad \text{Substitute } \sin x + \sin^2 x \text{ for } u.$$

8.6.30 Using the trigonometric substitution $v = 5\sin\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, it follows that $dv = 5\cos\theta\,d\theta$ and

$$\sqrt{25-v^2} = \sqrt{25 - 25\sin^2\theta} = |5\cos\theta| = 5\cos\theta.$$

The lower limit $v = \frac{5}{2}$ implies that $\sin\theta = \frac{1}{2}$ and it follows that $\theta = \frac{\pi}{6}$. Similarly, if $v = \frac{5\sqrt{3}}{2}$ then $\sin\theta = \frac{\sqrt{3}}{2}$ and so $\theta = \frac{\pi}{3}$.

$$\int_{5/2}^{5\sqrt{3}/2} \frac{dv}{v^2\sqrt{25-v^2}} = \int_{\pi/6}^{\pi/3} \frac{5\cos\theta}{(25\sin^2\theta)(5\cos\theta)}d\theta = \frac{1}{25}\int_{\pi/6}^{\pi/3} \csc^2\theta\,d\theta = -\frac{1}{25}\cot\theta\Big|_{\pi/6}^{\pi/3}$$
$$= -\frac{1}{25}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) = \frac{2}{25\sqrt{3}}.$$

8.6.31 Removing a factor of 3 from the square root, we have

$$\int \sqrt{36-9x^2}\,dx = \int \sqrt{9(4-x^2)}\,dx = 3\int \sqrt{4-x^2}\,dx.$$

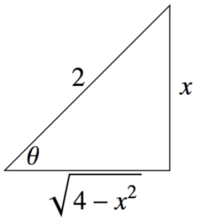
The expression $4-x^2$ suggests the substitution $x = 2\sin\theta$, for $\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. It follows that $dx = 2\cos\theta\,d\theta$ and

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = 2|\cos\theta| = 2\cos\theta.$$

Making these substitutions, we have

$$\begin{aligned}
\int \sqrt{36-9x^2}dx &= 3\int \sqrt{4-x^2}\,dx && \text{Factor.}\\
&= 12\int \cos^2\theta\,d\theta && \text{Trigonometric substitution}\\
&= 6\int (1+\cos 2\theta)\,d\theta && \cos^2\theta = \frac{1+\cos 2\theta}{2}\\
&= 6\theta + 3\sin 2\theta + C && \text{Integrate.}\\
&= 6\theta + 6\sin\theta\cos\theta + C. && \sin 2\theta = 2\sin\theta\cos\theta
\end{aligned}$$

From the equation $\sin\theta = \frac{x}{2}$ we have $\theta = \sin^{-1}\dfrac{x}{2}$. We use the following reference triangle to observe that $\cos\theta = \frac{\sqrt{4-x^2}}{2}$.



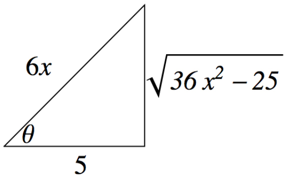
Therefore

$$\int \sqrt{36-9x^2}dx = \underbrace{6\sin^{-1}\frac{x}{2}}_{\theta} + 6\underbrace{\left(\frac{x}{2}\right)}_{\sin\theta}\underbrace{\left(\frac{\sqrt{4-x^2}}{2}\right)}_{\cos\theta} + C = 6\sin^{-1}\frac{x}{2} + \frac{3}{2}x\sqrt{4-x^2} + C.$$

8.6.32 We factor a 6 out of the square root, then proceed with a trigonometric substitution of $x = \frac{5}{6}\sec\theta$, where $0 < \theta < \frac{\pi}{2}$.

$$\begin{aligned}
\int \frac{dx}{\sqrt{36x^2-25}} &= \int \frac{dx}{6\sqrt{x^2-\frac{25}{36}}} && \text{Factor.}\\
&= \int \frac{\frac{5}{6}\sec\theta\tan\theta}{6\sqrt{\frac{25}{36}\sec^2\theta-\frac{25}{36}}}d\theta && \text{Trigonometric substitution}\\
&= \int \frac{\frac{5}{6}\sec\theta\tan\theta}{5\tan\theta}d\theta && \sqrt{\frac{25}{36}\sec^2\theta-\frac{25}{36}}=\frac{5}{6}|\tan\theta|=\frac{5}{6}\tan\theta\\
&= \frac{1}{6}\int \sec\theta\,d\theta && \text{Simplify.}\\
&= \frac{1}{6}\ln(\sec\theta+\tan\theta)+C && \text{Integrate.}
\end{aligned}$$

From our trigonometric substitution, we know that $\sec\theta = \frac{6x}{5}$ which is represented by the following reference triangle.



Using this triangle, we complete the integral:

$$\begin{aligned}
\int \frac{dx}{\sqrt{36x^2-25}} &= \frac{1}{6}\ln(\sec\theta+\tan\theta)+C\\
&= \frac{1}{6}\ln\left(\frac{6x}{5}+\frac{\sqrt{36x^2-25}}{5}\right)+C\\
&= \frac{1}{6}\ln\left(6x+\sqrt{36x^2-25}\right)-\frac{\ln 5}{6}+C\\
&= \frac{1}{6}\ln\left(6x+\sqrt{36x^2-25}\right)+C.
\end{aligned}$$

8.6.33 Letting $u=e^x$, we have $e^{2x}=(e^x)^2=u^2$ and $du=e^x dx$. Provided that $a\neq 0$,

$$\int \frac{e^x}{a^2+e^{2x}}dx = \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}+C \;=\frac{1}{a}\tan^{-1}\frac{e^x}{a}+C.$$

8.6.34 Let $u=\cos 3x+1$ which implies that $du=-3\sin 3x\,dx$, or $-\frac{1}{3}du=\sin 3x\,dx$.

$$\int_0^{\pi/9}\frac{\sin 3x}{\cos 3x+1}\,dx = -\frac{1}{3}\int_2^{3/2}\frac{du}{u} = -\frac{1}{3}\Big(\ln u\Big)\Big|_2^{3/2} = \frac{\ln 2-\ln\frac{3}{2}}{3} = \frac{1}{3}\ln\frac{4}{3}.$$

8.6.35 Let $u=\tan\theta$ which implies that $du=\sec^2\theta\,d\theta$. The lower limit of integration becomes $u=\tan 0=0$ and the upper limit becomes $u=\tan\pi/4=1$. Changing variables, we have

$$\int_0^{\pi/4}(\tan^2\theta+\tan\theta+1)\sec^2\theta\,d\theta = \int_0^1(u^2+u+1)du = \left(\frac{u^3}{3}+\frac{u^2}{2}+u\right)\Big|_0^1 = \frac{1}{3}+\frac{1}{2}+1=\frac{11}{6}$$

8.6.36 We apply integration by parts with the following choices for our variables.

$$\begin{aligned} u &= x & dv &= 10^x\,dx \\ du &= dx & v &= \tfrac{10^x}{\ln 10} \end{aligned}$$

We then have

$$\int x\cdot 10^x\,dx = \frac{x\cdot 10^x}{\ln 10} - \int \frac{10^x}{\ln 10}dx = \frac{x\cdot 10^x}{\ln 10} - \frac{10^x}{\ln^2 10} + C = \frac{10^x}{\ln 10}\left(x - \frac{1}{\ln 10}\right) + C.$$

8.6.37 We start by multiplying the integrand by $\dfrac{1+\sin 2x}{1+\sin 2x}$.

$$\begin{aligned} \int_0^{\pi/6} \frac{1}{1-\sin 2x}\cdot\frac{1+\sin 2x}{1+\sin 2x}\,dx &= \int_0^{\pi/6} \frac{1+\sin 2x}{1-\sin^2 2x}\,dx && \text{Simplify integrand.} \\ &= \int_0^{\pi/6} \frac{1+\sin 2x}{\cos^2 2x}\,dx && 1-\sin^2 2x = \cos^2 2x. \\ &= \int_0^{\pi/6} \big(\sec^2 2x + \sec 2x\tan 2x\big)\,dx && \text{Simplify integrand.} \\ &= \frac{1}{2}\big(\tan 2x + \sec 2x\big)\Big|_0^{\pi/6} && \text{Integrate.} \\ &= \frac{1}{2}\big(\sqrt{3}+2-1\big) = \frac{\sqrt{3}+1}{2}. && \text{Evaluate.} \end{aligned}$$

8.6.38 One approach for evaluating $\displaystyle\int_{\pi/6}^{\pi/2} \cos x\ln(\sin x)\,dx$ is to use a u-substitution by letting $u = \sin x$ and to then apply the integral formula for $\ln u$. Another approach that we use here is to use integration by parts with the following variable choices.

$$u = \ln(\sin x) \qquad dv = \cos x\,dx$$

$$du = \frac{\cos x}{\sin x}dx \qquad v = \sin x$$

Integrating by parts, we have

$$\int_{\pi/6}^{\pi/2} \cos x\ln(\sin x)\,dx = \sin x\ln(\sin x)\Big|_{\pi/6}^{\pi/2} - \int_{\pi/6}^{\pi/2} \cos x\,dx = -\frac{1}{2}\ln\frac{1}{2} - \sin x\Big|_{\pi/6}^{\pi/2}$$

$$= -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2} = \frac{\ln 2 - 1}{2}.$$

8.6.39 We integrate by parts with the following variable choices.

$$u = \ln(\sin x) \qquad dv = \sin x\,dx$$

$$du = \frac{\cos x}{\sin x}dx \qquad v = -\cos x$$

It follows that

$$\begin{aligned}
\int \sin x \ln(\sin x)\,dx &= -\cos x \ln(\sin x) + \int \frac{\cos^2 x}{\sin x} && \text{Integration by parts}\\
&= -\cos x \ln(\sin x) + \int \frac{1-\sin^2 x}{\sin x}\,dx && \cos^2 x = 1-\sin^2 x\\
&= -\cos x \ln(\sin x) + \int \Big(\csc x - \sin x\Big)\,dx && \text{Simplify integrand.}\\
&= -\cos x \ln(\sin x) - \ln|\csc x + \cot x| + \cos x + C. && \text{Evaluate.}
\end{aligned}$$

8.6.40 We integrate by parts with the following variable choices.

$$u = \ln(\sin x) \qquad dv = \sin 2x\,dx$$

$$du = \frac{\cos x}{\sin x}dx \quad v = -\tfrac{1}{2}\cos 2x$$

Then we have

$$\begin{aligned}
\int \sin 2x \ln(\sin x)\,dx &= -\frac{1}{2}\cos 2x \ln(\sin x) + \int \frac{\cos x \cos 2x}{2\sin x}dx && \text{Integration by parts}\\
&= -\frac{1}{2}\cos 2x \ln(\sin x) + \int \frac{\cos x(1-2\sin^2 x)}{2\sin x}dx && 1-2\sin^2 x = \cos 2x\\
&= -\frac{1}{2}\cos 2x \ln(\sin x) + \frac{1}{2}\int \cot x\ dx - \int \sin x \cos x dx && \text{Expand the integral.}\\
&= -\frac{1}{2}\cos 2x \ln(\sin x) + \frac{1}{2}\int \cot x\ dx - \int \frac{1}{2}\sin 2x dx && \sin x\cos x = \frac{1}{2}\sin 2x\\
&= -\frac{1}{2}\cos 2x \ln(\sin x) + \frac{1}{2}\ln|\sin x| + \frac{1}{4}\cos 2x + C. && \text{Evaluate.}
\end{aligned}$$

8.6.41 Split $\csc^4 x$ and use the identity $\csc^2 x = 1 + \cot^2 x$:

$$\begin{aligned}
\int \cot^{3/2} x \csc^4 x\,dx &= \int \cot^{3/2} x \csc^2 x \csc^2 x\,dx && \csc^4 x = \csc^2 x\csc^2 x\\
&= \int \cot^{3/2} x(1+\cot^2 x)\csc^2 x\,dx && \csc^2 x = 1+\cot^2 x\\
&= -\int u^{3/2}(1+u^2)\,du && u = \cot x,\ du = -\csc^2 x\,dx\\
&= -\int \left(u^{3/2}+u^{7/2}\right)du && \text{Expand integrand.}\\
&= -\frac{2}{5}u^{5/2} - \frac{2}{9}u^{9/2} + C && \text{Evaluate.}\\
&= -\frac{2}{5}\cot^{5/2} x - \frac{2}{9}\cot^{9/2} x + C. && \text{Replace } u \text{ with } \cot x.
\end{aligned}$$

8.6.42 Let $u = \sin^{-1} x$, which implies that $du = \dfrac{dx}{\sqrt{1-x^2}}$. We also change the limits of integration.

$$x = 0 \Rightarrow u = \sin^{-1} 0 = 0$$
$$x = \frac{1}{2} \Rightarrow u = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

Integrate:

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} u\,du = \frac{u^2}{2}\bigg|_0^{\pi/6} = \frac{\pi^2}{72}.$$

8.6.43 The integral $\int \frac{x^9}{\sqrt{1-x^{20}}}\,dx$ can be expressed as $\int \frac{x^9}{\sqrt{1-\left(x^{10}\right)^2}}\,dx$, so we let $u = x^{10}$ which implies that $du = 10x^9 dx$, or $x^9 dx = \frac{du}{10}$. Therefore

$$\int \frac{x^9}{\sqrt{1-\left(x^{10}\right)^2}}\,dx = \frac{1}{10}\int \frac{du}{\sqrt{1-u^2}} = \frac{\sin^{-1} u}{10} + C = \frac{\sin^{-1} x^{10}}{10} + C.$$

8.6.44 Factoring $x^3 - x^2$, an appropriate form of the partial fraction decomposition is

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

Multiplying both sides by $x^2(x-1)$ leads to

$$1 = Ax(x-1) + B(x-1) + Cx^2 = (A+C)x^2 + (B-A)x - B.$$

Equating coefficients of equal powers of x, we find that $A = -1$, $B = -1$, and $C = 1$. Therefore

$$\int \frac{dx}{x^3-x^2} = -\int \frac{1}{x}dx - \int \frac{dx}{x^2} + \int \frac{dx}{x-1} = -\ln|x| + \frac{1}{x} + \ln|x-1| + C.$$

8.6.45 Letting $u = 1 + e^x$, it follows that $du = e^x\,dx$, $e^x = u - 1$, and $dx = \frac{du}{e^x} = \frac{du}{u-1}$. The lower limit of integration is $u = 2$ and the upper limit is $u = 3$. We have

$$\int_0^{\ln 2} \frac{dx}{(1+e^x)^2} = \int_2^3 \frac{du}{u^2(u-1)}.$$

Partial fractions is used to evaluate the integral on the right. An appropriate form for the partial fraction decomposition of the integrand is

$$\frac{1}{u^2(u-1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1}.$$

Multiplying both sides by $u^2(u-1)$ leads to

$$1 = Au(u-1) + B(u-1) + Cu^2 = (A+C)u^2 + (B-A)u - B.$$

Equating coefficients of equal powers of u, we find that $A = -1$, $B = -1$, and $C = 1$. Therefore

$$\begin{aligned}\int_0^{\ln 2} \frac{dx}{(1+e^x)^2} = \int_2^3 \frac{du}{u^2(u-1)} = \int_2^3 \left(-\frac{1}{u} - \frac{1}{u^2} + \frac{1}{u-1}\right)du &= \left(-\ln|u| + \frac{1}{u} + \ln|u-1|\right)\bigg|_2^3 \\ &= \ln\frac{2}{3} - \frac{1}{6} + \ln 2 = \ln\frac{4}{3} - \frac{1}{6}.\end{aligned}$$

8.6.46 Letting $u = e^x$, we have $du = e^x\,dx$, $dx = \frac{du}{e^x} = \frac{du}{u}$, and $\int \frac{dx}{e^{2x}+1} = \int \frac{du}{u(u^2+1)}$. The form of the partial fraction decomposition of $\frac{1}{u(u^2+1)}$ is

$$\frac{1}{u(u^2+1)} = \frac{A}{u} + \frac{Bu+C}{u^2+1}.$$

Multiplying both sides by $u(u^2+1)$, we have

$$1 = A(u^2+1) + (Bu+C)u = (A+B)u^2 + Cu + A.$$

Comparing coefficients of equal powers, we find that $A = 1$, $B = -1$, and $C = 0$.

Inserting these constants into our partial fraction decomposition implies that

$$\int \frac{du}{u(u^2+1)} = \int \frac{du}{u} - \int \frac{u}{u^2+1} du.$$

In the second integral, we let $v = u^2+1$. Then $dv = 2u\,du$, or $\frac{1}{2}dv = u\,du$. It follows that

$$\int \frac{u}{u^2+1} du = \frac{1}{2}\int \frac{dv}{v} = \frac{1}{2}\ln|v| + C = \frac{1}{2}\ln(u^2+1) + C.$$

Assembling all our pieces, we have

$$\begin{aligned}\int \frac{dx}{e^{2x}+1} &= \int \frac{du}{u} - \int \frac{u}{u^2+1} du\\ &= \ln|u| - \frac{1}{2}\ln(u^2+1) + C\\ &= \ln e^x - \frac{1}{2}\ln(e^{2x}+1) + C\\ &= x - \frac{1}{2}\ln(e^{2x}+1) + C.\end{aligned}$$

8.6.47 Apply long division to obtain

$$\frac{2x^3+x^2-2x-4}{x^2-x-2} = 2x+3+\frac{5x+2}{x^2-x-2}.$$

Factor x^2-x-2 and then find the partial fraction decomposition of $\dfrac{5x+2}{x^2-x-2}$.

$$\frac{5x+2}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

Multiply both sides by $(x-2)(x+1)$.

$$5x+2 = A(x+1) + B(x-2).$$

Note that $x = 2$ implies that $A = 4$ and $x = -1$ implies that $B = 1$. Therefore

$$\begin{aligned}\int \Big(\frac{2x^3+x^2-2x-4}{x^2-x-2}\Big)dx &= \int \Big(2x+3+\frac{4}{x-2}+\frac{1}{x+1}\Big)dx\\ &= x^2+3x+4\ln|x-2| + \ln|x+1| + C.\end{aligned}$$

.

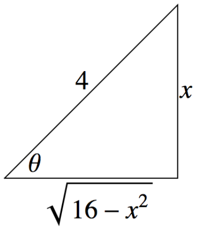
8.6.48 Let $x = 4\sin\theta$, for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$. This substitution implies that $dx = 4\cos\theta\,d\theta$ and

$$\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16\cos^2\theta} = |4\cos\theta| = 4\cos\theta.$$

With these substitutions, we have

$$\begin{aligned}\int \frac{\sqrt{16-x^2}}{x^2}\,dx &= \int \frac{4\cos\theta}{16\sin^2\theta}\,4\cos\theta\,d\theta\\ &= \int \cot^2\theta\,d\theta\\ &= \int (\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C.\end{aligned}$$

Given that $\sin\theta = \frac{x}{4}$, we can form the following right triangle that helps us evaluate the integral.



Therefore

$$\int \frac{\sqrt{16-x^2}}{x^2}\,dx = -\cot\theta - \theta + C = -\frac{\sqrt{16-x^2}}{x} - \sin^{-1}\frac{x}{4} + C.$$

8.6.49 We start by splitting off factors of $\tan x$ and $\sec x$ and then we replace $\tan^2 x$ with $\sec^2 x - 1$:

$$\begin{aligned}\int \tan^3 x \sec^9 x\,dx &= \int \tan^2 x \sec^8 x \sec x \tan x\,dx && \text{Split off } \tan x \text{ and } \sec x.\\ &= \int (\sec^2 x - 1)\sec^8 x \sec x \tan x\,dx && \tan^2 x = \sec^2 x - 1\\ &= \int (\sec^{10} x - \sec^8 x)\sec x \tan x\,dx && \text{Expand integrand.}\\ &= \int (u^{10} - u^8)du && u = \sec x,\ du = \sec x \tan x\,dx\\ &= \frac{u^{11}}{11} - \frac{u^9}{9} + C && \text{Integrate.}\\ &= \frac{\sec^{11} x}{11} - \frac{\sec^9 x}{9} + C. && \text{Replace } u \text{ with } \sec x.\end{aligned}$$

8.6.50 With an even power of $\sec x$, we split off a factor of $\sec^2 x$ and prepare the integral for a substitution of $u = \tan x$:

$$\begin{aligned}
\int \tan^7 x \sec^4 x\,dx &= \int \tan^7 x \sec^2 x \sec^2 x\,dx && \text{Split off } \sec^2 x.\\
&= \int \tan^7 x(\tan^2 x + 1)\sec^2 x\,dx && \sec^2 x = \tan^2 x + 1\\
&= \int u^7(u^2+1)\,du && u = \tan x,\ du = \sec^2 x\,dx\\
&= \int (u^9 + u^7)\,du && \text{Expand.}\\
&= \frac{u^{10}}{10} + \frac{u^8}{8} + C && \text{Integrate.}\\
&= \frac{\tan^{10} x}{10} + \frac{\tan^8 x}{8} + C. && \text{Replace } u \text{ with } \tan x.
\end{aligned}$$

8.6.51 Rewrite $\sec^{7/4} x$ as $\sec^{3/4} x \sec x$ and let $u = \sec x$. Then $du = \sec x \tan$ and after making this substitution, the lower limit of integration is $u = \sec 0 = 1$ and the upper limit is $u = \sec \frac{\pi}{3} = 2$. Integrating, we have

$$\int_0^{\pi/3} \tan x \sec^{7/4} x\,dx = \int_0^{\pi/3} \sec^{3/4} x \sec x \tan x\,dx = \int_1^2 u^{3/4} du = \frac{4}{7}u^{7/4}\Big|_1^2 = \frac{4}{7}\big(2^{7/4} - 1\big).$$

8.6.52 We apply the method of integration by parts using the following choices.

$$\begin{aligned} u &= t^2 & dv &= e^{3t}\,dt\\ du &= 2t\,dt & v &= \tfrac{e^{3t}}{3}\end{aligned}$$

The integral then becomes

$$\int t^2 e^{3t}\,dt = \frac{t^2e^{3t}}{3} - \frac{2}{3}\int te^{3t}dt.$$

Now we apply integration by parts again to the integral $\int te^{3t}dt$ with the following choices.

$$\begin{aligned} u &= t & dv &= e^{3t}dt\\ du &= dt & v &= \tfrac{e^{3t}}{3}\end{aligned}$$

We then have

$$\begin{aligned}
\frac{t^2e^{3t}}{3} - \frac{2}{3}\int te^{3t}dt &= \frac{t^2e^{3t}}{3} - \frac{2}{3}\left(\frac{te^{3t}}{3} - \frac{1}{3}\int e^{3t}dt\right)\\
&= \frac{t^2e^{3t}}{3} - \frac{2te^{3t}}{9} + \frac{2e^{3t}}{27} + C\\
&= e^{3t}\left(\frac{t^2}{3} - \frac{2t}{9} + \frac{2}{27}\right) + C.
\end{aligned}$$

8.6.53 We begin with the substitution $u = e^x$, $du = e^x\,dx$, and then split apart the integrand.

$$\int e^x \cot^3 e^x\,dx = \int \cot^3 u\,du = \int \cot^2 u \cot u\,du = \int (\csc^2 u - 1)\cot u\,du = \int \csc^2 u \cot u\,du - \int \cot u\,du$$

Using the substitution $v = \cot u$ in the first integral with $dv = -\csc^2 u\,du$ or $-dv = \csc^2 u\,du$, we find that

$$\int \csc^2 u \cot u\,du = -\int v\,dv = -\frac{v^2}{2} + C = -\frac{\cot^2 u}{2} + C.$$

Therefore

$$\int e^x \cot^3 e^x\,dx = \int \csc^2 u \cot u\,du - \int \cot u\,du = -\frac{\cot^2 u}{2} - \ln|\sin u| + C = -\frac{\cot^2 e^x}{2} - \ln|\sin e^x| + C.$$

8.6.54 The denominator of the integrand consists of a linear function and an irreducible quadratic factor. The form of the partial fraction decomposition is

$$\frac{2x^2+3x+26}{(x-2)(x^2+16)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+16}.$$

Multiplying both sides of this equation by $(x-2)(x^2+16)$ leads to

$$2x^2+3x+26 = A(x^2+16) + (Bx+C)(x-2) = (A+B)x^2 + (C-2B)x + (16A-2C)$$

Equating coefficients of equal powers of x results in the equations

$$A+B=2, C-2B=3, \text{ and } 16A-2C=26.$$

The solution to this system of equations is $A = 2, B = 0,$ and $C = 3$. Therefore

$$\int \frac{2x^2+3x+26}{(x-2)(x^2+16)}\,dx = \int \frac{2}{x-2}dx + \int \frac{3}{x^2+16}dx = 2\ln|x-2| + \frac{3}{4}\tan^{-1}\frac{x}{4} + C.$$

8.6.55 The denominator factors as $x^3 + x = x(x^2+1)$ and therefore the form of the partial fraction decomposition is

$$\frac{3x^2+3x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Multiplying both sides by $x(x^2+1)$ leads to

$$3x^2+3x+1 = A(x^2+1) + (Bx+C)x = (A+B)x^2 + Cx + A.$$

By equating coefficients of equal powers of x, we find that $A = 1$, $B = 2$, and $C = 3$. The original integral can be written as

$$\int \frac{3x^2+3x+1}{x^3+x}\,dx = \int \frac{dx}{x} + \int \frac{2x+3}{x^2+1}dx = \int \frac{dx}{x} + \int \frac{2x}{x^2+1}dx + \int \frac{3}{x^2+1}dx.$$

Using a substitution if $u = x^2 + 1$ in the second integral with $du = 2x\,dx$, we find that $\int \frac{2x}{x^2+1}dx = \ln(x^2+1) + C$. Completing the integration process we have

$$\int \frac{3x^2+3x+1}{x^3+x}\,dx = \int \frac{dx}{x} + \int \frac{2x}{x^2+1}dx + \int \frac{3}{x^2+1}dx = \ln|x| + \ln(x^2+1) + 3\tan^{-1}x + C.$$

8.6.56 Using the trigonometric identity $\sin 2x = 2\sin x\cos x$ the integral equals $\int_\pi^{3\pi/2} 2\sin x\cos x\, e^{\sin^2 x}\,dx$. Making the substitution $u = \sin^2 x$, we have $du = 2\sin x\cos x\,dx$; it is also the case that $x = \pi \Rightarrow u = \sin^2 0 = 0$ and $x = 3\pi/2 \Rightarrow u = \sin^2\left(\frac{3\pi}{2}\right) = 1$.

Therefore

$$\int_\pi^{3\pi/2} \sin 2x\, e^{\sin^2 x}\,dx = \int_\pi^{3\pi/2} 2\sin x\cos x\, e^{\sin^2 x}\,dx = \int_0^1 e^u du = e^u\big|_0^1 = e - 1.$$

8.6.57 Some observations are in order before we start integrating. Notice that

$$\int \sin\sqrt{x}\,dx = \int \sqrt{x}\,\frac{\sin\sqrt{x}}{\sqrt{x}}dx.$$

Also observe that $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = -2\cos\sqrt{x} + C$ because $\frac{d}{dx}\big(-2\cos\sqrt{x}\big) = \frac{\sin\sqrt{x}}{\sqrt{x}}$. These facts will help us evaluate the integral using integration by parts with the following variable choices.

$$\begin{array}{ll} u = \sqrt{x} & dv = \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx \\ du = \frac{dx}{2\sqrt{x}} & v = -2\cos\sqrt{x} \end{array}$$

Integrating by parts, we have

$$\begin{aligned} \int \sin\sqrt{x}dx &= -2\sqrt{x}\cos\sqrt{x} + \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx \\ &= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C. \end{aligned}$$

8.6.58 We apply integration by parts with the following choices for our variables.

$$\begin{array}{ll} u = \tan^{-1} w & dv = w^2\,dw \\ du = \frac{dw}{1+w^2} & v = \frac{w^3}{3} \end{array}$$

The integral then becomes

$$\int w^2\tan^{-1} w\,dw = \frac{w^3}{3}\tan^{-1} w - \frac{1}{3}\int \frac{w^3}{1+w^2}dw.$$

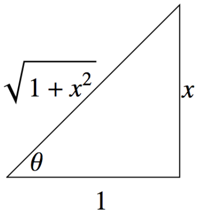
Using long division to obtain the equality $\frac{w^3}{1+w^2} = w - \frac{w}{1+w^2}$ and a u-substitution of $u = 1 + w^2$ to evaluate $\int \frac{w}{1+w^2}dw$, the integral is straightforward to complete:

$$\begin{aligned} \int w^2\tan^{-1} w\,dw &= \frac{1}{3}\left(w^3\tan^{-1} w - \int w\,dw + \int \frac{w}{1+w^2}dw\right) \\ &= \frac{1}{3}\left(w^3\tan^{-1} w - \frac{w^2}{2} + \frac{1}{2}\ln(1+w^2)\right) + C. \end{aligned}$$

8.6.59 Factor the denominator and make the trigonometric substitution $x = \tan\theta$, where $-\pi/2 < \theta < \pi/2$. Then $dx = \sec^2\theta\,d\theta$ and

$$\int \frac{dx}{x^4+x^2} = \int \frac{dx}{x^2(x^2+1)} = \int \frac{\sec^2\theta}{\tan^2\theta\underbrace{(\tan^2\theta+1)}_{\sec^2\theta}}d\theta = \int \cot^2\theta\,d\theta = \int(\csc^2\theta - 1)d\theta = -\cot\theta - \theta + C.$$

Because $\tan\theta = x$, we use the following reference triangle and the substitution $\theta = \tan^{-1} x$ to complete the integration.



$$\int \frac{dx}{x^4+x^2} = -\cot\theta - \theta + C = -\frac{1}{x} - \tan^{-1} x + C.$$

8.6.60 We evaluate the indefinite integral first. Apply integration by parts with the following variable choices.

$$\begin{aligned} u &= e^{-3x} & dv &= \cos x\,dx \\ du &= -3e^{-3x}dx & v &= \sin x \end{aligned}$$

The corresponding integral then becomes

$$\int e^{-3x}\cos x\,dx = e^{-3x}\sin x + 3\int e^{-3x}\sin x\,dx.$$

Integration by parts is applied again to the integral on the right side of the equation using the following choices.

$$\begin{aligned} u &= e^{-3x} & dv &= \sin x\,dx \\ du &= -3e^{-3x}dx & v &= -\cos x \end{aligned}$$

We then have

$$\int e^{-3x}\cos x\,dx = e^{-3x}\sin x + 3\int e^{-3x}\sin x\,dx = e^{-3x}\sin x + 3\Big(-e^{-3x}\cos x - 3\int e^{-3x}\cos x\,dx\Big).$$

Simplifying the equation, we find that

$$\int e^{-3x}\cos x\,dx = e^{-3x}\sin x - 3e^{-3x}\cos x - 9\int e^{-3x}\cos x\,dx.$$

The indefinite integral is then found by solving for $\int e^{-3x}\cos x\,dx$ and adding a constant of integration:

$$\int e^{-3x}\cos x\,dx = \frac{e^{-3x}}{10}\big(\sin x - 3\cos x\big) + C.$$

Using this result, we evaluate the definite integral:

$$\int_0^{\pi/2} e^{-3x}\cos x\,dx = \frac{e^{-3x}}{10}\big(\sin x - 3\cos x\big)\Big|_0^{\pi/2} = \frac{e^{-3\pi/2}+3}{10}.$$

8.6.61 Letting $u = \sin^{-1} x$ we have $x = \sin u$ and $dx = \cos u\,du$. In this case, $x = 0 \Rightarrow u = \sin^{-1} 0 = 0$ and $x = \frac{\sqrt{2}}{2} \Rightarrow u = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$. So the integral becomes

$$\int_0^{\sqrt{2}/2} e^{\sin^{-1} x}\,dx = \int_0^{\pi/4} e^u \cos u\,du.$$

Using integration by parts twice (similar to the solution in the previous problem), it can be shown that

$$\int e^u \cos u\,du = \frac{e^u}{2}\big(\cos u + \sin u\big) + C.$$

Now we summarize what we've done so far and then complete the exercise:

$$\begin{aligned}\int_0^{\sqrt{2}/2} e^{\sin^{-1} x}\,dx &= \int_0^{\pi/4} e^u \cos u\,du && u\text{-substitution}\\ &= \frac{e^u}{2}\big(\cos u + \sin u\big)\Big|_0^{\pi/4} && \text{Integraton by parts (twice)}\\ &= \frac{e^{\pi/4}}{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)-\frac{1}{2} && \text{Evaluate.}\\ &= \frac{\sqrt{2}e^{\pi/4}-1}{2}. && \text{Simplify.}\end{aligned}$$

8.6.62 Rearranging the identity

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

and replacing θ with $\theta/2$, we have $\sqrt{1+\cos\theta} = \sqrt{2\cos^2\frac{\theta}{2}} = \sqrt{2}\cos\frac{\theta}{2}$ (the last equality is justified because $\cos\frac{\theta}{2} \geq 0$ for $0 \leq \theta \leq \pi/2$). Therefore

$$\int_0^{\pi/2}\sqrt{1+\cos\theta}\,d\theta = \sqrt{2}\int_0^{\pi/2}\cos\frac{\theta}{2}\,d\theta = 2\sqrt{2}\sin\frac{\theta}{2}\Big|_0^{\pi/2} = 2.$$

8.6.63 We apply integration by parts with the following choices for our variables.

$$\begin{array}{ll} u = \ln x & dv = x^a\,dx\\ du = \frac{1}{x}dx & v = \dfrac{x^{a+1}}{a+1}\end{array}$$

The integration process is straightforward:

$$\begin{aligned}\int x^a \ln x\,dx &= \frac{x^{a+1}\ln x}{a+1} - \int\frac{x^a}{a+1}dx && \text{Integration by parts}\\ &= \frac{x^{a+1}\ln x}{a+1} - \frac{x^{a+1}}{(a+1)^2} + C && \text{Evaluate.}\\ &= \frac{x^{a+1}}{a+1}\left(\ln x - \frac{1}{a+1}\right) + C. && \text{Factor.}\end{aligned}$$

8.6.64 Letting $u = \ln ax$ so that $du = \frac{1}{ax}a\,dx = \frac{dx}{x}$, we have

$$\int\frac{\ln ax}{x}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{\ln^2 ax}{2} + C.$$

8.6.65 By factoring the denominator, we can apply the formula $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$:

$$\int_0^{1/6}\frac{dx}{\sqrt{1-9x^2}} = \int_0^{1/6}\frac{dx}{\sqrt{9(\frac{1}{9}-x^2)}} = \frac{1}{3}\int_0^{1/6}\frac{dx}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}} = \frac{1}{3}\sin^{-1}(3x)\Big|_0^{1/6} = \frac{\pi}{18}.$$

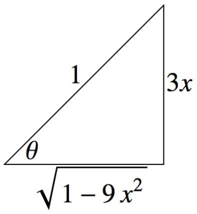
8.6.66 Although trigonometric substitution works here, it's easier to use a u-substitution of $u = 1-9x^2$. This implies that $du = -18x\,dx$ or $-\frac{1}{18}du = x\,dx$. The substitution may now be done:

$$\int\frac{x}{\sqrt{1-9x^2}}dx = -\frac{1}{18}\int\frac{du}{\sqrt{u}} = -\frac{1}{9}\sqrt{u} + C = -\frac{1}{9}\sqrt{1-9x^2} + C.$$

8.6.67 $\sqrt{1-9x^2} = \sqrt{1-(3x)^2}$ suggests the change of variables $3x = \sin\theta$, $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. This substitution implies that $3\,dx = \cos\theta\,d\theta$ and

$$\sqrt{1-9x^2} = \sqrt{1-(3x)^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta.$$

Using these substitutions and the following reference triangle, the integral is evaluated as follows.



$$\begin{aligned}
\int \frac{x^2}{\sqrt{1-(3x)^2}}dx &= \frac{1}{27}\int \frac{\sin^2\theta}{\cos\theta}\cdot\cos\theta\,d\theta && \text{Trigonometric substitution}\\
&= \frac{1}{27}\int \sin^2\theta\,d\theta && \text{Simplify.}\\
&= \frac{1}{54}\int (1-\cos 2\theta)\,d\theta && \sin^2\theta = \frac{1-\cos 2\theta}{2}\\
&= \frac{1}{54}(\theta - \frac{1}{2}\sin 2\theta) + C && \text{Evaluate.}\\
&= \frac{1}{54}(\theta - \sin\theta\cos\theta) + C && \frac{1}{2}\sin 2\theta = \sin\theta\cos\theta\\
&= \frac{1}{54}\left(\sin^{-1}3x - 3x\sqrt{1-9x^2}\right) + C && \text{Reference triangle}
\end{aligned}$$

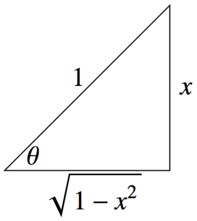
8.6.68 We begin by making a u-substitution and then we complete the square in the denominator.

$$\begin{aligned}
\int \frac{e^x}{e^{2x}+2e^x+17}\,dx &= \int \frac{du}{u^2+2u+17} && u = e^x,\ du = e^x dx\\
&= \int \frac{du}{(u+1)^2+16} && \text{Complete the square.}\\
&= \frac{1}{4}\tan^{-1}\frac{u+1}{4} + C && \text{Evaluate.}\\
&= \frac{1}{4}\tan^{-1}\frac{e^x+1}{4} + C && \text{Replace } u \text{ with } e^x.
\end{aligned}$$

8.6.69 The presence of the expression $1-x^2$ in the integrand suggests a trigonometric substitution of $x = \sin\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. For this choice, $dx = \cos\theta\,d\theta$, $1-x^2 = 1-\sin^2\theta = \cos^2\theta$ and $\sqrt{1-x^2} = |\cos\theta| = \cos\theta$. This trigonometric substitution gets the integration process started.

$$\begin{aligned}
\int \frac{dx}{1-x^2+\sqrt{1-x^2}} &= \int \frac{\cos\theta}{\cos^2\theta+\cos\theta}d\theta && \text{Trigonometric substitution} \\
&= \int \frac{d\theta}{\cos\theta+1} && \text{Simplify.} \\
&= \int \frac{1}{\cos\theta+1}\cdot\frac{1-\cos\theta}{1-\cos\theta}\,d\theta && \text{Multiply by } \frac{1-\cos\theta}{1-\cos\theta}. \\
&= \int \frac{1-\cos\theta}{\sin^2\theta}d\theta && \text{Simplify.} \\
&= \int (\csc^2\theta-\csc\theta\cot\theta)d\theta && \frac{1-\cos\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}-\frac{\cos\theta}{\sin^2\theta}=\csc^2\theta-\csc\theta\cot\theta \\
&= -\cot\theta+\csc\theta+C && \text{Evaluate.}
\end{aligned}$$

Given that $\sin\theta = x$, a geometric description of the relationship between x and θ is given by the following reference triangle.



Therefore

$$\int \frac{dx}{1-x^2+\sqrt{1-x^2}} = -\cot\theta+\csc\theta = -\frac{\sqrt{1-x^2}}{x}+\frac{1}{x}+C = \frac{1-\sqrt{1-x^2}}{x}+C.$$

8.6.70 We use integration by parts with the following choices for the variables.

$$\begin{array}{ll} u=\ln(x^2+a^2) & dv=dx \\ du=\frac{2x}{x^2+a^2}dx & v=x \end{array}$$

Applying integration by parts, we have

$$\int \ln(x^2+a^2)\,dx = x\ln(x^2+a^2)-2\int\frac{x^2}{x^2+a^2}dx.$$

Using long division, it can be shown that $\frac{x^2}{x^2+a^2}=1-\frac{a^2}{x^2+a^2}$. Therefore

$$\begin{aligned}
\int \ln(x^2+a^2)\,dx &= x\ln(x^2+a^2)-2\int\left(1-\frac{a^2}{x^2+a^2}\right)dx \\
&= x\ln(x^2+a^2)-2\int dx+2a^2\int\frac{dx}{x^2+a^2} \\
&= x\ln(x^2+a^2)-2x+2a\tan^{-1}\frac{x}{a}+C.
\end{aligned}$$

8.6.71 We start by multiplying the numerator and the denominator of the integrand by $(1-\cos x)$:

$$\begin{aligned}
\frac{1-\cos x}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x} &= \frac{1-2\cos x+\cos^2 x}{1-\cos^2 x} && \text{Multiply by } \frac{1-\cos x}{1-\cos x}.\\
&= \frac{1-2\cos x+\cos^2 x}{\sin^2 x} && \text{Trigonometric Identity.}\\
&= \csc^2 x-2\cot x\csc x+\cot^2 x && \text{Simplify.}\\
&= \csc^2 x-2\cot x\csc x+\csc^2 x-1 && \text{Replace } \cot^2 x \text{ with } \csc^2 x-1.\\
&= 2\csc^2 x-2\cot x\csc x-1 && \text{Simplify.}
\end{aligned}$$

With the transformed integrand, the integral is easier to evaluate:

$$\int \frac{1-\cos x}{1+\cos x}\,dx = \int (2\csc^2 x-2\cot x\csc x-1)dx = -2\cot x+2\csc x-x+C.$$

8.6.72 We use integration by parts with the following choices for the variables.

$$\begin{array}{ll} u = x^2 & dv = \sinh x\,dx \\ du = 2x\,dx & v = \cosh x \end{array}$$

$$\int x^2\sinh x\,dx = x^2\cosh x - \int 2x\cosh x\,dx$$

We apply integration by parts again with the following choices for the variables.

$$\begin{array}{ll} u = 2x & dv = \cosh x\,dx \\ du = 2\,dx & v = \sinh x \end{array}$$

$$\begin{aligned}
\int x^2\sinh x\,dx &= x^2\cosh x - \int 2x\cosh x\,dx\\
&= x^2\cosh x - 2x\sinh x + \int 2\sinh x\,dx + C\\
&= x^2\cosh x - 2x\sinh x + 2\cosh x + C.
\end{aligned}$$

8.6.73 Letting $u = 1+\sqrt{x}$ it follows that $du = \frac{dx}{2\sqrt{x}}$. Rearranging these equations, we have $\sqrt{x} = u-1$ and $dx = 2\sqrt{x}\,du = 2(u-1)du$. After the substitution is made, the lower limit of integration is $u = 4$ and the upper limit is $u = 5$. Therefore

$$\begin{aligned}
\int_9^{16}\sqrt{1+\sqrt{x}}\,dx &= 2\int_4^5 \sqrt{u}(u-1)du\\
&= 2\int_4^5 (u^{3/2}-u^{1/2})du\\
&= 2\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)\Big|_4^5\\
&= 4u^{3/2}\left(\frac{u}{5}-\frac{1}{3}\right)\Big|_4^5\\
&= \frac{4u^{3/2}}{15}(3u-5)\Big|_4^5\\
&= \frac{40\sqrt{5}}{3}-\frac{224}{15}.
\end{aligned}$$

8.6.74 If $u = e^x - 1$, then $du = e^x dx$, $e^x = u + 1$, and $e^{2x} = (u + 1)^2$. Therefore

$$\begin{aligned}
\int \frac{e^{3x}}{e^x - 1}\,dx &= \int \frac{e^{2x}e^x}{e^x - 1}\,dx \\
&= \int \frac{(u+1)^2}{u}du \\
&= \int \left(u + 2 + \frac{1}{u}\right)du \\
&= \frac{u^2}{2} + 2u + \ln|u| + C \\
&= \frac{(e^x - 1)^2}{2} + 2(e^x - 1) + \ln|e^x - 1| + C \\
&= \frac{e^{2x}}{2} - e^x + \frac{1}{2} + 2e^x - 2 + \ln|e^x - 1| + C \\
&= \frac{e^{2x}}{2} + e^x + \ln|e^x - 1| + C.
\end{aligned}$$

8.6.75 Letting $u = \tan^{-1}\sqrt{x}$, it follows that $du = \dfrac{1}{1 + (\sqrt{x})^2} \cdot \dfrac{1}{2\sqrt{x}}dx = \dfrac{dx}{2(\sqrt{x} + x^{3/2})}$. Then $2du = \dfrac{dx}{\sqrt{x} + x^{3/2}}$. The lower limit of integration is $u = \tan^{-1} 1 = \dfrac{\pi}{4}$ and the upper limit is $u = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$. Therefore

$$\int_1^3 \frac{\tan^{-1}\sqrt{x}}{x^{1/2} + x^{3/2}}\,dx = \int_{\pi/4}^{\pi/3} 2u\,du = u^2\Big|_{\pi/4}^{\pi/3} = \frac{7\pi^2}{144}.$$

8.6.76 By completing the square of the denominator of the integrand and then making a u-substitution we have

$$\begin{aligned}
\int \frac{x}{x^2 + 6x + 18}\,dx &= \int \frac{x}{(x+3)^2 + 9}\,dx && \text{Complete the square.} \\
&= \int \frac{u - 3}{u^2 + 9}\,du && u = x + 3,\ du = dx,\ x = u - 3 \\
&= \int \frac{u}{u^2 + 9}\,du - \int \frac{3}{u^2 + 9}du && \text{Split into two integrals.}
\end{aligned}$$

The second integral equals $\tan^{-1}\dfrac{u}{3}$; the first integral is evaluated by making a substitution of $w = u^2 + 9$, which implies that $dw = 2u\,du$ or $\frac{1}{2}dw = u\,du$:

$$\int \frac{u}{u^2 + 9}du = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(u^2 + 9) + C.$$

Therefore,

$$\int \frac{x}{x^2 + 6x + 18}\,dx = \frac{1}{2}\ln(u^2 + 9) - \tan^{-1}\frac{u}{3} + C = \frac{1}{2}\ln\left((x+3)^2 + 9\right) - \tan^{-1}\frac{x+3}{3} + C.$$

8.6.77 We use integration by parts with the following choices for the variables:

$$u = \cos^{-1} x \qquad dv = dx$$

$$du = -\frac{dx}{\sqrt{1-x^2}} \qquad v = x$$

Now we apply integration by parts followed by integration by substitution.

$$\begin{aligned}
\int \cos^{-1} x\,dx &= x\cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}}dx && \text{Integration by parts}\\
&= x\cos^{-1} x - \frac{1}{2}\int u^{-1/2}du && u = 1-x^2;\ du = -2x\,dx\\
&= x\cos^{-1} x - u^{1/2} + C && \text{Integrate.}\\
&= x\cos^{-1} x - \sqrt{1-x^2} + C. && \text{Replace } u \text{ with } 1-x^2.
\end{aligned}$$

8.6.78 Start with integration by parts, using the following variable choices.

$$u = \left(\cos^{-1} x\right)^2 \qquad dv = dx$$

$$du = -\frac{2\cos^{-1} x}{\sqrt{1-x^2}}dx \qquad v = x$$

It follows that

$$\int (\cos^{-1} x)^2\,dx = x(\cos^{-1} x)^2 + 2\int \frac{x\cos^{-1} x}{\sqrt{1-x^2}}dx.$$

Apply integration by parts again to the integral on the right using the following choices for the variables.

$$u = \cos^{-1} x \qquad dv = \frac{x}{\sqrt{1-x^2}}dx$$

$$du = -\frac{dx}{\sqrt{1-x^2}} \qquad v = -\sqrt{1-x^2}$$

The second integral becomes

$$\int \frac{x\cos^{-1} x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2}\cos^{-1} x - \int dx = -\sqrt{1-x^2}\cos^{-1} x - x + C.$$

Assembling our work, we have

$$\begin{aligned}
\int (\cos^{-1} x)^2\,dx &= x(\cos^{-1} x)^2 + 2\int \frac{x\cos^{-1} x}{\sqrt{1-x^2}}dx\\
&= x(\cos^{-1} x)^2 + 2\Big(-\sqrt{1-x^2}\cos^{-1} x - x\Big) + C\\
&= x(\cos^{-1} x)^2 - 2\sqrt{1-x^2}\cos^{-1} x - 2x + C.
\end{aligned}$$

8.6.79 We apply integration by parts with the following choices for the variables.

$$u = \sin^{-1} x \qquad dv = \frac{dx}{x^2}$$

$$du = \frac{dx}{\sqrt{1-x^2}} \qquad v = -\frac{1}{x}$$

Using these substitutions with integration by parts, along with the answer to Exercise 25, we evaluate the integral.

$$\int \frac{\sin^{-1} x}{x^2}\,dx = -\frac{\sin^{-1} x}{x} + \int \frac{dx}{x\sqrt{1-x^2}} = -\frac{\sin^{-1} x}{x} + \ln\left|\frac{x}{1+\sqrt{1-x^2}}\right| + C.$$

8.6.80 Start by completing the square of $-4x - x^2$ and then make a u-substitution of $u = x+2$. The lower limit of integration becomes 0 and the upper limit becomes 1.

$$\int_{-2}^{-1} \sqrt{-4x-x^2}\,dx = \int_{-2}^{-1} \sqrt{4-(x+2)^2}\,dx = \int_0^1 \sqrt{4-u^2}\,du$$

Using the trigonometric substitution $u = 2\sin\theta$, it follows that $du = 2\cos\theta\,d\theta$. With this substitution, $u = 0$ implies that $\theta = 0$ and $u = 1$ implies $\theta = \frac{\pi}{6}$. So we have

$$\int_{-2}^{-1} \sqrt{-4x-x^2}\,dx = \int_0^1 \sqrt{4-u^2}\,du = \int_0^{\pi/6} 4\cos^2\theta\,d\theta.$$

We evaluate the integral with the help of the identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$.

$$\int_{-2}^{-1} \sqrt{-4x-x^2}\,dx = \int_0^{\pi/6} 2(1+\cos 2\theta)\,d\theta = \big(2\theta + \sin 2\theta\big)\Big|_0^{\pi/6} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$

8.6.81 Factoring the denominator, we have

$$x^5 + 2x^3 + x = x(x^4 + 2x^2 + 1) = x(x^2+1)^2.$$

So the partial fraction decomposition of the integrand has the form

$$\frac{x^4 + 2x^3 + 5x^2 + 2x + 1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Multiplying by $x(x^2+1)^2$, we have

$$x^4 + 2x^3 + 5x^2 + 2x + 1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x.$$

To find the constants, we expand the right hand side of equation and group equal powers of x:

$$x^4 + 2x^3 + 5x^2 + 2x + 1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A.$$

Comparing coefficients of equal powers, we see that $A = 1$, $B = 0$, $C = 2$, $D = 3$, and $E = 0$. Using these values, the integral is evaluated with the help of a u-substitution in the third integral.

$$\begin{aligned}
\int \frac{x^4+2x^3+5x^2+2x+1}{x(x^2+1)^2}dx &= \int \frac{dx}{x} + \int \frac{2}{x^2+1}dx + \int \frac{3x}{(x^2+1)^2}dx && \text{Partial fractions.}\\
&= \int \frac{dx}{x} + \int \frac{2}{x^2+1}dx + \frac{3}{2}\int \frac{du}{u^2} && u = x^2+1,\ \frac{1}{2}du = x\,dx\\
&= \ln|x| + 2\tan^{-1} x - \frac{3}{2u} + C && \text{Evaluate.}\\
&= \ln|x| + 2\tan^{-1} x - \frac{3}{2(x^2+1)} + C. && \text{Replace } u \text{ with } x^2+1.
\end{aligned}$$

8.6.82 Letting $u = \tan x$ it follows that $du = \sec^2 x\,dx$, and

$$dx = \frac{du}{\sec^2 x} = \frac{du}{\tan^2 x + 1} = \frac{du}{u^2+1}.$$

With this change of variable,

$$\int \frac{dx}{1+\tan x} = \int \frac{du}{(u+1)(u^2+1)}.$$

Because the denominator on the right side of the equation contains a linear term and an irreducible quadratic factor, the form of the partial fraction decomposition is

$$\frac{1}{(u+1)(u^2+1)} = \frac{A}{u+1} + \frac{Bu+C}{u^2+1}$$

Multiplying by $(u+1)(u^2+1)$ leads to

$$1 = A(u^2+1) + (Bu+C)(u+1) = (A+B)u^2 + (B+C)u + A + C.$$

Comparing coefficients of equal powers, we obtain three equations with three unknowns:

$$A+B=0, B+C=0, \text{ and } A+C=1.$$

Solving this system gives

$$A = \frac{1}{2}, B = -\frac{1}{2}, \text{ and } C = \frac{1}{2}.$$

Substituting these values into the partial fraction decomposition, the integral can be evaluated as follows:

$$\begin{aligned}
\int \frac{dx}{1+\tan x} &= \int \frac{du}{(u+1)(u^2+1)} && u\text{-substitution}\\
&= \frac{1}{2}\int \frac{du}{u+1} - \frac{1}{2}\int \frac{u}{u^2+1}du + \frac{1}{2}\int \frac{du}{u^2+1} && \text{Partial fractions}\\
&= \frac{1}{2}\ln|u+1| - \frac{1}{4}\ln(u^2+1) + \frac{1}{2}\tan^{-1} u + C && \text{Integrate.}\\
&= \frac{1}{2}\ln|\tan x+1| - \frac{1}{4}\ln(\tan^2 x+1) + \frac{1}{2}\tan^{-1}\big(\tan x\big) + C && \text{Replace } u \text{ with } \tan x.\\
&= \frac{1}{2}\ln|\tan x+1| - \frac{1}{4}\ln(\sec^2 x) + \frac{x}{2} + C. && 1+\tan^2 x = \sec^2 x
\end{aligned}$$

8.6.83 We simplify the integrand first using the substitution $u = e^x$, so that $du = e^x dx$ and

$$\int e^x \sin^{998}(e^x)\cos^3(e^x)\,dx = \int \sin^{998} u \cos^3 u\,du.$$

We then continue the integration process, by first splitting $\cos^3 u$.

$$\begin{aligned}
\int e^x \sin^{998}(e^x)\cos^3(e^x)\,dx &= \int \sin^{998} u\cos^3 u\,du && u = e^x,\ du = e^x dx\\
&= \int \sin^{998} u\cos^2 u\cos u\,du && \text{Split } \cos^3 u.\\
&= \int \sin^{998} u(1-\sin^2 u)\cos u\,du && \cos^2 x = 1-\sin^2 x\\
&= \int (\sin^{998} u - \sin^{1000} u)\cos u\,du && \text{Expand.}\\
&= \int (w^{998} - w^{1000})\,dw && w = \sin u,\ dw = \cos u\,du\\
&= \frac{w^{999}}{999} - \frac{w^{1001}}{1000} + C && \text{Integrate.}\\
&= \frac{\sin^{999} u}{999} - \frac{\sin^{1001} u}{1001} + C && \text{Replace } w \text{ with } \sin u.\\
&= \frac{\sin^{999}(e^x)}{999} - \frac{\sin^{1001}(e^x)}{1001} + C && \text{Replace } u \text{ with } e^x.
\end{aligned}$$

8.6.84 Let $u = \tan\theta$ which implies that $du = \sec^2\theta\,d\theta$.

$$\begin{aligned}
\int \frac{\tan\theta + \tan^3\theta}{(1+\tan\theta)^{50}}\,d\theta &= \int \frac{\tan\theta(1+\tan^2\theta)}{(1+\tan\theta)^{50}}\,d\theta\\
&= \int \frac{\tan\theta\sec^2\theta}{(1+\tan\theta)^{50}}\,d\theta = \int \frac{u}{(1+u)^{50}}\,du.
\end{aligned}$$

Now let $w = u+1$ so that $dw = du$ and $u = w-1$. We then have

$$\begin{aligned}
\int \frac{w-1}{w^{50}}\,dw &= \int \left(w^{-49} - w^{-50}\right)\,dw\\
&= -\frac{w^{-48}}{48} + \frac{w^{-49}}{49} + C = \frac{(u+1)^{-49}}{49} - \frac{(u+1)^{-48}}{48} + C\\
&= \frac{1}{49(\tan\theta+1)^{49}} - \frac{1}{48(\tan\theta+1)^{48}} + C.
\end{aligned}$$

8.6.85

a True. For example, the method of partial fractions or a trigonometric substitution can be used to evaluate the integral.

b. True. See Example 4.

c. False. It is easiest to use the substitution $u = \tan 3x$.

d. False. The integral $\int \frac{u^2}{u-1}\,du$ would imply that $du = dx$, which is not the case here.

8.6.86 We need to compute

$$A = \int_0^2 \left(\frac{2}{x^2-2x+2} - \frac{x}{x^2-2x+2} \right) dx = \int_0^2 \frac{2-x}{(x-1)^2+1}\, dx.$$

Let $u = x-1$ so that $x = u+1$ and $du = dx$. Then we have

$$\int_{-1}^1 \frac{1-u}{u^2+1}\, du = \int_{-1}^1 \frac{1}{u^2+1}\, du - \int_{-1}^1 \frac{u}{u^2+1}\, du.$$

The first integral is that of an even function while the second is that of an odd function, so the sum is equal to

$$2\int_0^1 \frac{1}{u^2+1}\, du = 2\tan^{-1} u \Big|_0^1 = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}.$$

8.6.87 The surface area is given by $S = 2\pi \displaystyle\int_0^{\pi/2} \sin x\sqrt{1+\cos^2 x}\, dx$. Let $u = \cos x$ so that $du = -\sin x\, dx$. Then $S = -2\pi \displaystyle\int_1^0 \sqrt{1+u^2}\, du = 2\pi \int_0^1 \sqrt{1+u^2}\, du$. Now let $u = \tan\theta$ with $du = \sec^2\theta\, d\theta$. Substituting gives

$$2\pi \int_0^{\pi/4} \sec^3\theta\, d\theta = \pi\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4} = \pi\left(\sqrt{2}+\ln(1+\sqrt{2})\right).$$

The last evaluated integral came from the reduction formula for $\displaystyle\int \sec^3\theta\, d\theta$ together with the integral formula for $\displaystyle\int \sec\theta\, d\theta$.

8.6.88 We use the Shell method.

$$V = \int_0^1 2\pi x \cdot \frac{x}{\sqrt{4-x^2}}\, dx = 2\pi \int_0^1 \frac{x^2}{\sqrt{4-x^2}}\, dx.$$

Let $x = 2\sin\theta$ so that $dx = 2\cos\theta\, d\theta$ and $\sqrt{4-x^2} = 2\cos\theta$. We then have

$$\begin{aligned} V &= 4\pi \int_0^{\pi/6} 2\sin^2\theta\, d\theta = 4\pi \int_0^{\pi/6} (1-\cos 2\theta)\, d\theta \\ &= 4\pi\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/6} = 4\pi\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = 2\pi\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}\left(2\pi - 3\sqrt{3}\right). \end{aligned}$$

8.6.89 We use the Disk method.

$$\begin{aligned} V &= \int_0^{\pi/4} \pi \frac{dx}{(1-\sin x)^2} = \pi \int_0^{\pi/4} \frac{(1+\sin x)^2}{(1-\sin x)^2(1+\sin x)^2}\, dx = \pi \int_0^{\pi/4} \frac{1+2\sin x+\sin^2 x}{(1-\sin^2 x)^2}\, dx. \\ &= \pi \int_0^{\pi/4} \frac{1+2\sin x+\sin^2 x}{\cos^4 x}\, dx \\ &= \pi \int_0^{\pi/4} \sec^4 x\, dx + \pi \int_0^{\pi/4} 2\tan x\sec^3 x\, dx + \pi \int_0^{\pi/4} \tan^2 x\sec^2 x\, dx. \end{aligned}$$

For the first integral in the sum, write the integrand as $\sec^4 x = \sec^2 x\sec^2 x = (1+\tan^2 x)\sec^2 x$. Then let $u = \tan x$ so that $du = \sec^2 x\, dx$. The first integral is equal to

$$\pi \int_0^1 (1+u^2)\, du = \pi\left(u + \frac{u^3}{3}\right)\Big|_0^1 = \pi\left(1+\frac{1}{3}\right) = \frac{4\pi}{3}.$$

For the second integral in the above sum, let $u = \sec x$ so that $du = \sec x \tan x dx$. We have

$$\pi \int_1^{\sqrt{2}} 2u^2\, du = \frac{2\pi u^3}{3}\bigg|_1^{\sqrt{2}} = \frac{4\sqrt{2}\pi}{3} - \frac{2\pi}{3}.$$

For the third integral in the sum, let $u = \tan x$ so that $du = \sec^2 x\, dx$. Then that last integral is equal to

$$\pi \int_0^1 u^2\, du = \frac{\pi u^3}{3}\bigg|_0^1 = \frac{\pi}{3}.$$

Adding these three results together gives

$$V = \pi\left(\frac{4}{3} + \frac{4\sqrt{2}}{3} - \frac{2}{3} + \frac{1}{3}\right) = \pi\left(\frac{4\sqrt{2}}{3} + 1\right) = \frac{\pi}{3}\left(4\sqrt{2} + 3\right).$$

8.6.90 The work is given by

$$W = \int_0^2 1000(9.8)\pi\sqrt{y^2+4}(2-y)\, dy = 9800\pi\left(2\int_0^2 \sqrt{y^2+4}\, dy - \int_0^2 y\sqrt{y^2+4}\, dy\right).$$

For the first integral we let $y = 2\tan\theta$ so that $dy = 2\sec^2\theta\, d\theta$ and $\sqrt{y^2+4} = 2\sec\theta$. For the second integral let $u = y^2 + 4$ so that $du = 2y\, dy$. Then

$$\begin{aligned}
W &= 9800\pi\left(2\int_0^{\pi/4} 4\sec^3\theta\, d\theta - \frac{1}{2}\int_4^8 u^{1/2}\, du\right)\\
&= 9800\pi\left(8\left(\frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2}\right)\bigg|_0^{\pi/4} - \frac{1}{3}u^{3/2}\bigg|_4^8\right)\\
&= 9800\pi\left(4(\sqrt{2} + \ln|1+\sqrt{2}|) - \frac{1}{3}\left(16\sqrt{2} - 8\right)\right)\\
&= \frac{9800\pi}{3}\left(12\sqrt{2} + 12\ln(1+\sqrt{2}) - 16\sqrt{2} + 8\right) = \frac{9800\pi}{3}\left(12\ln(1+\sqrt{2}) + 8 - 4\sqrt{2}\right)\\
&= \frac{39200\pi}{3}\left(3\ln(1+\sqrt{2}) + 2 - \sqrt{2}\right).
\end{aligned}$$

8.6.91 The work is given by

$$W = \int_0^{\pi/3} 1000(9.8)\pi\sec^2 y\left(\frac{\pi}{3} - y\right)\, dy = 9800\pi\left(\int_0^{\pi/3} \frac{\pi}{3}\sec^2 y\, dy - \int_0^{\pi/3} y\sec^2 y\, dy\right).$$

For the second integral, we integrate by parts. Let $u = y$ and $dv = \sec^2 y\, dy$. Then $du = dy$ and $v = \tan y$. Then

$$\int \sec^2 y\, dy = y\tan y - \int \tan y\, dy = y\tan y + \ln|\cos y| + C.$$

Then

$$W = 9800\pi\left(\frac{\pi}{3}\tan y\bigg|_0^{\pi/3} - (y\tan y + \ln|\cos y|)\bigg|_0^{\pi/3}\right) = 9800\pi\left(\frac{\pi}{3}\sqrt{3} - \left(\frac{\pi}{3}\sqrt{3} + \ln\frac{1}{2}\right)\right) = 9800\pi\ln 2.$$

8.6.92 We evaluate the indefinite integral first. Starting with a substitution of $w = y^3$, it follows that $dw = 3y^2\, dy$ and $w^2 = y^6$. This substitution gives us

$$\int y^8 e^{y^3}\, dy = \frac{1}{3}\int w^2 e^w\, dw.$$

Apply integration by parts to the integral on the right with the following variable choices.

$$\begin{aligned} u &= w^2 & dv &= e^w dw \\ du &= 2w\,dw & v &= e^w \end{aligned}$$

Then we have

$$\int y^8 e^{y^3}\,dy = \frac{1}{3}\int w^2 e^w\,dw = \frac{1}{3}\left(w^2 e^w - 2\int w e^w dw\right).$$

We apply integration by parts to $\int w e^w dw$ with the following variable choices.

$$\begin{aligned} u &= w & dv &= e^w dw \\ du &= dw & v &= e^w \end{aligned}$$

After evaluating the last integral, we replace w with y^3 and factor.

$$\begin{aligned} \int y^8 e^{y^3}\,dy &= \frac{1}{3}\left(w^2 e^w - 2\int w e^w dw\right) \\ &= \frac{1}{3}\left(w^2 e^w - 2\left(w e^w - \int e^w dw\right)\right) \\ &= \frac{1}{3}\left(w^2 e^w - 2w e^w + 2e^w\right) + C \\ &= \frac{e^{y^3}}{3}\left(y^6 - 2y^3 + 2\right) + C. \end{aligned}$$

Evaluating the definite integral, we have

$$\int_1^{\sqrt[3]{2}} y^8 e^{y^3}\,dy = \frac{e^{y^3}}{3}\left(y^6 - 2y^3 + 2\right)\Bigg|_1^{\sqrt[3]{2}} = \frac{2e^2}{3} - \frac{e}{3} = \frac{e(2e-1)}{3}.$$

8.6.93 Start with a substitution of $u = e^x$. Then $du = e^x\,dx$, which implies that $dx = \dfrac{du}{e^x} = \dfrac{du}{u}$. Making this substitution, we have

$$\int \frac{68}{e^{2x} + 2e^x + 17}\,dx = \int \frac{68}{u(u^2 + 2u + 17)}\,du.$$

The denominator has one simple linear factor u and one irreducible quadratic factor $u^2 + 2u + 17$. Therefore the partial fraction decomposition has the form

$$\frac{68}{u(u^2 + 2u + 17)} = \frac{A}{u} + \frac{Bu + C}{u^2 + 2u + 17}.$$

Multiplying both sides by $u(u^2 + 2u + 17)$, we have

$$68 = A(u^2 + 2u + 17) + (Bu + C)u = (A + B)u^2 + (2A + C)u + 17A.$$

By equating coefficients of equal powers of x, we find that $A = 4$, $B = -4$, and $C = -8$. Using these values, we have

$$\int \frac{68}{u(u^2 + 2u + 17)}du = \int \left(\frac{4}{u} + \frac{-4u - 8}{u^2 + 2u + 17}\right)du = \int \frac{4}{u}du - \int \frac{4u + 8}{u^2 + 2u + 17}du.$$

The first integral on the right equals $4\ln|u| + C$. For the second integral on the right, we use a substitution of $w = u^2 + 2u + 17$, which implies that $dw = (2u + 2)du$. We write the numerator in the second integral as $4u + 8 = 2(2u + 2) + 4$ and split the integral:

$$\int \frac{68}{u(u^2+2u+17)}du = \int \frac{4}{u}du - 2\int \underbrace{\frac{2u+2}{u^2+2u+17}}_{w=u^2+2u+17}du - 4\int \underbrace{\frac{du}{u^2+2u+17}}_{(u+1)^2+16}$$
$$= 4\ln|u| - 2\ln|u^2+2u+17| - \tan^{-1}\left(\frac{u+1}{4}\right) + C.$$

Replacing u with e^x and then simplifying, we have

$$\int \frac{68}{e^{2x}+2e^x+17}\,dx = 4\ln e^x - 2\ln|e^{2x}+2e^x+17| - \tan^{-1}\left(\frac{e^x+1}{4}\right) + C$$

$$= 4x - 2\ln(e^{2x}+2e^x+17) - \tan^{-1}\left(\frac{e^x+1}{4}\right) + C.$$

8.6.94 Factoring the denominator, we find that

$$t^3+1 = (t+1)(t^2-t+1).$$

The partial fraction decomposition is

$$\frac{1}{(t+1)(t^2-t+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1},$$

which leads to the equation

$$1 = A(t^2-t+1) + (Bt+C)(t+1) = (A+B)t^2 + (-A+B+C)t + (A+C).$$

Comparing coefficients of equal powers, we have

$$A+B=0, -A+B+C=0, A+C=1.$$

The solution to this system is $A=1/3$, $B=-1/3$, and $C=2/3$. Using these constants, the partial fraction decomposition is

$$\frac{1}{(t+1)(t^2-t+1)} = \frac{1}{3(t+1)} + \frac{-t+2}{3(t^2-t+1)}.$$

By splitting up the second fraction on the right, we can use integration by substitution and trigonometric substitution.

$$\int \frac{1}{(t+1)(t^2-t+1)}dt = \int \frac{1}{3(t+1)}dt + \int \frac{-t+\frac{1}{2}}{3(t^2-t+1)}dt + \int \frac{\frac{3}{2}}{3(t^2-t+1)}dt.$$

Let $u = t^2-t+1$ in the second integral so that $du = (2t-1)dt$ or $-\frac{1}{2}du = (-t+\frac{1}{2})dt$:

$$\int \frac{-t+\frac{1}{2}}{3(t^2-t+1)}dt = -\frac{1}{6}\int \frac{du}{u} = -\frac{1}{6}\ln|u| + C = -\frac{1}{6}\ln|t^2-t+1| + C$$

For the third integral, we complete a square first:

$$\int \frac{\frac{3}{2}}{3(t^2-t+1)}dt = \frac{1}{2}\int \frac{dt}{t^2-t+1} = \frac{1}{2}\int \frac{dt}{(t-\frac{1}{2})^2+\frac{3}{4}} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C.$$

Assembling all the pieces, we have

$$\int \frac{dt}{t^3+1} = \int \frac{1}{3(t+1)}dt + \int \frac{-t+\frac{1}{2}}{3(t^2-t+1)}dt + \int \frac{\frac{3}{2}}{3(t^2-t+1)}dt$$
$$= \frac{1}{3}\ln|t+1| - \frac{1}{6}\ln|t^2-t+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C.$$

8.6.95 Let $u = \tan x$. Then $du = \sec^2 x\,dx$, and

$$dx = \frac{du}{\sec^2 x} = \frac{du}{\tan^2 x + 1} = \frac{du}{u^2+1}.$$

With this change of variable, the integral becomes

$$\int \frac{dx}{1-\tan^2 x} = \int \frac{du}{(1-u^2)(u^2+1)} = \int \frac{du}{(1+u)(1-u)(u^2+1)}.$$

The form of the partial fraction decomposition of the integrand on the right is

$$\frac{1}{(1+u)(1-u)(u^2+1)} = \frac{A}{1+u} + \frac{B}{1-u} + \frac{Cu+D}{u^2+1}.$$

Multiplying both sides by $(1+u)(1-u)(u^2+1)$ and then regrouping terms, we have

$$\begin{aligned} 1 &= A(1-u)(u^2+1) + B(1+u)(u^2+1) + (Cu+D)(1-u^2) \\ &= (-A+B-C)u^3 + (A+B-D)u^2 + (-A+B+C)u + A+B+D. \end{aligned}$$

By comparing coefficients of equal powers, we obtain the system of equations

$$-A+B-C=0, A+B-D=0, -A+B+C=0, A+B+D=1.$$

You can verify that the values satisfying this system are $A = 1/4$, $B = 1/4$, $C = 0$, $D = 1/2$. The partial fraction decomposition makes the integration process straightforward:

$$\begin{aligned} \int \frac{dx}{1-\tan^2 x} &= \int \frac{du}{(1-u^2)(u^2+1)} && u\text{-substitution} \\ &= \frac{1}{4}\int \frac{du}{1+u} + \frac{1}{4}\int \frac{du}{1-u} + \frac{1}{2}\int \frac{du}{u^2+1} && \text{Partial fractions} \\ &= \frac{1}{4}\ln|u+1| - \frac{1}{4}\ln|u-1| + \frac{1}{2}\tan^{-1} u + C && \text{Integrate.} \\ &= \frac{1}{4}\ln|\tan x+1| - \frac{1}{4}\ln|\tan x-1| + \frac{1}{2}\tan^{-1}(\tan x) + C && \text{Replace } u \text{ with } \tan x. \\ &= \frac{1}{4}\ln|\tan x+1| - \frac{1}{4}\ln|\tan x-1| + \frac{x}{2} + C && \text{Simplify.} \end{aligned}$$

8.6.96 Let $w = x^{1/3}$. Then $x = w^3$ and $dx = 3w^2dw$. Changing variables, we have

$$\int e^{(x^{1/3})}dx = \int 3w^2e^w dw.$$

Using integration by parts, with the following choices for the variables.

$$u = 3w^2 \qquad dv = e^w dw$$

$$du = 6w\,dw \quad v = e^w$$

Then we have

$$\int e^{(x^{1/3})}dx = \int 3w^2e^w dw = 3w^2e^w - \int 6we^w dw.$$

We apply integration by parts again to the integral on the right.

$$u = 6w \qquad dv = e^w dw$$

$$du = 6\,dw \quad v = e^w$$

Using these substitutions, we have

$$\int e^{(x^{1/3})}dx = 3w^2e^w - (6we^w - \int 6e^w dw) = 3w^2e^w - 6we^w + 6e^w + C.$$

Factoring and then replacing w with $x^{1/3}$, we obtain

$$\int e^{(x^{1/3})}dx = 3w^2e^w - 6we^w + 6e^w = 3e^w(w^2 - 2w + 2) = 3e^{(x^{1/3})}(x^{2/3} - 2x^{1/3} + 2) + C.$$

8.6.97 We integrate by parts with the following variable assignments.

$$u = \tan^{-1}\sqrt[3]{x} \qquad dv = dx$$

$$du = \frac{dx}{3x^{2/3}(1 + x^{2/3})} \quad v = x$$

Integrating by parts, we have

$$\int \tan^{-1}\sqrt[3]{x}\,dx = x\tan^{-1}\sqrt[3]{x} - \int \frac{x^{1/3}}{3(1 + x^{2/3})}dx.$$

For the integral on the right, we make a substitution of $v = x^{1/3}$ which implies that $dv = \frac{dx}{3x^{2/3}}$, and $dx = 3v^2dv$. With this change of variable, we have

$$\int \frac{x^{1/3}}{3(1 + x^{2/3})}dx = \int \frac{v}{3(1 + v^2)}3v^2dv = \int \frac{v^3}{1 + v^2}dv.$$

It follows by long division that $\dfrac{v^3}{1+v^2} = v - \dfrac{v}{1+v^2}$. Letting $w = 1 + v^2$, it follows that $dw = 2vdv$ and

$$\begin{aligned}\int \frac{v^3}{1 + v^2}dv &= \int v\,dv - \int \frac{v}{1 + v^2}dv\\ &= \int v\,dv - \frac{1}{2}\int \frac{dw}{w}\\ &= \frac{v^2}{2} - \frac{1}{2}\ln|w| + C\\ &= \frac{v^2}{2} - \frac{1}{2}\ln(v^2 + 1) + C.\end{aligned}$$

Summarizing and then completing our work, we have

$$\begin{aligned}
\int \tan^{-1}\sqrt[3]{x}\,d &= x\tan^{-1}\sqrt[3]{x} - \int \frac{x^{1/3}}{3(1+x^{2/3})}dx && \text{Integration by parts}\\
&= x\tan^{-1}\sqrt[3]{x} - \int \frac{v^3}{1+v^2}dv && v = x^{1/3},\ dv = \frac{dx}{3x^{2/3}}\\
&= x\tan^{-1}\sqrt[3]{x} - \left(\int v\,dv - \int \frac{v}{1+v^2}dv\right) && \frac{v^3}{1+v^2} = v - \frac{v}{1+v^2}\\
&= x\tan^{-1}\sqrt[3]{x} - \left(\frac{v^2}{2} - \frac{1}{2}\ln(1+v^2)\right) + C && \text{Integrate.}\\
&= x\tan^{-1}\sqrt[3]{x} - \frac{x^{2/3}}{2} + \frac{1}{2}\ln(1+x^{2/3}) + C. && \text{Replace } v \text{ with } x^{1/3}\text{; simplify.}
\end{aligned}$$

8.6.98 To evaluate this integral, we use the trick of multiplying by 1 by introducing a factor of $\frac{\sqrt{\sin x}}{\sqrt{\sin x}}$ in the integral and then make a substitution of $w = \sqrt{\sin x}$ with $dw = \dfrac{\cos x}{2\sqrt{\sin x}}dx$.

$$\int e^{\sqrt{\sin x}}\cos x\,dx = \int \frac{\sqrt{\sin x}}{\sqrt{\sin x}}e^{\sqrt{\sin x}}\cos x\,dx = \int 2we^w dw$$

Using integration by parts with the following substitutions,

$$u = 2w \qquad dv = e^w dw$$

$$du = 2dw \quad v = e^w$$

we have

$$\int e^{\sqrt{\sin x}}\cos x\,dx = \int 2we^w dw = 2we^w - \int 2e^w dw = 2we^w - 2e^w + C.$$

Finally, we factor and replace w with $\sqrt{\sin x}$:

$$\int e^{\sqrt{\sin x}}\cos x\,dx = 2e^w(w-1) + C = 2e^{\sqrt{\sin x}}(\sqrt{\sin x} - 1) + C.$$

8.6.99 The surface area is given by

$$S = \int_0^{\pi/4} 2\pi\tan x\sqrt{1+\sec^4 x}\,dx = 2\pi\int_0^{\pi/4} \frac{\sqrt{1+\sec^4 x}\tan x\sec x}{\sec x}\,dx.$$

Let $u = \sec x$ so that $du = \sec x\tan x\,dx$. Substituting gives $2\pi\displaystyle\int_1^{\sqrt{2}} \frac{\sqrt{1+u^4}}{u}\,du$. Now let $u^2 = \tan\theta$ so that $2u\,du = \sec^2\theta\,d\theta$. Substituting gives

$$2\pi\int_{\pi/4}^{\tan^{-1}2} \frac{\sec\theta}{\sqrt{\tan\theta}}\cdot\frac{\sec^2\theta}{2\sqrt{\tan\theta}}\,d\theta = \pi\int_{\pi/4}^{\tan^{-1}2} \frac{\sec^3\theta}{\tan\theta}\cdot\frac{\tan\theta}{\tan\theta}\,d\theta = \pi\int_{\pi/4}^{\tan^{-1}2} \frac{\sec^2\theta\sec\theta\tan\theta}{\sec^2\theta - 1}\,d\theta.$$

Now let $w = \sec\theta$ so that $dw = \sec\theta\tan\theta\,d\theta$. Then

$$S = \pi\int_{\sqrt{2}}^{\sqrt{5}} \frac{w^2}{w^2-1}\,dw.$$

Using long division and then a partial fraction decomposition, we can write this integrand as

$$\frac{w^2}{w^2-1} = 1 - \frac{1/2}{w+1} + \frac{1/2}{w-1}.$$

Therefore

$$S=\pi\int_{\sqrt{2}}^{\sqrt{5}}\left(1-\frac{1}{2(w+1)}+\frac{1}{2(w-1)}\right)dw=\pi\left(w-\frac{1}{2}\ln(w+1)+\frac{1}{2}\ln(w-1)\right)\bigg|_{\sqrt{2}}^{\sqrt{5}}$$
$$=\pi\left(\sqrt{5}-\sqrt{2}+\frac{1}{2}\ln\frac{\sqrt{5}-1}{\sqrt{5}+1}-\frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\approx 3.83908.$$

8.7 Other Methods of Integration

8.7.1 The power rule, substitution, integration by parts, and partial fraction decomposition are examples of analytical methods.

8.7.2 Many computer algebra systems can give exact answers, although sometimes they don't express the answers in elementary terms, and many don't add the arbitrary constant.

8.7.3 The computer algebra system may use a different algorithm than whoever prepared the table, so the results may look different – however, they should differ by a constant.

8.7.4 Using a reduction formula is an analytical method, however, when reduced, the final integral may need to be attacked by a numerical method.

8.7.5 Let $u=e^x$. Then $du=e^x\,dx$. Substituting yields

$$\int\cos^3u\,du=-\frac{1}{3}\sin^3u+\sin u+C=-\frac{1}{3}\sin^3(e^x)+\sin(e^x)+C.$$

8.7.6 Let $u=\sin x$. Then $du=\cos x\,dx$. Substituting gives

$$\int\sqrt{100-u^2}\,du=\frac{u}{2}\sqrt{100-u^2}+50\sin^{-1}\left(\frac{u}{10}\right)+C=\frac{\sin x}{2}\sqrt{100-\sin^2x}+50\sin^{-1}\left(\frac{\sin x}{10}\right)+C.$$

8.7.7 Using table entry 17, we have $\displaystyle\int\cos^{-1}(x)\,dx=x\cos^{-1}x-\sqrt{1-x^2}+C$.

8.7.8 Using table entry 48, we have $\displaystyle\int\sin(3x)\cos(2x)\,dx=-\frac{\cos 5x}{10}-\frac{\cos x}{2}+C$.

8.7.9 Using table entry 77 for $\displaystyle\int\frac{1}{\sqrt{x^2+a^2}}\,dx$, we see that $\displaystyle\int\frac{1}{\sqrt{x^2+16}}\,dx=\ln(x+\sqrt{x^2+16})+C$.

8.7.10 Using table entry 69 for $\displaystyle\int\frac{1}{\sqrt{x^2-a^2}}\,dx$, we see that $\displaystyle\int\frac{1}{\sqrt{x^2-25}}\,dx=\ln|x+\sqrt{x^2-25}|+C$.

8.7.11 Using table entry 90 for $\displaystyle\int\frac{x}{ax+b}\,dx$, we see that $\displaystyle\int\frac{3u}{2u+7}\,du=\frac{3u}{2}-\frac{21}{4}\ln|2u+7|+C$.

8.7.12 Using table entry 96 for $\displaystyle\int\frac{1}{x(ax+b)}\,dx$, we have $\displaystyle\int\frac{1}{y(2y+9)}\,dy=\frac{1}{9}\ln\left|\frac{y}{2y+9}\right|+C$.

8.7.13 Using table entry 47, we have $\displaystyle\int\frac{dx}{1-\cos 4x}=-\frac{1}{4}\cot(2x)+C$.

8.7.14 Using table entry 62 for $\displaystyle\int\frac{1}{x\sqrt{a^2-x^2}}\,dx$ we have that $\displaystyle\int\frac{1}{x\sqrt{81-x^2}}\,dx=\frac{-1}{9}\ln\left|\frac{9+\sqrt{81-x^2}}{x}\right|+C$.

8.7.15 Using table entry 94 for $\int \frac{x}{\sqrt{ax+b}}\,dx$, we have that $\int \frac{x\,dx}{\sqrt{4x+1}} = \frac{1}{24}(4x-2)\sqrt{4x+1}+C = \frac{1}{12}(2x-1)\sqrt{4x+1}+C$.

8.7.16 Using table entry 93 for $\int x\sqrt{ax+b}\,dx$ we have that $\int t\sqrt{4t+12}\,dt = \frac{2}{240}(12t-24)(4t+12)^{3/2}+C = \frac{1}{10}(t-2)(4t+12)^{3/2}+C = \frac{4}{5}(t-2)(t+3)^{3/2}+C$.

8.7.17 Using table entry 69 for $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$ we have $\int \frac{1}{\sqrt{9x^2-100}}\,du = \frac{1}{3}\int \frac{1}{\sqrt{u^2-100}}\,du$ where $u=3x$. This is then equal to $\frac{1}{3}\ln|u+\sqrt{u^2-100}|+C = \frac{1}{3}\ln|3x+\sqrt{9x^2-100}|+C$. This can be written as $\frac{1}{3}\ln|x+\sqrt{x^2-(100/9)}|+C$.

8.7.18 $\int \frac{1}{225-16x^2}\,dx = \frac{1}{16}\int \frac{1}{(225/16)-x^2}\,dx$. Using table entry 67 for $\int \frac{1}{a^2-x^2}\,dx$ we have

$$\begin{aligned}\frac{1}{16}\int \frac{1}{(15/4)^2-x^2}\,dx &= \frac{1}{16}\cdot\frac{2}{15}\ln\left|\frac{x+(15/4)}{x-(15/4)}\right|+C\\ &= \frac{1}{120}\ln\left|\frac{4x+15}{4x-15}\right|+C.\end{aligned}$$

8.7.19 $\int \frac{e^x}{\sqrt{e^{2x}+4}}\,dx = \int \frac{1}{\sqrt{u^2+4}}\,du$ where $u=e^x$. Then using table entry 77, we have $\ln(u+\sqrt{u^2+4})+C = \ln(e^x+\sqrt{e^{2x}+4})+C$.

8.7.20 $\int \frac{\sqrt{\ln^2 x+4}}{x}\,dx = \int \sqrt{u^2+4}\,du$ where $u=\ln x$. Then using table entry 76 we have $\frac{u}{2}\sqrt{u^2+4}+\frac{4}{2}\ln(u+\sqrt{u^2+4})+C = \frac{\ln x}{2}\sqrt{\ln^2 x+4}+2\ln(\ln x+\sqrt{\ln^2 x+4})+C$.

8.7.21 $\int \frac{\cos x}{\sin^2 x+2\sin x}\,dx = \int \frac{1}{u^2+2u}\,du = \int \frac{1}{(u+1)^2-1}\,du$ where $u=\sin x$. Then using table entry 74, we have

$$-\frac{1}{2}\ln\left|\frac{u+2}{u}\right|+C = -\frac{1}{2}\ln\left|\frac{\sin x+2}{\sin x}\right|+C.$$

8.7.22 Let $u=\sqrt{x}$, so that $du=\frac{1}{2\sqrt{x}}\,dx$. Substituting yields $2\int \cos^{-1} u\,du$. Then using table entry 17, we have $2\int \cos^{-1}(u)\,du = 2(u\cos^{-1}(u)-\sqrt{1-u^2})+C = 2\sqrt{x}\cos^{-1}\sqrt{x}-2\sqrt{1-x}+C$.

8.7.23 Let $u=\ln x$, so that $du=\frac{1}{x}\,dx$. Substituting yields $\int u\sin^{-1}(u)\,du = \frac{2u^2-1}{4}\sin^{-1}(u)+\frac{u\sqrt{1-u^2}}{4}+C = \frac{2\ln^2 x-1}{4}\sin^{-1}(\ln x)+\frac{\ln x\sqrt{1-\ln^2 x}}{4}+C$, where we used table entry 105.

8.7.24 Let $u=1+4e^t$, so that $du=4e^t\,dt$. Substituting yields $\int \frac{4e^t}{4e^t\sqrt{1+4e^t}}\,dt = \int \frac{1}{(u-1)\sqrt{u}}\,du$. Now let $v=\sqrt{u}$, so that $dv=\frac{1}{2\sqrt{u}}\,du$. Then we have $\int \frac{2}{v^2-1}\,dv = \ln\left|\frac{v-1}{v+1}\right|+C = \ln\left|\frac{v^2-1}{(v+1)^2}\right|+C = \ln\left|\frac{4e^t}{(\sqrt{1+4e^t}+1)^2}\right|+C$, where we used table entry 74.

8.7.25 Using table entry 84 for $\int \frac{1}{(a^2+x^2)^{3/2}}\,dx$, we have $\int \frac{1}{(16+9x^2)^{3/2}}\,dx = \frac{1}{3}\int \frac{1}{(16+u^2)^{3/2}}\,du = \frac{1}{3}\cdot\frac{3x}{16\sqrt{16+9x^2}} + C = \frac{x}{16\sqrt{16+9x^2}} + C$.

8.7.26 Using table entry 68 for $\int \sqrt{x^2-a^2}\,dx$, we have $\int \sqrt{4x^2-9}\,dx = \frac{1}{2}\int \sqrt{u^2-9}\,du = \frac{x}{2}\sqrt{4x^2-9} - \frac{9}{4}\ln|2x+\sqrt{4x^2-9}| + C$.

8.7.27 Using table entry 62 for $\int \frac{1}{x\sqrt{a^2-x^2}}\,dx$, we have $\int \frac{1}{x\sqrt{(12)^2-x^2}}\,dx = \frac{-1}{12}\ln\left|\frac{12+\sqrt{144-x^2}}{x}\right| + C$.

8.7.28 Using table entry 85 for $\int \frac{dx}{x(x^2+a^2)}$, we have $\int \frac{dv}{v(v^2+8)} = \frac{1}{16}\ln\left(\frac{v^2}{v^2+8}\right) + C$.

8.7.29 Using table entry 103 for $\int \ln^n x\,dx$ we have that $\int \ln^2 x\,dx = x\ln^2 x - 2\int \ln x\,dx = x\ln^2 x - 2(x\ln x - x) + C = 2x + x\ln^2 x - 2x\ln x + C$.

8.7.30 Using table entry 102 (twice) for $\int x^n e^{ax}\,dx$ we have that $\int x^2 e^{5x}\,dx = \frac{1}{5}x^2e^{5x} - \frac{2}{5}\int xe^{ax}\,dx = \frac{1}{5}x^2e^{5x} - \frac{2}{25}\left(xe^{5x} - \frac{1}{5}e^{5x}\right) + C$.

8.7.31 Note that $\sqrt{x^2+10x} = \sqrt{x^2+10x+25-25} = \sqrt{(x+5)^2-5^2}$. Thus, the given integral is equal to $\int \sqrt{u^2-5^2}\,du$ where $u = x+5$. Using table entry 68 we have that this is equal to $\frac{u}{2}\sqrt{u^2-25} - \frac{25}{2}\ln|u+\sqrt{u^2-25}| + C = \frac{x+5}{2}\sqrt{(x+5)^2-25} - \frac{25}{2}\ln|x+5+\sqrt{(x+5)^2-25}| + C$.

8.7.32 Note that $\sqrt{x^2-8x} = \sqrt{x^2-8x+16-16} = \sqrt{(x-4)^2-4^2}$. Thus, the given integral is equal to $\int \sqrt{u^2-4^2}\,du$ where $u = x-4$. Using table entry 68 we have that this is equal to $\frac{u}{2}\sqrt{u^2-16} - 8\ln|u+\sqrt{u^2-16}| + C = \frac{x-4}{2}\sqrt{(x-4)^2-16} - 8\ln|x-4+\sqrt{(x-4)^2-16}| + C$.

8.7.33 $\int \frac{1}{x^2+2x+10}\,dx = \int \frac{1}{(x+1)^2+9}\,dx$. Using table entry 14 this is equal to $\frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + C$.

8.7.34 $\int \sqrt{x^2-4x+8}\,dx = \int \sqrt{(x-2)^2+4}\,dx$. Using table entry 76, this is equal to

$$\frac{x-2}{2}\sqrt{(x-2)^2+4} + 2\ln(x-2+\sqrt{(x-2)^2+4}) + C.$$

8.7.35 $\int \frac{1}{x(x^{10}+1)}\,dx = \int \frac{10x^9}{10x^{10}(x^{10}+1)}\,dx$. Let $u = x^{10}+1$ so that $du = 10x^9\,dx$. Substituting yields $\int \frac{1}{10}\cdot\frac{1}{(u-1)u}\,du$. Using table entry 96 for $\int \frac{dx}{x(ax+b)}$ we have

$$\frac{1}{10}\ln\left|\frac{u-1}{u}\right| + C = \frac{1}{10}\ln\left|\frac{x^{10}}{x^{10}+1}\right| + C.$$

8.7.36 $\int \frac{1}{t(t^8-256)}\,dt = \int \frac{8t^7}{8t^8(t^8-256)}\,dt$. Let $u = t^8 - 256$ so that $du = 8t^7\,dt$. Substituting yields $\frac{1}{8}\int \frac{1}{u(u+256)}\,du$. Using table entry 96 for $\int \frac{dt}{t(at+b)}$ we have

$$\frac{1}{2048}\ln\left|\frac{u}{u+256}\right| + C = \frac{1}{2048}\ln\left|\frac{t^8-256}{t^8}\right| + C.$$

8.7.37 $\int \frac{1}{\sqrt{x^2-6x}}\,dx = \int \frac{1}{\sqrt{(x-3)^2-9}}\,dx$. Using table entry 69, this is equal to $\ln|x-3+\sqrt{(x-3)^2-9}| + C$. This can also be written as $2\ln(\sqrt{x-6}+\sqrt{x}) + C$.

8.7.38 $\int \frac{1}{\sqrt{x^2+10x}}\,dx = \int \frac{1}{\sqrt{(x+5)^2-25}}\,dx$. Using table entry 69, this is equal to

$$\ln|x+5+\sqrt{(x+5)^2-25}| + C.$$

8.7.39 Let $u = x^3$, so that $du = 3x^2\,dx$. Substituting yields

$$\frac{1}{3}\int \frac{\tan^{-1}(u)}{u^2}\,du = -\frac{1}{3}\left(\frac{1}{u}\tan^{-1}(u) - \int \frac{1}{u(1+u^2)}\,du\right) = -\frac{1}{3}\left(\frac{\tan^{-1}(x^3)}{x^3}\right) + \frac{1}{3}\int \frac{1}{u(1+u^2)}\,du,$$

where we used table entry 18. Now let $w = u^2+1$ so that $dw = 2u\,du$. This last integral is thus equal to

$$\frac{1}{3}\int \frac{2u\,du}{2u^2(1+u^2)} = \frac{1}{6}\int \frac{dw}{(w-1)w} = -\frac{1}{6}\ln\left|\frac{w}{w-1}\right| + C = -\frac{1}{6}\ln\left|\frac{u^2+1}{u^2}\right| + C = -\frac{1}{6}\ln\left|\frac{x^6+1}{x^6}\right| + C,$$

where we used table entry 96. Thus the original integral is equal to $-\frac{1}{3}\left(\frac{\tan^{-1}(x^3)}{x^3}\right) - \frac{1}{6}\ln\left|\frac{x^6+1}{x^6}\right| + C$. This can be written as $-\frac{\tan^{-1}(x^3)}{3x^3} + \ln\left|\frac{x}{(x^6+1)^{1/6}}\right| + C$.

8.7.40 Let $u = e^t$ so that $du = e^t\,dt$. Substituting yields $\int \frac{u^2}{\sqrt{4+u^2}}\,du$. Using table entry 82, this yields $-2\ln(u+\sqrt{4+u^2}) + \frac{u\sqrt{4+u^2}}{2} + C$, so the value of the integral is $-2\ln(e^t+\sqrt{4+e^{2t}}) + \frac{e^t\sqrt{4+e^{2t}}}{2} + C$

8.7.41 The integral which gives the length of the curve is $\frac{1}{2}\int_0^8 \sqrt{4+x^2}\,dx$. Using table entry 76, we have

$$\frac{1}{2}\left(\frac{x}{2}\sqrt{4+x^2} + \frac{4}{2}\ln(x+\sqrt{4+x^2})\right)\Bigg|_0^8 = 4\sqrt{17} + \ln(8+2\sqrt{17}) - \ln 2 = 4\sqrt{17} + \ln(4+\sqrt{17}) \approx 18.59.$$

8.7.42 The integral which gives the length of the curve is

$$\frac{1}{2}\int_0^2 \sqrt{4+9x}\,dx = \frac{1}{18}\left(\frac{2}{3}\cdot(4+9x)^{3/2}\right)\Bigg|_0^2 = \frac{1}{27}\left(\sqrt{22}^3 - 8\right) \approx 3.53.$$

We used table entry 86.

8.7.43 The integral which gives the length of the curve is $\int_0^{\ln 2} \sqrt{1+e^{2x}}\,dx = \int_1^2 \frac{\sqrt{1+u^2}}{u}\,du$ where $u = e^x$. This is equal to

$$\left(\sqrt{1+u^2} - \ln\left|\frac{1+\sqrt{1+u^2}}{u}\right|\right)\Bigg|_1^2 = \sqrt{5} - \sqrt{2} + \ln(1+\sqrt{2}) - \ln\left(\frac{1+\sqrt{5}}{2}\right) \approx 1.22.$$

Note that we used table entry 83.

8.7.44 The volume is given by $\pi \int_1^e x^4 \ln x \, dx$. We use table entry 100. This gives

$$\pi \left(\left(\frac{x^5 \ln x}{5} \right) - \frac{x^5}{25} \right) \Big|_1^e = \frac{\pi(4e^5+1)}{25}.$$

8.7.45 Using the method of shells, we have $\frac{V}{2\pi} = \int_0^{12} \frac{x}{\sqrt{x+4}} \, dx = \frac{2}{3}(x-8)\sqrt{x+4} \Big|_0^{12} = \frac{32}{3} + \frac{32}{3} = \frac{64}{3}$. Thus $V = \frac{128\pi}{3}$. We used table entry 94 to compute the integral.

8.7.46 The area is given by

$$\int_0^3 \frac{1}{\sqrt{x^2-2x+2}} \, dx = \int_0^3 \frac{1}{\sqrt{(x-1)^2+1}} \, dx = \int_{-1}^2 \frac{1}{\sqrt{u^2+1}} \, du = \ln(u + \sqrt{u^2+1}) \Big|_{-1}^2 = \ln \frac{2+\sqrt{5}}{\sqrt{2}-1}.$$

We used table entry 77 to compute the integral.

8.7.47 Using the method of disks, we have $\frac{V}{\pi} = \int_0^{\pi/2} \sin^2 y \, dy = \frac{y}{2} - \frac{\sin 2y}{4} \Big|_0^{\pi/2} = \frac{\pi}{4}$. Thus $V = \frac{\pi^2}{4}$. We used table entry 31 to compute the antiderivative.

8.7.48 The average value of f is

$$\frac{1}{2} \int_{-1}^1 \frac{2}{x^2+1} \, dx = \tan^{-1}(x) \Big|_{-1}^1 = \frac{\pi}{2} \approx 1.57.$$

The average value of g is

$$\frac{1}{2} \int_{-1}^1 \frac{7}{4\sqrt{x^2+1}} \, dx = \frac{7}{8} \int_{-1}^1 \frac{1}{\sqrt{x^2+1}} \, dx = \frac{7}{8} \left(\ln(x + \sqrt{x^2+1}) \Big|_{-1}^1 \right) = \frac{7}{8} \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right) \approx 1.542.$$

Thus, the average value of f is greater than the average value of g. We used table entries 14 and 77.

8.7.49 $\int \frac{x}{\sqrt{2x+3}} \, dx = \frac{1}{3}(x-3)\sqrt{3+2x} + C.$

8.7.50 $\int \sqrt{4x^2+36} \, dx = 2 \left(\frac{1}{2} x\sqrt{x^2+9} + \frac{9}{2} \sinh^{-1}\left(\frac{x}{3}\right) \right) + C = \left(x\sqrt{x^2+9} + 9\sinh^{-1}\left(\frac{x}{3}\right) \right) + C$

8.7.51 $\int \tan^2 3x \, dx = \frac{1}{3} \tan 3x - x + C.$

8.7.52 $\int_0^{\pi/2} \cos^6 x \, dx = \frac{5\pi}{32} \approx 0.491.$

8.7.53 $\int_0^4 (9+x^2)^{3/2} \, dx = \frac{1540 + 243 \ln 3}{8} \approx 225.870.$

8.7.54 $\int (a^2-t^2)^{-2} \, dt = \frac{1}{4} \frac{\ln(a+t)}{a^3} - \frac{1}{4} \frac{\ln(t-a)}{a^3} - \frac{1}{4a^2(t-a)} - \frac{1}{4a^2(t+a)} + C.$

8.7.55 $\int \frac{(x^2-a^2)^{3/2}}{x} \, dx = \frac{1}{3}(x^2-a^2)^{3/2} - a^2\sqrt{x^2-a^2} + a^3 \cos^{-1}(a/x) + C.$

8.7.56 $\int \frac{1}{x(a^2-x^2)^2} \, dx = \frac{\frac{a^2}{a^2-x^2} + 2\ln(x) - \ln(x^2-a^2)}{2a^4} + C.$

8.7.57 $\displaystyle\int_0^{\pi/2} \frac{1}{1+\tan^2 t}\,dt = \frac{\pi}{4} \approx 0.785.$

8.7.58 $\displaystyle\int_0^{2\pi} \frac{1}{(4+2\sin t)^2}\,dt = \frac{\pi\sqrt{3}}{9} \approx 0.605.$

8.7.59 $\displaystyle\int (a^2-x^2)^{3/2}\,dx = \frac{-x}{8}(2x^2-5a^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\sin^{-1}(x/a) + C.$

8.7.60 $\displaystyle\int (y^2+a^2)^{-5/2}\,dy = \frac{y(3a^2+2y^2)}{3(a^2+y^2)^{3/2}a^4} + C.$

8.7.61 $\displaystyle\int_0^1 \ln x \ln(1+x)\,dx = 2 - \frac{\pi^2}{12} - \ln 4 \approx -0.209.$

8.7.62 $\displaystyle\int_0^1 x^{31}e^{x^2}\,dx = 653{,}837{,}184{,}000 - 240{,}533{,}257{,}867e \approx 0.0802.$

8.7.63 $\displaystyle\int_1^2 \frac{x^{19}}{\sqrt{x^4-1}}\,dx = \frac{27{,}456\sqrt{15}}{7} \approx 15{,}190.9.$

8.7.64 Let $u = x+1$. Then $du = dx$ and $x = u-1$. We have

$$\begin{aligned}\int x(x+1)^8\,dx &= \int (u-1)u^8\,du = \int (u^9-u^8)\,du = \frac{u^{10}}{10} - \frac{u^9}{9} + C\\ &= \frac{(x+1)^{10}}{10} - \frac{(x+1)^9}{9} + C = (x+1)^9\left(\frac{x+1}{10} - \frac{1}{9}\right) + C\\ &= \frac{1}{90}(x+1)^9(9x-1) + C.\end{aligned}$$

8.7.65 $\displaystyle\int x^3e^{2x}\,dx = \frac{1}{2}x^3e^{2x} - \frac{3}{2}\int x^2e^{2x}\,dx = \frac{1}{2}x^3e^{2x} - \frac{3}{2}\left(\frac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx\right) = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} +$ $\displaystyle\frac{3}{2}\left(\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx\right) = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C.$

8.7.66

$$\begin{aligned}\int p^2e^{-3p}\,dp &= \frac{-1}{3}p^2e^{-3p} + \frac{2}{3}\int pe^{-3p}\,dp = -\frac{1}{3}p^2e^{-3p} + \frac{2}{3}\left(-\frac{1}{3}pe^{-3p} + \frac{1}{3}\int e^{-3p}\,dp\right)\\ &= -\frac{1}{3}p^2e^{-3p} - \frac{2}{9}pe^{-3p} - \frac{2}{27}e^{-3p} + C.\end{aligned}$$

8.7.67 Let $u = 3y$. Then $\displaystyle\int \tan^4 3y\,dy = \frac{1}{3}\int \tan^4 u\,du = \frac{1}{3}\left(\frac{1}{3}\tan^3 u - \int \tan^2 u\,du\right) = \frac{1}{9}\tan^3 u -$ $\displaystyle\frac{1}{3}\left(\tan u - \int du\right) = \frac{1}{9}\tan^3 3y - \frac{1}{3}\tan 3y + y + C.$

8.7.68 Let $u = 4t$. Then

$$\int \sec^4 4t\,dt = \frac{1}{4}\int \sec^4 u\,du = \frac{1}{4}\left(\frac{1}{3}\tan u\sec^2 u + \frac{2}{3}\int \sec^2 u\,du\right) = \frac{1}{12}\tan 4t\sec^2 4t + \frac{1}{6}\tan 4t + C.$$

8.7.69 By inspection, the curves intersect at $x = 1$ where they both have have value $\sqrt{2}$. The area we are seeking is given by $\displaystyle\int_0^1 \left(\frac{2}{\sqrt{\sqrt{x}+1}} - \sqrt{\sqrt{x}+x}\right)dx$. Using a computer algebra system, we find that

$$\int_0^1 \left(\frac{2}{\sqrt{\sqrt{x}+1}} - \sqrt{\sqrt{x}+x}\right)dx = \frac{1}{24}\left(128 - 78\sqrt{2} - 3\ln(3+2\sqrt{2})\right).$$

8.7.70 Note that $f'(x) = \cos x - \sin x$. The surface area is given by

$$\int_0^{\pi} 2\pi(1 + \sin x + \cos x)\sqrt{1 + (\cos x - \sin x)^2}\, dx \approx 41.8741.$$

8.7.71 Let $u = ax + b$, so that $du = a\, dx$. Then we have

$$\frac{1}{a}\int \frac{ax + b - b}{ax + b}\, dx = \frac{1}{a}\int \left(1 - \frac{b}{ax + b}\right) dx = \frac{1}{a}\left(x - \frac{b}{a}\int \frac{1}{u}\, du\right)$$
$$= \frac{1}{a}\left(x - \frac{b}{a}\ln|u|\right) + C = \frac{1}{a}\left(x - \frac{b}{a}\ln|ax + b|\right) + C.$$

8.7.72 Let $u = \sqrt{ax + b}$, so that $du = \dfrac{a}{2\sqrt{ax + b}}\, dx$. Then we have

$$\frac{2}{a}\int x \cdot \frac{a}{2\sqrt{ax + b}}\, dx = \frac{2}{a}\int \frac{u^2 - b}{a}\, du = \frac{2}{a^2}\int (u^2 - b)\, du = \frac{2}{a^2}\left(\frac{u^3}{3} - bu\right) + C$$
$$= \frac{2}{a^2}\left(\frac{(ax + b)^{3/2}}{3} - b\sqrt{ax + b}\right) + C.$$

8.7.73 Let $u = ax + b$, so that $du = a\, dx$. Then we have

$$\frac{1}{a}\int \frac{u - b}{a} u^n\, du = \frac{1}{a^2}\int (u^{n+1} - bu^n)\, du = \frac{1}{a^2}\left(\frac{u^{n+2}}{n + 2} - \frac{bu^{n+1}}{n + 1}\right) + C$$
$$= \frac{1}{a^2}\left(\frac{(ax + b)^{n+2}}{n + 2} - \frac{b(ax + b)^{n+1}}{n + 1}\right) + C.$$

8.7.74 Let $u = \sin^{-1} x$ and $dv = x^n\, dx$. Then $du = \dfrac{1}{\sqrt{1 - x^2}}\, dx$ and $v = \dfrac{x^{n+1}}{n + 1}$. Then the original integral is equal to

$$\frac{x^{n+1}}{n + 1}\sin^{-1} x - \int \frac{x^{n+1}}{n + 1} \cdot \frac{1}{\sqrt{1 - x^2}}\, dx = \frac{1}{n + 1}\left(x^{n+1}\sin^{-1} x - \int \frac{x^{n+1}}{\sqrt{1 - x^2}}\, dx\right).$$

8.7.75

a. True. This is because these are equal for $x > 1$.

b. True. Note that $\dfrac{1}{9} = 0.\overline{11}$.

8.7.76 Note that $\dfrac{d}{dx}\left(\dfrac{1}{2}\cos^{-1}\sqrt{x^{-4}}\right) = \dfrac{1}{2} \cdot \dfrac{-1}{\sqrt{1 - x^{-4}}} \cdot \dfrac{-2}{x^3} = \dfrac{1}{x^3} \cdot \dfrac{1}{\sqrt{1 - (1/x^4)}} = \dfrac{1}{x\sqrt{x^4 - 1}}$.

Also, $\dfrac{d}{dx}\left(\dfrac{1}{2}\cos^{-1} x^{-2}\right) = \dfrac{1}{2} \cdot \dfrac{-1}{\sqrt{1 - x^{-4}}} \cdot \dfrac{-2}{x^3} = \dfrac{1}{x\sqrt{x^4 - 1}}$.

And finally, $\dfrac{d}{dx}\left(\dfrac{1}{2}\tan^{-1}\sqrt{x^4 - 1}\right) = \dfrac{1}{2} \cdot \dfrac{1}{1 + x^4 - 1} \cdot \dfrac{4x^3}{2\sqrt{x^4 - 1}} = \dfrac{1}{x\sqrt{x^4 - 1}}$.

Thus all three answers are correct in the sense that they are all antiderivatives of the original integrand, and thus must differ by a constant.

8.7.77 The two answers differ by a constant, namely the constant one. This can be seen as follows:

$$\frac{2\sin(x/2)}{\cos(x/2) + \sin(x/2)} = \frac{2\tan(x/2)}{1 + \tan(x/2)} = 2\frac{\sin x}{1 + \cos x} \cdot \frac{1}{1 + \frac{\sin x}{1 + \cos x}}$$
$$= \frac{2\sin x}{1 + \cos x + \sin x} = \frac{2\sin x}{1 + \cos x + \sin x} \cdot \frac{1 - \cos x - \sin x}{1 - \cos x - \sin x}$$
$$= \frac{2\sin x(1 - \cos x - \sin x)}{-2\sin x\cos x} = \frac{\sin x + \cos x - 1}{\cos x} = \frac{\sin x - 1}{\cos x} + 1.$$

8.7.78 Note that $\frac{\ln|x-1|}{3}+\frac{\ln|x+2|}{6}-\frac{\ln|x|}{2}+C=\frac{1}{6}\left(2\ln|x-1|+\ln|x+2|-3\ln|x|\right)+C=$ $\frac{1}{6}\left(\ln(x-1)^2+\ln|x+2|-\ln|x|^3\right)+C=\frac{1}{6}\ln\left(\frac{(x-1)^2|x+2|}{|x|^3}\right)+C.$

8.7.79 Let $u=2x$. Then

$$\int x\sin^{-1}2x\,dx=\frac{1}{4}\int u\sin^{-1}u\,du=\frac{2u^2-1}{16}\sin^{-1}u+\frac{u\sqrt{1-u^2}}{16}+C=\frac{8x^2-1}{16}\sin^{-1}2x+\frac{x\sqrt{1-4x^2}}{8}+C.$$

8.7.80 Let $u=10x$. Then

$$\begin{aligned}\frac{1}{25}\int u\cos^{-1}u\,du&=\frac{1}{25}\left(\frac{2u^2-1}{4}\cos^{-1}u-\frac{u\sqrt{1-u^2}}{4}\right)+C\\&=\frac{1}{25}\left(\frac{200x^2-1}{4}\cos^{-1}10x-\frac{5x\sqrt{1-100x^2}}{2}\right)+C.\end{aligned}$$

8.7.81 $\int x^{-2}\tan^{-1}x\,dx=-(1/x)\tan^{-1}x+\int\frac{1}{x(1+x^2)}\,dx=\frac{-\tan^{-1}x}{x}+\ln\left(\frac{|x|}{\sqrt{1+x^2}}\right)+C.$

8.7.82 $\int\frac{\sin^{-1}ax}{x^2}\,dx=-a\left(\frac{\sin^{-1}u}{u}-\int\frac{1}{u\sqrt{1-u^2}}\,du\right)$, where $u=ax$. Then we have

$$\frac{-a\sin^{-1}u}{u}-a\ln\left|\frac{1+\sqrt{1-u^2}}{u}\right|+C=\frac{-\sin^{-1}ax}{x}-a\ln\left|\frac{1+\sqrt{1-(ax)^2}}{ax}\right|+C.$$

8.7.83

a. Note that $\int_0^b f(x)\,dx=\int_0^{b/2}f(x)\,dx+\int_{b/2}^b f(x)\,dx$. For the second integral, let $u=b-x$. Then $du=-dx$. The second integral is equal to $-\int_{b-(b/2)}^0 f(b-u)\,du=\int_0^{b/2}f(b-u)\,du$. Because this can be written as $\int_0^{b/2}f(b-x)\,dx$, we have $\int_0^b f(x)\,dx=\int_0^{b/2}(f(x)+f(b-x))\,dx$.

b.

$$\begin{aligned}\int_0^{\pi/4}\ln(1+\tan x)\,dx&=\int_0^{\pi/8}(\ln(1+\tan x)+\ln(1+\tan(\pi/4-x)))\,dx\\&=\int_0^{\pi/8}\left(\ln(1+\tan x)+\ln\left(1+\frac{\tan\pi/4-\tan x}{1+\tan\pi/4\tan x}\right)\right)\,dx\\&=\int_0^{\pi/8}\left(\ln(1+\tan x)+\ln\left(1+\frac{1-\tan x}{1+\tan x}\right)\right)\,dx\\&=\int_0^{\pi/8}\left(\ln(1+\tan x)+\ln\left(\frac{2}{1+\tan x}\right)\right)=\int_0^{\pi/8}\ln(2)\,dx=\frac{\pi\ln 2}{8}.\end{aligned}$$

8.7.84

a. Let $u=\ln x$. Then $du=\frac{dx}{x}$, so $dx=x\,du=e^u\,du$. Substituting gives $\int\cos(u)e^u\,du$. Table entry 98 gives $\frac{e^u(\cos u+\sin u)}{2}+C=\frac{x(\cos\ln x+\sin\ln x)}{2}+C.$

b. Let $u = \cos \ln x$ and $dv = dx$. Then $du = \dfrac{-\sin \ln x}{x}\,dx$ and $v = x$. Then we have $\int \cos \ln x\,dx = x \cos \ln x + \int \sin \ln x\,dx$. Now, focusing on the last integral, let $u = \sin \ln x$ and $dv = dx$. Then $du = \dfrac{\cos \ln x}{x}\,dx$ and $v = x$. Now we have $\int \cos \ln x\,dx = x \cos \ln x + \int \sin \ln x\,dx = x \cos \ln x + x \sin \ln x - \int \cos \ln x\,dx$ Adding $\int \cos \ln x\,dx$ to both sides of this equation and dividing by 2 gives

$$\int \cos \ln x\,dx = \frac{x \cos \ln x + x \sin \ln x}{2} + C.$$

8.7.85

a. Using a computer algebra system, we have

θ_0	T	Relative Error
0.1	6.27927	0.000623603
0.2	6.26762	0.0024778
0.3	6.24854	0.0051388
0.4	6.22253	0.00965413
0.5	6.19021	0.0147967
0.6	6.15236	0.0208215
0.7	6.10979	0.0275963
0.8	6.06338	0.0349831
0.9	6.01399	0.0428433
1.0	5.96247	0.0510427

b. All of these values are within 10 percent of 2π.

8.7.86

a. $L(c) = \int_0^c \sqrt{1+4x^2}\,dx$. Note that $L'(c) = \sqrt{1+4c^2} > 0$, so L is increasing.

b. Note that $L''(c) = \dfrac{4c}{\sqrt{1+4c^2}} \geq 0$ on $[0, \infty)$, so L is concave up on that interval.

c. We have $\lim_{c\to\infty} \dfrac{L(c)}{c^2} = \lim_{c\to\infty} \dfrac{L'(c)}{2c} = \lim_{c\to\infty} \dfrac{\sqrt{1+4c^2}}{2c} = 1$. Thus $L(c)$ increases as c^2 increases.

8.7.87

a. The result holds.

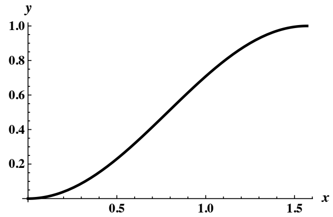
b. First note that $\int_0^{\pi/2} \cos^n x\,dx = -\int_{\pi/2}^0 \cos^n(\pi/2 - \theta)\,d\theta = \int_0^{\pi/2} \sin^n \theta\,d\theta$. So we only need to show the result for $\int_0^{\pi/2} \sin^{10} x\,dx$. Repeatedly using the reduction formula we have

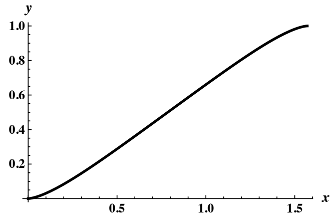
$$\begin{aligned}\int_0^{\pi/2} \sin^{10} x\,dx &= \frac{9}{10}\int_0^{\pi/2} \sin^8 x\,dx = \frac{9\cdot 7}{10\cdot 8}\int_0^{\pi/2} \sin^6 x\,dx \\ &= \frac{9\cdot 7\cdot 5}{10\cdot 8\cdot 6}\int_0^{\pi/2} \sin^4 x\,dx = \frac{9\cdot 7\cdot 5\cdot 3}{10\cdot 8\cdot 6\cdot 4}\int_0^{\pi/2} \sin^2 x\,dx \\ &= \frac{9\cdot 7\cdot 5\cdot 3}{10\cdot 8\cdot 6\cdot 4\cdot 2}\int_0^{\pi/2} dx = \frac{63\pi}{2^9}\end{aligned}$$

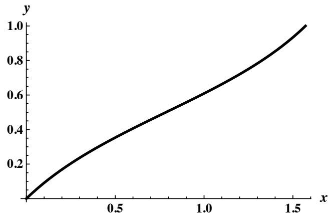
c. The values decrease as n increases.

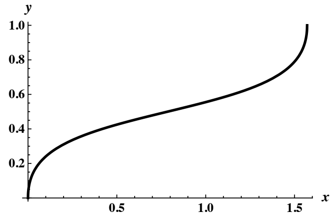
8.7.88

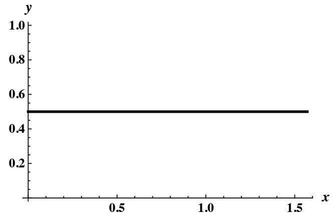
a. The graphs change shape, but the area under the curve remains constant.

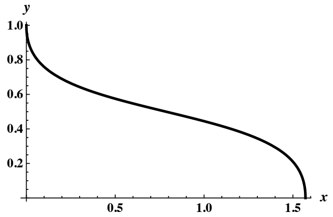


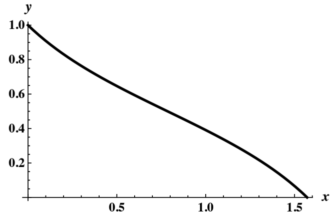


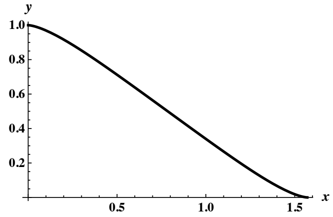


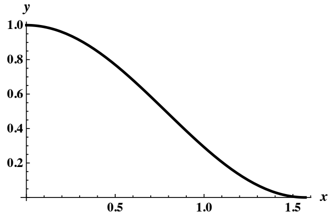












b. When $m = 0$, the integrand is equal to $\frac{1}{2}$ and the bounded region forms a rectangle of width $\frac{\pi}{2}$ and height $\frac{1}{2}$. For all values of m, the graph of $f(x) = \frac{1}{(1 + \tan^m x)}$ is symmetric with respect to the point $(\pi/2, 1/2)$. When $m \neq 0$, for any area located above $y = \frac{1}{2}$ and below f on one side of $x = \frac{\pi}{4}$, there is an equal amount an area below $y = \frac{1}{2}$ and above f on the other side of $x = \frac{\pi}{4}$. The result is easily verified with a computer algebra system.

8.8 Numerical Integration

8.8.1 $\Delta x = \frac{18 - 4}{28} = \frac{1}{2}$.

8.8.2 The Midpoint Rule uses the value of the function evaluated at the midpoint of each subinterval to determine the height of the approximating rectangle over each subinterval. The areas of these rectangles are added up to yield an approximation to the definite integral.

8.8.3 The Trapezoidal Rule approximates the definite integral by using a trapezoid over each subinterval rather than a rectangle.

8.8.4 It is evaluated at 1, 5, and 9, which are the midpoints of the 3 subinterval of length 4.

8.8.5 $M(4) = 2(2) + 5(2) + 6(2) + 8(2) = 42.$

8.8.6 $T(4) = \left(\frac{1}{2}(1) + 4 + 7 + 5 + \frac{1}{2}(5)\right) 2 = 38.$

8.8.7 $S(4) = (1 + 4(4) + 2(7) + 4(5) + 5)\,\frac{2}{3} = \frac{112}{3} \approx 37.33.$

8.8.8 $S(8) = ((1 + 4(2) + 2(4) + 4(5) + 2(7) + 4(6) + 2(5) + 4(8) + 5)\,\frac{1}{3} = \frac{122}{3} \approx 40.67.$

8.8.9 The endpoints of the subintervals are -1, 1, 3, 5, 7, and 9. The trapezoidal rule uses the value of f at each of these endpoints.

8.8.10 $S(4) = \frac{4T(4) - 6}{3} = 4.8.$

8.8.11 The absolute error is $|\pi - 3.14| \approx 0.0015926536$. The relative error is $\frac{|\pi - 3.14|}{\pi} \approx 5 \times 10^{-4}$.

8.8.12 The absolute error is $|\sqrt{2} - 1.414| \approx 2.14 \times 10^{-4}$. The relative error is $\frac{|\sqrt{2} - 1.414|}{\sqrt{2}} \approx 1.51 \times 10^{-4}$.

8.8.13 The absolute error is $|e - 2.72| \approx 0.0017181715$. The relative error is $\frac{|e - 2.72|}{e} \approx 6.32 \times 10^{-4}$.

8.8.14 The absolute error is $|e - 2.718| \approx 2.81 \times 10^{-4}$ and the relative error is $\frac{|e - 2.718|}{e} \approx 1.04 \times 10^{-4}$.

8.8.15

For $n = 1$, we have $f(6) \cdot 8 = 72 \cdot 8 = 576$.
For $n = 2$ we have $f(4) \cdot 4 + f(8) \cdot 4 = 32 \cdot 4 + 128 \cdot 4 = 640$.
For $n = 4$, we have $f(3) \cdot 2 + f(5) \cdot 2 + f(7) \cdot 2 + f(9) \cdot 2 = 18 \cdot 2 + 50 \cdot 2 + 98 \cdot 2 + 162 \cdot 2 = 656$.

8.8.16

For $n = 1$, we have $f(5) \cdot 8 = 125 \cdot 8 = 1000$.
For $n = 2$ we have $f(3) \cdot 4 + f(7) \cdot 4 = 27 \cdot 4 + 343 \cdot 4 = 1480$.
For $n = 4$, we have $f(2) \cdot 2 + f(4) \cdot 2 + f(6) \cdot 2 + f(8) \cdot 2 = 8 \cdot 2 + 64 \cdot 2 + 216 \cdot 2 + 512 \cdot 2 = 1600$.

8.8.17 We have

$$\frac{1}{6}\left(\sin(\pi/12) + \sin(\pi/4) + \sin(5\pi/12) + \sin(7\pi/12) + \sin(3\pi/4) + \sin(11\pi/12)\right) \approx 0.6439505509.$$

8.8.18 We have

$$\frac{1}{8}\left(e^{-1/16} + e^{-3/16} + e^{-5/16} + e^{-7/16} + e^{-9/16} + e^{-11/16} + e^{-13/16} + e^{-15/16}\right) \approx 0.6317092095.$$

8.8.19 For $n = 2$ we have $T(2) = \frac{4}{2}(f(2) + 2f(6) + f(10)) = 2(8 + 2(72) + 200) = 704$.

Using the results of number 15: For $n = 4$, note that $T(4) = \frac{T(2) + M(2)}{2} = \frac{704 + 640}{2} = 672$.

Using the results of number 15: For $n = 8$ we have that $T(8) = \frac{T(4) + M(4)}{2} = \frac{672 + 656}{2} = 664$.

8.8.20

$T(2) = \frac{T(1) + M(1)}{2}$. Using problem 16 if we compute $T(1)$.

We have $T(1) = \left(\frac{1}{2} + \frac{729}{2}\right) \cdot 8 = 2920$. Thus, $T(2) = \frac{2920 + 1000}{2} = 1960$.

For $n = 4$, we have $T(4) = \frac{T(2) + M(2)}{2} = \frac{1960 + 1480}{2} = 1720$.

For $n = 8$ we have $T(8) = \frac{T(4) + M(4)}{2} = \frac{1720 + 1600}{2} = 1660$.

8.8.21 We have

$$T(6) = \frac{1}{12}\left(\sin 0 + 2\sin\frac{\pi}{6} + 2\sin\frac{\pi}{3} + 2\sin\frac{\pi}{2} + 2\sin\frac{2\pi}{3} + 2\sin\frac{5\pi}{6} + \sin\pi\right)$$
$$= \frac{1}{6}\left(\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{1}{6}\left(2+\sqrt{3}\right).$$

8.8.22 $T(8) = \frac{1}{16}(e^{-0} + 2e^{-1/8} + 2e^{-1/4} + 2e^{-3/8} + 2e^{-1/2} + 2e^{-5/8} + 2e^{-3/4} + 2e^{-7/8} + e^{-1}) \approx 0.6329434182.$

8.8.23 $S(4) = \frac{\pi}{12}\left(\sqrt{\sin 0} + 4\sqrt{\sin\frac{\pi}{4}} + 2\sqrt{\sin\frac{\pi}{2}} + 4\sqrt{\sin\frac{3\pi}{4}} + \sqrt{\sin\pi}\right) \approx 2.28477.$

$$S(8) \approx \frac{\pi}{24}\left(\sqrt{\sin 0} + 4\sqrt{\sin\frac{\pi}{8}} + 2\sqrt{\sin\frac{\pi}{4}} + 4\sqrt{\sin\frac{3\pi}{8}} + \right.$$
$$\left. 2\sqrt{\sin\frac{\pi}{2}} + 4\sqrt{\sin\frac{5\pi}{8}}. + 2\sqrt{\sin\frac{3\pi}{4}} + 4\sqrt{\sin\frac{7\pi}{8}} + \sqrt{\sin\pi}\right) \approx 2.35646.$$

8.8.24 $S(4) = \frac{1}{3}\left(\sqrt{4} + 4\sqrt{5} + 2\sqrt{6} + 4\sqrt{7} + \sqrt{8}\right) \approx 9.75156.$

$$S(8) \approx \frac{1}{6}\left(\sqrt{4} + 4\sqrt{4.5} + 2\sqrt{5} + 4\sqrt{5.5} + 2\sqrt{6} + 4\sqrt{6.5} + 2\sqrt{7} + 4\sqrt{7.5} + \sqrt{8}\right) \approx 9.75161.$$

8.8.25

$$S(10) = \frac{1}{6}\left(e^{-4} + 4e^{-2.25} + 2e^{-1} + 4e^{-0.25} + 2e^{0} + 4e^{-0.25} + 2e^{-1}\right.$$
$$\left. + 4e^{-2.25} + 2e^{-4} + 4e^{-6.25} + e^{-9}\right) \approx 1.7680.$$

8.8.26

$$S(8) = \frac{1}{12}\left(\cos\sqrt{2} + 4\cos\sqrt{2.25} + 2\cos\sqrt{2.5} + 4\cos\sqrt{2.75}\right.$$
$$\left. + 2\cos\sqrt{3} + 4\cos\sqrt{3.25} + 2\cos\sqrt{3.5} + 4\cos\sqrt{3.75} + \cos\sqrt{4}\right) \approx -0.30082.$$

8.8.27 The width of each subinterval is 1/25, so

$$M(25) = \frac{1}{25}(\sin\pi/50 + \sin 3\pi/50 + \sin 5\pi/50 + \cdots + \sin 49\pi/50) \approx 0.63704.$$

Because $\int_0^1 \sin\pi x\,dx = \frac{2}{\pi}$, the absolute error is $|2/\pi - M(25)| \approx 4.19 \times 10^{-4}$ and the relative error is this number divided by $2/\pi$ which is approximately 6.58×10^{-4}. The Trapezoidal Rule yields approximately 0.63578, with a relative error of ≈ 0.00132.

8.8.28 The width of each subinterval is 1/50, so

$$M(50) = \frac{1}{50}(e^{-1/100} + (e^{-1/100})^3 + (e^{-1/100})^5 + \cdots + (e^{-1/100})^{99}) \approx 0.63211.$$

$$T(50) = \frac{1}{100}(e^0 + 2e^{-1/50} + 2e^{-2/50} + \cdots + 2e^{-49/50} + e^{-1}) \approx 0.63214.$$

The actual value of the integral is $1 - \frac{1}{e}$. The absolute error for $M(50)$ is $|1 - \frac{1}{e} - .63211| \approx 1 \times 10^{-5}$, and the relative error is that number divided by $1 - \frac{1}{e}$ which is about 1.66×10^{-5}. The absolute error for $T(50)$ is $|1 - \frac{1}{e} - .63214| \approx 2.1 \times 10^{-5}$ and the relative error is that number divided by $1 - \frac{1}{e}$ which is about 3.33×10^{-5}.

8.8.29

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	99	1	102	2
8	99.75	0.250	100.5	0.5
16	99.9375	0.0625	100.125	0.125
32	99.984375	0.0156	100.03125	0.03125

8.8.30

n	$T(n)$	Absolute Error	$M(n)$	Absolute Error
4	6	2	3	1
8	4.5	0.5	3.75	0.25
16	4.125	0.125	3.9375	0.0625
32	4.03125	0.03125	3.984375	0.015625

8.8.31

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	1.50968181	9.7×10^{-3}	1.48067370	1.9×10^{-2}
8	1.50241228	2.4×10^{-3}	1.49517776	4.8×10^{-3}
16	1.50060256	6.0×10^{-4}	1.49879502	1.2×10^{-3}
32	1.50015061	1.5×10^{-4}	1.49969879	3.0×10^{-4}

8.8.32

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	1.004785839	4.8×10^{-3}	0.99036501	9.6×10^{-3}
8	1.001210217	1.2×10^{-3}	0.99757542	2.4×10^{-3}
16	1.000303459	3.03×10^{-4}	0.99939282	6.07×10^{-4}
32	1.000075922	7.59×10^{-5}	0.99984814	1.52×10^{-4}

8.8.33 Because the given function has odd symmetry about the midpoint of the interval $[0, \pi]$, the midpoint rule calculates to be zero for all even values of n, as does the trapezoidal rule.

8.8.34

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	0.27572053	0.224	1.0373146	0.54
8	0.425459016	0.075	0.65651757	0.15
16	0.47975863	0.02	0.54098829	0.041
32	0.49482934	0.0052	0.51037346	0.01

8.8.35 $T(4) = 3\,(9.1 + 2(12) + 2(26) + 2(46) + 53) = 690.3\,\text{million}\,\text{ft}^3$.
$S(4) = 2(9.1 + 4(12) + 2(26) + 4(46) + 53) = 692.2\,\text{million ft}^3$.

8.8.36 $T(6) = 2(9.1 + 2(10.5) + 2(15) + 2(26) + 2(40) + 2(49) + 53) = 686.2\,\text{million}\,\text{ft}^3$.
$S(6) = \dfrac{4}{3}\,(9.1 + 4(10.5) + 2(15) + 4(26) + 2(40) + 4(49) + 53) = 685.5\,\text{million}\,\text{ft}^3$.

8.8.37 Answers may vary.

$$\begin{aligned} \overline{T} &= \frac{1}{12}\int_0^{12} T(t)\,dt \approx \frac{1}{12}\text{Trapezoid}(12) \\ &\approx \frac{1}{24}(47 + 2(50 + 46 + 45 + 48 + 52 + 54 + 61 + 62 + 63 + 63 + 59) + 55) = 54.5. \end{aligned}$$

8.8.38 Answers may vary.

$$\begin{aligned}\overline{T} &= \frac{1}{12}\int_0^{12} T(t)\,dt \approx \frac{1}{12}\text{Trapezoid}(12)\\ &\approx \frac{1}{24}(41+2(44+46+48+52+53+53+53+51+51+49+47)+47) = 49.25.\end{aligned}$$

8.8.39 Answers may vary.

$$\begin{aligned}\overline{T} &= \frac{1}{12}\int_0^{12} T(t)\,dt \approx \frac{1}{12}\text{Trapezoid}(12)\\ &\approx \frac{1}{24}(35+2(34+34+36+36+37+37+36+35+35+34+33)+32)) \approx 35.0.\end{aligned}$$

8.8.40 Answers may vary.

$$\begin{aligned}\overline{T} &= \frac{1}{12}\int_0^{12} T(t)\,dt \approx \frac{1}{12}\text{Trapezoid}(12)\\ &\approx \frac{1}{24}(9+2(11+11+12+14+15+17+19+20+22+24+24)+25) \approx 17.1667.\end{aligned}$$

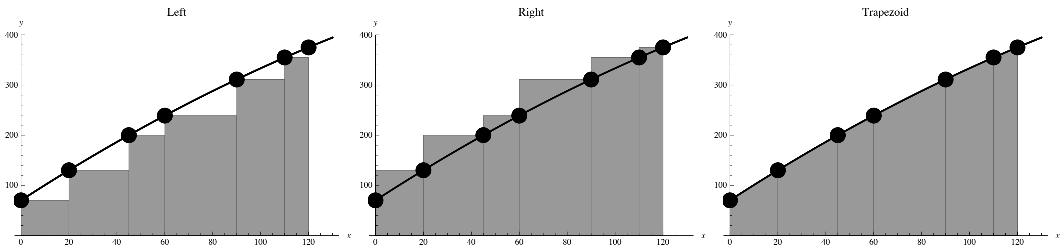
8.8.41

a. Left Riemann sum: $\frac{1}{120}\left(70\cdot 20+130\cdot 25+200\cdot 15+239\cdot 30+311\cdot 20+355\cdot 10\right) = 204.917$.
Right Riemann sum: $\frac{1}{120}\left(130\cdot 20+200\cdot 25+239\cdot 15+311\cdot 30+355\cdot 20+375\cdot 10\right) = 261.375$.
Trapezoidal rule:

$$\begin{aligned}&\frac{(70+130)\cdot 20}{2\cdot 120}+\frac{(130+200)\cdot 25}{2\cdot 120}+\frac{(200+239)\cdot 15}{2\cdot 120}+\frac{(239+311)\cdot 30}{2\cdot 120}\\ &\qquad+\frac{(311+355)\cdot 20}{2\cdot 120}+\frac{(355+375)\cdot 10}{2\cdot 120} = 233.146\end{aligned}$$

These are approximations to the average temperature of the curling iron on the interval $[0,120]$.

b. Because the function is increasing, the left Riemann sum is an underestimate and the right Riemann sum is an overestimate. The Trapezoidal rule appears to be a slight underestimate. The followings is a plot of the points with straight line segments connecting them. The shape suggestions that the actual function is concave down on the interval, so the trapezoidal rule will be an underestimate.



c. The change in temperature over the time interval is

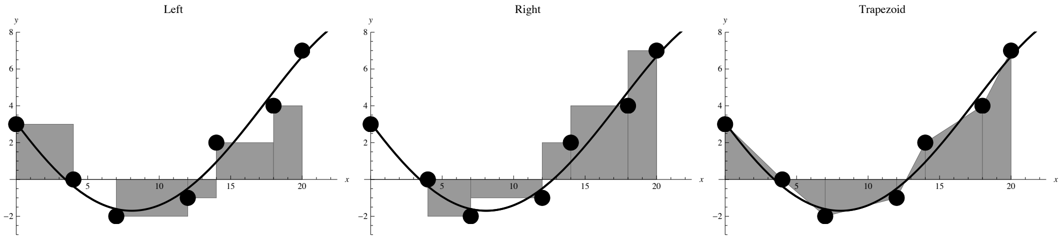
$$\int_0^{120} T'(t)\,dt = T(120) - T(0) = 375 - 70 = 305\,\text{degrees}.$$

8.8.42

a. Left Riemann sum: $(3 \cdot 4 + 0 \cdot 3 + -2 \cdot 5 + -1 \cdot 2 + 2 \cdot 4 + 4 \cdot 2) = 16$.
Right Riemann sum: $(0 \cdot 4 + -2 \cdot 3 + -1 \cdot 5 + 2 \cdot 2 + 4 \cdot 4 + 7 \cdot 2) = 23$.
Trapezoidal rule:

$$\frac{(3+0)\cdot 4}{2} + \frac{(0+-2)\cdot 3}{2} + \frac{(-2+-1)\cdot 5}{2} + \frac{(2+-1)\cdot 2}{2} + \frac{(2+4)\cdot 4}{2} + \frac{(4+7)\cdot 2}{2} = 19.5$$

b. Plots of all three approximations, together with the point themselves and a curve fitted through the points (a degree 4 curve, actually) are:



c. We have

$$\int_4^{12} (3f'(x) - 2)\, dx = 3\int_4^{12} f'(x)\, dx + \int_4^{12} 2\, dx = 3(f(12) - f(4)) + 2 \cdot (12 - 4) = 13.$$

8.8.43

a. The net change in elevation is $\displaystyle\int_0^5 v(t)\, dt$, which can be approximated by the trapezoidal rule to give

$$\begin{aligned} &\frac{(0+100)\cdot 1}{2} + \frac{(100+120)\cdot 0.5}{2} + \frac{(120+150)\cdot 1.5}{2} \\ &\qquad + \frac{(150+110)\cdot 0.5}{2} + \frac{(110+90)\cdot 0.5}{2} + \frac{(90+80)\cdot 1}{2} = 507.5. \end{aligned}$$

If we add the net change to the original elevation of 5400 feet, we have an elevation of approximately 5907.5 feet.

b. The right Riemann sum is

$$100 \cdot 1 + 120 \cdot 0.5 + 150 \cdot 1.5 + 110 \cdot 0.5 + 90 \cdot 0.5 + 80 \cdot 1 = 565.$$

If we add the net change to the original elevation of 5400 feet, we have an elevation of approximately 5965 feet.

c. The elevation can be estimated by

$$5400 + \int_0^5 g(t)\, dt = 5400 + \int_0^5 (3.49t^3 - 43.21t^2 + 142.43t - 1.75)\, dt \approx 5916.52,$$

so the elevation of the balloon is about 5917 feet.

8.8.44

a. The trapezoidal rule gives

$$\begin{aligned} &\frac{(0+2.5)\cdot 1}{2} + \frac{(2.5+3.2)\cdot 1}{2} + \frac{(3.2+4)\cdot 1}{2} + \frac{(4+6)\cdot 1}{2} \\ &\qquad + \frac{(6+7)\cdot 2}{2} + \frac{(7+5.3)\cdot 1.5}{2} + \frac{(3+0)\cdot 0.5}{2} = 35.675. \end{aligned}$$

b. The left Riemann sum gives

$$0 \cdot 1 + 2.5 \cdot 1 + 3.2 \cdot 1 + 4 \cdot 1 + 6 \cdot 2 + 7 \cdot 1.5 + 5.3 \cdot 0.5 = 34.85.$$

c. Although the surface area of the piece appears to be less than half of $81 = 9^2$ (the area of a 9×9 piece of wood), the shape prohibits the creation of two identical pieces.

8.8.45

n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
25	3.19623162	—	—	—
50	3.19495398	4.3×10^{-4}	3.19452809	4.5×10^{-8}

8.8.46

n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
30	6.411850535	—	—	—
60	6.402962881	.003	6.400000329	3.3×10^{-7}

8.8.47

n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
50	1.00008509	—	—	—
100	1.00002127	2.1×10^{-5}	1.00000000	4.6×10^{-9}

8.8.48

n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
64	0.6657662105	—	—	—
128	0.6657718648	1.9×10^{-6}	.665774	8.3×10^{-12}

8.8.49

n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
4	1820	284	—	—
8	1607.75	71.8	1537	1
16	1553.9844	18	1536.0625	6.3×10^{-2}
32	1540.4990	4.5	1536.0039	3.9×10^{-3}

8.8.50

n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
4	0.99036501	9.6×10^{-3}	—	—
8	0.99757542	2.4×10^{-3}	0.99997889	2.1×10^{-5}
16	0.99939282	6.1×10^{-4}	0.99999862	1.4×10^{-6}
32	0.99984814	1.5×10^{-4}	0.9999999	8.7×10^{-8}

8.8.51

n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
4	0.46911538	5.3×10^{-2}	—	—
8	0.50826998	1.3×10^{-2}	0.52132152	2.9×10^{-4}
16	0.51825968	3.4×10^{-3}	0.52158957	1.7×10^{-5}
32	0.52076933	8.4×10^{-4}	0.52160588	1.1×10^{-6}

8.8.52

n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
4	2.300552032	1.30	—	—
8	1.390088342	.39	1.086600445	8.7×10^{-2}
16	1.103308986	.1	1.007715867	7.7×10^{-3}
32	1.026229165	.026	1.000535892	5.4×10^{-4}

8.8.53

a. True. Note that $\Delta x = \dfrac{b-a}{18}$, and $E_S(18) \le \dfrac{1 \cdot (b-a)}{10 \cdot 18} \cdot (\Delta x)^4 = \dfrac{(\Delta x)^5}{10}$.

b. False. Because $E_M(n) \le \dfrac{k(b-a)^3}{24n^2}$, we have $E_M(3n) \le \dfrac{k(b-a)^3}{24(9n^2)}$, so the error decreases by a factor of about 9.

c. True. Because $E_T(n) \le \dfrac{k(b-a)^3}{12n^2}$, we have $E_T(4n) \le \dfrac{k(b-a)^3}{24(16n^2)}$, so the error decreases by a factor of about 16.

8.8.54 $\displaystyle\int_0^{\pi/2} \sin^6 x\,dx = \frac{5\pi}{32} \approx 0.4908738521.$

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	0.4908738521	2.341×10^{-11}	.4908738521	2.341×10^{-11}
8	0.4908738521	≈ 0	0.49087352	≈ 0
16	0.4908738521	≈ 0	0.49087352	≈ 0
32	0.4908738521	≈ 0	0.49087352	≈ 0

8.8.55 $\displaystyle\int_0^{\pi/2} \cos^9 x\,dx = \frac{128}{315}.$

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	0.40635058	1.4×10^{-6}	0.40634783	1.4×10^{-6}
8	0.40634921	7.6×10^{-10}	0.40634921	7.6×10^{-9}
16	0.40634921	6.6×10^{-13}	0.40634921	6.6×10^{-13}
32	0.40634921	8.9×10^{-16}	0.40634921	7.8×10^{-16}

8.8.56 $\displaystyle\int_0^{1} (8x^7 - 7x^8)\,dx = \frac{2}{9}.$

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	0.2192	2.2×10^{-3}	0.2257	3.4×10^{-3}
8	0.222	2.0×10^{-4}	0.2224	2.2×10^{-4}
16	0.2222	1.2×10^{-5}	0.2222	1.4×10^{-5}
32	0.2222	7.8×10^{-7}	0.2222	3.9×10^{-7}

8.8.57 $\displaystyle\int_0^{\pi} \ln(5 + 3\cos x)x\,dx = \pi\ln(9/2).$

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	4.72531820	1.2×10^{-4}	4.72507878	1.2×10^{-4}
8	4.72519851	9.1×10^{-9}	4.72519849	9.1×10^{-9}
16	4.72519850	0.	4.72519850	8.9×10^{-16}
32	4.72519850	0.	4.72519850	8.9×10^{-19}

8.8.58 $\displaystyle\int_0^{2\pi} \frac{1}{(5+3\sin x)^2}\,dx = \frac{5\pi}{32} \approx 0.4908738521.$

n	$S(n)$	Absolute Error
4	0.6400995032	0.149
8	0.475006	0.0159
16	0.49050507	3.7×10^{-4}

8.8.59 $\displaystyle\int_0^{\pi} \frac{\cos x}{(5/4)-\cos x}\,dx = \frac{2\pi}{3} \approx 2.094395102.$

n	$S(n)$	Absolute Error
4	1.916439963	0.178
8	2.080919302	0.0135
16	2.094341841	5.3×10^{-5}

8.8.60 $\displaystyle\int_0^{\pi} \ln(2+\cos x)\,dx = \pi \ln\left(\frac{2+\sqrt{3}}{2}\right) \approx 1.959759164.$

n	$S(n)$	Absolute Error
4	1.962437352	.0027
8	1.95976612	6.96×10^{-6}

8.8.61 $\displaystyle\int_0^{\pi} \sin 6x \cos 3x\,dx = \frac{4}{9} = 0.\overline{4}.$

n	$S(n)$	Absolute Error
8	0.0305049084	0.475
16	0.4540112289	0.0096
32	0.44487	0.00087

8.8.62 $\displaystyle T = \frac{4}{w}\int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\phi}}\,d\phi$, $w = \sqrt{\frac{g}{L}}$, $g = 9.8$, and $k^2 = \sin^2(\theta_0/2)$.

n	$S(n)$
4	2.08732001
8	2.08732001

8.8.63 We are computing $\displaystyle\frac{1}{3\sqrt{2\pi}}\int_{66}^{72} e^{-(x-69)^2/18}\,dx$. If we use Simpson's rule we obtain

n	$S(n)$
4	0.683
8	0.683

So about 68.3%.

8.8.64 We are computing $\displaystyle\frac{1}{22\sqrt{2\pi}}\int_{60}^{90} e^{-(x-110)^2/968}\,dx$. If we compute $S(8)$, we see that $S(8) \approx 0.17012657$.

8.8.65

a. For even n we have $S(n) = \dfrac{20}{n}(365)(f(a) + 4f(x_1) + 4f(x_2) + \ldots + 4f(x_{n-1}) + f(b))$. For $n = 6$ since there are 6 decades between 1940 and 2000, we have $S(6) \approx 160{,}000$ millions of barrels produced.

b. Following part (a) with $n = 6$ we have $S(n) \approx 68{,}000$ millions of barrels imported.

8.8.66

a. $M(30) \approx 1.80967$.

b. $f'(x) = -2x\sin x^2$, so $f''(x) = -2\sin x^2 + (-2x)(\cos x^2)2x = -2(\sin x^2 + 2x^2\cos x^2)$.

c. For $-1 \leq x \leq 1$,

$$|-2(2x^2\cos x^2 + \sin x^2| \leq 2(2x^2|\cos x^2| + |\sin x^2|) \leq 2(2(1)(1) + 1) = 6.$$

d. $E_M \leq \dfrac{k(b-a)}{24}(\Delta x)^2 = \dfrac{6(2)}{24}\left(\dfrac{1}{15}\right)^2 = \dfrac{1}{450} = 0.00\overline{2}$.

8.8.67

a. $M(50) \approx 35.43456$.

b. $f'(x) = \dfrac{1}{2}(x^3+1)^{-1/2}3x^2 = \dfrac{3x^2}{2(x^3+1)^{1/2}}$. Then $f''(x) = \dfrac{2(x^3+1)^{1/2}(6x) - 3x^2(x^3+1)^{-1/2}(3x^2)}{4(x^3+1)}$. Multiplying the numerator and denominator by $(x^3+1)^{1/2}$ yields

$$\frac{12x(x^3+1) - 9x^4}{4(x^3+1)^{3/2}} = \frac{3x^4 + 12x}{4(x^3+1)^{3/2}} = \frac{3x(x^3+4)}{4(x^3+1)^{3/2}}.$$

c. Because f'' is decreasing on $[1, 6]$, $|f''(x)| \leq f''(1) = \dfrac{15}{8\sqrt{2}}$.

d. Note that $\dfrac{15}{8\sqrt{2}} < 1.33$. We have $E_M \leq \dfrac{k(b-a)}{24}(\Delta x)^2 = \dfrac{1.33(5)}{24}\left(\dfrac{1}{10}\right)^2 \leq 0.0028$.

8.8.68

a. $T(50) = \dfrac{1}{100}(1 + 2\displaystyle\sum_{i=1}^{49} e^{i^2/50^2} + e) \approx 1.462832952$.

b. $f'(x) = 2xe^{x^2}$, so $f''(x) = 4x^2e^{x^2} + 2e^{x^2} = 2e^{x^2}(2x+1)$.

c. Because both e^{x^2} and $2x+1$ are increasing and positive on $[0, 1]$, we have that $|e^{x^2}| \leq e$ and $|2x+1| \leq 2 \cdot 1 + 1 = 3$. Thus $|f''(x)| = 2|e^{x^2}||2x+1| \leq 2e \cdot 3 < 2(3)^2 = 18$.

d. Because $E_T(n) \leq k(b-a)^3/12n^2 = 18/12(50^2) = 6 \times 10^{-4}$, $T(50)$ is accurate to at least 3 decimal places.

8.8.69

a. $T(40) = \dfrac{1}{80}(\sin 1 + 2\displaystyle\sum_{i=1}^{39} \sin e^{i/40} + \sin e) \approx 0.8748$.

b. $f(x) = \sin e^x$, so $f'(x) = e^x\cos e^x$. $f''(x) = -e^{2x}\sin e^x + e^x\cos e^x = e^x(\cos e^x - e^x\sin e^x)$.

c. $|f''(x)| = |e^x| \cdot |\cos e^x - e^x\sin e^x| \leq |e^x|(|\cos e^x| + |e^x\sin e^x|) \leq e(1+e) < 10.11$. However, the graph of the absolute value of $f''(x)$ reveals that is is actually bounded by 5.75 on the interval $[0, 1]$.

d. Because $E_T(n) \leq k(b-a)^3/12n^2 = 6/(12(40^2)) = 0.0003125$, $T(40)$ is accurate to at least 3 decimal places.

8.8.70

a. $S(30) \approx 0.886207336$.

b. $f'(x) = -2xe^{-x^2}$. $f''(x) = -2e^{-x^2} + (-2x)e^{-x^2}(-2x) = -2e^{-x^2}(1-2x^2) = 2e^{-x^2}(2x^2-1)$. $f'''(x) = -4xe^{-x^2}(2x^2-1) + 2e^{-x^2}(4x) = 4xe^{-x^2}(1-2x^2+2) = 4xe^{-x^2}(3-2x^2) = 4e^{-x^2}(3x-2x^3)$. $f^{(4)}(x) = -8xe^{-x^2}(3x-2x^3) + 4e^{-x^2}(3-6x^2) = 4e^{-x^2}(4x^4-6x^2+3-6x^2) = 4e^{-x^2}(4x^4-12x^2+3)$.

c. Using a graph of $f^{(4)}$, we see that $|f^{(4)}(x)| \le 12$ on $[0,3]$. So $E_S \le \dfrac{k(b-a)}{180}(\Delta x)^4 = \dfrac{12(3)}{180}\left(\dfrac{1}{10}\right)^4 = \dfrac{1}{50,000} = 0.00002$.

8.8.71

a. $S(20) \approx 0.97775$.

b. The error is bounded by $\dfrac{f''''(x)}{180}\left(\dfrac{1}{20}\right)^4 \le \dfrac{1}{180 \cdot 20^4} \le 3.5 \times 10^{-8}$.

8.8.72 $\displaystyle\int_a^b (mx+k)\,dx = \left[\frac{mx^2}{2} + kx\right]_a^b = \frac{mb^2}{2} + kb - \frac{ma^2}{2} - ka = \frac{m}{2}(b^2-a^2) + k(b-a) = \left(\frac{m}{2}(b+a)+k\right)(b-a) = \frac{f(a)+f(b)}{2}(b-a) = T_{[a,b]}(1)$. Now for any n, the above argument shows that the trapezoidal rule gives the exact value of $\displaystyle\int_{x_{i-1}}^{x_i} mx + k\,dx$. Thus

$$\int_a^b (mx+k)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (mx+k)\,dx = \sum_{i=1}^{n} T_{[x_{i-1},x_i]}(1) = T_{[a,b]}(n).$$

8.8.73 $\displaystyle\int_0^{2\pi} \sqrt{a^2\cos^2 t + b^2 \sin^2 t}\,dt$, $a = 4$, $b = 8$.

n	$S(n)$
4	41.88790205
8	39.05860599

8.8.74 $\displaystyle\mathrm{Si}(1) = \int_0^1 \frac{\sin t}{t}\,dt$. We have

n	$M(n)$
4	0.9468682055
8	0.9462791963
16	0.9461320920

For $\displaystyle\mathrm{Si}(10) = \int_0^{10} \frac{\sin t}{t}\,dt$, we have

n	$M(n)$
4	1.682149231
8	1.663648208
16	1.659636470

8.8.75

a. The exact value is $\displaystyle\int_0^4 x^3\,dx = \left.\frac{x^4}{4}\right|_0^4 = 64$. Simpson's rule gives $S(2) = (0 + 4\cdot 8 + 64)\dfrac{2}{3} = 64$. The values match exactly.

b. $S(4) = (0 + 4\cdot 1 + 2\cdot 8 + 4 \cdot 27 + 64)\dfrac{1}{3} = 64$. This also matches the exact value exactly.

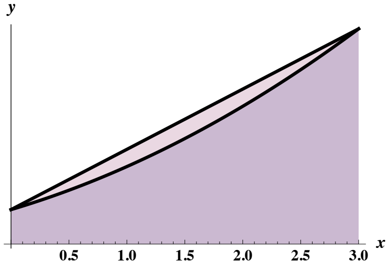
c. The 4th derivative of x^3 is 0, so the value of K in the theorem can be taken to be 0, so the error is 0.

d. Any polynomial of degree 3 or less has 4th derivative equal to 0, so the value of K in the theorem can be taken to be 0, so the error is 0.

8.8.76 Let $x_j = a + j(b-a)/2n$ for $1 \le j \le 2n$ and let $x_i = a + i(b-a)/n$ and let $\overline{x_i}$ be the midpoint of the interval $[x_{i-1}, x_i]$. Then

$$\begin{aligned}
T(2n) &= \frac{b-a}{4n}(f(a) + 2\sum_{j=1}^{2n-1} f(x_j) + f(b)) \\
&= \frac{1}{2}\cdot\frac{b-a}{2n}(f(a) + 2\sum_{1\le j\le 2n-1, j \text{ even}} f(x_j) + f(b) + 2\sum_{1\le j\le 2n-1, j \text{ odd}} f(x_j)) \\
&= \frac{1}{2}\left(\frac{b-a}{2n}(f(a) + 2\sum_{i-1}^{n-1} f(x_i) + f(b)) + \frac{b-a}{n}\sum_{i=1}^{n} f(\overline{x_i})\right) = \frac{1}{2}(T(n) + M(n)).
\end{aligned}$$

8.8.77 The trapezoidal rule will be an overestimate in this case. This is because of the fact that if the function is above the axis and concave up on the given interval, then each trapezoid on each subinterval lies over the area under the curve for that corresponding subinterval.



8.8.78 Let $x_i = a + i(b-a)/n$ for $1 \le i \le n$ and let $x_j = a + j(b-a)/2n$ for $1 \le j \le 2n$. We have $4T(2n) - T(n)$

$$\begin{aligned}
&= \frac{b-a}{2n}(2(f(a) + 2\sum_{j=1}^{2n-1} f(x_j) + f(b)) - (f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)) \\
&= \frac{b-a}{2n}((f(a) + 4\sum_{1\le j\le 2n-1, j \text{ odd}} f(x_j) + 4\sum_{2\le j\le 2n-2, j \text{ even}} f(x_j) - 2\sum_{i=1}^{n-1} f(x_i) + f(b))) \\
&= \frac{b-a}{2n}(f(a) + 4\sum_{1\le j\le 2n-1, j \text{ odd}} f(x_j) + 4\sum_{2\le j\le 2n-2} f(x_j) - 2\sum_{2\le j\le 2n-1} f(x_j) + f(b)) \\
&= \frac{b-a}{2n}(f(a) + 4\sum_{1\le j\le 2n-1, j \text{ odd}} f(x_j) + 2\sum_{2\le j\le 2n-2, j \text{ even}} f(x_j) + f(b)) = 3S(2n).
\end{aligned}$$

8.8.79 Using the given formula, we have

$$S(2n) = \frac{2M(n) + T(n)}{3}.$$

So $S(20) = \frac{1}{3}(2M(10) + T(10)) \approx 1.000001$.

8.9 Improper Integrals

8.9.1 The interval of integration is infinite or the integrand is unbounded on the interval of integration.

8.9.2 $\lim_{b\to\infty} \int_2^b \frac{dx}{x^3}$. We have

$$\lim_{b\to\infty} -\frac{1}{2}\left(\frac{1}{x^2}\right)\Big|_2^b = \lim_{b\to\infty} -\frac{1}{2}\left(\frac{1}{b^2} - \frac{1}{4}\right) = \frac{1}{8}.$$

8.9.3 $\lim_{b\to\infty}\int_2^b \frac{dx}{x^{1/5}}$. We have

$$\lim_{b\to\infty}\frac{5}{4}\cdot x^{4/5}\bigg|_2^b = \lim_{b\to\infty}\left(\frac{5}{4}\cdot b^{4/5} - \frac{5}{4}\cdot 2^{4/5}\right) = \infty.$$

The integral diverges.

8.9.4 $\int_0^1 \frac{1}{x^{1/5}}\,dx = \lim_{b\to 0^+}\int_b^1 \frac{dx}{x^{1/5}}$. We have

$$\lim_{b\to 0^+}\frac{5}{4}\cdot x^{4/5}\bigg|_b^1 = \lim_{b\to 0^+}\left(\frac{5}{4} - \frac{5}{4}\cdot b^{4/5}\right) = \frac{5}{4}.$$

8.9.5 This is $\int_{-\infty}^{\infty} f(x)\,dx$.

8.9.6 As shown in Example 2, this integral converges if and only if $p > 1$.

8.9.7 $\int_3^\infty x^{-2}\,dx = \lim_{b\to\infty}\int_3^b x^{-2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{x}\right)\bigg|_3^b = \lim_{b\to\infty}\left(\frac{1}{3} - \frac{1}{b}\right) = \frac{1}{3}.$

8.9.8 $\lim_{b\to\infty}\frac{dx}{x} = \lim_{b\to\infty}\ln x\bigg|_2^b = \lim_{b\to\infty}(\ln b - \ln 2) = \infty.$ This integral diverges.

8.9.9 $\int_2^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\int_2^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} 2\sqrt{x}\bigg|_2^b = \lim_{b\to\infty} 2(\sqrt{b} - \sqrt{2}) = \infty$, so the integral diverges.

8.9.10 $\int_0^\infty e^{-2x}\,dx = \lim_{b\to\infty}\int_0^b e^{-2x}\,dx = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-2x}\right)\bigg|_0^b = \lim_{b\to\infty}\frac{1}{2}\left(1 - e^{-2b}\right) = \frac{1}{2}.$

8.9.11 $\int_0^\infty e^{-ax}\,dx = \lim_{b\to\infty}\int_0^b e^{-ax}\,dx = \lim_{b\to\infty}\frac{e^{-ax}}{-a}\bigg|_0^b = \lim_{b\to\infty}\left(-\frac{1}{ae^{ab}} + \frac{1}{a}\right) = \frac{1}{a}.$

8.9.12 $\lim_{b\to-\infty}\int_b^{-1}\frac{dx}{\sqrt[3]{x}} = \lim_{b\to-\infty}\frac{3}{2}x^{2/3}\bigg|_b^{-1} = \lim_{b\to-\infty}\frac{3}{2}\left(1 - b^{2/3}\right) = -\infty.$ This integral diverges.

8.9.13 $\int_0^\infty \cos x\,dx = \lim_{b\to\infty}\int_0^b \cos x\,dx = \lim_{b\to\infty}\sin x\bigg|_0^b = \lim_{b\to\infty}\sin b$, which does not exist so the integral diverges.

8.9.14 $\lim_{b\to-\infty}\int_b^{-1}\frac{dx}{x^3} = \lim_{b\to-\infty} -\left(\frac{1}{2x^2}\right)\bigg|_b^{-1} = \lim_{b\to-\infty}\left(-\frac{1}{2} + \frac{1}{2b^2}\right) = -\frac{1}{2}.$

8.9.15 The given integral is equal to $\lim_{b\to-\infty}\int_b^0 \frac{dx}{x^2+100} + \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+100}$, if both limits exist.

$$\lim_{b\to-\infty}\int_b^0 \frac{dx}{x^2+100} = \lim_{b\to-\infty}\frac{1}{10}\tan^{-1}\frac{x}{10}\bigg|_b^0 = \frac{1}{10}\left(0 + \frac{\pi}{2}\right) = \frac{\pi}{20}.$$

$$\lim_{b\to\infty}\int_0^b \frac{dx}{x^2+100} = \lim_{b\to\infty}\frac{1}{10}\tan^{-1}\frac{x}{10}\bigg|_0^b = \frac{1}{10}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{20}.$$

So the given integral has value $\frac{\pi}{20} + \frac{\pi}{20} = \frac{\pi}{10}$.

8.9.16 The given integral is equal to $\lim\limits_{b\to-\infty}\int_b^0 \frac{dx}{x^2+a^2} + \lim\limits_{b\to\infty}\int_0^b \frac{dx}{x^2+a^2}$, if both limits exist.

$$\lim_{b\to-\infty}\int_b^0 \frac{dx}{x^2+a^2} = \lim_{b\to-\infty}\frac{1}{a}\tan^{-1}\frac{x}{a}\bigg|_b^0 = \frac{1}{a}\left(0+\frac{\pi}{2}\right) = \frac{\pi}{2a}.$$

$$\lim_{b\to\infty}\int_0^b \frac{dx}{x^2+a^2} = \lim_{b\to\infty}\frac{1}{a}\tan^{-1}\frac{x}{a}\bigg|_0^b = \frac{1}{a}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{2a}.$$

So the given integral has value $\frac{\pi}{2a}+\frac{\pi}{2a}=\frac{\pi}{a}$.

8.9.17 $\lim\limits_{b\to\infty}\int_7^b \frac{dx}{(x+1)^{1/3}} = \lim\limits_{b\to\infty}\frac{3}{2}(x+1)^{2/3}\bigg|_7^b = \lim\limits_{b\to\infty}\left(\frac{3}{2}(b+1)^{2/3}-\frac{3}{2}7^{2/3}\right)=\infty$.
The integral diverges.

8.9.18 $\int_2^\infty \frac{dx}{(x+2)^2} = \lim\limits_{b\to\infty}\int_2^b \frac{dx}{(x+2)^2} = \lim\limits_{b\to\infty} -\frac{1}{x+2}\bigg|_2^b = \lim\limits_{b\to\infty} -\frac{1}{b+2}+\frac{1}{4}=\frac{1}{4}$.

8.9.19 Let $u=x^3+x$ so that $du=(3x^2+1)\,dx$ Substituting gives

$$\int_2^\infty \frac{1}{u}\,du = \lim_{b\to\infty}\int_2^b \frac{1}{u}\,du = \lim_{b\to\infty}\ln u\,\bigg|_2^b = \lim_{b\to\infty}(\ln b-\ln 2)=\infty.$$

The given integral diverges.

8.9.20 $\int_1^\infty 2^{-x}\,dx = \lim\limits_{b\to\infty}\int_1^b 2^{-x}\,dx = \lim\limits_{b\to\infty}\left(\frac{-1}{\ln 2(2^x)}\right)\bigg|_1^b = \lim\limits_{b\to\infty}\left(\frac{-1}{\ln 2(2^b)}+\frac{1}{2\ln 2}\right)=\frac{1}{2\ln 2}$.

8.9.21 $\int_2^\infty \frac{\cos(\pi/x)}{x^2}\,dx = \lim\limits_{b\to\infty}\int_2^b \frac{\cos(\pi/x)}{x^2}\,dx = \lim\limits_{b\to\infty}\left(-\frac{1}{\pi}\sin(\pi/x)\right)\bigg|_2^b = \lim\limits_{b\to\infty}\frac{1}{\pi}(1-\sin(\pi/b))=\frac{1}{\pi}$.

8.9.22 Note that $\int \frac{1}{z^2}\sin(\pi/z)\,dz = -\frac{1}{\pi}\int \sin u\,du = \frac{1}{\pi}\cos u + C = \frac{1}{\pi}\cos(\pi/z)+C$. Thus,

$$\lim_{b\to-\infty}\int_b^{-2}\frac{1}{z^2}\sin(\pi/z)\,dz = \lim_{b\to-\infty}\frac{1}{\pi}\cos(\pi/z)\bigg|_b^{-2} = \lim_{b\to-\infty}\frac{1}{\pi}(0-\cos(\pi/b)) = -\frac{1}{\pi}.$$

8.9.23 $\int_0^\infty \frac{e^u}{e^{2u}+1}\,du = \lim\limits_{b\to\infty}\int_0^b \frac{e^u}{e^{2u}+1}\,du$. Let $u=e^u$ so that $du=e^u\,du$. After substitution we have $\lim\limits_{b\to\infty}\int_1^{e^b}\frac{1}{u^2+1}\,du = \lim\limits_{b\to\infty}\left(\tan^{-1}(u)\right)\bigg|_1^{e^b} = \lim\limits_{b\to\infty}(\tan^{-1}(e^b)-\tan^{-1}(1)) = \pi/2-\pi/4=\pi/4$.

8.9.24 $\int_{-\infty}^a \sqrt{e^x}\,dx = \lim\limits_{b\to-\infty}\int_b^a e^{x/2}\,dx = \lim\limits_{b\to-\infty}\left(2e^{x/2}\right)\bigg|_b^a = \lim\limits_{b\to-\infty}2(e^{a/2}-e^{b/2})=2e^{a/2}$.

8.9.25 $\int_{-\infty}^\infty \frac{e^{3x}}{1+e^{6x}}\,dx = \lim\limits_{a\to-\infty}\int_a^0 \frac{e^{3x}}{1+e^{6x}}\,dx + \lim\limits_{b\to\infty}\int_0^b \frac{e^{3x}}{1+e^{6x}}\,dx$. Let $u=e^{3x}$ so that $du=3e^{3x}$. Note that

$$\int \frac{e^{3x}}{1+e^{6x}}\,dx = \frac{1}{3}\int\frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}u+C = \frac{1}{3}\tan^{-1}e^{3x}+C.$$

So we have

$$\frac{1}{3}\lim_{a\to-\infty}\left(\tan^{-1}1-\tan^{-1}e^{3a}\right)+\frac{1}{3}\lim_{b\to\infty}\left(\tan^{-1}e^{3b}-\tan^{-1}1\right) = \frac{1}{3}\left(\frac{\pi}{4}-0\right)+\frac{1}{3}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\frac{\pi}{6}.$$

8.9.26 $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{x}{(x^2+1)^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{x}{(x^2+1)^2}\,dx$. Let $u = x^2+1$ so that $du = 2x\,dx$. Note that

$$\int \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int u^{-2}\,du = -\frac{1}{2}\cdot\frac{1}{u} + C = -\frac{1}{2}\cdot\frac{1}{x^2+1} + C.$$

Then our integral is given by

$$\lim_{a\to-\infty} -\frac{1}{2}\left(1 - \frac{1}{a^2}\right) + \lim_{b\to\infty} -\frac{1}{2}\left(\frac{1}{b^2+1} - 1\right) = -\frac{1}{2} + \frac{1}{2} = 0.$$

8.9.27

$$\begin{aligned}
\int_{-\infty}^{\infty} xe^{-x^2}\,dx &= \lim_{b\to-\infty}\int_b^0 xe^{-x^2} + \lim_{b\to\infty}\int_0^b xe^{-x^2}\\
&= \lim_{b\to-\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Bigg|_b^0 + \lim_{b\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Bigg|_0^b\\
&= \lim_{b\to-\infty}\left(-\frac{1}{2} + \frac{1}{2}e^{-b^2}\right) + \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}\right) = -\frac{1}{2} + \frac{1}{2} = 0.
\end{aligned}$$

8.9.28 $\displaystyle\int_1^{\infty} \frac{\tan^{-1} s}{s^2+1}\,ds = \lim_{b\to\infty}\int_1^b \frac{\tan^{-1} s}{s^2+1}\,ds = \lim_{b\to\infty}\left(\frac{1}{2}(\tan^{-1} s)^2\right)\Bigg|_1^b = \lim_{b\to\infty}\frac{1}{2}\left((\tan^{-1} b)^2 - \left(\frac{\pi}{4}\right)^2\right) = \frac{1}{2}\left(\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right) = \frac{3\pi^2}{32}$.

8.9.29 By symmetry, the given integral is equal to $\displaystyle 2\int_0^{\infty} \frac{(\tan^{-1} t)^2}{t^2+1}\,dt$ if the integral exists. Let $u = \tan^{-1} t$ so that $du = \dfrac{dt}{t^2+1}$. We have

$$2\lim_{b\to\infty}\int_0^{\tan^{-1} b} u^2\,du = 2\lim_{b\to\infty}\frac{u^3}{3}\Bigg|_0^{\tan^{-1} b} = 2\lim_{b\to\infty}\frac{(\tan^{-1} b)^3}{3} = \frac{2(\pi/2)^3}{3} = \frac{\pi^3}{12}.$$

8.9.30 $\displaystyle\int_{-\infty}^0 e^x\,dx = \lim_{b\to-\infty}\int_b^0 e^x\,dx = \lim_{b\to-\infty} e^x\Big|_b^0 = \lim_{b\to-\infty}\left(1 - e^b\right) = 1.$

8.9.31 $\displaystyle\int_1^{\infty} \frac{1}{v(v+1)}\,dv = \lim_{b\to\infty}\int_1^b \left(\frac{1}{v} - \frac{1}{v+1}\right)dv = \lim_{b\to\infty}\ln\left|\frac{v}{v+1}\right|\,\Bigg|_1^b = \lim_{b\to\infty}\ln\left|\frac{b}{b+1}\right| - \ln(1/2) = 0 - \ln(1/2) = \ln 2.$

8.9.32 Write

$$\int_1^{\infty} \frac{1}{x^2(x-1)}\,dx = \int_1^2 \frac{1}{x^2(x-1)}\,dx + \int_2^{\infty} \frac{1}{x^2(x-1)}dx,$$

if both integrals converge.

However,

$$\begin{aligned}
\lim_{c\to1^+}\int_c^2 \frac{1}{x^2(x-1)}\,dx &= \lim_{c\to1^+}\int_c^2 \left(-\frac{1}{x^2} + \frac{1}{x-1} - \frac{1}{x}\right)dx\\
&= \lim_{c\to1^+}\left(\frac{1}{x} + \ln\left|\frac{x-1}{x}\right|\Bigg|_c^2\right) = \lim_{c\to1^+}\left(\frac{1}{2} + \ln\left(\frac{1}{2}\right) - \frac{1}{c} - \ln\left(\frac{c-1}{c}\right)\right)
\end{aligned}$$

which diverges. Therefore, $\displaystyle\int_1^{\infty} \frac{1}{x^2(x-1)}\,dx$ diverges.

8.9.33 $\displaystyle\int_2^\infty \frac{dy}{y\ln y} = \lim_{b\to\infty}\int_2^b \frac{dy}{y\ln y} = \lim_{b\to\infty} (\ln(\ln y))\Big|_2^b = \lim_{b\to\infty} (\ln(\ln b) - \ln(\ln 2)) = \infty$, so the integral diverges.

8.9.34 $\displaystyle\int_{-\infty}^{-4/\pi} \frac{\sec^2(1/x)}{x^2}\,dx = \lim_{b\to-\infty}\int_b^{-4/\pi} \frac{\sec^2(1/x)}{x^2}\,dx$. Let $u = 1/x$ so that $du = -1/x^2\,dx$. Then we have $\displaystyle\lim_{b\to-\infty}\int_{1/b}^{-\pi/4} (-\sec^2 u)\,du = -\lim_{b\to-\infty} \tan u\,\Big|_{1/b}^{-\pi/4} = -\lim_{b\to-\infty}(-1-\tan(1/b)) = 1$.

8.9.35

$$\int_{-\infty}^0 \frac{dx}{\sqrt[3]{2-x}} = \lim_{b\to-\infty}\int_b^0 (2-x)^{-1/3}\,dx = \lim_{b\to-\infty}\left(-\frac{3}{2}(2-x)^{2/3}\right)\Big|_b^0$$
$$= \lim_{b\to-\infty}\left(\frac{3}{2}\left(-2^{2/3}+(2-b)^{2/3}\right)\right) = \infty,$$

so the integral diverges.

8.9.36

$$\int_{e^2}^\infty \frac{dx}{x\ln^p x} = \lim_{b\to\infty}\int_{e^2}^b \frac{dx}{x\ln^p x} = \lim_{b\to\infty}\left(\frac{1}{1-p}\ln^{1-p} x\right)\Big|_{e^2}^b = \lim_{b\to\infty}\frac{1}{p-1}\left(2^{1-p}-\ln^{1-p} b\right) = \frac{1}{(p-1)2^{p-1}}.$$

8.9.37 $\displaystyle\int_0^8 \frac{dx}{\sqrt[3]{x}} = \lim_{c\to0^+}\int_c^8 x^{-1/3}\,dx = \lim_{c\to0^+}\left(\frac{3}{2}x^{2/3}\right)\Big|_c^8 = \frac{3}{2}\lim_{c\to0^+}(4-c^{2/3}) = 6.$

8.9.38 $\displaystyle\lim_{c\to1^+}\int_c^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{c\to1^+}\left(2\sqrt{x-1}\right)\Big|_c^2 = \lim_{c\to1^+}\left(2-2\sqrt{c-1}\right) = 2.$

8.9.39 $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta = \lim_{c\to\pi/2^-}\int_0^c \tan\theta\,d\theta = \lim_{c\to\pi/2^-}(\ln\sec\theta)\Big|_0^c = \lim_{c\to\pi/2^-}\ln\sec c = \infty$, so the integral diverges.

8.9.40 $\displaystyle\lim_{c\to-3^+}\int_c^1 (2x+6)^{-2/3}\,dx = \lim_{c\to-3^+}\left(\frac{3}{2}\sqrt[3]{2x+6}\right)\Big|_c^1 = \lim_{c\to-3^+}\frac{3}{2}\left(2-\sqrt[3]{2c+6}\right) = 3.$

8.9.41 $\displaystyle\lim_{b\to(\pi/2)^-}\int_0^b \sec x\tan x\,dx = \lim_{b\to(\pi/2)^-}(\sec x)\Big|_0^b = \lim_{b\to(\pi/2)^-}(\sec b - 1) = \infty$. The given integral diverges.

8.9.42 $\displaystyle\lim_{c\to3^+}\int_c^4 (z-3)^{-3/2}\,dz = \lim_{c\to3^+}\left(-2(z-3)^{-1/2}\right)\Big|_c^4 = \lim_{c\to3^+}\left(-2-(-2(c-3)^{-1/2}\right) = \infty$. The given integral diverges.

8.9.43 Note that $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}} + C$. Thus,

$$\lim_{c\to0^+}\int_c^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{c\to0^+}\left(2e^{\sqrt{x}}\right)\Big|_c^1 = \lim_{c\to0^+}\left(2e-2e^{\sqrt{c}}\right) = 2e-2.$$

8.9.44 Use the substitution $u = e^y - 1$ so that $du = e^y\,dy$. Then $y = 0$ corresponds to $u = 0$ and $y = \ln 3$ corresponds to $u = 2$, so we have

$$\int_0^{\ln 3} \frac{e^y}{(e^y-1)^{2/3}}\,dy = \int_0^2 u^{-2/3}\,du = \lim_{b\to0^+}\int_b^2 u^{-2/3}\,du = \lim_{b\to0^+} 3u^{1/3}\Big|_b^2 = \lim_{b\to0^+}(3\cdot2^{1/3}-3b^{1/3}) = 3\cdot2^{1/3}.$$

8.9.45 $\displaystyle\int_0^1 \frac{x^3}{x^4-1}\,dx = \lim_{c\to 1^-}\int_0^c \frac{x^3}{x^4-1}\,dx = \lim_{c\to 1^-}\left(\frac{1}{4}\ln|x^4-1|\right)\Big|_0^c = \frac{1}{4}\lim_{c\to 1^-}\ln|c^4-1| = -\infty$, so the integral diverges.

8.9.46 This integral is improper at both limits, so we split it as $\displaystyle\int_1^\infty \frac{dx}{\sqrt[3]{x-1}} = \int_1^2 \frac{dx}{\sqrt[3]{x-1}} + \int_2^\infty \frac{dx}{\sqrt[3]{x-1}}$. If we let $u = x-1$ in the second integral on the right we obtain

$$\int_2^\infty \frac{dx}{\sqrt[3]{x-1}} = \int_1^\infty \frac{du}{u^{1/3}},$$

which diverges using the result in Example 2. Therefore the original integral diverges.

8.9.47

$$\begin{aligned}\int_0^{10}\frac{dx}{\sqrt[4]{10-x}} &= \lim_{c\to 10^-}\int_0^c (10-x)^{-1/4}\,dx = \lim_{c\to 10^-}\left(-\frac{4}{3}(10-x)^{3/4}\right)\Big|_0^c\\ &= \frac{4}{3}\lim_{c\to 10^-}\left(10^{3/4}-(10-c)^{3/4}\right) = \frac{4}{3}10^{3/4}.\end{aligned}$$

8.9.48 This integral is improper at the point $x=3$, so we split it as $\displaystyle\int_1^{11}\frac{dx}{(x-3)^{2/3}} = \int_1^3 \frac{dx}{(x-3)^{2/3}} + \int_3^{11}\frac{dx}{(x-3)^{2/3}}$ and evaluate each integral separately:

$$\int_1^3 \frac{dx}{(x-3)^{2/3}} = \lim_{c\to 3^-}\int_1^c (x-3)^{-2/3}\,dx = \lim_{c\to 3^-}\left(3(x-3)^{1/3}\right)\Big|_1^c = 3\lim_{c\to 3^-}\left(2^{1/3}-(3-c)^{1/3}\right) = 3\cdot 2^{1/3},$$

and

$$\int_3^{11}\frac{dx}{(x-3)^{2/3}} = \lim_{c\to 3^+}\int_c^{11}(x-3)^{-2/3}\,dx = \lim_{c\to 3^+}\left(3(x-3)^{1/3}\right)\Big|_c^{11} = 3\lim_{c\to 3^+}\left(8^{1/3}-(c-3)^{1/3}\right) = 6,$$

so $\displaystyle\int_1^{11}\frac{dx}{(x-3)^{2/3}} = 6 + 3\cdot 2^{1/3}$.

8.9.49 The given integrand isn't defined for $x=1$, so we write the given integral as

$$\int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}.$$

We will show that the given integral diverges by showing that $\displaystyle\int_0^1 \frac{dx}{(x-1)^2}$ diverges. We have

$$\lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^2} = \lim_{b\to 1^-} -\frac{1}{x-1}\Big|_0^b = \lim_{b\to 1^-}\left(-\frac{1}{b-1}+1\right) = \infty.$$

8.9.50 We can write this as

$$\begin{aligned}\int_0^1 \frac{1}{(x-1)^{1/3}}\,dx + \int_1^9 \frac{1}{(x-1)^{1/3}}\,dx &= \lim_{b\to 1^-}\int_0^b \frac{1}{(x-1)^{1/3}}\,dx + \lim_{c\to 1^+}\int_c^9 \frac{1}{(x-1)^{1/3}}\,dx\\ &= \lim_{b\to 1^-}\frac{3}{2}\left((x-1)^{2/3}\right)\Big|_0^b + \lim_{c\to 1^+}\frac{3}{2}\left((x-1)^{2/3}\right)\Big|_c^9 = -\frac{3}{2}+6 = \frac{9}{2}.\end{aligned}$$

8.9.51 By symmetry,

$$\begin{aligned}\int_{-2}^{2}\frac{dp}{\sqrt{4-p^2}} &= 2\int_0^2\frac{dp}{\sqrt{4-p^2}} = 2\lim_{c\to 2^-}\int_0^c\frac{dp}{\sqrt{4-p^2}}\\ &= 2\lim_{c\to 2^-}\left(\sin^{-1}(p/2)\right)\Big|_0^c = 2\lim_{c\to 2^-}\left(\sin^{-1}(c/2)-\sin^{-1}0\right)\\ &= 2(\sin^{-1}1-0)=\pi.\end{aligned}$$

8.9.52 Integration by parts gives $\int xe^{-x}\,dx = -(x+1)e^{-x}+C$, so $\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx = \lim_{b\to\infty}\left(-(x+1)e^{-x}\right)\big|_0^b = \lim_{b\to\infty}(1-(b+1)e^{-b}) = 1$.

8.9.53 Integration by parts gives $\int \ln x\,dx = x\ln x - x + C$, so

$$\int_0^1 \ln x\,dx = \lim_{c\to 0^+}\int_c^1 \ln x\,dx = \lim_{c\to 0^+}(x\ln x - x)\Big|_c^1 = \lim_{c\to 0^+}(-1-(c\ln c - c)) = -1,$$

because $\lim_{c\to 0^+} c\ln c = 0$.

8.9.54 Integration by parts gives $\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x+1}{x}+C$, so

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b\frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{\ln x+1}{x}\right)\Big|_1^b = \lim_{b\to\infty}\left(1-\frac{\ln b+1}{b}\right) = 1,$$

because $\lim_{b\to\infty}\frac{\ln b+1}{b}=0$.

8.9.55 Write the given integral as $\lim_{a\to 0^+}\int_a^{\ln 2}\frac{e^x}{\sqrt{e^{2x}-1}}\,dx$ and then let $u=e^x$ so that $du=e^x\,dx$. Substituting gives $\lim_{a\to 0^+}\int_{e^a}^2\frac{du}{\sqrt{u^2-1}}$. Then we let $u=\sec\theta$ with $du=\sec\theta\tan\theta\,d\theta$. Substituting again gives

$$\begin{aligned}\lim_{a\to 0^+}\int_{\sec^{-1}e^a}^{\pi/3}\sec\theta\,d\theta &= \lim_{a\to 0^+}\ln|\sec\theta+\tan\theta|\Big|_{\sec^{-1}e^a}^{\pi/3}\\ &= \lim_{a\to 0^+}\left(\ln(2+\sqrt{3})-\ln\left(e^a+\sqrt{e^{2a}-1}\right)\right) = \ln(2+\sqrt{3}).\end{aligned}$$

8.9.56 Write the given integral as $\lim_{a\to 0^+}\int_a^1\frac{dx}{x+\sqrt{x}} = \lim_{a\to 0^+}\int_a^1\frac{dx}{(\sqrt{x}+1)\sqrt{x}}$. Let $u=\sqrt{x}+1$ so that $du=\frac{1}{2\sqrt{x}}\,dx$. Substituting gives

$$\lim_{a\to 0^+}\int_{1+\sqrt{a}}^2\frac{2du}{u} = \lim_{a\to 0^+}2\ln u\Big|_{1+\sqrt{a}}^2 = \lim_{a\to 0^+}\left(2\ln 2-2\ln(1+\sqrt{a})\right). = 2\ln 2.$$

8.9.57 The function $e^{-|x|}$ is even, so $\int_{-\infty}^\infty e^{-|x|}\,dx = 2\int_0^\infty e^{-x}\,dx$. We have

$$2\lim_{b\to\infty}\int_0^b e^{-x}\,dx = 2\lim_{b\to\infty}-e^{-x}\Big|_0^b = 2\lim_{b\to\infty}\left(-e^{-b}+1\right) = 2.$$

8.9.58 Note that $x^2+2x+5=(x^2+2x+1)+4=(x+1)^2+4$. Also recall that $\int \frac{1}{(x+1)^2+4}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$.

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{dx}{(x+1)^2+4} &= \lim_{b\to-\infty}\int_b^0\frac{dx}{(x+1)^2+4}+\lim_{b\to\infty}\int_0^b\frac{dx}{(x+1)^2+4}\\
&= \lim_{b\to-\infty}\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\Big|_b^0+\lim_{b\to\infty}\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\Big|_0^b\\
&= \lim_{b\to-\infty}\frac{1}{2}\left(\tan^{-1}(1/2)-\tan^{-1}\left(\frac{b+1}{2}\right)\right)+\lim_{b\to\infty}\frac{1}{2}\left(\tan^{-1}\left(\frac{b+1}{2}\right)-\tan^{-1}(1/2)\right)\\
&= \frac{1}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=\frac{\pi}{2}.
\end{aligned}$$

8.9.59 We have the relation $B = I\int_0^\infty e^{-rt}\,dt = \frac{I}{r}$, using the result that $\int_0^\infty e^{-ax}\,dx=\frac{1}{a}$ for $a>0$. Therefore $B=\frac{5000}{0.12}=\$41{,}666.67$.

8.9.60 The total amount of water drained is $W=100\int_0^\infty e^{-0.05t}\,dt=\frac{100}{0.05}=2000\,\text{gal}$ (here we use the fact that $\int_0^\infty e^{-ax}\,dx=\frac{1}{a}$ for $a>0$).

8.9.61 As in Example 6, we have

$$\text{AUC}_i=\int_0^\infty C_i(t)\,dt=250\int_0^\infty e^{-0.08t}\,dt=\frac{250}{0.08}=3125$$

and

$$\text{AUC}_o=\int_0^\infty C_0(t)\,dt=200\int_0^\infty(e^{-0.08t}-e^{-1.8t})\,dt=200\left(\frac{1}{0.08}-\frac{1}{1.8}\right)=\frac{21{,}500}{9}\approx 2389,$$

(here we use the fact that $\int_0^\infty e^{-ax}\,dx=\frac{1}{a}$ for $a>0$). Therefore the bioavailability of the drug is

$$F=\frac{\text{AUC}_o}{\text{AUC}_i}=\frac{21{,}500}{9\cdot 3125}\approx 0.764.$$

8.9.62

a. We will make use of the result $\int_b^\infty e^{-at}\,dt=\frac{e^{-ab}}{a}$ for $a>0$ (this is derived in problem 73). The probability that a chip fails after 15,000 hours of operation (or equivalently, lasts at least 15,000 hours) is $p=0.00005\int_{15{,}000}^\infty e^{-0.00005t}\,dt=e^{-0.00005\cdot 15{,}000}\approx 0.472$.

b. The probability that a chip fails after 30,000 hours of operation is $p=0.00005\int_{30{,}000}^\infty e^{-0.00005t}\,dt=e^{-0.00005\cdot 30{,}000}\approx 0.223$. Of the chips that are still operating at 15,000 hours, the fraction that operate for at least another 15,000 hours is $\approx 0.223/0.472=0.472$.

c. From part a, we see that $0.00005\int_0^\infty e^{-0.00005t}\,dt=1$, which can be interpreted as meaning that all the chips are working initially.

8.9.63 Evaluate the improper integral

$$\int_0^\infty te^{-at}\,dt = \lim_{b\to\infty}\int_0^b te^{-at}\,dt = \lim_{b\to\infty}\left(-\frac{e^{-at}(at+1)}{a^2}\right)\Bigg|_0^b = \frac{1}{a^2}\lim_{b\to\infty}\left(1-e^{-ab}(ab+1)\right) = \frac{1}{a^2},$$

provided $a>0$. Therefore $0.00005\displaystyle\int_0^\infty te^{-0.00005t}\,dt = \frac{0.00005}{0.00005^2} = 20{,}000\,\text{hrs}.$

8.9.64 The maximum distance is

$$D = 10\int_0^\infty (t+1)^{-2}\,dt = 10\lim_{b\to\infty}\int_0^b (t+1)^{-2}\,dt = 10\lim_{b\to\infty}\left(-(t+1)^{-1}\right)\Big|_0^b = 10\,\text{mi}.$$

8.9.65 Using the result from Example 2, we see that the volume is given by $V = \pi\displaystyle\int_1^\infty x^{-4}\,dx = \frac{\pi}{4-1} = \frac{\pi}{3}$.

8.9.66 $V = \pi\displaystyle\int_2^\infty \frac{dx}{x^2+1} = \pi\lim_{b\to\infty}\int_2^b \frac{dx}{x^2+1} = \pi\lim_{b\to\infty}\left(\tan^{-1}x\right)\Big|_2^b = \pi\left(\frac{\pi}{2}-\tan^{-1}2\right) \approx 1.457.$

8.9.67 Using the result from Example 2, we see that the volume is given by

$$V = \pi\int_1^\infty\left(\frac{1}{x^2}+\frac{1}{x^3}\right)dx = \frac{\pi}{2-1}+\frac{\pi}{3-1} = \frac{3\pi}{2}.$$

8.9.68 $V = 2\pi\displaystyle\int_0^\infty \frac{x}{(x+1)^3}\,dx$; using the method of partial fractions, we see that $\dfrac{x}{(x+1)^3} = \dfrac{1}{(x+1)^2} - \dfrac{1}{(x+1)^3}$, so

$$V = 2\pi\left(\int_0^\infty \frac{dx}{(x+1)^2} - \int_0^\infty \frac{dx}{(x+1)^3}\right).$$

Now make the substitution $u = x+1$ and use the result from Example 2 to obtain

$$V = 2\pi\left(\int_1^\infty \frac{du}{u^2} - \int_1^\infty \frac{du}{u^3}\right) = 2\pi\left(\frac{1}{2-1}-\frac{1}{3-1}\right) = \pi.$$

8.9.69 $V = \pi\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^2} = \pi\lim_{b\to\infty}\int_2^b \frac{dx}{x(\ln x)^2} = \pi\lim_{b\to\infty}\left(-\frac{1}{\ln x}\right)\Bigg|_2^b = \pi\lim_{b\to\infty}\left(\frac{1}{\ln 2}-\frac{1}{\ln b}\right) = \frac{\pi}{\ln 2}.$

8.9.70 The volume is given by

$$\begin{aligned} V &= \pi\int_0^\infty \frac{x}{(x^2+1)^{2/3}}\,dx = \pi\lim_{b\to\infty}\int_0^b \frac{x}{(x^2+1)^{2/3}}\,dx \\ &= \pi\lim_{b\to\infty}\left(\frac{3}{2}(x^2+1)^{1/3}\right)\Bigg|_0^b = \pi\lim_{b\to\infty}\frac{3}{2}\left((b^2+1)^{1/3}-1\right) = \infty, \end{aligned}$$

so the volume is infinite.

8.9.71 Using disks, we have

$$V = \pi\int_1^2 (x-1)^{-1/2}\,dx = \pi\lim_{c\to1^+}\int_c^2 (x-1)^{-1/2}\,dx = \pi\lim_{c\to1^+}\left(2(x-1)^{1/2}\right)\Big|_c^2 = 2\pi\lim_{c\to1^+}(1-\sqrt{c-1}) = 2\pi.$$

8.9.72 Using disks, we have

$$V = \pi\int_{-1}^1\left(1+(x+1)^{-3/2}\right)^2\,dx = \pi\int_0^2(1+u^{-3/2})^2\,du = \pi\int_0^2(1+2u^{-3/2}+u^{-3})\,du$$

via the substitution $u = x+1$. Therefore

$$V = \pi\lim_{c\to0+}\int_c^2(1+2u^{-3/2}+u^{-3})\,du = \pi\lim_{c\to0^+}\left(u-4u^{-1/2}-\frac{1}{2}u^{-2}\right)\Bigg|_c^2 = \infty,$$

so the volume is infinite.

8.9.73 $V = \pi \int_0^{\pi/2} \tan^2 x\,dx = \lim_{a\to\frac{\pi}{2}^-} \pi \int_0^a \tan^2 x\,dx = \lim_{a\to\frac{\pi}{2}^-} \pi \int_0^a (\sec^2 x - 1)\,dx = \lim_{a\to\frac{\pi}{2}^-} \pi\,(\tan x - x)\Big|_0^a =$ $\lim_{a\to\frac{\pi}{2}^-} \pi\,(\tan a - a) = \infty$, so the volume doesn't exist.

8.9.74 First note that $\int \ln^2 x\,dx = 2x + x\ln^2 x - 2x\ln x$. This is obtained by integration by parts with $u = \ln x$ and $dv = \ln x\,dx$. This gives $du = \frac{1}{x}\,dx$ and $v = x\ln x - x$. Then we have $\int \ln^2 x\,dx = x\ln^2 x - x\ln x - \int(\ln x - 1)\,dx = x\ln^2 x - x\ln x - (x\ln x - x - x) + C = x\ln^2 x - 2x\ln x + 2x + C$.

Then the volume is $V = \pi \int_0^1 (-\ln x)^2\,dx = \lim_{a\to 0^+} \pi \int_a^1 \ln^2 x\,dx = \lim_{a\to 0^+} \pi\left(x\ln^2 x - 2x\ln x + 2x\right)\Big|_a^1$ $= \lim_{a\to 0^+} \pi\left(2 - (a\ln^2 a - 2a\ln a + 2a)\right) = 2\pi$. Note that

$$\lim_{a\to 0^+} a\ln a = \lim_{a\to 0^+} \frac{\ln a}{1/a} = \lim_{a\to 0^+} \frac{1/a}{-1/a^2} = \lim_{a\to 0^+} -a = 0,$$

by l'Hôpital's rule. Also

$$\lim_{a\to 0^+} a\ln^2 a = \lim_{a\to 0^+} \frac{\ln^2 a}{1/a} = \lim_{a\to 0^+} \frac{2\ln a/a}{-1/a^2} = \lim_{a\to 0^+} -2a\ln a = 0.$$

8.9.75 Using shells, we have

$$V = 2\pi \int_0^4 x(4-x)^{-1/3}\,dx = 2\pi \int_0^4 (4-u)u^{-1/3}\,du = 2\pi \int_0^4 (4u^{-1/3} - u^{2/3})\,du$$

via the substitution $u = 4 - x$. Therefore

$$V = 2\pi \lim_{c\to 0+} \int_c^4 (4u^{-1/3} - u^{2/3})\,du = 2\pi \lim_{c\to 0^+} \left(6u^{2/3} - \frac{3}{5}u^{5/3}\right)\Big|_c^4 = \frac{72\cdot 2^{1/3}\pi}{5}.$$

8.9.76 Using shells, we have

$$V = 2\pi \int_1^2 x(x^2-1)^{-1/4}\,dx = 2\pi \lim_{c\to 1^+} \int_c^2 x(x^2-1)^{-1/4}\,dx$$
$$= \pi \lim_{c\to 1^+} \left(\frac{4}{3}(x^2-1)^{3/4}\right)\Big|_c^2 = \frac{4}{3}\pi \lim_{c\to 1^+} (3^{3/4} - (c^2-1)^{3/4}) = \frac{4\pi}{3^{1/4}}.$$

8.9.77 $x^3 + 1 > x^3$ for $x \ge 1$, so $\frac{1}{x^3+1} < \frac{1}{x^3}$ for $x \ge 1$. Because $\int_1^\infty \frac{1}{x^3}\,dx$ converges, $\int_1^\infty \frac{1}{x^3+1}\,dx$ converges as well by the Comparison Test. Note that the fact that $\int_1^\infty \frac{1}{x^3}\,dx$ converges to $\frac{1}{2}$ follows from Example 2, Case 1.

8.9.78 $e^x + x + 1 > e^x$ for $x \ge 1$, so $\frac{1}{e^x + x + 1} < \frac{1}{e^x}$ for $x \ge 1$. Note that

$$\int_0^\infty \frac{dx}{e^x} = \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_0^b = \lim_{b\to\infty} \left(-e^{-b} + 1\right) = 1.$$

Thus $\int_0^\infty \frac{dx}{e^x + x + 1}$ converges by the Comparison Test.

8.9.79 For $x \ge 3$, $0 < \ln x \le x$, so $0 < \frac{1}{x} \le \frac{1}{\ln x}$. Note that

$$\int_3^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_3^b \frac{dx}{x} = \lim_{b\to\infty} \ln x\Big|_3^b = \lim_{b\to\infty} (\ln b - \ln 3) = \infty.$$

Using the Comparison Test, because $\int_3^\infty \frac{dx}{x}$ diverges, $\int_3^\infty \frac{dx}{\ln x}$ diverges as well.

8.9.80 For $x \geq 2$, $0 < x^4 - x - 1 \leq x^4 - 1$, so $\dfrac{x^3}{x^4 - x - 1} \geq \dfrac{x^3}{x^4 - 1} > 0$. We will show that $\displaystyle\int_2^\infty \frac{x^3}{x^4 - 1}\, dx$ is divergent, and then conclude by the Comparison Test that $\displaystyle\int_2^\infty \frac{x^3}{x^4 - x - 1}\, dx$ is divergent. In order to show that $\displaystyle\int_2^\infty \frac{x^3}{x^4 - 1}\, dx$ is divergent, we let $u = x^4 - 1$ so that $du = 4x^3\, dx$. Then we have

$$\lim_{b\to\infty} \int_{15}^{b^4-1} \frac{1}{4}\left(\frac{1}{u}\right) du = \lim_{b\to\infty} \frac{1}{4} \ln u \Big|_{15}^{b^4-1} = \frac{1}{4} \lim_{b\to\infty} \big(\ln(b^4 - 1) - \ln 15\big) = \infty.$$

8.9.81 $0 \leq \sin^2 x \leq 1$, so $0 \leq \dfrac{\sin^2 x}{x^2} \leq \dfrac{1}{x^2}$ for $x \geq 1$. By Example 2, $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges. Therefore $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\, dx$ converges as well by the Comparison Test.

8.9.82 $e^x > 1$ for $x \geq 1$, so $e^x(1 + x^2) \geq 1 + x^2$ for $x \geq 1$. It follows that

$$0 \leq \frac{1}{e^x(1 + x^2)} \leq \frac{1}{1 + x^2},$$

for $x \geq 1$. We will show that $\displaystyle\int_1^\infty \frac{1}{1 + x^2}\, dx$ converges to $\dfrac{\pi}{4}$ and then conclude that $\displaystyle\int_1^\infty \frac{1}{e^x(1 + x^2)}\, dx$ converges. We have

$$\lim_{b\to\infty} \int_1^b \frac{1}{1 + x^2}\, dx = \lim_{b\to\infty} \tan^{-1} x \Big|_1^b = \lim_{b\to\infty} \big(\tan^{-1} b - \tan^{-1} 1\big) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

8.9.83 $2 + \cos x \geq 1$, so $\dfrac{2 + \cos x}{\sqrt{x}} \geq \dfrac{1}{\sqrt{x}} \geq 0$. By Example 2, $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges, so $\displaystyle\int_1^\infty \frac{2 + \cos x}{\sqrt{x}}\, dx$ diverges by the Comparison Test.

8.9.84 $0 \leq 2 + \cos x \leq 3$, which implies that $0 \leq \dfrac{2 + \cos x}{x^2} \leq \dfrac{3}{x^2}$. By Example 2, $\displaystyle\int_1^\infty \frac{3}{x^2}\, dx$ converges, so $\displaystyle\int_1^\infty \frac{2 + \cos x}{x^2}\, dx$ converges as well by the Comparison Test.

8.9.85 $\sqrt{x^{1/3} + x} \geq \sqrt{x^{1/3}} = x^{1/6}$. So for $0 \leq x \leq 1$, $0 < \dfrac{1}{\sqrt{x^{1/3} + x}} \leq \dfrac{1}{x^{1/6}}$. We will show that $\displaystyle\int_0^1 \frac{dx}{x^{1/6}}$ converges and conclude that $\displaystyle\int_0^1 \frac{1}{\sqrt{x^{1/3} + x}}\, dx$ converges by the Comparison Test. We have

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{x^{1/6}} = \lim_{a\to 0^+} \frac{6}{5} x^{5/6} \Big|_a^1 = \lim_{a\to 0^+} \frac{6}{5}\left(1 - a^{5/6}\right) = \frac{6}{5}.$$

8.9.86 $\sin x + 1 \geq 1$, so $\dfrac{\sin x + 1}{x^5} \geq \dfrac{1}{x^5} > 0$ for all $x > 0$. We will show that $\displaystyle\int_0^1 \frac{1}{x^5}$ diverges, and then conclude that $\displaystyle\int_0^1 \frac{\sin x + 1}{x^5}\, dx$ diverges as well by the Comparison Test. We have

$$\lim_{a\to 0^+} \int_a^1 \frac{1}{x^5}\, dx = \lim_{a\to 0^+} -\frac{1}{4} \cdot \frac{1}{x^4} \Big|_a^1 = \lim_{a\to 0^+} -\frac{1}{4}\left(1 - \frac{1}{a^4}\right) = \infty.$$

8.9.87

a. True. The area under the curve $y = f(x)$ from 0 to ∞ is less than the area under $y = g(x)$ on this interval, which by assumption is finite.

b. False. For example, take $f(x) = 1$; then $\displaystyle\int_0^\infty f(x)\,dx = \infty$.

c. False. For example, take $p = 1/2$ and $q = 1$.

d. True. The area under the curve $y = x^{-q}$ from 1 to ∞ is less than the area under $y = x^{-p}$ on this interval, which by assumption is finite.

e. True. Using the result in Example 2, we see that this integral exists if and only if $3p + 2 > 1$, which is equivalent to $p > -1/3$.

8.9.88

a The fundamental theorem cannot be applied because the function $\dfrac{1}{x}$ is not continuous or bounded on $[-1, 1]$.

b. We will show that $\displaystyle\int_0^1 \frac{dx}{x}$ diverges, and then we may conclude that $\displaystyle\int_{-1}^1 \frac{dx}{x}$ diverges as well. We have

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{x} = \lim_{a\to 0^+} \ln x\Big|_a^1 = \lim_{a\to 0^+} (\ln 1 - \ln a) = \infty.$$

8.9.89 The region R has area $A = \displaystyle\int_0^\infty e^{-bx}\,dx - \int_0^\infty e^{-ax}\,dx = \frac{1}{b} - \frac{1}{a}$.

8.9.90 The region R has area $A = \displaystyle\int_1^\infty x^{-p}\,dx - \int_1^\infty x^{-q}\,dx = \frac{1}{p-1} - \frac{1}{q-1}$, where we use the result in Example 2.

8.9.91

a. We have

$$A(a, b) = \int_b^\infty e^{-ax}\,dx = \lim_{c\to\infty} \int_b^c e^{-ax}\,dx = \lim_{c\to\infty} \left(-\frac{1}{a}e^{-ax}\right)\Big|_b^c = \frac{1}{a}\lim_{c\to\infty}(e^{-ab} - e^{-ac}) = \frac{e^{-ab}}{a}.$$

b. Solving $e^{-ab} = 2a$ for b gives $b = g(a) = -\dfrac{1}{a}\ln 2a$.

c. The function g has $g'(x) = \dfrac{1}{x^2}\ln 2x - \dfrac{1}{x^2} = \dfrac{\ln 2x - 1}{x^2}$, so g has a critical point at $x = e/2$, and the first derivative test shows that g takes a minimum at this point. Hence $b^* = g(e/2) = -\dfrac{2}{e}$.

8.9.92 We first rewrite the integrand as $e^{-x\ln x}(\ln x + 1)$, and note that if $u = x\ln x$, then $du = (\ln x + 1)\,dx$. Note that as $x \to 0^+$, $u \to 0^-$, and as $x \to \infty$, $u \to \infty$, and when $x = 1$, $u = 0$. We consider the integral as

$$\int_0^1 x^{-x}(\ln x + 1)\,dx = \int_0^0 e^{-u}du = 0.$$

Now we consider

$$\int_1^\infty x^{-x}(\ln x + 1)\,dx = \int_0^\infty e^{-u}\,du = \lim_{b\to\infty} \int_0^b e^{-u}\,du = \lim_{b\to\infty} -e^{-u}\Big|_0^b = -\lim_{b\to\infty}\left(e^{-b} - 1\right) = 1.$$

8.9.93 In the following, we will make the substitution $u = \sqrt{x-1}$ so that $du = \dfrac{dx}{2\sqrt{x-1}}$ and $x = u^2+1$.

$$\begin{aligned}
\int_1^\infty \frac{dx}{x\sqrt{x-1}} &= \int_1^2 \frac{dx}{x\sqrt{x-1}}. + \int_2^\infty \frac{dx}{x\sqrt{x-1}} \\
&= \lim_{a\to 1^+} \int_a^2 \frac{dx}{x\sqrt{x-1}} + \lim_{b\to\infty} \int_2^b \frac{dx}{x\sqrt{x-1}} \\
&= 2\lim_{a\to 1^+} \int_{\sqrt{a-1}}^1 \frac{du}{u^2+1} + 2\lim_{b\to\infty} \int_1^{\sqrt{b-1}} \frac{du}{u^2+1} \\
&= 2\lim_{a\to 1^+} \tan^{-1} u\Big|_{\sqrt{a-1}}^1 + 2\lim_{b\to\infty} \tan^{-1} u\Big|_1^{\sqrt{b-1}} \\
&= 2\lim_{a\to 1^+} \left(\tan^{-1} 1 - \tan^{-1}\sqrt{a-1}\right) + 2\lim_{b\to\infty} \left(\tan^{-1}\sqrt{b-1} - \tan^{-1} 1\right) \\
&= 2\left(\frac{\pi}{4} - 0\right) + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \pi.
\end{aligned}$$

8.9.94 First, assume $p \neq 1$. Then

$$\begin{aligned}
\int_0^1 x^{-p}\,dx = \lim_{c\to 0^+} \int_c^1 x^{-p}\,dx &= \lim_{c\to 0^+} \left(\frac{x^{1-p}}{1-p}\right)\Big|_c^1 \\
&= \frac{1}{1-p} \lim_{c\to 0^+} \left(1 - c^{1-p}\right) = \frac{1}{1-p}
\end{aligned}$$

when $p < 1$ and is infinite otherwise. In the case $p = 1$ we have

$$\int_0^1 x^{-1}\,dx = \lim_{c\to 0^+} \int_c^1 x^{-1}\,dx = \lim_{c\to 0^+} (\ln x)\Big|_c^1 = \lim_{c\to 0^+} (-\ln c) = \infty,$$

so the integral exists if and only if $p < 1$.

8.9.95 These integrals cannot be evaluated by finding an antiderivative of their integrands. However, if we make the substitution $u = \pi/2 - x$ we find that $\int_0^{\pi/2} \ln\sin x\,dx = -\int_{\pi/2}^0 \ln\cos u\,du = \int_0^{\pi/2} \ln\cos x\,dx$. We also have

$$\int_{\pi/2}^\pi \ln\sin x\,dx = \int_0^{\pi/2} \ln\sin(x+\pi/2)\,dx = \int_0^{\pi/2} \ln\cos x\,dx,$$

and therefore

$$\int_0^{\pi/2} \ln\sin x\,dx = \frac{1}{2}\int_0^\pi \ln\sin x\,dx = \int_0^{\pi/2} \ln\sin 2y\,dy = \int_0^{\pi/2} (\ln\sin y + \ln\cos y + \ln 2)\,dy.$$

This implies $\int_0^{\pi/2} \ln\cos x\,dx = -\int_0^{\pi/2} \ln 2\,dx = -\dfrac{\pi\ln 2}{2}$.

8.9.96 This integral cannot be evaluated by finding an antiderivative of the integrand; however the result may be verified by numerical approximation.

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8.9.99

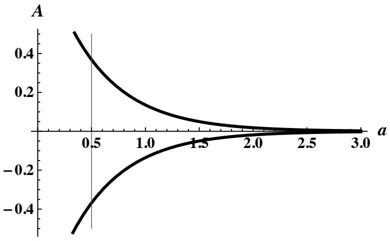
a. We have $\int_0^\infty e^{-ax}\cos bx\,dx = \lim_{c\to\infty}\int_0^c e^{-ax}\cos bx\,dx = \lim_{c\to\infty}\left(\frac{e^{-ax}(b\sin bx - a\cos bx)}{a^2+b^2}\right)\Big|_0^c =$ $\lim_{c\to\infty}\frac{a + e^{-ac}(b\sin bc - a\cos bc)}{a^2+b^2} = \frac{a}{a^2+b^2}$.

b. We have $\int_0^\infty e^{-ax}\sin bx\,dx = \lim_{c\to\infty}\int_0^c e^{-ax}\sin bx\,dx = \lim_{c\to\infty}\left(-\frac{e^{-ax}(a\sin bx + b\cos bx)}{a^2+b^2}\right)\Big|_0^c =$ $\lim_{c\to\infty}\frac{b - e^{-ac}(a\sin bc + b\cos bc)}{a^2+b^2} = \frac{b}{a^2+b^2}$.

8.9.100

a. Evaluate the improper integral

$\int_0^\infty xe^{-cx}\,dx = \lim_{b\to\infty}\int_0^b xe^{-cx}\,dx = \lim_{b\to\infty}\left(-\frac{e^{-cx}(cx+1)}{c^2}\right)\Big|_0^b = \frac{1}{c^2}\lim_{b\to\infty}\left(1 - e^{-cb}(cb+1)\right) = \frac{1}{c^2}$, provided $c > 0$. We also have $\int_0^\infty e^{-cx}\,dx = \frac{1}{c}$ for $c > 0$. Hence $\overline{x} = 1/c = 1/2$ for the case $c = 2$.



b. The curve $y = e^{-2x}$ has slope -2 at $x = 0$, so the equation of the tangent line at $(0,1)$ is $y = 1 - 2x$. Similarly, the equation of the tangent line to $y = -e^{-2x}$ at $(0,-1)$ is $y = -1 + 2x$.

c. Both tangent lines intersect the x-axis at $x = 1/2$.

d. More generally, in the case $a = 0$ the center of mass is $\overline{x} = 1/c$ and the tangent lines at (0 ± 1) have equations $y = \pm(1 - cx)$, so both tangent lines meet the x-axis at the center of mass. The case for arbitrary a can be reduced to the case $a = 0$ by a horizontal shift.

8.9.101

a. Integrate by parts with $u = \sqrt{x}\ln x$ and $v = -1/(1+x)$:

$$\int_0^\infty \frac{\sqrt{x}\ln x}{(1+x)^2}\,dx = \frac{1}{2}\int_0^\infty \frac{\ln x + 2}{\sqrt{x}(x+1)}\,dx = \frac{1}{2}\int_0^\infty \frac{\ln x}{\sqrt{x}(x+1)}\,dx + \int_0^\infty \frac{dx}{\sqrt{x}(x+1)}.$$

(The integration by parts is legitimate for this improper integral because the product uv has limit 0 as $x \to \infty$ and as $x \to 0^+$).

b. Let $y = 1/x$. Then $dy = -\frac{1}{x^2}\,dx$.

c. We have

$$\int_0^1 \frac{\ln x}{\sqrt{x}(x+1)}\,dx = \int_\infty^1 \frac{-\ln y}{\frac{1}{\sqrt{y}}\left(\frac{1}{y}+1\right)}\left(-\frac{dy}{y^2}\right) = -\int_1^\infty \frac{\ln y}{\sqrt{y}(1+y)}\,dy,$$

and hence $\int_0^\infty \frac{\ln x}{\sqrt{x}(x+1)}\,dx = 0$.

d. The change of variables $z = \sqrt{x}$ gives $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = 2\int_0^\infty \frac{dz}{z^2+1} = \pi$.

8.9.102 The Laplace transform of $f(t) = 1$ is given by $F(s) = \displaystyle\int_0^\infty e^{-st}\,dt = \frac{1}{s}$.

8.9.103 The Laplace transform of $f(t) = e^{at}$ is given by $F(s) = \displaystyle\int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}$, using the formula $\displaystyle\int_0^\infty e^{-cx}\,dx = \frac{1}{c}$ for $c > 0$.

8.9.104 The Laplace transform of $f(t) = t$ is given by $F(s) = \displaystyle\int_0^\infty te^{-st}\,dt = \frac{1}{s^2}$ (this formula is derived in the solution to problem 100).

8.9.105 The Laplace transform of $f(t) = \sin at$ is given by $F(s) = \displaystyle\int_0^\infty e^{-st}\sin at\,dt = \frac{a}{s^2+a^2}$ (this formula is derived in the solution to problem 99 b).

8.9.106 The Laplace transform of $f(t) = \cos at$ is given by $F(s) = \displaystyle\int_0^\infty e^{-st}\cos at\,dt = \frac{s}{s^2+a^2}$ (this formula is derived in the solution to problem 99 a).

8.9.107

a. Make the substitution $x = y + 2$; then

$$\int_1^3 \frac{dx}{\sqrt{(x-1)(3-x)}} = \int_{-1}^1 \frac{dy}{\sqrt{1-y^2}} = 2\int_0^1 \frac{dy}{\sqrt{1-y^2}} = 2\lim_{c\to 1^-}\left(\sin^{-1}x\right)\Big|_0^c = \pi.$$

b. The substitution $y = e^x$ gives

$$\int_1^\infty \frac{dx}{e^{x+1}+e^{3-x}} = \frac{1}{e}\int_1^\infty \frac{e^x dx}{e^{2x}+e^2} = \frac{1}{e}\int_e^\infty \frac{dy}{y^2+e^2} = \frac{1}{e}\lim_{b\to\infty}\left(\frac{1}{e}\tan^{-1}\left(\frac{y}{e}\right)\right)\Big|_e^b = \frac{\pi}{4e^2}.$$

8.9.108 The rate at which water is draining from the tank is given by $r(t) = 100(0.95)^t = 100e^{(\ln 0.95)t}$, so the total amount of water drained from the tank is

$$W = 100\int_0^\infty e^{(\ln 0.95)t}\,dt = \frac{100}{-\ln 0.95} \approx 1950\,\text{gal},$$

using the result that $\displaystyle\int_0^\infty e^{-ax}\,dx = \frac{1}{a}$ for $a > 0$. Therefore the full 3,000 gallon tank cannot be emptied at this rate.

8.9.109

a. We have $W = GMm\displaystyle\int_R^\infty x^{-2}\,dx = GMm\lim_{b\to\infty}\left(-\frac{1}{x}\right)\Big|_R^b = \frac{GMm}{R} \approx 6.279\times 10^7\,\text{m J}$.

b. Solve $\frac{1}{2}v_e^2 = 6.279\times 10^7$ to obtain $v_e \approx 11.207$ km/s.

c. We need $\displaystyle\frac{GM}{R} \geq \frac{1}{2}c^2 \iff R \leq \frac{2GM}{c^2} \approx 9\,\text{mm}$.

8.9.110 Let r_0 be the radius of the nucleus. The work required to bring a free proton to the edge of the nucleus is given by

$$W = kQq\int_{r_0}^\infty r^{-2}\,dr = kQq\lim_{b\to\infty}\left(-\frac{1}{r}\right)\Big|_{r_0}^b = \frac{kQq}{r_0} = \frac{50kq^2}{r_0} = 1.92\times 10^{-16}\,\text{N m}.$$

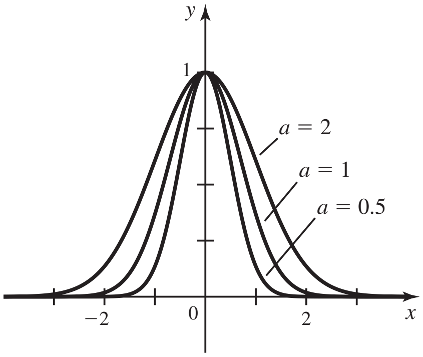
8.9.111

a. Repeatedly applying this reduction formula gives $\Gamma(p+1) = p(p-1)\cdots 2\cdot 1\cdot \int_0^\infty e^{-x}\,dx = p!\cdot 1 = p!$.

b. We have $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-1/2}e^{-x}\,dx = 2\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}$.

8.9.112

a.



b. The areas are $\sqrt{2\pi}$, $\sqrt{\pi}$, $\sqrt{\pi/2}$ respectively.

c. Completing the square gives

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a},$$

and therefore

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}\,dx = e^{(b^2-4ac)/(4a)}\int_{-\infty}^{\infty} e^{-a(x+(b/2a))^2}\,dx.$$

Make the substitution $y = x + b/(2a)$ to obtain

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}\,dx = e^{(b^2-4ac)/(4a)}\int_{-\infty}^{\infty} e^{-ay^2}\,dy = e^{(b^2-4ac)/(4a)}\sqrt{\frac{\pi}{a}}.$$

Chapter Eight Review

1

a. True. Two applications of integration by parts are needed to reduce to $\int x^2e^{2x}\,dx$.

b. False. This integral can be done using a trigonometric substitution.

c. False. Both are correct, since $-\cos^2 x = \sin^2 x - 1$, so $\sin^2 x$ and $-\cos^2 x$ differ by a constant.

d. True. Recall that $2\sin x\cos x = \sin 2x$.

e. False. Use long division to write the integrand as the sum of a polynomial and a proper rational function.

2 Let $u = x/2 + \pi/3$. Then $du = \frac{1}{2}\,dx$. Substituting gives

$$2\int \cos u\,du = 2\sin u + C = 2\sin(x/2 + \pi/3) + C.$$

3 Let $u = x + 4$ so that $du = dx$. Note that $x = u - 4$. Substituting gives

$$\begin{aligned}\int \frac{3(u-4)}{\sqrt{u}}\,du &= 3\int (u^{1/2} - 4u^{-1/2})\,du = 3((2/3)u^{3/2} - 8u^{1/2}) + C\\ &= 2(x+4)^{3/2} - 24\sqrt{x+4} + C = 2\sqrt{x+4}(x+4-12) + C = 2\sqrt{x+4}(x-8) + C.\end{aligned}$$

4 Let $u = 3t$ and $dv = e^{-t}\,dt$ Then $du = 3dt$ and $v = -e^{-t}$. Thus $\displaystyle\int_{-1}^{\ln 2} 3te^{-t}\,dt$ is equal to

$$-3te^{-t}\Big|_{-1}^{\ln 2} - \int_{-1}^{\ln 2}(-3e^{-t})\,dt = -3te^{-t}\Big|_{-1}^{\ln 2} - 3e^{-t}\Big|_{-1}^{\ln 2} = -\frac{3\ln 2}{2} - 3e - (3/2 - 3e) = -\frac{3}{2}(\ln 2 + 1).$$

5 Let $u = x$ and $dv = (1/2)(x+2)^{-1/2}\,dx$. Then $du = dx$ and $v = (x+2)^{1/2}$. We have

$$\begin{aligned}\int \frac{x}{2\sqrt{x+2}}\,dx &= x\sqrt{x+2} - \int \sqrt{x+2}\,dx = x\sqrt{x+2} - \frac{2}{3}(x+2)^{3/2} + C\\ &= \frac{1}{3}\sqrt{x+2}(3x - 2x - 4) + C = \frac{1}{3}\sqrt{x+2}(x-4) + C.\end{aligned}$$

6 $\displaystyle\int \left(2\sec^2 2\theta - \tan 2\theta \sec 2\theta\right)\,d\theta = \tan 2\theta - \frac{1}{2}\sec 2\theta + C.$

7

$$\int_{-2}^{1} \frac{3}{(x+2)^2 + 9}\,dx = \left(\tan^{-1}((x+2)/3)\right)\Big|_{-2}^{1} = \pi/4 - 0 = \pi/4.$$

8

$$\int_{\pi}^{2\pi} \frac{\cos(x/3)}{\sin(x/3)}\,dx = 3\,\left(\ln|\sin(x/3)|\right)\Big|_{\pi}^{2\pi} = 3\ln\left|\frac{\sin(2\pi/3)}{\sin(\pi/3)}\right| = 3\ln 1 = 0.$$

9 $\displaystyle\int_0^{\pi/4} \cos^5 2x \sin^2 2x\,dx = \int_0^{\pi/4} \cos(2x)(1 - \sin^2 2x)^2 \sin^2 2x\,dx$. Let $u = \sin 2x$ so that $du = 2\cos 2x\,dx$. Substituting gives

$$\frac{1}{2}\int_0^1 (1-u^2)^2u^2\,du = \frac{1}{2}\int_0^1 (u^2 - 2u^4 + u^6)\,du = \frac{1}{2}\left(u^3/3 - 2u^5/5 + u^7/7\right)\Big|_0^1 = \frac{1}{2}(1/3 - 2/5 + 1/7) - 0 = \frac{4}{105}.$$

10 By long division, $\dfrac{x^3 + 3x^2 + 1}{x^3 + 1} = 1 + \dfrac{3x^2}{x^3 + 1}$. Thus

$$\int \frac{x^3 + 3x^2 + 1}{x^3 + 1}\,dx = \int \left(1 + \frac{3x^2}{x^3 + 1}\right)\,dx = x + \ln|x^3 + 1| + C.$$

11 Let $u = \sqrt{t-1}$. Then $u^2 + 1 = t$, and $2u\,du = dt$. Substituting gives

$$\int \frac{u^2}{u^2+1}\,du = \int \left(1 - \frac{1}{u^2+1}\right)\,du = u - \tan^{-1} u + C = \sqrt{t-1} - \tan^{-1}\sqrt{t-1} + C.$$

12 Write $\dfrac{8x+5}{2x^2 + 3x + 1} = \dfrac{A}{2x+1} + \dfrac{B}{x+1}$. Then $8x + 5 = A(x+1) + B(2x+1)$. Letting $x = -1$ gives $B = 3$ and letting $x = -1/2$ gives $A = 2$. Thus $\dfrac{8x+5}{2x^2+3x+1} = \dfrac{2}{2x+1} + \dfrac{3}{x+1}$. Our integral is therefore equal to

$$\int \left(\frac{2}{2x+1} + \frac{3}{x+1}\right)\,dx = \ln|(2x+1)(x+1)^3| + C.$$

13 We integrate by parts. Let $u = e^{3x}$ and $dv = \sin 6x\,dx$. Then $du = 3e^{3x}\,dx$ and $v = -\dfrac{1}{6}\cos 6x$. Our integral is then equal to

$$I = \int_0^\pi e^{3x}\sin 6x\,dx = -\frac{1}{6}e^{3x}\cos 6x\bigg|_0^\pi + \frac{1}{2}\int_0^\pi e^{3x}\cos 6x\,dx. = -\frac{1}{6}e^{3\pi} + \frac{1}{6} + \frac{1}{2}\int_0^\pi e^{3x}\cos 6x\,dx.$$

We integrate by parts again, letting $u = e^{3x}$ and $dv = \cos 6x\,dx$. Then $du = 3e^{3x}\,dx$ and $v = \frac{1}{6}\sin 6x$. We have

$$\begin{aligned} I &= -\frac{1}{6}e^{3\pi} + \frac{1}{6} + \frac{1}{2}\left(\frac{1}{6}e^{3x}\sin 6x\bigg|_0^\pi - \frac{1}{2}\int_0^\pi e^{3x}\sin 6x\,dx\right) \\ &= -\frac{1}{6}e^{3\pi} + \frac{1}{6} - \frac{1}{4}I. \end{aligned}$$

So

$$\frac{5}{4}I = \frac{1}{6}\left(1 - e^{3\pi}\right),$$

and therefore

$$I = \frac{2}{15}\left(1 - e^{3\pi}\right).$$

14 We integrate by parts with $u = 6w - 10$ and $dv = e^{3w}\,dw$. Then $du = 6\,dw$ and $v = \dfrac{1}{3}e^{3w}$. We have

$$(6w-10)\frac{1}{3}e^{3w}\bigg|_2^3 - \int_2^3 2e^{3w}\,dw = \frac{8}{3}e^9 - \frac{2}{3}e^6 - \left(\frac{2}{3}e^{3w}\bigg|_2^3\right) = 2e^9.$$

15 We simplify the integrand via long division and then partial fractions. We have

$$\frac{3x^5 + 48x^3 + 3x^2 + 16}{x^3 + 16x} = 3x^2 + \frac{3x^2 + 16}{x(x^2 + 16)} = 3x^2 + \frac{1}{x} + \frac{2x}{x^2 + 16}.$$

Therefore our integral is equal to

$$\int_1^2 \left(3x^2 + \frac{1}{x} + \frac{2x}{x^2 + 16}\right)dx = \left(x^3 + \ln|x| + \ln|x^2 + 16|\right)\bigg|_1^2 = (8 + \ln 2 + \ln 20) - (1 + \ln 17) = 7 + \ln 40 - \ln 17.$$

16 We simplify the integrand by long division. We have

$$\frac{x^6 + 2x^4 + x^2 + 1}{(x^2+1)^2} = x^2 + \frac{1}{(x^2+1)^2}.$$

We have

$$\int \left(x^2 + \frac{1}{(x^2+1)^2}\right)dx = \frac{x^3}{3} + \int \frac{1}{(x^2+1)^2}\,dx.$$

For this last integral, we let $x = \tan u$ so that $dx = \sec^2 u\,du$ and $x^2 + 1 = \sec^2 u$. We have

$$\begin{aligned} \frac{x^3}{3} + \int \frac{\sec^2 u}{\sec^4 u}\,du &= \frac{x^3}{3} + \int \cos^2 u\,du = \frac{x^3}{3} + \frac{1}{2}\int (1 + \cos 2u)\,du \\ &= \frac{x^3}{3} + \frac{1}{2}u + \frac{1}{4}\sin 2u + C = \frac{x^3}{3} + \frac{1}{2}\tan^{-1}x + \frac{x}{2(x^2+1)} + C. \end{aligned}$$

In the last equation, we used the fact that $\dfrac{1}{4}\sin 2u = \dfrac{1}{2}\sin u\cos u = \dfrac{1}{2}\cdot\dfrac{x}{\sqrt{x^2+1}}\cdot\dfrac{1}{\sqrt{x^2+1}}$.

17 Write $\frac{2x^2+7x+4}{x^3+2x^2+2x} = \frac{Ax+B}{x^2+2x+2} + \frac{C}{x}$. Then $2x^2+7x+4 = Ax^2+Bx+C(x^2+2x+2)$. Letting $x=0$ gives $C=2$. Letting $x=1$ gives $13 = A+B+10$, so $A+B=3$. Letting $x=-1$ gives $-1 = A-B+2$, so $A-B=-3$. Solving this system of linear equations gives $A=0$ and $B=3$. Thus our integral is equal to

$$\int \left(\frac{3}{x^2+2x+2} + \frac{2}{x} \right) dx = \int \left(\frac{3}{(x+1)^2+1} + \frac{2}{x} \right) dx = 3\tan^{-1}(x+1) + 2\ln|x| + C.$$

18

$$\begin{aligned} \int_0^{\sqrt{2}} \frac{x+1}{3x^2+6}\,dx &= \int_0^{\sqrt{2}} \frac{x}{3x^2+6}\,dx + \frac{1}{3}\int_0^{\sqrt{2}} \frac{1}{x^2+2}\,dx \\ &= \frac{1}{6}\ln|3x^2+6|\Big|_0^{\sqrt{2}} + \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\Big|_0^{\sqrt{2}} \\ &= \frac{1}{6}(\ln 12 - \ln 6) + \frac{1}{3\sqrt{2}}\left(\frac{\pi}{4}\right) \\ &= \frac{1}{6}\ln 2 + \frac{\pi}{12\sqrt{2}} = \frac{1}{6}\left(\ln 2 + \frac{\pi\sqrt{2}}{4}\right). \end{aligned}$$

19 If we write

$$\frac{x^2+4x+7}{(x+3)^3} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3}$$

and solve, we obtain $A=1$, $B=-2$, and $C=4$. We then have

$$\int \frac{1}{x+3}\,dx - \int \frac{2}{(x+3)^2}\,dx + \int \frac{4}{(x+3)^3}\,dx = \ln|x+3| + \frac{2}{x+3} - \frac{2}{(x+3)^2} + C.$$

20 Let $x = \sin\theta$ so that $dx = \cos\theta\,d\theta$ and $\sqrt{1-x^2} = \cos\theta$. Substituting gives

$$\begin{aligned} \int \frac{\cos^2\theta}{\sin\theta}\,d\theta &= \int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta \\ &= \int (\csc\theta - \sin\theta)\,d\theta = -\ln|\csc\theta + \cot\theta| + \cos\theta + C \\ &= -\ln|\csc\sin^{-1}x + \cot\sin^{-1}x| + \cos\sin^{-1}x + C \\ &= -\ln|(1/x) + \sqrt{1-x^2}/x| + \sqrt{1-x^2} + C. \end{aligned}$$

21 Let $x = \sec\theta$ so that $dx = \sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-1} = \tan\theta$. Substituting gives

$$\int_{\pi/4}^{\pi/3} \tan^2\theta\,d\theta = \int_{\pi/4}^{\pi/3} (\sec^2\theta - 1)\,d\theta = (\tan\theta - \theta)\Big|_{\pi/4}^{\pi/3} = \sqrt{3} - \pi/3 - (1-\pi/4) = \sqrt{3} - 1 - \frac{\pi}{12}.$$

22 $\int \tan^3\theta\,d\theta = \int \tan\theta(\sec^2\theta - 1)\,d\theta = \int \tan\theta\sec^2\theta\,d\theta - \int \tan\theta\,d\theta$. For the first integral, let $u = \tan\theta$ and for the second, let $w = \cos\theta$. Substituting gives

$$\int u\,du + \int \frac{1}{w}\,dw = u^2/2 + \ln|w| + C = \frac{\tan^2\theta}{2} + \ln|\cos\theta| + C.$$

23 $\int \tan^4 t\sec^2 t\,dt$. Let $u = \tan t$ so that $du = \sec^2 t\,dt$. Substituting gives

$$\int u^4\,du = u^5/5 + C = \frac{1}{5}\tan^5 t + C.$$

24 We complete the square in the denominator of the integrand to obtain

$$\int \frac{1}{\sqrt{81-(x-9)^2}}\,dx = \int \frac{1}{9\sqrt{1-\left(\frac{x-9}{9}\right)^2}}\,dx.$$

Then let $u = \dfrac{x-9}{9}$ so that $du = \dfrac{1}{9}\,dx$. Substituting gives

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C = \sin^{-1}\left(\frac{x-9}{9}\right) + C.$$

25 Complete the square in the denominator of the integrand as follows:

$$4x^2+12x+10 = 4(x^2+3x+9/4-9/4)+10 = (4x^2+12x+9)-9+10 = (2x+3)^2+1.$$

Then we have

$$\int_{-3/2}^{-1} \frac{dx}{(2x+3)^2+1}$$

. Let $u = 2x+3$ and $du = 2\,dx$. Substituting gives

$$\frac{1}{2}\int_0^1 \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1} u\Big|_0^1 = \frac{\pi}{8}.$$

26 Let $u = \cos x$ so that $du = -\sin x\,dx$. We have

$$\int \tan 10x\,dx = \int -\frac{1}{10}\frac{1}{u}\,du = -\frac{1}{10}\ln|u| + C = -\frac{1}{10}\ln|\cos x| + C.$$

27 Note that

$$w^2+2w-8 = w^2+2w+1-9 = (w+1)^2-9.$$

Let $u = w+1$ with $du = dw$. Substituting gives $\displaystyle\int \frac{1}{u^2\sqrt{u^2-9}}\,du$. Now let $u = 3\sec\theta$ with $du = 3\sec\theta\tan\theta\,d\theta$. Now we have

$$\begin{aligned}\int \frac{1}{9\sec^2\theta\sqrt{9\sec^2\theta-9}}3\sec\theta\tan\theta\,d\theta &= \int \frac{3\tan\theta}{27\sec\theta\tan\theta}\,d\theta\\ &= \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C = \frac{1}{9}\cdot\frac{\sqrt{u^2-9}}{u} + C\\ &= \frac{1}{9}\cdot\frac{\sqrt{(w+1)^2-9}}{w+1} + C = \frac{\sqrt{w^2+2w-8}}{9(w+1)} + C.\end{aligned}$$

28 The integral can be written as $\displaystyle\int \tan^3 x\sec^2 x\,dx$. Let $u = \tan x$ so that $du = \sec^2 x\,dx$. Substituting gives

$$\int u^3\,du = \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C.$$

29 The integral can be written as $\displaystyle\int \cot^4 x\csc^2 x\,dx$. Let $u = \cot x$ so that $du = -\csc^2 x\,dx$. Substituting gives

$$-\int u^4\,du = -\frac{u^5}{5} + C = -\frac{\cot^5 x}{5} + C.$$

30 Let $u = \tan^{-1} 7x$ and $dv = x\,dx$. Then $du = \dfrac{7dx}{(7x)^2+1}$ and $v = \dfrac{x^2}{2}$. We have

$$\begin{aligned}\int x\tan^{-1} 7x\,dx &= \frac{1}{2}x^2\tan^{-1} 7x - \frac{7}{2}\int \frac{x^2}{49x^2+1}\,dx = \frac{1}{2}x^2\tan^{-1} 7x - \frac{1}{14}\int\left(1-\frac{1}{(7x)^2+1}\right)dx\\ &= \frac{1}{2}x^2\tan^{-1} 7x - \frac{1}{14}\left(x-\frac{\tan^{-1} 7x}{7}\right) + C = \frac{1}{2}x^2\tan^{-1} 7x - \frac{x}{14} + \frac{\tan^{-1} 7x}{98} + C.\end{aligned}$$

31 Let $u = x$ and $dv = \sinh x\,dx$. Then $du = dx$ and $v = \cosh x$. We have

$$\int x\sinh x\,dx = x\cosh x - \int \cosh x\,dx = x\cosh x - \sinh x + C.$$

32 $\displaystyle\int \csc^2 6x\cot 6x\,dx$. Let $u = \cot 6x$ so that $du = -6\csc^2 6x\,dx$. Substituting gives

$$-\frac{1}{6}\int u\,du = -\frac{u^2}{12} + C = \frac{-\cot^2 6x}{12} + C.$$

33 $\displaystyle\int \tan^2 3\theta\sec^2 3\theta(\sec 3\theta\tan 3\theta)\,d\theta = \int(\sec^2 3\theta - 1)\sec^2 3\theta(\sec 3\theta\tan 3\theta)\,d\theta.$
Let $u = \sec 3\theta$ so that $du = 3\sec 3\theta\tan 3\theta\,d\theta$. Substituting gives

$$\frac{1}{3}\int(u^2-1)u^2\,du = \frac{1}{3}\int(u^4-u^2)\,du = \frac{u^5}{15} - \frac{u^3}{9} + C = \frac{1}{15}\sec^5 3\theta - \frac{1}{9}\sec^3 3\theta + C.$$

34 Let $w = 2\sin\theta$ so that $dw = 2\cos\theta\,d\theta$ and $\sqrt{36-9w^2} = 3\sqrt{4-w^2} = 6\cos\theta$. Substituting gives

$$\int \frac{8\sin^3\theta}{6\cos\theta}\cdot 2\cos\theta\,d\theta = \int \frac{8}{3}\sin\theta(1-\cos^2\theta)\,d\theta.$$

Let $u = \cos\theta$ so that $du = -\sin\theta\,d\theta$. Substituting again gives

$$\begin{aligned}\frac{8}{3}\int(u^2-1)\,du &= \frac{8u^3}{9} - \frac{8u}{3} + C = \frac{8}{9}\cos^3\theta - \frac{8}{3}\cos\theta + C\\ &= \frac{8}{9}\cdot(\cos(\sin^{-1}(w/2)))^3 - \frac{8}{3}\cdot\cos(\sin^{-1}(w/2)) + C = \frac{1}{9}\cdot(\sqrt{4-w^2})^3 - \frac{12}{9}\cdot\sqrt{4-w^2} + C\\ &= \frac{1}{9}\cdot\sqrt{4-w^2}(4-w^2-12) = -\frac{1}{9}\sqrt{4-w^2}(w^2+8) + C.\end{aligned}$$

35 Let $x = 2\tan\theta$ so that $dx = 2\sec^2\theta\,d\theta$ and $\sqrt{4x^2+16} = 2\sqrt{x^2+4} = 4\sec\theta$. Substituting gives

$$\int \frac{8\tan^3\theta}{4\sec\theta}2\sec^2\theta\,d\theta = 4\int\tan\theta\sec\theta(\sec^2\theta - 1)\,d\theta.$$

Let $u = \sec\theta$ so that $du = \sec\theta\tan\theta\,d\theta$. Substituting again gives

$$\begin{aligned}4\int(u^2-1)\,du &= \frac{4}{3}u^3 - 4u + C = \frac{4}{3}\sec^3\theta - 4\sec\theta + C\\ &= \frac{4}{3}(\sec(\tan^{-1}(x/2)))^3 - 4\sec\tan^{-1}(x/2) + C = \frac{1}{6}\sqrt{x^2+4}(x^2+4) - 2\sqrt{x^2+4} + C\\ &= \frac{1}{6}\sqrt{x^2+4}(x^2+4-12) + C = \frac{1}{6}\sqrt{x^2+4}(x^2-8) + C.\end{aligned}$$

36 Note that $\dfrac{u^2+1}{u^2-1} = 1 + \dfrac{2}{u^2-1} = 1 + \dfrac{1}{u-1} - \dfrac{1}{u+1}$. Our integral is therefore equal to

$$\int_{-1/2}^{1/2}\left(-\frac{1}{u+1} + \frac{1}{u-1} + 1\right)du = \left(\ln\left|\frac{u-1}{u+1}\right| + u\right)\Bigg|_{-1/2}^{1/2} = \ln(1/3) + 1/2 - (\ln 3 - 1/2) = 1 - 2\ln 3.$$

37 Write $\dfrac{3x^3+4x^2+6x}{(x+1)^2(x^2+4)} = \dfrac{Ax+B}{x^2+4} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2}$. Then $3x^3+4x^2+6x = (Ax+B)(x+1)^2 + C(x+1)(x^2+4) + D(x^2+4)$. Letting $x = -1$ gives $D = -1$. Letting $x = 0$ gives $0 = B + 4C - 4$, so $B + 4C = 4$. Letting $x = 1$ gives $13 = 4A + 4B + 10C - 5$, so $9 = 2A + 2B + 5C$. Letting $x = 2$ gives $52 = 18A + 9B + 24C - 8$, so $60 = 18A + 9B + 24C$, so $20 = 6A + 3B + 8C$. Solving the system of linear equations gives $A = 2$, $B = 0$, and $C = 1$. Our integral is thus equal to

$$\int\left(\frac{2x}{x^2+4} + \frac{1}{x+1} - \frac{1}{(x+1)^2}\right)dx = \ln\left|(x^2+4)(x+1)\right| + \frac{1}{x+1} + C.$$

38 We integrate by parts with $u = x$ and $dv = \csc^2 x\, dx$. We have $du = dx$ and $v = -\cot x$.

$$\begin{aligned}\int_{\pi/4}^{\pi/2} x\csc^2 x\, dx &= -x\cot x\Big|_{\pi/4}^{\pi/2} + \int_{\pi/4}^{\pi/2} \cot x\, dx \\ &= -x\cot x\Big|_{\pi/4}^{\pi/2} + \ln(\sin x)\Big|_{\pi/4}^{\pi/2} \\ &= \left(-\frac{\pi}{2}\cot\frac{\pi}{2} + \ln\left(\sin\frac{\pi}{2}\right)\right) - \left(-\frac{\pi}{4}\cot\frac{\pi}{4} + \ln\left(\sin\frac{\pi}{4}\right)\right) \\ &= (0 + \ln 1) - \left(-\frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} - \ln\frac{1}{\sqrt{2}} = \frac{\pi}{4} - \ln\frac{\sqrt{2}}{2}.\end{aligned}$$

39 $\int \frac{1}{2+e^t}\, dt = \int \frac{e^t}{(e^t+2)e^t}\, dt$. Let $u = e^t$, so that $du = e^t\, dt$. Then we have $\int \frac{1}{(u+2)u}\, du$. If we write

$$\frac{1}{(u+2)u} = \frac{A}{u} + \frac{B}{u+2},$$

then $A(u+2) + Bu = 1$, and we find that $A = 1/2$ and $B = -1/2$. Then we integrate

$$\int \left(\frac{1/2}{u} - \frac{1/2}{u+2}\right) du = \frac{1}{2}\left(\ln|u| - \ln|u+2|\right) + C = \frac{1}{2}\left(t - \ln(2+e^t)\right) + C.$$

40 Note that by long division $\frac{x^2-4}{x+4} = x - 4 + \frac{12}{x+4}$. The original integral is therefore equal to

$$\int \left(x - 4 + \frac{12}{x+4}\right) dx = x^2/2 - 4x + 12\ln|x+4| + C.$$

41

$$\int \frac{1}{1+\cos 4\theta}\cdot\frac{1-\cos 4\theta}{1-\cos 4\theta}\, d\theta = \int \frac{1-\cos 4\theta}{\sin^2 4\theta}\, d\theta = \int (\csc^2 4\theta - \csc 4\theta\cot 4\theta)\, d\theta = \frac{1}{4}\left(-\cot 4\theta + \csc 4\theta\right) + C.$$

42 Two applications of integration by parts gives

$$\int x^2\cos x\, dx = x^2\sin x - 2\int x\sin x\, dx = x^2\sin x - 2\left(x(-\cos x) + \int \cos x\, dx\right) = (x^2-2)\sin x + 2x\cos x + C.$$

43 Two applications of integration by parts gives

$$\int e^x\sin x\, dx = e^x\sin x - \int e^x\cos x\, dx = e^x\sin x - e^x\cos x - \int e^x\sin x\, dx.$$

Thus,

$$2\int e^x\sin x\, dx = e^x\sin x - e^x\cos x,$$

so

$$\int e^x\sin x\, dx = \frac{1}{2}(e^x\sin x - e^x\cos x) + C.$$

44 Integration by parts gives

$$\int x^2\ln x\, dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x}\, dx = \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C.$$

Thus,

$$\int_1^e x^2\ln x\, dx = \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right)\Big|_1^e = \frac{e^3}{3}\cdot\frac{2}{3} - \frac{1}{3}\left(-\frac{1}{3}\right) = \frac{2e^3+1}{9}.$$

45 We decompose the integrand via partial fractions. If we write $\frac{2x^3+5x^2+13x+9}{x^2(x^2+4x+9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4x+9}$, we can solve and obtain $A = B = C = 1$ and $D = 0$. Then we have

$$\int \frac{1}{x}\,dx + \int \frac{1}{x^2}\,dx + \int \frac{x}{x^2+4x+9}\,dx = \ln|x| - \frac{1}{x} + \int \frac{x}{(x+2)^2+5}\,dx = \ln|x| - \frac{1}{x} + \int \frac{u-2}{u^2+5}\,du,$$

where $u = x+2$ so that $du = dx$. Then we have

$$\begin{aligned}\ln|x| - \frac{1}{x} + \int \frac{u}{u^2+5}\,du - \int \frac{2}{u^2+5}\,du &= \ln|x| - \frac{1}{x} + \frac{1}{2}\ln|u^2+5| - \frac{2}{\sqrt{5}}\tan^{-1}\frac{u}{\sqrt{5}} + C \\ &= \ln|x| - \frac{1}{x} + \frac{1}{2}\ln|x^2+4x+9| - \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C.\end{aligned}$$

46 We decompose the integrand via partial fractions. If we write

$$\frac{x^3+4x^2+12x+4}{(x^2+4x+10)^2} = \frac{Ax+B}{x^2+4x+10} + \frac{Cx+D}{(x^2+4x+10)^2},$$

and solve we obtain $A = 1$, $B = 0$, $C = 2$ and $D = 4$. We therefore have

$$\begin{aligned}\int \frac{x}{x^2+4x+10}\,dx + \int \frac{2(x+2)}{(x^2+4x+10)^2}\,dx &= \int \frac{x+2-2}{(x+2)^2+6}\,dx + \int \frac{2(x+2)}{(x^2+4x+10)^2}\,dx \\ &= \int \frac{x+2}{(x+2)^2+6}\,dx - \int \frac{2}{(x+2)^2+6}\,dx + \int \frac{2(x+2)}{(x^2+4x+10)^2}\,dx.\end{aligned}$$

For the first two integrals let $u = x+2$ and for the last let $w = x^2+4x+10$. Then we have

$$\begin{aligned}\int \frac{u}{u^2+6}\,du - \int \frac{2}{u^2+6}\,du + \int \frac{1}{w^2}\,dw &= \frac{1}{2}\ln|u^2+6| - \frac{2}{\sqrt{6}}\tan^{-1}\frac{u}{\sqrt{6}} - \frac{1}{w} + C \\ &= \frac{1}{2}\ln|x^2+4x+10| - \frac{2}{\sqrt{6}}\tan^{-1}\left(\frac{x+2}{\sqrt{6}}\right) - \frac{1}{x^2+4x+10} + C.\end{aligned}$$

47 Let $u = 4\theta$, Then

$$\int \cos^2 4\theta\,d\theta = \frac{1}{4}\int \cos^2 u\,du = \frac{1}{4}\int \frac{1+\cos 2u}{2}\,du = \frac{u}{8} + \frac{\sin 2u}{16} + C = \frac{\theta}{2} + \frac{\sin 8\theta}{16} + C.$$

48 Let $u = \cos 3x$. Then $du = -3\sin 3x\,dx$ and

$$\int \sin 3x \cos^6 3x\,dx = -\frac{1}{3}\int u^6\,du = -\frac{u^7}{21} + C = -\frac{\cos^7 3x}{21} + C.$$

49 Let $u = \sec 2z$. Then $du = 2\sec 2z \tan 2z\,dz$ and

$$\int \sec^{49} 2z \tan 2z\,dz = \frac{1}{2}\int u^{48}\,du = \frac{1}{2}\cdot\frac{u^{49}}{49} + C = \frac{\sec^{49} 2z}{98} + C.$$

50 Using the identity $\cos^2 x = \frac{1+\cos 2x}{2}$, we obtain

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \cos^4 3x\,dx &= \int_0^{\frac{\pi}{6}} \left(\frac{1+\cos 6x}{2}\right)^2 dx = \frac{1}{4}\int_0^{\frac{\pi}{6}} (1 + 2\cos 6x + \cos^2 6x)\,dx \\ &= \frac{1}{4}\int_0^{\frac{\pi}{6}} \left(1 + 2\cos 6x + \frac{1+\cos 12x}{2}\right) dx = \frac{1}{4}\int_0^{\frac{\pi}{6}} \left(\frac{3}{2} + 2\cos 6x + \frac{1}{2}\cos 12x\right) dx \\ &= \left(\frac{3x}{8} + \frac{1}{12}\sin 6x + \frac{1}{96}\sin 12x\right)\Bigg|_0^{\frac{\pi}{6}} = \frac{\pi}{16}.\end{aligned}$$

51 Let $u = \cos 4\theta$ so that $du = -4\sin\theta\, d\theta$. Then

$$\begin{aligned}\int_0^{\pi/4} \sin^5 4\theta\, d\theta &= \int_0^{\pi/4} \sin^4 4\theta \cdot \sin 4\theta\, d\theta = -\frac{1}{4}\int_1^{-1} (1-u^2)^2\, du = \frac{1}{4}\int_{-1}^{1} (1-u^2)^2\, du \\ &= \frac{1}{4}\int_{-1}^{2} (1-2u^2+u^4)\, du = \frac{1}{4}\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right)\Bigg|_{-1}^{1} \\ &= \frac{1}{4}\left(1 - \frac{2}{3} + \frac{1}{5} - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right) = \frac{4}{15}.\end{aligned}$$

52 Use the identity $\tan^2 u = \sec^2 u - 1$ and the substitution $v = \tan 2u$, we have

$$\begin{aligned}\int \tan^4 2u\, du &= \int \tan^2 2u \sec^2 2u\, du - \int \tan^2 2u\, du = \frac{1}{2}\int v^2\, dv - \int (\sec^2 2u - 1)\, du \\ &= \frac{v^3}{6} - \frac{1}{2}\tan 2u + u + C = \frac{1}{6}\tan^3 2u - \frac{1}{2}\tan 2u + u + C.\end{aligned}$$

53 If we let $u^6 = x$, then $6u^5\, du = dx$. Substituting yields

$$\begin{aligned}\int \frac{6u^5}{u^3+u^2}\, du &= 6\int \frac{u^3}{u+1}\, du = 6\int \left(u^2 - u + 1 - \frac{1}{u+1}\right)\, du \\ &= 2u^3 - 3u^2 + 6u - 6\ln|u+1| + C = 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln(\sqrt[6]{x}+1) + C.\end{aligned}$$

54 Let $x = (5/3)\sec\theta$ so that $dx = (5/3)\sec\theta\tan\theta\, d\theta$. Then

$$\int \frac{dx}{\sqrt{9x^2-25}} = \frac{5}{3}\int \frac{\sec\theta\tan\theta\, d\theta}{5\tan\theta} = \frac{1}{3}\int \sec\theta\, d\theta = \frac{1}{3}\ln|\sec\theta + \tan\theta| + C.$$

Now substitute $\sec\theta = (3/5)x$, $\tan\theta = (1/5)\sqrt{9x^2-25}$ to obtain

$$\int \frac{dx}{\sqrt{9x^2-25}} = \frac{1}{3}\ln(3x + \sqrt{9x^2-25}) + C$$

(note that we absorb the constant $-\ln 5$ into C, and we don't need absolute values because we were given $x > 5/3$).

55 We may write the given integral as $\frac{1}{\sqrt{2}}\int \frac{dy}{y^2\sqrt{9-y^2}}$. Let $y = 3\sin\theta$ so that $dy = 3\cos\theta\, d\theta$. Then

$$\frac{1}{\sqrt{2}}\int \frac{dy}{y^2\sqrt{9-y^2}} = \frac{1}{\sqrt{2}}\int \frac{3\cos\theta\, d\theta}{9\sin^2\theta \cdot 3\cos\theta} = \frac{1}{9\sqrt{2}}\int \csc^2\theta\, d\theta = -\frac{1}{9\sqrt{2}}\cot\theta + C.$$

Now use $\sin\theta = y/3$, $\cos\theta = (1/3)\sqrt{9-y^2}$ to obtain

$$\frac{1}{\sqrt{2}}\int \frac{dy}{y^2\sqrt{9-y^2}} = -\frac{1}{9\sqrt{2}y}\sqrt{9-y^2} + C.$$

56 Let $x = \sin\theta$ so that $dx = \cos\theta\, d\theta$. Then

$$\begin{aligned}\int_0^{\sqrt{3}/2} \frac{x^2}{(1-x^2)^{3/2}} dx &= \int_0^{\pi/3} \frac{\sin^2\theta}{\cos^3\theta}\cos\theta\, d\theta = \int_0^{\pi/3} \tan^2\theta\, d\theta = \int_0^{\pi/3} (\sec^2\theta - 1)\, d\theta \\ &= (\tan\theta - \theta)\Big|_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3}.\end{aligned}$$

57 Let $x = (3/2)\tan\theta$ so that $dx = (3/2)\sec^2\theta\, d\theta$, and note that $(3/2)\tan(\pi/6) = \sqrt{3}/2$. Then

$$\int_0^{\sqrt{3}/2} \frac{4}{9+4x^2} dx = \int_0^{\pi/6} \frac{4}{9\sec^2\theta} \cdot \frac{3}{2}\sec^2\theta\, d\theta = \frac{2}{3}\int_0^{\pi/6} d\theta = \frac{\pi}{9}.$$

58 Let $u = \sin\theta$ so that $du = \cos\theta\, d\theta$. Then

$$\int \frac{(1-u^2)^{5/2}}{u^8}\, du = \int \frac{\cos^5\theta}{\sin^8\theta}\cos\theta\, d\theta = \int \cot^6\theta\csc^2\theta\, d\theta.$$

Now make the substitution $v = \cot\theta$ so that $dv = -\csc^2\theta\, d\theta$ to obtain

$$\int \cot^6\theta\csc^2\theta\, d\theta = -\int v^6\, dv = -\frac{v^7}{7} + C,$$

so

$$\int \frac{(1-u^2)^{5/2}}{u^8}\, du = -\frac{1}{7}\left(\frac{\sqrt{1-u^2}}{u}\right)^7 + C.$$

59 Let $u = \cosh x$ so that $du = \sinh x\, dx$. Then

$$\int \frac{\sinh x}{\cosh^2 x}\, dx = \int u^{-2}\, du = \frac{-1}{u} + C = \frac{-1}{\cosh x} + C = -\operatorname{sech} x + C.$$

60 Let $u = x^2$ and $dv = \cosh x\, dx$. Then $du = 2x\, dx$ and $v = \sinh x$. The given integral is equal to $x^2\sinh x - \int 2x\sinh x\, dx$. Now let $u = x$ and $dv = \sinh x\, dx$, so that $du = dx$ and $v = \cosh x$. Then we have

$$x^2\sinh x - 2\left(x\cosh x - \int \cosh x\, dx\right) = x^2\sinh x - 2x\cosh x + 2\sinh x + C = (x^2+2)\sinh x - 2x\cosh x + C.$$

61 Let $u = \sinh x$ so that $du = \cosh x\, dx$. Substituting gives

$$\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-u^2}}\, du = \left(\sin^{-1}(u/2)\right)\Big|_0^{\sqrt{3}} = \frac{\pi}{3}.$$

62 Let $u = \sinh^{-1}(x)$ and $dv = dx$, so that $du = \frac{1}{\sqrt{1+x^2}}\, dx$ and $v = x$. Then

$$\int \sinh^{-1} x\, dx = x\sinh^{-1}(x) - \int \frac{x}{\sqrt{1+x^2}}\, dx = x\sinh^{-1}(x) - \sqrt{1+x^2} + C.$$

63 Using the method of partial fractions, we express $\dfrac{1}{x^2-2x-15} = \dfrac{1}{(x-5)(x+3)} = \dfrac{A}{x-5} + \dfrac{B}{x+3}$. Clearing denominators gives $1 = A(x+3) + B(x-5)$ and comparing coefficients gives $A+B=0$, $3A-5B=1$ which has solution $A = 1/8$, $B = -1/8$. Hence

$$\int \frac{dx}{x^2-2x-15} = \frac{1}{8}\int\left(\frac{1}{x-5} - \frac{1}{x+3}\right) dx = \frac{1}{8}\ln\left|\frac{x-5}{x+3}\right| + C.$$

64 Using the method of partial fractions, we express $\dfrac{1}{x^3-2x^2} = \dfrac{1}{x^2(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2}$. Clearing denominators gives $1 = Ax(x-2) + B(x-2) + Cx^2$ and comparing coefficients gives $A+C=0$, $-2A+B=0$ and $-2B=1$, which has solution $A=-1/4$, $B=-1/2$, $C=1/4$. Hence

$$\int \frac{dx}{x^2-2x-15} = \frac{1}{4}\int\left(-\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x-2}\right) dx = \frac{1}{2x} + \frac{1}{4}\ln\left|\frac{x-2}{x}\right| + C.$$

65 Using the method of partial fractions, we express $\dfrac{1}{(y+1)(y^2+1)} = \dfrac{A}{y+1} + \dfrac{By+C}{y^2+1}$. Clearing denominators gives $1 = A(y^2+1) + (By+C)(y+1)$ and comparing coefficients gives $A+B=0$, $B+C=0$ and $A+C=1$, which has solution $A=1/2$, $B=-1/2$, $C=1/2$. Hence

$$\int_0^1 \frac{dy}{(y+1)(y^2+1)} = \frac{1}{2}\int_0^1\left(\frac{1}{y+1} + \frac{1-y}{y^2+1}\right) dy = \frac{1}{2}\left(\ln(y+1) + \tan^{-1} y - \frac{1}{2}\ln(y^2+1)\right)\Big|_0^1 = \frac{1}{4}\ln 2 + \frac{\pi}{8}.$$

66 First we consider $\int \frac{6x}{1+x^6}\,dx$. Let $u = x^2$, so that $du = 2x\,dx$. Then we have $3\int \frac{1}{1+u^3}\,du = \int \left(\frac{1}{1+u} + \frac{2-u}{u^2-u+1}\right) du$ where this is obtained through a partial fraction decomposition. The first term evaluates to $\ln|u+1|$ while the second can be written as

$$-\frac{1}{2}\int \frac{2u-4}{u^2-u+1}\,du = -\frac{1}{2}\int \left(\frac{2u-1}{u^2-u+1} - \frac{3}{u^2-u+1}\right) du = -\frac{1}{2}\ln|u^2-u+1| + \frac{3}{2}\int \frac{1}{u^2-u+1}\,du.$$

This last integral can be written as $\frac{3}{2}\int \frac{1}{(u-1/2)^2+3/4}\,du = \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{2u-1}{\sqrt{3}}\right) + C$. Putting this all together, we have $\int \frac{6x}{1+x^6}\,dx = \ln(x^2+1) - \frac{1}{2}\ln(x^4-x^2+1) + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2-1}{\sqrt{3}}\right) + C$. If we let $G(x) = \ln(x^2+1) - \frac{1}{2}\ln(x^4-x^2+1) + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2-1}{\sqrt{3}}\right)$, then $G(1) = \frac{\sqrt{3}\pi}{6} + \ln 2$, and $\lim_{b\to\infty} G(b) = \frac{\sqrt{3}\pi}{2}$. This limit is computed as follows:

$$\lim_{b\to\infty} G(b) = \lim_{b\to\infty}\left(\ln\left(\frac{b^2+1}{\sqrt{b^4-b^2+1}}\right) + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{2b^2-1}{\sqrt{3}}\right)\right) = \ln 1 + \frac{3}{\sqrt{3}}\cdot\frac{\pi}{2} = \frac{\sqrt{3}}{2}.$$

Thus we have $\int_1^\infty \frac{6x}{1+x^6}\,dx = \frac{\sqrt{3}\pi}{3} - \ln 2$.

67 First consider $\int_0^1 \frac{dx}{\sqrt[3]{|x-1|}} = \lim_{a\to 1^-}\int \frac{dx}{\sqrt[3]{1-x}} = \lim_{a\to 1^-} -\frac{3}{2}(1-x)^{2/3}\Big|_0^a = \lim_{a\to 1^-} -\frac{3}{2}\left((1-a)^{2/3}-1\right) = \frac{3}{2}$.

Now consider $\int_1^2 \frac{dx}{\sqrt[3]{|x-1|}} = \lim_{b\to 1^+}\int_b^2 \frac{dx}{\sqrt[3]{x-1}} = \lim_{b\to 1^+} \frac{3}{2}(x-1)^{2/3}\Big|_b^2 = \lim_{b\to 1^+} \frac{3}{2}\left(1-(b-1)^{2/3}\right) = \frac{3}{2}$.

Thus, $\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}} = \frac{3}{2} + \frac{3}{2} = 3$.

68 Let $u = x+1$. Then we have

$$\int_{-1}^1 \frac{dx}{x^2+2x+5} = \int_0^2 \frac{du}{u^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\Big|_0^2 = \frac{\pi}{8}.$$

69 Factor $x^2-x-2 = (x-2)(x+1)$ and use the method of partial fractions: $\frac{1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$. Clearing denominators gives $1 = A(x+1) + B(x-2)$ and equating coefficients gives $A+B=0$, $A-2B=1$, so $A = 1/3$, $B = -1/3$. Therefore

$$\int \frac{dx}{x^2-x-2} = \frac{1}{3}\int \left(\frac{1}{x-2} - \frac{1}{x+1}\right) dx = \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + C.$$

70 As a preliminary step, express $\frac{3x^2+x-3}{x^2-1} = \frac{3(x^2-1+1)+x-3}{x^2-1} = 3 + \frac{x}{x^2-1}$; hence

$$\int \frac{3x^2+x-3}{x^2-1}\,dx = 3x + \int \frac{x}{x^2-1}\,dx = 3x + \frac{1}{2}\ln|x^2-1| + C,$$

where we make the substitution $u = x^2-1$ for the final integral above.

71 As a preliminary step, observe that

$$\frac{2x^2-4x}{x^2-4} = \frac{2x(x-2)}{(x-2)(x+2)} = \frac{2x}{x+2} = \frac{2(x+2-2)}{x+2} = 2 - \frac{4}{x+2}.$$

Hence

$$\int \frac{2x^2-4x}{x^2-4}\,dx = 2x - 4\ln|x+2| + C = 2(x - 2\ln|x+2|) + C.$$

72 Make the preliminary substitution $u = 2\sqrt{x}$; then $du = dx/\sqrt{x}$ and

$$\int_{1/12}^{1/4} \frac{dx}{\sqrt{x}(1+4x)} = \int_{1/\sqrt{3}}^{1} \frac{du}{1+u^2} = \tan^{-1} u\Big|_{1/\sqrt{3}}^{1} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$$

73 Make the preliminary substitution $x = e^{2t}$, $dx = 2e^{2t}$. Then we have

$$\int \frac{e^{2t}}{(1+e^{4t})^{3/2}}\,dt = \frac{1}{2}\int \frac{dx}{(1+x^2)^{3/2}}.$$

Now let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$ and

$$\frac{1}{2}\int \frac{dx}{(1+x^2)^{3/2}} = \frac{1}{2}\int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{2}\int \cos\theta\,d\theta = \frac{1}{2}\sin\theta + C.$$

Now

$$\sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{x}{\sqrt{1+x^2}} = \frac{e^{2t}}{\sqrt{1+e^{4t}}}.$$

Therefore

$$\int \frac{e^{2t}}{(1+e^{4t})^{3/2}}\,dt = \frac{1}{2}\frac{e^{2t}}{\sqrt{1+e^{4t}}} + C.$$

74 Let $v = 1+\sqrt{x}$. Then $dv = \dfrac{1}{2\sqrt{x}}\,dx$. We have

$$\begin{aligned}\int \frac{dx}{\sqrt{\sqrt{1+\sqrt{x}}}} &= \int \frac{1}{(1+\sqrt{x})^{1/4}}\cdot\frac{\sqrt{x}}{\sqrt{x}}\,dx\\ &= 2\int \frac{v-1}{v^{1/4}}\,dv = 2\int \left(v^{3/4} - v^{-1/4}\right)\,dv = 2\left(\frac{4}{7}v^{7/4} - \frac{4}{3}v^{3/4}\right) + C\\ &= \frac{8}{21}v^{3/4}(3v-7) + C = \frac{8}{21}(1+\sqrt{x})^{3/4}(3\sqrt{x}-4) + C.\end{aligned}$$

75

a. Let $u = e^x$. Then $du = e^x\,dx$ and we have

$$\int \sec u \tan u\,du = \sec u + C = \sec e^x + C.$$

b. We integrate by parts with $u = e^x$ and $dv = e^x \sec e^x \tan e^x\,dx$. Then $du = e^x\,dx$ and $v = \sec e^x$ (by part a). Then we have

$$e^x \sec e^x - \int e^x \sec e^x\,dx = e^x \sec e^x - \ln|\sec e^x + \tan e^x| + C.$$

76 Using the table of integrals entry 84, we find that

$$\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+1}(a(n+1)x-b)}{a^2(n+1)(n+2)} + C.$$

Therefore

$$\int x(2x+3)^5\,dx = \frac{(2x+3)^6(12x-3)}{168} + C.$$

77 Using the table of integrals entry 77, we find that

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}}\tan^{-1}\left(\sqrt{\frac{ax-b}{b}}\right) + C.$$

Therefore

$$\int \frac{dx}{x\sqrt{4x-6}} = \frac{\sqrt{6}}{3}\tan^{-1}\left(\sqrt{\frac{2x-3}{3}}\right) + C.$$

78 Using the table of integrals entry 34, we find that

$$\int_0^{\pi/2} \frac{d\theta}{1+\sin 2\theta} = \left(\frac{-1}{2}\tan\left(\pi/4-\theta\right)\right)\Big|_0^{\pi/2} = \frac{-1}{2}(\tan(-\pi/4)-\tan(\pi/4)) = \frac{1}{2}(1-(-1)) = 1.$$

79 Using the table of integrals reduction formula (number 45) twice, we have

$$\begin{aligned}\int \sec^5 x\,dx &= \frac{\sec^3 x\tan x}{4} + \frac{3}{4}\int \sec^3 x\,dx \\ &= \frac{\sec^3 x\tan x}{4} + \frac{3}{4}\left(\frac{\sec x\tan x}{2} + \frac{1}{2}\int \sec x\ dx\right) \\ &= \frac{\sec^3 x\tan x}{4} + \frac{3}{4}\left(\frac{\sec x\tan x}{2} + \frac{1}{2}(\ln|\sec x+\tan x|\right) + C \\ &= \frac{\sec^3 x\tan x}{4} + \frac{3\sec x\tan x}{8} + \frac{3}{8}(\ln|\sec x+\tan x| + C.\end{aligned}$$

80 Let $u = e^x$. Then $du = e^x\,dx$ and we have

$$\int_1^e \frac{1}{u^2\sqrt{16-u^2}}\,du = \left(-\frac{\sqrt{16-u^2}}{16u}\right)\Bigg|_1^e = \frac{\sqrt{15}}{16} - \frac{\sqrt{16-e^2}}{16e} = \frac{\sqrt{15}e-\sqrt{16-e^2}}{16e}.$$

81 Let $u = 1+\sin x$ so that $du = \cos x\,dx$. We have

$$\begin{aligned}\int_1^2 \ln^3 u\,du &= \left(u\ln^3 u - 3u\ln^2 u + 6u\ln u - 6u\right)\Big|_1^2 \\ &= 2\ln^3 2 - 6\ln^2 2 + 12\ln 2 - 12 - (0-6) = 2(\ln^3 2 - 3\ln^2 2 + 6\ln 2 - 3).\end{aligned}$$

82 This can be written as

$$\lim_{b\to-\infty}\int_b^{-1} \frac{1}{(x-1)^4}\,dx = \lim_{b\to-\infty}\left(\frac{-1}{3(x-1)^3}\right)\Bigg|_b^{-1} = \lim_{b\to-\infty}\left(\frac{1}{24} + \frac{1}{3(b-1)^3}\right) = \frac{1}{24}.$$

83 First evaluate

$$\int_0^b xe^{-x}\,dx = -e^{-x}(x+1)\Big|_0^b = 1-(b+1)e^{-b}.$$

Then

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}(1-(b+1)e^{-b}) = 1.$$

84 First consider $\int_0^{\pi/2} \sec^2 x\,dx$. We have $\int_0^{\pi/2} \sec^2 x\,dx = \lim_{a\to\pi/2^-}\int_0^a \sec^2 x\,dx = \lim_{a\to\pi/2^-} \tan x\Big|_0^a = \lim_{a\to\pi/2^-} \tan a - 0 = \infty$. This integral diverges, and thus $\int_0^\pi \sec^2 x\,dx$ diveges.

85 First take $0 < c < 3$ and evaluate

$$\int_0^c \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right)\Big|_0^c = \sin^{-1}\left(\frac{c}{3}\right).$$

Then

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \lim_{c\to 3^-} \sin^{-1}\left(\frac{c}{3}\right) = \frac{\pi}{2}.$$

86

$$\int_{-\infty}^{\infty} \frac{x^3}{1+x^8}\,dx = \int_{-\infty}^{0} \frac{x^3}{1+x^8}\,dx + \int_0^{\infty} \frac{x^3}{1+x^8}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{x^3}{1+x^8}\,dx + \lim_{b\to\infty}\int_0^b \frac{x^3}{1+x^8}\,dx.$$

Let $u = x^4$. Then $du = 4x^3\,dx$. We have

$$\int \frac{x^3}{1+x^8}\,dx = \frac{1}{4}\int \frac{1}{u^2}\,du = \frac{1}{4}\tan^{-1} u + C = \frac{1}{4}\tan^{-1} x^4 + C$$

Then the original integral is equal to

$$\lim_{a\to-\infty} \frac{1}{4}\tan^{-1} x^4\Big|_a^0 + \lim_{b\to\infty} \frac{1}{4}\tan^{-1} x^4\Big|_0^b = -\frac{\pi}{8} + \frac{\pi}{8} = 0.$$

87

$$\begin{aligned}
&\int_1^{\infty} \frac{dx}{x(x-1)^{1/3}}\,dx = \int_1^2 \frac{dx}{x(x-1)^{1/3}}\,dx + \int_2^{\infty} \frac{dx}{x(x-1)^{1/3}}\,dx \\
&= \lim_{a\to1^+}\int_a^2 \frac{dx}{x(x-1)^{1/3}}\,dx + \lim_{b\to\infty}\int_2^b \frac{dx}{x(x-1)^{1/3}}\,dx \\
&= \lim_{a\to1^-}\left(-\log\left(\sqrt[3]{x-1}+1\right) + \frac{1}{2}\log\left((x-1)^{2/3} - \sqrt[3]{x-1}+1\right) + \sqrt{3}\tan^{-1}\left(\frac{2\sqrt[3]{x-1}-1}{\sqrt{3}}\right)\right)\Bigg|_a^2 \\
&+ \lim_{b\to\infty}\left(-\log\left(\sqrt[3]{x-1}+1\right) + \frac{1}{2}\log\left((x-1)^{2/3} - \sqrt[3]{x-1}+1\right) + \sqrt{3}\tan^{-1}\left(\frac{2\sqrt[3]{x-1}-1}{\sqrt{3}}\right)\right)\Bigg|_2^b \\
&= \frac{\pi}{\sqrt{3}} - \ln 2 + \frac{\pi}{\sqrt{3}} + \ln 2 = \frac{2\pi}{\sqrt{3}}.
\end{aligned}$$

88 Completing the square in the denominator gives $x^2+2x+1+1 = (x+1)^2+1$. Let $u = x+1$ so that $du = dx$. We have

$$\lim_{a\to-\infty}\int_a^0 \frac{2}{u^2+1}\,du + \lim_{b\to\infty}\int_0^b \frac{2}{u^2+1}\,du = \lim_{a\to-\infty}\left(0 - 2\tan^{-1} a\right) + \lim_{b\to\infty} 2\tan^{-1} b = \pi + \pi = 2\pi.$$

89 $x^5+x^4+x^3+1 > x^5$ for $x \ge 1$, so $0 < \dfrac{1}{x^5+x^4+x^3+1} < \dfrac{1}{x^5}$. Because $\displaystyle\int_1^{\infty} \frac{1}{x^5}\,dx$ converges, so does the given integral by the Comparison Test.

90 $\sqrt{x+\sin x} > \sqrt{x}$ for $0 < x \le 1$, so $0 < \dfrac{1}{\sqrt{x+\sin x}} < \dfrac{1}{\sqrt{x}}$ for $0 < x \le 1$. Because $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges, so does $\displaystyle\int_0^1 \frac{dx}{\sqrt{x+\sin x}}$ by the Comparison Test.

91 $\dfrac{x^3}{\sqrt{x^7-1}} \cdot \dfrac{\frac{1}{x^3}}{\frac{1}{\sqrt{x^6}}} = \dfrac{1}{\sqrt{x-\frac{1}{x^6}}} > \dfrac{1}{\sqrt{x}}$ for $x \ge 3$. Then because $\displaystyle\int_3^{\infty} \frac{dx}{\sqrt{x}}$ diverges, so does $\displaystyle\int_3^{\infty} \frac{x^3}{\sqrt{x^7-1}}\,dx$ by the Comparison Test.

92 Let $u = \ln x$, so that $du = \frac{1}{x}\,dx$. Then the related indefinite integral is

$$\int \frac{du}{u^p} = \frac{1}{p-1}u^{p-1} + C = \frac{1}{(p-1)\ln^{p-1} x} + C.$$

So

$$\begin{aligned}\int_2^\infty \frac{dx}{x\ln^p x} &= \lim_{b\to\infty} \frac{1}{(p-1)\ln^{p-1} x}\bigg|_2^b \\ &= \lim_{b\to\infty}\left(\frac{1}{(p-1)\ln^{p-1} b} - \frac{1}{(p-1)\ln^{p-1} 2}\right) = -\frac{1}{(p-1)\ln^{p-1} 2} \text{ for } p > 1.\end{aligned}$$

For $p < 1$ this limit doesn't exist so the integral diverges. For $p = 1$, we have

$$\lim_{b\to\infty}\int_2^b \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \ln\ln x\bigg|_2^b = \lim_{b\to\infty}(\ln\ln b - \ln\ln 2) = \infty.$$

The integral converges to $-\frac{1}{(p-1)\ln^{p-1} 2}$ for $p > 1$ and diverges for $p \le 1$.

93 Using a Computer Algebra System, we find that

$$\int_{-1}^1 e^{-2x^2}\,dx \approx 1.196.$$

94 Using a Computer Algebra System, we find that

$$\int_1^{\sqrt{e}} x^3(\ln x)^3\,dx \approx 0.081.$$

95 $\Delta x = \frac{8-0}{4} = 2$. $M(4) = 2(4+4+6+8) = 44$. $T(4) = \frac{2}{2}(3+2(5)+2(7)+2(5)+5) = 42$. $S(4) = \frac{2}{3}(3+4(5)+2(7)+4(5)+5) = \frac{124}{3}$.

96 $\Delta x = \frac{3-1}{4} = \frac{1}{2}$. $M(4) = \frac{1}{2}(f(1.25)+f(1.75)+f(2.25)+f(2.75)) \approx 0.235137$. $T(4) = \frac{1/2}{2}(f(1) + 2f(1.5)+2f(2)+2f(2.5)+f(3)) \approx 0.248103$. $S(4) = \frac{1/2}{3}(f(1)+4f(1.5)+2f(2)+4f(2.5)+f(3)) \approx 0.239568$.

97 $\Delta x = \frac{1}{40}$. $M(40) \approx 0.398236$, $T(40) \approx 0.398771$, and $S(40) \approx 0.398416$.

98 $\Delta x = \frac{8-1}{60} = \frac{7}{60}$. $M(60) \approx 9.82388$, $T(60) \approx 9.82108$, and $S(60) \approx 9.82296$.

99

n	$T_2(n)$	$T_4(n)$	$T_8(n)$
4	0.880619	0.886319	1.036632
8	0.881704	0.886227	0.886319
16	0.881986	0.886227	0.886227
32	0.882058	0.886227	0.886227

Based on these results, we conclude that $\int_0^\infty e^{-x^2}\,dx \approx 0.886227$.

100 The area is given by the improper integral

$$\int_0^{\infty} ae^{-ax}\,dx = \lim_{b\to\infty}\int_0^b ae^{-ax}\,dx = \lim_{b\to\infty}\left(-e^{-ax}\Big|_0^b\right) = \lim_{b\to\infty}(1-e^{-ab}) = 1$$

as long as $a > 0$.

101 The volume generated by revolving around the x-axis is

$$V_x = \pi\int_0^{\pi}\sin^2 x\,dx = \pi\left(\frac{x}{2}-\frac{\sin 2x}{4}\right)\Big||_0^{\pi} = \frac{\pi^2}{2},$$

and the volume generated by revolving around the y-axis is

$$V_y = 2\pi\int_0^{\pi} x\sin x\,dx = 2\pi(\sin x - x\cos x)\Big||_0^{\pi} = 2\pi^2,$$

so the greater volume is obtained by revolving around the y-axis.

102 The volume is

$$V = 2\pi\int_1^e x\ln x\,dx = \frac{\pi}{2}x^2(2\ln x-1)\Big|_1^e = \frac{\pi}{2}\left(e^2+1\right).$$

103 The volume is

$$V = \pi\int_1^e (\ln x)^2\,dx = \pi x((\ln x)^2 - 2\ln x + 2)\Big|_1^e = \pi(e-2).$$

104 The volume is

$$V = \pi\int_1^e (1-(1-\ln x)^2)\,dx = \pi\int_1^e (2\ln x - \ln^2 x)\,dx = \pi\left(-4x + 4x\ln x - x\ln^2 x\right)\Big|_1^e = \pi(4-e).$$

105 The volume is

$$V = 2\pi\int_1^e (x-1)\ln x\,dx = \frac{\pi}{2}x\left(2(x-2)\ln x - x + 4\right)\Big|_1^e = \frac{\pi}{2}\left(e^2-3\right).$$

106 Note that

$$y' = \frac{1}{2}\sqrt{3-x^2} + \frac{x}{2}\frac{-2x}{2\sqrt{3-x^2}} + \frac{3}{2\sqrt{3}}\frac{1}{\sqrt{1-x^2/3}} = \frac{(6-2x^2)-2x^2+6}{4\sqrt{3-x^2}} = \frac{3-x^2}{\sqrt{3-x^2}} = \sqrt{3-x^2}.$$

$$L = \int_0^1 \sqrt{1+3-x^2}\,dx = \int_0^1 \sqrt{4-x^2}\,dx.$$

Using the table of integrals entry 51, we have

$$L = \left((x/2)\sqrt{4-x^2} + 2\sin^{-1}(x/2)\right)\Big|_0^1 = \frac{\sqrt{3}}{2}+\frac{\pi}{3}.$$

107

a. Observe that

$$\int_{1/2}^b \ln x\,dx = (x\ln x - x)\Big|_{1/2}^b \approx b\ln b - b + 0.847;$$

solve $b\ln b - b + 0.847 = 0$ numerically to obtain $b \approx 1.603$.

b. Similarly, we have
$$\int_{1/3}^{b} \ln x\,dx = (x\ln x - x)\Big|_{1/3}^{b} \approx b\ln b - b + 0.700;$$
solve $b\ln b - b + 0.700 = 0$ numerically to obtain $b \approx 1.870$.

c. In general, the pair (a, b) must satisfy the equation
$$\int_{a}^{b} \ln x\,dx = (b\ln b - b) - (a\ln a - a) = 0,$$
which gives $b\ln b - b = a\ln a - a$.

d. As a increases there is less negative area to the right of $x = 1$, so $b = g(a)$ is a decreasing function of a.

108 The arc length is given by the integral $\int_1^{e^2} \sqrt{1+\frac{1}{x^2}}\,dx = \int_1^{e^2} \frac{\sqrt{1+x^2}}{x}\,dx$, which can be evaluated by the trigonometric substitution $x = \tan\theta$ or using a CAS. We obtain
$$\int_{1}^{e^2} \frac{\sqrt{1+x^2}}{x}\,dx = \left(\sqrt{x^2+1} - \ln\left(2\left(\sqrt{x^2+1}+1\right)\right) + \ln x\right)\Big|_{1}^{e^2} \approx 6.789.$$

109 The average velocity is
$$\bar{v} = \frac{1}{\pi}\int_{0}^{\pi} 10\sin 3t\,dt = -\frac{10}{3\pi}\cos 3t\Big|_{0}^{\pi} = \frac{20}{3\pi}.$$

110

a. The distance traveled by car A after 2 hrs is
$$\int_{0}^{2} \frac{40}{t+1}\,dt = 40\ln(t+1)\Big|_{0}^{2} = 40\ln 3 \approx 43.944 \text{ mi}$$
and the distance traveled by car B after 2 hrs is
$$\int_{0}^{2} 40e^{-t/2}\,dt = -80e^{-t/2}\Big|_{0}^{2} = 80(1-e^{-1}) \approx 50.570 \text{ mi},$$
so car B traveled farther.

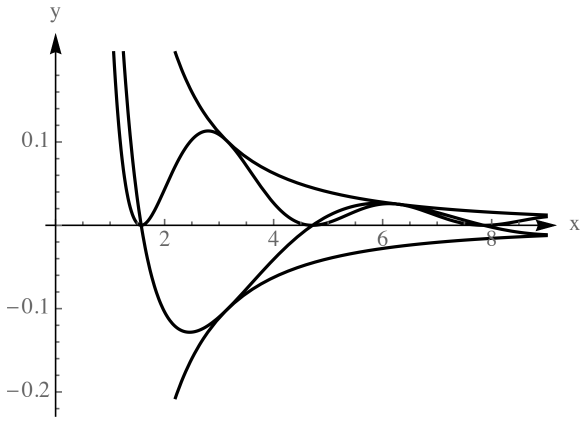
b. The distance traveled by car A after 3 hrs is
$$\int_{0}^{3} \frac{40}{t+1}\,dt = 40\ln(t+1)\Big|_{0}^{3} = 40\ln 4 \approx 55.452 \text{ mi}$$
and the distance traveled by car B after 2 hrs is
$$\int_{0}^{3} 40e^{-t/2}\,dt = -80e^{-t/2}\Big|_{0}^{3} = 80(1-e^{-3/2}) \approx 62.150 \text{ mi},$$
so car B traveled farther.

c. The distance traveled by car A after t hrs is
$$\int_{0}^{t} \frac{40}{s+1}\,ds = 40\ln(s+1)\Big|_{0}^{t} = 40\ln(t+1)$$
and the distance traveled by car B after t hrs is
$$\int_{0}^{t} 40e^{-s/2}\,ds = -80e^{-s/2}\Big|_{0}^{t} = 80(1-e^{-t/2}).$$
The distance traveled by car A increases without bound, whereas the distance traveled by car B approaches 80 mi as $t \to \infty$.

111 The number of cars is given by the integral

$$\int_0^4 800te^{-t/2}\,dt = -1600(t+2)e^{-t/2}\Big|_0^4 = 3200(1-3e^{-2}) \approx 1901.$$

112 Observe that both $g(x)$ and $h(x)$ lie between the functions $\pm 1/x^2$, which have finite area from 1 to ∞. Therefore the improper integrals of both $g(x)$ and $h(x)$ from 1 to ∞ are finite.



113

a. Using integration by parts, we find that

$$I(p) = \int_1^e \frac{\ln x}{x^p}\,dx = -\frac{x^{1-p}}{(p-1)^2}\left((p-1)\ln x + 1\right)\Big|_1^e = \frac{1}{(p-1)^2}(1-pe^{1-p})$$

for $p \neq 1$, and using the substitution $u = \ln x$ gives

$$I(1) = \int_1^e \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}\Big|_1^e = \frac{1}{2}.$$

b. We have

$$\lim_{p\to\infty} I(p) = \lim_{p\to\infty} \frac{1}{(p-1)^2}(1-pe^{1-p}) = \lim_{p\to\infty}\left(\frac{1}{(p-1)^2} - \frac{pe}{(p-1)^2}e^{-p}\right) = 0,$$

and

$$\lim_{p\to-\infty} I(p) = \lim_{p\to-\infty}\left(\frac{1}{(p-1)^2} - \frac{pe}{(p-1)^2}e^{-p}\right) = \infty,$$

since e^{-p} grows much faster than $(p-1)^2/p$ as $p \to -\infty$.

c. By inspection we see that $I(0) = 1$.

114

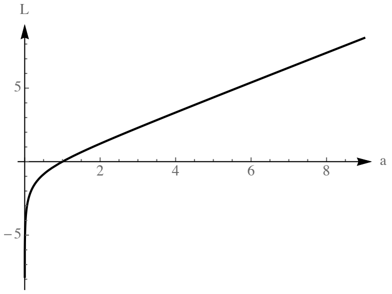
a. If $y = \ln x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$, so $\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \dfrac{\sqrt{x^2+1}}{x}$. Thus the arc length is

$$L(a) = \int_1^a \frac{\sqrt{x^2+1}}{x}\,dx.$$

If we let $u^2 = x^2 + 1$, then $2u\,du = 2x\,dx$. Substituting gives

$$\begin{aligned}\int_{\sqrt{2}}^{\sqrt{a^2+1}} \frac{u}{\sqrt{u^2-1}} \cdot \frac{u}{\sqrt{u^2-1}}\,du &= \int_{\sqrt{2}}^{\sqrt{a^2+1}} \frac{u^2}{u^2-1}\,du \\ &= \int_{\sqrt{2}}^{\sqrt{a^2+1}} \left(1 + \frac{1}{u^2-1}\right) du \\ &= \int_{\sqrt{2}}^{\sqrt{a^2+1}} \left(1 + \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right)\right) du \\ &= u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| \,\Bigg|_{\sqrt{2}}^{\sqrt{a^2+1}} \\ &= \sqrt{a^2+1} - \sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{a^2+1}-1}{\sqrt{a^2+1}+1}\right) + \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).\end{aligned}$$

b.



c. Because the only term in $L(a)$ which increases as $a \to \infty$ is $\sqrt{a^2+1}$, and this term increases like a (because $\lim\limits_{x\to\infty} \dfrac{\sqrt{a^2+1}}{a} = 1$), this function increases like a.

115 Use a calculator program for, say, Simpson's rule with $n = 100$, but replace the limits with 0.00000001 and 0.99999999 to avoid errors coming from trying to evaluate the function at $x = 0$ or $x = 1$; we obtain

$$\int_0^1 \frac{x^2 - x}{\ln x}\,dx \approx 0.4054651.$$

116 Numerically approximate the integral to obtain

$$\int_0^{1/2} \frac{\ln(1+2x)}{x}\,dx \approx 0.8224;$$

therefore $n = \pi^2/0.8224 = 12$. (When approximating the integral, replace the lower limit 0 with a small positive number like 0.00001 to avoid an error from evaluating the integrand at $x = 0$.)

117 Numerically approximate the integral to obtain

$$\int_0^1 \frac{\sin^{-1} x}{x}\,dx \approx 1.0889;$$

therefore $n = \pi \ln 2/1.0889 = 2$. (When approximating the integral, replace the lower limit 0 with a small positive number like 0.00001 to avoid an error from evaluating the integrand at $x = 0$.)

118

a. Following the hint, write

$$I(a) = \int_0^1 \frac{dx}{(1+x^a)(1+x^2)} + \int_1^\infty \frac{dx}{(1+x^a)(1+x^2)}$$

and make the substitution $u = 1/x$ in the second integral to obtain

$$\begin{aligned} I(a) &= \int_0^1 \frac{dx}{(1+x^a)(1+x^2)} - \int_1^0 \frac{du}{(1+u^{-a})(1+u^{-2})u^2} \\ &= \int_0^1 \frac{dx}{(1+x^a)(1+x^2)} + \int_0^1 \frac{u^a du}{(1+u^a)(1+u^2)} \\ &= \int_0^1 \frac{1+x^a}{(1+x^a)(1+x^2)}\,dx = \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x = \frac{\pi}{4}. \end{aligned}$$

b. Let

$$I = \int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x) + f(\sin x)}\,dx$$

and make the substitution $u = \pi/2 - x$ to obtain

$$I = -\int_{\pi/2}^0 \frac{f(\sin u)}{f(\sin u) + f(\cos u)}\,du = \int_0^{\pi/2} \frac{f(\sin u)}{f(\sin u) + f(\cos u)}\,du.$$

Therefore

$$2I = \int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x) + f(\sin x)}\,dx + \int_0^{\pi/2} \frac{f(\sin x)}{f(\cos x) + f(\sin x)}\,dx = \int_0^{\pi/2} dx = \frac{\pi}{2},$$

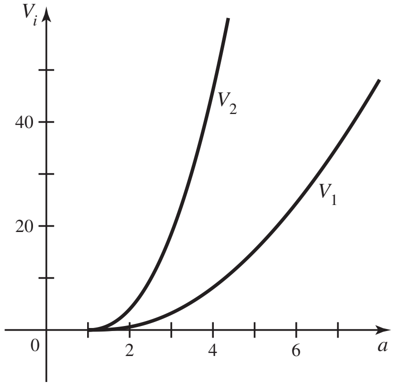
so $I = \pi/4$.

119

a. Using integration by parts or a CAS, we find that the volume is $V_1(a) = \pi \int_1^a (\ln x)^2\,dx = \pi x((\ln x)^2 - 2\ln x + 2)\Big|_1^a = \pi[(a\ln^2 a - 2a\ln a + 2(a-1)]$.

b. Using integration by parts or a CAS, we find that the volume is $V_2(a) = 2\pi \int_1^a x\ln x\,dx = \frac{\pi}{2}x^2(2\ln x - 1)\Big|_1^a = \frac{\pi}{2}(2a^2\ln a - a^2 + 1)$.

c. As shown by the graph, $V_2(a) > V_1(a)$ for all $a > 1$.



120

a. We have $V_1 = \pi \int_1^\infty x^{-2p}\,dx = \dfrac{\pi}{2p-1}$ for $p > \dfrac{1}{2}$, and $V_1 = \infty$ for $p \le 1/2$ (see Example 2 in Section 7.4 for the evaluation of this improper integral.) Similarly $V_2 = 2\pi \int_1^\infty x^{1-p}\,dx = \dfrac{2\pi}{p-2}$ for $p > 2$, and $V_2 = \infty$ for $p \le 2$. Observe that $V_1 = \dfrac{2\pi}{4p-2} < \dfrac{2\pi}{p-2} = V_2$ for all $p > 2$. Therefore $V_1 = V_2$ only when both are infinite.

b. We will use the fact that $\int_0^1 x^a\,dx = \dfrac{1}{a+1}$ for $a > -1$ and is infinite otherwise. We have $V_1 = \pi \int_0^1 x^{-2p}\,dx = \dfrac{\pi}{1-2p}$ for $p < \dfrac{1}{2}$, and $V_1 = \infty$ for $p \geq 1/2$. Similarly

$$V_2 = 2\pi \int_0^1 x^{1-p}\,dx = \frac{2\pi}{2-p} \text{ for } p < 2,$$

and $V_2 = \infty$ for $p \geq 2$. As above, $V_1 = V_2$ only when both are infinite.

121 We have

$$V_1 = \pi \int_0^b e^{-2ax}\,dx = -\frac{\pi}{2a}e^{-2ax}\Big|_0^b = \frac{\pi}{2a}\left(1 - e^{-2ab}\right),$$

and

$$V_2 = \pi \int_b^\infty e^{-2ax}\,dx = \lim_{c\to\infty}\left(-\frac{\pi}{2a}e^{-2ax}\Big|_b^c\right) = \frac{\pi}{2a}e^{-2ab}.$$

Equating and solving gives $e^{-2ab} = 1/2$, which is equivalent to $ab = (1/2)\ln 2$.

122 $R_1 = \int_0^{\pi/3} \tan x\,dx = -\ln|\cos x|\Big|_0^{\pi/3} = -\ln(1/2) = \ln 2.$

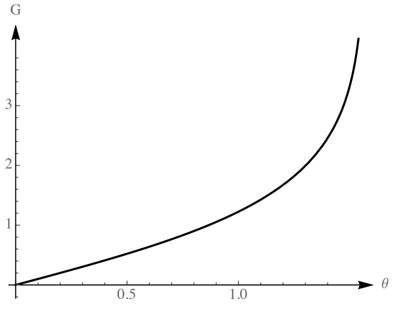
$R_2 = \int_0^{\pi/6} \sec x\,dx = \ln|\sec x + \tan x|\Big|_0^{\pi/6} = \ln\sqrt{3} = \dfrac{\ln 3}{2}.$

$R_1 \approx 0.693$ and $R_2 \approx 0.549$ so $R_1 > R_2$.

123

$$\begin{aligned} A &= \int_0^{\pi/4} (\sec x - \tan x)\,dx = (\ln|\sec x + \tan x| + \ln|\cos x|)\Big|_0^{\pi/4} \\ &= \ln\left(\sqrt{2}+1\right) + \ln\left(\frac{\sqrt{2}}{2}\right) - (0+0) = \ln\left(1 + \frac{\sqrt{2}}{2}\right). \end{aligned}$$

124 Note that the function is equivalent to $\ln|\sec\theta + \tan\theta|$. Its derivative is $\sec\theta$ which is positive on the interval $[0, \pi/2)$ as its second derivative $\sec\theta\tan\theta$, so it is increasing and concave up on the given interval.



125

a. From the reduction formula in Section 8.3 we have

$$\int_0^\pi \sin^m x\,dx = \frac{-\sin^{m-1} x\cos x}{m}\Big|_0^\pi + \frac{m-1}{m}\int_0^\pi \sin^{m-2}\,dx = \frac{m-1}{m}\int_0^\pi \sin^{m-2}\,dx.$$

b. We proceed by strong induction on n. For $n = 1$, we have

$$\int_0^\pi \sin^2 x\,dx = \int_0^\pi \left(\frac{1}{2} - \frac{\sin 2x}{2}\right)\,dx = \left(\frac{1}{2}x + \frac{\cos 2x}{4}\right)\Big|_0^\pi = \frac{\pi}{2}.$$

Assume the result holds for $k < n$. Then by the reduction formula, we have

$$\int_0^\pi \sin^{2n} x\,dx = \frac{2n-1}{2n}\int_0^\pi \sin^{2n-2} x\,dx = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{1}{2}\pi.$$

c. For $n = 1$ we have $\int_0^\pi \sin 3x\,dx = \int_0^\pi \sin x(1 - \cos^2 x)\,dx = \int_0^\pi \sin x\,dx - \int_0^\pi \sin x\cos^2 x\,dx. = -\cos x\Big|_0^\pi - \int_0^\pi \sin x\cos^2 x\,dx = 2 - \int_0^\pi \sin x\cos^2 x\,dx$. Let $u = \cos x$. Then $du = -\sin x\,dx$. Then we have

$$2 + \int_1^{-1} u^2\,du = 2 + \frac{u^3}{3}\Big|_1^{-1} = 2 - \frac{2}{3} = \frac{4}{3}.$$

If the result holds for integers less than n, then by the reduction formula we have

$$\int_0^\pi \sin^{2n+1} x\,dx = \frac{-\sin^{2n}(-\cos x)}{2n+1}\Big|_0^\pi + \frac{2n}{2n+1}\int_0^\pi \sin^{2n-1} x\,dx = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{2}{3}\cdot 2.$$

d. Because $0 \le \sin x \le 1$, if we multiply through by $\sin^m x$ we have $0 \le \sin^{m+1} x \le \sin^m x$ for $0 \le x \le \pi$. Therefore $\int_0^\pi 0\,dx \le \int_0^\pi \sin^{m+1} x\,dx \le \int_0^\pi \sin^m x\,dx$. So $0 \le \int_0^\pi \sin^{m+1} x\,dx \le \int_0^\pi \sin^m x\,dx$.

e. By part d we have

$$\int_0^\pi \sin^{2n+1} x\,dx \le \int_0^\pi \sin^{2n} x\,dx \le \int_0^\pi \sin^{2n-1} x\,dx.$$

Therefore

$$\frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{2}{3}\cdot 2 \le \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\pi \le \frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\frac{2}{3}\cdot 2$$

for any integer $n \ge 2$.

f. Let $A_n = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{2}{3}\cdot 2$ and let $B_n = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{3}{4}$ and let $C_n = \frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\frac{2}{3}\cdot 2$. The inequality in part e can be written as

$$A_n \le B_n\left(\frac{\pi}{2}\right) \le C_n.$$

From this it follows that $\frac{A_n}{B_n} \le \frac{\pi}{2}$ and that $\frac{\pi}{2} \le \frac{C_n}{B_n}$. The inequality $\frac{A_n}{B_n} \le \frac{\pi}{2}$ can be written as

$$\frac{(2n)^2}{2n+1}\cdot\frac{(2n-2)^2}{(2n-1)^2}\cdots\frac{2^2}{3^2} \le \frac{\pi}{2},$$

and the inequality $\frac{\pi}{2} \le \frac{C_n}{B_n}$ can be written as

$$\frac{\pi}{2} \le 2n\cdot\frac{(2n-2)^2}{(2n-1)^2}\cdot\frac{(2n-4)^2}{(2n-3)^2}\cdots\frac{2^2}{3^2},$$

so

$$\frac{2n}{2n+1}\cdot\frac{\pi}{2} \le \frac{(2n)^2}{2n+1}\cdot\frac{(2n-2)^2}{(2n-1)^2}\cdots\frac{2^2}{3^2}.$$

Putting these together gives the desired result.

g. Because $\lim_{n\to\infty}\frac{2n}{2n+1}\cdot\frac{\pi}{2} = \frac{\pi}{2}$, the result follows immediately from the Squeeze Theorem.

Chapter 9

Differential Equations

9.1 Basic Ideas

9.1.1

a This differential equation is order 1, so there is 1 arbitrary constant.

b. It is linear.

9.1.2 We have $y(0) = C + 10 = 5$, so $C = -5$. The solution is $y(t) = -5e^{-3t} + 10$.

9.1.3 Yes. Note that $y'''(t) = 0$ and $y'(t) = 2$.

9.1.4 Yes. $y'(t) = -18e^{-3t}$, so $y'(t) - 3y(t) = -18e^{3t} - (-18e^{-3t}) = 0$, and $y(0) = 6$.

9.1.5 The portion of the domain of $y = 2\sec t$ that contains the point $(\pi, -2)$ is the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

9.1.6 By the definition of a solution to an initial value problem, a solution to an initial value problem is differentiable on an interval and satisfies the differential equation on that interval. If the solution's graph crossed the line $t = 1$, then the domain of the solution would be an interval that includes $t = 1$, and yet $y'(t)$ is not defined at $t = 1$.

9.1.7 Yes, it is a solution. Note that $y'(t) = -5Ce^{-5t}$, so $y'(t) + 5y(t) = 0$.

9.1.8 Yes, it is a solution. $y'(t) = 3Ct^2$, so $ty'(t) - 3y(t) = 3Ct^3 - 3Ct^3 = 0$.

9.1.9 Yes, it is a solution. $y'(t) = 4C_1 \cos 4t - 4C_2 \sin 4t$, so $y''(t) = -16C_1 \sin 4t - 16C_2 \cos 4t$, so $y''(t) + 16y(t) = 0$.

9.1.10 Yes, it is a solution. $y'(x) = -C_1e^{-x} + C_2e^x$, so $y''(x) = C_1e^{-x} + C_2e^x$, so $y''(x) - y(x) = 0$.

9.1.11 $u'(t) = Ce^{1/(4t^4)}\left(-\frac{4}{4}\right)t^{-5} = -\frac{u(t)}{t^5}$. Thus $u'(t) + \frac{u(t)}{t^5} = -\frac{u(t)}{t^5} + \frac{u(t)}{t^5} = 0$.

9.1.12

$$u'(t) = C_1e^t + C_2e^t + C_2te^t$$

, and

$$u''(t) = C_1e^t + C_2e^t + C_2e^t + C_2te^t = C_1e^t + 2C_2e^t + C_2te^t.$$

Thus,

$$u''(t) - 2u'(t) + u(t) = (C_1e^t + 2C_2e^t + C_2te^t) - 2(C_1e^t + C_2e^t + C_2te^t) + C_1e^t + C_2te^t = 0.$$

9.1.13
$$g'(x) = -2C_1e^{-2x} + C_2e^{-2x} + -2C_2xe^{-2x},$$

so
$$g''(x) = 4C_1e^{-2x} - 2C_2e^{-2x} + -2C_2e^{-2x} + 4C_2xe^{-2x} = 4C_1e^{-2x} - 4C_2e^{-2x} + 4C_2xe^{-2x}.$$

Thus,
$$\begin{aligned} g''(x) + 4g'(x) + 4g(x) = 4C_1e^{-2x} - 4C_2e^{-2x} + 4C_2xe^{-2x} + 4(-2C_1e^{-2x} \\ + C_2e^{-2x} + -2C_2xe^{-2x}) + 4(C_1e^{-2x} + C_2xe^{-2x} + 2) = 8. \end{aligned}$$

9.1.14 $u'(t) = 2C_1t + 3C_2t^2$, so $u''(t) = 2C_1 + 6C_2t$. Thus,
$$t^2u''(t) - 4tu'(t) + 6u(t) = 2C_1t^2 + 6C_2t^3 - 4(2C_1t^2 + 3C_2t^3) + 6C_1t^2 + 6C_2t^3 = 0.$$

9.1.15 $u'(t) = 5C_1t^4 - 4C_2t^{-5} - 3t^2$, so $u''(t) = 20C_1t^3 + 20C_2t^{-6} - 6t$. Thus,
$$t^2u''(t) - 20u(t) = 20C_1t^5 + 20C_2t^{-4} - 6t^3 - 20\left(C_1t^5 + C_2t^{-4} - t^3\right) = 14t^3.$$

9.1.16 $z'(t) = -C_1e^{-t} + 2C_2e^{2t} - 3C_3e^{-3t} - e^t$. So
$$z''(t) = C_1e^{-t} + 4C_2e^{2t} + 9C_3e^{-3t} - e^t,$$

and
$$z'''(t) = -C_1e^{-t} + 8C_2e^{2t} - 27C_3e^{-3t} - e^t,$$

Thus
$$\begin{aligned} z'''(t) + 2z''(t) - 5z'(t) - 6z(t) &= -C_1e^{-t} + 8C_2e^{2t} - 27C_3e^{-3t} - e^t \\ &\quad + 2C_1e^{-t} + 8C_2e^{2t} + 18C_3e^{-3t} - 2e^t \\ &\quad + 5C_1e^{-t} - 10C_2e^{2t} + 15C_3e^{-3t} + 5e^t \\ &\quad - 6C_1e^{-t} - 6C_2e^{2t} - 6C_3e^{-3t} + 6e^t \\ &= 8e^t \end{aligned}$$

9.1.17 Yes, it is a solution. $y'(t) = 32e^{2t}$, so $y'(t) - 2y(t) = 32e^{2t} - (32e^{2t} - 20) = 20$. Also, $y(0) = 16 - 10 = 6$.

9.1.18 Yes, it is a solution. $y'(t) = 48t^5$, so $ty'(t) - 6y(t) = 48t^6 - 48t^6 + 18 = 18$. Also, $y(1) = 8 - 3 = 5$.

9.1.19 Yes, it is a solution. $y'(t) = 9\sin 3t$, so $y''(t) = 27\cos 3t$. Thus, $y''(t) + 9y(t) = 27\cos 3t - 27\cos 3t = 0$. Also, $y'(0) = 0$ and $y(0) = -3$.

9.1.20 Yes, it is a solution. $y'(x) = \dfrac{1}{4}(2e^{2x} + 2e^{-2x})$ and $y''(x) = \frac{1}{4}(4e^{2x} - 4e^{-2x})$. So $y''(x) - 4y(x) = 0$. Also, $y(0) = 0$ and $y'(0) = 1$.

9.1.21 $y(t) = \displaystyle\int (3 + e^{-2t})\,dt = 3t - \frac{1}{2}e^{-2t} + C.$

9.1.22 $y(t) = \displaystyle\int (12t^5 - 20t^4 + 2 - 6t^{-2})\,dt = 2t^6 - 4t^5 + 2t + \frac{6}{t} + C.$

9.1.23 $y(x) = \displaystyle\int (4\tan 2x - 3\cos x)\,dx = -2\ln|\cos 2x| - 3\sin x + C = 2\ln|\sec 2x| - 3\sin x + C.$

9.1.24 $p(x) = \displaystyle\int (16x^{-9} - 5 + 14x^6)\,dx = -2x^{-8} - 5x + 2x^7 + C.$

9.1.25 $y'(t) = \int (60t^4 - 4 + 12t^{-3})\,dt = 12t^5 - 4t - 6t^{-2} + C$. $y(t) = \int (12t^5 - 4t - 6t^{-2} + C)\,dt = 2t^6 - 2t^2 + 6t^{-1} + C_1t + C_2$.

9.1.26 $y'(t) = \int (15e^{3t} + \sin 4t)\,dt = 5e^{3t} - \frac{1}{4}\cos 4t + C_1$. $y(t) = \int (5e^{3t} - \frac{1}{4}\cos 4t + C_1)\,dt = \frac{5}{3}e^{3t} - \frac{1}{16}\sin 4t + C_1t + C_2$.

9.1.27 $u'(x) = \int (55x^9 + 36x^7 - 21x^5 + 10x^{-3})\,dx = 5.5x^{10} + \frac{9}{2}x^8 - \frac{7}{2}x^6 - 5x^{-2} + C_1$.
$u(x) = \int (5.5x^{10} + \frac{9}{2}x^8 - \frac{7}{2}x^6 - 5x^{-2} + C)\,dx = \frac{1}{2}x^{11} + \frac{1}{2}x^9 - \frac{1}{2}x^7 + 5x^{-1} + C_1x + C_2$.

9.1.28 $v'(x) = \int \frac{5}{x^2}\,dx = -\frac{5}{x} + C_1$, so $v(x) = \int \left(-\frac{5}{x} + C_1\right)\,dx = -5\ln|x| + C_1x + C_2$.

9.1.29 $u(x) = \int \frac{2x}{x^2+4}\,dx - \int \frac{2}{x^2+4}\,dx = \ln(x^2+4) - \tan^{-1}(x/2) + C$.

9.1.30 $y(t) = \int (t\ln t + 1)\,dt = t + \int t\ln t\,dt$. Let $u = \ln t$ and $dv = t$, so that $du = \frac{dt}{t}$ and $v = t^2/2$. Then $y(t) = t + (t^2\ln t)/2 - \int t/2\,dt = t + (t^2\ln t)/2 - t^2/4 + C$.

9.1.31 $y'(x) = \int \frac{x}{(1-x^2)^{3/2}}\,dx$. Let $u = 1 - x^2$, so that $du = -2x\,dx$. Substituting gives
$y'(x) = -\frac{1}{2}\int u^{-3/2}\,du = u^{-1/2} + C_1 = \frac{1}{\sqrt{1-x^2}} + C_1\,dx$. $y(x) = \int \left(\frac{1}{\sqrt{1-x^2}} + C_1\right)\,dx = \sin^{-1}(x) + C_1x + C_2$.

9.1.32 Note that $\frac{4}{t^2-4} = \frac{1}{t-2} - \frac{1}{t+2}$. Thus, $v(t) = \int \frac{4}{t^2-4}\,dt = \int \left(\frac{1}{t-2} - \frac{1}{t+2}\right)\,dt = \ln\left|\frac{t-2}{t+2}\right| + C$.

9.1.33 $y(t) = \int (1 + e^t)\,dt = t + e^t + C$. Because $y(0) = 4 = 1 + C$, we have $C = 3$. Thus, $y(t) = t + e^t + 3$.

9.1.34 $y(t) = \int (\sin t + \cos 2t)\,dt = -\cos t + \frac{1}{2}\sin 2t + C$. Because $y(0) = 4 = -1 + C$, we have $C = 5$. Thus, $y(t) = -\cos t + \frac{1}{2}\sin 2t + 5$.

9.1.35 $y(x) = \int (3x^2 - 3x^{-4})\,dx = x^3 + x^{-3} + C$. Because $y(1) = 0 = 1 + 1 + C$, we have $C = -2$. So $y(x) = x^3 + x^{-3} - 2$, $x > 0$.

9.1.36 $y(x) = \int 4\sec^2 2x\,dx = 2\tan 2x + C$. Because $y(0) = 8 = 0 + C$, we have $C = 8$. Thus, $y(x) = 2\tan 2x + 8$, $-\frac{\pi}{4} < x < \frac{\pi}{4}$.

9.1.37 $y'(t) = \int (12t - 20t^3)\,dt = 6t^2 - 5t^4 + C_1$. Because $y'(0) = 0 = 0 + C_1$, we have $C_1 = 0$. $y(t) = \int (6t^2 - 5t^4)\,dt = 2t^3 - t^5 + C_2$. Because $y(0) = 1 = 0 - 0 + C_2$, we have $C_2 = 1$. Thus, $y(t) = 2t^3 - t^5 + 1$.

9.1.38 $u'(x) = \int (4e^{2x} - 8e^{-2x})\,dx = 2e^{2x} + 4e^{-2x} + C_1$. Because $u'(0) = 3 = 2 + 4 + C_1$, we have $C_1 = -3$. $u(x) = \int (2e^{2x} + 4e^{-2x} - 3)\,dx = e^{2x} - 2e^{-2x} - 3x + C_2$. Because $u(0) = 1 = 1 - 2 - 0 + C_2$, we have $C_2 = 2$. Thus, $u(x) = e^{2x} - 2e^{-2x} - 3x + 2$.

9.1.39 Using the result of number 40 below, we have $y'(t) = te^t - e^t + C_1$, and because $y'(0) = 1 = 0 - 1 + C_1$, we have $C_1 = 2$. Thus $y'(t) = te^t - e^t + 2$. $y(t) = \int y'(t)\,dt = \int (te^t - e^t + 2)\,dt = te^t - e^t - e^t + 2t + C_2 = te^t - 2e^t + 2t + C_2$. Because $y(0) = 0 = 0 - 2 + 0 + C_2$, we have $C_2 = 2$. Thus, $y(t) = te^t - 2e^t + 2t + 2$.

9.1.40 Let $u = t$ and $dv = e^t\,dt$. Then $du = dt$ and $v = e^t$. Thus, $y(t) = \int te^t\,dt = te^t - \int e^t\,dt = te^t - e^t + C$. Because $y(0) = -1 = 0 - 1 + C$, we have $C = 0$. Thus $y(t) = te^t - e^t$.

9.1.41 $u(x) = \int \left(\frac{1}{x^2 + 4^2} - 4\right) dx = \frac{1}{4}\tan^{-1}(x/4) - 4x + C$. Because $u(0) = 2 = 0 - 0 + C$, we have $C = 2$. Thus, $u(x) = \frac{1}{4}\tan^{-1}(x/4) - 4x + 2$.

9.1.42 $p(x) = \int \frac{2}{x(x+1)}\,dx = \int \left(\frac{2}{x} - \frac{2}{x+1}\right) dx = 2\ln\left|\frac{x}{x+1}\right| + C$. Because $p(1) = 0 = 2\ln(1/2) + C$, we have $C = -2\ln(1/2) = 2\ln 2$. Thus, $p(x) = 2\ln\left|\frac{x}{x+1}\right| + 2\ln 2$, $x > 0$.

9.1.43

a. $v(t) = -9.8t + 29.4$. $s(t) = -4.9t^2 + 29.4t + 30$. The domain of each function is approximately $0 \le t \le 6.89$.

b. The object reaches its high point when $-9.8t + 29.4 = 0$, or $t = \frac{29.4}{9.8} = 3$. At that time its position is $s(3) \approx 74.1$ meters.

9.1.44

a. $v(t) = -9.8t + 49$. $s(t) = -4.9t^2 + 49t + 60$. The domain of each function is approximately $0 \le t \le 11.1$.

b. The object reaches its high point when $-9.8t + 49 = 0$, or $t = \frac{49}{9.8} = 5$. At that time its position is $s(5) = 182.5$ meters.

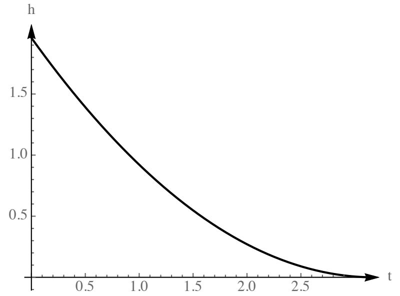
9.1.45 We have $p(t) = (1500 - 20H)e^{.05t} + 20H$. The amount of resource is increasing when $1500 - 20H > 0$, which occurs for $H < 75$. The amount of resource is constant when $1500 - 20H = 0$, which occurs for $H = 75$. If $H = 100$, the resource is zero when $(1500 - 2000)e^{.05t} + 2000 = 0$, which occurs for $t = 20\ln 4 \approx 28$.

9.1.46 We have $p(t) = (p_0 - 10000)e^{.05t} + 10000$. The amount of resource is decreasing when $p_0 - 10000 < 0$, or $p_0 < 10,000$. The amount of resource is constant when $p_0 = 10,000$. If $p_0 = 9000$, the resource vanishes when $-1000e^{.05t} = -10000$, or $t = 20\ln 10 \approx 46$.

9.1.47 The height function is given by

$$h(t) = \left(\sqrt{1.96} - \frac{0.3\sqrt{2 \cdot 9.8}}{1.5} \cdot \frac{t}{2}\right)^2 \approx (1.4 - 0.44t)^2 \text{ for } 0 \le t \le 3.16.$$

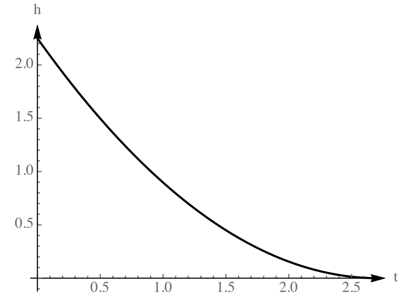
The tank is empty when $h(t) = 0$, which occurs after about 3.16 seconds.



9.1.48 The height function is given by

$$h(t) = \left(\sqrt{2.25} - \frac{0.5\sqrt{2 \cdot 9.8}}{2} \cdot \frac{t}{2}\right)^2 \approx (1.5 - 0.553t)^2 \text{ for } 0 \le t \le 2.71.$$

The tank is empty when $h(t) = 0$, which occurs after about 2.71 seconds.



9.1.49

a. False. That is a specific solution. The general solution is $t + C$.

b. False. It is second order, but is not linear.

c. True. First find the general solution, and then find the specific solution which satisfies the initial condition.

9.1.50

a. $y'(t) = C_1e^t - C_2e^{-t}$, so $y''(t) = C_1e^t + C_2e^{-t}$. Thus, $y''(t) - y(t) = 0$.

b. $y'(t) = 2C_1e^{2t} - 2C_2e^{-2t}$, so $y''(t) = 4C_2e^{2t} + 4C_2e^{-2t}$. Thus, $y''(t) - 4y(t) = 0$.

c. It appears that a general solution should be $C_1e^{kt} + C_2e^{-kt}$. Then $y'(t) = kC_1e^{kt} - kC_2e^{-kt}$, and $y''(t) = k^2C_1e^{kt} + k^2C_2e^{-kt}$. Thus, $y''(t) - k^2y(t) = 0$.

d. If $y(t) = C_1 \cosh kt + C_2 \sinh kt$, then $y'(t) = kC_1 \sinh kt + kC_2 \cosh kt$ and $y''(t) = k^2C_1 \cosh kt + k^2C_2 \sinh kt$. Thus $y''(t) - k^2y(t) = 0$.

9.1.51

a. $y'(t) = C_1 \cos t - C_2 \sin t$, so $y''(t) = -C_1 \sin t - C_2 \cos t$. Thus, $y''(t) + y(t) = 0$.

b. $y'(t) = 2C_2 \cos 2t - 2C_2 \sin 2t$, so $y''(t) = -4C_2 \sin 2t - 4C_2 \cos 2t$. Thus, $y''(t) + 4y(t) = 0$.

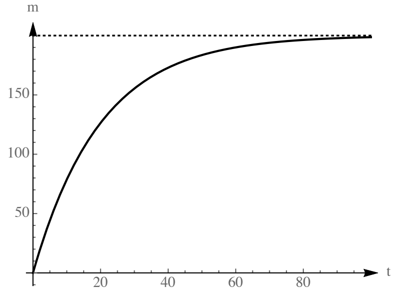
c. A general solution appears to be $y(t) = C_1 \sin kt + C_2 \cos kt$. Then $y'(t) = kC_1 \cos kt - kC_2 \sin kt$, so $y''(t) = -k^2C_1 \sin kt - k^2C_2 \cos kt$. And then $y''(t) + k^2y(t) = 0$.

9.1.52

a. Let $m(t) = \frac{I}{k}(1 - e^{-kt})$. Note that $m(0) = 0$. Then $m'(t) = \frac{I}{k}(ke^{-kt})$. Therefore,

$$m'(t) + km(t) = \frac{I}{k}(ke^{-kt}) + \frac{kI}{k}(1 - e^{-kt}) = Ie^{-kt} + I - Ie^{-kt} = I.$$

b. We have $m(t) = 200(1 - e^{-0.05t})$.



c. It appears that $\lim_{t\to\infty} m(t) = 200$.

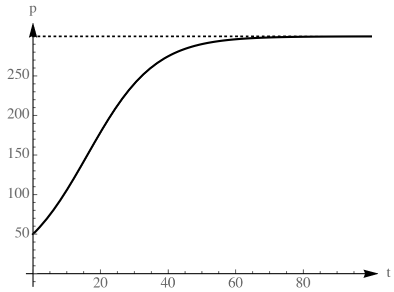
9.1.53

a. Let $p(t) = \dfrac{K}{1+Ce^{-rt}}$. Note that $1 - \dfrac{P}{K} = 1 - \dfrac{1}{1+Ce^{-rt}} = \dfrac{Ce^{-rt}}{1+Ce^{-rt}}$. We have

$$p'(t) = \frac{KCre^{-rt}}{(1+Ce^{-rt})^2} = r \cdot \frac{K}{1+Ce^{-rt}} \cdot \frac{Ce^{-rt}}{1+Ce^{-rt}} = rp\left(1 - \frac{p}{K}\right).$$

b. If $p(0) = 50 = \dfrac{K}{1+C}$, then $50 + 50C = K$, so $C = \dfrac{K-50}{50}$.

c. We have $p(t) = \dfrac{300}{1+5e^{-0.1t}}$.



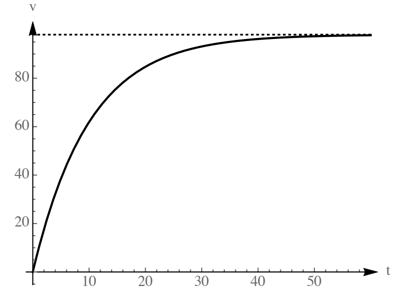
d. $\lim_{t\to\infty} \dfrac{300}{1+5e^{-0.1t}} = \dfrac{300}{1+0} = 300$, which is consistent with the graph from part c.

9.1.54

a. Let $v(t) = \dfrac{g}{b}(1 - e^{-bt})$. Then $v(0) = 0$, and

$$v'(t) = \frac{g}{b} \cdot be^{-bt} = ge^{-bt} = g - b \cdot \frac{g}{b}(1 - e^{-bt}) = g - bv.$$

b. With $b = 0.1$, we have $v(t) = 98(1 - e^{-0.1t})$.



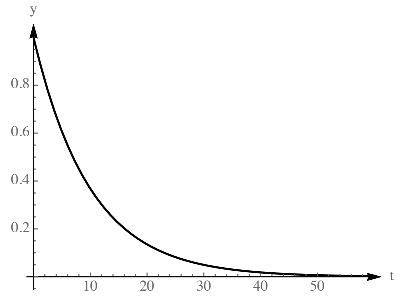
c. $\lim_{t\to\infty} v(t) = 98$.

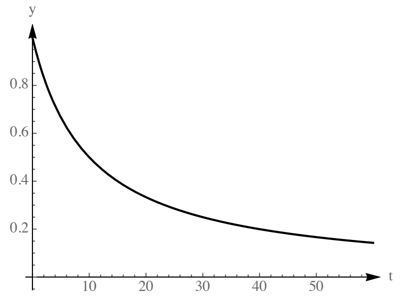
9.1.55

a. If $y(t) = y_0e^{-kt}$, then $y(0) = y_0$, and $y'(t) = -ky_0e^{-kt}$, so $y'(t) = -ky(t)$.

b. Let $y(t) = \dfrac{y_0}{y_0kt+1}$. Then $y(0) = y_0$, and $y'(t) = \dfrac{-y_0^2k}{(y_0kt+1)^2} = -k(y(t))^2$.

c. The first order reaction decays more quickly.



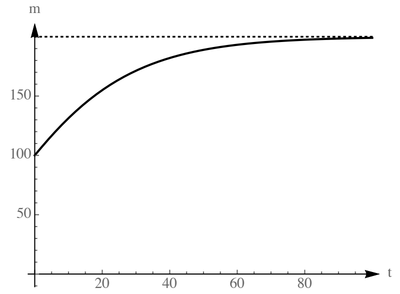


9.1.56

a. Let $M(t) = K\left(\dfrac{M_0}{K}\right)^{e^{-rt}}$. Note that $\ln(M(t)/K) = e^{-rt}\ln(M_0/K)$.

$$M'(t) = K\left(\frac{M_0}{K}\right)^{e^{-rt}} \ln(M_0/K)\left(-re^{-rt}\right) = -rM(t)\ln(M(t)/K). \text{ Also, } M(0) = K(M_0/K) = M_0.$$

b. Using $K = 200$, $M_0 = 100$, and $r = 0.05$, we have $M(t) = K\left(\dfrac{M_0}{K}\right)^{e^{-rt}} = 200(1/2)^{e^{-0.05t}}$.



c. $\lim_{t\to\infty} M(t) = 200 = K$.

9.2 Direction Fields and Euler's Method

9.2.1 Choose a regular grid of points in the ty-plane, and for each point P, make a small line segment with slope $f(t,y)$.

9.2.2 It will have slope $3^2 - 3(1)^2 = 6$.

9.2.3 $u_0 = y(3) = 1$. $u_1 = u_0 + f(3,1)(.1) = 1 + .6 = 1.6$.

9.2.4 Because the differential equation is giving the slope at a given point, we can approximate the solution to the differential equation by starting at the initial point, and using the slope to guide where the next iteration should be. In essence, we are numerically "following the direction field" to estimate the solution to the differential equation.

9.2.5

a. This matches with D.

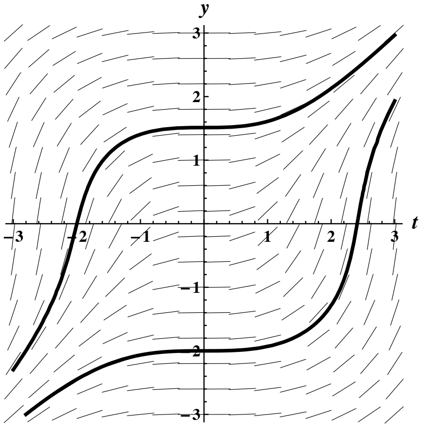
b. This matches with B.

c. This matches with A.

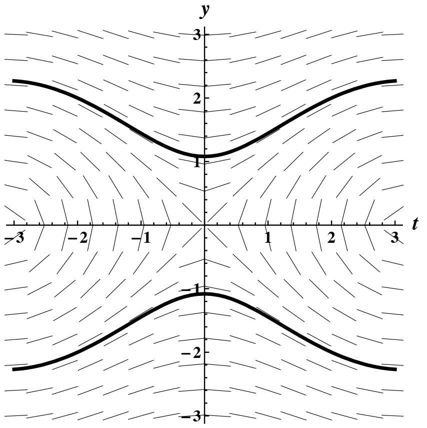
d. This matches with C.

9.2.6 Note that the slopes are zero when $y = -1$ and when $t = 1$. Also, they are positive when both $y > -1$ and $t > 1$. The only differential equation with this property is choice a.

9.2.7

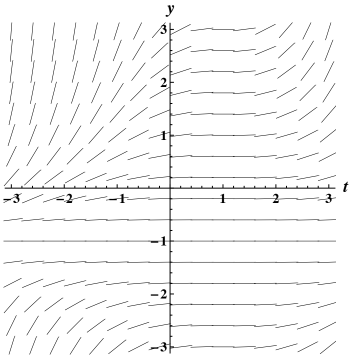


9.2.8



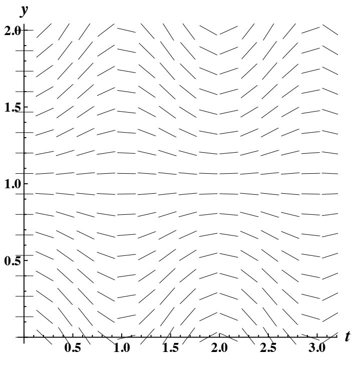
9.2.9

An initial condition of $y(0) = -1$ leads to a constant solution. For any other initial condition, the solutions are increasing over time.



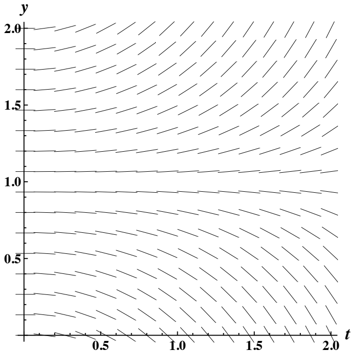
9.2.10

An initial condition of $y(0) = 1$ leads to a constant solution. For any other initial condition, the solutions oscillate between increasing and decreasing over intervals of time length one.

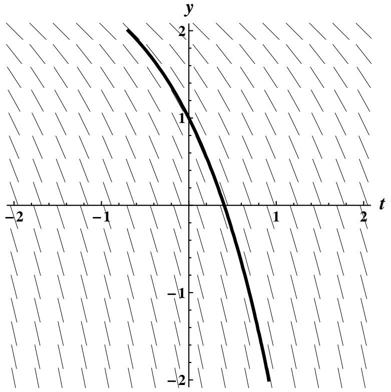


9.2.11

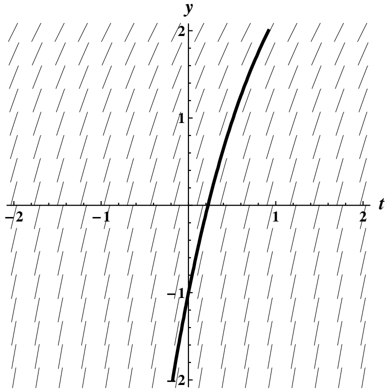
An initial condition of $y(0) = 1$ leads to a constant solution. Initial conditions $y(0) = A$ lead to solutions that are increasing over time if $A > 1$ and solutions that are decreasing over time if $A < 1$.



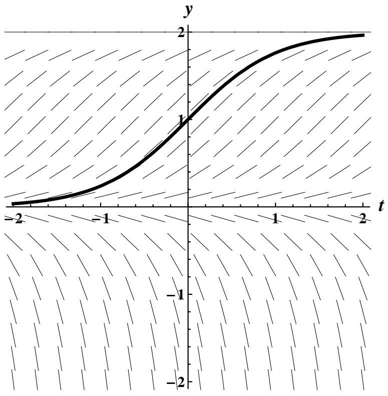
9.2.12



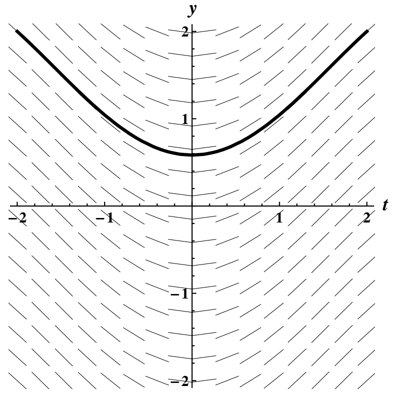
9.2.13



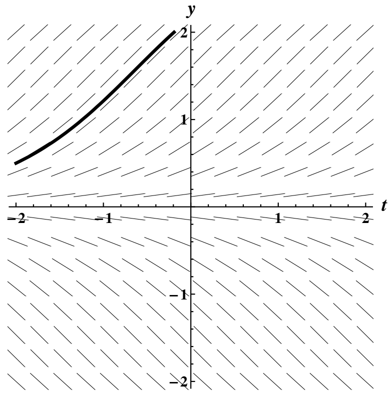
9.2.14



9.2.15



9.2.16



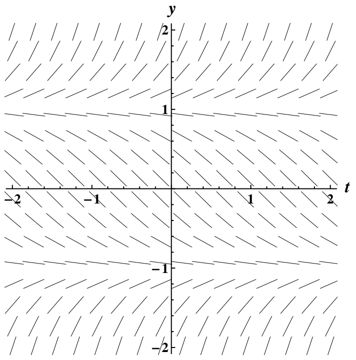
9.2.17

a. The solutions $y = 1$ and $y = -1$ are constant.

b. Solutions are increasing when both $y > 1$ and $y > -1$, (so for $y > 1$) and when both $y < 1$ and $y < -1$ (so for $y < -1$). In other words, for $|y| > 1$. Solutions are decreasing for $|y| < 1$.

c. Initial condition $y(0) = A$ leads to solutions that are increasing over time if $|A| > 1$ and decreasing over time if $|A| < 1$.

d.



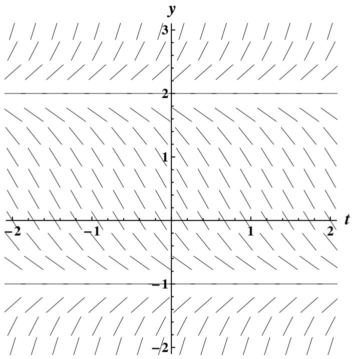
9.2.18

a. The solutions $y = 2$ and $y = -1$ are constant.

b. Solutions are increasing when both $y > 2$ and $y > -1$, (so for $y > 2$) or when both $y < 2$ and $y < -1$ (so for $y < -1$). Solutions are decreasing for $-1 < y < 2$.

c. Initial condition $y(0) = A$ leads to solutions that are increasing over time if $A > 2$ or $A < -1$, and decreasing over time if $-1 < A < 2$.

d.



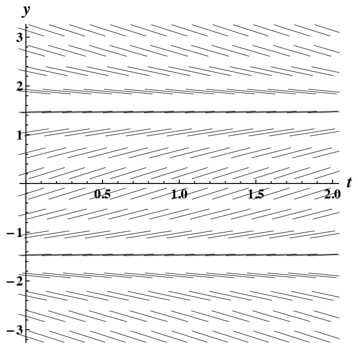
9.2.19

a. The solutions $y = \pi/2$ and $y = -\pi/2$ are constant.

b. Solutions are increasing when $|y| < \pi/2$ and decreasing when $\pi/2 < |y| < \pi$.

c. Initial condition $y(0) = a$ leads to solutions that are increasing over time if $|A| < \pi/2$ and decreasing over time if $|A| > \pi/2$.

d.



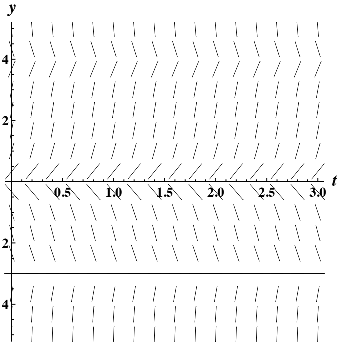
9.2.20

a. The solutions $y = 0$, $y = -3$, and $y = 4$ are constant.

b. Solutions are increasing when $y < -3$ and when $0 < y < 4$. Solutions are decreasing when $-3 < y < 0$ and when $y > 4$.

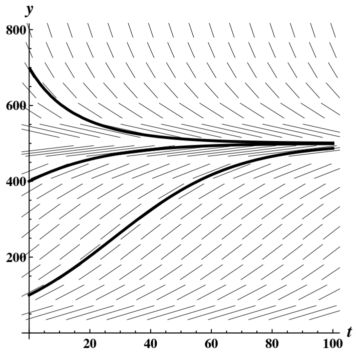
c. Initial condition $y(0) = A$ leads to solutions that are increasing over time if $A < -3$ or $0 < A < 4$, and decreasing over time if $-3 < A < 0$ or $A > 4$.

d.



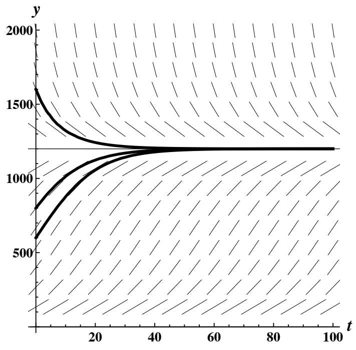
9.2.21

Equilibrium solutions are $P(t) = 0$ and $P(t) = 500$.



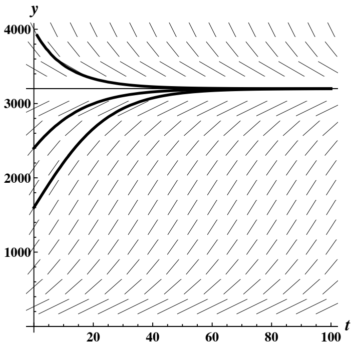
9.2.22

Equilibrium solutions are $P(t) = 0$ and $P(t) = 1200$.



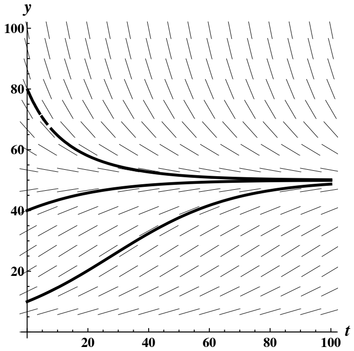
9.2.23

Equilibrium solutions are $P(t) = 0$ and $P(t) = 3200$.



9.2.24

Equilibrium solutions are $P(t) = 0$ and $P(t) = 50$.



9.2.25 $u_0 = 2$. $u_1 = 2 + f(0,2)(0.5) = 2 + (4)(0.5) = 4$. $u_2 = 4 + f(0.5,4)(0.5) = 4 + (8)(0.5) = 8$. So $y(0.5) \approx 4$ and $y(1) \approx 8$.

9.2.26 $u_0 = -1$. $u_1 = -1 + f(0,-1)(0.2) = -1 + 1(0.2) = -0.8$. $u_2 = -0.8 + f(0.2,-0.8)(0.2) = -0.8 + (0.8)(0.2) = -0.64$. So $y(0.2) \approx -0.8$ and $y(0.4) \approx -0.64$.

9.2.27 $u_0 = 1$. $u_1 = 1 + f(0,1)(0.1) = 1 + 0.1 = 1.1$. $u_2 = 1.1 + f(0.1,1.1)(0.1) = 1.1 + (0.9)(0.1) = 1.19$. So $y(0.1) \approx 1.1$ and $y(0.2) \approx 1.19$.

9.2.28 $u_0 = 4$. $u_1 = 4 + f(0,4)(0.5) = 4 + (4)(0.5) = 6$. $u_2 = 6 + f(0.5,6)(0.5) = 6 + (6.5)(0.5) = 9.25$. So $y(0.5) \approx 6$ and $y(1) \approx 9.25$.

9.2.29

a.

Δt	approximation of $y(0.2)$	approximation of $y(0.4)$
0.2	0.8	0.64
0.1	0.81	0.65610
0.05	0.81451	0.66342
0.025	0.81665	0.66692

b. $e^{-0.2} \approx 0.818731$ and $e^{-0.4} \approx 0.67032$.

Δt	error in approximation of $y(0.2)$	error in approximation of $y(0.4)$
0.2	0.018371	0.03032
0.1	0.00873	0.01422
0.05	0.00422	0.00690
0.025	0.00208	0.00340

c. The time step $\Delta t = 0.025$ has the smallest errors. A smaller time step generally produces more accurate results.

d. Halving the time steps results in approximately halving the error.

9.2.30

a.

Δt	approximation of $y(0.2)$	approximation of $y(0.4)$
0.2	2.2	2.42
0.1	2.205	2.43101
0.05	2.20763	2.43681
0.025	2.20897	2.43987

b. $2e^{0.1} \approx 2.21034$ and $2e^{0.2} \approx 2.44281$.

Δt	error in approximation of $y(0.2)$	error in approximation of $y(0.4)$
0.2	0.0103418	0.0228055
0.1	0.00534184	0.011793
0.05	0.00271605	0.00599972
0.025	0.00136963	0.00302642

c. The time step $\Delta t = 0.025$ has the smallest errors. A smaller time step generally produces more accurate results.

d. Halving the time steps results in approximately halving the error.

9.2.31

a.

Δt	approximation of $y(0.2)$	approximation of $y(0.4)$
0.2	3.2	3.36
0.1	3.19	3.3439
0.05	3.18549	3.33658
0.025	3.18335	3.33308

b. $4 - e^{-0.2} \approx 3.18127$ and $4 - e^{-0.4} \approx 3.32968$.

Δt	error in approximation of $y(0.2)$	error in approximation of $y(0.4)$
0.2	0.0187308	0.03032
0.1	0.00873075	0.01422
0.05	0.0042245	0.00689961
0.025	0.0020789	0.00339988

c. The time step $\Delta t = 0.025$ has the smallest errors. A smaller time step generally produces more accurate results.

d. Halving the time steps results in approximately halving the error.

9.2.32

a.

Δt	approximation of $y(0.2)$	approximation of $y(0.4)$
0.2	0.2	0.48
0.1	0.22	0.52
0.05	0.23	0.54
0.025	0.235	0.55

b. $0.2^2 + 0.2 = 0.24$ and $0.4^2 + 0.4 = 0.56$.

Δt	error in approximation of $y(0.2)$	error in approximation of $y(0.4)$
0.2	0.04	0.08
0.1	0.02	0.04
0.05	0.01	0.02
0.025	0.005	0.01

c. The time step $\Delta t = 0.025$ has the smallest errors. A smaller time step generally produces more accurate results.

d. Halving the time steps results in approximately halving the error.

9.2.33

a. The computations yield:

t_k	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
u_k	1	0.6	0.36	0.216	0.1296	0.07776	0.046656	0.0279936	0.0167962	0.0100777	0.00604662

So $y(2) \approx 0.00604662$.

b. $y(2) = e^{-4} \approx 0.0183156$, so the error is about $0.0183156 - 0.00604662 = 0.012269$.

c. The computations yield:

t_k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
u_k	1	0.8	0.64	0.512	0.4096	0.32768	0.262144	0.209715	0.167772	0.134218	0.107374

t_k	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
u_k	0.0859	0.0687	0.0550	0.0440	0.0352	0.02815	0.02252	0.01801	0.01441	0.01153

So $y(2) \approx 0.01153$ The error is about $0.0183156 - 0.01153 = 0.0067856$.

d. The error with twice as many steps is about half the other error.

9.2.34

a. The computations yield:

t_k	0	0.2	0.4	0.6	0.8	1	1.2	1.4
u_k	-1	0.6	1.56	2.136	2.4816	2.68896	2.81338	2.88803

t_k	1.6	1.8	2	2.2	2.4	2.6	2.8	3
u_k	2.93282	2.95969	2.97581	2.98549	2.99129	2.99478	2.99687	2.99812

So $y(3) \approx 2.99812$.

b. $y(3) = (3 - 4e^{-6}) \approx 2.99008$, so the error is about $2.99812 - 2.99008 = 0.00804$.

c. The computations yield:

t_k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
u_k	-1	-0.2	0.44	0.952	1.3616	1.68928	1.95142	2.16114	2.32891	2.46313	2.5705

t_k	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
u_k	2.6564	2.72512	2.7801	2.82408	2.85926	2.88741	2.90993	2.92794	2.94235	2.95388

t_k	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3
u_k	2.96311	2.97049	2.97639	2.98111	2.98489	2.98791	2.99033	2.99226	2.99381	2.99505

So $y(3) \approx 2.99505$ The error is about $2.99812 - 2.99505 = 0.00307$.

d. The error with twice as many steps is less than half the other error.

9.2.35

a. After many calculations, we arrive at $y(4) \approx 3.05765$.

b. $y(4) = 3 + 5e^{-4} \approx 3.09158$, so the error is about $3.09158 - 3.05765 = .03393$.

c. After many calculations, we arrive at $y(4) = 3.0739$. The error is about $3.09158 - 3.0739 = 0.01768$.

d. The error with twice as many steps is about half the other error.

9.2.36

a. After many calculations, we arrive at $y(2) \approx 4.45125$.

b. $y(4) = \sqrt{20} \approx 4.47214$, so the error is about $4.47214 - 4.45125 = .02089$.

c. After many calculations, we arrive at $y(2) = 4.46173$. The error is about $4.47214 - 4.46173 = 0.01041$.

d. The error with twice as many steps is about half the other error.

9.2.37

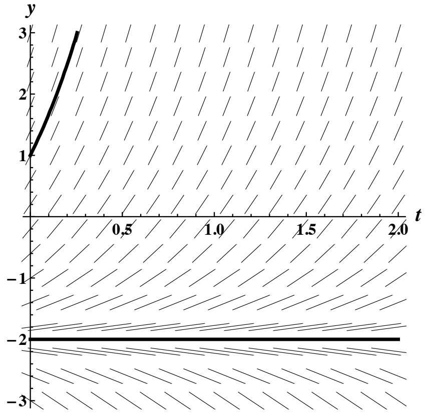
a. True.

b. False. It allows you to compute approximations.

9.2.38

a. $y = -2$ is an equilibrium solution, because $2(-2) + 4 = 0$.

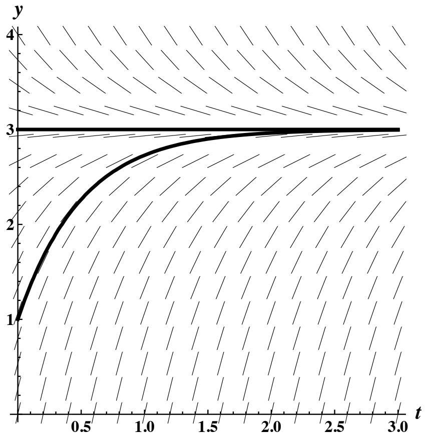
b and c.



9.2.39

a. $y = 3$ is an equilibrium solution, because $6 - 2(3) = 0$.

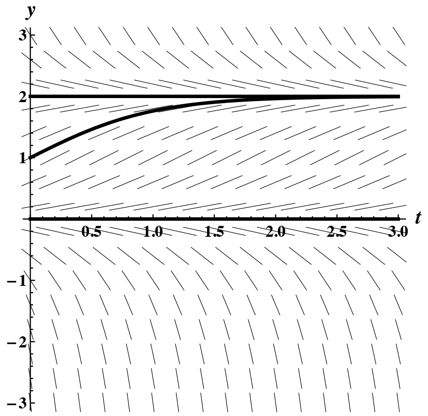
b and c.



9.2.40

a. Solve $y(2-y) = 0$ to get equilibrium solutions $y = 0$ and $y = 2$.

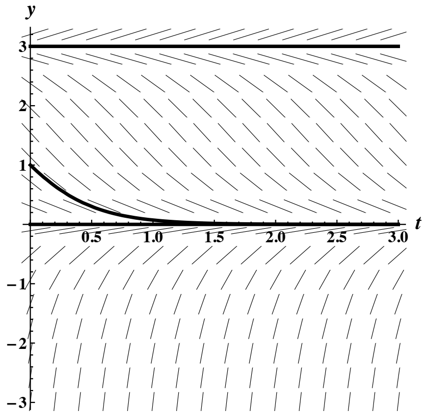
b and c.



9.2.41

a. Solve $y(y-3) = 0$ to get equilibrium solutions $y = 0$ and $y = 3$.

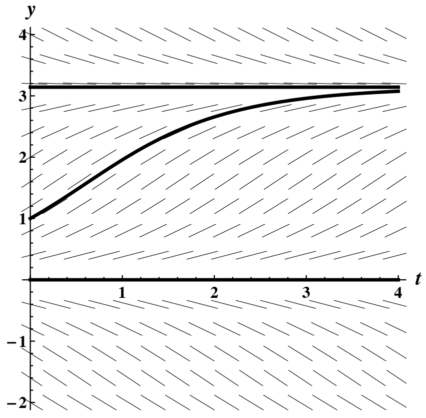
b and c.



9.2.42

a. Solve $\sin y = 0$ to get equilibrium solutions $y = k\pi$, where k is any integer.

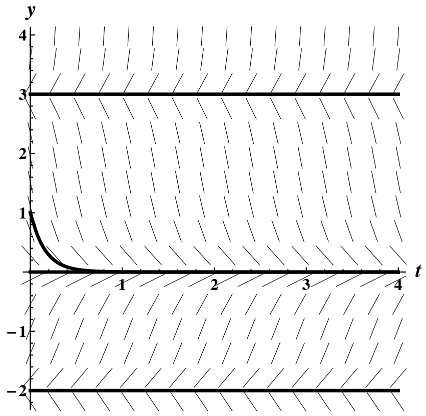
b and c.



9.2.43

a. The equilibrium solutions are $y = 0$, $y = -2$ and $y = 3$.

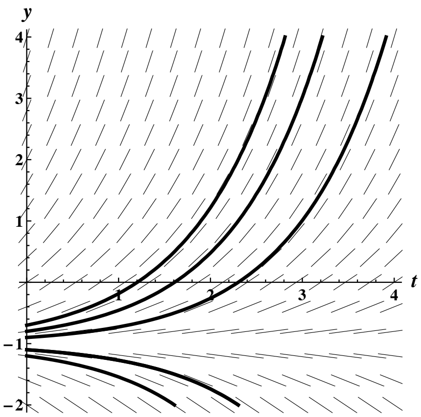
b and c.



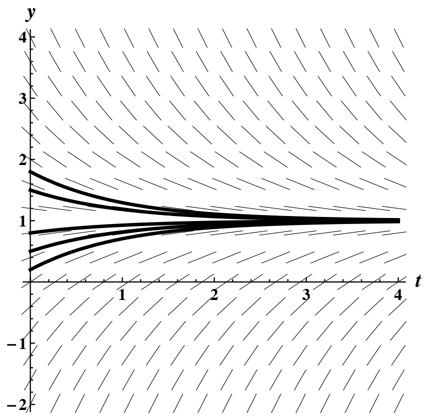
9.2.44

a. Solving $y' = 0$ gives the equilibrium solution $y = -b/a$, which is a horizontal line.

b.



c.



Note that the general solutions $y = \left(A + \dfrac{b}{a}\right)e^{at} - \dfrac{b}{a}$ increases without bound if $a > 0$ and $A > -\dfrac{b}{a}$, and decreases without bound if $a > 0$ but $A < -\dfrac{b}{a}$. But if $a < 0$, the general solutions have limit $-\dfrac{b}{a}$, but must increase to it if $-A < \dfrac{b}{a}$ and decrease to it if $A > -\dfrac{b}{a}$.

9.2.45

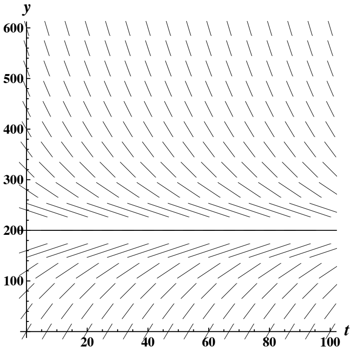
a. $\Delta t = \dfrac{b-a}{N}$.

b. Recall that $u_0 = A$ and $t_0 = a$. So $u_1 = A + f(a, A)\left(\dfrac{b-a}{N}\right)$.

c. $u_{k+1} = u_k + f(t_k, u_k)\left(\dfrac{b-a}{N}\right)$, where $t_k = a + k\left(\dfrac{b-a}{N}\right)$ for $k = 0, 1, 2, \ldots, N-1$.

9.2.46

a.

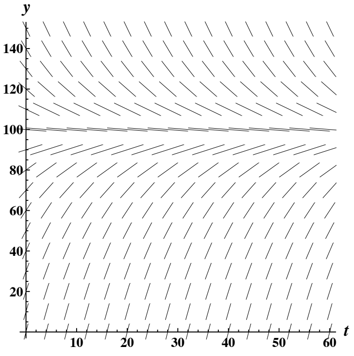


b. The equilibrium solution is $m(t) = 200$.

c. The solutions are increasing for $A < 200$ and decreasing for $A > 200$.

9.2.47

a.

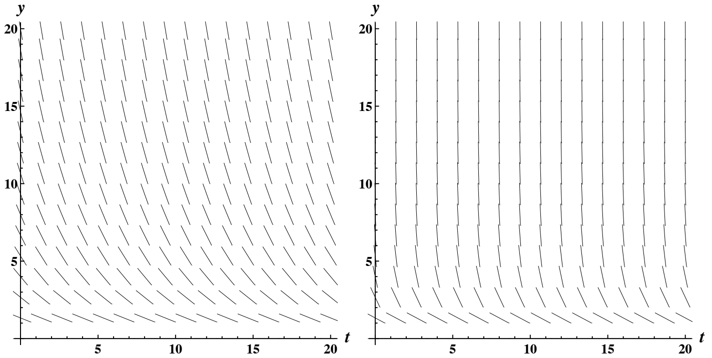


b. The solutions are increasing for $A < 98$ and decreasing for $A > 98$.

c. The equilibrium solution is $v(t) = 98$.

9.2.48

a. In both cases, the equilibrium is 0.



b. The second order approaches more quickly.

9.2.49

a. We have $u_0 = y(0) = 1$, and $u_{k+1} = u_k + f(t_k, u_k)h = u_k + au_kh = u_k(1 + ah)$ for $k = 0, 1, 2 \ldots$.

b. Suppose $u_k = (1 + ah)^k$. Then $u_{k+1} = u_k(1 + ah) = (1 + ah)^k(1 + ah) = (1 + ah)^{k+1}$, $k = 0, 1, 2, \ldots$.

c. $\lim\limits_{h\to 0} u_k = \lim\limits_{h\to 0}(1 + ah)^k = \lim\limits_{h\to 0}(1 + ah)^{t_k/h} = \lim\limits_{h\to 0}\left((1 + ah)^{1/h}\right)^{t_k} = (e^a)^{t_k} = e^{at_k} = y(t_k)$.

9.2.50

a. We have $u_0 = y(0) = 1$, and $u_{k+1} = u_k + f(t_k, u_k)h = u_k - au_kh = u_k(1 - ah)$ for $k = 0, 1, 2 \ldots$.

b. Suppose $u_k = (1 - ah)^k$. Then $u_{k+1} = u_k(1 - ah) = (1 - ah)^k(1 - ah) = (1 - ah)^{k+1}$, $k = 0, 1, 2, \ldots$.

c. The function r^z increases as z increases exactly when $|r| > 1$ and decreases exactly when $|r| < 1$. So $u_k = (1-ah)^k$ will increase when k increases exactly when $|1-ah| > 1$ and will decrease as k increases exactly when $|1 - ah| < 1$.

d. Suppose $|1 - ah| < 1$. Then $-1 < 1 - ah < 1$. Subtracting one from everything gives $-2 < -ah < 0$. Dividing through by $-a$ (which is a negative number) gives $0 < h < 2/a$, as desired.

9.3 Separable Differential Equations

9.3.1 A separable first-order differential equation is one that can be written in the form $g(y)y'(t) = h(t)$, where the factor $g(y)$ is a function of y and $h(t)$ is a function of t.

9.3.2 Yes, this equation is separable because it can be written in the form $y^2y'(t) = t^{-2}(t + 4)$.

9.3.3 No, this equation cannot be written in the required form.

9.3.4 Integrate both sides with respect to t and convert the integral on the left side to an integral with respect to y.

9.3.5 We have $y'(t) = t^3$, so $\displaystyle\int \frac{dy}{dt}\,dt = \int t^3\,dt$, so $y = \dfrac{t^4}{4} + C$.

9.3.6 $y'(t) = 5e^{-4t}$, so $\displaystyle\int y'(t)\,dt = \int 5e^{-4t}\,dt$, so $y = -\dfrac{5}{4}e^{-4t} + C$.

9.3.7 $y\dfrac{dy}{dt} = 3t^2$, so $\displaystyle\int y\,dy = \int 3t^2\,dt$. Thus, $\dfrac{y^2}{2} = t^3 + C$, and thus $y = \pm\sqrt{2t^3 + C}$.

9.3.8 We have $\dfrac{1}{y}\dfrac{dy}{dx} = x^2 + 1$, so $\displaystyle\int \frac{1}{y}\frac{dy}{dx}\,dx = \int (x^2 + 1)\,dx$. Thus, $\ln y = \dfrac{x^3}{3} + x + C$, and $y = Ae^{(x^3/3)+x}$.

9.3.9 We have $\int e^{-y/2}\,dy = \int \sin t\,dt$, and so $-2e^{-y/2} = -\cos t + C$. Thus, $y = -2\ln\left(\frac{1}{2}\cos t + C\right)$.

9.3.10 We have $\int w^{-1/2}\,dw = \int \frac{3x+1}{x^2}\,dx = \int \left(\frac{3}{x} + \frac{1}{x^2}\right)dx$, so $2w^{1/2} = 3\ln|x| - \frac{1}{x} + C$, and thus $w = \left(\frac{3}{2}\ln|x| - \frac{1}{2x} + C\right)^2$.

9.3.11 $\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{x^2}$, so $\int \frac{1}{y^2}\frac{dy}{dx}\,dx = \int \frac{1}{x^2}\,dx$, so $-\frac{1}{y} = -\frac{1}{x} + C = \frac{Cx-1}{x}$. Thus, $y = \frac{x}{1-Cx}$. If we replace the arbitrary constant C by its opposite, this can be written as $y = \frac{x}{1+Cx}$.

9.3.12 $\frac{y}{y^2+4}\,y'(t) = \frac{t}{(t^2+1)^3}$. Therefore, $\int \frac{y}{y^2+4}\,dy = \int \frac{t}{(t^2+1)^3}\,dt$. We have

$$\frac{1}{2}\ln(y^2+4) = -\frac{1}{4}\frac{1}{(t^2+1)^2} + C,$$

so

$$\ln(y^2+4) = C_1 - \frac{1}{2(t^2+1)^2}.$$

This can be written as $y^2 + 4 = Ae^{-(t^2+1)^{-2}/2}$, so $|y| = \sqrt{Ae^{-(t^2+1)^{-2}/2} - 4}$.

9.3.13 $-\frac{2}{y^3}\frac{dy}{dt} = \sin t$, so $\int \frac{-2}{y^3}\frac{dy}{dt}\,dt = \int \sin t\,dt$. Thus, $\frac{1}{y^2} = -\cos t + C$. Solving for y gives $y = \pm\frac{1}{\sqrt{C - \cos t}}$.

9.3.14 $\frac{1}{y^2+4}y'(t) = e^{-t/2}$, so $\int \frac{1}{y^2+4}\,dy = \int e^{-t/2}\,dt$, so

$$\frac{1}{2}\tan^{-1}(y/2) = -2e^{-t/2} + C.$$

Thus, $\tan^{-1}(y/2) = C_1 - 4e^{-t/2}$, and $y = 2\tan(C_1 - 4e^{-t/2})$.

9.3.15 $e^u u'(x) = e^{2x}$, so $\int e^u\,du = \int e^{2x}\,dx$, and $e^u = \frac{1}{2}e^{2x} + C$. Thus, $u = \ln\left(\frac{e^{2x}}{2} + C\right)$.

9.3.16 $\frac{1}{u^2-4}u'(x) = \frac{1}{x}$, so $\int \frac{1}{u^2-4}\,du = \int \frac{1}{x}\,dx$, and thus

$$\frac{1}{4}\ln\left|\frac{u-2}{u+2}\right| = \ln|x| + C,$$

so

$$\ln\left|\frac{u-2}{u+2}\right| = 4\ln|x| + C_1.$$

Therefore, $\left|\frac{u-2}{u+2}\right| = Ax^4$, so $1 - \frac{4}{u+2} = \pm Ax^4$, and $\frac{4}{u-2} = 1 \pm Ax^4$, so $u - 2 = \frac{4}{1 \pm Ax^4}$, and $u = 2 + \frac{4}{1 \pm Ax^4}$.

9.3.17 This is separable, and is already written in the desired form. We have $\int 2y\,dy = \int 3t^2\,dt$, so $y^2 = t^3 + C$. Because $y(0) = 9$, we have $81 = C$, so $y = \sqrt{t^3 + 81}$.

9.3.18 This equation is not separable.

9.3.19 This equation is not separable.

9.3.20 This equation is separable. We have $\int \frac{dy}{y} = \int (4t^3+1)\,dt$, and thus $\ln|y| = t^4+t+C$. Therefore, $y = \pm e^{\left(t^4+t+C\right)} = Ae^{t^4+t}$. Substituting $y(0)=4$ gives $A=4$, so the solution to this initial value problem is $y = 4e^{t^4+t}$.

9.3.21 The equation is separable. We have $\int \frac{1}{y}\,dy = \int e^t\,dt$, so $\ln|y| = e^t + C$, and $|y| = e^{e^t}e^C = Ae^{e^t}$. Then $|y(0)| = 1 = Ae$, so $A = \frac{1}{e}$ and $y = -\frac{1}{e}e^{e^t} = -e^{-1}e^{e^t} = -e^{e^t-1}$.

9.3.22 The equation is separable. We have $\int \frac{1}{y}\,dy = \int \cos x\,dx$, so $\ln|y| = \sin x + C$, and $|y| = e^{\sin x}e^C$. Because $y(0)=3$ we have $3 = e^C$. So $y(x) = 3e^{\sin x}$.

9.3.23 This equation is separable. We have $\int e^y\,dy = \int e^x\,dx$, and thus $e^y = e^x + C$. Therefore, $y = \ln(e^x+C)$. Substituting $y(0) = \ln 3$ gives $\ln 3 = \ln(1+C)$, so $C=2$ and the solution to this initial value problem is $y = \ln(e^x+2)$.

9.3.24 This equation is separable. We have $\int \sec^2 y\,dy = \int dt$, so $\tan y = t + C$. Because $y(1) = \pi/4$ we have $1 = 1 + C$, so $C = 0$. Thus, $y = \tan^{-1}(t)$.

9.3.25 This equation is separable. We have $\int e^y\,dy = \int \frac{\ln^3 t}{t}\,dt$. For the integral on the right-hand side, let $u = \ln t$ so that $du = \frac{1}{t}\,dt$. Then $\int \frac{\ln^3 t}{t}\,dt = \int u^3\,du = \frac{u^4}{4} + C = \frac{\ln^4 t}{4} + C$. Thus we have

$$e^y = \frac{\ln^4 t}{4} + C.$$

Because $y(1) = 0$, we have $1 = 0 + C$, so $y = \ln\left(\frac{\ln^4 t}{4} + 1\right)$.

9.3.26 The equation is separable. We have $\int \frac{dy}{y(y+1)} = \int \frac{1}{t}\,dt$. For the integral on the left-hand side, we have $\int \frac{dy}{y(y+1)} = \int \left(\frac{1}{y} - \frac{1}{1+y}\right)dy = \ln|y| - \ln|y+1| + C$. Thus we have $\ln\left|\frac{y}{y+1}\right| = \ln|t| + C$, so $\frac{y}{y+1} = e^C t$, and because $y(3) = 1$ we have $e^C = \frac{1}{6}$, so

$$\frac{y}{y+1} = \frac{t}{6}.$$

Cross multiplying gives $6y = ty + t$, so $6y - ty = y(6-t) = t$, and $y = \frac{t}{6-t}$, for $t < 6$.

9.3.27 The equation is separable. We have $\int 2y\,dy = \int \sec^2 t\,dt$, so $y^2 = \tan t + C$. Because $y(\pi/4) = 1$ we have $C = 0$, and $y = \sqrt{\tan t}$, Note that the original differential equation is undefined for $y = 0$, which corresponds to $t = 0$ and $t = \pi$, so the solution is $y = \sqrt{\tan t}$ for $0 < t < \pi/2$.

9.3.28 The equation is separable. Assume $y > -3$ and $t > -6/5$. $\int \left(\frac{1}{y+3}\right) y'(t)\, dt = \int \frac{1}{5t+6}\, dt$, so $\ln(y+3) = \frac{\ln(5t+6)}{5} + C$. We can write this as $5\ln(y+3) = \ln(5t+6) + 5C$, so $(y+3)^5 = A(5t+6)$. Because $y(2) = 0$, we have $3^5 = 16A$, so $A = 3^5/16$. We have $(y+3) = \frac{3\sqrt[5]{2}}{2}(5t+6)^{1/5}$. Thus, $y = -3 + \frac{3\sqrt[5]{2}}{2}(5t+6)^{1/5} = \frac{3}{2}(-2 + 2^{1/5}(6+5t)^{1/5})$ for $t > -\frac{6}{5}$.

9.3.29 The equation is separable.

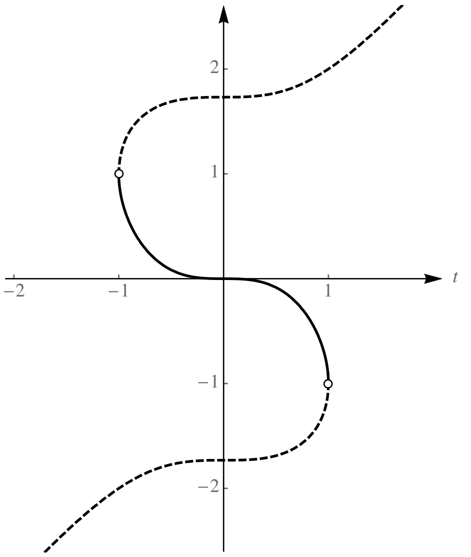
$yy'(t) = t$, so $\int y\, dy = \int t\, dt$, so $\frac{y^2}{2} = \frac{t^2}{2} + C$. Because $y(1) = 2$, we have $2 = 1/2 + C$, so $C = 3/2$. So $y^2 = t^2 + 3$, and $y = \sqrt{t^2+3}$.

9.3.30 The equation is separable. We have $\int \frac{1}{y^3}\, dy = \int \sin t\, dt$, so $-\frac{1}{2y^2} = -\cos t + C$. Because $y(0) = 1$ we have $C = \frac{1}{2}$ and $\frac{1}{2y^2} = \cos t - \frac{1}{2}$. Then $\frac{1}{y^2} = 2\cos t - 1$, and $y^2 = \frac{1}{2\cos t - 1}$, so $y = \frac{1}{\sqrt{2\cos t - 1}}$ for $-\frac{\pi}{3} < t < \frac{\pi}{3}$.

9.3.31 This is separable, and can be written as $y'(t) = \frac{1}{t}$. Thus, $\int y'(t)\, dt = \int \frac{dt}{t} = \ln t + C$, so $y(t) = \ln t + C$. Because $y(1) = 2 = 0 + C$, we have $C = 2$. Thus, $y(t) = \ln t + 2$.

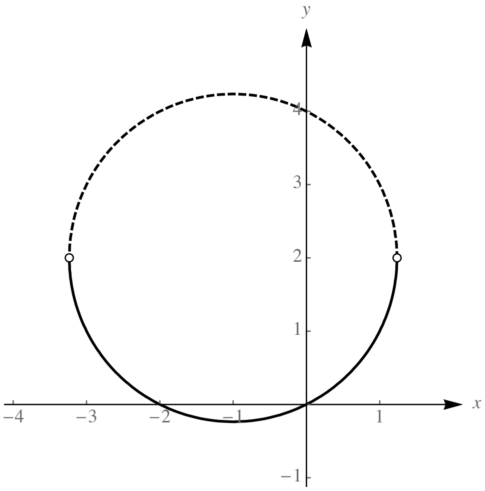
9.3.32 This is separable, and can be written as $y'(t) = \cos t$. Thus $y(t) = \sin t + C$, and because $y(0) = 1 = 0 + C$, we have $C = 1$. Thus, $y(t) = \sin t + 1$. Note that the original differential equation only makes sense for $t \neq \frac{k\pi}{2}$ for odd integers k. So our solution is $y(t) = \sin t + 1$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

9.3.33 We may separate variables and write $\int (y^2 - 1)\, dy = \int 2t^2\, dt$. Then $\frac{y^3}{3} - y = \frac{2t^3}{3} + C$. Because $y(0) = 0$, it follows that $C = 0$. So we have $y^3 - 3y = 2t^3$. From the original differential equation, we see that $y'(t)$ is undefined for $y = \pm 1$. Using the solution, $y = 1$ for $2t^3 = -2$, or $t = -1$, and likewise $y = -1$ implies that $t = 1$. So the solution is $y^3 - 3y = 2t^3$ for $-1 < t < 1$, and the graph is the portion of the curve passing through the origin.



9.3.34 $(2-y)y'(x) = 1+x$, so $\int (2-y)\,dy = \int (1+x)\,dx$, and $2y - \frac{y^2}{2} = x + \frac{x^2}{2} + C$. Because $y(1) = 1$, we have $2 - 1/2 = 1 + 1/2 + C$, so $C = 0$. Thus, $2y - \frac{y^2}{2} = x + \frac{x^2}{2}$ describes the solution. This can be written as $x + \frac{x^2}{2} + \frac{y^2}{2} - 2y = 0$. The original differential equation is undefined for $y = 2$ which corresponds to $x + \frac{x^2}{2} - 2 = 0$ or $x^2 + 2x - 4 = 0$. Using the quadratic formula, this gives $x = -1 \pm \sqrt{5}$. So the solution is

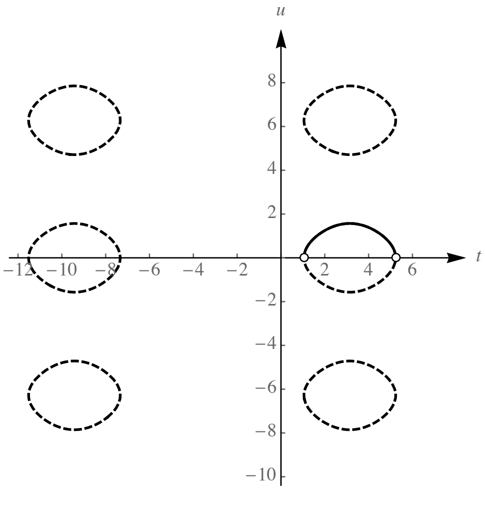
$$x + \frac{x^2}{2} + \frac{y^2}{2} - 2y = 0, \text{ for } -1-\sqrt{5} < x < -1+\sqrt{5}.$$



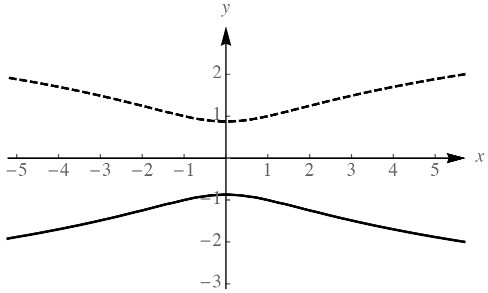
9.3.35 $(\sin u)u'(x) = \cos(x/2)$, so $\int \sin u\,du = \int \cos(x/2)\,dx$. We have $-\cos u = 2\sin(x/2) + C$. Because $u(\pi) = \pi/2$, we have $-\cos \pi/2 = 2\sin(\pi/2) + C$, so $0 = 2(1) + C$, so $C = -2$. Thus, $-\cos u = 2\sin(x/2) - 2$ describes the solution. Note that the original differential equation is undefined for $u = 0$ which corresponds to $x = \pi/3$ and $x = 5\pi/3$. So the solution is

$$-\cos u = 2\sin\left(\frac{x}{2}\right) - 2, \text{ for } \frac{\pi}{3} < x < \frac{5\pi}{3}.$$

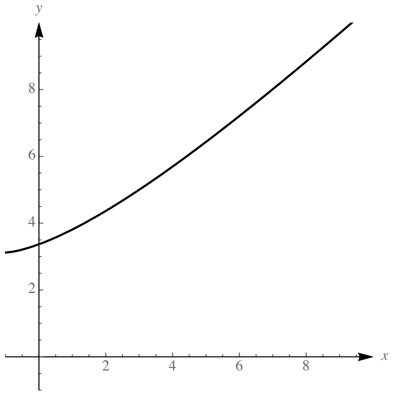
Not that the non-dotted curve contains the point $(\pi, \pi/2)$.



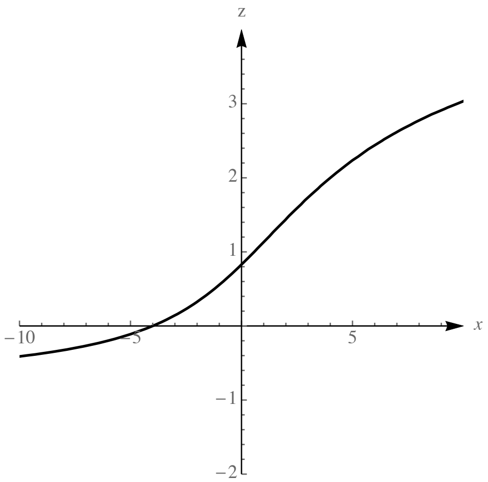
9.3.36 $(2+y^2)^2yy'(x) = 2x$, so $\int (2+y^2)^2yy'(x)\,dx = \int 2x\,dx$, and thus $\dfrac{(2+y^2)^3}{6} = x^2 + C$. Because $y(1) = -1$, we have $C = \frac{21}{6}$. Thus, $(2+y^2)^3 = 6x^2 + 21$, and thus $y^2 = \sqrt[3]{6x^2+21} - 2$.



9.3.37 $\sqrt{y+4}y'(x) = \sqrt{x+1}$, so $\int \sqrt{y+4}\,dy = \int \sqrt{x+1}\,dx$, so $\dfrac{2}{3}(y+4)^{3/2} = \dfrac{2}{3}(x+1)^{3/2} + C$. Because $y(3) = 5$, we have $\dfrac{2}{3}(27) = \dfrac{16}{3} + C$, so $C = \dfrac{38}{3}$. Thus, $\dfrac{2}{3}(y+4)^{3/2} = \dfrac{2}{3}(x+1)^{3/2} + \dfrac{38}{3}$, so $(y+4)^{3/2} = (x+1)^{3/2} + 19$

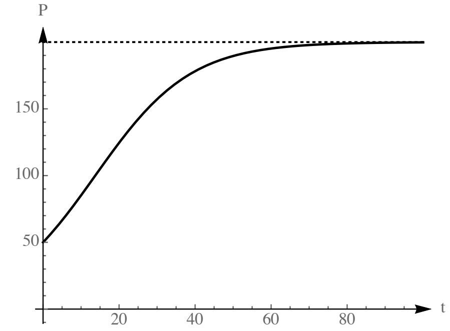


9.3.38 $\dfrac{1}{z^2+4}z'(x) = \dfrac{1}{x^2+16}$, so $\int \dfrac{dz}{z^2+4} = \int \dfrac{dx}{x^2+16}$. Thus, $\dfrac{1}{2}\tan^{-1}(z/2) = \dfrac{1}{4}\tan^{-1}(x/4) + C$. Because $z(4) = 2$, we have $\pi/8 = \pi/16 + C$, so $C = \pi/16$. Thus, $2\tan^{-1}(z/2) = \tan^{-1}(x/4) + \pi/4$ describes the solution.



9.3.39

a. This equation is separable, so we have $\int \frac{200}{P(200-P)}\,dP = \int 0.08\,dt$, so $\int \left(\frac{1}{P} + \frac{1}{200-P}\right) dP = 0.08t + C$. Therefore, $\ln\left|\frac{P}{200-P}\right| = 0.08t + C$. Substituting $P(0) = 50$ gives $-\ln 3 = C$, and solving for P gives $P(t) = \frac{200}{3e^{-0.08t}+1}$.

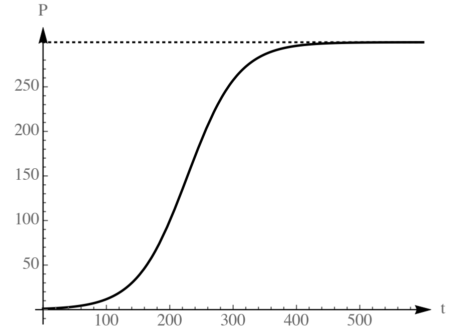


b. The steady-state population is $\lim_{t\to\infty} P(t) = 200$.

9.3.40

a. This equation is separable, so we have $\int \frac{A}{P(A-P)}\,dP = \int k\,dt$, so $\int \left(\frac{1}{P} + \frac{1}{A-P}\right) dP = kt + D$. Therefore, $\ln\left|\frac{P}{A-P}\right| = kt + D$, which is equivalent to $\frac{P}{A-P} = Ce^{kt}$. Substituting $P(0) = P_0$ gives $C = P_0/(A-P_0)$, and solving for P gives $P(t) = \frac{AP_0}{P_0 + (A-P_0)e^{-kt}}$.

b.



c. The denominator in $P(t)$ above is positive for all $t \geq 0$ when $0 < P_0 < A$, so $P(t)$ is defined for all $t \geq 0$; we have $\lim_{t\to\infty} P(t) = A$, which is the steady-state solution.

9.3.41

a. True. It can be written as $u^7 u'(x) = x^{-2}$.

b. False.

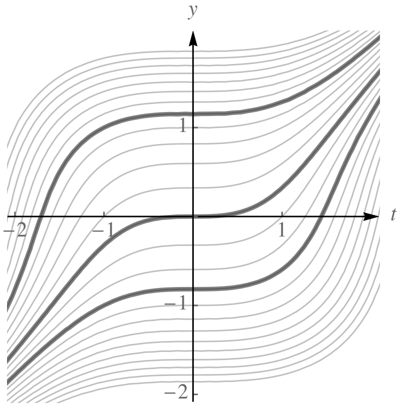
c. True. When separated, we have $ye^y y'(x) = x$, and the left-hand side of the equation can be integrated by parts.

9.3.42

a $\int (y^2+1)\,dy = \int t^2\,dt$, so $\dfrac{y^3}{3} + y = \dfrac{t^3}{3} + C_1$, and $y^3 + 3y = t^3 + C$, where $C = 3C_1$.

b. For $y(-1) = 1$ we have $C = 5$. For $y(0) = 0$ we have $C = 0$. For $y(-1) = -1$ we have $C = -3$.

c.

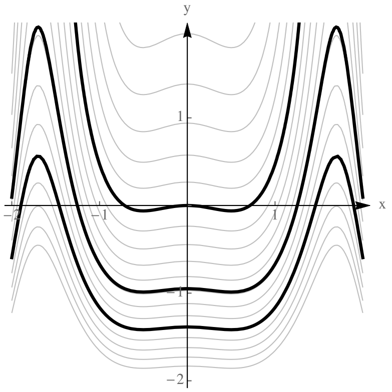


9.3.43

a. $\int e^{-y/2}\,dy = \int (4x\sin x^2 - x)\,dx$. Thus, $-2e^{-y/2} = -2\cos x^2 - \dfrac{x^2}{2} + K$. Then $e^{-y/2} = \cos x^2 + \dfrac{x^2}{4} + C$, so $-\dfrac{y}{2} = \ln\left(\cos x^2 + \dfrac{x^2}{4} + C\right)$, and $y = -2\ln\left(\cos x^2 + \dfrac{x^2}{4} + C\right)$.

b. When $y(0) = 0$ we have $0 = -2\ln(1+C)$, so $C = 0$. When $y(0) = \ln(1/4)$, we have $\ln(1/4) = -2\ln(1+C)$, so $\ln 2 = \ln(1+C)$, so $C = 1$. When $y(\sqrt{\pi/2}) = 0$, we have $0 = -2\ln(0 + \pi/8 + C$, so $C = 1 - \dfrac{\pi}{8}$.

c.



9.3.44

a. $4x + 2y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{4x}{2y} = -\dfrac{2x}{y}$.

b. Curves are orthogonal when their slopes are negative reciprocals of each other, and the negative reciprocal of $-\dfrac{2x}{y}$ is $\dfrac{y}{2x}$.

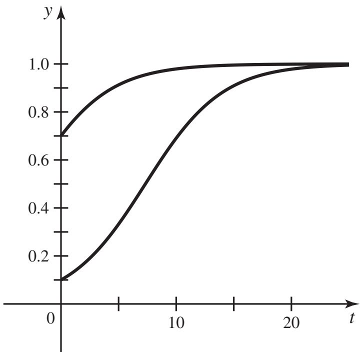
c. We have $\dfrac{2\,dy}{y} = \dfrac{dx}{x}$, so $2\ln|y| = \ln y^2 = \ln|x| + C$. Thus, $y^2 = e^C|x|$. We can write $\pm e^C = k$, so we have $y^2 = kx$.

9.3.45 Differentiating implicitly gives $2x + 2yy' = 0$, so $y' = -\dfrac{x}{y}$. We are thus seeking curves so that $\dfrac{dy}{dx} = \dfrac{y}{x}$. We have $\dfrac{dy}{y} = \dfrac{dx}{x}$, so $\ln|y| = \ln|x| + C$ so $y = e^C|x| = kx$. So the family of curves we are seeking is the collection of curves $y = kx$.

9.3.46

a. This equation is separable, so we have $\displaystyle\int \frac{1}{y(1-y)}\,dy = \int k\,dt$, so $\displaystyle\int \left(\frac{1}{y} + \frac{1}{1-y}\right)dy = kt + D$. Therefore, $\ln\left|\dfrac{y}{1-y}\right| = kt + D$, which is equivalent to $\dfrac{y}{1-y} = Ce^{kt}$. Substituting $y(0) = y_0$ gives $C = y_0/(1-y_0)$, and thus $y = \dfrac{y_0}{(1-y_0)e^{-kt} + y_0}$.

b.



c. The denominator in $y(t)$ above is positive for all $t \ge 0$ when $0 < y_0 < 1$, so $y(t)$ is defined for all $t \ge 0$; we have $\lim_{t\to\infty} y(t) = 1$, which is the steady-state solution.

9.3.47

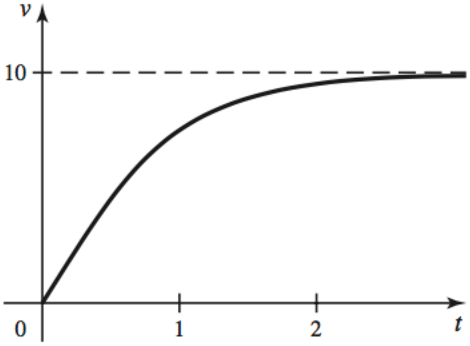
a. We have $mv'(t) = mg - kv^2$, so $v'(t) = g - av^2$ with $a = k/m$.

b. We solve $av^2 = g$ to obtain the terminal velocity $\tilde{v} = \sqrt{g/a} = \sqrt{gm/k}$.

c. This equation is separable, so we have $\displaystyle\int \frac{1}{g - av^2}dv = \int dt$, so $\displaystyle -\frac{1}{a}\int \frac{1}{v^2 - \tilde{v}^2}\,dv = t + D$.

Thus, $\displaystyle -\frac{1}{2a\tilde{v}}\int\left(\frac{1}{v-\tilde{v}} - \frac{1}{v+\tilde{v}}\right)dv = t + D$, and $-\dfrac{1}{2a\tilde{v}}\ln\left|\dfrac{v-\tilde{v}}{v+\tilde{v}}\right| = t + D$, hence $\dfrac{v-\tilde{v}}{v+\tilde{v}} = Ce^{-2a\tilde{v}t}$ The initial condition $v(0) = 0$ gives $C = -1$, and solving for v gives $v = \dfrac{1 - e^{-2a\tilde{v}t}}{1 + e^{-2a\tilde{v}t}}\tilde{v}$. This can be written as $v = (\sqrt{g/a})\dfrac{e^{2\sqrt{ag}t} - 1}{e^{2\sqrt{ag}t} + 1}$.

d. We have $a = 0.1$, $\tilde{v} \approx 9.90$ m/s



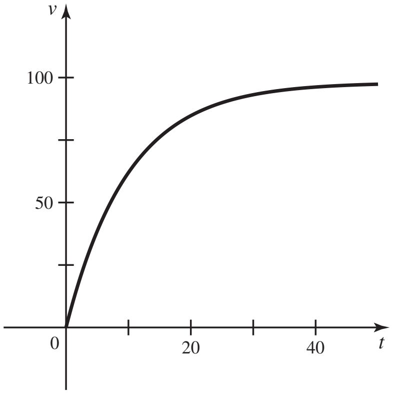
9.3.48

a. We have $mv'(t) = mg - Rv$, so $v'(t) = g - bv$ with $b = R/m$.

b. Solve $bv = g$ to obtain terminal velocity $\tilde{v} = g/b = mg/R$.

c. The equation $v' = g - bv$ is first-order linear, with general solution $v = Ce^{-bv} + \tilde{v}$. The initial condition $v(0) = 0$ gives $C = -\tilde{v}$, which gives $v = \tilde{v}(1 - e^{-bt})$.

d. We have $b = 0.1$, $\tilde{v} = 98$ m/s.



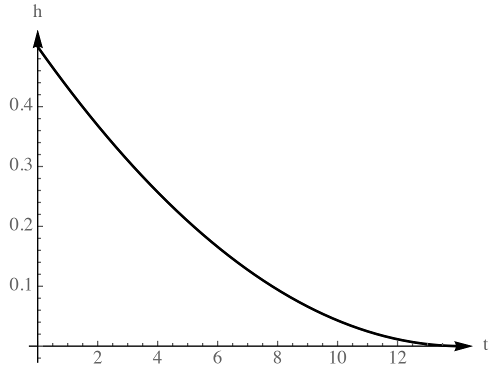
9.3.49

a. The equation $h' = -k\sqrt{h}$ is separable, so we have $\int \frac{dh}{\sqrt{h}} = \int -k\,dt$, so $2\sqrt{h} = -kt + C$. The initial condition $h(0) = H$ gives $C = 2\sqrt{H}$, so the solution is $h = \left(\sqrt{H} - \frac{kt}{2}\right)^2$.

b. The solution for $k = 0.1$ and $H = 0.5$ is $h = \left(\sqrt{0.5} - 0.05t\right)^2$, $0 \le t \le 14.1$.

c. The tank is drained when $h(t) = 0$, which gives $t = \frac{\sqrt{0.5}}{0.05} \approx 14.1$ s.

d.

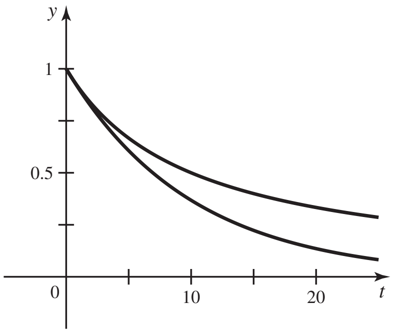


9.3.50

a. The general solution to $y' = -ky$ is $y = Ce^{-kt}$, $t \geq 0$.

b. The equation $y' = -ky^2$ is separable, so we have $-\displaystyle\int \frac{dy}{y^2} = \int k\,dt$, so $\dfrac{1}{y} = kt + C$. The initial condition $y(0) = y_0$ gives $C = 1/y_0$, and solving for y gives $y = \dfrac{1}{kt + 1/y_0} = \dfrac{y_0}{1 + ky_0 t}$., for $t \geq 0$.

c.

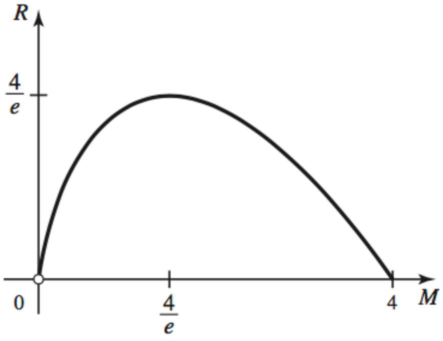


9.3.51

a. The growth rate is positive when $0 < M < 4$. The function $R(M)$ has derivative

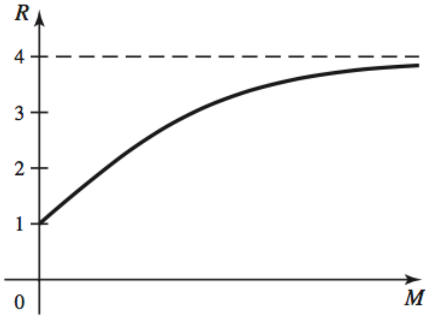
$$R'(M) = -r\left(\ln\left(\frac{M}{K}\right) + M \cdot \frac{K}{M} \cdot \frac{1}{K}\right) = -r\left(\ln\left(\frac{M}{K}\right) + 1\right),$$

which is 0 when $M/4 = 1/e$ or $M = 4/e$. We also observe that $\displaystyle\lim_{M\to 0^+} R(M) = 0$ and $R(4) = 0$, so $R(M)$ takes its maximum at the critical point $M = 4/e$.



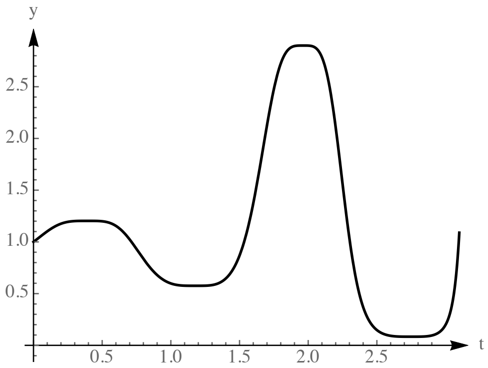
b. The equation is separable, so we have $\int \frac{dM}{M(\ln M - \ln K)} = -\int r\,dt$, so $\ln|\ln M - \ln K| = -rt + D$, and thus $\ln\left(\frac{M}{K}\right) = Ce^{-rt}$. Therefore $M = Ke^{Ce^{-rt}}$.

The conditions $r = 1$, $K = 4$ and $M_0 = 1$ give $C = -\ln 4$ and $M = 4e^{(-\ln 4)e^{-t}} = 4^{1-e^{-t}}$. Observe that $\lim_{t\to\infty} M(t) = 4e^0 = 4$, so the limiting size of the tumor is 4. In general, the limiting size of the tumor is $\lim_{t\to\infty} Ke^{Ce^{-rt}} = K$, because $r > 0$.



9.3.52 Solving with a CAS gives

$$y = e^{\left(\frac{e^t(1740\sin(4t) + 204\sin(12t) + 435\cos(4t) + 17\cos(12t))}{9860} - \frac{113}{2465}\right)}.$$



9.3.53

a. $\frac{1}{y^2}y'(t) = 1$, so $\int \frac{dy}{y^2} = \int 1\,dt$. Thus $-\frac{1}{y} = t + C$. Because $y_0 = 1$ we have $-1 = C$. Thus, $y = \frac{1}{1-t}$ for $t < 1$.

b. $\frac{1}{y^3}y'(t) = 1$, so $\int \frac{dy}{y^3} = \int 1\,dt$. Thus $-\frac{1}{2y^2} = t + C$. Because $y_0 = \frac{1}{\sqrt{2}}$ we have $-1 = C$. Thus, $\frac{1}{2y^2} = 1 - t$ and $y = \frac{1}{\sqrt{2}\sqrt{1-t}}$ for $t < 1$.

c. $\frac{1}{y^{n+1}}y'(t) = 1$, so $\int \frac{dy}{y^{n+1}} = \int 1\,dt$. Thus $-\frac{1}{ny^n} = t + C$. Because $y_0 = n^{-1/n}$ we have $-1 = C$. Thus, $\frac{1}{ny^n} = 1 - t$, and $ny^n = \frac{1}{1-t}$. Thus, $y = \frac{1}{(n(1-t))^{1/n}}$ for $t < 1$.

We have $\lim_{t\to 1^-} \frac{1}{(n(1-t))^{1/n}} = \infty$.

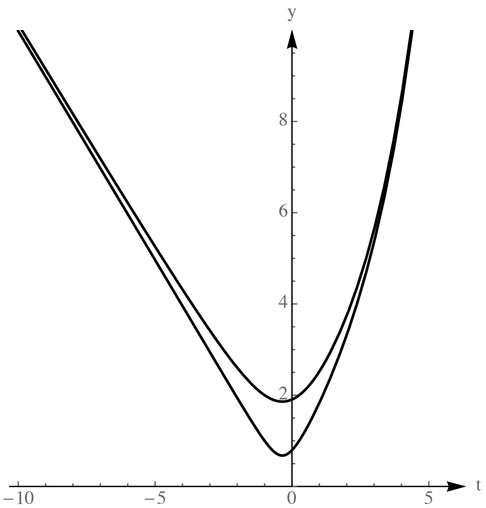
9.3.54

a. $\int yy'(t)\,dt = \int (e^t/2 + t)\,dt$, so $y^2/2 = e^t/2 + t^2/2 + C$, so $y^2 = e^t + t^2 + C_1$. So $y = \pm\sqrt{e^t + t^2 + C_1}$.

b. If $y(-1) = 1$, then $1 = \sqrt{1/e + 1 + C_1}$ so $C_1 = -1/e$, so $y = \sqrt{e^t + t^2 - 1/e}$.

If $y(-1) = 2$, then $2 = \sqrt{1/e + 1 + C_1}$ so $C_1 = 3 - 1/e$, so $y = \sqrt{e^t + t^2 + 3 - 1/e}$.

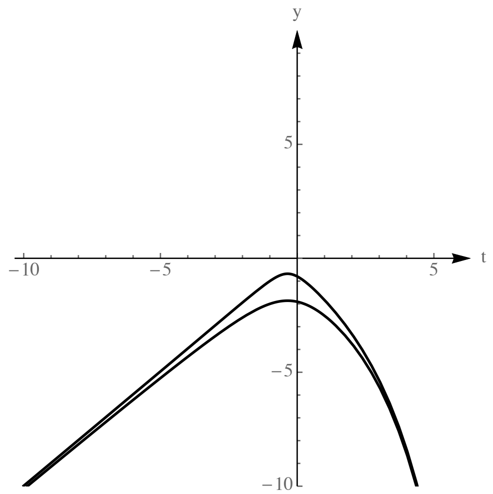
c. For $t > 0$, the solutions increase without bound as t increases.



d. If $y(-1) = -1$, then $-1 = -\sqrt{1/e + 1 + C_1}$, so $C_1 = -1/e$ and $y = -\sqrt{e^t + t^2 - 1/e}$.

If $y(-1) = -2$, then $-2 = -\sqrt{1/e + 1 + C_1}$ so $C_1 = 3 - 1/e$, so $y = -\sqrt{e^t + t^2 + 3 - 1/e}$.

e. For $t > 0$, the solutions decrease without bound as t increases.



9.4 Special First-Order Linear Differential Equations

9.4.1 Because $y(0) = 4$, we have $4 = C - 13$, so $C = 17$. Thus, the solution is $y = 17e^{-10t} - 13$.

9.4.2 The general solution is $y = Ce^{3t} - \left(-\dfrac{12}{3}\right) = Ce^{3t} + 4$.

9.4.3 The general solution is $y = Ce^{-4t} - \left(\dfrac{6}{-4}\right) = Ce^{-4t} + \dfrac{3}{2}$.

9.4.4 The equilibrium solution is $y = 3$. It is unstable.

9.4.5 Because $k = 3$ and $b = -4$, we have $y = Ce^{3t} + \dfrac{4}{3}$.

9.4.6 Because $k = -1$ and $b = 2$, we have $y = Ce^{-x} + 2$.

9.4.7 Because $k = -2$ and $b = -4$, we have $y = Ce^{-2x} - 2$.

9.4.8 Because $k = 2$ and $b = 6$, we have $y = Ce^{2x} - 3$.

9.4.9 Because $k = -12$ and $b = 15$, we have $u = Ce^{-12t} + \dfrac{5}{4}$.

9.4.10 Because $k = \frac{1}{2}$ and $b = 14$, we have $v = Ce^{y/2} - 28$.

9.4.11 Because $k = 3$ and $b = -6$, we have $y = Ce^{3t} + 2$. Because $y(0) = 9$, we have $9 = C + 2$, so $C = 7$. Thus, the solution is $y = 7e^{3t} + 2$.

9.4.12 Because $k = -1$ and $b = 2$, we have $y = Ce^{-x} + 2$. Because $y(0) = -2$, we have $-2 = C + 2$, so $C = -4$. Thus, the solution is $y = -4e^{-x} + 2$.

9.4.13 Because $k = 2$ and $b = 8$, we have $y = Ce^{2t} - 4$. Because $y(0) = 0$, we have $0 = C - 4$, so $C = 4$. Thus, the solution is $y = 4e^{2t} - 4$.

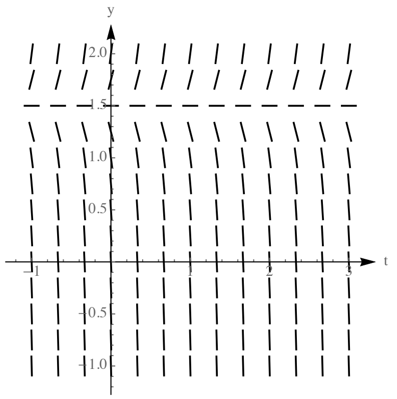
9.4.14 Because $k = 2$ and $b = 6$, we have $u = Ce^{2x} - 3$. Because $u(1) = 6$, we have $6 = Ce^{2} - 3$, so $C = 9e^{-2}$. Thus, the solution is $u = 9e^{2x-2} - 3$.

9.4.15 Because $k = 3$ and $b = 12$, we have $y = Ce^{3t} - 4$. Because $y(1) = 4$, we have $4 = Ce^3 - 4$, so $C = 8e^{-3}$. Thus, the solution is $y = 8e^{3t-3} - 4$.

9.4.16 Because $k = -1/2$ and $b = 6$, we have $z = Ce^{-t/2} + 12$. Because $z(-1) = 0$, we have $0 = Ce^{1/2} + 12$, so $C = -12e^{-1/2}$. Thus, the solution is $z = -12e^{-t/2-1/2} + 12$.

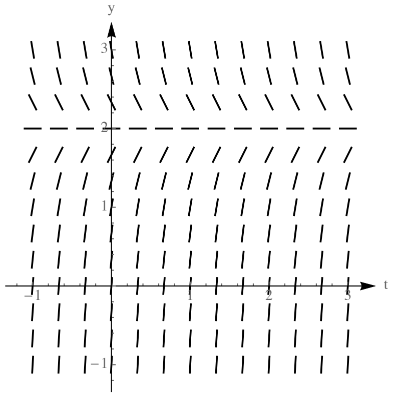
9.4.17

The equilibrium solution is $y = \frac{18}{12} = \frac{3}{2}$. The solution is unstable.



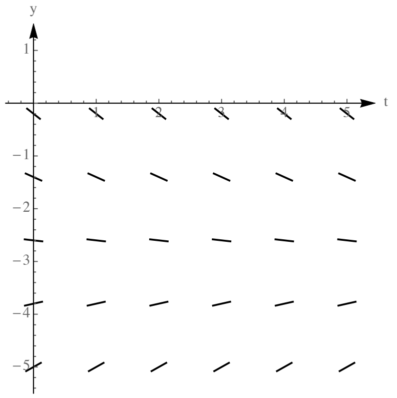
9.4.18

The equilibrium solution is $y = 2$. The solution is stable.



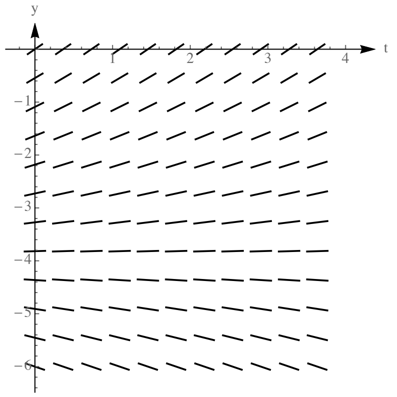
9.4.19

The equilibrium solution is $y = -3$. The solution is stable.



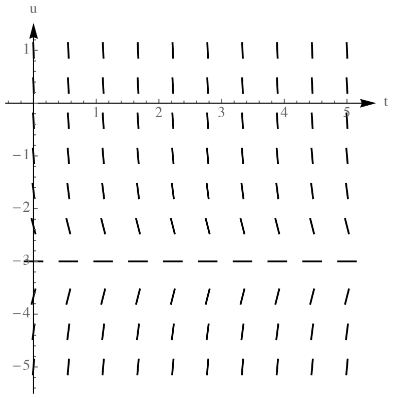
9.4.20

The equilibrium solution is $y = -4$. The solution is unstable.



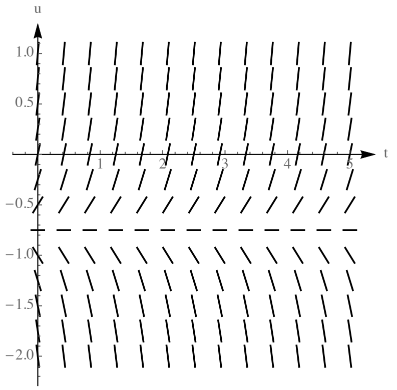
9.4.21

The equilibrium solution is $u = -3$. The solution is stable.



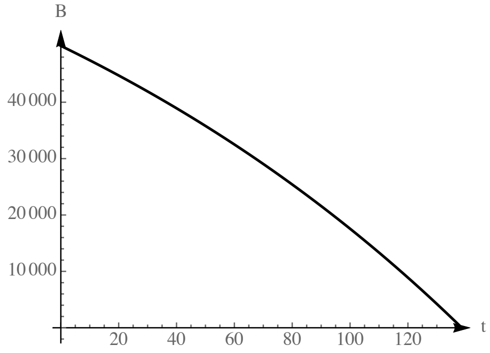
9.4.22

The equilibrium solution is $u = -3/4$. The solution is unstable.



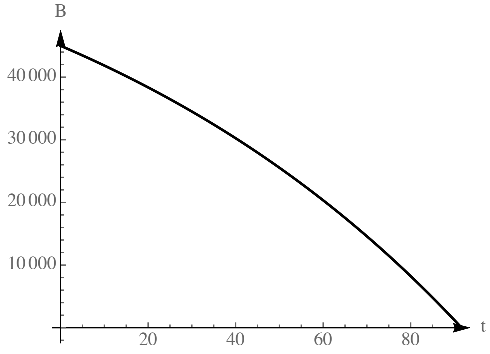
9.4.23 Because $k = 0.005$ and $b = -500$, we have $B = Ce^{0.005t} + \dfrac{500}{0.005} = Ce^{0.005t} + 100000$. Because $B(0) = 50,000$, we have $50000 = C + 100000$, so $C = -50000$. Thus, $B = -50000e^{0.005t} + 100000$.

The balance is zero when $t = \ln(2)/0.005 \approx 139$ months.



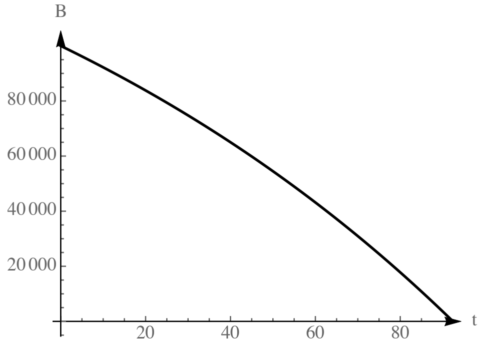
9.4.24 Because $k = 0.01$ and $b = -750$, we have $B = Ce^{0.01t} + \dfrac{750}{0.01} = Ce^{0.01t} + 75000$. Because $B(0) = 45,000$, we have $45000 = C + 75000$, so $C = -30000$. Thus, $B = -30000e^{0.01t} + 75000$.

The balance is zero when $t = \ln(5/2)/0.01 \approx 92$ months.



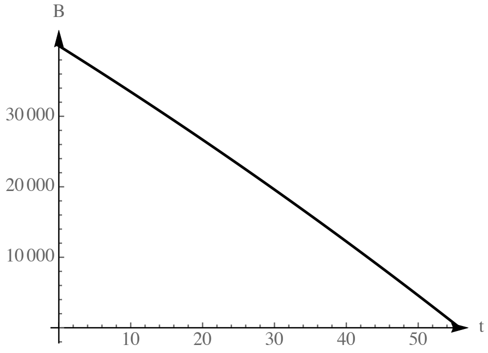
9.4.25 Because $k = 0.0075$ and $b = -1500$, we have $B = Ce^{0.0075t} + \dfrac{1500}{0.0075} = Ce^{0.0075t} + 200000$. Because $B(0) = 100,000$, we have $100000 = C + 200000$, so $C = -100000$. Thus, $B = -100000e^{0.0075t} + 200000$.

The balance is zero when $t = \ln(2)/0.0075 \approx 93$ months.



9.4.26 Because $k = 0.004$ and $b = -800$, we have $B = Ce^{0.004t} + \dfrac{800}{0.004} = Ce^{0.004t} + 200000$. Because $B(0) = 40,000$, we have $40000 = C + 200000$, so $C = -160000$. Thus, $B = -160000e^{0.004t} + 200000$.

The balance is zero when $t = \ln(5/4)/0.004 \approx 56$ months.



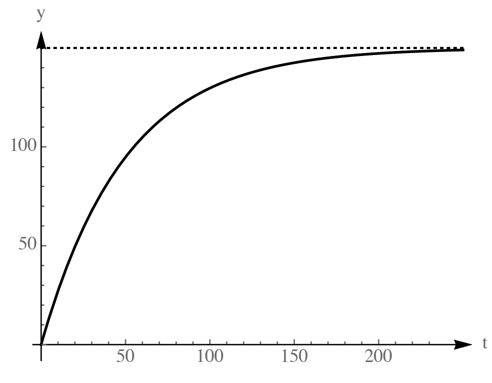
9.4.27 We know that the solution has the form $T(t) = (90 - 25)e^{-kt} + 25 = 65e^{-kt} + 25$. Because $T(1) = 85$, we have $65e^{-k} + 25 = 85$. Thus, $k = -\ln(60/65) \approx 0.08$. Thus, $T(t) = 65e^{-0.08t} + 25$. We have $T(t) = 30$ when $65e^{-0.08t} + 25 = 30$, or when $e^{-0.08t} = \dfrac{1}{13}$. This occurs when $t = \ln(13)/0.08 \approx 32$ minutes after the coffee is first poured.

9.4.28 We have $T(t) = (900 - 30)e^{-0.02t} + 30$. We have $T(t) = 100$ when $870e^{-0.02t} + 30 = 100$, which occurs when $e^{-0.02t} = \dfrac{7}{87}$. This occurs when $t = \dfrac{\ln(7/87)}{-0.02} \approx 126$.

9.4.29 $T(t) = (5 - 20)e^{-kt} + 20 = -15e^{-kt} + 20$. Because $T(1) = 7$, we have $-15e^{-k} + 20 = 7$, so $k = \ln(15/13) \approx 0.143$. Thus $T(t) = -15e^{-0.143t} + 20$. The milk will reach 18 degrees when $-15e^{-0.143t} + 20 = 18$, which occurs when $e^{-0.143t} = 2/15$. Thus, when $t \approx \dfrac{\ln(2/15)}{-0.143} \approx 14$ minutes.

9.4.30 $T(t) = (100 - 10)e^{-kt} + 10 = 90e^{-kt} + 10$. We are given that $T(30) = 80$, so $80 = 90e^{-30k} + 10$, so $e^{-30k} = 7/9$. Thus $k = \dfrac{\ln(7/9)}{-30} \approx 0.0084$. The soup will reach 30 degrees when $30 = 90e^{-0.0084t} + 10$, which occurs when $e^{-0.0084t} = 2/9$, or $t = \dfrac{\ln(2/9)}{-0.0084} \approx 179$ minutes.

9.4.31



a. The general solution is $y(t) = Ce^{-0.02t} + 150$; substitute $y(0) = 0$ to obtain $C + 150 = 0$, so $C = -150$; hence $y(t) = 150(1 - e^{-0.02t})$.

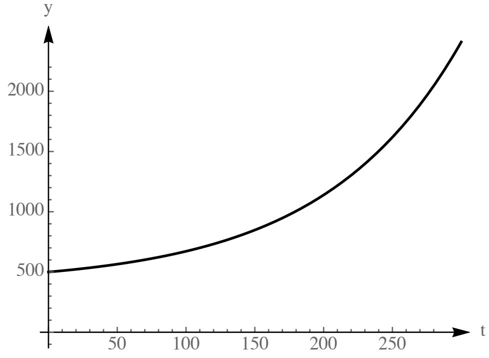
b. The steady-state level is $\lim\limits_{t\to\infty} 150(1 - e^{-0.02t}) = 150\,\text{mg}$.

c. We have $150(1 - e^{-0.02t}) = 0.9 \cdot 150$, so $e^{-0.02t} = 0.1$, and thus $t = \dfrac{\ln 10}{0.02} \approx 115.1$ hours.

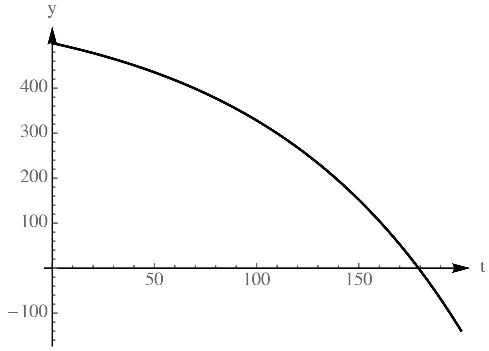
9.4.32

a. The general solution is $y(t) = Ce^{0.01t} + 10b$; substitute $y(0) = 500$ to obtain $C + 10b = 500$, so $C = 500 - 10b$; hence $y(t) = (500 - 10b)e^{0.01t} + 10b$.

b. In this case the solution is $y(t) = 100e^{0.01t} + 400$, which approaches ∞ as $t \to \infty$.



c. In this case the solution is $y(t) = -100e^{0.01t} + 600$, which reaches 0 when $e^{0.01t} = 6$ or $t = 100 \ln 6 \approx 179$ yrs.



9.4.33

a. The equation $y'(t) = 0.008y - h$ has steady-state solution $y = h/0.008 = 125h$, so we solve $y_0 = 2000 = 125h$ to obtain $h = 16$.

b. If $h = 200$, then the steady-state solution is $y = 125 \cdot 200 = 25{,}000$.

9.4.34

a. The equation $B' = rB - m$ is first-order linear, with general solution $B = Ce^{rt} + m/r$; in this case $r = 0.05$, $m/r = 20{,}000$, and the initial condition $B_0 = 15{,}000$ gives $C = -5000$, so $B = 20{,}000 - 5000e^{0.05t}$. The balance decreases.

b. The steady-state (constant balance) solution is $B = m/r = \$50{,}000$, which gives $m = 0.05 \cdot 50{,}000 = \$2500$.

9.4.35

a. False. It is $y(t) = Ce^{2t} + 9$ where C is an arbitrary constant.

b. True. Note that if $y(t) = 0$, then $y'(t) = 0$, not $k(0) - b = -b$ as dictated by the equation.

c. False. It is not separable.

d. False. It approaches it asymptotically.

9.4.36 First note that if $ky+b<0$, then $|ky+b| = -ky-b$. We will use this fact in what follows. We have $\frac{1}{ky+b}y'(t) = 1$, so $\int \frac{dy}{ky+b} = \int dt$, so $\frac{1}{k}\ln(-ky-b) = t + C_1$. Thus $\ln(-ky-b) = kt + C_2$, so $-ky-b = C_3e^{kt}$, and $y = Ce^{-kt} - \frac{b}{k}$.

9.4.37

a. Note that $k = 0.03$ and $b = -600$. Thus, $B(t) = Ce^{0.03t} + 20,000$. If $B_0 = B(0)$ is the amount borrowed, then $B_0 = C + 20,000$, so $C = B_0 - 20,000 = 20,000$.

Thus, $B(t) = 20,000e^{0.03t} + 20,000$. This is an increasing function because its derivative is positive. It is occurring because the amount paid monthly is less than the monthly accruing interest.

b. Because $20000 \cdot 0.03 = 600$, the amount borrowed should be less than 20000 if the balance is to be decreasing.

c. The maximum amount that can be borrowed and not have the unpaid balance increase is B_0 when $rB_0 - m = 0$, or $B_0 = \frac{m}{r}$.

9.4.38

a. $T(t) = (T_0 - A)e^{-kt} + A$. We are seeking $t_{1/2}$ so that $T(t_{1/2}) = \frac{T_0}{2}$. Thus we have $T_0/2 - A = (T_0 - A)e^{-k(t_{1/2})}$, so $\frac{T_0 - 2A}{2(T_0 - A)} = e^{-k(t_{1/2})}$. Thus, $t_{1/2} = -\frac{1}{k}\ln\left|\frac{T_0 - 2A}{2(T_0 - A)}\right|$.

b. As k increases, $t_{1/2}$ decreases.

c. If $A > T_0/2$, then the equation $T_0/2 - A = (T_0 - A)e^{-k(t_{1/2})}$ would have no solution, because the left-hand side is negative and the right-hand side is positive.

9.4.39 $\int (ty'(t) + y(t))\,dt = \int (1+t)\,dt$, so $ty(t) = t + \frac{t^2}{2} + C$ for $t > 0$. Thus, $y(t) = 1 + \frac{t}{2} + \frac{C}{t}$. Because $y(1) = 4$, we have $4 = 1 + 1/2 + C$, so $C = 5/2$. Thus, $y(t) = 1 + \frac{t}{2} + \frac{5}{2t}$ for $t > 0$.

9.4.40 $\int (t^3y'(t) + 3t^2y(t))\,dt = \int (1/t + 1)\,dt$. Thus, $t^3y(t) = \ln|t| + t + C$ for $t > 0$. Therefore, $y(t) = \frac{\ln|t|}{t^3} + \frac{1}{t^2} + \frac{C}{t^3}$. Because $y(1) = 6$, we have $6 = 0+1+C$, so $C = 5$. The solution is thus $y(t) = \frac{\ln|t|}{t^3} + \frac{1}{t^2} + \frac{5}{t^3}$ for $t > 0$.

9.4.41 $\int (e^{-t}y'(t) - e^{-t}y(t))\,dt = \int e^{2t}\,dt$, so $e^{-t}y(t) = e^{2t}/2 + C$. Thus, $y(t) = e^{3t}/2 + Ce^t$. Because $y(0) = 4$, we have $4 = 1/2 + C$, so $C = 7/2$. The solution is therefore $y(t) = e^{3t}/2 + 7e^t/2$.

9.4.42 $\int ((t^2+1)y'(t) + 2ty(t))\,dt = \int 3t^2\,dt$, so $(t^2+1)y(t) = t^3 + C$. Thus, $y(t) = \frac{t^3}{t^2+1} + \frac{C}{t^2+1}$. Because $y(2) = 8$, we have $8 = 8/5 + C/5$, so $C = 32$. The solution is therefore $y(t) = \frac{t^3+32}{t^2+1}$.

9.4.43

a. Let $v = y^{1-p}$. Then $v'(t) = (1-p)y^{-p}y'(t)$, so $y'(t) = \frac{y^p}{1-p}v'(t)$.

b. Given $y'(t) + ay = by^p$, we have $\frac{y^p}{1-p}v'(t) + ay = by^p$, so $\frac{1}{1-p}v'(t) + ay^{1-p} = b$, so

$$v'(t) = (p-1)av(t) + b(1-p).$$

Then $v(t) = Ce^{(p-1)at} - \frac{b(1-p)}{(p-1)a} = Ce^{(p-1)at} + \frac{b}{a}$. Therefore,

$$y(t) = \left(Ce^{(p-1)at} + \frac{b}{a}\right)^{1/(1-p)}.$$

9.4.44

a. Using the results of the previous problem, we have $a = 1$, $b = 2$, and $p = 2$. Thus

$$y(t) = \left(Ce^t + 2\right)^{-1}.$$

b. We have $a = -2$, $p = -1$, and $b = 3$, so

$$y(t) = \left(Ce^{4t} + \frac{-3}{2}\right)^{1/2}.$$

c. We have $a = 1$, $b = 1$, and $p = 1/2$, so

$$y(t) = \left(Ce^{-t/2} + 1\right)^2.$$

9.4.45 Let $p(t) = \exp\left(\int 1/t\,dt\right) = e^{\ln t} = t$. The original differential equation can be written as $t(y'(t) + (1/t)y(t)) = 0$, or $ty'(t) + y(t) = 0$. Integrating both sides with respect to t gives $ty(t) = C$, so $y(t) = \frac{C}{t}$. Because $y(1) = 6$, we have $C = 6$,and $y = \frac{6}{t}$ for $t > 0$.

9.4.46 Let $p(t) = \exp\left(\int 3/t\,dt\right) = e^{3\ln t} = t^3$. The original differential equation can be written as $t^3(y'(t) + (3/t)y(t)) = t^3 - 2t^4$, or $t^3y'(t) + 3t^2y(t) = t^3 - 2t^4$. Integrating both sides with respect to t gives $t^3y(t) = t^4/4 - 2t^5/5 + C$, so $y(t) = t/4 - 2t^2/5 + C/t^3$. Because $y(2) = 0$, we have $0 = \frac{1}{2} - \frac{8}{5} + \frac{C}{8}$, and solving for C gives $C = \frac{44}{5}$. Thus,

$$y(t) = \frac{t}{4} - \frac{2t^2}{5} + \frac{44}{5t^3}$$

for $t > 0$.

9.4.47 Let $p(t) = \exp\left(\int \frac{2t}{t^2+1}\,dt\right) = \exp(\ln(t^2 + 1)) = t^2 + 1$. The original differential equation can be written as $(t^2 + 1)y'(t) + (2t)y(t) = (t^2 + 1)(1 + 3t^2)$. Integrating both sides with respect to t gives

$$(t^2 + 1)y(t) = \int (3t^4 + 4t^2 + 1)\,dt = 3t^5/5 + 4t^3/3 + t + C.$$

Because $y(1) = 4$, we have $8 = 3/5 + 4/3 + 1 + C$, so $C = \frac{76}{15}$. Thus,

$$y(t) = \frac{3t^5/5 + 4t^3/3 + t + \frac{76}{15}}{t^2 + 1} = \frac{9t^5 + 20t^3 + 15t + 76}{15(t^2 + 1)}.$$

9.4.48 Let $p(t) = \exp\left(\int 2t\,dt\right) = e^{t^2}$. The original differential equation can be written as $e^{t^2}y'(t) + e^{t^2}(2t)y(t) = 3te^{t^2}$. Integrating both sides with respect to t gives $e^{t^2}y(t) = \frac{3}{2}e^{t^2} + C$. Because $y(0) = 1$, we have $1 = \frac{3}{2} + C$, so $C = -\frac{1}{2}$. Thus, $y = \frac{\frac{3}{2}e^{t^2} - \frac{1}{2}}{e^{t^2}} = \frac{3e^{t^2} - 1}{2e^{t^2}}$.

9.5 Modeling with Differential Equations

9.5.1 The growth rate function specifies the rate of growth of the population. If the growth rate function is positive, then the population is increasing, while the population is decreasing when the growth rate function is negative.

9.5.2 The carrying capacity is the upper limit of the size of a population, due to limitations in resources. Mathematically, it appears as a horizontal asymptote as $t \to \infty$.

9.5.3 If the growth rate function is positive and decreasing, then the population is increasing. Whether or not the population is increasing is completely determined by whether the growth rate function is positive or negative.

9.5.4 A stirred tank reaction takes place in a tank that is filled with a soluble substance like salt or sugar. The tank has an inflow and an outflow pipe, so the volume of the solution is constant. The tank is assumed to be completely stirred at all times, so the solution is uniform. The problem is to find the mass of the substance in the tank at all times.

9.5.5 Is is a linear, first-order differential equation.

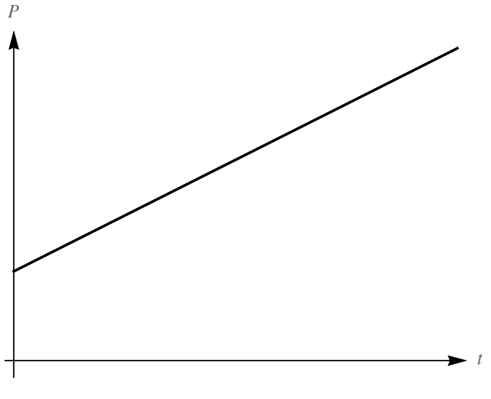
9.5.6

- In the absence of prey, the predator population deceases exponentially, while encounters between prey and predators increase the predator population (the prey are the food supply).
- In the absence of predators, the prey population increases exponentially, while encounters between the prey and predators deplete the prey population (the predators eat the prey).

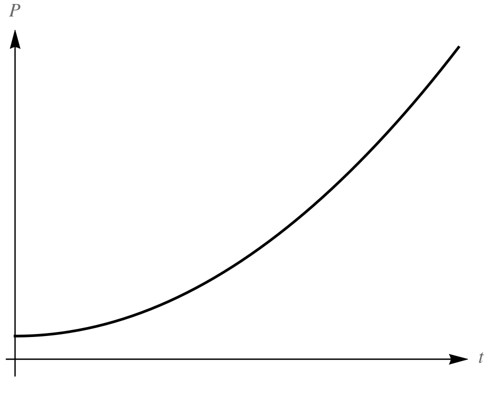
9.5.7 The solution curves in the FH-plane are closed curves that circulate around the equilibrium point.

9.5.8 They both oscillate cyclically, with the prey population peaking slightly before the predator population.

9.5.9



9.5.10



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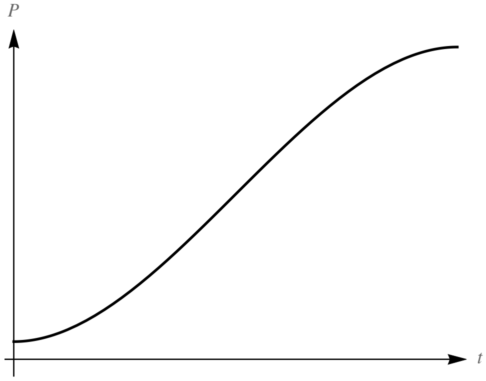
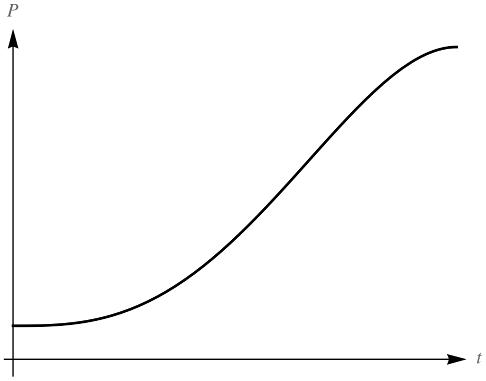
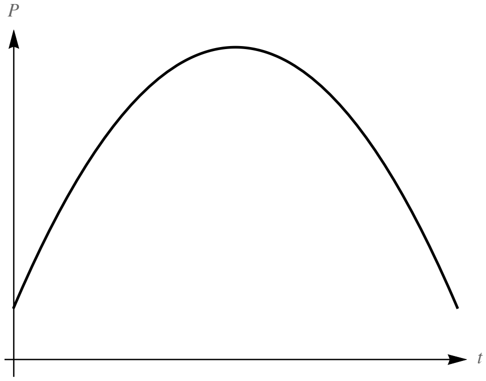


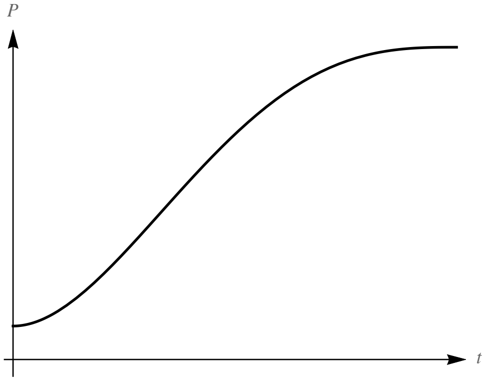
9.5.12



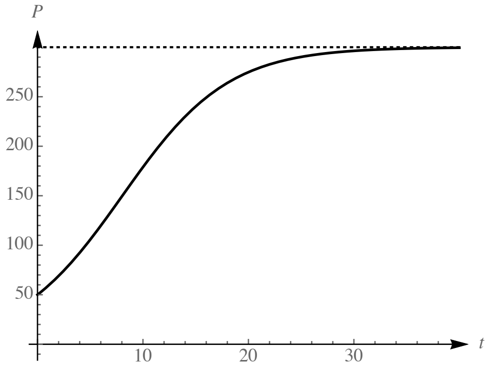
9.5.13



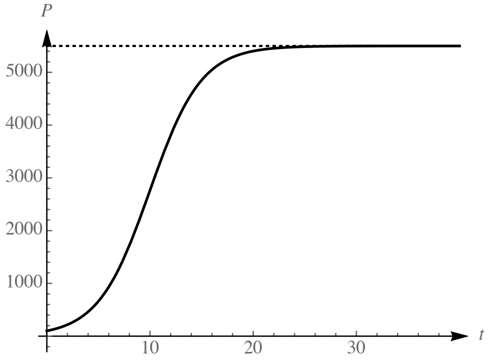
9.5.14



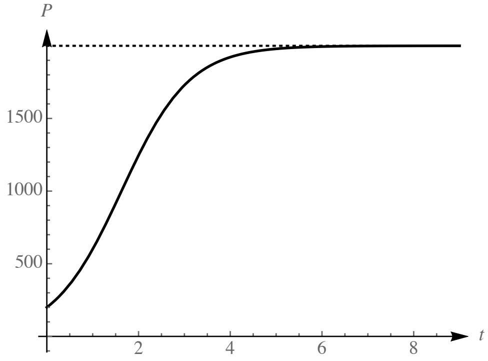
9.5.15 We must solve $P'(t) = 0.2P\left(1 - \frac{P}{300}\right)$. We have $\int \frac{300P'(t)}{P(300-P)}\,dt = \int 0.2\,dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{300 - P(t)}\right) dt = \int 0.2\,dt$. Thus, $\ln\left(\frac{P(t)}{300 - P(t)}\right) = 0.2t + C$. Taking the exponential of both sides and reciprocating gives $\frac{300 - P(t)}{P(t)} = Ae^{-0.2t}$. Because $P(0) = 50$, we have $A = 5$. Thus $\frac{300}{P(t)} = 5e^{-0.2t} + 1$, so $P(t) = \frac{300}{5e^{-0.2t} + 1}$ for $t \geq 0$.



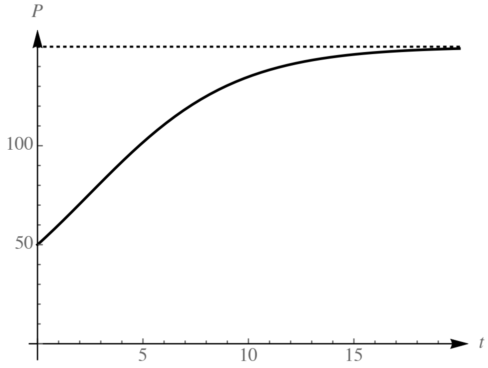
9.5.16 We must solve $P'(t) = 0.4P\left(1 - \frac{P}{5500}\right)$. We have $\int \frac{5500P'(t)}{P(5500-P)}\,dt = \int 0.4\,dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{5500 - P(t)}\right) dt = \int 0.4\,dt$. Thus, $\ln\left(\frac{P(t)}{5500 - P(t)}\right) = 0.4t + C$. Taking the exponential of both sides and reciprocating gives $\frac{5500 - P(t)}{P(t)} = Ae^{-0.4t}$. Because $P(0) = 100$, we have $A = 54$. Thus $\frac{5500}{P(t)} = 54e^{-0.4t} + 1$, so $P(t) = \frac{5500}{54e^{-0.4t} + 1}$ for $t \geq 0$.



9.5.17 We must solve $P'(t) = rP\left(1 - \frac{P}{2000}\right)$. We have $\int \frac{2000P'(t)}{P(2000 - P)}\,dt = \int r\,dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{2000 - P(t)}\right)dt = \int r\,dt$. Thus, $\ln\left(\frac{P(t)}{2000 - P(t)}\right) = rt + C$. Taking the exponential of both sides and reciprocating gives $\frac{2000 - P(t)}{P(t)} = Ae^{-rt}$. Because $P(0) = 200$, we have $A = 9$. Thus $\frac{2000}{P(t)} = 9e^{-rt} + 1$, so $P(t) = \frac{2000}{9e^{-rt} + 1}$. Now because $P(1) = 600$, we have $r = \ln(27/7)$. So $P(t) = \frac{2000}{9e^{\ln(7/27)t} + 1}$ for $t \geq 0$.



9.5.18 We must solve $P'(t) = rP\left(1 - \frac{P}{150}\right)$. We have $\int \frac{150P'(t)}{P(150 - P)}\,dt = \int r\,dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{150 - P(t)}\right)dt = \int r\,dt$. Thus, $\ln\left(\frac{P(t)}{150 - P(t)}\right) = rt + C$. Taking the exponential of both sides and reciprocating gives $\frac{150 - P(t)}{P(t)} = Ae^{-rt}$. Because $P(0) = 50$, we have $A = 2$. Thus $\frac{150}{P(t)} = 2e^{-rt} + 1$, so $P(t) = \frac{150}{2e^{-rt} + 1}$. Now because $P(1) = 60$, we have $r = \ln(4/3)$. So $P(t) = \frac{150}{2e^{\ln(3/4)t} + 1}$ for $t \geq 0$.



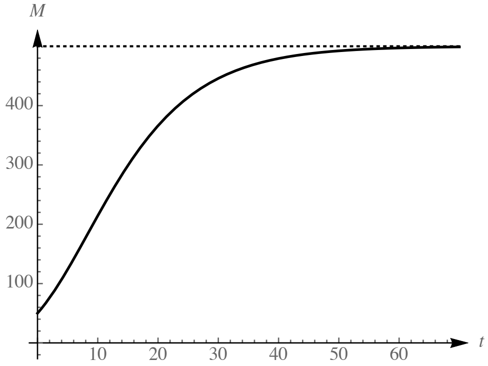
9.5.19 We have $\dfrac{M'(t)}{M\ln(M/K)} = -r$, so $\displaystyle\int \frac{M'(t)}{M\ln(M/K)}\,dt = \int -r\,dt$.

Note that $\dfrac{d}{dt}\ln(M/K) = (K/M)(1/K)M'(t) = M'(t)/M$. Thus integrating gives $\ln|\ln(M/K)| = -rt + C$, and thus $\ln(M/K) = Ae^{-rt}$. Because $M(0) = M_0$, we have $A = \ln(M_0/K)$.

Thus $M(t) = K(\exp(\ln(M_0/K)e^{-rt})) = K\left(\dfrac{M_0}{K}\right)^{e^{-rt}}$ for $t \geq 0$.

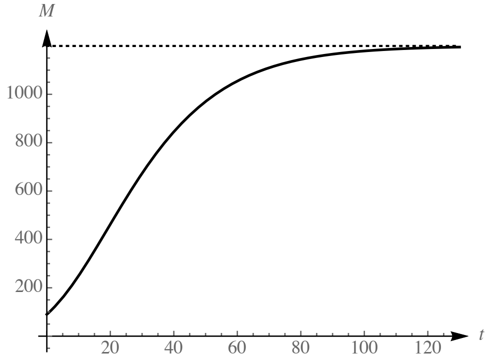
9.5.20 With $r = 0.1$, $K = 500$, and $M_0 = 50$, we have

$$M(t) = 500\left(\frac{1}{10}\right)^{e^{-0.1t}} \quad \text{for } t \geq 0.$$



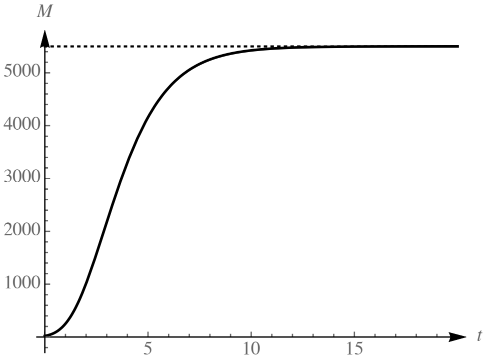
9.5.21 With $r = 0.05$, $K = 1200$, and $M_0 = 90$, we have

$$M(t) = 1200\left(\frac{3}{40}\right)^{e^{-0.05t}} \quad \text{for } t \geq 0.$$



9.5.22 With $r = 0.6$, $K = 5500$, and $M_0 = 20$, we have

$$M(t) = 5500\left(\frac{1}{275}\right)^{e^{-0.6t}} \quad \text{for } t \geq 0.$$



9.5.23

a. The initial mass of copper sulfate is $m_0 = 0$. We have $m'(t) = -\dfrac{4}{500}m(t) + 20 \cdot 4 = -\dfrac{1}{125}m(t) + 80$.

b. This is an equation of the form $m'(t) = km+b$, so the solution is $m(t) = Ce^{kt} - \dfrac{b}{k} = Ce^{-0.008t} + 10,000$. Because $m(0) = 0$, we have $C = -10,000$. Thus, $m(t) = -10,000e^{-0.008t} + 10,000$ for $t \geq 0$.

9.5.24

a. The initial mass of the salt is $m_0 = 3000$. We have $m'(t) = -\dfrac{3}{1500}m(t) + 20 \cdot 3 = -\dfrac{1}{500}m(t) + 60$.

b. This is an equation of the form $m'(t) = km+b$, so the solution is $m(t) = Ce^{kt} - \dfrac{b}{k} = Ce^{-0.002t} + 30,000$. Because $m(0) = 3000$, we have $C = -27,000$. Thus, $m(t) = -27,000e^{-0.002t} + 30,000$ for $t \geq 0$.

9.5.25

a. The initial mass of the sugar is $m_0 = 2000 \cdot 40 = 80,000$. We have $m'(t) = -\dfrac{10}{2000}m(t) + 10 \cdot 10 = -\dfrac{1}{200}m(t) + 100$.

b. This is an equation of the form $m'(t) = km+b$, so the solution is $m(t) = Ce^{kt} - \dfrac{b}{k} = Ce^{-0.005t} + 20,000$. Because $m(0) = 80000$, we have $C = 60,000$. Thus, $m(t) = 60,000e^{-0.005t} + 20,000$ for $t \geq 0$.

9.5.26

a. The initial mass of the pollutant is $m_0 = 1000000 \cdot 20 = 20,000,000$. We have $m'(t) = -\dfrac{1200}{1000000}m(t) + 0 \cdot 1200 = -\dfrac{3}{2500}m(t)$.

b. This is an equation of the form $m'(t) = km+b$, so the solution is $m(t) = Ce^{kt} - \dfrac{b}{k} = Ce^{-0.0012t}$. Because $m(0) = 20,000,000$, we have $C = 20,000,000$. Thus, $m(t) = 20,000,000e^{-0.0012t}$ for $t \geq 0$. The pond will have a mass of 10 percent of the initial value when $0.1 = e^{-0.0012t}$, so $t = -\ln(0.1)/0.0012 \approx 1919$ hours.

9.5.27

a. x is the predator and y is the prey.

b. $-3x + 6xy = 0$ when $x = 0$ or $y = 1/2$. $y - 4xy = 0$ when $y = 0$ or $x = 1/4$.

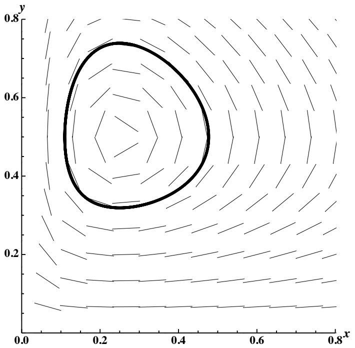
   The desired lines are $x = 1/4$ and $y = 1/2$.

c. The equilibrium points are where both equations are zero simultaneously, so they are $(0, 0)$ and $(1/4, 1/2)$.

d. Note that $x'(t) = 3x(2y - 1)$ and $y'(t) = y(1 - 4x)$.

   - For $0 < x < 1/4$ and $0 < y < 1/2$, we have $x' < 0$ and $y' > 0$.
   - For $0 < x < 1/4$ and $y > 1/2$, we have $x' > 0$ and $y' > 0$.
   - For $x > 1/4$ and $0 < y < 1/2$, we have $x' < 0$ and $y' < 0$.
   - For $x > 1/4$ and $y > 1/2$, we have $x' > 0$ and $y' < 0$.

e. The direction of the solution is clockwise.



9.5.28

a. x is the prey and y is the predator.

b. $2x - 4xy = 0$ when $x = 0$ or $y = 1/2$. $-y + 2xy = 0$ when $y = 0$ or $x = 1/2$.

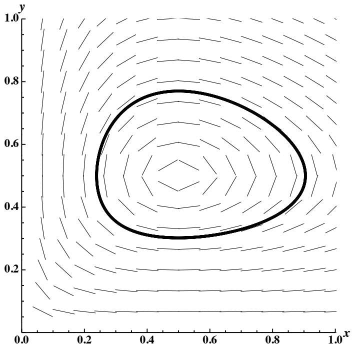
   The desired lines are $x = 1/2$ and $y = 1/2$.

c. The equilibrium points are where both equations are zero simultaneously, so they are $(0, 0)$ and $(1/2, 1/2)$.

d. Note that $x'(t) = 2x(1 - 2y)$ and $y'(t) = y(2x - 1)$.

   - For $0 < x < 1/2$ and $0 < y < 1/2$, we have $x' > 0$ and $y' < 0$.
   - For $0 < x < 1/2$ and $y > 1/2$, we have $x' < 0$ and $y' < 0$.
   - For $x > 1/2$ and $0 < y < 1/2$, we have $x' > 0$ and $y' > 0$.
   - For $x > 1/2$ and $y > 1/2$, we have $x' < 0$ and $y' > 0$.

e. The direction of the solution is counterclockwise.



9.5.29

a. x is the predator and y is the prey.

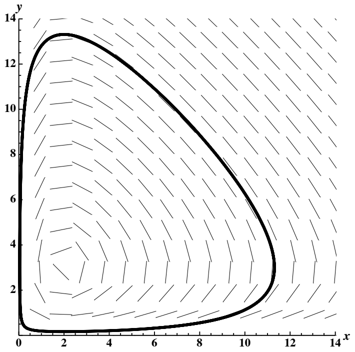
b. $-3x + xy = 0$ when $x = 0$ or $y = 3$. $2y - xy = 0$ when $y = 0$ or $x = 2$.
The desired lines are $x = 2$ and $y = 3$.

c. The equilibrium points are where both equations are zero simultaneously, so they are $(0, 0)$ and $(2, 3)$.

d. Note that $x'(t) = x(y - 3)$ and $y'(t) = y(2 - x)$.

  - For $0 < x < 2$ and $0 < y < 3$, we have $x' < 0$ and $y' > 0$.
  - For $0 < x < 2$ and $y > 3$, we have $x' > 0$ and $y' > 0$.
  - For $x > 2$ and $0 < y < 3$, we have $x' < 0$ and $y' < 0$.
  - For $x > 2$ and $y > 3$, we have $x' > 0$ and $y' < 0$.

e. The direction of the solution is clockwise.



9.5.30

a. x is the prey and y is the predator.

b. $2x - xy = 0$ when $x = 0$ or $y = 2$. $-y + xy = 0$ when $y = 0$ or $x = 1$.
The desired lines are $x = 1$ and $y = 2$.

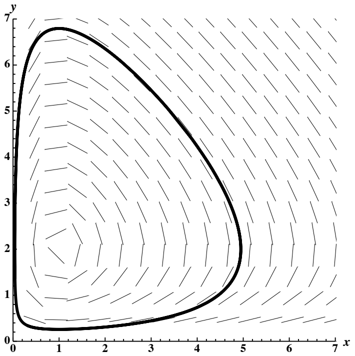
c. The equilibrium points are where both equations are zero simultaneously, so they are $(0, 0)$ and $(1, 2)$.

d. Note that $x'(t) = x(2 - y)$ and $y'(t) = y(x - 1)$.

  - For $0 < x < 1$ and $0 < y < 2$, we have $x' > 0$ and $y' < 0$.
  - For $0 < x < 1$ and $y > 2$, we have $x' < 0$ and $y' < 0$.
  - For $x > 1$ and $0 < y < 2$, we have $x' > 0$ and $y' > 0$.

  - For $x > 1$ and $y > 2$, we have $x' < 0$ and $y' > 0$.

e. The direction of the solution is counterclockwise.



9.5.31

a. True. The growth rate function is the derivative, so where it is positive, the population is increasing.

b. True. In the limit, the solution in the tank is the same as the solution being poured in.

c. True. In the absence of predators, we assume that the prey population increases exponentially.

9.5.32

a. $f'(P) = r\left(1 - \frac{P}{K}\right) + rP\left(\frac{-1}{K}\right) = r\left(1 - \frac{2P}{K}\right)$. This is zero when $P = \frac{K}{2}$, and an analysis of the sign of f' shows that $f' > 0$ for $P < \frac{K}{2}$ and $f' < 0$ when $P > \frac{K}{2}$. Thus there is a maximum of $f(K/2) = r(K/2)(1/2) = rK/4$ at $P = K/2$.

b. $f'(M) = -r\ln(M/K) - rM(K/M)(1/K) = -r(\ln(M/K) + 1)$. This is zero when $\ln(M/K) = -1$, or $M/K = e^{-1}$, which occurs for $M = \frac{K}{e}$. Note that $f(K/e) = -r(K/e)\ln(1/e) = rK/e$. An analysis of the sign of f' shows that $f' > 0$ for $M < \frac{K}{e}$ and $f' < 0$ when $M > \frac{K}{e}$. Thus there is a maximum at $M = \frac{K}{e}$.

9.5.33 We must solve $P'(t) = rP\left(1 - \frac{P}{K}\right)$. We have $\int \frac{KP'(t)}{P(K-P)}\,dt = \int r\,dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{K - P(t)}\right)dt = \int r\,dt$. Thus, $\ln\left(\frac{P(t)}{K-P(t)}\right) = rt + C$. Taking the exponential of both sides and reciprocating gives $\frac{K-P(t)}{P(t)} = Ae^{-rt}$. Because $P(0) = P_0$, we have $A = \frac{K-P_0}{P_0}$. Thus $\frac{K}{P(t)} = (\frac{K-P_0}{P_0})e^{-rt} + 1$, so $P(t) = \frac{K}{\frac{K-P_0}{P_0}e^{-rt} + 1} = \frac{KP_0}{(K-P_0)e^{-rt} + P_0}$.

9.5.34 $M(0) = K\left(\frac{M_0}{K}\right)^{e^0} = K\left(\frac{M_0}{K}\right) = M_0$. Because $\lim_{t\to\infty} e^{-rt} = 0$, we have

$$\lim_{t\to\infty} M(t) = K\left(\frac{M_0}{K}\right)^0 = K.$$

9.5.35

a. Note that this differential equation is first-order linear with $k = -\dfrac{R}{V}$ and $b = C_iR$, so the solution is $m(t) = Ce^{(-R/V)t} - \dfrac{C_iR}{-R/V} = Ce^{(-R/V)t} + C_iV$. Because $m(0) = m_0$, we have $m_0 = C + C_iV$, so $C = m_0 - C_iV$. Thus, the solution is $m(t) = (m_0 - C_iV)e^{-Rt/V} + C_iV$.

b. $m(0) = (m_0 - C_iV) + C_iV = m_0$.

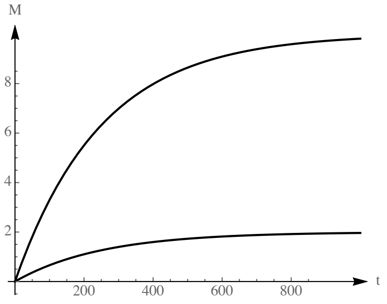
c. Note that $\lim\limits_{t\to\infty} e^{-Rt/V} = 0$, so $\lim\limits_{t\to\infty} m(t) = C_iV$. In the limit, the solution in the tank is the incoming solution, so the amount of material in the tank is the amount per unit volume times the volume of the tank.

d. Increasing R causes the graph to approach the asymptote more quickly.

9.5.36

a. The initial mass of the drug in the body is 0. We have $m'(t) = -\dfrac{0.06}{4}m(t)+(0.06)(500) = -0.015m(t)+30$.

b. This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \dfrac{b}{k} = Ce^{-0.015t} + 2000$. Because $m(0) = 0$, we have $C = -2000$. Thus, $m(t) = -2000e^{-0.015t} + 2000$.



c. Because $\lim\limits_{t\to\infty} e^{-.015t} = 0$, $\lim\limits_{t\to\infty} m(t) = 2000$.

d. The drug mass reaches 1800 mg when $1800 = -2000e^{-.015t} + 2000$, so when $0.1 = e^{-0.015t}$, so $t = -\ln(0.1)/0.015 \approx 153.5$ minutes.

9.5.37

a. We can write the equation as $I'(t) = -\dfrac{1}{RC}I(t)$. This is a first-order linear equation with $k = -\dfrac{1}{RC}$ and $b = 0$. Thus, the solution is $I(t) = Ce^{kt} - \dfrac{b}{k} = Ae^{-t/RC}$. Because $I(0) = \dfrac{V}{R}$, we have $A = \dfrac{V}{R}$. Thus, $I(t) = \dfrac{V}{R}e^{-t/RC}$. The current decays exponentially with decay constant $-\dfrac{1}{RC}$.

b. We can write the equation as $Q'(t) = -\dfrac{1}{RC}Q + \dfrac{V}{R}$. This is a first-order linear equation with $k = -\dfrac{1}{RC}$ and $b = \dfrac{V}{R}$. Thus, the solution is $Q(t) = Ce^{kt} - \dfrac{b}{k} = Ae^{-t/RC} - \dfrac{(V/R)}{(-1/RC)} = Ae^{-t/RC} + VC$. Because $Q(0) = 0$, we have $A = -VC$. Thus,

$$Q(t) = (-VC)\,e^{-t/RC} + VC.$$

In the long run, the charge has limit VC.

9.5.38

a. $P(t) = P_0e^{rt}$. $P(0) = P_0 = 296$ and $P(10) = 321$, so $321 = 296e^{10r}$, so $r = \frac{1}{10}\ln\left(\frac{321}{296}\right) \approx 0.0081$.

b. $P(t) = \frac{K}{1 + Ce^{-0.0081}t}$. Because $P(0) = 296 = \frac{K}{1+C}$, we have $1 + C = \frac{K}{296}$, so $C = \frac{K-296}{296}$. Then we have

$$P(t) = \frac{K}{1 + \left(\frac{K-296}{296}\right)e^{-0.0081t}} = \frac{296K}{296 + (K-296)e^{-0.0081t}}.$$

Because $P(45) = 398$, we have

$$398 = \frac{296K}{296 + (K-296)e^{-0.0081t}},$$

and solving for K gives $K \approx 1838.65$.

c. It will reach 95 percent of 1838.65 when $\frac{296(1838.65)}{296 + (1838.65 - 296)e^{-0.0081t}} = 0.95(1838.65)$. Solving for t gives $t \approx 567.3$ years.

d. Solving $410 = \frac{296}{296 + (K-296)e^{-0.0081(45)}}$ for K gives $K \approx 3289.9$.

e. Solving $380 = \frac{296}{296 + (K-296)e^{-0.0081(45)}}$ for K gives $K \approx 1071.22$.

f. Small differences in the 35-year projection result in large differences in the estimated carrying capacity.

9.5.39

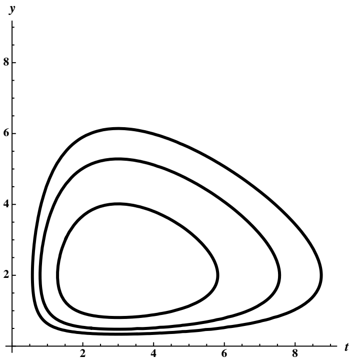
a. We have $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{cy - dxy}{-ax + bxy} = \frac{y(c - dx)}{x(by - a)}$. Thus we can write $\frac{by - a}{y}\frac{dy}{dx} = \frac{c - dx}{x}$, or

$$\left(b - \frac{a}{y}\right)\frac{dy}{dx} = \frac{c}{x} - d.$$

Thus, the equation is separable.

b. We have $\int\left(b - \frac{a}{y}\right)\frac{dy}{dx}\,dx = \int\left(\frac{c}{x} - d\right)dx$, so $by - a\ln y = c\ln x - dx + K$, so $dx + by = c\ln x + a\ln y + K$. Taking the exponential function of both sides gives $e^{dx+by} = Cx^cy^a$ for an arbitrary constant C.

c.



Chapter Nine Review

1

a. False. It is a first-order, linear differential equation, but it isn't separable.

b. False. It is a first-order, separable differential equation, but it isn't linear.

c. True. Note that $y' = 1 - t^{-2}$, so $ty' = t - t^{-1}$. Thus, $ty' + y = t - t^{-1} + (t + t^{-1}) = 2t$. Also, $y(1) = 2$.

d. True.

e. False. In general, Euler's method gives approximate solutions.

2 $y'(t) = -3y$, so $\int \frac{y'(t)}{y}\,dt = \int (-3)\,dt$. Thus, $\ln|y| = -3t + K$, and exponentiating both sides gives $y = Ce^{-3t}$.

3 $y'(t) = -2y + 6$, so $\int \frac{y'(t)}{-2y+6}\,dt = \int 1\,dt$, and therefore $-\frac{1}{2}\ln|-2y+6| = t + K$. It follows that $|-2y+6| = Ae^{-2t}$. We can write $-2y + 6 = \pm Ae^{-2t}$, so $y(t) = Ce^{-2t} + 3$.

4 $p'(x) = 4p + 8 = 4(p+2)$, so $\int \frac{p'(x)}{p+2}\,dx = \int 4\,dx$, and thus $\ln|p+2| = 4x + K$. Thus $|p+2| = Ae^{4x}$, so $p + 2 = \pm Ae^{4x}$, and thus $p(x) = Ce^{4x} - 2$.

5 $y'(t) = 2ty$, so $\int \frac{y'(t)}{y}\,dt = \int 2t\,dt$, and therefore, $\ln|y| = t^2 + K$. It follows that $y = Ce^{t^2}$.

6 $y'(t) = \frac{\sqrt{y}}{\sqrt{t}}$, so $\int y^{-1/2}y'(t)\,dt = \int t^{-1/2}\,dt$. Integrating gives $2\sqrt{y} = 2\sqrt{t} + K$, so $\sqrt{y} = \sqrt{t} + C$, so $y(t) = (\sqrt{t} + C)^2$.

7 $\frac{y'(t)}{y} = \frac{1}{t^2+1}$. Integrating both sides with respect to t gives $\ln|y| = \tan^{-1}(t) + K$, so $y = Ce^{\tan^{-1}(t)}$.

8 $2yy'(x) = \sin x$, so $y^2 = -\cos x + C$, and thus $y = \pm\sqrt{C - \cos x}$.

9 $\int \frac{y'(t)}{y^2+1}\,dt = \int (2t+1)\,dt$, so $\tan^{-1}(y) = t^2 + t + C$, and thus $y = \tan(t^2 + t + C)$.

10 $\frac{z'(t)}{z} = \frac{t}{t^2+1}$. Integrating both sides with respect to t gives $\ln|z| = \frac{1}{2}\ln|t^2+1| + K$. Thus $z = C\sqrt{t^2+1}$.

11 $\int y'(t)\,dt = \int (2t + \cos t)\,dt$, so $y(t) = t^2 + \sin t + C$. Because $y(0) = 1$, we have $1 = 0 + 0 + C$, so $C = 1$. Thus, $y(t) = t^2 + \sin t + 1$.

12 $y'(t) = -3(y-3)$, so $\int \frac{y'(t)}{y-3}\,dt = \int (-3)\,dt$, so $\ln|y-3| = -3t + K$, and thus $y - 3 = Ce^{-3t}$. Because $y(0) = 4$, we have $C = 1$. Thus $y = e^{-3t} + 3$.

13 $\int \frac{Q'(t)}{Q-8}\,dt = \int 1\,dt$, so $\ln|Q-8| = t + K$. Thus, $Q - 8 = Ce^t$. Because $Q(1) = 0$, we have $-8 = Ce$, so $C = -\frac{8}{e}$. Thus, $Q = -8e^{t-1} + 8$.

14 $yy'(x) = x$, so $\int yy'(x)\,dx = \int x\,dx$, so $y^2/2 = x^2/2 + C$. Because $y(2) = 4$, we have $8 = 2 + C$, so $C = 6$. Thus, $y^2 = x^2 + 12$, and $y = \sqrt{x^2+12}$.

15 $u^{-1/3}u'(t) = t^{-1/3}$. Thus $\int u^{-1/3}u'(t)\,dt = \int t^{-1/3}\,dt$. We have $\frac{3}{2}u^{2/3} = \frac{3}{2}t^{2/3} + C$. Because $u(1) = 8$, we have $6 = \frac{3}{2} + C$, so $C = \frac{9}{2}$. Thus, $u^{2/3} = t^{2/3} + 3$, and $u = (t^{2/3} + 3)^{3/2}$ for $t > 0$.

16 $\int (\sin y)y'(x)\,dx = \int 4x\,dx$, so $-\cos y = 2x^2 + C$. Because $y(0) = \pi/2$, we have $0 = 0 + C$, so $C = 0$. Thus $\cos y = -2x^2$, and $y = \cos^{-1}(-2x^2)$.

17 $\frac{s'(t)}{s} = \frac{1}{t(t^2+1)}$, so $\int \frac{ds}{s} = \int \frac{1}{t(t^2+1)}\,dt$, which can be written as $\int \frac{ds}{s} = \int \left(\frac{1}{t} - \frac{t}{t^2+1}\right)\,dt$. Then we have

$$\ln|s| = \ln|t| - \frac{1}{2}\ln(1+t^2) + C,$$

Because $s(1) = 1$, we have $C = \ln\sqrt{2}$. Then $\ln|s| = \ln\left(\frac{\sqrt{2}|t|}{\sqrt{1+t^2}}\right)$, and

$$|s| = \frac{\sqrt{2}|t|}{\sqrt{1+t^2}}.$$

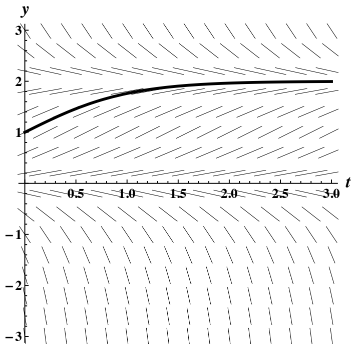
Because $s(1) = 1$, we have

$$s = \frac{\sqrt{2}t}{\sqrt{1+t^2}}.$$

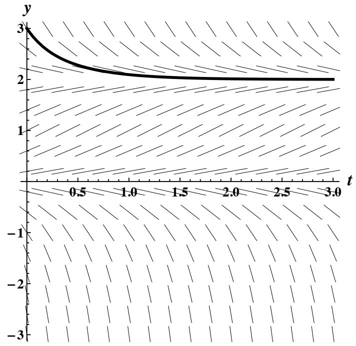
18 $\int (\sec^2\theta)\theta'(x)\,dx = \int 4x\,dx$. Thus, $\tan\theta = 2x^2 + C$. Because $\theta(0) = \pi/4$, we have $1 = 0 + C$, so $C = 1$. It follows that $\theta = \tan^{-1}(2x^2 + 1)$.

19

a.



b.



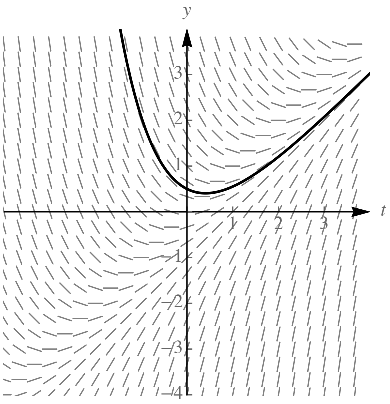
c. The solutions are increasing when $0 < A < 2$.

d. The solutions are decreasing when $A < 0$ or when $A > 2$.

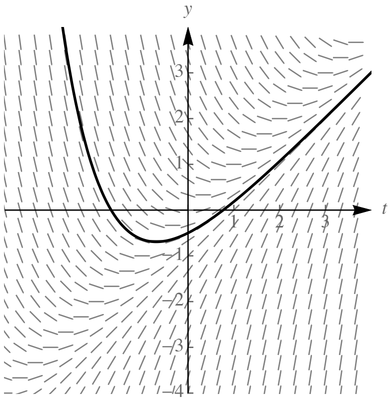
e. The equilibrium solutions are $y = 0$ and $y = 2$.

20

a.



b.



c. The solutions are increasing for $t > y$ and decreasing for $t < y$.

d. $y = t$.

21

a. $u_0 = 1$. $u_1 = 1 + f(0, 1)(0.1) = 1 + 1/20 = 1.05$. Also, $u_2 = 1.05 + f(0.1, 1.05)(0.1) = 1.05 + 0.047619 = 1.09762$. Thus, $y(0.1) \approx 1.05$ and $y(0.2) \approx 1.09762$.

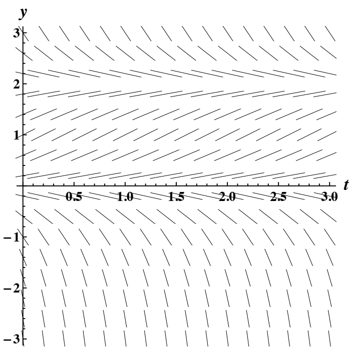
b. $u_0 = 1$. $u_1 = 1 + f(0, 1)(0.05) = 1 + 0.025 = 1.025$. Also, $u_2 = 1.025 + f(0.05, 1.025)(0.05) = 1.04939$. $u_3 = 1.04939 + f(0.1, 1.04939)(.05) = 1.07321$. $u_4 = 1.07321 + f(0.15, 1.07321)(.05) = 1.0961$. Thus, $y(0.1) \approx 1.04939$ and $y(0.2) \approx 1.09651$.

c. For part a, we have $\dfrac{|1.09762 - \sqrt{1.2}|}{\sqrt{1.2}} = 0.00198539$.

For part b, we have $\dfrac{|109651 - \sqrt{1.2}|}{\sqrt{1.2}} = 0.000972103$. Part b is more accurate.

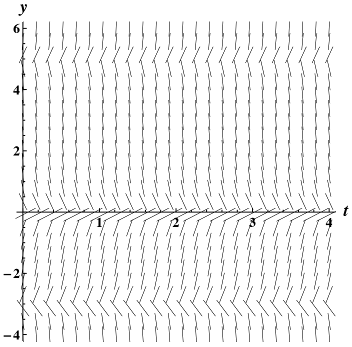
22

The equilibrium solutions are $y = 0$ (unstable) and $y = 2$ (stable).



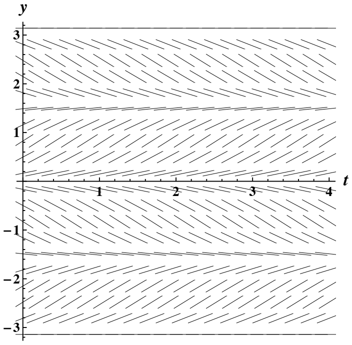
23

The equilibrium solutions are $y = -3$ (unstable) and $y = 0$ (stable), and $y = 5$ (unstable).



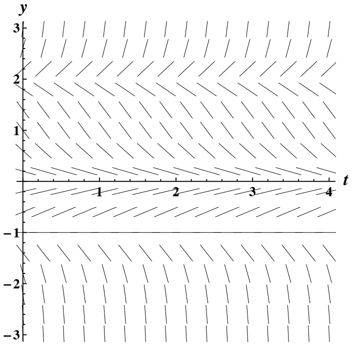
24

The equilibrium solutions are $y = -\pi/2$ (stable) and $y = 0$ (unstable), and $y = \pi/2$ (stable).



25

The equilibrium solutions are $y = -1$ (unstable) and $y = 0$ (stable), and $y = 2$ (unstable).



26

a. The equilibrium solutions occur where $P'(t) = 0$, so they are $P = 0$ and $P = 1200$.

b. We must solve $P'(t) = 0.2P\left(1 - \frac{P}{1200}\right)$. We have $\displaystyle\int \frac{1200P'(t)}{P(1200-P)}\,dt = \int 0.2\,dt$, which can be written as $\displaystyle\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{1200-P(t)}\right)dt = \int 0.2\,dt$. Thus, $\ln\left(\dfrac{P(t)}{1200-P(t)}\right) = 0.2t + C$. Taking the exponential of both sides and reciprocating gives $\frac{1200-P(t)}{P(t)} = Ae^{-0.2t}$. Because $P(0) = 50$, we have $A = 23$. Thus $\dfrac{1200}{P(t)} = 23e^{-0.2t} + 1$, so $P(t) = \dfrac{1200}{23e^{-0.2t}+1} = \dfrac{60000}{1150e^{-0.2t}+50}$.

c. The carrying capacity is $\lim_{t\to\infty} \dfrac{1200}{23e^{-0.2t}+1} = 1200$.

d. $P''(t) = 0.2(1-(P/1200)) + 0.2P(-1/1200) = 0.2 - 0.2(P/600)$. This is zero when $P = 600$. Also, note that $P'''(600) < 0$, so there is a maximum for P' at 600 by the Second Derivative Test.

27

a. The logistic equation has solution $P(t) = \dfrac{KP_0}{(K-P_0)e^{-rt}+P_0} = \dfrac{32000}{1580e^{-rt}+20}$. We are seeking r so that $80 = \dfrac{32000}{1580e^{-20r}+20}$. Solving for r gives $r = \dfrac{\ln(79/19)}{20} \approx 0.0713$.

b. The solution is $P(t) = \dfrac{32000}{1580e^{-0.0713t}+20} = \dfrac{1600}{79e^{-0.0713t}+1}$ for $t \geq 0$.

c. We are seeking t so that $800 = \dfrac{1600}{79e^{-0.0713t}+1}$. Solving for t gives $t \approx 61$ hours.

28

a. Assuming $P(t) = 435e^{kt}$ and $P(5) = 435e^{5k} = 487$, we have $k = \ln(487/435)/5 \approx 0.0226$.

b. The logistic equation has solution $P(t) = \dfrac{KP_0}{(K-P_0)e^{-rt}+P_0} = \dfrac{K\cdot 435}{(K-435)e^{-0.0226t}+435}$. We are seeking K so that $1570 = \dfrac{K\cdot 435}{(K-435)e^{-0.0226\cdot 90}+435}$. Solving for K gives $K \approx 2585$.

c. $P(t) = \dfrac{2585\cdot 435}{2150e^{-0.0226t}+435}$, $t \geq 0$.

d. Solving $2000 = \dfrac{2585\cdot 435}{2150e^{-0.0226t}+435}$ for t gives $t \approx 125$, which would be in about 2085.

e. It is very hard to predict the actual carrying capacity. Perhaps the estimate for the population in 2050 isn't accurate. Also, small changes in these kinds of estimates can lead to large changes in the actual population sizes.

29

a. The initial mass of the sugar is $m_0 = 0$. We have $m'(t) = -\dfrac{0.5}{100}m(t) + 100\cdot 0.5 = -\dfrac{1}{200}m(t) + 10$. This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \dfrac{b}{k} = Ce^{-0.005t} + 2000$. Because $m(0) = 0$, we have $C = -2000$. Thus, $m(t) = -2000e^{-0.005t} + 2000$.

b. The steady-state mass is $\lim_{t\to\infty} m(t) = 2000$.

c. The mass reaches $0.95(2000)$ when $0.05 = e^{-0.005t}$, or $t = \ln(0.05)/(-0.005) \approx 599$ minutes.

30

a. $T(t) = (80-25)e^{-kt} + 25$. Because $T(5) = 60$, we have $60 = 55e^{-5k} + 25$, so $35/55 = e^{-5k}$, and thus $k = -\ln(35/55)/5 \approx 0.0904$.

b. $T(t) = 55e^{-0.0904t} + 25$.

c. The coffee reaches 50 degrees when $50 = 55e^{-0.0904t} + 25$, or when $e^{-0.0904t} = 25/55$. Thus $t = -\ln(5/11)/0.0904 \approx 8.7$ minutes.

31

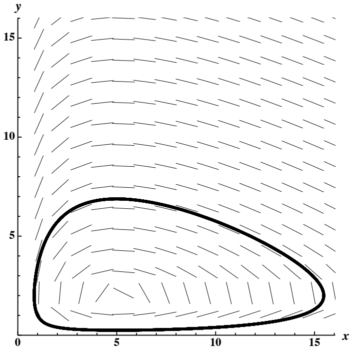
a. x represents the predator and y represents the prey.

b. $x'(t) = 2x(-2+y)$ is zero when $x = 0$ and when $y = 2$. $y'(t) = y(5-x)$ is zero when $y = 0$ and when $x = 5$.

c. The equilibrium points occur when both equations are zero simultaneously, which occurs at $(0, 0)$ and $(5, 2)$.

d.
   - For $0 < x < 5$ and $0 < y < 2$, we have $x' < 0$ and $y' > 0$.
   - For $0 < x < 5$ and $y > 2$, we have $x' > 0$ and $y' > 0$.
   - For $x > 5$ and $0 < y < 2$, we have $x' < 0$ and $y' < 0$.
   - For $x > 5$ and $y > 2$, we have $x' > 0$ and $y' < 0$.

e. The solution evolves in the clockwise direction.



32

a. $t^2y'(t) + (2t)y(t) = \dfrac{d}{dt}(t^2y(t))$.

b. $\displaystyle\int (t^2y'(t) + (2t)y(t))\,dt = \int e^{-t}\,dt$, so $t^2y(t) = -e^{-t} + C$. Thus, $y(t) = \dfrac{-e^{-t} + C}{t^2}$, $t > 0$.

c. If $y(1) = 0$, then $0 = C - 1/e$, so $C = 1/e$. Thus, $y(t) = \dfrac{-e^{-t} + 1/e}{t^2}$, $t > 0$.

33

a. If $y(t) = t^p$ is a solution, then $p(p-1)t^p + 2pt^p - 12t^p = 0$, so $p^2 - p + 2p - 12 = 0$, so $p^2 + p - 12 = (p-3)(p+4) = 0$. Thus, $p_1 = 3$ and $p_2 = -4$.

b. If $y(t) = C_1t^3 + C_2t^{-4}$, and if $y(1) = 0$, then $C_1 + C_2 = 0$. Note that $y'(t) = 3C_1t^2 - 4C_2t^{-5}$, so if $y'(1) = 7$, we have $7 = 3c_1 - 4C_2$. Solving the system of linear equations in C_1 and C_2 gives $C_1 = 1$ and $C_2 = -1$. Thus the solution we seek is $y(t) = t^3 - t^{-4}$, $t > 0$.

Chapter 10

Sequences and Infinite Series

10.1 An Overview

10.1.1 A *sequence* is an ordered list of numbers $a_1, a_2, a_3, \dots,$ often written $\{a_1, a_2, \dots\}$ or $\{a_n\}$. For example, the natural numbers $\{1, 2, 3, ...\}$ are a sequence where $a_n = n$ for every n.

10.1.2 $a_1 = \frac{1}{1} = 1$; $a_2 = \frac{1}{2}$; $a_3 = \frac{1}{3}$; $a_4 = \frac{1}{4}$; $a_5 = \frac{1}{5}$.

10.1.3 $a_1 = 1$ (given); $a_2 = 1 \cdot a_1 = 1$; $a_3 = 2 \cdot a_2 = 2$; $a_4 = 3 \cdot a_3 = 6$; $a_5 = 4 \cdot a_4 = 24$.

10.1.4 $a_n = (-1)^n$ for $n = 1, 2, 3, \dots.$ Or, we could write $a_n = (-1)^{n+1}$ for $n = 0, 1, 2 \dots.$ Other answers are possible.

10.1.5 $a_n = (-1)^{n+1}n$ for $n = 1, 2, 3, \dots.$ Or we could write $a_n = (-1)^n(n+1)$ for $n = 0, 1, 2, \dots.$ Other answer are possible.

10.1.6 The sequence has limit π.

10.1.7 The sequence has limit e.

10.1.8 No. The terms of the sequence do not approach a unique number.

10.1.9 $S_1 = \sum_{k=1}^{1} k^2 = 1$; $S_2 = \sum_{k=1}^{2} k^2 = 1+4 = 5$; $S_3 = \sum_{k=1}^{3} k^2 = 1+4+9 = 14$; $S_4 = \sum_{k=1}^{4} k^2 = 1+4+9+16 = 30$.

10.1.10 $S_1 = \sum_{k=1}^{1} k = 1$; $S_2 = \sum_{k=1}^{2} k = 1+2 = 3$; $S_3 = \sum_{k=1}^{3} k = 1+2+3 = 6$; $S_4 = \sum_{k=1}^{4} k = 1+2+3+4 = 10$.

10.1.11 One possible answer is $\sum_{k=1}^{\infty} 10$.

10.1.12 $S_1 = \sum_{k=1}^{1} \frac{1}{k} = \frac{1}{1} = 1$; $S_2 = \sum_{k=1}^{2} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$; $S_3 = \sum_{k=1}^{3} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$; $S_4 = \sum_{k=1}^{4} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$.

10.1.13 $a_1 = \frac{1}{10}; a_2 = \frac{1}{100}; a_3 = \frac{1}{1000}; a_4 = \frac{1}{10000}$.

10.1.14 $a_1 = 3(1) + 1 = 4$. $a_2 = 3(2) + 1 = 7$, $a_3 = 3(3) + 1 = 10$, $a_4 = 3(4) + 1 = 13$.

10.1.15 $a_1 = -\frac{1}{2}$, $a_2 = \frac{1}{2^2} = \frac{1}{4}$. $a_3 = -\frac{2}{2^3} = -\frac{1}{8}$, $a_4 = \frac{1}{2^4} = \frac{1}{16}$.

10.1.16 $a_1 = 2 - 1 = 1$. $a_2 = 2 + 1 = 3$, $a_3 = 2 - 1 = 1$, $a_4 = 2 + 1 = 3$.

10.1.17 $a_1 = \frac{2^2}{2+1} = \frac{4}{3}$. $a_2 = \frac{2^3}{2^2+1} = \frac{8}{5}$. $a_3 = \frac{2^4}{2^3+1} = \frac{16}{9}$. $a_4 = \frac{2^5}{2^4+1} = \frac{32}{17}$.

10.1.18 $a_1 = 1 + \frac{1}{1} = 2$; $a_2 = 2 + \frac{1}{2} = \frac{5}{2}$; $a_3 = 3 + \frac{1}{3} = \frac{10}{3}$; $a_4 = 4 + \frac{1}{4} = \frac{17}{4}$.

10.1.19 $a_1 = 1 + \sin(\pi/2) = 2$; $a_2 = 1 + \sin(2\pi/2) = 1 + \sin(\pi) = 1$; $a_3 = 1 + \sin(3\pi/2) = 0$; $a_4 = 1 + \sin(4\pi/2) = 1 + \sin(2\pi) = 1$.

10.1.20 $a_1 = 1$; $a_2 = 2(1) = 2$; $a_3 = 3(2)(1) = 6$; $a_4 = 4(3)(2)(1) = 24$.

10.1.21 $a_1 = 2$, $a_2 = 2(2) = 4$, $a_3 = 2(4) = 8$, $a_4 = 2(8) = 16$.

10.1.22 $a_1 = 32$, $a_2 = 32/2 = 16$, $a_3 = 16/2 = 8$, $a_4 = 8/2 = 4$.

10.1.23 $a_1 = 10$ (given); $a_2 = 3 \cdot a_1 - 12 = 30 - 12 = 18$; $a_3 = 3 \cdot a_2 - 12 = 54 - 12 = 42$; $a_4 = 3 \cdot a_3 - 12 = 126 - 12 = 114$.

10.1.24 $a_1 = 1$ (given); $a_2 = a_1^2 - 1 = 0$; $a_3 = a_2^2 - 1 = -1$; $a_4 = a_3^2 - 1 = 0$.

10.1.25 $a_0 = 1$(given); $a_2 = \frac{1}{1+1} = \frac{1}{2}$; $a_3 = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$; $a_4 = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}$.

10.1.26 $a_0 = 1$ (given); $a_1 = 1$ (given); $a_2 = a_1 + a_0 = 2$; $a_3 = a_2 + a_1 = 3$; $a_4 = a_3 + a_2 = 5$.

10.1.27

a. $\frac{1}{32}, \frac{1}{64}$.

b. $a_1 = 1$; $a_{n+1} = \frac{a_n}{2}$.

c. $a_n = \frac{1}{2^{n-1}}$.

10.1.29

a. 32, 64.

b. $a_1 = 1$; $a_{n+1} = 2a_n$.

c. $a_n = 2^{n-1}$.

10.1.31

a. 243, 729.

b. $a_1 = 1$; $a_{n+1} = 3a_n$.

c. $a_n = 3^{n-1}$.

10.1.28

a. 14, 17.

b. $a_1 = 2$; $a_{n+1} = a_n + 3$.

c. $a_n = -1 + 3n$.

10.1.30

a. 2, 1.

b. $a_1 = 64$; $a_{n+1} = \frac{a_n}{2}$.

c. $a_n = \frac{64}{2^{n-1}}$.

10.1.32

a. 36, 49.

b. $a_1 = 1$; $a_{n+1} = (\sqrt{a_n} + 1)^2$.

c. $a_n = n^2$.

10.1.33

a. -5, 5.

b. $a_1 = -5$, $a_{n+1} = -a_n$.

c. $a_n = (-1)^n 5$.

10.1.34

a. 0, 1

b. $a_1 = 1$, $a_{n+1} = 1 - a_n$ for $n = 1, 2, 3, \ldots$.

c. $a_n = \dfrac{1-(-1)^n}{2}$, for $n = 1, 2, 3, \ldots$.

10.1.35 $a_1 = 9$, $a_2 = 99$, $a_3 = 999$, $a_4 = 9999$. This sequence diverges, because the terms get larger without bound.

10.1.36 $a_1 = 1$(given); $a_2 = \dfrac{10}{1} = 10$; $a_3 = \dfrac{10}{10} = 1$; $a_4 = \dfrac{10}{1} = 10$. This sequence diverges.

10.1.37 $a_1 = \dfrac{1}{10}$, $a_2 = \dfrac{1}{100}$, $a_3 = \dfrac{1}{1000}$, $a_4 = \dfrac{1}{10,000}$. This sequence converges to zero.

10.1.38 $a_0 = 1$(given); $a_1 = \dfrac{1}{10}$; $a_2. = \dfrac{1}{1/10} = 10$; $a_3 = \dfrac{1}{10}$; $a_4 = \dfrac{1}{1/10} = 10$. This sequence diverges.

10.1.39 $a_1 = 3 + \cos\pi = 2$. $a_2 = 3 + \cos 2\pi = 4$. $a_3 = 3 + \cos 3\pi = 2$. $a_4 = 3 + \cos 4\pi = 4$. This sequence diverges.

10.1.40 $a_1 = 0.9$, $a_2 = 0.99$, $a_3 = 0.999$, $a_4 = 0.9999$. This sequence converges to 1.

10.1.41 $a_1 = 1 + 1 = 2$, $a_2 = 1 + 1 = 2$, $a_3 = 2$, $a_4 = 2$. This constant sequence converges to 2.

10.1.42 $a_1 = 1 - \dfrac{1}{2} \cdot \dfrac{2}{3} = \dfrac{2}{3}$. Similarly, $a_2 = a_3 = a_4 = \frac{2}{3}$. This constant sequences converges to $\dfrac{2}{3}$.

10.1.43 $a_0 = 100$, $a_1 = 0.5 \cdot 100 + 50 = 100$, $a_2 = 0.5 \cdot 100 + 50 = 100$, $a_3 = 0.5 \cdot 100 + 50 = 100$, $a_4 = 0.5 \cdot 100 + 50 = 100$. This constant sequence converges to 100.

10.1.44 $a_1 = 0 - 1 = -1$. $a_2 = -10 - 1 = -11$, $a_3 = -110 - 1 = -111$, $a_4 = -1110 - 1 = -1111$. This sequence diverges.

10.1.45

n	a_n
1	0.83333
2	0.96154
3	0.99206
4	0.99840
5	0.99968
6	0.99994
7	0.99999
8	0.99999
9	0.99999
10	0.99999

This sequence appears to converge to 1.

10.1.46

n	a_n
1	0.95885
2	0.98961
3	0.99740
4	0.9995
5	0.99984
6	0.99996
7	0.99999
8	0.99999
9	0.99999
10	1.0000

This sequence appears to converge to 1.

10.1.47

n	1	2	3	4	5	6	7	8	9	10
a_n	2	6	12	20	30	42	56	72	90	110

This sequence appears to diverge.

10.1.48

n	1	2	3	4	5	6	7	8	9	10
a_n	9.9	9.95	9.9667	9.975	9.98	9.9833	9.9857	9.9875	9.9889	9.99

This sequence appears to converge to 10.

10.1.49

a. 2.5, 2.25, 2.125, 2.0625.

b. The limit appears to be 2.

10.1.50

a. 1.33333, 1.125, 1.06667, 1.04167.

b. The limit appears to be 1.

10.1.51

n	1	2	3	4	5	6	7	8	9	10
a_n	3	3.5000	3.7500	3.8750	3.9375	3.9688	3.9844	3.9922	3.9961	3.9980

The limit appears to be 4.

10.1.52

n	0	1	2	3	4	5	6	7	8	9
a_n	1	-2.75	-3.6875	-3.9219	-3.9805	-3.9951	-3.9988	-3.9997	-3.9999	-4.00

The limit appears to be -4.

10.1.53

n	0	1	2	3	4	5	6	7	8	9
a_n	1	5	21	85	341	1365	5461	21,845	87,381	349,525

This sequence appears to diverge.

10.1.54

n	0	1	2	3	4	5	6	7	8	9	10
a_n	32	16	8	4	2	1	0.5	0.25	0.125	0.0625	0.03125

The limit appears to be 0.

10.1.55

n	a_n
1	8
2	4.41421356
3	4.05050149
4	4.00629289
5	4.00078630
6	4.00009828
7	4.00001229
8	4.00000154
9	4.00000019
10	4.00000002

The limit appears to be 4.

10.1.56

n	a_n
1	10
2	9.4339811
3	9.1908568
4	9.0944292
5	9.0374462
6	9.0166274
7	9.0073869
8	9.0032825
9	9.0014588
10	9.0006483

The limit appears to be 9.

10.1.57

a. 20, 10, 5, 2.5.

b. $h_n = 20(0.5)^n$.

10.1.58

a. 10, 9, 8.1, 7.29.

b. $h_n = 10(0.9)^n$.

10.1.59

a. 30, 7.5, 1.875, 0.46875.

b. $h_n = 30(0.25)^n$.

10.1.60

a. 20, 15, 11.25, 8.4375

b. $h_n = 20(0.75)^n$.

10.1.61 $S_1 = 0.3$, $S_2 = 0.33$, $S_3 = 0.333$, $S_4 = 0.3333$. It appears that the infinite series has a value of $0.3333\ldots = \frac{1}{3}$.

10.1.62 $S_1 = 0.6$, $S_2 = 0.66$, $S_3 = 0.666$, $S_4 = 0.6666$. It appears that the infinite series has a value of $0.6666\ldots = \frac{2}{3}$.

10.1.63 $S_1 = 4$, $S_2 = 4.9$, $S_3 = 4.99$, $S_4 = 4.999$. The infinite series has a value of $4.999\cdots = 5$.

10.1.64 $S_1 = 10$, $S_2 = 10+100 = 110$, $S_3 = 10+100+1000 = 1110$, $S_4 = 10+100+1000+10000 = 11110$. The infinite series diverges.

10.1.65 $S_1 = \frac{6}{10} = 0.6$, $S_2 = \frac{66}{100} = 0.66$, $S_3 = \frac{666}{1000} = 0.666$, $S_4 = \frac{6666}{10000} = 0.6666$. The limit is $\frac{2}{3}$.

10.1.66 $S_1 = -1$, $S_2 = 0$, $S_3 = -1$, $S_4 = 0$. The limit does not exist.

10.1.67

a. $S_1 = \frac{2}{3}$, $S_2 = \frac{4}{5}$, $S_3 = \frac{6}{7}$, $S_4 = \frac{8}{9}$.

b. It appears that $S_n = \frac{2n}{2n+1}$. $S_5 = \frac{10}{11}$, $S_6 = \frac{12}{13}$, $S_7 = \frac{14}{15}$, $S_8 = \frac{16}{17}$.

c. The series has a value of 1 (the partial sums converge to 1).

10.1.68

a. $S_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, $S_3 = \frac{7}{8}$, $S_4 = \frac{15}{16}$.

b. $S_n = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$. $S_5 = \frac{31}{32}$, $S_6 = \frac{63}{64}$, $S_7 = \frac{127}{128}$, $S_8 = \frac{255}{256}$.

c. The partial sums converge to 1, so that is the value of the series.

10.1.69

a. $S_1 = 9$, $S_2 = 9.9$, $S_3 = 9.99$, $S_4 = 9.999$.

b. $S_n = 10 - (0.1)^{n-1}$. $S_5 = 9.9999$, $S_6 = 9.99999$, $S_7 = 9.999999$, $S_8 = 9.9999999$.

c. The partial sums converge to 10, which is the value of the series.

10.1.70

a. $S_1 = 2$, $S_2 = \frac{8}{3}$, $S_3 = \frac{26}{9}$, $S_4 = \frac{80}{27}$.

b. $S_n = \frac{3^n - 1}{3^{n-1}}$. $S_5 = \frac{242}{81} \approx 2.98765$, $S_6 = \frac{728}{243} \approx 2.99588$, $S_7 = \frac{2186}{729} \approx 2.99863$, $S_8 = \frac{6560}{2187} \approx 2.99954$.

c. The partial sums converge to 3, which is the value of the series.

10.1.71

a. True. For example, $S_2 = 1 + 2 = 3$, and $S_4 = a_1 + a_2 + a_3 + a_4 = 1 + 2 + 3 + 4 = 10$.

b. False. For example, $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \cdots$ where $a_n = 1 - \frac{1}{2^n}$ converges to 1, but each term is greater than the previous one.

c. True. In order for the partial sums to converge, they must get closer and closer together. In order for this to happen, the difference between successive partial sums, which is just the value of a_n, must approach zero.

10.1.72

a. $p_0 = 250$, $p_1 = 250 \cdot 1.03 = 258$, $p_2 = 250 \cdot 1.03^2 = 265$, $p_3 = 250 \cdot 1.03^3 = 273$, $p_4 = 250 \cdot 1.03^4 = 281$.

b. The initial population is 250, so that $p_0 = 250$. Then $p_n = 250 \cdot (1.03)^n$, because the population increases by 3 percent each month.

c. $p_{n+1} = p_n \cdot 1.03$.

d. The population increases without bound.

10.1.73

a. $M_0 = 20$, $M_1 = 20 \cdot 0.5 = 10$, $M_2 = 20 \cdot 0.5^2 = 5$, $M_3 = 20 \cdot 0.5^3 = 2.5$, $M_4 = 20 \cdot 0.5^4 = 1.25$

b. $M_n = 20 \cdot 0.5^n$.

c. The initial mass is $M_0 = 20$. We are given that 50% of the mass is gone after each decade, so that $M_{n+1} = 0.5 \cdot M_n$, $n \geq 0$.

d. The amount of material goes to 0.

10.1.74

a. $c_0 = 100$, $c_1 = 103$, $c_2 = 106.09$, $c_3 = 109.27$, $c_4 = 112.55$.

b. $c_n = 100(1.03)^n$, $n \geq 0$.

c. We are given that $c_0 = 100$ (where year 0 is 1984); because it increases by 3% per year, $c_{n+1} = 1.03 \cdot c_n$.

d. The sequence diverges.

10.1.75

a. $d_0 = 200$, $d_1 = 200 \cdot .95 = 190$, $d_2 = 200 \cdot .95^2 = 180.5$, $d_3 = 200 \cdot .95^3 = 171.475$, $d_4 = 200 \cdot .95^4 = 162.90125$.

b. $d_n = 200(0.95)^n$, $n \geq 0$.

c. We are given $d_0 = 200$; because 5% of the drug is washed out every hour, that means that 95% of the preceding amount is left every hour, so that $d_{n+1} = 0.95 \cdot d_n$.

d. The sequence converges to 0.

10.1.76 The height at the n^{th} bounce is given by the recurrence $h_n = r \cdot h_{n-1}$; an explicit form for this sequence is $h_n = h_0 \cdot r^n$. The distance traveled by the ball during the n^{th} bounce is thus $2h_n = 2h_0 \cdot r^n$, so that $S_n = \sum_{i=0}^{n} 2h_0 \cdot r^n$.

a. Here $h_0 = 20$, $r = 0.5$, so $S_0 = 40$, $S_1 = 40 + 40 \cdot 0.5 = 60$, $S_2 = S_1 + 40 \cdot (0.5)^2 = 70$, $S_3 = S_2 + 40 \cdot (0.5)^3 = 75$, $S_4 = S_3 + 40 \cdot (0.5)^4 = 77.5$

b.

n	0	1	2	3	4	5
a_n	40	60	70	75	77.5	78.75
n	6	7	8	9	10	11
a_n	79.375	79.6875	79.8438	79.9219	79.9609	79.9805
n	12	13	14	15	16	17
a_n	79.9902	79.9951	79.9976	79.9988	79.9994	79.9997
n	18	19	20	21	22	23
a_n	79.9998	79.9999	79.9999	80.0000	80.0000	80.0000

The sequence converges to 80.

10.1.77 Using the work from the previous problem:

a. Here $h_0 = 20$, $r = 0.75$, so $S_0 = 40$, $S_1 = 40 + 40 \cdot 0.75 = 70$, $S_2 = S_1 + 40 \cdot (0.75)^2 = 92.5$, $S_3 = S_2 + 40 \cdot (0.75)^3 = 109.375$, $S_4 = S_3 + 40 \cdot (0.75)^4 = 122.03125$

b.

n	0	1	2	3	4	5
a_n	40	70	92.5	109.375	122.0313	131.5234
n	6	7	8	9	10	11
a_n	138.6426	143.9819	147.9865	150.9898	153.2424	154.9318
n	12	13	14	15	16	17
a_n	156.1988	157.1491	157.8618	158.3964	158.7973	159.0980
n	18	19	20	21	22	23
a_n	159.3235	159.4926	159.6195	159.715	159.786	159.839

The sequence converges to 160.

10.1.78

a. Using the recurrence $a_{n+1} = \frac{1}{2}\left(a_n + \frac{10}{a_n}\right)$, we build a table:

n	0	1	2	3	4	5
a_n	10	5.5	3.659090909	3.196005081	3.162455622	3.162277665

The true value is $\sqrt{10} \approx 3.162277660$, so the sequence converges with an error of less than 0.01 after only 4 iterations, and is within 0.0001 after only 5 iterations.

b. The recurrence is now $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$

n	0	1	2	3	4	5	6
a_n	2	1.5	1.416666667	1.414215686	1.414213562	1.414213562	1.414213562

The true value is $\sqrt{2} \approx 1.414213562$, so the sequence converges with an error of less than 0.01 after 2 iterations, and is within 0.0001 after only 3 iterations.

10.1.79

n	a_n
0	0.800
1	0.697
2	0.767
3	0.720
4	0.752
5	0.730
6	0.745
7	0.735
8	0.742
9	0.737
10	0.740
11	0.738
12	0.740
13	0.739
14	0.739

The limit appears to be 0.739.

10.1.80

n	a_n
0	5.000
1	0.561
2	0.054
3	0.050
4	0.050

The limit appears to be 9.

10.2 Sequences

10.2.1 There are many examples; one is $a_n = \frac{1}{n}$. This sequence is nonincreasing (in fact, it is decreasing) and has a limit of 0.

10.2.2 Again there are many examples; one is $a_n = \ln(n)$. It is increasing, and has no limit.

10.2.3 There are many examples; one is $a_n = \frac{1}{n}$. This sequence is nonincreasing (in fact, it is decreasing), is bounded above by 1 and below by 0, and has a limit of 0.

10.2.4 For example, $a_n = (-1)^n$. For all values of n we have $|a_n| = 1$, so it is bounded. All the odd terms are -1 and all the even terms are 1, so the sequence does not have a limit.

10.2.5 $\{r^n\}$ converges for $-1 < r \leq 1$. It diverges for all other values of r (see Theorem 10.3).

10.2.6 Because $0.2 < 1$, this sequence converges to 0. Because $0.2 > 0$, the convergence is monotone.

10.2.7 Because $1.00001 > 1$, the sequence diverges; because $1.00001 > 0$, the divergence is monotone.

10.2.8 Because $|-2.5| > 1$, the sequence diverges; because $-2.5 < 0$, the divergence is not monotone. The sequence diverges by oscillation.

10.2.9 Because $|-0.7| < 1$, the sequence converges to 0; because $-0.7 < 0$, it does not do so monotonically. The sequence converges by oscillation.

10.2.10 Because $\lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 1$, we have $\lim_{n\to\infty} a_n = 1$ by the Squeeze Theorem.

10.2.11 $\{e^{n/100}\}$ grows faster than $\{n^{100}\}$ as $n \to \infty$.

10.2.12 $\lim_{n\to\infty} \frac{n^{100}}{n^n} = 0$, because $n^{100} << n^n$.

10.2.13 Divide numerator and denominator by n^4 to get $\lim_{n\to\infty} \frac{1/n}{1 + \frac{1}{n^4}} = 0$.

10.2.14 Divide numerator and denominator by n^{12} to get $\lim_{n\to\infty} \frac{1}{3 + \frac{4}{n^{12}}} = \frac{1}{3}$.

10.2.15 Divide numerator and denominator by n^3 to get $\lim_{n\to\infty} \frac{3 - n^{-3}}{2 + n^{-3}} = \frac{3}{2}$.

10.2.16 Divide numerator and denominator by n^3 to get $\lim_{n\to\infty} \frac{n^2 + 3/n^2}{10 + 1/n^2} = \infty$. The sequence diverges.

10.2.17 Because $\lim_{n\to\infty} \frac{10n}{10n + 4} = 1$, and because the inverse tangent function is continuous, the given sequence has limit $\tan^{-1} 1 = \pi/4$.

10.2.18 $\lim_{n\to\infty} \frac{n\pi}{2n + 2} = \frac{\pi}{2}$, using L'Hôpital's rule. Thus the sequence converges to $\cot(\pi/2) = 0$.

10.2.19 $\lim_{n\to\infty} (1 + \cos(1/n)) = 1 + \cos(0) = 2$.

10.2.20 $\ln(n^3 + 1) - \ln(3n^3 + 10n) = \ln\left(\frac{n^3 + 1}{3n^3 + 10n}\right) = \ln\left(\frac{1 + n^{-3}}{3 + 10n^{-2}}\right)$, so the limit is $\ln(1/3) = -\ln 3$.

10.2.21 $\ln(\sin(1/n)) + \ln n = \ln(n \sin(1/n)) = \ln\left(\frac{\sin(1/n)}{1/n}\right)$. As $n \to \infty$, $\sin(1/n)/(1/n) \to 1$, so the limit of the original sequence is $\ln 1 = 0$.

10.2.22 Divide numerator by k and denominator by $k = \sqrt{k^2}$ to get $\lim_{k\to\infty} \frac{1}{\sqrt{9 + (1/k^2)}} = \frac{1}{3}$.

10.2.23 Divide numerator by $n^2 = \sqrt{n^4}$ and the denominator by n^2 to obtain

$$\lim_{n\to\infty} \frac{\sqrt{4 + 3/n^3}}{8 + 1/n^2} = \frac{\sqrt{4 + 0}}{8 + 0} = \frac{2}{8} = \frac{1}{4}.$$

10.2.24 Divide numerator and denominator by e^n to get $\lim_{n\to\infty} \frac{2+(1/e^n)}{1} = 2$.

10.2.25 The sequence is equivalent to $\left\{\frac{2\ln n}{\ln 3 + \ln n}\right\}$. Divide numerator and denominator by $\ln n$ to obtain

$$\lim_{n\to\infty} \frac{2}{(\ln 3/\ln n)+1} = \frac{2}{1} = 2.$$

10.2.26 Because $|-1.01| > 1$, the sequence diverges.

10.2.27 This is the sequence

$$\frac{2^{n+1}}{3^n} = 2\cdot\left(\frac{2}{3}\right)^n;$$

because $0 < \frac{2}{3} < 1$, the sequence converges monotonically to zero.

10.2.28 $|-0.003| < 1$, so the sequence converges to zero.

10.2.29 Because $|0.5| < 1$ and $|0.75| < 1$, the sequence converges to $0 + 3\cdot 0 = 0$.

10.2.30 $\lim_{n\to\infty} \frac{e^n+\pi^n}{e^n} = \lim_{n\to\infty}\left(1+\left(\frac{\pi}{e}\right)^n\right) = 1 + \lim_{n\to\infty}\left(\frac{\pi}{e}\right)^n$ which doesn't exist, because $\left|\frac{\pi}{e}\right| > 1$.

10.2.31 Divide numerator and denominator by 3^n to get $\lim_{n\to\infty}\frac{3+(1/3^{n-1})}{1} = 3$.

10.2.32 Divide numerator and denominator by 4^n to get $\lim_{n\to\infty}\frac{(3/4)^n}{(3/4)^n+1} = \frac{0}{0+1} = 0$.

10.2.33 Because $\frac{(n+1)!}{n!} = n+1$, this sequence diverges.

10.2.34 $\frac{(2n)!n^2}{(2n+2)!} = \frac{(2n)!n^2}{(2n+2)(2n+1)(2n)!} = \frac{n^2}{4n^2+6n+2}$. Now divide numerator and denominator by n^2 to obtain $\lim_{n\to\infty}\frac{1}{4+6/n+2/n^2} = \frac{1}{4}$.

10.2.35 $\lim_{n\to\infty}\tan^{-1} n = \frac{\pi}{2}$.

10.2.36 Note that $e^{1/10} = \sqrt[10]{e} \approx 1.1$. Let $r = \frac{e^{1/10}}{2}$ and note that $0 < r < 1$. Thus $\lim_{n\to\infty}\frac{e^{n/10}}{2^n} = \lim_{n\to\infty} r^n = 0$.

10.2.37 Multiply by $\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$ to obtain

$$\lim_{n\to\infty}\left(\sqrt{n^2+1}-n\right) = \lim_{n\to\infty}\frac{\left(\sqrt{n^2+1}-n\right)\left(\sqrt{n^2+1}+n\right)}{\sqrt{n^2+1}+n} = \lim_{n\to\infty}\frac{1}{\sqrt{n^2+1}+n} = 0.$$

10.2.38 Let $y = n^{2/n}$. Then $\ln y = \frac{2\ln n}{n}$. By L'Hôpital's rule we have $\lim_{x\to\infty}\frac{2\ln x}{x} = \lim_{x\to\infty}\frac{2}{x} = 0$, so $\lim_{n\to\infty} n^{2/n} = e^0 = 1$.

10.2.39 Find the limit of the logarithm of the expression, which is $n\ln\left(1+\frac{2}{n}\right)$. Using L'Hôpital's rule:

$$\lim_{n\to\infty} n\ln\left(1+\frac{2}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{2}{n}\right)}{1/n} = \lim_{n\to\infty}\frac{\frac{1}{1+(2/n)}\left(-\frac{2}{n^2}\right)}{-1/n^2} = \lim_{n\to\infty}\frac{2}{1+(2/n)} = 2.$$

Thus the limit of the original expression is e^2.

10.2.40 Using L'Hôpital's rule, we have $\lim_{n\to\infty}\frac{\ln n}{n^{1.1}} = \lim_{n\to\infty}\frac{1/n}{(1.1)n^{.1}} = \lim_{n\to\infty}\frac{1}{(1.1)n^{1.1}} = 0.$

10.2.41 Let $y = \sqrt[n]{e^{3n+4}} = e^{\frac{3n+4}{n}}$. Then $\ln y = \frac{3n+4}{n}$. Then $\lim_{n\to\infty}\ln y = \lim_{n\to\infty}\frac{3n+4}{n} = 3.$ So $\lim_{n\to\infty}\sqrt[n]{e^{3n+4}} = e^3$.

10.2.42 Take the logarithm of the expression and use L'Hôpital's rule:

$$\lim_{n\to\infty} n\ln\left(\frac{n}{n+5}\right) = \lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+5}\right)}{1/n} = \lim_{n\to\infty}\frac{\frac{n+5}{n}\cdot\frac{5}{(n+5)^2}}{-1/n^2} = \lim_{n\to\infty}\frac{-5n}{n+5} = -5.$$

Thus the original limit is e^{-5}.

10.2.43 Take the logarithm of the expression and use L'Hôpital's rule:

$$\lim_{n\to\infty}\frac{n}{2}\ln\left(1+\frac{1}{2n}\right) = \lim_{n\to\infty}\frac{\ln(1+(1/2n))}{2/n} = \lim_{n\to\infty}\frac{\frac{1}{1+(1/2n)}\cdot\frac{-1}{2n^2}}{-2/n^2} = \lim_{n\to\infty}\frac{1}{4(1+(1/2n))} = \frac{1}{4}.$$

Thus the original limit is $e^{1/4}$.

10.2.44 Find the limit of the logarithm of the expression, which is $3n\ln\left(1+\frac{4}{n}\right)$. Using L'Hôpital's rule:

$$\lim_{n\to\infty} 3n\ln\left(1+\frac{4}{n}\right) = \lim_{n\to\infty}\frac{3\ln\left(1+\frac{4}{n}\right)}{1/n} = \lim_{n\to\infty}\frac{\frac{1}{1+(4/n)}\left(-\frac{12}{n^2}\right)}{-1/n^2} = \lim_{n\to\infty}\frac{12}{1+(4/n)} = 12.$$

Thus the limit of the original expression is e^{12}.

10.2.45 Using L'Hôpital's rule: $\lim_{n\to\infty}\frac{n}{e^n+3n} = \lim_{n\to\infty}\frac{1}{e^n+3} = 0.$

10.2.46 $\ln\frac{1}{n} = -\ln n$, so this is $-\lim_{n\to\infty}\frac{\ln n}{n}$. By L'Hôpital's rule, we have $-\lim_{n\to\infty}\frac{\ln n}{n} = -\lim_{n\to\infty}\frac{1}{n} = 0.$

10.2.47 Taking logs, we have $\lim_{n\to\infty}\frac{1}{n}\ln(1/n) = \lim_{n\to\infty} -\frac{\ln n}{n} = \lim_{n\to\infty} -\frac{1}{n} = 0$ by L'Hôpital's rule. Thus the original sequence has limit $e^0 = 1$.

10.2.48 Find the limit of the logarithm of the expression, which is $n\ln\left(1-\frac{4}{n}\right)$, using L'Hôpital's rule: $\lim_{n\to\infty} n\ln\left(1-\frac{4}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1-\frac{4}{n}\right)}{1/n} = \lim_{n\to\infty}\frac{\frac{1}{1-(4/n)}\left(\frac{4}{n^2}\right)}{-1/n^2} = \lim_{n\to\infty} -\frac{4}{1-(4/n)} = -4$. Thus the limit of the original expression is e^{-4}.

10.2.49 Except for a finite number of terms, this sequence is just $a_n = ne^{-n}$, so it has the same limit as this sequence. Note that $\lim_{n\to\infty}\frac{n}{e^n} = \lim_{n\to\infty}\frac{1}{e^n} = 0$, by L'Hôpital's rule.

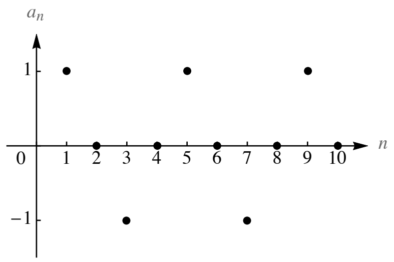
10.2.50 Using L'Hôpital's rule:

$$\lim_{n\to\infty} n(1-\cos(1/n)) = \lim_{n\to\infty}\frac{1-\cos(1/n)}{1/n} = \lim_{n\to\infty}\frac{-\sin(1/n)(-1/n^2)}{-1/n^2} = -\sin(0) = 0.$$

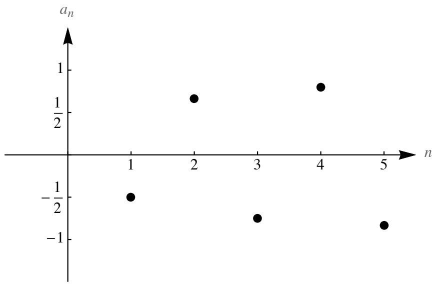
10.2.51 $\lim_{n\to\infty} n\sin(6/n) = \lim_{n\to\infty}\frac{\sin(6/n)}{1/n} = \lim_{n\to\infty}\frac{\frac{-6\cos(6/n)}{n^2}}{(-1/n^2)} = \lim_{n\to\infty} 6\cos(6/n) = 6\cdot\cos 0 = 6.$

10.2.52 Dividing the numerator and denominator by n^8 gives $a_n = \dfrac{1+(1/n)}{(1/n)+\ln n}$. Because $1+(1/n) \to 1$ as $n \to \infty$ and $(1/n)+\ln n \to \infty$ as $n \to \infty$, we have $\lim\limits_{n\to\infty} a_n = 0$.

10.2.53 When n is an integer, $\sin\left(\dfrac{n\pi}{2}\right)$ oscillates between the values ± 1 and 0, so this sequence does not converge.



10.2.54 The even terms form a sequence $a_{2n} = \dfrac{2n}{2n+1}$, which converges to 1 (e.g. by L'Hôpital's rule); the odd terms form the sequence $a_{2n+1} = -\dfrac{2n+1}{2n+2}$, which converges to -1. Thus the sequence as a whole does not converge.



10.2.55 Because

$$-\frac{1}{2^n} \le \frac{(-1)^n}{2^n} \le \frac{1}{2^n},$$

and because $\lim\limits_{n\to\infty} -\dfrac{1}{2^n} = \lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$, the given sequence has limit zero by the Squeeze Theorem.

10.2.56 Note that

$$-\frac{1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}.$$

Because $\lim\limits_{n\to\infty} -\dfrac{1}{n} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, The given sequence has limit zero by the Squeeze Theorem.

10.2.57 Ignoring the factor of $(-1)^n$ for the moment, we see, taking logs, that $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$, so that $\lim\limits_{n\to\infty} \sqrt[n]{n} = e^0 = 1$. Taking the sign into account, the odd terms converge to -1 while the even terms converge to 1. Thus the sequence does not converge.

10.2.58 Because $-1 \le \cos(n\pi/2) \le 1$ for all n, we have $-\dfrac{1}{\sqrt{n}} \le \dfrac{\cos(n\pi/2)}{\sqrt{n}} \le \dfrac{1}{\sqrt{n}}$ and because both $\pm\dfrac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, the given sequence converges to 0 as well by the Squeeze Theorem.

10.2.59 This sequence diverges. To see this, call the given sequence a_n, and assume it converges to limit L. Then because the sequence $b_n = \dfrac{n}{n+1}$ converges to 1, the sequence $c_n = \dfrac{a_n}{b_n}$ would converge to L as well. But $c_n = \sin^3 \dfrac{\pi n}{2}$ doesn't converge (because it is $1, -1, 1, -1 \cdots$), so the given sequence doesn't converge either.

10.2.60 Because $-\frac{n^2}{2n^3+n} \le \frac{(-1)^{n+1}n^2}{2n^3+n} \le \frac{n^2}{2n^3+n}$, and because both $-\frac{n^2}{2n^3+n}$ and $\frac{n^2}{2n^3+n}$ have limit 0 as $n \to \infty$, the limit of the given sequence is also 0 by the Squeeze Theorem. Note that $\lim_{n\to\infty} \frac{n^2}{2n^3+n} = \lim_{n\to\infty} \frac{1/n}{2+1/n^2} = \frac{0}{2} = 0$.

10.2.61 This is the sequence $\frac{\cos n}{e^n}$; the numerator is bounded in absolute value by 1 and the denominator increases without bound, so the limit is zero.

10.2.62 By L'Hôpital's rule we have: $\lim_{n\to\infty} \frac{e^{-n}}{2\sin(e^{-n})} = \lim_{n\to\infty} \frac{-e^{-n}}{2\cos(e^{-n})(-e^{-n})} = \frac{1}{2\cos 0} = \frac{1}{2}$.

10.2.63 Because $\lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}$, $\lim_{n\to\infty} \frac{\tan^{-1} n}{n} = 0$.

10.2.64 $\lim_{n\to\infty} \frac{(2n^3+n)\tan^{-1} n}{n^3+4} = \lim_{n\to\infty} \frac{2n^3+n}{n^3+4} \cdot \lim_{n\to\infty} \tan^{-1} n = 2 \cdot \frac{\pi}{2} = \pi$.

10.2.65 Because $-1 \le \cos n \le 1$, we have $-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$. Because both $-\frac{1}{n}$ and $\frac{1}{n}$ have limit 0 as $n \to \infty$, the given sequence does as well.

10.2.66 Because $-1 \le \sin 6n \le 1$, we have $-\frac{1}{5n} \le \frac{\sin 6n}{5n} \le \frac{1}{5n}$. Because both $-\frac{1}{5n}$ and $\frac{1}{5n}$ have limit 0 as $n \to \infty$, the given sequence does as well.

10.2.67 Because $-1 \le \sin n \le 1$ for all n, the given sequence satisfies $-\frac{1}{2^n} \le \frac{\sin n}{2^n} \le \frac{1}{2^n}$, and because both $\pm\frac{1}{2^n} \to 0$ as $n \to \infty$, the given sequence converges to zero as well by the Squeeze Theorem.

10.2.68 If $f(t) = \int_1^t x^{-2}dx$, then $\lim_{t\to\infty} f(t) = \lim_{n\to\infty} a_n$. But

$$\lim_{t\to\infty} f(t) = \int_1^\infty x^{-2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\bigg|_1^b\right] = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1.$$

10.2.69 Evaluate the limit of each term separately: $\lim_{n\to\infty} \frac{75^{n-1}}{99^n} = \frac{1}{99}\lim_{n\to\infty}\left(\frac{75}{99}\right)^{n-1} = 0$, while $-\frac{5^n}{8^n} \le \frac{5^n \sin n}{8^n} \le \frac{5^n}{8^n}$, so by the Squeeze Theorem, this second term converges to 0 as well. Thus the sum of the terms converges to zero.

10.2.70 Because $\lim_{n\to\infty} 0.99^n = 0$, and because cosine is continuous, the first term converges to $\cos 0 = 1$. The limit of the second term is $\lim_{n\to\infty} \frac{7^n+9^n}{63^n} = \lim_{n\to\infty}\left(\frac{7}{63}\right)^n + \lim_{n\to\infty}\left(\frac{9}{63}\right)^n = 0$. Thus the sum converges to 1.

10.2.71

a. After the n^{th} dose is given, the amount of drug in the bloodstream is $d_n = 0.5 \cdot d_{n-1} + 80$, because the half-life is one day. The initial condition is $d_1 = 80$.

b. The limit of this sequence is 160 mg.

c. Let $L = \lim_{n\to\infty} d_n$. Then from the recurrence relation, we have $d_n = 0.5 \cdot d_{n-1} + 80$, and thus $\lim_{n\to\infty} d_n = 0.5 \cdot \lim_{n\to\infty} d_{n-1} + 80$, so $L = 0.5 \cdot L + 80$, and therefore $L = 160$.

10.2.72

a.

$$\begin{aligned}
B_0 &= \$20,000\\
B_1 &= 1.005\cdot B_0 - \$200 = \$19,900\\
B_2 &= 1.005\cdot B_1 - \$200 = \$19,799.50\\
B_3 &= 1.005\cdot B_2 - \$200 = \$19,698.50\\
B_4 &= 1.005\cdot B_3 - \$200 = \$19,596.99\\
B_5 &= 1.005\cdot B_4 - \$200 = \$19,494.97
\end{aligned}$$

b. $B_n = 1.005\cdot B_{n-1} - \$200$

c. Using a calculator or computer program, B_n becomes negative after the 139^{th} payment, so 139 months or almost 11 years.

10.2.73

a.

$$\begin{aligned}
B_0 &= 0\\
B_1 &= 1.0075\cdot B_0 + \$100 = \$100\\
B_2 &= 1.0075\cdot B_1 + \$100 = \$200.75\\
B_3 &= 1.0075\cdot B_2 + \$100 = \$302.26\\
B_4 &= 1.0075\cdot B_3 + \$100 = \$404.52\\
B_5 &= 1.0075\cdot B_4 + \$100 = \$507.56
\end{aligned}$$

b. $B_n = 1.0075\cdot B_{n-1} + \$100$.

c. Using a calculator or computer program, $B_n > \$5,000$ during the 43^{rd} month.

10.2.74

a. Let D_n be the *total number* of liters of alcohol in the mixture after the n^{th} replacement. At the next step, 2 liters of the 100 liters is removed, thus leaving $0.98\cdot D_n$ liters of alcohol, and then $0.1\cdot 2 = 0.2$ liters of alcohol are added. Thus $D_n = 0.98\cdot D_{n-1} + 0.2$. Now, $C_n = D_n/100$, so we obtain a recurrence relation for C_n by dividing this equation by 100: $C_n = 0.98\cdot C_{n-1} + 0.002$.

$$\begin{aligned}
C_0 &= 0.4\\
C_1 &= 0.98\cdot 0.4 + 0.002 = 0.394\\
C_2 &= 0.98\cdot C_1 + 0.002 = 0.38812\\
C_3 &= 0.98\cdot C_2 + 0.002 = 0.38236\\
C_4 &= 0.98\cdot C_3 + 0.002 = 0.37671\\
C_5 &= 0.98\cdot C_4 + 0.002 = 0.37118
\end{aligned}$$

The rounding is done to five decimal places.

b. Using a calculator or a computer program, $C_n < 0.15$ after the 89^{th} replacement.

c. If the limit of C_n is L, then taking the limit of both sides of the recurrence equation yields $L = 0.98L + 0.002$, so $.02L = .002$, and $L = .1 = 10\%$.

10.2.75 Because $n! \ll n^n$ by Theorem 10.6, we have $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$.

10.2.76 $\{3^n\} \ll \{n!\}$ because $\{b^n\} \ll \{n!\}$ in Theorem 10.6. Thus, $\lim_{n\to\infty} \frac{3^n}{n!} = 0$.

10.2.77 Theorem 10.6 indicates that $\ln^q n \ll n^p$, so $\ln^{20} n \ll n^{10}$, so $\lim_{n\to\infty} \frac{n^{10}}{\ln^{20} n} = \infty$.

10.2.78 Theorem 10.6 indicates that $\ln^q n \ll n^p$, so $\ln^{1000} n \ll n^{10}$, so $\lim_{n\to\infty} \frac{n^{10}}{\ln^{1000} n} = \infty$.

10.2.79 By Theorem 10.6, $n^p \ll b^n$, so $n^{1000} \ll 2^n$, and thus $\lim_{n\to\infty} \frac{n^{1000}}{2^n} = 0$.

10.2.80 Dividing the numerator and denominator by $n!$ gives $a_n = \frac{(4^n/n!)+5}{1+(2^n/n!)}$. By Theorem 10.6, we have $4^n \ll n!$ and $2^n \ll n!$. Thus, $\lim_{n\to\infty} a_n = \frac{0+5}{1+0} = 5$.

10.2.81 Dividing the numerator and denominator by 6^n gives $a_n = \frac{1+(1/2)^n}{1+(n^{100}/6^n)}$. By Theorem 10.6, $n^{100} \ll 6^n$. Thus $\lim_{n\to\infty} a_n = \frac{1+0}{1+0} = 1$.

10.2.82 We can write $a_n = \frac{(7/5)^n}{n^7}$. Theorem 10.6 indicates that $n^7 \ll b^n$ for $b > 1$, so $\lim_{n\to\infty} a_n = \infty$.

10.2.83

a. True. See Theorem 10.2 part 4.

b. False. For example, if $a_n = 1/n$ and $b_n = e^n$, then $\lim_{n\to\infty} a_n b_n = \infty$.

c. True. The definition of the limit of a sequence involves only the behavior of the n^{th} term of a sequence as n gets large (see the Definition of Limit of a Sequence). Thus suppose a_n, b_n differ in only finitely many terms, and that M is large enough so that $a_n = b_n$ for $n > M$. Suppose a_n has limit L. Then for $\varepsilon > 0$, if N is such that $|a_n - L| < \varepsilon$ for $n > N$, first increase N if required so that $N > M$ as well. Then we also have $|b_n - L| < \varepsilon$ for $n > N$. Thus a_n and b_n have the same limit. A similar argument applies if a_n has no limit.

d. True. Note that a_n converges to zero. Intuitively, the nonzero terms of b_n are those of a_n, which converge to zero. More formally, given ϵ, choose N_1 such that for $n > N_1$, $a_n < \epsilon$. Let $N = 2N_1 + 1$. Then for $n > N$, consider b_n. If n is even, then $b_n = 0$ so certainly $b_n < \epsilon$. If n is odd, then $b_n = a_{(n-1)/2}$, and $(n-1)/2 > ((2N_1+1)-1)/2 = N_1$ so that $a_{(n-1)/2} < \epsilon$. Thus b_n converges to zero as well.

e. False. If $\{a_n\}$ happens to converge to zero, the statement is true. But consider for example $a_n = 2 + \frac{1}{n}$. Then $\lim_{n\to\infty} a_n = 2$, but $(-1)^n a_n$ does not converge (it oscillates between positive and negative values increasingly close to ± 2).

f. True. Suppose $\{0.000001a_n\}$ converged to L, and let $\epsilon > 0$ be given. Choose N such that for $n > N$, $|0.000001a_n - L| < \epsilon \cdot 0.000001$. Dividing through by 0.000001, we get that for $n > N$, $|a_n - 1000000L| < \epsilon$, so that a_n converges as well (to $1000000L$).

10.2.84

a. The first three terms of the sequence are 1, 2.5, and 3.25. Because we are given that the sequence is monotonic, the sequence is nondecreasing.

b. Assuming the sequence converges, let its limit be L. Then $\lim_{n\to\infty} a_{n+1} = \frac{1}{2} \lim_{n\to\infty} a_n + 2$, so $L = \frac{1}{2}L + 2$, and thus $\frac{1}{2}L = 2$, so $L = 4$.

10.2.85

a. The first three terms of the sequence are 0.3, 0.42, and 0.4872. Because we are given that the sequence is monotonic, the sequence is nondecreasing.

b. Let its limit be L, then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}(2a_n(1-a_n)) = 2(\lim_{n\to\infty} a_n)(1-\lim_{n\to\infty} a_n)$, so $L = 2L(1-L) = 2L - 2L^2$, and thus $2L^2 - L = 0$, so $L = 0, \frac{1}{2}$. Thus the limit appears to be either 0 or 1/2; with the given initial condition, doing a few iterations by hand (as above) confirms that the sequence converges to 1/2.

10.2.86

a. The first 3 terms of the sequence are 2, 1.5, and 1.416667. Because we are given that the sequence is monotonic, it is nonincreasing.

b. lLet the limit be L. Then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{1}{2}\lim_{n\to\infty} a_n + \frac{1}{\lim_{n\to\infty} a_n}$, so $L = \frac{1}{2}L + \frac{1}{L}$, and therefore $L^2 = \frac{1}{2}L^2 + 1$. So $L^2 = 2$, and thus $L = \sqrt{2}$.

10.2.87

a. The first 3 terms of the sequence are 3, 2.236068, and 2.058171. Because we are given that the sequence is monotonic, it is nonincreasing.

b. Let its limit be L. Then $\lim_{n\to\infty} a_{n+1} = \sqrt{2 + \lim_{n\to\infty} a_n}$, so $L = \sqrt{2+L}$. Thus we have $L^2 = 2 + L$, so $L^2 - L - 2 = 0$, and thus $L = -1, 2$. A square root can never be negative, so this sequence must converge to 2.

10.2.88

a. Rounded to the nearest fish, the populations are

$$\begin{aligned}
F_0 &= 4000\\
F_1 &= 1.015F_0 - 80 = 3980\\
F_2 &= 1.015F_1 - 80 \approx 3960\\
F_3 &= 1.015F_2 - 80 \approx 3939\\
F_4 &= 1.015F_3 - 80 \approx 3918\\
F_5 &= 1.015F_4 - 80 \approx 3897
\end{aligned}$$

b. $F_n = 1.015F_{n-1} - 80$

c. The population decreases and eventually reaches zero.

d. With an initial population of 5500 fish, the population increases without bound.

e. If the initial population is less than 5333 fish, the population will decline to zero. This is essentially because for a population of less than 5333, the natural increase of 1.5% does not make up for the loss of 80 fish.

10.2.89

a. $d_{n+1} = 0.4d_n + 75$; $d_1 = 75$.

b. By examining the first few terms of the sequence, it appears to be increasing and bounded above by 125. To confirm this, observe that $d_2 = 105$ and so $d_2 > d_1$. Assume that $d_{k+1} > d_k$, for some positive integer k. Then $d_{k+2} = 0.4d_{k+1} + 75 > 0.4d_k + 75 = d_{k+1}$. Therefore $d_{k+2} > d_{k+1}$. So by mathematical induction, it follows that $\{d_n\}$ is an increasing sequence. Because $\{d_n\}$ is an increasing sequence and $d_1 = 75$, the sequence is bounded below by 75. To show that the sequence is indeed bounded above by 125, first observe that $d_1 = 75 < 125$. Now assume that $d_k < 125$, for some positive integer k for $k \geq 1$. Then $d_{k+1} = 0.4d_k + 75 \leq 0.4(125) + 75 = 125$, and therefore $d_{k+1} < 125$. So by mathematical induction, the sequence is bounded above by 125. Because $\{d_n\}$ is bounded below and above, it is bounded. So by Theorem 10.5, it converges.

c. Assume that the sequence $\{d_n\}$ converges to L. Taking limits of the recurrence relation, we have $\lim\limits_{n\to\infty} d_n = 0.4 \lim\limits_{n\to\infty} d_n + 75 = 0.4L + 75$. So the limit is $L = 75/0.6 = 125$. In the long run there will be approximately 125 mg of medication in the blood

10.2.90

a. $x_0 = 7$, $x_1 = 6$, $x_2 = 6.5 = \dfrac{13}{2}$, $x_3 = 6.25$, $x_4 = 6.375 = \dfrac{51}{8}$, $x_5 = 6.3125 = \dfrac{101}{16}$, $x_6 = 6.34375 = \dfrac{203}{32}$.

b. For the formula given in the problem, we have $x_0 = \dfrac{19}{3} + \dfrac{2}{3}\left(-\dfrac{1}{2}\right)^0 = 7$, $x_1 = \dfrac{19}{3} + \dfrac{2}{3} \cdot \dfrac{-1}{2} = \dfrac{19}{3} - \dfrac{1}{3} = 6$,

so that the formula holds for $n = 0, 1$. Now assume the formula holds for all integers $\leq k$; then

$$\begin{aligned}
x_{k+1} = \frac{1}{2}(x_k + x_{k-1}) &= \frac{1}{2}\left(\frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^k + \frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k-1}\right) \\
&= \frac{1}{2}\left(\frac{38}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k-1}\left(-\frac{1}{2} + 1\right)\right) \\
&= \frac{1}{2}\left(\frac{38}{3} + 4 \cdot \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1} \cdot \frac{1}{2}\right) \\
&= \frac{1}{2}\left(\frac{38}{3} + 2 \cdot \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1}\right) \\
&= \frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1}.
\end{aligned}$$

c. As $n \to \infty$, $(-1/2)^n \to 0$, so that the limit is 19/3, or 6 1/3.

10.2.91 The approximate first few values of this sequence are:

n	0	1	2	3	4	5	6
c_n	0.7071	0.6325	0.6136	0.6088	0.6076	0.6074	0.6073

The value of the constant appears to be around 0.607.

10.2.92 Note that $1 - \dfrac{1}{i} = \dfrac{i-1}{i}$, so that the product is $\dfrac{1}{2} \cdot \dfrac{2}{3} \cdot \dfrac{3}{4} \cdot \dfrac{4}{5} \cdots$, so that $a_n = \dfrac{1}{n}$ for $n \geq 2$. The sequence $\left\{\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dots\right\}$ has limit zero.

10.2.93

a. We begin by showing that $a_{n+1} > a_n$, for $n \geq 1$. This inequality holds for $n = 0$ because $a_1 = 7 > a_0$. Assume that $a_{n+1} > a_n$ for a nonnegative integer $n = k$. Then $a_{k+2} = (1/3)a_{k+1} + 6 > (1/3)a_k + 6 = a_{k+1}$. Therefore $a_{n+1} > a_n$ holds for $n = k + 1$. So by mathematical induction, we have established that $a_{n+1} > a_n$, for $n \geq 1$, which implies that $\{a_n\}$ is a monotonic sequence. Next, we establish that $a_n < 9$, for $n \geq 1$. This result holds for $n = 0$ because $a_0 = 3 < 9$. Now assume that $a_k < 9$ for a nonnegative integer k. Then $a_{k+1} = (1/3)a_k + 6 < (1/3)(9) + 6 = 9$. So by mathematical induction, we've shown that $a_n < 9$, for $n \geq 1$. Because $\{a_n\}$ is an increasing sequence and $a_0 = 3$, it follows that $3 \leq a_n < 9$. Therefore $\{a_n\}$ is bounded. Because $\{a_n\}$ is monotonic and bounded, it converges.

b. Because $\{a_n\}_{n=0}^{\infty}$ is bounded above and is increasing, it converges. Assume that L is the limit of the sequence. Taking the limit of both sides of the recurrence relation gives $L = \frac{1}{3}L + 6$, or $3L = L + 18$, so $L = 9$.

10.2.94

a. To show the sequence is decreasing, notice that $a_1 = \sqrt{26} < \sqrt{36} = 6 = a_0$. So $a_1 < a_0$. Assume that $a_{k+1} < a_k$ for some nonnegative integer k. Then $a_{k+2} = \sqrt{a_{k+1} + 20} < \sqrt{a_k + 20} = a_{k+1}$. So $a_{k+2} < a_{k+1}$, and so by mathematical induction, it has been shown that $a_{n+1} < a_n$ for $n \geq 0$. Observe that $a_1 = \sqrt{26} > 5$. Assume that $a_k > 5$ for some nonnegative integer k. Then $a_{k+1} = \sqrt{a_k + 20} > \sqrt{5 + 20} = 5$ and so $a_{k+1} > 5$. So by mathematical induction, we've established that $a_n > 5$ for $n \geq 0$. Because the sequence is decreasing, it is bounded above by the the first term in the sequence, which is 6. So the sequence is bounded.

b. Because $\{a_n\}_{n=0}^{\infty}$ is bounded below and is decreasing, it converges. Assume that L is the limit of the sequence. Taking the limit of both sides of the recurrence relation gives $L = \sqrt{L + 20}$, so $L^2 - L - 20 = (L - 5)(L + 4) = 0$, so $L = 5$ or $L = -4$. Because the sequence consists of all positive terms, we must have $L = 5$.

10.2.95

a. $a_0 = \sqrt{2} \approx 1.41421$, $a_1 = \sqrt{2 + \sqrt{2}} \approx 1.84776$, $a_2 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.96157$,

$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \approx 1.99037$, and $a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \approx 1.99759$.

b. We have $a_1 = \sqrt{2 + \sqrt{2}}$, so it is the case that $a_1 > a_0$. Now assume that $a_k > a_{k-1}$ for some $k \geq 1$. Then $a_{k+1} = \sqrt{2 + a_k} > \sqrt{2 + a_{k-1}} = a_k$. So we have $a_{k+1} > a_k$. So by mathematical induction, it follows that a_n is increasing. Because the first term in the sequence is $\sqrt{2}$ and the sequence is increasing, the sequence is bounded below by $\sqrt{2}$. It appears that all of the terms in the sequence are less than 2 and we now use mathematical induction to prove this. We have $a_0 = \sqrt{2} < 2$. Suppose $a_k < 2$ for some $k \geq 1$. Then $a_{k+1} = \sqrt{2 + a_k} < \sqrt{2 + 2} = 2$. So it follows that $\sqrt{2} < a_n < 2$ for $n \geq 0$.

c. Because $\{a_n\}_{n=0}^{\infty}$ is bounded and increasing, it converges. Assume that L is the limit of the sequence. Taking the limit of both sides of the recurrence relation gives $L = \sqrt{2 + L}$, so $L^2 = 2 + L$, or $L^2 - L - 2 = (L - 2)(L + 1) = 0$. So $L = 2$ or $L = -1$ and because terms of the sequence are all positive, the limit is 2.

10.2.96

a. With $p = 0.5$ we have for $a_{n+1} = a_n^p$:

n	1	2	3	4	5	6	7
a_n	0.707	0.841	0.971	08	0.979	0.989	0.995

Experimenting with recurrence (1) one sees that for $0 < p \le 1$ the sequence converges to 1, while for $p > 1$ the sequence diverges to ∞.

b. With $p = 1.2$ and $a_n = p^{a_{n-1}}$ we obtain

n	1	2	3	4	5	6	7	8	9	10
a_n	1.2	1.2446	1.2547	1.2570	1.2577	1.2577	1.2577	1.2577	1.2577	1.2577

With recurrence (2), in addition to converging for $p < 1$ it also converges for values of p less than approximately 1.444. Here is a table of approximate values for different values of p:

p	1.1	1.2	1.3	1.4	1.44	1.444	1.445
$\lim_{n\to\infty} a_n$	1.1118	1.25776	1.471	1.887	2.39385	2.587	Diverges

It appears that the upper limit of convergence is about 1.444.

10.2.97

a. $f_0 = f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 5, f_5 = 8, f_6 = 13, f_7 = 21, f_8 = 34, f_9 = 55, f_{10} = 89.$

b. The sequence is clearly not bounded.

c. $\frac{f_{10}}{f_9} \approx 1.61818$

d. We use induction. Note that

$$\frac{1}{\sqrt{5}}\left(\varphi + \frac{1}{\varphi}\right) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} + \frac{2}{1+\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\left(\frac{1+2\sqrt{5}+5+4}{2(1+\sqrt{5})}\right) = 1 = f_1.$$

Also note that

$$\frac{1}{\sqrt{5}}\left(\varphi^2 - \frac{1}{\varphi^2}\right) = \frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2} - \frac{2}{3+\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\left(\frac{9+6\sqrt{5}+5-4}{2(3+\sqrt{5})}\right) = 1 = f_2.$$

Now note that

$$\begin{aligned} f_{n-1} + f_{n-2} &= \frac{1}{\sqrt{5}}(\varphi^{n-1} - (-1)^{n-1}\varphi^{1-n} + \varphi^{n-2} - (-1)^{n-2}\varphi^{2-n}) \\ &= \frac{1}{\sqrt{5}}((\varphi^{n-1} + \varphi^{n-2}) - (-1)^n(\varphi^{2-n} - \varphi^{1-n})). \end{aligned}$$

Now, note that $\varphi - 1 = \frac{1}{\varphi}$, so that

$$\varphi^{n-1} + \varphi^{n-2} = \varphi^{n-1}\left(1 + \frac{1}{\varphi}\right) = \varphi^{n-1} \cdot \varphi = \varphi^n$$

and

$$\varphi^{2-n} - \varphi^{1-n} = \varphi^{-n}(\varphi^2 - \varphi) = \varphi^{-n}(\varphi(\varphi - 1)) = \varphi^{-n}.$$

Making these substitutions, we get

$$f_n = f_{n-1} + f_{n-2} = \frac{1}{\sqrt{5}}(\varphi^n - (-1)^n\varphi^{-n})$$

10.2.98

a. We show that the arithmetic mean of any two positive numbers exceeds their geometric mean. Let a, $b > 0$; then $\frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(a - 2\sqrt{ab} + b) = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 \geq 0$. Because in addition $a_0 > b_0$, we have $a_n > b_n$ for all n.

b. To see that $\{a_n\}$ is decreasing, note that

$$a_{n+1} = \frac{a_n + b_n}{2} < \frac{a_n + a_n}{2} = a_n.$$

Similarly,

$$b_{n+1} = \sqrt{a_n b_n} > \sqrt{b_n b_n} = b_n,$$

so that $\{b_n\}$ is increasing.

c. $\{a_n\}$ is monotone and nonincreasing by part (b), and bounded below by part (a) (it is bounded below by any of the b_n), so it converges by the monotone convergence theorem. Similarly, $\{b_n\}$ is monotone and nondecreasing by part (b) and bounded above by part (a), so it too converges.

d.

$$a_{n+1} - b_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n} = \frac{1}{2}(a_n - 2\sqrt{a_n b_n} + b_n) < \frac{1}{2}(a_n - 2\sqrt{b_n^2} + b_n) = \frac{1}{2}(a_n - b_n).$$

Thus the difference between a_{n+1} and b_{n+1} is less than half the difference between a_n and b_n, so that difference goes to zero and the two limits are the same.

e. The AGM of 12 and 20 is approximately 15.745; Gauss' constant is $\frac{1}{\text{AGM}(1,\sqrt{2})} \approx 0.8346$.

10.2.99

a. Define a_n as given in the problem statement. Then we can *define* the value of the continued fraction to be $\lim_{n\to\infty} a_n$.

b. $a_0 = 1$, $a_1 = 1 + \dfrac{1}{a_0} = 2$, $a_2 = 1 + \dfrac{1}{a_1} = \dfrac{3}{2} = 1.5$, $a_3 = 1 + \dfrac{1}{a_2} = \dfrac{5}{3} \approx 1.667$, $a_4 = 1 + \dfrac{1}{a_3} = \dfrac{8}{5} = 1.6$, $a_5 = 1 + \dfrac{1}{a_4} = \dfrac{13}{8} = 1.625$.

c. From the list above, the values of the sequence alternately decrease and increase, so we would expect that the limit is somewhere between 1.6 and 1.625.

d. Assume that the limit is equal to L. Then from $a_{n+1} = 1 + \dfrac{1}{a_n}$, we have $\lim_{n\to\infty} a_{n+1} = 1 + \dfrac{1}{\lim_{n\to\infty} a_n}$, so $L = 1 + \dfrac{1}{L}$, and thus $L^2 - L - 1 = 0$. Therefore, $L = \dfrac{1 \pm \sqrt{5}}{2}$, and because L is clearly positive, it must be equal to $\dfrac{1 + \sqrt{5}}{2} \approx 1.618$.

e. Here $a_0 = a$ and $a_{n+1} = a + \dfrac{b}{a_n}$. Assuming that $\lim_{n\to\infty} a_n = L$ we have $L = a + \dfrac{b}{L}$, so $L^2 = aL + b$, and thus $L^2 - aL - b = 0$. Therefore, $L = \dfrac{a \pm \sqrt{a^2 + 4b}}{2}$, and because $L > 0$ we have $L = \dfrac{a + \sqrt{a^2 + 4b}}{2}$.

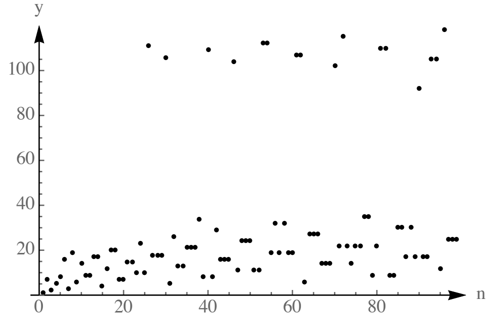
10.2.100

a.

2 : 1
3 : 10, 5, 16, 8, 4, 2, 1
4 : 2, 1
5 : 16, 8, 4, 2, 1
6 : 3, 10, 5, 16, 8, 4, 2, 1
7 : 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1
8 : 4, 2, 1
9 : 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1
10 : 5, 16, 8, 4, 2, 1

b. From the above, $H_2 = 1, H_3 = 7$, and $H_4 = 2$.

c. This plot is for $1 \leq n \leq 100$. Like hailstones, the numbers in the sequence a_n rise and fall but eventually crash to the earth. The conjecture appears to be true.



10.2.101 Let $\varepsilon > 0$ be given and let N be an integer with $N > \frac{1}{\varepsilon}$. Then if $n > N$, we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

10.2.102 Let $\varepsilon > 0$ be given. We wish to find N such that $|(1/n^2) - 0| < \varepsilon$ if $n > N$. This means that $\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} < \varepsilon$. So choose N such that $\frac{1}{N^2} < \varepsilon$, so that $N^2 > \frac{1}{\varepsilon}$, and then $N > \frac{1}{\sqrt{\varepsilon}}$. This shows that such an N always exists for each ε and thus that the limit is zero.

10.2.103 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left|\frac{3n^2}{4n^2+1} - \frac{3}{4}\right| = \left|\frac{-3}{4(4n^2+1)}\right| = \frac{3}{4(4n^2+1)} < \varepsilon$. But this means that $3 < 4\varepsilon(4n^2+1)$, or $16\varepsilon n^2 + (4\varepsilon - 3) > 0$. Solving the quadratic, we get $n > \frac{1}{4}\sqrt{\frac{3}{\varepsilon} - 4}$, provided $\varepsilon < 3/4$. So let $N = \frac{1}{4}\sqrt{\frac{3}{\varepsilon}}$ if $\epsilon < 3/4$ and let $N = 1$ otherwise.

10.2.104 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $|b^{-n} - 0| = b^{-n} < \varepsilon$, so that $-n \ln b < \ln \varepsilon$. So choose N to be any integer greater than $-\frac{\ln \varepsilon}{\ln b}$.

10.2.105 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left|\frac{cn}{bn+1} - \frac{c}{b}\right| = \left|\frac{-c}{b(bn+1)}\right| = \frac{c}{b(bn+1)} < \varepsilon$. But this means that $\varepsilon b^2 n + (b\varepsilon - c) > 0$, so that $N > \frac{c}{b^2\varepsilon}$ will work.

10.2.106 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \varepsilon$. Thus we want $n < \varepsilon(n^2+1)$, or $\varepsilon n^2 - n + \varepsilon > 0$. Whenever n is larger than the larger of the two roots of this quadratic, the desired inequality will hold. The roots of the quadratic are $\frac{1 \pm \sqrt{1-4\varepsilon^2}}{2\varepsilon}$, so we choose N to be any integer greater than $\frac{1+\sqrt{1-4\varepsilon^2}}{2\varepsilon}$.

10.2.107 First note that for $a = 1$ we already know that $\{n^n\}$ grows fast than $\{n!\}$. So if $a > 1$, then $n^{an} \ge n^n$, so that $\{n^{an}\}$ grows faster than $\{n!\}$ for $a > 1$ as well. To settle the case $a < 1$, recall Stirling's formula which states that for large values of n,

$$n! \sim \sqrt{2\pi n}n^n e^{-n}.$$

Thus

$$\begin{aligned}
\lim_{n\to\infty} \frac{n!}{n^{an}} &= \lim_{n\to\infty} \frac{\sqrt{2\pi n}n^n e^{-n}}{n^{an}} \\
&= \sqrt{2\pi} \lim_{n\to\infty} n^{\frac{1}{2}+(1-a)n} e^{-n} \\
&\ge \sqrt{2\pi} \lim_{n\to\infty} n^{(1-a)n} e^{-n} \\
&= \sqrt{2\pi} \lim_{n\to\infty} e^{(1-a)n\ln n} e^{-n} \\
&= \sqrt{2\pi} \lim_{n\to\infty} e^{((1-a)\ln n - 1)n}.
\end{aligned}$$

If $a < 1$ then $(1-a)\ln n - 1 > 0$ for large values of n because $1 - a > 0$, so that this limit is infinite. Hence $\{n!\}$ grows faster than $\{n^{an}\}$ exactly when $a < 1$.

10.2.108 $\{2n-3\}_{n=3}^{\infty}$.

10.2.109 $\{(n-2)^2 + 6(n-2) - 9\}_{n=3}^{\infty} = \{n^2 + 2n - 17\}_{n=3}^{\infty}$.

10.2.110 $\{a_n\} \ll \{b_n\}$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. But $\lim_{n\to\infty} \frac{ca_n}{db_n} = \frac{c}{d} \lim_{n\to\infty} \frac{a_n}{b_n} = 0$, so that $\{ca_n\} \ll \{db_n\}$.

10.2.111 For $b = 2$, $2^3 > 3!$ but $16 = 2^4 < 4! = 24$, so the crossover point is $n = 4$. For e, $e^5 \approx 148.41 > 5! = 120$ while $e^6 \approx 403.4 < 6! = 720$, so the crossover point is $n = 6$. For 10, $24! \approx 6.2 \times 10^{23} < 10^{24}$, while $25! \approx 1.55 \times 10^{25} > 10^{25}$, so the crossover point is $n = 25$.

10.3 Infinite Series

10.3.1 The ratio is the common ratio between successive terms in the sum.

10.3.2 A geometric sum is the sum of a finite number of terms which have a constant ratio; a geometric series is the sum of an infinite number of such terms.

10.3.3 No. For example, the geometric series with $a_n = 3 \cdot 2^n$ does not have a finite sum.

10.3.4 Yes, because there are only a finite number of terms.

10.3.5

a. $a = \frac{2}{3}$; $r = \frac{1}{5}$.

b. $a = \frac{1}{27}$; $r = -\frac{1}{3}$.

10.3.6 The series converges if and only if $|r| < 1$.

10.3.7 $S_n = \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \ldots + \left(\frac{1}{n+3} - \frac{1}{n+4}\right) = \frac{1}{4} - \frac{1}{n+4}$. $S_{36} = \frac{1}{4} - \frac{1}{40} = \frac{10-1}{40} = \frac{9}{40}$.

10.3.8

$$\sum_{k=5}^{\infty} \frac{3}{4k^2 - 63} = \sum_{k=1}^{\infty} \frac{3}{4(k+4)^2 - 63} = \sum_{k=1}^{\infty} \frac{3}{4k^2 + 32k + 1}.$$

10.3.9 $S = 1 \cdot \frac{1 - 3^9}{1 - 3} = \frac{19682}{2} = 9841$.

10.3.10 $S = 1 \cdot \frac{1 - (1/4)^{11}}{1 - (1/4)} = \frac{4^{11} - 1}{3 \cdot 4^{10}} = \frac{4194303}{3 \cdot 1048576} = \frac{1398101}{1048576} \approx 1.333$.

10.3.11 $S = 1 \cdot \frac{1 - (4/25)^{21}}{1 - 4/25} = \frac{25^{21} - 4^{21}}{25^{21} - 4 \cdot 25^{20}} \approx 1.1905$.

10.3.12 $S = 16 \cdot \frac{1 - 2^9}{1 - 2} = 511 \cdot 16 = 8176$.

10.3.13 $S = 1 \cdot \frac{1 - (-3/4)^{10}}{1 + 3/4} = \frac{4^{10} - 3^{10}}{4^{10} + 3 \cdot 4^9} = \frac{141361}{262144} \approx 0.5392$.

10.3.14 $S = 1 \cdot \frac{1 - \pi^7}{1 - \pi} = \frac{\pi^7 - 1}{\pi - 1} \approx 1409.84$.

10.3.15 $\frac{1}{4} \sum_{k=0}^{6} \frac{1}{3^k} = \frac{1}{4} \left(\frac{1 - (1/3)^7}{1 - (1/3)} \right) = \frac{1093}{2916}$.

10.3.16 $A_{60} = 100 + 100(1.0015) + \cdots + 100(1.0015)^{59} = 100 \left(\frac{1 - 1.0015^{60}}{1 - 1.0015} \right) = 6273.37$.

10.3.17 $A_{60} = 250 + 250(1.002) + \cdots + 250(1.002)^{59} = 250 \left(\frac{1 - 1.002^{60}}{1 - 1.002} \right) = 15,920.22$.

10.3.18

a. The nth partial sum is $S_n = a \left(\frac{1 - r^n}{1 - r} \right) = \frac{1 - (2/5)^n}{1 - (2/5)} = \frac{5}{3} \left(1 - \left(\frac{2}{5} \right)^n \right)$. So the sum of the series is $S = \lim_{n\to\infty} S_n = \frac{5}{3}(1 - 0) = \frac{5}{3}$.

b. $S = \sum_{k=0}^{\infty} \left(\frac{2}{5} \right)^k = \frac{1}{1 - (2/5)} = \frac{5}{3}$.

10.3.19

a. The nth partial sum is $S_n = a \left(\frac{1 - r^n}{1 - r} \right) = \frac{1 - (-2/7)^n}{1 - (-2/7)} = \frac{7}{9} \left(1 - \left(-\frac{2}{7} \right)^n \right)$. So the sum of the series is $S = \lim_{n\to\infty} S_n = \frac{7}{9}(1 - 0) = \frac{7}{9}$.

b. $S = \sum_{k=0}^{\infty} \left(-\frac{2}{7} \right)^k = \frac{1}{1 - (-2/7)} = \frac{7}{9}$.

10.3.20

a. The series can be written as $2\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k$.

The nth partial sum is $S_n = a\left(\frac{1-r^n}{1-r}\right) = 2\left(\frac{1-(2/3)^n}{1-(2/3)}\right) = 6\left(1-\left(\frac{2}{3}\right)^n\right)$. So the sum of the series is $S = \lim_{n\to\infty} S_n = 6(1-0) = 6$.

b. $S = 2\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = \frac{2}{1-(2/3)} = 6$.

10.3.21 $\frac{1}{1-1/4} = \frac{4}{3}$.

10.3.22 $\frac{1}{1-3/5} = \frac{5}{2}$.

10.3.23 $\frac{1}{1+9/10} = \frac{10}{19}$.

10.3.24 $-\frac{2/3}{1+2/3} = -\frac{2}{5}$.

10.3.25 $\frac{1}{1-0.9} = 10$.

10.3.26 $\frac{1}{1-2/7} = \frac{7}{5}$.

10.3.27 Divergent, because $r > 1$.

10.3.28 $\frac{1}{1-1/\pi} = \frac{\pi}{\pi-1}$.

10.3.29 $\frac{e^{-2}}{1-e^{-2}} = \frac{1}{e^2-1}$.

10.3.30 $\frac{5/4}{1-1/2} = \frac{5}{2}$.

10.3.31 $\frac{2^{-3}}{1-2^{-3}} = \frac{1}{7}$.

10.3.32 $\frac{3\cdot 4^3/7^3}{1-4/7} = \frac{64}{49}$.

10.3.33 $\frac{1/625}{1-1/5} = \frac{1}{500}$.

10.3.34 Note that this is the same as $\sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^k$.
Then $S = \frac{1}{1-3/4} = 4$.

10.3.35 $3\cdot\frac{1}{1+1/\pi} = \frac{3\pi}{\pi+1}$.

10.3.36 $\sum_{k=1}^{\infty}\left(-\frac{1}{e}\right)^k = -\frac{1/e}{1+1/e} = -\frac{1}{e+1}$.

10.3.37 $\frac{1}{1-e/\pi} = \frac{\pi}{\pi-e}$.

10.3.38 $\frac{1/16}{1-3/4} = \frac{1}{4}$.

10.3.39 $\frac{0.15^2}{1.15} = \frac{9}{460} \approx 0.0196$.

10.3.40 $-\frac{3/8^3}{1+1/8^3} = -\frac{1}{171}$.

10.3.41 $\frac{4/12}{1-(1/2)} = \frac{4/12}{11/12} = \frac{4}{11}$.

10.3.42 $3\left(\frac{1/e^2}{1-(1/e)}\right) = \frac{3}{e(e-1)}$.

10.3.43 $A_5 = 200 + 200(0.25) + \cdots + 200(0.25)^4 = 200\left(\frac{1-0.25^5}{1-0.25}\right) = 266.406$.

$A_{10} = 200 + 200(0.25) + \cdots + 200(0.25)^9 = 200\left(\frac{1-0.25^{10}}{1-0.25}\right) = 266.666$.

$A_{30} = 200 + 200(0.25) + \cdots + 200(0.25)^{29} = 200\left(\frac{1-0.25^{30}}{1-0.25}\right) = 266.667$.

$\lim_{n\to\infty} 200\left(\frac{1}{1-0.25}\right) = 266.\overline{66}\,\text{mg}$.

10.3.44 $A_5 = 150 + 150(0.4) + \cdots + 150(0.4)^4 = 150\left(\dfrac{1-0.4^5}{1-0.4}\right) = 247.44$.
$A_{10} = 150 + 150(0.4) + \cdots + 150(0.4)^9 = 150\left(\dfrac{1-0.4^{10}}{1-0.4}\right) = 249.974$.
$A_{30} = 150 + 150(0.4) + \cdots + 150(0.4)^{29} = 150\left(\dfrac{1-0.4^{30}}{1-0.4}\right) = 250$.
$\lim_{n\to\infty} A_n = 150\left(\dfrac{1}{1-0.4}\right) = 250\,\text{mg}$.

10.3.45 $200 + 200(0.5) + 200(0.5)^2 + \cdots = \dfrac{200}{1-0.5} = 400\,\text{mg}$.

10.3.46 $0.\overline{6} = 0.666\ldots = \sum_{k=1}^{\infty} 6(0.1)^k = \dfrac{0.6}{1-0.1} = \dfrac{0.6}{0.9} = \dfrac{2}{3}$.

10.3.47 $0.\overline{3} = 0.333\ldots = \sum_{k=1}^{\infty} 3(0.1)^k = \dfrac{0.3}{1-0.1} = \dfrac{0.3}{0.9} = \dfrac{1}{3}$.

10.3.48 $0.\overline{09} = 0.0909\ldots = \sum_{k=1}^{\infty} 9(0.01)^k = \dfrac{0.09}{1-0.01} = \dfrac{0.09}{.99} = \dfrac{9}{99} = \dfrac{1}{11}$.

10.3.49 $0.\overline{037} = 0.037037037\ldots = \sum_{k=1}^{\infty} 37(0.001)^k = \dfrac{0.037}{1-0.001} = \dfrac{0.037}{0.999} = \dfrac{1}{27}$.

10.3.50 $1.\overline{25} = 1.252525\ldots = 1 + \sum_{k=0}^{\infty} 0.25 \cdot 10^{-2k} = 1 + \dfrac{0.25}{1-1/100} = 1 + \dfrac{25}{99} = \dfrac{124}{99}$.

10.3.51 $0.\overline{456} = 0.456456456\ldots = \sum_{k=0}^{\infty} 0.456 \cdot 10^{-3k} = \dfrac{0.456}{1-1/1000} = \dfrac{456}{999} = \dfrac{152}{333}$.

10.3.52 $5.12\overline{83} = 5.12838383\ldots = 5.12 + \sum_{k=0}^{\infty} 0.0083 \cdot 10^{-2k} = 5.12 + \dfrac{0.0083}{1-1/100} = \dfrac{512}{100} + \dfrac{0.83}{99} = \dfrac{128}{25} + \dfrac{83}{9900} = \dfrac{50771}{9900}$.

10.3.53 $0.00\overline{952} = 0.00952952\ldots = \sum_{k=0}^{\infty} 0.00952 \cdot 10^{-3k} = \dfrac{0.00952}{1-1/1000} = \dfrac{9.52}{999} = \dfrac{952}{99900} = \dfrac{238}{24975}$.

10.3.54 The second part of each term cancels with the first part of the succeeding term, so $S_n = \dfrac{1}{1+2} - \dfrac{1}{n+3} = \dfrac{n}{3n+9}$, and $\lim_{n\to\infty} \dfrac{n}{3n+9} = \dfrac{1}{3}$.

10.3.55 The second part of each term cancels with the first part of the succeeding term, so $S_n = \dfrac{1}{1+1} - \dfrac{1}{n+2} = \dfrac{n}{2n+4}$, and $\lim_{n\to\infty} \dfrac{n}{2n+4} = \dfrac{1}{2}$.

10.3.56 $\dfrac{20}{25k^2+15k-4} = \dfrac{20}{(5k-1)(5k+4)} = \dfrac{4}{5k-1} - \dfrac{4}{5k+4}$, so the series given is the same as $\sum_{k=1}^{\infty}\left(\dfrac{4}{5k-1} - \dfrac{4}{5k+4}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term, so $S_n = 1 - \dfrac{4}{5n+4}$. Thus $\lim_{n\to\infty} S_n = 1$.

10.3.57 $\frac{1}{(k+6)(k+7)} = \frac{1}{k+6} - \frac{1}{k+7}$, so the series given is the same as $\sum_{k=1}^{\infty}\left(\frac{1}{k+6} - \frac{1}{k+7}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term, so $S_n = \frac{1}{1+6} - \frac{1}{n+7}$. Thus $\lim_{n\to\infty} S_n = \frac{1}{7}$.

10.3.58

$$\frac{10}{4k^2+32k+63} = \frac{10}{(2k+7)(2k+9)}.$$

Our series can be written as $\sum_{k=-3}^{\infty} \frac{10}{(2k+7)(2k+9)} = \sum_{k=1}^{\infty} \frac{10}{(2k-1)(2k+1)}$ by reindexing. Then, using a partial fraction decomposition, we have $\sum_{k=1}^{\infty}\left(\frac{5}{2k-1} - \frac{5}{2k+1}\right)$. In this last series, the second part of each term cancels with the first part of the succeeding term, so $S_n = \frac{5}{1} - \frac{5}{2n+1}$. Thus $\lim_{n\to\infty} S_n = 5$.

10.3.59 Note that $\frac{4}{(4k-3)(4k+1)} = \frac{1}{4k-3} - \frac{1}{4k+1}$.

Thus the given series is the same as $\sum_{k=3}^{\infty}\left(\frac{1}{4k-3} - \frac{1}{4k+1}\right) = \sum_{k=1}^{\infty}\left(\frac{1}{4k+5} - \frac{1}{4k+9}\right)$. In this last series, the second part of each term cancels with the first part of the succeeding term, so we have $S_n = \frac{1}{9} - \frac{1}{4n+9}$, and thus $\lim_{n\to\infty} S_n = \frac{1}{9}$.

10.3.60 $\frac{1}{(3k+1)(3k+4)} = \frac{1}{3}\left(\frac{1}{3k+1} - \frac{1}{3k+4}\right)$, so the series given can be written $\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{1}{3k+1} - \frac{1}{3k+4}\right) = \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{1}{3k-2} - \frac{1}{3k+1}\right)$. In this last series, the second part of each term cancels with the first part of the succeeding term, so we are left with $S_n = \frac{1}{3}\left(\frac{1}{1} - \frac{1}{(3n-1)}\right)$ and $\lim_{n\to\infty} S_n = \frac{1}{3}\lim_{n\to\infty}\left(1 - \frac{1}{3n-1}\right) = \frac{1}{3}$.

10.3.61 $\ln\left(\frac{k+1}{k}\right) = \ln(k+1) - \ln k$, so the series given is the same as $\sum_{k=1}^{\infty}(\ln(k+1) - \ln k)$, in which the first part of each term cancels with the second part of the next term, so we have $S_n = \ln(n+1) - \ln 1 = \ln(n+1)$, and thus the series diverges.

10.3.62 Note that $\frac{2}{(2k-1)(2k+1)} = \frac{1}{2k-1} - \frac{1}{2k+1}$.

Thus the given series is the same as $\sum_{k=3}^{\infty}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \sum_{k=1}^{\infty}\left(\frac{1}{2k+3} - \frac{1}{2k+5}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term, so we have $S_n = \frac{1}{5} - \frac{1}{2n+5}$. Thus, $\lim_{n\to\infty} S_n = \frac{1}{5}$.

10.3.63 $\frac{1}{(k+p)(k+p+1)} = \frac{1}{k+p} - \frac{1}{k+p+1}$, so that $\sum_{k=1}^{\infty} \frac{1}{(k+p)(k+p+1)} = \sum_{k=1}^{\infty}\left(\frac{1}{k+p} - \frac{1}{k+p+1}\right)$ and this series telescopes to give $S_n = \frac{1}{p+1} - \frac{1}{n+p+1} = \frac{n}{n(p+1)+(p+1)^2}$ so that $\lim_{n\to\infty} S_n = \frac{1}{p+1}$.

10.3.64 $\frac{1}{(ak+1)(ak+a+1)} = \frac{1}{a}\left(\frac{1}{ak+1} - \frac{1}{ak+a+1}\right)$, so that $\sum_{k=1}^{\infty} \frac{1}{(ak+1)(ak+a+1)} =$ $\frac{1}{a}\sum_{k=1}^{\infty}\left(\frac{1}{ak+1} - \frac{1}{ak+a+1}\right)$. This series telescopes - the second term of each summand cancels with the first term of the succeeding term – so that $S_n = \frac{1}{a}\left(\frac{1}{a+1} - \frac{1}{an+a+1}\right)$, and thus the limit of the sequence is $\frac{1}{a(a+1)}$.

10.3.65 Let $a_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+3}}$. Then the second term of a_n cancels with the first term of a_{n+2}, so the series telescopes and $S_n = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{n-1+3}} - \frac{1}{\sqrt{n+3}}$ and thus the sum of the series is the limit of S_n, which is $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}$.

10.3.66 $\frac{6}{k(k+2)} = \frac{3}{k} - \frac{3}{k+2}$. So our series can be written as $\sum_{k=1}^{\infty}\left(\frac{3}{k} - \frac{3}{k+2}\right)$. Starting with S_3 the second part of each term cancels the first part of the term that is two ahead, so

$$S_n = \frac{3}{1} + \frac{3}{2} - \frac{3}{n+1} - \frac{3}{n+2}.$$

So $\lim_{n\to\infty} S_n = 3 + \frac{3}{2} = \frac{9}{2}$.

10.3.67 $\frac{3}{k^2+5k+4} = \frac{3}{(k+1)(k+4)} = \frac{1}{k+1} - \frac{1}{k+4}$. So our series can be written as

$$\sum_{k=1}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+4}\right).$$

Starting with S_4, the second part of each term cancels the first part of the term that is 3 terms ahead, so

$$S_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4},$$

and $\lim_{n\to\infty} S_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$.

10.3.68 Note that $S_n = (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + \cdots + (\sqrt{n+1}-\sqrt{n})$. The second part of each term cancels with the first part of the previous term. Thus, $S_n = \sqrt{n+1} - 1$. and because $\lim_{n\to\infty} \sqrt{n+1} - 1 = \infty$, the series diverges.

10.3.69 This series telescopes to give

$$S_n = \tan^{-1}(n+1) - \tan^{-1}(1) = \tan^{-1}(n+1) - \frac{\pi}{4}.$$

Then because $\lim_{n\to\infty} \tan^{-1}(n+1) = \frac{\pi}{2}$, the sum of the series is equal to $\lim_{n\to\infty} S_n = \frac{\pi}{4}$.

10.3.70

a. Because the first part of each term cancels the second part of the previous term, the nth partial sum telescopes to be $S_n = \frac{1}{2} - \frac{1}{2^{n+1}}$. Thus, the sum of the series is $\lim_{n\to\infty} S_n = \frac{1}{2}$.

b. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{2^k}-\frac{1}{2^{k+1}}\right)=\sum_{k=1}^{\infty}\frac{1}{2^k}-\sum_{k=1}^{\infty}\frac{1}{2^{k+1}}=\frac{1/2}{1-1/2}-\frac{1/4}{1-1/2}=1-\frac{1}{2}=\frac{1}{2}$.

10.3.71

a. Because the first part of each term cancels the second part of the previous term, the nth partial sum telescopes to be $S_n=\frac{4}{3}-\frac{4}{3^{n+1}}$. Thus, the sum of the series is $\displaystyle\lim_{n\to\infty}S_n=\frac{4}{3}$.

b. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{4}{3^k}-\frac{4}{3^{k+1}}\right)=\sum_{k=1}^{\infty}\frac{4}{3^k}-\sum_{k=1}^{\infty}\frac{4}{3^{k+1}}=\frac{4/3}{1-1/3}-\frac{4/9}{1-1/3}=2-\frac{2}{3}=\frac{4}{3}$.

10.3.72 $\displaystyle\frac{1}{9k^2+39k+40}=\frac{1}{(3k+5)(3k+8)}=\frac{1/3}{3k+5}-\frac{1/3}{3k+8}$. We have

$$S_n=\frac{1}{3}\left(\left(\frac{1}{8}-\frac{1}{11}\right)+\left(\frac{1}{11}-\frac{1}{14}\right)+\cdots+\left(\frac{1}{3n+5}-\frac{1}{3n+8}\right)\right)=\frac{1}{3}\left(\frac{1}{8}-\frac{1}{3n+8}\right).$$

Thus $\displaystyle S=\lim_{n\to\infty}S_n=\frac{1}{24}$.

10.3.73 $16k^2+8k-3=(4k+3)(4k-1)$, so $\displaystyle\frac{1}{16k^2+8k-3}=\frac{1}{(4k+3)(4k-1)}=\frac{1}{4}\left(\frac{1}{4k-1}-\frac{1}{4k+3}\right)$. Thus the series given is equal to $\displaystyle\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{1}{4k-1}-\frac{1}{4k+3}\right)$. This series telescopes, so $\displaystyle S_n=\frac{1}{4}\left(-1-\frac{1}{4n-1}\right)$, so the sum of the series is equal to $\displaystyle\lim_{n\to\infty}S_n=-\frac{1}{4}$.

10.3.74 The first term of the k^{th} summand is $\sin\left(\dfrac{k\pi}{2k-1}\right)$; the second term of the $(k+1)^{\text{st}}$ summand is $-\sin\left(\dfrac{k\pi}{2k-1}\right)$; these two are equal except for sign, so they cancel. Thus

$$S_n=-\sin 0+\sin\left(\frac{n\pi}{2n-3}\right)=\sin\left(\frac{n\pi}{2n-3}\right).$$

Because $\dfrac{n\pi}{2n-3}$ has limit $\dfrac{\pi}{2}$ as $n\to\infty$, and because the sine function is continuous, $\displaystyle\lim_{n\to\infty}S_n$ is $\sin\left(\frac{\pi}{2}\right)=1$.

10.3.75 $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k 5^{3-k}=5^3\sum_{k=0}^{\infty}\left(\frac{1}{20}\right)^k=5^3\cdot\frac{1}{1-1/20}=\frac{5^3\cdot 20}{19}=\frac{2500}{19}$.

10.3.76 $\displaystyle\frac{3^6/8^6}{1-(3/8)^3}=\frac{729}{248320}$

10.3.77 This can be written as $\displaystyle\frac{1}{3}\sum_{k=1}^{\infty}\left(-\frac{2}{3}\right)^k$. This is a geometric series with ratio $r=-\dfrac{2}{3}$ so the sum is

$$\frac{1}{3}\cdot\frac{-2/3}{1-(-2/3)}=\frac{1}{3}\cdot\left(-\frac{2}{5}\right)=-\frac{2}{15}.$$

10.3.78 We have

$$S_n=\left(\sin^{-1}1-\sin^{-1}\frac{1}{2}\right)+\left(\sin^{-1}\frac{1}{2}-\sin^{-1}\frac{1}{3}\right)+\cdots+\left(\sin^{-1}\frac{1}{n}-\sin^{-1}\frac{1}{n+1}\right).$$

Note that the first part of each term cancels the second part of the previous term, so the nth partial sum telescopes to be $\sin^{-1}1-\sin^{-1}\dfrac{1}{n+1}$. Because $\sin^{-1}1=\dfrac{\pi}{2}$ and $\displaystyle\lim_{n\to\infty}\sin^{-1}\frac{1}{n+1}=\sin^{-1}0=0$, we have $\displaystyle\lim_{n\to\infty}S_n=\frac{\pi}{2}$.

10.3.79 Note that

$$\frac{\ln((k+1)k^{-1})}{(\ln k)\ln(k+1)} = \frac{\ln(k+1)}{(\ln k)\ln(k+1)} - \frac{\ln k}{(\ln k)\ln(k+1)} = \frac{1}{\ln k} - \frac{1}{\ln(k+1)}.$$

In the partial sum S_n, the first part of each term cancels the second part of the preceding term, so we have $S_n = \frac{1}{\ln 2} - \frac{1}{\ln(n+2)}$. Thus we have $\lim_{n\to\infty} S_n = \frac{1}{\ln 2}$.

10.3.80 This can be written as $\frac{1}{e}\sum_{k=1}^{\infty}\left(\frac{\pi}{e}\right)^k$. This is a geometric series with $r = \frac{\pi}{e} > 1$, so the series diverges.

10.3.81 $\sum_{k=0}^{\infty}\left(3\left(\frac{2}{5}\right)^k - 2\left(\frac{5}{7}\right)^k\right) = 3\sum_{k=0}^{\infty}\left(\frac{2}{5}\right)^k - 2\sum_{k=0}^{\infty}\left(\frac{5}{7}\right)^k = 3\left(\frac{1}{3/5}\right) - 2\left(\frac{1}{2/7}\right) = 5 - 7 = -2.$

10.3.82 $\sum_{k=1}^{\infty}\left(2\left(\frac{3}{5}\right)^k + 3\left(\frac{4}{9}\right)^k\right) = 2\sum_{k=1}^{\infty}\left(\frac{3}{5}\right)^k + 3\sum_{k=1}^{\infty}\left(\frac{4}{9}\right)^k = 2\left(\frac{3/5}{2/5}\right) + 3\left(\frac{4/9}{5/9}\right) = 3 + \frac{12}{5} = \frac{27}{5}.$

10.3.83 $\sum_{k=1}^{\infty}\left(\frac{1}{3}\left(\frac{5}{6}\right)^k + \frac{3}{5}\left(\frac{7}{9}\right)^k\right) = \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^k + \frac{3}{5}\sum_{k=1}^{\infty}\left(\frac{7}{9}\right)^k = \frac{1}{3}\left(\frac{5/6}{1/6}\right) + \frac{3}{5}\left(\frac{7/9}{2/9}\right) = \frac{5}{3} + \frac{21}{10} = \frac{113}{30}.$

10.3.84 $\sum_{k=0}^{\infty}\left(\frac{1}{2}(0.2)^k + \frac{3}{2}(0.8)^k\right) = \frac{1}{2}\sum_{k=0}^{\infty}(0.2)^k + \frac{3}{2}\sum_{k=0}^{\infty}(0.8)^k = \frac{1}{2}\left(\frac{1}{0.8}\right) + \frac{3}{2}\left(\frac{1}{0.2}\right) = \frac{5}{8} + \frac{15}{2} = \frac{65}{8}.$

10.3.85 $\sum_{k=1}^{\infty}\left(\left(\frac{1}{6}\right)^k + \left(\frac{1}{3}\right)^{k-1}\right) = \sum_{k=1}^{\infty}\left(\frac{1}{6}\right)^k + \sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^{k-1} = \frac{1/6}{5/6} + \frac{1}{2/3} = \frac{17}{10}.$

10.3.86 $\sum_{k=0}^{\infty}\frac{2-3^k}{6^k} = \sum_{k=0}^{\infty}\left(\frac{2}{6^k} - \frac{3^k}{6^k}\right) = 2\sum_{k=0}^{\infty}\left(\frac{1}{6}\right)^k - \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = 2\left(\frac{1}{5/6}\right) - \frac{1}{1/2} = \frac{2}{5}.$

10.3.87

a. True. The two series differ by a finite amount $(\sum_{k=1}^{9} a_k)$, so if one converges, so does the other.

b. True. The same argument applies as in part (a).

c. False. If $\sum a_k$ converges, then $a_k \to 0$ as $k \to \infty$, so that $a_k + 0.0001 \to 0.0001$ as $k \to \infty$, so that $\sum(a_k + 0.0001)$ cannot converge.

d. False. Suppose $p = -1.0001$. Then $\sum p^k$ diverges but $p + .0001 = -0.9991$ so that $\sum(p + .0001)^k$ converges.

e. True. $\left(\frac{\pi}{e}\right)^{-k} = \left(\frac{e}{\pi}\right)^k$; because $e < \pi$, this is a geometric series with ratio less than 1.

f. False. For example, let $0 < a < 1$ and $b > 1$.

g. True. Suppose $a > \frac{1}{2}$. Then we want $a = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. Solving for r gives $r = 1 - \frac{1}{a}$. Because $a > 0$ we have $r < 1$; because $a > \frac{1}{2}$ we have $r > 1 - \frac{1}{1/2} = -1$. Thus $|r| < 1$ so that $\sum_{k=0}^{\infty} r^k$ converges, and it converges to a.

10.3.88 $0.010101\ldots = \dfrac{1}{2^2} + \dfrac{1}{2^4} + \dfrac{1}{2^6} + \cdots = \dfrac{1/4}{1-1/4} = \dfrac{1}{3}$.

10.3.89

a. $0.00110011\ldots = \dfrac{1}{2^3} + \dfrac{1}{2^4} + \dfrac{1}{2^7} + \dfrac{1}{2^8} + \dfrac{1}{2^{11}} + \dfrac{1}{2^{12}} + \cdots = \dfrac{3}{2^4} + \dfrac{3}{2^8} + \dfrac{3}{2^{12}} + \cdots = \dfrac{3/2^4}{1-1/2^4} = \dfrac{3}{15} = \dfrac{1}{5}$.

b. $0.0011001100110011 = \dfrac{3}{2^4} + \dfrac{3}{2^8} + \dfrac{3}{2^{12}} + \dfrac{3}{2^{16}} \approx 0.19999695$.

10.3.90 We seek r so that $\dfrac{1}{1-r} = 10$, so $r - 1 = -\dfrac{1}{10}$, so $r = \dfrac{9}{10}$.

10.3.91 We seek r so that $\dfrac{r^6}{1-r^2} = 10$, so $r^6 = 10 - 10r^2$, so $r^6 + 10r^2 - 10 = 0$. Using a computer algebra system, we obtain $r \approx 0.96$.

10.3.92

a. Note that $\dfrac{3^k}{(3^{k+1}-1)(3^k-1)} = \dfrac{1}{2} \cdot \left(\dfrac{1}{3^k-1} - \dfrac{1}{3^{k+1}-1}\right)$. Then

$$\sum_{k=1}^{\infty} \frac{3^k}{(3^{k+1}-1)(3^k-1)} = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{3^k-1} - \frac{1}{3^{k+1}-1}\right).$$

This series telescopes to give $S_n = \dfrac{1}{2}\left(\dfrac{1}{3-1} - \dfrac{1}{3^{n+1}-1}\right)$, so that the sum of the series is $\lim_{n\to\infty} S_n = \dfrac{1}{4}$.

b. We mimic the above computations. First, $\dfrac{a^k}{(a^{k+1}-1)(a^k-1)} = \dfrac{1}{a-1} \cdot \left(\dfrac{1}{a^k-1} - \dfrac{1}{a^{k+1}-1}\right)$, so we see that we cannot have $a = 1$, because the fraction would then be undefined. Continuing, we obtain $S_n = \dfrac{1}{a-1}\left(\dfrac{1}{a-1} - \dfrac{1}{a^{n+1}-1}\right)$. Now, $\lim_{n\to\infty} \dfrac{1}{a^{n+1}-1}$ converges if and only if the denominator grows without bound; this happens if and only if $|a| > 1$. Thus, the original series converges for $|a| > 1$, when it converges to $\dfrac{1}{(a-1)^2}$. Note that this is valid even for a negative.

10.3.93

$$\begin{aligned}
A_n &= A_{n-1}(1+r) - M \\
&= (A_{n-2}(1+r) - M)(1+r) - M \\
&= (A_{n-3}(1+r) - M)(1+r)^2 - (M + M(1+r)) \\
&= A_{n-3}(1+r)^3 - (M + M(1+r) + M(1+r)^2) = \cdots \\
&= A_0(1+r)^n - (M + M(1+r) + M(1+r)^2 + \cdots + M(1+r)^{n+1}) \\
&= A_0(1+r)^n - M\left(\frac{1-(1+r)^n}{1-(1+r)}\right) \\
&= A_0(1+r)^n - \frac{M}{r}\left((1+r)^n - 1\right)
\end{aligned}$$

10.3.94 $20000(1.0075)^n - \dfrac{600}{0.0075}(1.0075^n - 1) = 0$, so $20000(1.0075)^n - 80000((1.0075)^n - 1) = 0$, so $-60000(1.0075)^n + 80000 = 0$. Then $(1.0075)^n = \dfrac{80000}{60000} = \dfrac{4}{3}$, so $n = \dfrac{\ln(4/3)}{\ln 1.0075} \approx 38.5$. Rounding up, we have 39 payments.

10.3.95 $180000(1.005)^n - \dfrac{1000}{0.005}((1.005)^n - 1) = 0$, so $-20000(1.005)^n + 200000 = 0$. Then $(1.005)^n = 10$, so $n = \dfrac{\ln 10}{\ln 1.005} \approx 461.67$. Rounding up we have 462 months.

10.3.96 $\displaystyle\sum_{k=1}^{\infty}(a_k \pm b_k) = \lim_{n\to\infty}\sum_{k=1}^{n}(a_k \pm b_k) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k\right) = \lim_{n\to\infty}\sum_{k=1}^{n} a_k \pm \lim_{n\to\infty}\sum_{k=1}^{n} b_k = A \pm B.$

10.3.97 $\displaystyle\sum_{k=1}^{\infty} ca_k = \lim_{n\to\infty}\sum_{k=1}^{n} ca_k = \lim_{n\to\infty} c\sum_{k=1}^{n} a_k = c\lim_{n\to\infty}\sum_{k=1}^{n} a_k$, so that one sum diverges if and only if the other one does.

10.3.98 It will take Achilles 1/5 hour to cover the first mile. At this time, the tortoise has gone 1/5 mile more, and it will take Achilles 1/25 hour to reach this new point. At that time, the tortoise has gone another 1/25 of a mile, and it will take Achilles 1/125 hour to reach this point. Adding the times up, we have

$$\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots = \frac{1/5}{1 - 1/5} = \frac{1}{4},$$

so it will take Achilles 1/4 of an hour (15 minutes) to catch the tortoise.

10.3.99 At the n^{th} stage, there are 2^{n-1} triangles of area $A_n = \dfrac{1}{8}A_{n-1} = \dfrac{1}{8^{n-1}}A_1$, so the total area of the triangles formed at the n^{th} stage is $\dfrac{2^{n-1}}{8^{n-1}}A_1 = \left(\dfrac{1}{4}\right)^{n-1} A_1$. Thus the total area under the parabola is

$$\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1} A_1 = A_1\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1} = A_1\frac{1}{1-1/4} = \frac{4}{3}A_1.$$

10.3.100 Let L_n be the amount of light transmitted through the window the n^{th} time the beam hits the second pane. Then the amount of light that was available before the beam went through the pane was $\dfrac{L_n}{1-p}$, so $\dfrac{pL_n}{1-p}$ is reflected back to the first pane, and $\dfrac{p^2L_n}{1-p}$ is then reflected back to the second pane. Of that, a fraction equal to $1-p$ is transmitted through the window. Thus

$$L_{n+1} = (1-p)\frac{p^2L_n}{1-p} = p^2L_n.$$

The amount of light transmitted through the window the first time is $(1-p)^2$. Thus the total amount is

$$\sum_{i=0}^{\infty} p^{2n}(1-p)^2 = \frac{(1-p)^2}{1-p^2} = \frac{1-p}{1+p}.$$

10.3.101

a. I_{n+1} is obtained by I_n by dividing each edge into three equal parts, removing the middle part, and adding two parts equal to it. Thus 3 equal parts turn into 4, so $L_{n+1} = \dfrac{4}{3}L_n$. This is a geometric sequence with a ratio greater than 1, so the n^{th} term grows without bound.

b. As the result of part (a), I_n has $3\cdot 4^n$ sides of length $\dfrac{1}{3^n}$; each of those sides turns into an added triangle in I_{n+1} of side length 3^{-n-1}. Thus the added area in I_{n+1} consists of $3\cdot 4^n$ equilateral triangles with side 3^{-n-1}. The area of an equilateral triangle with side x is $\dfrac{x^2\sqrt{3}}{4}$. Thus $A_{n+1} = A_n + 3\cdot 4^n\cdot\dfrac{3^{-2n-2}\sqrt{3}}{4} = A_n + \dfrac{\sqrt{3}}{12}\cdot\left(\dfrac{4}{9}\right)^n$, and $A_0 = \dfrac{\sqrt{3}}{4}$. Thus $A_{n+1} = A_0 + \displaystyle\sum_{i=0}^{n}\frac{\sqrt{3}}{12}\cdot\left(\frac{4}{9}\right)^i$, so that

$$A_\infty = A_0 + \frac{\sqrt{3}}{12}\sum_{i=0}^{\infty}\left(\frac{4}{9}\right)^i = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\frac{1}{1-4/9} = \frac{\sqrt{3}}{4}(1+\frac{3}{5}) = \frac{2}{5}\sqrt{3}.$$

10.3.102

a. Clearly for $k < n$, h_k is a leg of a right triangle whose hypotenuse is r_k and whose other leg is formed where the vertical line (in the picture) meets a diameter of the next smaller sphere; thus the other leg of the triangle is r_{k+1}. The Pythagorean theorem then implies that $h_k^2 = r_k^2 - r_{k+1}^2$.

b. The height is $H_n = \sum_{i=1}^{n} h_i = r_n + \sum_{i=1}^{n-1} \sqrt{r_i^2 - r_{i+1}^2}$ by part (a).

c. From part (b), because $r_i = a^{i-1}$,

$$\begin{aligned} H_n &= r_n + \sum_{i=1}^{n-1} \sqrt{r_i^2 - r_{i+1}^2} = a^{n-1} + \sum_{i=1}^{n-1} \sqrt{a^{2i-2} - a^{2i}} \\ &= a^{n-1} + \sum_{i=1}^{n-1} a^{i-1}\sqrt{1-a^2} = a^{n-1} + \sqrt{1-a^2}\sum_{i=1}^{n-1} a^{i-1} \\ &= a^{n-1} + \sqrt{1-a^2}\left(\frac{1-a^{n-1}}{1-a}\right). \end{aligned}$$

d. $\lim_{n\to\infty} H_n = \lim_{n\to\infty} a^{n-1} + \sqrt{1-a^2} \lim_{n\to\infty} \frac{1-a^{n-1}}{1-a} = 0 + \sqrt{1-a^2}\left(\frac{1}{1-a}\right) = \sqrt{\frac{1-a^2}{(1-a)(1-a)}} = \sqrt{\frac{1+a}{1-a}}$.

10.3.103 $|S - S_n| = \left|\sum_{i=n}^{\infty} r^k\right| = \left|\frac{r^n}{1-r}\right|$ because the latter sum is simply a geometric series with first term r^n and ratio r.

10.3.104

a. Solve $\frac{0.6^n}{0.4} < 10^{-6}$ for n to get $n = 29$.

b. Solve $\frac{0.15^n}{0.85} < 10^{-6}$ for n to get $n = 8$.

10.3.105

a. Solve $\left|\frac{(-0.8)^n}{1.8}\right| = \frac{0.8^n}{1.8} < 10^{-6}$ for n to get $n = 60$.

b. Solve $\frac{0.2^n}{0.8} < 10^{-6}$ for n to get $n = 9$.

10.3.106

a. Solve $\frac{0.72^n}{0.28} < 10^{-6}$ for n to get $n = 46$.

b. Solve $\left|\frac{(-0.25)^n}{1.25}\right| = \frac{0.25^n}{1.25} < 10^{-6}$ for n to get $n = 10$.

10.3.107

a. Solve $\frac{1/\pi^n}{1-1/\pi} < 10^{-6}$ for n to get $n = 13$.

b. Solve $\frac{1/e^n}{1-1/e} < 10^{-6}$ for n to get $n = 15$.

10.3.108

a. $f(x) = \displaystyle\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$; because f is represented by a geometric series, $f(x)$ exists only for $|x| < 1$. Then $f(0) = 1$, $f(0.2) = \dfrac{1}{0.8} = 1.25$, $f(0.5) = \dfrac{1}{1-0.5} = 2$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

10.3.109

a. $f(x) = \displaystyle\sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{1+x}$; because f is a geometric series, $f(x)$ exists only when the ratio, $-x$, is such that $|-x| = |x| < 1$. Then $f(0) = 1$, $f(0.2) = \dfrac{1}{1.2} = \dfrac{5}{6}$, $f(0.5) = \dfrac{1}{1+.05} = \dfrac{2}{3}$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

10.3.110

a. $f(x) = \displaystyle\sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2}$. f is a geometric series, so $f(x)$ is defined only when the ratio, x^2, is less than 1, which means $|x| < 1$. Then $f(0) = 1$, $f(0.2) = \dfrac{1}{1-.04} = \dfrac{25}{24}$, $f(0.5) = \dfrac{1}{1-0.25} = \dfrac{4}{3}$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

10.3.111 $f(x)$ is a geometric series with ratio $\dfrac{1}{1+x}$; thus $f(x)$ converges when $\left|\dfrac{1}{1+x}\right| < 1$. For $x > -1$, $\left|\dfrac{1}{1+x}\right| = \dfrac{1}{1+x}$ and $\dfrac{1}{1+x} < 1$ when $1 < 1+x$, $x > 0$. For $x < -1$, $\left|\dfrac{1}{1+x}\right| = \dfrac{1}{-1-x}$, and this is less than 1 when $1 < -1-x$, i.e. $x < -2$. So $f(x)$ converges for $x > 0$ and for $x < -2$. When $f(x)$ converges, its value is $\dfrac{1}{1-\frac{1}{1+x}} = \dfrac{1+x}{x}$, so $f(x) = 3$ when $1+x = 3x$, $x = \dfrac{1}{2}$.

10.4 The Divergence and Integral Tests

10.4.1 The series diverges.

10.4.2 No. For example, the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges although $\dfrac{1}{k} \to 0$ as $k \to \infty$.

10.4.3 We can be sure that $\displaystyle\lim_{k\to\infty} a_k = 0$.

10.4.4 It converges for $p > 1$, and diverges for all other values of p.

10.4.5 For the same values of p as in the previous problem – it converges for $p > 1$, and diverges for all other values of p.

10.4.6 Let S_n be the partial sums. Then $S_{n+1} - S_n = a_{n+1} > 0$ because $a_{n+1} > 0$. Thus the sequence of partial sums is increasing.

10.4.7 The remainder of an infinite series is the error in approximating a convergent infinite series by a finite number of terms.

10.4.8 Yes. Suppose $\sum a_k$ converges to S, and let the sequence of partial sums be $\{S_n\}$. Then for any $\epsilon > 0$ there is some N such that for any $n > N$, $|S - S_n| < \epsilon$. But $|S - S_n|$ is simply the remainder R_n when the series is approximated to n terms. Thus $R_n \to 0$ as $n \to \infty$.

10.4.9 $a_k = \dfrac{k}{2k+1}$ and $\lim\limits_{k\to\infty} a_k = \dfrac{1}{2}$, so the series diverges.

10.4.10 $a_k = \dfrac{k}{k^2+1}$ and $\lim\limits_{k\to\infty} a_k = 0$, so the divergence test is inconclusive.

10.4.11 $a_k = \dfrac{1}{1000+k}$ and $\lim\limits_{k\to\infty} a_k = 0$, so the divergence test is inconclusive.

10.4.12 $a_k = \dfrac{k^3}{k^3+1}$ and $\lim\limits_{k\to\infty} a_k = 1$, so the series diverges.

10.4.13 $a_k = \dfrac{k}{\ln k}$ and $\lim\limits_{k\to\infty} a_k = \infty$, so the series diverges.

10.4.14 $a_k = \dfrac{k^2}{2^k}$ and $\lim\limits_{k\to\infty} a_k = 0$, so the divergence test is inconclusive.

10.4.15 $a_k = \dfrac{\sqrt{k}}{\ln^{10} k}$ and $\lim\limits_{k\to\infty} a_k = \infty$, so the series diverges.

10.4.16 By Theorem 10.6 $k^3 \ll k!$, so $\lim\limits_{k\to\infty} \dfrac{k^3}{k!} = 0$. The divergence test is inconclusive.

10.4.17 Let $f(x) = \dfrac{1}{\sqrt{x+8}}$. $f(x)$ is obviously continuous and decreasing for $x \geq 1$. Because

$$\int_1^\infty \frac{1}{\sqrt{x+8}}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x+8}}\,dx = \lim_{b\to\infty} 2\sqrt{x+8}\bigg|_1^b = \lim_{b\to\infty} (2\sqrt{b+8} - 6) = \infty,$$

the series diverges.

10.4.18 The function $f(x) = \dfrac{1}{(2x+4)^2}$ is positive, decreasing, and continuous for $x \geq 1$. The relevant integral is

$$\int_1^\infty \frac{dx}{(2x+4)^2} = \lim_{b\to\infty} -\frac{1}{2(2x+4)}\bigg|_1^b = -\frac{1}{2}\lim_{b\to\infty}\left(\frac{1}{2b+4} - \frac{1}{6}\right) = \frac{1}{12}.$$

Because the integral converges, the given series also converges by the Integral Test.

10.4.19 The function $f(x) = \dfrac{1}{\sqrt[3]{5x+3}}$ is positive, decreasing, and continuous for $x \geq 1$. The relevant integral is

$$\int_1^\infty \frac{dx}{\sqrt[3]{5x+3}} = \lim_{b\to\infty} \frac{3}{10}(5x+3)^{2/3}\bigg|_1^b = \frac{3}{10}\lim_{b\to\infty}\left((5b+3)^{2/3} - 4\right) = \infty.$$

Because the integral diverges, the given series also diverges by the Integral Test.

10.4.20 The function $f(x) = \dfrac{e^x}{1+e^{2x}}$ is positive and continuous for $x \geq 1$. Note that $f'(x) = \dfrac{e^x(1-e^{2x})}{(e^{2x}+1)^2} \leq 0$ for $x \geq 1$, so f is decreasing. The relevant integral is

$$\int_1^\infty \frac{e^x}{1+e^{2x}}\,dx = \lim_{b\to\infty} \tan^{-1} e^x\bigg|_1^b = \lim_{b\to\infty}\left(\tan^{-1} e^b - \tan^{-1} e\right) = \frac{\pi}{2} - \tan^{-1} e.$$

Because the integral converges, the given series also converges by the Integral Test.

10.4.21 Let $f(x) = x \cdot e^{-2x^2}$. This function is continuous for $x \geq 1$. Its derivative is $e^{-2x^2}(1 - 4x^2) < 0$ for $x \geq 1$, so $f(x)$ is decreasing. The relevant integral is

$$\int_1^\infty x \cdot e^{-2x^2}\, dx = -\frac{1}{4}\lim_{b\to\infty} e^{-2x^2}\Big|_1^b = -\frac{1}{4}\lim_{b\to\infty}\left(e^{-2b^2} - e^{-2}\right) = \frac{1}{4e^2}.$$

Because the integral converges, the given series also converges by the Integral Test.

10.4.22 Let $f(x) = \dfrac{1}{x(\ln x)^2}$. $f(x)$ is continuous and decreasing for $x \geq 2$. The relevant integral is

$$\int_2^\infty f(x)\, dx = \lim_{b\to\infty}\int_2^b \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} -\frac{1}{\ln x}\Big|_2^b = \lim_{b\to\infty}\left(\frac{1}{\ln 2} - \frac{1}{\ln b}\right) = \frac{1}{\ln 2}.$$

Because the integral converges, the given series also converges by the Integral Test.

10.4.23 $\displaystyle\sum_{k=1}^\infty \frac{1}{k^{1/5}}$ is a p-series with $p = \dfrac{1}{5} < 1$. The series diverges by the p-series test.

10.4.24 $\displaystyle\sum_{k=1}^\infty \frac{1}{k^{3/2}}$ is a p-series with $p = \dfrac{3}{2} > 1$. The series converges by the p-series test.

10.4.25 We will use the Integral Test. $f(x) = \dfrac{x^3}{e^{x^4}}$ is continuous and positive, and $f'(x) = \dfrac{x^2(3 - 4x^4)}{e^{x^4}} < 0$ for $x > 1$. Thus f is decreasing for $x > 1$. The relevant integral is

$$\int_1^\infty \frac{x^3}{e^{x^4}}\, dx = \lim_{b\to\infty}\left(-\frac{e^{-x^4}}{4}\right)\Big|_1^b = \frac{1}{4e}.$$

Because the integral converges, the given series also converges by the Integral Test.

10.4.26 We will use the Integral Test. $f(x) = \dfrac{x^2}{x^3 + 5}$ is continuous and positive, and $f'(x) = \dfrac{x(10 - x^3)}{(x^3 + 5)^2} < 0$ for $x \geq 3$. Thus f is decreasing for $x \geq 3$. The relevant integral is

$$\int_1^\infty \frac{x^2}{x^3 + 5}\, dx = \frac{1}{3}\lim_{b\to\infty}\left(\ln(x^3 + 5)\right)\Big|_1^b = \infty.$$

Because the integral diverges, the given series also diverges by the Integral Test.

10.4.27 $a_k = k^{1/k}$. In order to compute $\displaystyle\lim_{k\to\infty} a_k$, we let $y_k = \ln a_k = \dfrac{\ln k}{k}$. By Theorem 10.6, (or by L'Hôpital's rule) $\displaystyle\lim_{k\to\infty} y_k = 0$, so $\displaystyle\lim_{k\to\infty} a_k = e^0 = 1$. The given series thus diverges by the Divergence Test.

10.4.28 $a_k = \dfrac{\sqrt{k^2 + 1}}{k}$ and $\displaystyle\lim_{k\to\infty} a_k = 1$, so the series diverges by the Divergence Test.

10.4.29 This is a p-series with $p = 10$, so this series converges by the p-series test.

10.4.30 $\displaystyle\sum_{k=2}^\infty \frac{k^e}{k^\pi} = \sum_{k=2}^\infty \frac{1}{k^{\pi - e}}$. Note that $\pi - e \approx 3.1416 - 2.71828 < 1$, so this series diverges by the p-series test.

10.4.31 $\displaystyle\sum_{k=3}^\infty \frac{1}{(k - 2)^4} = \sum_{k=1}^\infty \frac{1}{k^4}$, which is a p-series with $p = 4$, thus convergent by the p-series test.

10.4.32 $\sum_{k=1}^{\infty} 2k^{-3/2} = 2\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is a p-series with $p = 3/2$, thus convergent by the p-series test.

10.4.33 Let $f(x) = \frac{x}{e^x}$. $f(x)$ is clearly continuous for $x > 1$, and its derivative, $f'(x) = \frac{e^x - xe^x}{e^{2x}} = (1-x)\frac{e^x}{e^{2x}}$, is negative for $x > 1$ so that $f(x)$ is decreasing. Because $\int_1^{\infty} f(x)\,dx = 2e^{-1}$, the series converges by the Integral Test.

10.4.34 Let $f(x) = \frac{1}{x \cdot \ln x \cdot \ln\ln x}$. $f(x)$ is continuous and decreasing for $x > 3$, and

$$\int_3^{\infty} \frac{1}{x \cdot \ln x \cdot \ln\ln x}\,dx = \lim_{b\to\infty} \int_3^b \frac{1}{x \cdot \ln x \cdot \ln\ln x}\,dx = \lim_{b\to\infty} (\ln\ln\ln b - \ln\ln\ln 3) = \infty.$$

The given series therefore diverges by the Integral Test.

10.4.35 Because $\lim_{k\to\infty} \left(\frac{k}{k+10}\right)^k = \lim_{k\to\infty} \frac{1}{(1+(10/k))^k} = \frac{1}{e^{10}} \neq 0$, the series diverges by the Divergence Test.

10.4.36 We use the Integral Test. Note that $f(x) = \frac{x}{(x^2+1)^3}$ is continuous and positive on $(1, \infty)$. Also, $f'(x) = \frac{1-5x^2}{(x^2+1)^4} < 0$ for $x > 1$, so is decreasing. The relevant integral is

$$\int_1^{\infty} \frac{x}{(x^2+1)^3}\,dx = -\frac{1}{4} \lim_{b\to\infty} \left.\frac{1}{(x^2+1)^2}\right|_1^b = -\frac{1}{4} \lim_{b\to\infty} \left(\frac{1}{(b^2+1)^2} - \frac{1}{4}\right) = \frac{1}{16}.$$

Thus, the given series converges by the Integral Test.

10.4.37 $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$ is a p-series with $p = 1/3$, thus divergent by the p-series test.

10.4.38 $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{27k^2}} = \frac{1}{3}\sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$ is a p-series with $p = 2/3$, thus divergent by the p-series test.

10.4.39

a. $S \approx S_2 = 1 + \frac{1}{2^7} = 1.0078125.$

b. $R_2 < \int_2^{\infty} \frac{dx}{x^7} = \lim_{b\to\infty} \int_2^b x^{-7}\,dx = \lim_{b\to\infty} \left.-\frac{1}{6x^6}\right|_2^b = \lim_{b\to\infty} -\left(\frac{1}{6b^6} - \frac{1}{384}\right) \approx 0.0026042.$

c.

$$\begin{aligned}
L_2 &= S_2 + \int_3^{\infty} \frac{dx}{x^7} = S_2 + \lim_{b\to\infty} \int_3^b x^{-7}\,dx \\
&= S_2 + \lim_{b\to\infty} \left.-\frac{1}{6x^6}\right|_3^b = S_2 + \lim_{b\to\infty} -\left(\frac{1}{6b^6} - \frac{1}{4374}\right) \\
&\approx 1.0078125 + 0.000228624 \approx 1.0080411.
\end{aligned}$$

$$\begin{aligned}
U_2 &= S_2 + \int_2^{\infty} \frac{dx}{x^7} = S_2 + \lim_{b\to\infty} \int_2^b x^{-7}\,dx \\
&= S_2 + \lim_{b\to\infty} \left.-\frac{1}{6x^6}\right|_2^b = S_2 + \lim_{b\to\infty} -\left(\frac{1}{6b^6} - \frac{1}{384}\right) \\
&\approx 1.0078125 + 0.0026042 \approx 1.0104167.
\end{aligned}$$

So

$$1.0080411 \le S \le 1.0104167.$$

10.4.40

a. $S \approx S_5 = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} \approx 1.185662.$

b. $R_5 < \int_5^\infty \frac{dx}{x^3} = \lim_{b\to\infty} \int_5^b x^{-3}\,dx = \lim_{b\to\infty} -\frac{1}{2x^2}\bigg|_5^b = \lim_{b\to\infty} -\left(\frac{1}{2b^2} - \frac{1}{2(25)}\right) = \frac{1}{50} = 0.02.$

c.

$$\begin{aligned} L_5 &= S_5 + \int_6^\infty \frac{dx}{x^3} = S_5 + \lim_{b\to\infty} \int_6^b x^{-3}\,dx \\ &= S_5 + \lim_{b\to\infty} -\frac{1}{2x^2}\bigg|_6^b = S_5 + \lim_{b\to\infty} -\left(\frac{1}{2b^2} - \frac{1}{72}\right) \\ &\approx 1.185662 + 0.013888889 \approx 1.1995509 \end{aligned}$$

$$\begin{aligned} U_5 &= S_5 + \int_5^\infty \frac{dx}{x^3} = S_5 + \lim_{b\to\infty} -\frac{1}{2x^2}\bigg|_5^b \\ &= S_5 + \lim_{b\to\infty} -\left(\frac{1}{2b^2} - \frac{1}{2(25)}\right) = S_5 + \frac{1}{50} \\ &\approx 1.1856620 + 0.02 \approx 1.2056620. \end{aligned}$$

So

$$1.1995509 \le S \le 1.205662.$$

10.4.41

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^6}\,dx = \frac{1}{5n^5}$.

b. We solve $\frac{1}{5n^5} < 10^{-3}$ to get $n = 3$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^6}\,dx = S_n + \frac{1}{5(n+1)^5}$, and $U_n = S_n + \int_n^\infty \frac{1}{x^6}\,dx = S_n + \frac{1}{5n^5}$.

d. $S_{10} \approx 1.017341512$, so $L_{10} \approx 1.017341512 + \frac{1}{5\cdot 11^5} \approx 1.017342754$, and $U_{10} \approx 1.017341512 + \frac{1}{5\cdot 10^5} \approx 1.017343512$.

10.4.42

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x\ln^2 x}\,dx = \frac{1}{\ln n}$.

b. We solve $\frac{1}{\ln n} < 10^{-3}$ to get $n = e^{1000} \approx 10^{434}$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x\ln^2 x}\,dx = S_n + \frac{1}{\ln(n+1)}$, and $U_n = S_n + \int_n^\infty \frac{1}{x\ln^2 x}\,dx = S_n + \frac{1}{\ln n}$.

d. $S_{10} = \sum_{k=2}^{11} \frac{1}{k\ln^2 k} \approx 1.700396385$, so $L_{10} \approx 1.700396385 + \frac{1}{\ln 12} \approx 2.102825989$, and $U_{10} \approx 1.700396385 + \frac{1}{\ln 11} \approx 2.117428776$.

10.4.43

a. The remainder R_n is bounded by $\displaystyle\int_n^\infty \frac{1}{3^x}\,dx = \frac{1}{3^n \ln(3)}$.

b. We solve $\dfrac{1}{3^n \ln(3)} < 10^{-3}$ to obtain $n = 7$.

c. $L_n = S_n + \displaystyle\int_{n+1}^\infty \frac{1}{3^x}\,dx = S_n + \frac{1}{3^{n+1}\ln(3)}$, and $U_n = S_n + \displaystyle\int_n^\infty \frac{1}{3^x}\,dx = S_n + \frac{1}{3^n \ln(3)}$.

d. $S_{10} \approx 0.4999915325$, so $L_{10} \approx 0.4999915325 + \dfrac{1}{3^{11}\ln 3} \approx 0.4999966708$, and $U_{10} \approx 0.4999915325 + \dfrac{1}{3^{10}\ln 3} \approx 0.5000069475$.

10.4.44

a. The remainder R_n is bounded by $\displaystyle\int_n^\infty xe^{-x^2}\,dx = \frac{1}{2e^{n^2}}$.

b. We solve $\dfrac{1}{2e^{n^2}} < 10^{-3}$ to get $n = 3$.

c. $L_n = S_n + \displaystyle\int_{n+1}^\infty xe^{-x^2}\,dx = S_n + \frac{1}{2e^{(n+1)^2}}$, and $U_n = S_n + \displaystyle\int_n^\infty xe^{-x^2}\,dx = S_n + \frac{1}{2e^{n^2}}$.

d. $S_{10} \approx 0.4048813986$, so $L_{10} \approx 0.4048813986 + \dfrac{1}{2e^{11^2}} \approx 0.4048813986$, and $U_{10} \approx 0.4048813986 + \dfrac{1}{2e^{10^2}} \approx 0.4048813986$.

10.4.45 We first try to determine what n will suffice. We need $R_n < \dfrac{1}{10^4}$. We have $R_n = \displaystyle\int_n^\infty \frac{dx}{x^7} = \frac{1}{6n^6}$. Setting this less than $\dfrac{1}{10^4}$ and solving for n we have $n > \left(\dfrac{10^4}{6}\right)^{1/6} \approx 3.44$, so $n = 4$ should suffice. We have

$$S_4 = 1 + \frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{4^7} \approx 1.0083.$$

10.4.46 We first try to determine what n will suffice. We need $R_n < \dfrac{1}{10^3}$. We have $R_n = \displaystyle\int_n^\infty \frac{dx}{(3x+2)^2} = \frac{1}{9n+6}$. Setting this less than $\dfrac{1}{10^3}$ and solving for n we have $9n + 6 > 10^3$, so $n > \dfrac{10^3 - 6}{9} \approx 110.4$. So we need $n = 111$.

We have

$$S_{111} = \sum_{k=1}^{111} \frac{1}{(3k+2)^2} \approx 0.08944$$

10.4.47

a. False. This is a geometric series.

b. True. If we reindex the series, we obtain $\displaystyle\sum_{k=1}^\infty \frac{1}{\sqrt{k}}$, which is a p-series with $p = \dfrac{1}{2}$.

c. False. Both series converge, but not necessarily to the same value.

d. True. The partial sums are an increasing sequence.

e. False. Let $p = 1.0005$; then $-p + .001 = -(p - .001) = -0.9995$, so that $\sum k^{-p}$ converges (p-series) but $\sum k^{-p+.001}$ diverges.

f. False. Let $a_k = \dfrac{1}{k}$, the harmonic series.

10.4.48 $\lim\limits_{k\to\infty} \ln\left(\dfrac{2k^6}{1+k^6}\right) = \ln 2 \neq 0$, so the given series diverges by the Divergence Test.

10.4.49 This is a geometric series with $r = \dfrac{1}{e}$ and $\left|\dfrac{1}{e}\right| < 1$, so the series converges the Geometric Series Test.

10.4.50 Diverges by the Divergence Test because $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \sqrt{\dfrac{k+1}{k}} = 1 \neq 0$.

10.4.51 Converges by the Integral Test because $\displaystyle\int_1^\infty \frac{1}{(3x+1)(3x+4)}\,dx = \int_1^\infty \frac{1}{3(3x+1)} - \frac{1}{3(3x+4)}\,dx = \lim_{b\to\infty}\int_1^b \left(\frac{1}{3(3x+1)} - \frac{1}{3(3x+4)}\right)dx = \lim_{b\to\infty} \frac{1}{9}\left(\ln\left(\frac{3x+1}{3x+4}\right)\right)\Big|_1^b = \lim_{b\to\infty} = -\frac{1}{9}\cdot\ln(4/7) \approx 0.06217 < \infty$.

10.4.52 Converges by the Integral Test because $\displaystyle\int_0^\infty \frac{10}{x^2+9}\,dx = \frac{10}{3}\lim_{b\to\infty}\left(\tan^{-1}(x/3)\big|_0^b\right) = \frac{10}{3}\frac{\pi}{2} \approx 5.236 < \infty$.

10.4.53 Diverges by the Divergence Test because $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \dfrac{k}{\sqrt{k^2+1}} = 1 \neq 0$.

10.4.54 Converges because it is the sum of two geometric series. In fact, $\displaystyle\sum_{k=1}^\infty \frac{2^k+3^k}{4^k} = \sum_{k=1}^\infty (2/4)^k + \sum_{k=1}^\infty (3/4)^k = \frac{1/2}{1-(1/2)} + \frac{3/4}{1-(3/4)} = 1+3 = 4$.

10.4.55 We use the Integral Test. The function $f(x) = \dfrac{4}{x\sqrt{\ln x}}$ is positive, decreasing, and continuous for $x \geq 3$. The relevant integral is

$$\lim_{b\to\infty}\int_3^b \frac{4}{x\sqrt{\ln x}}\,dx = 8\lim_{b\to\infty}\sqrt{\ln x}\Big|_3^b = 8\lim_{b\to\infty}(\sqrt{\ln b} - \sqrt{\ln 3}) = \infty.$$

The given series therefore diverges by the Integral Test.

10.4.56 Note that

$$\frac{1}{k^2+7k+12} = \frac{1}{(k+3)(k+4)} = \frac{1}{k+3} - \frac{1}{k+4}.$$

The sequence of partial sums telescopes:

$$S_n = \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n+3} - \frac{1}{n+4}\right) = \frac{1}{4} - \frac{1}{n+4}.$$

The the series is convergent with sum $S = \lim\limits_{n\to\infty}\left(\dfrac{1}{4} - \dfrac{1}{n+4}\right) = \dfrac{1}{4}$.

10.4.57 The series can be written as $\displaystyle\sum_{k=1}^\infty \left(\frac{6}{5}\right)^k$, which is a divergent geometric series because $r = \dfrac{6}{5} > 1$.

10.4.58 The series diverges by the Divergence Test. We have

$$\lim_{k\to\infty}\left(\frac{k+6}{k}\right)^k = \lim_{k\to\infty}\left(1+\frac{6}{k}\right)^k = e^6 \neq 0.$$

10.4.59 Reindexing gives $\sum_{k=4}^{\infty} \frac{3}{(k-3)^4} = 3\sum_{k=1}^{\infty} \frac{1}{k^4}$, which is convergent because $\sum_{k=1}^{\infty} \frac{1}{k^4}$ is a convergent p-series with $p = 4$ by the p-series test.

10.4.60 The series diverges by the Divergence Test. Note that

$$\lim_{k\to\infty} \frac{3^k}{k^2+1} = \lim_{k\to\infty} \frac{3^k \ln 3}{2k} = \lim_{k\to\infty} \frac{3^k \ln^2 3}{2} = \infty \neq 0.$$

10.4.61 This series can be written as $2\sum_{k=4}^{\infty} \frac{1}{k^2}$, so it converges, because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series with $p = 2$.

10.4.62 This series can be written as $\sum_{k=1}^{\infty} \frac{1}{2k+1}$. We use the Integral Test. Note that $f(x) = \frac{1}{2x+1}$ is positive, continuous, and decreasing for $x \ge 1$. The relevant integral is

$$\int_1^{\infty} \frac{dx}{2x+1} = \frac{1}{2} \lim_{b\to\infty} \ln(2x+1)\Big|_1^b = \frac{1}{2} \lim_{b\to\infty} (\ln(2b+1) - \ln 3) = \infty.$$

The given integral therefore diverges by the Integral Test.

10.4.63 This can be written in the form $9\sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^k$ so it is convergent because $\sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^k$ is a convergent geometric series. In fact, its value is $9\left(\frac{3/5}{1-3/5}\right) = 9\left(\frac{3}{2}\right) = \frac{27}{2}$.

10.4.64

a. In order for the series to converge, the integral $\int_2^{\infty} \frac{1}{x(\ln x)^p}\, dx$ must exist. But

$$\int \frac{1}{x(\ln x)^p}\, dx = \frac{1}{1-p}(\ln x)^{1-p},$$

so in order for this improper integral to exist, we must have that $1 - p < 0$ or $p > 1$.

b. The series converges faster for $p = 3$ because the terms of the series get smaller faster.

10.4.65

a. Note that $\int \frac{1}{x \ln x (\ln\ln x)^p}\, dx = \frac{1}{1-p}(\ln\ln x)^{1-p}$, and thus the improper integral with bounds n and ∞ exists only if $p > 1$ because $\ln\ln x > 0$ for $x > e$. So this series converges for $p > 1$.

b. For large values of z, clearly $\sqrt{z} > \ln z$, so that $z > (\ln z)^2$. Write $z = \ln x$; then for large x, $\ln x > (\ln\ln x)^2$; multiplying both sides by $x \ln x$ we have that $x \ln^2 x > x \ln x (\ln\ln x)^2$, so that the first series converges faster because the terms get smaller faster.

10.4.66 $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges by the Integral Test, because $\int_2^{\infty} \frac{1}{x \ln x} = \lim_{b\to\infty} \left(\ln\ln x \Big|_2^b \right) = \infty$.

10.4.67 To approximate the sequence for $\zeta(x)$, note that the remainder R_n after n terms is bounded by

$$\int_n^{\infty} \frac{1}{z^x}\, dz = \frac{1}{x-1} n^{1-x}.$$

For $x = 3$, if we wish to approximate the value to within 10^{-3}, we must solve $\frac{1}{2}n^{-2} < 10^{-3}$, so that $n = 23$, and $\sum_{k=1}^{23} \frac{1}{k^3} \approx 1.201151926$. The true value is ≈ 1.202056903.
For $x = 5$, if we wish to approximate the value to within 10^{-3}, we must solve $\frac{1}{4}n^{-4} < 10^{-3}$, so that $n = 4$, and $\sum_{k=1}^{4} \frac{1}{k^5} \approx 1.036341789$. The true value is ≈ 1.036927755.

10.4.68

a. Starting with $\cot^2 x < \frac{1}{x^2} < 1 + \cot^2 x$, substitute $k\theta$ for x:

$$\begin{aligned} \cot^2(k\theta) &< \frac{1}{k^2\theta^2} < 1 + \cot^2(k\theta), \\ \sum_{k=1}^{n} \cot^2(k\theta) &< \sum_{k=1}^{n} \frac{1}{k^2\theta^2} < \sum_{k=1}^{n}(1 + \cot^2(k\theta)), \\ \sum_{k=1}^{n} \cot^2(k\theta) &< \frac{1}{\theta^2}\sum_{k=1}^{n} \frac{1}{k^2} < n + \sum_{k=1}^{n} \cot^2(k\theta). \end{aligned}$$

Note that the identity is valid because we are only summing for k up to n, so that $k\theta < \frac{\pi}{2}$.

b. Substitute $\frac{n(2n-1)}{3}$ for the sum, using the identity:

$$\begin{aligned} \frac{n(2n-1)}{3} &< \frac{1}{\theta^2}\sum_{k=1}^{n} \frac{1}{k^2} < n + \frac{n(2n-1)}{3}, \\ \theta^2\frac{n(2n-1)}{3} &< \sum_{k=1}^{n} \frac{1}{k^2} < \theta^2\frac{n(2n+2)}{3}, \\ \frac{n(2n-1)\pi^2}{3(2n+1)^2} &< \sum_{k=1}^{n} \frac{1}{k^2} < \frac{n(2n+2)\pi^2}{3(2n+1)^2}. \end{aligned}$$

c. By the Squeeze Theorem, if the expressions on either end have equal limits as $n \to \infty$, the expression in the middle does as well, and its limit is the same. The expression on the left is

$$\pi^2\frac{2n^2 - n}{12n^2 + 12n + 3} = \pi^2\frac{2 - n^{-1}}{12 + 12n^{-1} + 3n^{-2}},$$

which has a limit of $\frac{\pi^2}{6}$ as $n \to \infty$. The expression on the right is

$$\pi^2\frac{2n^2 + 2n}{12n^2 + 12n + 3} = \pi^2\frac{2 + 2n^{-1}}{12 + 12n^{-1} + 3n^{-3}},$$

which has the same limit. Thus $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

10.4.69 $\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$, splitting the series into even and odd terms. But $\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2}$. Thus $\frac{\pi^2}{6} = \frac{1}{4} \cdot \frac{\pi^2}{6} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$, so that the sum in question is $\frac{3\pi^2}{24} = \frac{\pi^2}{8}$.

10.4.70

a. The lateral surface area of the cake is

$$2\pi + 2\pi\left(\frac{1}{2}\right) + 2\pi\left(\frac{1}{3}\right) + \cdots = 2\pi\sum_{k=1}^{\infty}\frac{1}{k},$$

which diverges because the Harmonic Series diverges.

b. The volume of the cake is

$$\pi + \pi\left(\frac{1}{2}\right)^2 + \pi\left(\frac{1}{3}\right)^2 + \cdots = \pi\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^3}{6}.$$

c. The cake has infinite surface area and yet it has finite volume.

10.4.71 Let $S_n = \sum_{k=1}^{n}\frac{1}{\sqrt{k}}$. Then this looks like a left Riemann sum for the function $y = \frac{1}{\sqrt{x}}$ on $[1, n+1]$. Because each rectangle lies above the curve itself, we see that S_n is bounded below by the integral of $\frac{1}{\sqrt{x}}$ on $[1, n+1]$. Now,

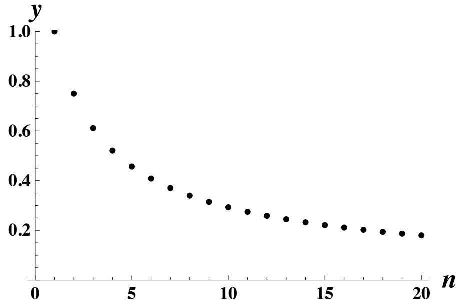
$$\int_1^{n+1}\frac{1}{\sqrt{x}}\,dx = \int_1^{n+1}x^{-1/2}\,dx = 2\sqrt{x}\Big|_1^{n+1} = 2\sqrt{n+1} - 2.$$

This integral diverges as $n \to \infty$, so the series does as well by the bound above.

10.4.72

a. $\{F_n\}$ is a decreasing sequence because each term in F_n is smaller than the corresponding term in F_{n-1} and thus the sum of terms in F_n is smaller than the sum of terms in F_{n-1}.

b.



c. It appears that $\lim_{n\to\infty} F_n = 0$.

10.4.73

a. $x_1 = \sum_{k=2}^{2}\frac{1}{k} = \frac{1}{2}$, $x_2 = \sum_{k=3}^{4}\frac{1}{k} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$, $x_3 = \sum_{k=4}^{6}\frac{1}{k} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{37}{60}$.

b. x_n has n terms. Each term is bounded below by $\frac{1}{2n}$ and bounded above by $\frac{1}{n+1}$. Thus $x_n \geq n\cdot\frac{1}{2n} = \frac{1}{2}$, and $x_n \leq n \cdot \frac{1}{n+1} < n \cdot \frac{1}{n} = 1$.

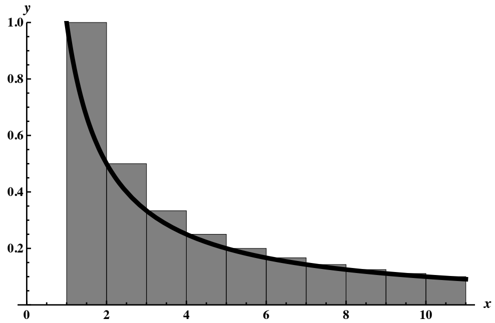
c. The right Riemann sum for $\int_1^2 \frac{dx}{x}$ using n subintervals has n rectangles of width $\frac{1}{n}$; the right edges of those rectangles are at $1 + \frac{i}{n} = \frac{n+i}{n}$ for $i = 1, 2, \ldots, n$. The height of such a rectangle is the value of $\frac{1}{x}$ at the right endpoint, which is $\frac{n}{n+i}$. Thus the area of the rectangle is $\frac{1}{n}\cdot\frac{n}{n+i} = \frac{1}{n+i}$. Adding up over all the rectangles gives x_n.

d. The limit $\lim_{n\to\infty} x_n$ is the limit of the right Riemann sum as the width of the rectangles approaches zero. This is precisely $\displaystyle\int_1^2 \frac{dx}{x} = \ln x\Big|_1^2 = \ln 2$.

10.4.74

The first diagram is a left Riemann sum for $f(x) = \frac{1}{x}$ on the interval $[1, 11]$ (we assume $n = 10$ for purposes of drawing a graph). The area under the curve is $\int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$, and the sum of the areas of the rectangles is obviously $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Thus

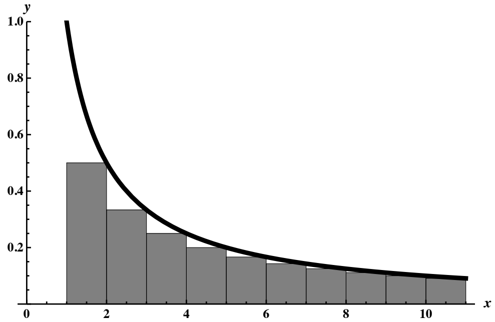
$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$



a. The second diagram is a right Riemann sum for the same function on the same interval. Considering only $[1, n]$, we see that, comparing the area under the curve and the sum of the areas of the rectangles, that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n$$

Adding 1 to both sides gives the desired inequality.



b. According to part (a), $\ln(n+1) < S_n$ for $n = 1, 2, 3, \ldots,$, so that $E_n = S_n - \ln(n+1) > 0$.

c. Using the second figure above and assuming $n = 9$, the final rectangle corresponds to $\frac{1}{n+1}$, and the area under the curve between $n+1$ and $n+2$ is clearly $\ln(n+2) - \ln(n+1)$.

d. $E_{n+1} - E_n = S_{n+1} - \ln(n+2) - (S_n - \ln(n+1)) = \frac{1}{n+1} - (\ln(n+2) - \ln(n+1))$. But this is positive because of the bound established in part (c).

e. Using part (a), $E_n = S_n - \ln(n+1) < 1 + \ln(n) - \ln(n+1) < 1$.

f. E_n is a monotone (increasing) sequence that is bounded, so it has a limit.

g. The first ten values (E_1 through E_{10}) are

$$0.3068528194, 0.401387711, 0.447038972, 0.473895421, 0.491573864,$$
$$0.504089851, 0.513415601, 0.520632565, 0.526383161, 0.531072981.$$

$E_{1000} \approx 0.576716082$.

h. For $S_n > 10$ we need $10 - 0.5772 = 9.4228 > \ln(n+1)$. Solving for n gives $n \approx 12366.16$, so $n = 12367$.

10.4.75

a. Dividing both sides of the recurrence equation by f_n gives $\dfrac{f_{n+1}}{f_n} = 1 + \dfrac{f_{n-1}}{f_n}$. Let the limit of the ratio of successive terms be L. Taking the limit of the previous equation gives $L = 1 + \dfrac{1}{L}$. Thus $L^2 = L + 1$, so $L^2 - L - 1 = 0$. The quadratic formula gives $L = \dfrac{1 \pm \sqrt{1 - 4 \cdot (-1)}}{2}$, but we know that all the terms are positive, so we must have $L = \dfrac{1 + \sqrt{5}}{2} = \phi \approx 1.618$.

b. Write the recurrence in the form $f_{n-1} = f_{n+1} - f_n$ and divide both sides by f_{n+1}. Then we have $\frac{f_{n-1}}{f_{n+1}} = 1 - \frac{f_n}{f_{n+1}}$. Taking the limit gives $1 - \frac{1}{\phi}$ on the right-hand side.

c. Consider the harmonic series with the given groupings, and compare it with the sum of $\frac{f_{k-1}}{f_{k+1}}$ as shown. The first three terms match exactly. The sum of the next two are $\frac{1}{4} + \frac{1}{5} > \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$. The sum of the next three are $\frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$. The sum of the next five are $\frac{1}{9} + \cdots + \frac{1}{13} > 5 \cdot \frac{1}{13} = \frac{5}{13}$. Thus the harmonic series is bounded below by the series $\sum_{k=1}^{\infty} \frac{f_{k-1}}{f_{k+1}}$.

d. The result above implies that the harmonic series diverges, because the series $\sum_{k=1}^{\infty} \frac{f_{k-1}}{f_{k+1}}$ diverges, since its general term has limit $1 - \frac{1}{\phi} \neq 0$.

10.4.76

a. Note that the center of gravity of any stack of dominoes is the average of the locations of their centers. Define the midpoint of the zeroth (top) domino to be $x = 0$, and stack additional dominoes down and to its right (to increasingly positive x-coordinates.) Let $m(n)$ be the x-coordinate of the midpoint of the n^{th} domino. Then in order for the stack not to fall over, the left edge of the n^{th} domino must be placed directly under the center of gravity of dominos 0 through $n-1$, which is $\frac{1}{n}\sum_{i=0}^{n-1} m(i)$, so that $m(n) = 1 + \frac{1}{n}\sum_{i=0}^{n-1} m(i)$. Claim that in fact $m(n) = \sum_{k=1}^{n} \frac{1}{k}$. Use induction. This is certainly true for $n = 1$. Note first that $m(0) = 0$, so we can start the sum at 1 rather than at 0. Now, $m(n) = 1 + \frac{1}{n}\sum_{i=1}^{n-1} m(i) = 1 + \frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=1}^{i} \frac{1}{j}$. Now, 1 appears $n-1$ times in the double sum, 2 appears $n-2$ times, and so forth, so we can rewrite this sum as $m(n) = 1 + \frac{1}{n}\sum_{i=1}^{n-1} \frac{n-i}{i} = 1 + \frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{n}{i} - 1\right) = 1 + \frac{1}{n}\left(n\sum_{i=1}^{n-1}\frac{1}{i} - (n-1)\right) = \sum_{i=1}^{n-1}\frac{1}{i} + 1 - \frac{n-1}{n} = \sum_{i=1}^{n}\frac{1}{i}$, and we are done by induction (noting that the statement is clearly true for $n = 0$, $n = 1$). Thus the maximum overhang is $\sum_{k=2}^{n}\frac{1}{k}$.

b. For an infinite number of dominos, because the overhang is the harmonic series, the distance is potentially infinite.

10.5 Comparison Tests

10.5.1 Given a series of positive terms $\sum a_k$ that you suspect converges, find a series $\sum b_k$ that you know converges, for which $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ where $L \geq 0$ is a finite number. If you are successful, you will have shown that the series $\sum a_k$ converges.

Given a series of positive terms $\sum a_k$ that you suspect diverges, find a series $\sum b_k$ that you know diverges, for which $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ where $L > 0$ (including the case $L = \infty$). If you are successful, you will have shown that $\sum a_k$ diverges.

10.5.2 The difference between successive partial sums is a term in the sequence. Because the terms are positive, differences between successive partial sums are as well, so the sequence of partial sums is increasing.

10.5.3 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$.

10.5.4 No. The problem is that $k^4 - 1 < k^4$, so $\dfrac{1}{k^4-1} > \dfrac{1}{k^4}$. Having series whose terms are smaller that converges doesn't tell you anything about the series whose terms are bigger.

10.5.5 $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$.

10.5.6 Note that $\sqrt{k} - 1 < \sqrt{k}$ for $k \ge 2$, so $\dfrac{1}{\sqrt{k}-1} > \dfrac{1}{\sqrt{k}}$ for $k \ge 2$. Because $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is a divergent p-series, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}}$ diverges and it follows that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}-1}$ diverges by the Comparison Theorem.

10.5.7 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$.

10.5.8 We have $\displaystyle\lim_{k\to\infty} \frac{\frac{k+1}{3k^3+2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^3+k^2}{3k^3+2} = \frac{1}{3}$. Because $0 < \dfrac{1}{3} < \infty$, and because $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series, it follows that the series $\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{3k^3+2}$ converges by the Limit Comparison Test.

10.5.9 $\dfrac{1}{k^2+4} < \dfrac{1}{k^2}$, and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+4}$ converges as well, by the Comparison Test.

10.5.10 Use the Limit Comparison Test with $\left\{\dfrac{1}{k^2}\right\}$. The ratio of the terms of the two series is $\dfrac{k^4+k^3-k^2}{k^4+4k^2-3}$ which has limit 1 as $k \to \infty$. Because the comparison series converges, the given series does as well.

10.5.11 Use the Limit Comparison Test with $\left\{\dfrac{1}{k}\right\}$. The ratio of the terms of the two series is $\dfrac{k^3-k}{k^3+4}$ which has limit 1 as $k \to \infty$. Because the comparison series diverges, the given series does as well.

10.5.12 Use the Limit Comparison Test with $\left\{\dfrac{1}{k}\right\}$. The ratio of the terms of the two series is $\dfrac{0.0001k}{k+4}$ which has limit 0.0001 as $k \to \infty$. Because the comparison series diverges, the given series does as well.

10.5.13 Because $\dfrac{\sqrt{k}}{k^2+3} < \dfrac{\sqrt{k}}{k^2} = \dfrac{1}{k^{3/2}}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is a convergent p-series, $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+3}$ converges by the Comparison Test.

10.5.14 Because $\dfrac{1}{5^k+3} < \dfrac{1}{5^k}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k}$ is a convergent geometric series, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k+3}$ converges by the Comparison Test.

10.5.15 We use the Limit Comparison Test with the convergent series $\sum_{k=1}^{\infty}\left(\frac{4}{5}\right)^k$. We have

$$\lim_{k\to\infty}\frac{\frac{4^k}{5^k-3}}{\left(\frac{4}{5}\right)^k}=\lim_{k\to\infty}\frac{5^k}{5^k-3}=\lim_{k\to\infty}\frac{1}{1-\frac{3}{5^k}}=1.$$

Because $0<1<\infty$, we have that the given series is convergent by the Limit Comparison Test.

10.5.16 Use the Comparison Test. Note that $\sin^2 k\le 1$ for all k, so $\frac{\sin^2 k}{k^2}\le\frac{1}{k^2}$ for all k. Because $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges, so does the given series.

10.5.17 For all k, $\frac{1}{k^{3/2}+1}<\frac{1}{k^{3/2}}$. The series whose terms are $\frac{1}{k^{3/2}}$ is a p-series which converges, so the given series converges as well by the Comparison Test.

10.5.18 Note that $k\cdot 10^k>10^k$ for $k\ge 1$. Taking reciprocals of each side, we have $\frac{1}{k\cdot 10^k}\le\frac{1}{10^k}$ for $k\ge 1$. Because the series $\sum_{k=1}^{\infty}\left(\frac{1}{10}\right)^k$ converges (it is a geometric series with $r=\frac{1}{10}$), the given series converges as well by the Comparison Test.

10.5.19 Note that $\frac{1+\cos^2 k}{k-3}>\frac{1}{k-3}>\frac{1}{k}$ for $k\ge 4$. Because $\sum_{k=1}^{\infty}\frac{1}{k}$ is the divergent harmonic series, we have that $\sum_{k=4}^{\infty}\frac{1+\cos^2 k}{k-3}$ diverges as well by the Comparison Test.

10.5.20 We use the Limit Comparison Test with the convergent geometric series $\sum_{k=1}^{\infty}\frac{1}{6^k}$. We have

$$\lim_{k\to\infty}\frac{\frac{k^2+k+2}{6^k(k^2+1)}}{\frac{1}{6^k}}=\lim_{k\to\infty}\frac{k^2+k+2}{k^2+1}=1.$$

Because $0<1<\infty$, the given series converges by the Limit Comparison Test.

10.5.21 We use the Limit Comparison Test with the divergent harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}$. We have

$$\lim_{k\to\infty}\frac{\frac{21k^5+3k^3+28k^2+4}{8k^6+2k^3-1}}{\frac{1}{k}}=\lim_{k\to\infty}\frac{21k^6+3k^4+28k^3+4k}{8k^6+2k^3-1}=\frac{21}{8}.$$

Because $0<\frac{21}{8}<\infty$, the given series diverges by the Limit Comparison Test.

10.5.22 Use the Limit Comparison Test with $\sum_{k=1}^{\infty}\frac{1}{k}$. The ratio of the terms of the two series is $k\sqrt{\frac{k}{k^3+1}}=\sqrt{\frac{k^3}{k^3+1}}$, which has limit 1 as $k\to\infty$. Because the comparison series diverges, the given series does as well by the Limit Comparison Test.

10.5.23 $\sin(1/k) > 0$ for $k \geq 1$, so we can apply the Comparison Test with $\sum_{k=1}^{\infty} \frac{1}{k^2}$. $\sin(1/k) < 1$, so $\frac{\sin(1/k)}{k^2} < \frac{1}{k^2}$. Because the comparison series converges, the given series converges as well by the Comparison Test.

10.5.24 Use the Limit Comparison Test with $\sum_{k=1}^{\infty} \frac{1}{3^k}$. The ratio of the terms of the two series is $\frac{3^k}{3^k - 2^k} = \frac{1}{1 - \left(\frac{2^k}{3^k}\right)}$, which has limit 1 as $k \to \infty$. Because the comparison series converges, the given series does as well by the Limit Comparison Test.

10.5.25 Use the Limit Comparison Test with $\sum_{k=1}^{\infty} \frac{1}{k}$. The ratio of the terms of the two series is $\frac{k}{2k - \sqrt{k}} = \frac{1}{2 - 1/\sqrt{k}}$, which has limit $1/2$ as $k \to \infty$. Because the comparison series diverges, the given series does as well by the Limit Comparison Test.

10.5.26 $\frac{1}{k\sqrt{k+2}} < \frac{1}{k\sqrt{k}} = \frac{1}{k^{3/2}}$. Because the series whose terms are $\frac{1}{k^{3/2}}$ is a $p-$series with $p > 1$, it converges. Because the comparison series converges, the given series converges as well by the Comparison Test.

10.5.27 Note that $2 + (-1)^k \leq 3$ for $k \geq 1$. It follows that $\frac{2 + (-1)^k}{k^2} \leq \frac{3}{k^2}$ for $k \geq 1$. Because $\sum_{k=1}^{\infty} \frac{3}{k^2} = 3\sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series, $\sum_{k=1}^{\infty} \frac{3}{k^2}$ converges. It follows that the given series converges by the Comparison Test.

10.5.28 Note that $2 + \sin k > 1$ for $k \geq 1$. It follows that $\frac{2 + \sin k}{k} > \frac{1}{k}$. Because $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, it follows that the given series diverges by the Comparison Test.

10.5.29 Use the Limit Comparison Test with $\frac{k^{2/3}}{k^{3/2}}$. The ratio of corresponding terms of the two series is

$$\frac{\sqrt[3]{k^2+1}}{\sqrt{k^3+1}} \cdot \frac{k^{3/2}}{k^{2/3}} = \frac{\sqrt[3]{k^2+1}}{\sqrt[3]{k^2}} \cdot \frac{\sqrt{k^3}}{\sqrt{k^3+1}},$$

which has limit 1 as $k \to \infty$. The comparison series is the series whose terms are $k^{2/3-3/2} = k^{-5/6}$, which is a p-series with $p < 1$, so it, and the given series, both diverge.

10.5.30 For all $k \geq 3$, $\frac{1}{(k \ln k)^2} < \frac{1}{k^2}$. Because the series whose terms are $\frac{1}{k^2}$ converges, the given series converges as well.

10.5.31 We use the Limit Comparison Test with the divergent series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$. We have

$$L = \lim_{k\to\infty} \frac{\frac{20}{\sqrt[3]{k} + \sqrt{k}}}{\frac{1}{\sqrt{k}}} = \lim_{k\to\infty} \frac{20k^{1/2}}{k^{1/3} + k^{1/2}} = \lim_{k\to\infty} \frac{20}{(1/k^{1/6}) + 1} = 20.$$

Because $0 < L < \infty$, we have that the given series diverges by the Limit Comparison Test.

10.5.32 We have $\frac{\sqrt{\ln k}}{k} \geq \frac{1}{k}$ for $k \geq 3$ and because $\sum_{k=1}^{\infty} \frac{1}{k}$ is the divergent harmonic series, $\sum_{k=1}^{\infty} \frac{\sqrt{\ln k}}{k}$ diverges as well by the Comparison Test.

10.5.33 We use the Limit Comparison Test with the divergent series $\sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$L = \lim_{k\to\infty} \frac{\frac{1}{2^{\ln k}}}{\frac{1}{k}} = . \lim_{k\to\infty} \frac{k}{2^{\ln k}} = \lim_{k\to\infty} \frac{e^{\ln k}}{2^{\ln k}} = \lim_{k\to\infty} \left(\frac{e}{2}\right)^{\ln k} = \infty.$$

Because $L = \infty$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, then $\sum_{k=1}^{\infty} \frac{1}{2^{\ln k}}$ diverges by part 3 of the Limit Comparison Test.

10.5.34 We use the Limit Comparison Test with the convergent p-series $\sum_{k=2}^{\infty} \frac{1}{k^3}$. We have

$$L = \lim_{k\to\infty} \frac{\frac{\ln^2 k}{k^4}}{\frac{1}{k^3}} = \lim_{k\to\infty} \frac{\ln^2 k}{k} = \lim_{k\to\infty} \frac{2\ln k}{k} = \lim_{k\to\infty} \frac{2}{k} = 0.$$

Because $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges, the given series converges by part 2 of the Limit Comparison Test.

10.5.35 We use the Limit Comparison Test with the convergent $p$series $\sum_{k=2}^{\infty} \frac{1}{k^{5/4}}$. We have

$$L = \lim_{k\to\infty} \frac{\frac{1}{4^{\ln k}}}{\frac{1}{k^{5/4}}} = \lim_{k\to\infty} \frac{k^{5/4}}{4^{\ln k}} = \lim_{k\to\infty} \frac{e^{\frac{5}{4}\ln k}}{4^{\ln k}} = \lim_{k\to\infty} \left(\frac{e^{5/4}}{4}\right)^{\ln k} = 0.$$

The last limit follows from the fact that $\frac{e^{5/4}}{4} \approx 0.87 < 1$. Because $L = 0$ and $\sum_{k=1}^{\infty} \frac{1}{k^{5/4}}$ converges, the given series converges by the Limit Comparison Test.

10.5.36 We use the Limit Comparison Test with the convergent geometric series $\sum_{k=1}^{\infty} \frac{1}{\pi^k}$. We have

$$L = \lim_{k\to\infty} \frac{\frac{5}{\pi^k + e^k - 2}}{\frac{1}{\pi^k}} = \lim_{k\to\infty} \frac{5\pi^k}{\pi^k + e^k - 2} = \lim_{k\to\infty} \frac{5}{1 + (e/\pi)^k - 2/\pi^k} = 5.$$

Because $0 < L < \infty$, the given series converges by the Limit Comparison Test.

10.5.37

a. False. For example, let $\{a_k\}$ be all zeros, and $\{b_k\}$ be all 1's.

b. True. This is a result of the Comparison Test.

c. True. Both of these statements follow from the Comparison Test.

d. True. The limit of the ratio is 1 in this case, so the two series both converge.

10.5.38

Method 1: We use the Comparison Test with the convergent geometric series $\sum_{k=1}^{\infty}\frac{1}{(e^2)^k}$. Note that $\frac{e^{2k}}{e^{4k}+1}\le\frac{e^{2k}}{e^{4k}}=\frac{1}{e^{2k}}$. So $\sum_{k=1}^{\infty}\frac{e^{2k}}{e^{4k}+1}$ converges by the Comparison Test because $\sum_{k=1}^{\infty}\frac{1}{(e^2)^k}$ converges.

Method 2: Let $f(x)=\frac{e^{2x}}{e^{4x}+1}$. Then f is continuous and positive for $x\ge 1$,and $f'(x)=\frac{2e^{2x}(1-e^{4x})}{(e^{4x}+1)^2}<0$ for $x\ge 1$, so f is decreasing for $x\ge 1$. We have

$$\int_1^{\infty}\frac{e^{2x}}{e^{4x}+1}\,dx=\lim_{b\to\infty}\frac{1}{2}\tan^{-1}e^{2x}\Big|_1^b=\frac{1}{2}\left(\frac{\pi}{2}-\tan^{-1}e^2\right).$$

Because the integral converges, the given series also converges by the Integral Test.

10.5.39 Method 1: We use the Comparison Test with the convergent p-series $\sum_{k=1}^{\infty}\frac{1}{k^2}$. Note that $\frac{1}{k^2+2k+1}\le\frac{1}{k^2}$. Thus the given series converges by the Comparison Test.

Method 2: We rewrite the summand. We have $\sum_{k=1}^{\infty}\frac{1}{k^2+2k+1}=\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}=\sum_{k=2}^{\infty}\frac{1}{k^2}$. Because $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is a convergent series, $\sum_{k=2}^{\infty}\frac{1}{k^2}$ converges as well.

10.5.40 Use the Divergence Test: $\lim_{k\to\infty}a_k=\lim_{k\to\infty}\left(1-\frac{1}{k}\right)^k=\frac{1}{e}\ne 0$, so the given series diverges.

10.5.41 Use the Divergence Test: $\lim_{k\to\infty}a_k=\lim_{k\to\infty}\left(1+\frac{2}{k}\right)^k=e^2\ne 0$, so the given series diverges.

10.5.42 We use the Limit Comparison Test with the convergent geometric series $\sum_{k=1}^{\infty}\frac{1}{(e^3)^k}$. We have

$$L=\lim_{k\to\infty}\frac{\frac{e^{2k}+k}{e^{5k}-k^2}}{\frac{1}{e^{3k}}}=\lim_{k\to\infty}\frac{e^{5k}+ke^{3k}}{e^{5k}-k^2}=\lim_{k\to\infty}\frac{1+\frac{k}{e^{2k}}}{1-\frac{k^2}{e^{5k}}}=1.$$

Thus the given series converges by the Limit Comparison Test.

10.5.43 Use the Divergence Test: $\lim_{k\to\infty}\frac{k^2+2k+1}{3k^2+1}=\frac{1}{3}\ne 0$, so the given series diverges.

10.5.44 Use the Limit Comparison Test with the series whose kth term is $\left(\frac{2}{e}\right)^k$. Note that

$$L=\lim_{k\to\infty}\frac{2^k}{e^k-1}\cdot\frac{e^k}{2^k}=\lim_{k\to\infty}\frac{e^k}{e^k-1}=1.$$

The given series thus converges because $\sum_{k=1}^{\infty}\left(\frac{2}{e}\right)^k$ converges (because it is a geometric series with $r=\frac{2}{e}<1$).

10.5.45 Use the Limit Comparison Test with the harmonic series. Note that $\lim_{k\to\infty} \dfrac{\frac{1}{\ln k}}{\frac{1}{k}} = \lim_{k\to\infty} \dfrac{k}{\ln k} = \infty$, and because the harmonic series diverges, the given series does as well.

10.5.46 Use the Limit Comparison Test with the series whose kth term is $\frac{1}{5^k}$. Note that $\lim_{k\to\infty} \dfrac{1}{5^k-1} \cdot \dfrac{5^k}{1} = 1$, and the series $\sum_{k=1}^{\infty} \dfrac{1}{5^k}$ converges because it is a geometric series with $r = \dfrac{1}{5}$. Thus, the given series also converges.

10.5.47 Use the Limit Comparison Test with the series whose kth term is $\dfrac{1}{k^{3/2}}$. Note that $\lim_{k\to\infty} \dfrac{1}{\sqrt{k^3-k+1}} \cdot \dfrac{\sqrt{k^3}}{1} = \lim_{k\to\infty} \sqrt{\dfrac{k^3}{k^3-k+1}} = \sqrt{1} = 1$, and the series $\sum_{k=1}^{\infty} \dfrac{1}{k^{3/2}}$ converges because it is a p-series with $p = \dfrac{3}{2}$. Thus, the given series also converges.

10.5.48 Use the Limit Comparison Test with the series whose kth term is $\dfrac{1}{5^k}$. Note that $\lim_{k\to\infty} \dfrac{1}{5^k-3^k} \cdot \dfrac{5^k}{1} = \lim_{k\to\infty} \dfrac{1}{1-(3/5)^k} = 1$, and the series $\sum_{k=3}^{\infty} \dfrac{1}{5^k}$ converges because it is a geometric series with $r = \dfrac{1}{5}$. Thus, the given series also converges.

10.5.49 Use the Comparison Test. Note that $\dfrac{1}{k} + 2^{-k} > \dfrac{1}{k}$ for $k \ge 1$. Because the harmonic series $\sum_{k=1}^{\infty} \dfrac{1}{k}$ diverges, so does the given series.

10.5.50 Use the Comparison Test with $\sum_{k=1}^{\infty} \dfrac{5}{k} = 5\sum_{k=1}^{\infty} \dfrac{1}{k}$. Note that $\dfrac{5\ln k}{k} > \dfrac{5}{k}$ for $k \ge 3$. Because $\sum_{k=3}^{\infty} \dfrac{1}{k}$ diverges, so does $5\sum_{k=1}^{\infty} \dfrac{1}{k}$, and therefore so does the given series.

10.5.51 Use the Limit Comparison Test with the convergent p-series $\sum_{k=1}^{\infty} \dfrac{1}{k^3}$. The ratio of corresponding terms is $\dfrac{k^{11}}{k^{11}+3}$, which has limit 1 as $k \to \infty$. Therefore the given series converges by the Limit Comparison Test.

10.5.52 Use the Comparison Test:: $\dfrac{1}{k^2 \ln k} < \dfrac{1}{k^2}$ for $k \ge 3$. Because $\sum_{k=3}^{\infty} \dfrac{1}{k^2}$ converges, so does $\sum_{k=3}^{\infty} \dfrac{1}{k^2 \ln k}$, and therefore so does $\sum_{k=2}^{\infty} \dfrac{1}{k^2 \ln k}$.

10.5.53 This is a p-series with exponent greater than 1, so it converges.

10.5.54 We use the Limit Comparison Test with the convergent p-series $\sum_{k=1}^{\infty} \dfrac{1}{k^2}$. We have

$$L = \lim_{k\to\infty} \frac{\frac{\tan^{-1} k}{k^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \tan^{-1} k = \frac{\pi}{2}.$$

Because $0 < L < \infty$, the given series is convergent.

10.5.55 $\ln\left(\frac{k+2}{k+1}\right) = \ln(k+2) - \ln(k+1)$, so this series telescopes. We get $\sum_{k=1}^{n} \ln\left(\frac{k+2}{k+1}\right) = \ln(n+2) - \ln 2$. Because $\lim_{n\to\infty} (\ln(n+2) - \ln 2) = \infty$, the sequence of partial sums diverges, so the given series is divergent.

10.5.56 Use the Divergence Test. Note that $\lim_{k\to\infty} k^{-1/k} = \lim_{k\to\infty} \frac{1}{\sqrt[k]{k}} = 1 \neq 0$, so the given series diverges.

10.5.57 For $k > 7$, $\ln k > 2$ so note that $\frac{1}{k^{\ln k}} < \frac{1}{k^2}$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the given series converges as well by the Comparison Test.

10.5.58 Use the Limit Comparison Test with the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Note that $\frac{\sin^2(1/k)}{1/k^2} = \left(\frac{\sin(1/k)}{1/k}\right)^2$. Because $\lim_{x\to 0} \frac{\sin x}{x} = 1$, the limit of this expression is $1^2 = 1$ as $k \to \infty$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the given series does as well.

10.5.59 Use the Limit Comparison Test with the harmonic series. $\frac{\tan(1/k)}{1/k}$ has limit 1 as $k \to \infty$ because $\lim_{x\to 0} \frac{\tan x}{x} = 1$. Thus the original series diverges.

10.5.60 We use the Limit Comparison Test with the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$L = \lim_{k\to\infty} \frac{\sqrt{\frac{1}{2^{2k}} + \frac{1}{k^2}}}{\frac{1}{k}} = \lim_{k\to\infty} \sqrt{\frac{k^2}{4^k} + 1} = 1.$$

Therefore the given series diverges by the Limit Comparison Test.

10.5.61 Note that $\frac{1}{(2k+1)\cdot(2k+3)} = \frac{1}{2}\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right)$. Thus this series telescopes.

$$\sum_{k=0}^{n} \frac{1}{(2k+1)(2k+3)} = \frac{1}{2}\sum_{k=0}^{n}\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right) = \frac{1}{2}\left(-\frac{1}{2n+3} + 1\right),$$

so the given series converges to 1/2, because that is the limit of the sequence of partial sums.

10.5.62 This series is $\sum_{k=1}^{\infty} \frac{k-1}{k^2} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k^2}\right)$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, if the original series also converged, we would have that $\sum_{k=1}^{\infty} \frac{1}{k}$ converged, which is false. Thus the original series diverges.

10.5.63 Use the Limit Comparison Test: $\lim_{k\to\infty} \frac{a_k^2}{a_k} = \lim_{k\to\infty} a_k = 0$, because $\sum a_k$ converges. By the Limit Comparison Test part 2, the series $\sum a_k^2$ must converge as well.

10.5.64

a. Use the Limit Comparison Test with the divergent harmonic series. Note that $\lim_{k\to\infty} \dfrac{\sin(1/k)}{1/k} = 1$, because $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$. Because the comparison series diverges, the given series does as well.

b. We use the Limit Comparison Test with the convergent series $\sum \dfrac{1}{k^2}$. Note that $\lim_{k\to\infty} \dfrac{(1/k)\sin(1/k)}{1/k^2} = \lim_{k\to\infty} \dfrac{\sin(1/k)}{1/k} = 1$, so the given series converges.

10.5.65 To prove case (2), assume $L = 0$ and that $\sum b_k$ converges. Because $L = 0$, for every $\varepsilon > 0$, there is some N such that for all $n > N$, $|\frac{a_k}{b_k}| < \varepsilon$. Take $\varepsilon = 1$; this then says that there is some N such that for all $n > N$, $0 < a_k < b_k$. By the Comparison Test, because $\sum b_k$ converges, so does $\sum a_k$. To prove case (3), because $L = \infty$, then $\lim_{k\to\infty} \frac{b_k}{a_k} = 0$, so by the argument above, we have $0 < b_k < a_k$ for sufficient large k. But $\sum b_k$ diverges, so by the Comparison Test, $\sum a_k$ does as well.

10.5.66

a. Let P_n be the n^{th} partial product of the a_k: $P_n = \prod_{k=1}^{n} a_k$.

We have
$$\prod_{k=1}^{n} \frac{k}{k+1} = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n}{n+1} = \frac{1}{n+1}.$$
This has limit 0 as $n \to \infty$.

b. Because $\sum_{k=1}^{\infty} \ln a_k = L$, the nth partial sum S_n of $\sum_{k=1}^{\infty} \ln a_k$ converges to L. Then
$$e^{S_n} = e^{\ln a_1 + \ln a_2 + \cdots \ln a_n} = e^{\ln a_1} e^{\ln a_2} \cdots e^{\ln a_n} = a_1 a_2 \cdots a_n.$$
Therefore $\lim_{n\to\infty} (a_1 a_2 \cdots a_n) = \lim_{n\to\infty} e^{S_n} = e^L$.

c. Because $\sum_{k=0}^{\infty} \ln e^{1/2^k} = \sum_{k=1}^{\infty} \ln e^{1/2^{k-1}} = \sum_{k=1}^{\infty} \dfrac{1}{2^{k-1}} = \dfrac{1}{1-\frac{1}{2}} = 2$, we must have $\prod_{k=0}^{\infty} e^{1/2^k} = e^2$.

10.5.67 The sum on the left is simply the left Riemann sum over n equal intervals between 0 and 1 for $f(x) = x^p$. The limit of the sum is thus $\int_0^1 x^p dx = \dfrac{1}{p+1} x^{p+1}\Big|_0^1 = \dfrac{1}{p+1}$, because p is positive.

10.6 Alternating Series

10.6.1 Because $S_{n+1} - S_n = a_{n+1}$ alternates signs.

10.6.2 Check that the terms of the series are nonincreasing in magnitude after some finite number of terms, and that the $\lim_{k\to\infty} a_k = 0$.

10.6.3 Because
$$S = S_{2n+1} + (a_{2n} - a_{2n+1}) + (a_{2n+2} - a_{2n+3}) + \cdots$$
and each term of the form $a_{2k} - a_{2k+1} > 0$, so that $S_{2n+1} < S$. Also
$$S = S_{2n} + (-a_{2n+1} + a_{2n+2}) + (-a_{2n+3} + a_{2n+4}) + \cdots$$
and each term of the form $-a_{2k+1} + a_{2k+2} < 0$, so that $S < S_{2n}$. Thus the sum of the series is trapped between the odd partial sums and the even partial sums.

10.6.4 The difference between L and S_n is bounded in magnitude by a_{n+1}.

10.6.5 The remainder is less than the first neglected term because

$$S - S_n = (-1)^{n+1}(a_{n+1} + (-a_{n+2} + a_{n+3}) + \cdots)$$

so that the sum of the series *after* the first disregarded term has the opposite sign from the first disregarded term.

10.6.6 The alternating harmonic series $\sum (-1)^k \dfrac{1}{k}$ converges, but not absolutely.

10.6.7 No. If the terms are positive, then the absolute value of each term is the term itself, so convergence and absolute convergence would mean the same thing in this context.

10.6.8 Yes. The series $\displaystyle\sum_{k=0}^{\infty} \left|(-0.5)^k\right| = \sum_{k=0}^{\infty} 0.5^k$ is also a convergent geometric series.

10.6.9 Yes. For example, $\displaystyle\sum \frac{(-1)^k}{k^3}$ converges absolutely and thus not conditionally (see the definition).

10.6.10 The series $\displaystyle\sum_{k=0}^{\infty} \left|\frac{(-1)^k}{k^p}\right| = \sum_{k=0}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \le 1$, which implies that $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k^p}$ converges absolutely for $p > 1$. If $0 < p < 1$, the sequence $\left\{\dfrac{1}{k^p}\right\}$ is decreasing and converges to 0, and $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^p}$ is a divergent p-series for $p \le 1$. So $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k^p}$ is conditionally convergent for $0 < p \le 1$.

10.6.11 The terms of the series decrease in magnitude, and $\displaystyle\lim_{k\to\infty} \frac{1}{2k+1} = 0$, so the given series converges.

10.6.12 The terms of the series decrease in magnitude, and $\displaystyle\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$, so the given series converges.

10.6.13 $\displaystyle\lim_{k\to\infty} \frac{k}{3k+2} = \frac{1}{3} \ne 0$, so the given series diverges.

10.6.14 $\displaystyle\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e \ne 0$, so the given series diverges.

10.6.15 The terms of the series decrease in magnitude, and $\lim_{k\to\infty} \frac{1}{k^3} = 0$, so the given series converges.

10.6.16 The terms of the series decrease in magnitude, and $\lim_{k\to\infty} \frac{1}{k^2+10} = 0$, so the given series converges.

10.6.17 The terms of the series decrease in magnitude, and $\lim_{k\to\infty} \frac{k^2}{k^3+1} = \lim_{k\to\infty} \frac{1/k}{1+1/k^3} = 0$, so the given series converges. In order to see that the terms decrease in magnitude, consider $f(x) = \dfrac{x^2}{x^3+1}$. Then $f'(x) = \dfrac{x(2-x^3)}{(1+x^3)^2}$, which is negative for $x \ge 2$.

10.6.18 The terms of the series eventually decrease in magnitude, because if $f(x) = \dfrac{\ln(x)}{x^2}$, then $f'(x) = \dfrac{x(1-2\ln x)}{x^4} = \dfrac{1-2\ln x}{x^3}$, which is negative for large enough x. Further, $\lim_{k\to\infty} \frac{\ln k}{k^2} = \lim_{k\to\infty} \frac{1/k}{2k} = \lim_{k\to\infty} \frac{1}{2k^2} = 0$. Thus the given series converges.

10.6.19 $\lim_{k\to\infty} \frac{k^2-1}{k^2+3} = 1$, so the terms of the series do not tend to zero and thus the given series diverges.

10.6.20 $\sum_{k=0}^{\infty}\left(-\frac{1}{5}\right)^k = \sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{5}\right)^k$. Note that $(1/5)^k$ is decreasing, and tends to zero as $k \to \infty$, so the given series converges.

10.6.21 $\lim_{k\to\infty}\left(1+\frac{1}{k}\right) = 1$, so the given series diverges.

10.6.22 Note that $\cos(\pi k) = (-1)^k$, and so the given series is alternating. Because $\lim_{k\to\infty} \frac{1}{k^2} = 0$ and $\dfrac{1}{k^2}$ is decreasing, the given series is convergent.

10.6.23 Because $\lim_{k\to\infty} \frac{k^{11}+2k^5+1}{4k(k^{10}+1)} = \frac{1}{4}$, $\lim_{k\to\infty}\left((-1)^{k+1}\frac{k^{11}+2k^5+1}{4k(k^{10}+1)}\right) \neq 0$. So the given series diverges by the Divergence Test.

10.6.24 Clearly $\dfrac{1}{k\ln^2 k}$ is nonincreasing, and $\lim_{k\to\infty} \frac{1}{k\ln^2 k} = 0$, so the given series converges.

10.6.25 $\lim_{k\to\infty} k^{1/k} = 1$ (for example, take logs and apply L'Hôpital's rule), so the given series diverges by the Divergence Test.

10.6.26 $a_{k+1} < a_k$ because $\dfrac{a_{k+1}}{a_k} = \dfrac{(k+1)!}{(k+1)^{k+1}} \cdot \dfrac{k^k}{k!} = \left(\dfrac{k}{k+1}\right)^k < 1$. Additionally, $\dfrac{k!}{k^k} \to 0$ as $k \to \infty$, so the given series converges.

10.6.27 $\dfrac{1}{\sqrt{k^2+4}}$ is decreasing and tends to zero as $k \to \infty$, so the given series converges.

10.6.28 $S_3 = \dfrac{1}{2} - \dfrac{1}{9} + \dfrac{1}{28} \approx 0.424603$. $|S - S_3| \le |a_4| = \dfrac{1}{65} \approx 0.0153846$.

10.6.29 $S_4 = \dfrac{-}{1} + \dfrac{1}{16} - \dfrac{1}{81} + \dfrac{1}{256} \approx -0.945939$. $|S - S_4| \le |a_5| = \dfrac{1}{625} \approx 0.0016$.

10.6.30 $S_2 = 1 - \dfrac{1}{256} \approx 0.996094$. $|S - S_2| \le |a_2| = \dfrac{1}{2401} \approx 0.000416$.

10.6.31 $S_5 = 1 - \dfrac{1}{3} + \dfrac{1}{21} - \dfrac{1}{91} + \dfrac{1}{273} \approx 0.70696$. $|S - S_5| \le |a_5| = \dfrac{1}{25+625+1} \approx 0.001536$.

10.6.32 $S_3 = 1 - \dfrac{1}{6} + \dfrac{1}{120} \approx 0.841667$. $|S - S_3| = |a_3| = \dfrac{1}{7!} \approx 0.000198$.

10.6.33 We want $\dfrac{1}{n+1} < 10^{-4}$, or $n+1 > 10^4$, so $n = 10^4$.

10.6.34 The series starts with $k = 0$, so we want $\dfrac{1}{n!} < 10^{-4}$, or $n! > 10^4 = 10000$. This happens for $n = 8$.

10.6.35 The series starts with $k = 0$, so we want $\dfrac{1}{2n+1} < 10^{-4}$, or $2n+1 > 10^4$, $n = 5000$.

10.6.36 We want $\dfrac{1}{(n+1)^2} < 10^{-4}$, or $(n+1)^2 > 10^4$, so $n = 100$.

10.6.37 We want $\dfrac{1}{(n+1)^4} < 10^{-4}$, or $(n+1)^4 > 10^4$, so $n = 10$.

10.6.38 The series starts with $k = 0$, so we want $\dfrac{1}{(2n+1)^3} < 10^{-4}$, or $2n + 1 > 10^{4/3}$, so $n = 11$.

10.6.39 To figure out how many terms we need to sum, we must find n such that $\dfrac{1}{(n+1)^5} < 10^{-3}$, so that $(n + 1)^5 > 1000$; this occurs first for $n = 3$. Thus $-\dfrac{1}{1} + \dfrac{1}{2^5} - \dfrac{1}{3^5} \approx -0.972865$.

10.6.40 To figure out how many terms we need to sum, we must find n such that $\dfrac{1}{(2(n+1)+1)^3} < 10^{-3}$, or $(2n + 3)^3 > 10^3$, so $2n + 3 > 10$ and $n = 4$. Thus the approximation is $\displaystyle\sum_{k=1}^{4} \frac{(-1)^n}{(2n+1)^3} \approx -0.3058074682$.

10.6.41 To figure out how many terms we need to sum, we must find n so that $\dfrac{n}{n^2+1} < 10^{-3}$, so that $\dfrac{n^2+1}{n} = n + (1/n) > 1000$. This occurs first for $n = 1000$. We have $\displaystyle\sum_{k=1}^{1000} \frac{(-1)^k k}{k^2+1} \approx -0.269111$.

10.6.42 To figure out how many terms we need to sum, we must find n such that $\frac{n}{n^4+1} < 10^{-3}$, so that $\frac{n^4+1}{n} = n^3 + 1/n > 1000$, which occurs for $n = 10$. We have $\displaystyle\sum_{k=1}^{10} \frac{(-1)^k k}{k^4+1} \approx -0.408068$.

10.6.43 To figure how many terms we need to sum, we must find n such that $\dfrac{1}{(n+1)^{n+1}} < 10^{-3}$, or $(n + 1)^{n+1} > 1000$, so $n = 4$ ($5^5 = 3125$). Thus the approximation is $\displaystyle\sum_{k=1}^{4} \frac{(-1)^n}{n^n} \approx -.7831307870$.

10.6.44 To figure how many terms we need to sum, we must find n such that $\dfrac{1}{(2(n+1)+1)!} < 10^{-3}$, or $(2n + 3)! > 1000$, so $2n + 3 \geq 7$ and $n = 2$. The approximation is $\displaystyle\sum_{k=1}^{2} \frac{(-1)^{n+1}}{(2n+1)!} \approx 0.1583333333$.

10.6.45 The series of absolute values is a p-series with $p = 2/3$, so it diverges. The given alternating series does converge, though, by the Alternating Series Test. Thus, the given series is conditionally convergent.

10.6.46 Because $\displaystyle\lim_{k\to\infty} \frac{2k}{\sqrt{k^2+9}} = \lim_{k\to\infty} 2\sqrt{\frac{k^2}{k^2+9}} = 2\sqrt{1} = 2 \neq 0$, the given series diverges by the Divergence Test.

10.6.47 Note that the absolute value of the kth term is $\dfrac{1}{(k+1)!}$, and also note that for $k \geq 1$, $(k+1)! > k^2$, and thus $\dfrac{1}{(k+1)!} < \dfrac{1}{k^2}$. Now because $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series, the given series converges absolutely by the Comparison Test.

10.6.48 The series of absolute values is $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k}$, which converges, so the series converges absolutely.

10.6.49 This series of all positive terms is convergent, because it is a geometric series with $r = \dfrac{3}{4}$. Therefore it converges absolutely.

10.6.50 Note that $\sum_{k=1}^{\infty} \frac{1}{k^{0.99}}$ is a divergent p-series because $p < 1$. However, $\left\{ \frac{1}{k^{0.99}} \right\}$ is a decreasing sequence with limit 0, so the given series converges by the Alternating Series Test. Therefore the given series is conditionally convergent.

10.6.51 The series of absolute values is $\sum_{k=1}^{\infty} \frac{|\cos(k)|}{k^3}$, which converges by the Comparison Test because $\frac{|\cos(k)|}{k^3} \le \frac{1}{k^3}$. Thus the series converges absolutely.

10.6.52 The series of absolute values is $\sum_{k=1}^{\infty} \frac{k^2}{\sqrt{k^6+1}}$. The limit comparison test with $\sum_{k=1}^{\infty} \frac{1}{k}$ gives $\lim_{k\to\infty} \frac{k^3}{\sqrt{k^6+1}} = \lim_{k\to\infty} \sqrt{\frac{k^6}{k^6+1}} = 1$. Because the comparison series diverges, so does the series of absolute values. The original series converges conditionally, however, because the terms are nonincreasing and $\lim_{k\to\infty} \frac{k^2}{\sqrt{k^6+1}} = \lim_{k\to\infty} \sqrt{\frac{k^4}{k^6+1}} = 0$.

10.6.53 The absolute value of the kth term of this series has limit $\pi/2$ as $k \to \infty$, so the given series is divergent by the Divergence Test.

10.6.54 The series of absolute values is a geometric series with $r = \frac{1}{e}$ and $|r| < 1$, so the given series converges absolutely

10.6.55 Note that the absolute value of the kth term is $\frac{1}{2\sqrt{k}-1}$. We will use the Limit Comparison Test with the divergent p-series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$. We have

$$\lim_{k\to\infty} \frac{\frac{1}{2\sqrt{k}-1}}{\frac{1}{\sqrt{k}}}. = \lim_{k\to\infty} \frac{\sqrt{k}}{2\sqrt{k}-1} = \frac{1}{2}.$$

Therefore the given series is not absolutely convergent. However, it is the case that $\left\{ \frac{1}{2\sqrt{k}-1} \right\}$ is decreasing and has limit 0, so by the Alternating Series Test, the given series is convergent. Therefore, the given series is conditionally convergent.

10.6.56 The absolute value of the kth term is $\frac{1}{3^k-1}$. We use the Limit Comparison Test with the convergent geometric series $\sum_{k=1}^{\infty} \frac{1}{3^k}$. We have

$$\lim_{k\to\infty} \frac{\frac{1}{3^k-1}}{\frac{1}{3^k}} = \lim_{k\to\infty} \frac{3^k}{3^k-1} = 1.$$

So the given series is absolutely convergent.

10.6.57 The series of absolute values is $\sum_{k=1}^{\infty} \frac{k}{2k+1}$, but $\lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2}$, so by the Divergence Test, this series diverges. The original series does not converge conditionally, either, because $\lim_{k\to\infty} a_k = \frac{1}{2} \neq 0$.

10.6.58 The series of absolute values is $\displaystyle\sum_{k=3}^{\infty}\frac{1}{\ln k}$. Note that for $k \geq 3$, $\ln k \leq k$, so $\dfrac{1}{\ln k} > \dfrac{1}{k}$. Therefore $\displaystyle\sum_{k=3}^{\infty}\frac{1}{\ln k}$ diverges by the Comparison Test. However, because $\lim_{k\to\infty}\frac{1}{\ln k} = 0$ and the terms are nonincreasing, the series does converge by the Alternating Series Test, so the series converges conditionally.

10.6.59 The series of absolute values is $\displaystyle\sum_{k=1}^{\infty}\frac{\tan^{-1}(k)}{k^3}$, which converges by the Comparison Test because $\dfrac{\tan^{-1}(k)}{k^3} < \dfrac{\pi}{2}\cdot\dfrac{1}{k^3}$, and $\displaystyle\sum_{k=1}^{\infty}\frac{\pi}{2}\cdot\frac{1}{k^3}$ converges because it is a constant multiple of a convergent $p-$series. So the original series converges absolutely.

10.6.60 Note that the absolute value of the kth term is $\dfrac{k^2+1}{3k^4+3}$. We use the Limit Comparison Test with the convergent p-series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$. We have

$$L = \lim_{k\to\infty}\frac{\frac{k^2+1}{3k^4+3}}{\frac{1}{k^2}} = \lim_{k\to\infty}\frac{k^4+k^2}{3k^4+3} = \frac{1}{3}.$$

Because $0 < L < \infty$, the given series converges absolutely by the Limit Comparison Test.

10.6.61 The absolute value of the kth term is $\dfrac{k^2+1}{k^3-1}$. We use the Limit Comparison Test with the divergent harmonic series. We have

$$L = \lim_{k\to\infty}\frac{\frac{k^2+1}{k^3-1}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{k^3+k}{k^3-1} = 1.$$

The given series therefore does not converge absolutely. However, the given series does converge by the Alternating Series Test. Note that if $f(x) = \dfrac{x^2+1}{x^3-1}$, then $f'(x) = \dfrac{-(2x+3x^2+x^4)}{(x^3-1)^2}$ which is less than 0 for $x \geq 1$. Therefore the absolute value of the sequence of terms of the original series is decreasing and has limit 0, so the given series is conditionally convergent.

10.6.62 Note that $\left|\dfrac{\sin k}{3^k+4^k}\right| \leq \left|\dfrac{1}{3^k+4^k}\right| < \dfrac{1}{3^k}$. Also note that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{3^k}$ is a convergent geometric series. The given series is therefore absolutely convergent by the Comparison Test.

10.6.63 For $k \geq 3$,

$$\frac{k!}{k^k} = \frac{1\cdot 2\cdot 3\cdots k}{k\cdot k\cdot k\cdots k} = \frac{1}{k}\cdot\frac{2}{k}\cdot\frac{3}{k}\cdots\frac{k-1}{k}\cdot\frac{k}{k} \leq \frac{1}{k}\cdot\frac{2}{k}\cdot\frac{k}{k}\cdot\frac{k}{k}\cdots\frac{k}{k} = \frac{2}{k^2}.$$

Because the series $\displaystyle\sum_{k=3}^{\infty}\frac{2}{k^2}$ is a convergent series (it is a constant multiple of a p-series), it follows that the given series converges absolutely by the Comparison Test.

10.6.64 Neither condition is satisfied. $\dfrac{a_{k+1}}{a_k} = \dfrac{(k+1)(2k+1)}{(2k+3)k} = \dfrac{2k^2+3k+1}{2k^2+3k} > 1$, and $\lim_{k\to\infty} a_k = \frac{1}{2}$.

10.6.65

a. False. For example, consider the alternating harmonic series.

b. True. This is part of Theorem 10.21.

c. True. This statement is simply saying that a convergent series converges.

d. True. This is part of Theorem 10.21.

e. False. Let $a_k = \dfrac{1}{k}$.

f. True. Use the Comparison Test: $\displaystyle\lim_{k\to\infty} \frac{a_k^2}{a_k} = \lim_{k\to\infty} a_k = 0$ because $\sum a_k$ converges, so $\sum a_k^2$ and $\sum a_k$ converge or diverge together. Because the latter converges, so does the former.

g. True, by definition. If $\sum |a_k|$ converged, the original series would converge absolutely, not conditionally.

10.6.66 Let $S = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \cdots$. Then

$$\begin{array}{ccccccccccc} S & = & \left(1-\frac{1}{2}\right) & + & \left(\frac{1}{3}-\frac{1}{4}\right) & + & \left(\frac{1}{5}-\frac{1}{6}\right) & + & \left(\frac{1}{7}-\frac{1}{8}\right) & + & \dots \\ \frac{1}{2}S & = & \frac{1}{2} & - & \frac{1}{4} & + & \frac{1}{6} & - & \frac{1}{8} & + & \dots \end{array}$$

Add these two series together to get

$$\frac{3}{2}S = \frac{3}{2}\ln 2 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \cdots$$

To see that the results are as desired, consider a collection of four terms:

$$\begin{array}{cccccc} \cdots + & \left(\frac{1}{4k+1} - \frac{1}{4k+2}\right) & + & \left(\frac{1}{4k+3} - \frac{1}{4k+4}\right) & + & \dots \\ \dots & + & \frac{1}{4k+2} & - & \frac{1}{4k+4} & + \ \cdots \end{array}$$

Adding these results in the desired sign pattern. This repeats for each group of four elements.

10.6.67 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = 2 \cdot \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2}$, and thus $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{6} - \frac{1}{2} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{12}$.

10.6.68 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} = 2\sum_{k=1}^{\infty} \frac{1}{(2k)^4} = 2 \cdot \frac{1}{16} \sum_{k=1}^{\infty} \frac{1}{k^4}$, and thus $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} = \frac{\pi^4}{90} - \frac{1}{8} \cdot \frac{\pi^4}{90} = \frac{7\pi^4}{720}$.

10.6.69 Both series diverge, so comparisons of their values are not meaningful.

10.6.70

a. The sequences $\{a_{2n}\} = \left\{\dfrac{1}{n}\right\}$ and $\{a_{2n-1}\} = \left\{\dfrac{2}{n}\right\}$ both converge to 0, so $\{a_n\}$ converges to 0.

b. $\displaystyle S_{2n} = (2-1) + \left(1 - \frac{1}{2}\right) + \left(\frac{2}{3} - \frac{1}{3}\right) + \cdots + \left(\frac{2}{n} - \frac{1}{n}\right) = \sum_{k=1}^{n} \frac{1}{k}$. Because the harmonic series diverges to ∞, $\displaystyle\lim_{n\to\infty} S_{2n} = \infty$.

3. Because $\{S_{2n}\}$ diverges, $\{S_n\}$ diverges. This implies that $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ diverges. To apply the Alternating Series Test, it must be the case that $\{a_n\}$ is nonincreasing and converges to 0. Observe that

$$a_{2n} - a_{2n+1} = \frac{2}{2n} - \frac{4}{2n+2} = \frac{1-n}{n(n+1)} < 0$$

for $n \geq 2$. So it is the case that $a_{2n+1} > a_{2n}$ for $n \geq 2$, so $\{a_n\}$ is not a nonincreasing sequence. So the conditions of the Alternating Series Test are violated.

10.6.71

a. The sequences $\{a_{2n}\} = \left\{\frac{1}{n}\right\}$ and $\{a_{2n-1}\} = \left\{\frac{1}{n^2}\right\}$ both converge to 0, so $\{a_n\}$ converges to 0.

b.

$$S_{2n} = \left(\frac{1}{1^2} - \frac{1}{1}\right) + \left(\frac{1}{2^2} - \frac{1}{2}\right) + \left(\frac{1}{3^2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n^2} - \frac{1}{n}\right) = \sum_{k=1}^{\infty}\left(\frac{1}{k^2} - \frac{1}{k}\right).$$

c. Recall that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges while $\sum_{k=1}^{\infty} \frac{1}{k^2}$ diverges. Therefore, $S_{2n} = \sum_{k=1}^{\infty}\left(\frac{1}{k^2} - \frac{1}{k}\right)$ diverges. So $\{S_n\}$ diverges, which means that the given alternating series diverges. This does not violate the Alternating Series Test because the sequence $\{a_n\}$ is not a nonincreasing sequence.

10.7 The Ratio and Root Tests

10.7.1 Given a series $\sum a_k$ of terms, compute $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ and call it r. If $r < 1$, the given series converges absolutely. If $r > 1$ (including $r = \infty$), the given series diverges. If $r = 1$, the test is inconclusive.

10.7.2 Given a series $\sum a_k$ of terms, compute $\lim_{k\to\infty} \sqrt[k]{|a_k|}$ and call it ρ. If $\rho < 1$, the given series converges absolutely. If $\rho > 1$ (including $\rho = \infty$), the given series diverges. If $\rho = 1$, the test is inconclusive.

10.7.3 $\frac{1000!}{998!} = \frac{1000(999)(998!)}{998!} = 1000(999) = 999,000.$

10.7.4 $\frac{(100!)^2}{(99!)^2} = \frac{100(99!)(100)(99!)}{(99!)(99!)} = 100^2 = 10,000.$

10.7.5 $\frac{k!}{(k+2)!} = \frac{k!}{(k+2)(k+1)(k!)} = \frac{1}{(k+2)(k+1)}.$

10.7.6 $\frac{(2k+1)!}{(2k-1)!} = \frac{(2k+1)(2k)((2k-1)!)}{(2k-1)!} = (2k+1)(2k).$

10.7.7 The Ratio Test.

10.7.8 No. These tests only indicate whether the series converges absolutely or diverges.

10.7.9 The absolute value of the ratio between successive terms is $\left|\frac{a_{k+1}}{a_k}\right| = \frac{1}{(k+1)!} \cdot \frac{(k)!}{1} = \frac{1}{k+1}$, which goes to zero as $k \to \infty$, so the given series converges absolutely by the Ratio Test.

10.7.10 The absolute value of the ratio between successive terms is $\left|\frac{a_{k+1}}{a_k}\right| = \frac{2^{k+1}}{(k+1)!} \cdot \frac{(k)!}{2^k} = \frac{2}{k+1}$; the limit of this ratio is zero, so the given series converges absolutely by the Ratio Test.

10.7.11 The kth root of the absolute value of the kth term is $\frac{10k^3+k}{9k^3+k+1}$. The limit of this as $k \to \infty$ is $\frac{10}{9} > 1$, so the given series diverges by the Root Test.

10.7.12 The kth root of the absolute value of the kth term is $\frac{2k}{k+1}$. The limit of this as $k \to \infty$ is $2 > 1$, so the given series diverges by the Root Test.

10.7.13 The absolute value of the ratio between successive terms is $\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)^2}{4^(k+1)} \cdot \frac{4^k}{(k)^2} = \frac{1}{4}\left(\frac{k+1}{k}\right)^2$. The limit is $1/4$ as $k \to \infty$, so the given series converges absolutely by the Ratio Test.

10.7.14 The absolute value of the ratio between successive terms is

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{2^{(k+1)}}{(k+1)^{(k+1)}} \cdot \frac{(k)^k}{2^k} = \frac{2}{k+1}\left(\frac{k}{k+1}\right)^k.$$

Note that $\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k = \lim_{k\to\infty}\left(1+\frac{-1}{k+1}\right)^k = \frac{1}{e}$, so the limit of the ratio is $0 \cdot \frac{1}{e} = 0$, so the given series converges absolutely by the Ratio Test.

10.7.15 The absolute value of the ratio between successive terms is $\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)e^{-(k+1)}}{(k)e^{-(k)}} = \frac{k+1}{(k)e}$. The limit of this ratio as $k \to \infty$ is $1/e < 1$, so the given series converges absolutely by the Ratio Test.

10.7.16 The absolute value of the ratio between successive terms is $\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)!}{(k+1)^{(k+1)}} \cdot \frac{(k)^k}{(k)!} = \left(\frac{k}{k+1}\right)^k$. This is the reciprocal of $\left(\frac{k+1}{k}\right)^k$ which has limit e as $k \to \infty$, so the limit of the ratio of successive terms is $1/e < 1$, so the given series converges absolutely by the Ratio Test.

10.7.17 The absolute value of the ratio between successive terms is $\left|\frac{a_{k+1}}{a_k}\right| = \frac{k^2 7^{k+1}}{(k+1)^2 7^k} = \frac{7k^2}{(k+1)^2}$ which has limit $7 > 1$ as $k \to \infty$. Therefore the given series diverges by the Ratio Test.

10.7.18 The kth root of the absolute value of the kth term is $\left(1+\frac{3}{k}\right)^k$. The limit of this as $k \to \infty$ is $= e^3 > 1$, so the given series diverges by the Root Test.

10.7.19 The absolute value of the ratio between successive terms is

$$\frac{2^{k+1}}{(k+1)^{99}} \cdot \frac{(k)^{99}}{2^k} = 2\left(\frac{k}{k+1}\right)^{99};$$

the limit as $k \to \infty$ is 2, so the given series diverges by the Ratio Test.

10.7.20 The absolute value of the ratio between successive terms is

$$\frac{(k+1)!}{(k+1)^6} \cdot \frac{k^6}{k!} = \frac{k+1}{1}\left(\frac{k}{k+1}\right)^6;$$

the limit as $k \to \infty$ is ∞, so the given series diverges by the Ratio Test.

10.7.21 The absolute value of the ratio between successive terms is

$$\frac{((k+1)!)^2}{(2(k+1))!} \cdot \frac{(2k)!}{((k)!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)};$$

the limit as $k \to \infty$ is $1/4$, so the given series converges absolutely by the Ratio Test.

10.7.22 The absolute value of the ratio between successive terms is

$$\frac{(k+1)^4 2^{-(k+1)}}{(k)^4 2^{-k}} = \frac{1}{2}\left(\frac{k+1}{k}\right)^4;$$

the limit as $k \to \infty$ is $\frac{1}{2}$, so the given series converges absolutely by the Ratio Test.

10.7.23 The kth root of the absolute value of the kth term is $\left(\frac{k}{k+1}\right)^{2k}$. The limit of this as $k \to \infty$ is $e^{-2} < 1$, so the given series converges absolutely by the Root Test.

10.7.24 The kth root of the absolute value of the kth term is $\frac{3^k}{k}$, which has limit ∞ as $k \to \infty$, so the given series diverges by the Root Test.

10.7.25 The kth root of the absolute value of the kth term is $\frac{3k^2+4k}{2k^2+1}$ which has limit $\frac{3}{2} > 1$ as $k \to \infty$, so the given series diverges by the Root Test.

10.7.26 The kth root of the absolute value of the kth term is $\frac{1}{\ln(k+1)}$. The limit of this as $k \to \infty$ is 0, so the given series converges absolutely by the Root Test.

10.7.27 The kth root of the absolute value of the kth term is $\frac{1}{k}$. The limit of this as $k \to \infty$ is 0, so the given series converges absolutely by the Root Test.

10.7.28 The kth root of the absolute value of the kth term is $\frac{k^{1/k}}{e}$. The limit of this as $k \to \infty$ is $\frac{1}{e} < 1$, so the given series converges absolutely by the Root Test.

10.7.29 We use the Ratio Test. The limit of the absolute values of the ratio of the terms is

$$L = \lim_{k\to\infty} \frac{k!k!((k+1)^{2k+2}}{(k+1)!(k+1)!k^{2k}} = \lim_{k\to\infty} \frac{(k+1)^{2k+2}}{(k+1)(k+1)k^{2k}} = \lim_{k\to\infty} \frac{(k+1)^{2k}}{k^{2k}} = \lim_{k\to\infty} \left(\left(1+\frac{1}{k}\right)^k\right)^2 = e^2 > 1.$$

So the given series diverges by the Ratio Test.

10.7.30 The kth root of the absolute value of the kth term is $\left|\frac{k}{1-5k}\right|$, which has limit $\frac{1}{5}$ as $k \to \infty$, so the given series converges absolutely by the Root Test.

10.7.31

a. False. For example, when $n = 1$ we have $1!1! = 1$ but $(2 \cdot 1)! = 2! = 2$.

b. True. $\frac{(2n)!}{(2n-1)!} = \frac{2n(2n-1)!}{(2n-1)!} = 2n$.

c. True. By the Root Test, $\sum a_k$ converges, so therefore $\sum 10a_k$ converges.

d. True. In this case, the limit of the ratio of terms is 1.

10.7.32 The kth root of the kth term is $\frac{4k^3+7}{9k^2-1}$ which has limit ∞ as $k \to \infty$. Therefore the given series diverges by the Root Test.

10.7.33 Note that $\sum_{k=1}^{\infty} \left((-1)^k \frac{2^{k+1}}{9^{k-1}}\right) = \sum_{k=1}^{\infty} 18 \left(-\frac{2}{9}\right)^k$ is a geometric series with $r = -\frac{2}{9}$. Because $|r| < 1$, the given series converges absolutely.

10.7.34 We use the Ratio Test. We have

$$\lim_{k\to\infty} \frac{\frac{(k+1)!}{(2k+3)!}}{\frac{k!}{(2k+1)!}} = \lim_{k\to\infty} \frac{(2k+1)!(k+1)!}{(2k+3)!k!} = \lim_{k\to\infty} \frac{k+1}{(2k+3)(2k+2)} = 0.$$

Therefore the given series converges absolutely by the Ratio Test.

10.7.35 We use the Ratio Test. We have

$$\lim_{k\to\infty} \frac{\dfrac{(2k+3)!}{((k+1)!)^2}}{\dfrac{(2k+1)!}{(k!)^2}} = \lim_{k\to\infty} \frac{(2k+3)!(k!)^2}{(2k+1)!((k+1)!)^2} = \lim_{k\to\infty} \frac{(2k+3)(2k+2)}{(k+1)^2} = 4 > 1.$$

Therefore the given series diverges by the Ratio Test.

10.7.36 Use the Root Test: The kth root of the absolute value of the kth term is $\dfrac{k^2}{2k^2+1}$. The limit of this as $k \to \infty$ is $\dfrac{1}{2} < 1$, so the given series converges absolutely by the Root Test.

10.7.37 Use the Ratio Test: the ratio of successive terms is $\dfrac{(k+1)^{100}}{(k+2)!} \cdot \dfrac{(k+1)!}{k^{100}} = \left(\dfrac{k+1}{k}\right)^{100} \cdot \dfrac{1}{k+2}$. This has limit $1^{100} \cdot 0 = 0$ as $k \to \infty$, so the given series converges absolutely by the Ratio Test.

10.7.38 $\lim\limits_{k\to\infty} k^3 \sin(1/k^3) = \lim\limits_{k\to\infty} \dfrac{\sin(1/k^3)}{1/k^3} = 1 \neq 0$, so the given series diverges by the Divergence Test.

10.7.39 We use the Limit Comparison Test with the divergent series $\sum\limits_{k=1}^{\infty} \dfrac{1}{k}$ on the corresponding series with all positive terms to obtain

$$L = \lim_{k\to\infty} \frac{\dfrac{k^3}{\sqrt{k^8+1}}}{\dfrac{1}{k}} = \lim_{k\to\infty} \frac{k^4}{\sqrt{k^8+1}} = \lim_{k\to\infty} \frac{1}{\sqrt{1+1/k^8}} = 1.$$

Therefore the given series is not absolutely convergent. To see that it is convergent (and therefore conditionally convergent), note that $\lim\limits_{k\to\infty} \dfrac{k^3}{\sqrt{k^8+1}} = 0$, and that the sequence is decreasing, so by the Alternating Series Test, the series is convergent. To see that the sequence decreases, note that if $f(x) = \dfrac{x^3}{\sqrt{x^8+1}}$, that $f'(x) = \dfrac{x^2(3-x^8)}{(x^8+1)^{3/2}}$ which is less than 0 for $x \geq 2$. This shows that the the sequence $|a_k|$ is decreasing, at least for $k \geq 2$.

10.7.40 Use the Root Test. $\lim\limits_{k\to\infty} \dfrac{1}{1+p} = \dfrac{1}{1+p} < 1$ because $p > 0$, so the given series converges absolutely.

10.7.41 Use the Root Test. The kth root of the absolute value of the kth term is $(k^{1/k} - 1)^2$, which has limit 0 as $k \to \infty$, so the given series converges absolutely by the Root Test.

10.7.42 Use the Ratio Test: $\left|\dfrac{a_{k+1}}{a_k}\right| = \dfrac{((k+1)!)^3}{((3k+3)!} \cdot \dfrac{(3k)!}{(k!)^3} = \dfrac{(k+1)^3}{(3k+1)(3k+2)(3k+3)}$, which has limit $1/27$ as $k \to \infty$. Thus the given series converges absolutely.

10.7.43 Use the Ratio Test. $\left|\dfrac{a_{k+1}}{a_k}\right| = \dfrac{2^{k+1}(k+1)!}{(k+1)^{k+1}} \cdot \dfrac{(k)^k}{2^k(k)!} = 2\left(\dfrac{k}{k+1}\right)^k$, which has limit $\dfrac{2}{e}$ as $k \to \infty$, so the given series converges absolutely.

10.7.44 Use the Root Test. $\lim\limits_{k\to\infty} \left(1 - \dfrac{1}{k}\right)^k = e^{-1} < 1$, so the given series converges absolutely.

10.7.45 We have $|a_k| = \frac{1}{k^{0.99}}$, and $\sum_{k=1}^{\infty} \frac{1}{k^{0.99}}$ is a p-series with $p = 0.99$. Because $p < 1$, the series $\sum |a_k|$ diverges. However, because the sequence $\left\{\frac{1}{k^{0.99}}\right\}$ decreases to 0, the given series converges by the Alternating Series Test. Therefore, the given series is conditionally convergent.

10.7.46 Use the Root Test. $\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{100} \cdot \frac{1}{k} = 0$, so the given series converges absolutely by the Root Test.

10.7.47 This series is $\sum_{k=1}^{\infty} \frac{k^2}{k!}$. By the Ratio Test, $\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)^2}{(k+1)!} \cdot \frac{k!}{k^2} = \frac{1}{k+1}\left(\frac{k+1}{k}\right)^2$, which has limit 0 as $k \to \infty$, so the given series converges absolutely.

10.7.48 Note that $\lim_{k\to\infty} \frac{1}{\tan^{-1} k} = \frac{2}{\pi} \neq 0$, so $\lim_{k\to\infty} \frac{(-1)^k}{\tan^{-1} k} \neq 0$, and thus the given series diverges by the Divergence Test.

10.7.49 For the corresponding series of all positive terms, we use the Limit Comparison Test with the divergent series $\sum_{k=1}^{\infty} \frac{1}{k^{3/4}}$. We have

$$L = \lim_{k\to\infty} \frac{\frac{1}{\sqrt{k^{3/2}+k}}}{\frac{1}{k^{3/4}}} = \lim_{k\to\infty} \sqrt{\frac{k^{3/2}}{k^{3/2}+k}} = 1.$$

Therefore the given series does not converge absolutely. However, because $\left\{\frac{1}{\sqrt{k^{3/2}+k}}\right\}$ is decreasing to 0, the given series is convergent by the Alternating Series Test. Thus, the given series is conditionally convergent.

10.7.50 For any p, if k is sufficiently large then $k^{1/p} > \ln k$ because powers grow faster than logs, so that $k > (\ln k)^p$ and thus $1/k < 1/(\ln k)^p$. Because $\sum_{k=1}^{\infty} 1/k$ diverges, we see that the original series diverges for all p.

10.7.51 For $p \le 1$ and $k > e$, $\frac{\ln k}{k^p} > \frac{1}{k^p}$. The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges, so the given series diverges. For $p > 1$, let $q < p - 1$; then for sufficiently large k, $\ln k < k^q$, so that by the Comparison Test, $\frac{\ln k}{k^p} < \frac{k^q}{k^p} = \frac{1}{k^{p-q}}$. But $p - q > 1$, so that $\sum_{k=1}^{\infty} \frac{1}{k^{p-q}}$ is a convergent p-series. Thus the original series is convergent precisely when $p > 1$.

10.7.52 $\int_2^{\infty} \frac{dx}{x \ln x (\ln\ln x)^p} = \lim_{b\to\infty} \left(\left.\frac{(\ln\ln x)^{1-p}}{1-p}\right|_2^b \right)$. This improper integral converges if and only if $p > 1$. Thus the original series converges for $p > 1$.

10.7.53 For $p \le 1$, $\frac{(\ln k)^p}{k^p} > \frac{1}{k^p}$, and $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges for $p \le 1$, so the original series diverges. For $p > 1$, let $q < p - 1$; then for sufficiently large k, $(\ln k)^p < k^q$. Note that $\frac{(\ln k)^p}{k^p} < \frac{k^q}{k^p} = \frac{1}{k^{p-q}}$. But $p - q > 1$, so $\sum_{k=1}^{\infty} \frac{1}{k^{p-q}}$ converges, so the given series converges. Thus, the given series converges exactly for $p > 1$.

10.7.54 Using the Ratio Test, $\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)!p^{k+1}}{(k+2)^{k+1}} \cdot \frac{(k+1)^k}{(k)!p^k} = \frac{(k+1)p(k+1)^k}{(k+2)^{k+1}} = p\left(\frac{k+1}{k+2}\right)^{k+1} =$ $p \cdot \left(\frac{1}{1+\frac{1}{k+1}}\right)^{k+1}$, which has limit pe^{-1}. The series converges if the ratio limit is less than 1, so if $p < e$. If $p > e$, the given series diverges by the Ratio Test. If $p = e$, the given series diverges by the Divergence Test.

10.7.55 Use the Ratio Test:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)p^{k+1}}{k+2} \cdot \frac{k+1}{kp^k} = p \cdot \frac{k^2+2k+1}{k^2+2k},$$

and this expression has limit p as $k \to \infty$. Thus the series converges for $p > 1$.

10.7.56 $\ln\left(\frac{k}{k+1}\right)^p = p(\ln(k) - \ln(k+1))$, so

$$\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)^p = p\sum_{k=1}^{\infty}(\ln(k) - \ln(k+1))$$

which telescopes, and the n^{th} partial sum is $-p\ln(n+1)$, and $\lim_{n\to\infty} -p\ln(n+1)$ is not a finite number for any value of p other than 0. The given series diverges for all values of p other than $p = 0$.

10.7.57 $\lim_{k\to\infty} a_k = \lim_{k\to\infty}\left(1 - \frac{p}{k}\right)^k = e^{-p} \neq 0$, so this sequence diverges for all p by the Divergence Test.

10.7.58 $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k}\right| = \frac{|x|}{k+1}$. This has limit 0 as $k \to \infty$ for any value of x, so the series converges absolutely for all x.

10.7.59 $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}}{x^k}\right| = |x|$. This has limit $|x|$ as $k \to \infty$, so the series converges absolutely for $|x| < 1$. It clearly does not converge for $x = \pm 1$.

10.7.60 $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}}{k+1} \cdot \frac{k}{x^k}\right| = |x| \cdot \frac{k}{k+1}$, which has limit $|x|$ as $k \to \infty$. Thus this series converges for $|x| < 1$; additionally, for $x = 1$ (where the Ratio Test is inconclusive), the series is the harmonic series which diverges and for $x = -1$ the series is the alternating harmonic series which converges. So the series converges for $-1 \leq x < 1$.

10.7.61 $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}}{(k+1)^2} \cdot \frac{k^2}{x^k}\right| = |x|\left(\frac{k}{k+1}\right)^2$, which has limit $|x|$ as $k \to \infty$. Thus the series converges for $|x| < 1$. When $x = 1$, the series is $\sum_{k=1}^{\infty} \frac{1}{k^2}$, which converges, and for $x = -1$, the series is $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ which converges. Thus the original series converges for $|x| \leq 1$.

10.7.62 $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{2k+2}}{(k+1)^2} \cdot \frac{k^2}{x^{2k}}\right| = x^2\left(\frac{k}{k+1}\right)^2$, which has limit x^2 as $k \to \infty$, so the series converges for $|x| < 1$. When $x = \pm 1$, the series is $\frac{1}{k^2}$, which converges. Thus this series converges for $-1 \leq x \leq 1$.

10.7.63 $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}}{2^{k+1}} \cdot \frac{2^k}{x^k}\right| = \frac{|x|}{2}$, which has limit $\frac{|x|}{2}$ as $k \to \infty$. Thus the series converges for $|x| < 2$. For $x = \pm 2$, it is obviously divergent. So the series converges for $-2 < x < 2$.

10.7.64

a. Use the Ratio Test:

$$\frac{a_{k+1}}{a_k} = \frac{1\cdot 3\cdot 5\cdots(2k+1)}{p^{k+1}(k+1)!}\cdot\frac{p^k(k)!}{1\cdot 3\cdot 5\cdots(2k-1)} = \frac{(2k+1)}{(k+1)p}$$

and this expression has limit $\frac{2}{p}$ as $k \to \infty$. Thus the series converges for $p > 2$.

b. Following the hint, when $p = 2$ we have $\sum_{k=1}^{\infty}\frac{(2k)!}{2^k k!(2\cdot 4\cdot 6\cdots 2k)} = \sum_{k=1}^{\infty}\frac{(2k)!}{(2^k)^2(k!)^2}$. Using Stirling's formula, the numerator is asymptotic to $2\sqrt{\pi}\sqrt{k}(2k)^{2k}e^{-2k} = 2\sqrt{\pi}\sqrt{k}(2^k)^2(k^k)^2e^{-2k}$ while the denominator is asymptotic to $(2^k)^2 2\pi k(k^k)^2e^{-2k}$, so the quotient is asymptotic to $\frac{1}{\sqrt{\pi}\sqrt{k}}$. Thus the original series diverges for $p = 2$ by the Limit Comparison Test with the divergent p-series $\sum_{k=1}^{\infty}\frac{1}{k^{1/2}}$.

10.8 Choosing a Convergence Test

10.8.1 The presence of the power k makes the Root Test a good candidate for determining convergence.

10.8.2 The Limit Comparison Test is likely the easiest way to do determine convergence. However, we could factor the denominator as $(k+1)(k-2)$ and then obtain the partial fraction decomposition of the fraction in the sum, and then recognize that we have a telescoping series.

10.8.3 The Divergence Test is appropriate because $\lim_{k\to\infty}\frac{2k^2}{k^2-k-2} \neq 0$.

10.8.4 Let $f(x) = \frac{1}{x\ln^7 x}$ and apply the Integral Test.

10.8.5 Reindex the sum $\sum_{k=10}^{\infty}\frac{1}{(k-9)^5}$ to obtain $\sum_{k=1}^{\infty}\frac{1}{k^5}$ and apply the p-series Test. Another approach is to use the Limit Comparison Test with the comparison series $\sum_{k=10}^{\infty}\frac{1}{k^5}$.

10.8.6 The Ratio Test is an appropriate choice due to the presence of the factorial term.

10.8.7 Use the Comparison Test or Limit Comparison Test with the comparison series $\sum_{k=10}^{\infty}\frac{1}{k^2}$.

10.8.8 By writing out the first few terms of the series or by rewriting the series $\sum_{k=1}^{\infty}\frac{(-3)^k}{4^{k+1}}$ as $\sum_{k=1}^{\infty}\frac{1}{4}\left(-\frac{3}{4}\right)^k$, we can see that this is a geometric series. The value of r will determine whether the series converges or diverges.

10.8.9 The series is an alternating series and the function $f(x) = \frac{1}{\sqrt{2^x+\ln x}}$ is nonincreasing for $x \geq 1$, so the Alternating Series Test is an appropriate test to apply to this series.

10.8.10 Use a telescoping series argument to determine whether this series converges.

10.8.11 Taking the limit of the kth-term of the series, we have $\lim_{k\to\infty} \frac{2k^4+k}{4k^4-8k} = \frac{1}{2}$. The terms of the series $\sum_{k=1}^{\infty} \frac{2k^4+k}{4k^4-8k}$ do not approach 0, so the series diverges by the Divergence Test.

10.8.12 By rewriting the series, we see that it is a geometric series with $r = \frac{1}{49}$:

$$\sum_{k=1}^{\infty} 7^{-2k} = \sum_{k=1}^{\infty} \frac{1}{7^{2k}} = \sum_{k=1}^{\infty} \frac{1}{49^k}.$$

Because $|r| < 1$, the series converges.

10.8.13 We apply the Limit Comparison Test using the series $\sum_{k=3}^{\infty} \frac{1}{k}$ as the comparison series. This comparison series diverges because it is the tail of the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Calculating L, with the help of l'Hôpital's Rule, we have

$$L = \lim_{k\to\infty} \frac{5/(2+\ln k)}{1/k} = \lim_{k\to\infty} \frac{5k}{2+\ln k} = \lim_{k\to\infty} \frac{5}{1/k} = \lim_{k\to\infty} 5k = \infty.$$

By case (3) of the Limit Comparison Test, the series $\sum_{k=3}^{\infty} \frac{5}{2+\ln k}$ diverges because the comparison series $\sum_{k=3}^{\infty} \frac{1}{k}$ diverges.

10.8.14 We use the Limit Comparison Test using the comparison series $\sum_{k=1}^{\infty} \frac{k^2}{k^4} = \sum_{k=1}^{\infty} \frac{1}{k^2}$. Note that

$$L = \lim_{k\to\infty} \frac{\frac{7k^2-k-2}{4k^4-3k+1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{7k^4-k^3-2k^2}{4k^4-3k+1} = \frac{7}{4},$$

and therefore $0 < L < \infty$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series, it follows that $\sum_{k=1}^{\infty} \frac{7k^2-k-2}{4k^4-3k+1}$ converges by the Limit Comparison Test.

10.8.15 The presence of $k!$ in the sum suggests that the Ratio Test might be an appropriate approach here.

$$r = \lim_{k\to\infty} \left| \frac{(-7)^{k+1}/(k+1)!}{(-7)^k/k!} \right| = \lim_{k\to\infty} \frac{7^{k+1}}{(k+1)!} \cdot \frac{k!}{7^k} = \lim_{k\to\infty} \frac{7}{k+1} = 0$$

Because $r < 1$, the series $\sum_{k=1}^{\infty} \frac{(-7)^k}{k!}$ converges by the Ratio Test.

10.8.16 Because $\sum_{k=1}^{\infty} \frac{7^k}{k!}$ converges (previous solution) and $\frac{7^k}{k!+10} < \frac{7^k}{k!}$, it follows from the Comparison Test that $\sum_{k=1}^{\infty} \frac{7^k}{k!+10}$ converges.

10.8.17 The limit of the kth-term of the series is $\lim_{k\to\infty} \frac{(-k)^3}{3k^3+2} = -\frac{1}{3}$. Therefore $\sum_{k=1}^{\infty} \frac{(-k)^3}{3k^3+2}$ diverges by the Divergence Test because $\lim_{k\to\infty} \frac{(-k)^3}{3k^3+2} \neq 0$.

10.8.18 We apply the Divergence Test by showing that the limit $\lim_{k\to\infty}\left(1+a/k\right)^k$ is nonzero. Noting that $\lim_{k\to\infty}\left(1+a/k\right)^k = \lim_{k\to\infty} e^{\ln\left(1+a/k\right)^k} = e^{\lim_{k\to\infty} k\ln(1+a/k)}$, the first step is to evaluate the limit

$$L = \lim_{k\to\infty} k\ln\left(1+\frac{a}{k}\right).$$

The limit has the form $\infty\cdot 0$, which we first put in the form $0/0$ so that l'Hôpital's Rule may be applied.

$$\begin{aligned} L = \lim_{k\to\infty} k\ln\left(1+\frac{a}{k}\right) &= \lim_{k\to\infty}\frac{\ln(1+a/k)}{1/k} && k=\frac{1}{1/k} \\ &= \lim_{k\to\infty}\frac{\frac{1}{1+a/k}\cdot\left(-\frac{a}{k^2}\right)}{\left(-\frac{1}{k^2}\right)} && \text{L'Hôpital's Rule} \\ &= \lim_{k\to\infty}\frac{a}{1+a/k} = a && \text{Simplify and evaluate.} \end{aligned}$$

Therefore

$$\lim_{k\to\infty}\left(1+\frac{a}{k}\right)^k = e^L = e^a.$$

This limit does not equal 0 because $e^a \neq 0$ for any constant a. It follows from the Divergence Test that the series $\sum_{k=1}^{\infty}\left(1+\frac{a}{k}\right)^k$ diverges for any constant a.

10.8.19 Writing out the first few terms of the series, we have

$$\sum_{k=0}^{\infty}\frac{3^{k+4}}{5^{k-2}} = \frac{3^4}{5^{-2}}+\frac{3^5}{5^{-1}}+\frac{3^6}{5^0}+\frac{3^7}{5^1}+\cdots.$$

This is a geometric series with $r = 3/5$. Because $r < 1$, the series converges.

10.8.20 We use the Limit Comparison Test using the comparison series $\sum_{j=1}^{\infty}\frac{j^{10}}{j^{11}} = \sum_{j=1}^{\infty}\frac{1}{j}$.

$$L = \lim_{j\to\infty}\frac{\frac{4j^{10}}{j^{11}+1}}{\frac{1}{j}} = \lim_{j\to\infty}\frac{4j^{11}}{j^{11}+1} = 4$$

Therefore $0 < L < \infty$. The series $\sum_{j=1}^{\infty}\frac{4j^{10}}{j^{11}+1}$ diverges by the Limit Comparison Test because the comparison series $\sum_{j=1}^{\infty}\frac{1}{j}$ diverges.

10.8.21 The Alternating Series can be used here to establish convergence of the series. Another approach is to first establish that the positive-term series $\sum_{k=1}^{\infty}\frac{k}{k^3+1}$ converges. Notice that

$$\frac{k}{k^3+1} < \frac{k}{k^3} = \frac{1}{k^2}.$$

It follows that $\sum_{k=1}^{\infty}\frac{k}{k^3+1}$ converges by the Comparison Test because $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is a convergent p-series. This implies that the alternating series $\sum_{k=1}^{\infty}\frac{(-1)^k k}{k^3+1}$ is absolutely convergent. Therefore $\sum_{k=1}^{\infty}\frac{(-1)^k k}{k^3+1}$ converges because every absolutely convergent series converges.

10.8.22 The series $\sum_{k=1}^{\infty}\left(\frac{e+1}{\pi}\right)^k$ is a geometric series with $r = \frac{e+1}{\pi}$. Because $r > 1$, the series diverges.

10.8.23 Apply the Ratio Test.

$$r = \lim_{k\to\infty}\left|\frac{(k+1)^5/5^{k+1}}{k^5/5^k}\right| = \lim_{k\to\infty}\frac{(k+1)^5}{5^{k+1}}\cdot\frac{5^k}{k^5} = \lim_{k\to\infty}\frac{1}{5}\left(\frac{k+1}{k}\right)^5 = \frac{1}{5}$$

Because $r < 1$, the series $\sum_{k=1}^{\infty}\frac{k^5}{5^k}$ converges by the Ratio Test.

10.8.24 Reindex the series:

$$\sum_{k=1}^{\infty}\frac{4}{(k+3)^3} = \frac{4}{4^3}+\frac{4}{5^3}+\frac{4}{6^3}+\frac{4}{7^3}+\cdots = \sum_{k=4}^{\infty}\frac{4}{k^3}.$$

The series $\sum_{k=4}^{\infty}\frac{1}{k^3}$ converges because it is the tail of the convergent p-series $\sum_{k=1}^{\infty}\frac{1}{k^3}$. It follows that $\sum_{k=1}^{\infty}\frac{4}{(k+3)^3}$ converges.

10.8.25 The terms of the series are positive and decreasing. We apply the Integral Test, with the help of the substitution $u = \sqrt{x}$.

$$\int_1^{\infty}\frac{dx}{\sqrt{x}e^{\sqrt{x}}} = \lim_{b\to\infty}\int_1^b\frac{dx}{\sqrt{x}e^{\sqrt{x}}} = \lim_{b\to\infty}\int_1^{\sqrt{b}}\frac{2}{e^u}du = \lim_{b\to\infty}\left(-\frac{2}{e^u}\right)\Bigg|_1^{\sqrt{b}} = \lim_{b\to\infty}\left(\frac{2}{e}-\frac{2}{e^{\sqrt{b}}}\right) = \frac{2}{e}$$

Because the integral converges, it follows from the Integral Test that the series converges.

10.8.26 Adding 5 to the inequality $-1 \le \sin k$, and then dividing by $\sqrt{k}$, it follows that for $k \ge 1$,

$$\frac{4}{\sqrt{k}} \le \frac{5+\sin k}{\sqrt{k}}.$$

Because $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ is a divergent p-series, $\sum_{k=1}^{\infty}\frac{4}{\sqrt{k}}$ diverges. So by the Comparison Test, $\sum_{k=1}^{\infty}\frac{5+\sin k}{\sqrt{k}}$ diverges.

10.8.27 Adding 3 to the inequality $\cos 5k \le 1$, and then dividing by k^3, we obtain

$$\frac{3+\cos 5k}{k^3} \le \frac{4}{k^3}.$$

Because $\sum_{k=1}^{\infty}\frac{1}{k^3}$ is a convergent p-series, $\sum_{k=1}^{\infty}\frac{4}{k^3}$ converges. So by the Comparison Test, $\sum_{k=1}^{\infty}\frac{3+\cos 5k}{k^3}$ converges.

10.8.28 For $k \geq 3$, the expression $\dfrac{\ln k}{k^{1/3}}$ is positive, which implies that $\displaystyle\sum_{k=3}^{\infty} \frac{(-1)^k \ln k}{k^{1/3}}$ is an alternating series. We apply the Alternating Series Test to show that this series converges. It is not obvious that the terms of the series are decreasing in magnitude, so we use calculus to show that the derivative of function $f(x) = \dfrac{\ln x}{x^{1/3}}$ is negative after a fixed number of terms of the series. By the Quotient Rule,

$$f'(x) = \frac{x^{1/3} \cdot \frac{1}{x} - (\ln x)\frac{1}{3}x^{-2/3}\cdot}{x^{2/3}} = \frac{3 - \ln x}{3x^{4/3}}.$$

For $x \geq 27$, $\ln x \geq 3$ because $\ln x \geq \ln 3^3 > \ln e^3 = 3$. So $f'(x)$ is negative for $x \geq 27$ which implies that the terms of the series decrease for $x \geq 27$. By l'Hôpital's Rule,

$$\lim_{k\to\infty} \frac{\ln k}{k^{1/3}} = \lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{1}{3}k^{-2/3}} = \lim_{k\to\infty} \frac{3}{k^{1/3}} = 0.$$

The conditions of the Alternating Series Test are met which implies that the series converges.

10.8.29 We first examine the series $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{k^{10}}$ using the Ratio Test.

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{10^{k+1}/(k+1)^{10}}{10^k/k^{10}} = \lim_{k\to\infty} \frac{10^{k+1}}{(k+1)^{10}} \cdot \frac{k^{10}}{10^k} = 10 \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^{10} = 10.$$

Because $r > 1$, the series $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{k^{10}}$ diverges by the Ratio Test. Because $\dfrac{10^k+1}{k^{10}} > \dfrac{10^k}{k^{10}}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{k^{10}}$ diverges, the series $\displaystyle\sum_{k=1}^{\infty} \frac{10^k+1}{k^{10}}$ diverges by the Comparison Test.

10.8.30 For $k \geq 3$, $\ln k > 1$. Multiplying both sides of this inequality by k^3, we have $k^3 \ln k > k^3$ for $k \geq 3$. Inverting both sides we have $\dfrac{1}{k^3 \ln k} < \dfrac{1}{k^3}$ for $k \geq 3$. Because $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k^3}$ is a convergent p-series, it follows that $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k^3 \ln k}$ converges by the Comparison Test.

10.8.31 Note that $\dfrac{1}{j^2+4} < \dfrac{1}{j^2}$. Because $\displaystyle\sum_{j=1}^{\infty} \frac{1}{j^2}$ is a convergent p-series, it follows that $\displaystyle\sum_{j=1}^{\infty} \frac{1}{j^2+4}$ converges by the Comparison Test. Therefore $\displaystyle\sum_{j=1}^{\infty} \frac{5}{j^2+4}$ converges.

10.8.32 In the solution to Exercise 18, we showed that $\lim\limits_{k\to\infty} \left(1 + a/k\right)^k = e^a$. It follows that

$$\lim_{k\to\infty} \frac{k^k}{(k+2)^k} = \lim_{k\to\infty} \left(\frac{k}{k+2}\right)^k = \frac{1}{\lim\limits_{k\to\infty} \left(\frac{k+2}{k}\right)^k} = \frac{1}{\lim\limits_{k\to\infty} \left(1 + \frac{2}{k}\right)^k} = \frac{1}{e^2}.$$

Because the terms of the series $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{(k+2)^k}$ do not approach 0 as $k \to \infty$, the series diverges by the Divergence Test.

10.8.33 Using the comparison series $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k}$, we calculate the limit L using L'Hôpital's Rule.

$$L = \lim_{k\to\infty} \frac{1/(k^{1/3}\ln k)}{1/k} = \lim_{k\to\infty} \frac{1}{k^{1/3}\ln k} \cdot \frac{k}{1} = \lim_{k\to\infty} \frac{k^{2/3}}{\ln k} = \lim_{k\to\infty} \frac{\frac{2}{3}k^{-1/3}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{2}{3}k^{2/3} = \infty$$

The series $\sum_{k=3}^{\infty} \frac{1}{k}$ diverges because it is the tail of the divergent harmonic series. Because $L = \infty$, it follows from case (3) of the Limit Comparison Test that $\sum_{k=3}^{\infty} \frac{1}{k^{1/3}\ln k}$ diverges.

10.8.34 We apply the Alternating Series Test. The first step is to show that the terms of the series decrease in magnitude after a fixed number of terms of the series. Let $f(x) = \frac{5x^2}{\sqrt{3x^5+1}}$. Then

$$f'(x) = \frac{10x\sqrt{3x^5+1} - 5x^2\frac{15x^4}{2\sqrt{3x^5+1}}}{3x^5+1} = \frac{20x(3x^5+1) - 75x^6}{2(3x^5+1)^{3/2}} = \frac{20x - 15x^6}{2(3x^5+1)^{3/2}} = \frac{5x(4-3x^5)}{2(3x^5+1)^{3/2}}.$$

Because $f'(x) < 0$ for $x \geq 2$, it follows that the terms of the series are decreasing in magnitude for $k \geq 2$. Furthermore,

$$\lim_{k\to\infty} \frac{5k^2}{\sqrt{3k^5+1}} = \lim_{k\to\infty} \frac{5k^2/k^2}{(\sqrt{3k^5+1})/k^2} = \lim_{k\to\infty} \frac{5}{\sqrt{3k+1/k^4}} = 0.$$

The conditions of the Alternating Series Test are met which implies that the series $\sum_{k=1}^{\infty} \frac{(-1)^k 5k^2}{\sqrt{3k^5+1}}$ converges.

10.8.35 We apply the Root Test.

$$\rho = \lim_{k\to\infty} \left|\frac{2^k 3^k}{k^k}\right|^{1/k} = \lim_{k\to\infty} \frac{6}{k} = 0$$

Because $\rho < 1$, the series $\sum_{k=1}^{\infty} \frac{2^k 3^k}{k^k}$ converges by the Root Test.

10.8.36 Applying the Ratio Test, we have

$$\begin{aligned} r &= \lim_{k\to\infty} \left|\frac{(-1)^{k+2}\big(2(k+1)+1\big)/(k+1)!}{(-1)^{k+1}(2k+1)/(k!)}\right| = \lim_{k\to\infty} \frac{(2k+3)k!}{(k+1)!(2k+1)} \\ &= \lim_{k\to\infty} \frac{(2k+3)}{(k+1)(2k+1)} = \lim_{k\to\infty} \frac{2k+3}{2k^2+3k+1} = 0. \end{aligned}$$

Because $r < 1$, the series $\sum_{k=1}^{\infty} \frac{2k+1}{k!}$ converges by the Ratio Test.

10.8.37 We apply the Root Test.

$$\rho = \lim_{k\to\infty} \left|(-1)^k \left(\frac{5k}{3k+7}\right)^k\right|^{1/k} = \lim_{k\to\infty} \frac{5k}{3k+7} = \frac{5}{3}$$

Because $\rho > 1$, the series $\sum_{k=1}^{\infty} \left(\frac{5k}{3k+7}\right)^k$ diverges by the Root Test.

10.8.38 Simplifying the terms of the series, we have $\sum_{k=1}^{\infty} \frac{2^k(k-1)!}{k!} = \sum_{k=1}^{\infty} \frac{2^k(k-1)!}{k(k-1)!} = \sum_{k=1}^{\infty} \frac{2^k}{k}$. Applying l'Hôpital's Rule, we have

$$\lim_{k\to\infty} \frac{2^k}{k} = \lim_{k\to\infty} (2^k \ln 2) = \infty.$$

Because the terms of this series do not approach 0, the series diverges by the Divergence Test.

10.8.39 Applying the Ratio Test, we have

$$r = \lim_{k\to\infty} \left| \frac{\frac{5^{k+1}((k+1)!)^2}{(2k+2)!}}{\frac{5^k(k!)^2}{(2k)!}} \right| = \lim_{k\to\infty} \frac{5^k 5(k+1)!(k+1)!}{(2k+2)(2k+1)(2k)!} \cdot \frac{(2k)!}{5^k(k!)(k!)} = \lim_{k\to\infty} \frac{5(k+1)}{2(2k+1)} = \frac{5}{4}.$$

Because $r > 1$, the series $\sum_{k=1}^{\infty} \frac{5^k(k!)^2}{(2k)!}$ diverges by the Ratio Test.

10.8.40 We simplify the series first by observing that $\cos\big((2j+1)\pi\big) = -1$ for any integer j. Therefore

$$\sum_{j=1}^{\infty} \frac{\cos\big((2j+1)\pi\big)}{j^2+1} = \sum_{j=1}^{\infty} \frac{-1}{j^2+1}.$$

Because $\sum_{j=1}^{\infty} \frac{1}{j^2}$ is a convergent p-series and $\frac{1}{j^2+1} \le \frac{1}{j^2}$, it follows that $\sum_{j=1}^{\infty} \frac{1}{j^2+1}$ converges by the Comparison Test. Therefore $\sum_{j=1}^{\infty} \frac{\cos\big((2j+1)\pi\big)}{j^2+1} = \sum_{j=1}^{\infty} \frac{-1}{j^2+1}$ converges.

10.8.41 We apply the Ratio Test.

$$r = \lim_{k\to\infty} \left| \frac{2^{k+1}/(3^{k+1}-2^{k+1})}{2^k/(3^k-2^k)} \right| = \lim_{k\to\infty} \frac{2\big(3^k-2^k\big)}{3^{k+1}-2^{k+1}} = \lim_{k\to\infty} \frac{2\big(3^k/3^k-2^k/3^k\big)}{3^{k+1}/3^k-2^{k+1}/3^k} = \lim_{k\to\infty} \frac{2\big(1-(2/3)^k\big)}{3-2(2/3)^k} = \frac{2}{3}$$

Because $r < 1$, it follows that $\sum_{k=1}^{\infty} \frac{2^k}{3^k-2^k}$ converges.

10.8.42 Apply the Comparison test with the comparison series $\sum_{k=1}^{\infty} \frac{3}{2^k}$. The inequality $2+(-1)^k \le 3$ implies that

$$\frac{2+(-1)^k}{2^k} \le \frac{3}{2^k}.$$

Because $\sum_{k=1}^{\infty} \frac{3}{2^k}$ is a convergent geometric series, it follows from the Comparison Test $\sum_{k=1}^{\infty} \frac{2+(-1)^k}{2^k}$ converges.

10.8.43 The values of $\cos\frac{(3k-1)\pi}{3}$ oscillate between $-1/2$ and $1/2$, which implies that $\lim_{k\to\infty} \cos\frac{(3k-1)\pi}{3} \neq 0$. So by the Divergence Test, the series $\sum_{k=1}^{\infty} \cos\frac{(3k-1)\pi}{3}$ diverges.

10.8.44 Apply the Limit Comparison test using $\sum_{k=10}^{\infty} \frac{1}{k}$, which is a tail of the divergent harmonic series.

$$L = \lim_{k\to\infty} \frac{\frac{1}{\sqrt{k(k-9)}}}{\frac{1}{k}} = \lim_{k\to\infty} \sqrt{\frac{k^2}{k^2-9k}} = \sqrt{\lim_{k\to\infty} \frac{1}{1-9/k}} = 1$$

We see that $0 < L < \infty$. This implies that $\sum_{k=10}^{\infty} \frac{1}{\sqrt{k(k-9)}}$ diverges by the Limit Comparison Test.

10.8.45 We will use the Integral Test, and we first show that the terms of the series $\sum_{k=1}^{\infty} \frac{k^4}{e^{k^5}}$ are decreasing by demonstrating that the positive and continuous function $f(x) = \frac{x^4}{e^{x^5}}$ is decreasing. In this case,

$$f'(x) = \frac{e^{x^5}4x^3 - x^4e^{x^5}5x^4}{e^{2x^5}} = \frac{x^3e^{x^5}(4-5x^5)}{e^{2x^5}} = \frac{x^3(4-5x^5)}{e^{x^5}}.$$

Therefore $f'(x) < 0$ for $x \geq 1$ which implies that the terms of the series $\sum_{k=1}^{\infty} \frac{k^4}{e^{k^5}}$ are decreasing. Next, we integrate with the help of the substitution $u = x^5$, which implies that $du = 5x^4dx$.

$$\begin{aligned}\int_1^{\infty} \frac{x^4}{e^{x^5}}dx &= \lim_{b\to\infty} \int_1^b \frac{x^4}{e^{x^5}}dx \\ &= \lim_{b\to\infty} \frac{1}{5}\int_1^{b^5} \frac{1}{e^u}dx \\ &= \lim_{b\to\infty} \frac{1}{5}\left(-\frac{1}{e^u}\right)\bigg|_1^{b^5} \\ &= \lim_{b\to\infty} \frac{1}{5}\left(\frac{1}{e} - \frac{1}{e^{b^5}}\right) = \frac{1}{5e}\end{aligned}$$

Because the integral converges, it follows by the Integral Test that the series $\sum_{k=1}^{\infty} \frac{k^4}{e^{k^5}}$ converges.

10.8.46 The denominator $(k+1)! - k! = (k+1)k! - k! = k!(k+1-1) = k! \cdot k$. Therefore

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)! - k!} = \sum_{k=1}^{\infty} \frac{1}{k! \cdot k}.$$

Apply the Ratio Test by taking the limit of successive terms of the series.

$$r = \lim_{k\to\infty} \left| \frac{\frac{1}{(k+1)!(k+1)}}{\frac{1}{k!\cdot k}} \right| = \lim_{k\to\infty} \frac{k!\cdot k}{(k+1)!(k+1)} = \lim_{k\to\infty} \frac{k!\cdot k}{(k+1)k!(k+1)} = \lim_{k\to\infty} \frac{k}{k^2+2k+1} = 0$$

Because $r < 1$, it follows by the Ratio Test that the series $\sum_{k=1}^{\infty} \frac{1}{(k+1)! - k!}$ converges.

10.8.47 We apply the Ratio Test.

$$\begin{aligned} r &= \lim_{k\to\infty} \left| \frac{\frac{(4k+4)!}{((k+1)!)^4}}{\frac{(4k)!}{(k!)^4}} \right| \\ &= \lim_{k\to\infty} \frac{(4k+4)!}{\big((k+1)!\big)^4} \cdot \frac{(k!)^4}{(4k)!} \\ &= \lim_{k\to\infty} \frac{(4k+4)(4k+3)(4k+2)(4k+1)(4k)!}{\big((k+1)k!\big)^4} \cdot \frac{(k!)^4}{(4k)!} \\ &= \lim_{k\to\infty} \frac{(4k+4)(4k+3)(4k+2)(4k+1)}{(k+1)^4} = 4^4 \end{aligned}$$

Because $r > 1$, it follows from the Ratio Test that the series $\displaystyle\sum_{k=1}^{\infty} \frac{(4k)!}{(k!)^4}$ diverges.

10.8.48 The series $\frac{2}{3} + \frac{3}{8} + \frac{4}{15} + \frac{5}{24} + \frac{6}{35} + \cdots$ is represented in summation notation as $\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{k(k+2)}$. We apply the Limit Comparison Test using the divergent comparison series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$.

$$L = \lim_{k\to\infty} \frac{\frac{k+1}{k(k+2)}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k+1}{k+2} = 1$$

Because $0 < L < \infty$, the series $\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{k(k+2)}$ diverges by the Limit Comparison Test.

10.8.49 We apply the Comparison Test using the comparison series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{6/5}}$. Note that

$$\frac{\sqrt[5]{k}}{\sqrt[5]{k^7+1}} < \frac{\sqrt[5]{k}}{\sqrt[5]{k^7}} = \frac{1}{k^{6/5}}.$$

Because $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{6/5}}$ is a convergent p-series, $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[5]{k}}{\sqrt[5]{k^7+1}}$ converges by the Comparison Test.

10.8.50 Note that $e^{k^3} > k^3$ for $k > 1$, so that $\frac{1}{e^{k^3}} < \frac{1}{k^3}$. Because $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series, it follows by the Comparison Test that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{e^{k^3}}$ is a convergent series.

10.8.51 The limit of the kth-term of the series is

$$\lim_{k\to\infty} \frac{7^k + 11^k}{11^k} = \lim_{k\to\infty} \left(\frac{7^k}{11^k} + \frac{11^k}{11^k} \right) = \lim_{k\to\infty} \left(\left(\frac{7}{11} \right)^k + 1 \right) = 1$$

Because $\displaystyle\lim_{k\to\infty} \frac{7^k + 11^k}{11^k} \neq 0$, it follows that $\displaystyle\sum_{k=1}^{\infty} \frac{7^k + 11^k}{11^k}$ diverges by the Divergence Test.

10.8.52 The series $\sum_{k=1}^{\infty} \frac{7^k + 11^k}{13^k}$ can be written as the sum of two geometric series:

$$\sum_{k=1}^{\infty} \frac{7^k + 11^k}{13^k} = \sum_{k=1}^{\infty} \frac{7^k}{13^k} + \sum_{k=1}^{\infty} \frac{11^k}{13^k} = \sum_{k=1}^{\infty} \left(\frac{7}{13}\right)^k + \sum_{k=1}^{\infty} \left(\frac{11}{13}\right)^k.$$

Because both $\frac{7}{13}$ and $\frac{11}{13}$ are less than one, the series $\sum_{k=1}^{\infty} \frac{7^k + 11^k}{13^k}$ converges because it is the sum of two convergent geometric series.

10.8.53 We apply the Limit Comparison Test using the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^9}$ as the comparison series. Observe that

$$L = \lim_{k\to\infty} \frac{\sin \frac{1}{k^9}}{\frac{1}{k^9}} = 1.$$

Because $0 < L < \infty$, we conclude that the series $\sum_{k=1}^{\infty} \sin \frac{1}{k^9}$ converges by the Limit Comparison Test.

10.8.54 Observe that

$$L = \lim_{j\to\infty} j^9 \sin \frac{1}{j^9} = \lim_{j\to\infty} \frac{\sin \frac{1}{j^9}}{\frac{1}{j^9}} = 1.$$

Because $\lim_{j\to\infty} j^9 \sin \frac{1}{j^9} \neq 0$, the series $\sum_{j=1}^{\infty} j^9 \sin \frac{1}{j^9}$ diverges by the Divergent Test.

10.8.55 Because $\lim_{k\to\infty} \cos \frac{1}{k^9} = 1 \neq 0$, $\sum_{k=1}^{\infty} \cos \frac{1}{k^9}$ diverges by the Divergence Test.

10.8.56 We apply the Root Test with the help of the solution to Exercise 18 where we showed that $\lim_{k\to\infty} \left(1 + a/k\right)^k = e^a$.

$$\rho = \lim_{k\to\infty} \sqrt[k]{\left|\left(\frac{k}{k+10}\right)^{-k^2}\right|} = \lim_{k\to\infty} \left(\frac{k+10}{k}\right)^k = \lim_{k\to\infty} \left(1 + \frac{10}{k}\right)^k = e^{10}$$

Because $\rho > 1$, the series $\sum_{k=1}^{\infty} \left(\frac{k}{k+10}\right)^{-k^2}$ diverges.

10.8.57 We note that

$$\sum_{k=1}^{\infty} 5^{1-2k} = \sum_{k=1}^{\infty} 5 \cdot 5^{-2k} = \sum_{k=1}^{\infty} \frac{5}{25^k}.$$

So $\sum_{k=1}^{\infty} 5^{1-2k}$ is a geometric series with $r = \frac{1}{25}$. Because $|r| < 1$, the series converges.

10.8.58 Observe that $\frac{1}{\sqrt{\ln(k+2)}}$ decreases as k increases and $\lim_{k\to\infty} \frac{1}{\sqrt{\ln(k+2)}} = 0$. So by the Alternating Series Test, $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{\ln(k+2)}}$ converges.

10.8.59 We first establish that $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ is a convergent series using the Ratio Test. Taking the limit of successive terms of $\sum_{k=1}^{\infty} \frac{k!}{k^k}$, we have

$$r = \lim_{k\to\infty} \left| \frac{\frac{(k+1)!}{(k+1)^{k+1}}}{\frac{k!}{k^k}} \right| = \lim_{k\to\infty} \frac{(k+1)k!}{(k+1)^k(k+1)} \cdot \frac{k^k}{k!} = \lim_{k\to\infty} \frac{k^k}{(k+1)^k} = \lim_{k\to\infty} \frac{1}{\left(1+\frac{1}{k}\right)^k} = \frac{1}{e},$$

where the last equality follows from the fact that $\lim_{k\to\infty}(1+a/k)^k = e^a$ for any real number a (see the solution to Exercise 18). Because $r < 1$, the series $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ converges by the Ratio Test. It is also the case that $\frac{k!}{k^k+3} < \frac{k!}{k^k}$ which implies that $\sum_{k=1}^{\infty} \frac{k!}{k^k+3}$ converges by the Comparison Test.

10.8.60 The series equals the sum of two convergent p-series

$$\sum_{k=1}^{\infty} \left(\frac{1}{k^2} + \frac{1}{k^5}\right) = \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{1}{k^5}.$$

Therefore $\sum_{k=1}^{\infty} \left(\frac{1}{k^2} + \frac{1}{k^5}\right)$ converges.

10.8.61 We apply the Limit Comparison Test by using the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ as our comparison series. With the help of l'Hôpital's Rule, we calculate L.

$$L = \lim_{k\to\infty} \frac{\frac{1}{\ln(e^k+1)}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{\ln(e^k+1)} = \lim_{k\to\infty} \frac{1}{\frac{e^k}{e^k+1}} = \lim_{k\to\infty} \left(1 + \frac{1}{e^k}\right) = 1$$

Because $0 < L < \infty$, $\sum_{k=1}^{\infty} \frac{1}{\ln(e^k+1)}$ diverges by the Limit Comparison Test.

10.8.62 Apply the Root Test.

$$\rho = \lim_{k\to\infty} \sqrt[k]{\left| \left(\frac{\tan^{-1} k}{\pi}\right)^k \right|} = \frac{1}{\pi} \lim_{k\to\infty} \tan^{-1} k = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}$$

Because $\rho < 1$, $\sum_{k=0}^{\infty} \left(\frac{\tan^{-1} k}{\pi}\right)^k$ converges.

10.8.63 In the solution to Exercise 18, we showed that $\lim_{k\to\infty} \left(1 + a/k\right)^k = e^a$. We use this result in the following limit calculation. Applying the Root Test, we have

$$\rho = \lim_{k\to\infty} \sqrt[k]{\left| \left(\frac{k+a}{k}\right)^{k^2} \right|} = \lim_{k\to\infty} \left(1 + \frac{a}{k}\right)^k = e^a.$$

Because $a > 0$, $\rho = e^a > 1$. So the series diverges by the Root Test.

10.8.64 Rewrite the series using summation notation.

$$\frac{1}{1\cdot 4}+\frac{1}{2\cdot 7}+\frac{1}{3\cdot 10}+\frac{1}{4\cdot 13}+\cdots=\sum_{k=1}^{\infty}\frac{1}{k(3k+1)}$$

Using the convergent series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ as a comparison series, we apply the Limit Comparison Test:

$$L=\lim_{k\to\infty}\frac{\frac{1}{k(3k+1)}}{\frac{1}{k^2}}=\lim_{k\to\infty}\frac{k^2}{k(3k+1)}=\frac{1}{3}.$$

Because $0<L<\infty$, the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(3k+1)}$ converges.

10.8.65 The nth-partial sum of $\displaystyle\sum_{k=1}^{\infty}\left(\cos\frac{1}{k}-\cos\frac{1}{k+1}\right)$ is

$$S_n=\left(\cos 1-\cos\frac{1}{2}\right)+\left(\cos\frac{1}{2}-\cos\frac{1}{3}\right)+\cdots\left(\cos\frac{1}{n}-\cos\frac{1}{n+1}\right)=\cos 1-\cos\frac{1}{n+1}.$$

It follows that $\displaystyle\lim_{n\to\infty}S_n=\cos 1-\cos 0=\cos 1-1$. Because the sequence of partial sums converges, the series $\displaystyle\sum_{k=1}^{\infty}\left(\cos\frac{1}{k}-\cos\frac{1}{k+1}\right)$ converges.

10.8.66 Use the Ratio Test with the help of l'Hôpital's Rule.

$$r=\lim_{k\to\infty}\left|\frac{\frac{4^{(k+1)^2}}{(k+1)!}}{\frac{4^{k^2}}{k!}}\right|=\lim_{k\to\infty}\frac{4^{((k+1)^2-k^2)}k!}{(k+1)!}=\lim_{k\to\infty}\frac{4^{2k+1}}{(k+1)}=\lim_{k\to\infty}\frac{4^{2k+1}\cdot 2\ln 4}{1}=\infty$$

Because $r>1$, the series $\displaystyle\sum_{k=1}^{\infty}\frac{4^{k^2}}{k!}$ diverges by the Ratio Test.

10.8.67 Apply the Limit Comparison Test using the convergent geometric series $\displaystyle\sum_{j=1}^{\infty}\frac{1}{2^j}$ as the comparison series.

$$L=\lim_{j\to\infty}\frac{\frac{\cot^{-1}\frac{1}{j}}{2^j}}{\frac{1}{2^j}}=\lim_{j\to\infty}\cot^{-1}\frac{1}{j}=\frac{\pi}{2}$$

Because $0<L<\infty$, the series $\displaystyle\sum_{j=1}^{\infty}\frac{\cot^{-1}\frac{1}{j}}{2^j}$ converges by the Limit Comparison Test.

10.8.68 Multiplying numerator and denominator by $\sqrt{k+1}+\sqrt{k}$, we have

$$\sum_{k=2}^{\infty}\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2-k}}\cdot\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}}=\sum_{k=2}^{\infty}\frac{1}{\sqrt{k^2-k}(\sqrt{k+1}+\sqrt{k})}$$

Using the convergent p-series $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^{3/2}}$ as the comparison series, we use the Limit Comparison Test.

$$\begin{aligned}
L &= \lim_{k\to\infty} \frac{\left(\frac{1}{\sqrt{k^2-k}(\sqrt{k+1}+\sqrt{k})}\right)}{\frac{1}{k^{3/2}}} \\
&= \lim_{k\to\infty} \frac{k^{3/2}}{\sqrt{k^2-k}(\sqrt{k+1}+\sqrt{k})} \\
&= \lim_{k\to\infty} \frac{k^{3/2}/k^{3/2}}{\frac{\sqrt{k^2-k}}{k}\,\frac{\sqrt{k+1}+\sqrt{k}}{k^{1/2}}} \\
&= \lim_{k\to\infty} \frac{1}{\sqrt{1-\frac{1}{k}}\left(\sqrt{1+\frac{1}{k}}+1\right)} = \frac{1}{2}
\end{aligned}$$

Because $0 < L < \infty$, $\displaystyle\sum_{k=2}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2-k}}$ converges by the Limit Comparison Test.

10.8.69 Note that

$$\lim_{k\to\infty}\left(1+\frac{1}{2k}\right)^k = e^{1/2}.$$

Because the limit does not equal 0, the series $\displaystyle\sum_{k=1}^{\infty}\left(1+\frac{1}{2k}\right)^k$ diverges by the Divergence Test.

10.8.70 We have

$$\lim_{k\to\infty} e^{k/100} = \infty.$$

Because the limit does not equal 0, the series $\displaystyle\sum_{k=1}^{\infty} e^{k/100}$ diverges by the Divergence Test.

10.8.71 We apply the Limit Comparison Test with the comparison series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{5/4}}$.

$$L = \lim_{k\to\infty}\frac{\ln^2 k/k^{3/2}}{1/k^{5/4}} = \lim_{k\to\infty}\frac{\ln^2 k}{k^{1/4}} = \lim_{k\to\infty}\frac{2(\ln k)/k}{\frac{1}{4}k^{-3/4}} = 8\lim_{k\to\infty}\frac{\ln k/k}{k^{-3/4}} = 8\lim_{k\to\infty}\frac{\ln k}{k^{1/4}} = 32\lim_{k\to\infty}\frac{1}{k^{1/4}} = 0.$$

Because the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{5/4}}$ converges, it follows from case (2) of the Limit Comparison Test that $\displaystyle\sum_{k=1}^{\infty}\frac{\ln^2 k}{k^{3/2}}$ converges.

10.8.72 Calculate the following limit.

$$\lim_{k\to\infty}\cos\frac{1}{k} = \cos 0 = 1$$

It follows that $\displaystyle\lim_{k\to\infty}(-1)^k\cos\frac{1}{k} \neq 0$. So by the Divergence Test, $\displaystyle\sum_{k=1}^{\infty}(-1)^k\cos\frac{1}{k}$ diverges.

10.8.73 Use the Ratio Test.

$$r = \lim_{k\to\infty}\left|\frac{(k+1)^2\cdot 1.0001^{-k-1}}{k^2\cdot 1.0001^{-k}}\right| = \frac{1}{1.001}$$

Because $r < 1$, the series $\displaystyle\sum_{k=0}^{\infty} k^2\cdot 1.001^{-k}$ converges by the Ratio Test.

10.8.74 Using the Ratio Test, we have

$$r = \lim_{k\to\infty} \left| \frac{(k+1)\cdot 0.999^{-k-1}}{k\cdot 0.999^{-k}} \right| = \frac{1}{0.999}$$

Because $r > 1$, the series $\sum_{k=0}^{\infty} k \cdot 0.999^{-k}$ diverges by the Ratio Test.

10.8.75 The terms of the series decrease in magnitude, for $k \geq 1$. Furthermore $\lim_{k\to\infty} \frac{1}{k^k} = 0$. Therefore, the series converges by the Alternating Series Test.

10.8.76 We first establish that the related positive-term series $\sum_{k=3}^{\infty} \frac{1}{k(k-2)}$ converges using the Limit Comparison Test with the help of the convergent comparison series $\sum_{k=3}^{\infty} \frac{1}{k^2}$.

$$L = \lim_{k\to\infty} \frac{1/\big(k(k-2)\big)}{1/k^2} = \lim_{k\to\infty} \frac{k^2}{k^2-2k} = 1$$

The value of L is positive and finite which implies that $\sum_{k=3}^{\infty} \frac{1}{k(k-2)}$ converges. It follows that the series $\sum_{k=3}^{\infty} \frac{2}{k(2-k)}$ converges because it equals $\sum_{k=3}^{\infty} \frac{-2}{k(k-2)}$.

10.8.77 The series diverges by the Divergence Test because

$$\lim_{k\to\infty} \frac{3k}{\sqrt[4]{k^4+3}} = \lim_{k\to\infty} \frac{3k/k}{(\sqrt[4]{k^4+3})/k} = \lim_{k\to\infty} \frac{3}{\sqrt[4]{k^4/k^4+3/k^4}} = \lim_{k\to\infty} \frac{3}{\sqrt[4]{1+3/k^4}} \neq 0.$$

10.8.78 We use the Root Test.

$$\rho = \lim_{k\to\infty} \sqrt[k]{|k\cdot 3^{-k^2}|} = \lim_{k\to\infty} \frac{k^{1/k}}{3^k} = \frac{\lim_{k\to\infty} k^{1/k}}{\lim_{k\to\infty} 3^k} = \frac{1}{\lim_{k\to\infty} 3^k} = 0$$

Because $\rho < 1$ it follows that $\sum_{k=0}^{\infty} k\cdot 3^{-k^2}$ converges.

10.8.79 Apply the Limit Comparison Test using the divergent p-series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ as the comparison series.

$$L = \lim_{k\to\infty} \frac{\tan^{-1}\frac{1}{\sqrt{k}}}{\frac{1}{\sqrt{k}}} = \lim_{k\to\infty} \frac{\frac{1}{1+(1/\sqrt{k})^2}\left(-\frac{1}{2}k^{-3/2}\right)}{-\frac{1}{2}k^{-3/2}} = \lim_{k\to\infty} \frac{1}{1+(1/\sqrt{k})^2} = 1$$

Because $0 < L < \infty$, $\sum_{k=1}^{\infty} \tan^{-1} \frac{1}{\sqrt{k}}$ diverges by the Limit Comparison Test.

10.8.80 Apply the Root Test.

$$\rho = \lim_{k\to\infty} \sqrt[k]{\left| \left(\frac{3k^8-2}{3k^9+2} \right)^k \right|} = \lim_{k\to\infty} \frac{3k^8-2}{3k^9+2} = 0$$

Because $\rho < 1$, the series $\sum_{k=1}^{\infty} \left(\frac{3k^8-2}{3k^9+2} \right)^k$ converges.

10.8.81 The nth-partial sum of the series $\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k+2}}-\frac{1}{\sqrt{k}}\right)$ is

$$S_n=\left(\frac{1}{\sqrt{3}}-1\right)+\left(\frac{1}{2}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{3}}\right)+\left(\frac{1}{\sqrt{6}}-\frac{1}{2}\right)+\cdots+\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n-1}}\right)+\left(\frac{1}{\sqrt{n+2}}-\frac{1}{\sqrt{n}}\right).$$

For $n\geq 4$, S_n simplifies to $S_n=-1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}$ So the limit of the sequence of partial sums is $\lim_{n\to\infty}S_n=-1-\frac{1}{\sqrt{2}}$. Therefore the series $\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k+2}}-\frac{1}{\sqrt{k}}\right)$ converges (to this limit).

10.8.82 The series can be rewritten as

$$\sum_{k=0}^{\infty}\left(\left(\frac{2}{3}\right)^{k+1}-\left(\frac{3}{2}\right)^{-k+1}\right)=\sum_{k=0}^{\infty}\left(\left(\frac{2}{3}\right)^{k+1}-\left(\frac{2}{3}\right)^{k-1}\right).$$

Because $\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{k+1}$ and $\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{k-1}$ are geometric series, both with $r=2/3$, they both converge. Therefore the series converges.

10.8.83 We apply the Integral Test using the function $f(x)=\frac{1}{x\ln^{10}x}$. This function is continuous, positive, and

decreasing for $x\geq 2$. Integrating and using the substitution $u=\ln x$, we have

$$\begin{aligned}\int_2^{\infty}\frac{dx}{x\ln^{10}x}&=\lim_{b\to\infty}\int_2^{b}\frac{dx}{x\ln^{10}x}\\&=\lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{du}{u^{10}}\\&=\lim_{b\to\infty}\left(-\frac{1}{9u^9}\right)\Big|_{\ln 2}^{\ln b}\\&=\lim_{b\to\infty}\left(-\frac{1}{9\ln^9 b}+\frac{1}{9\ln^9 2}\right)=\frac{1}{9\ln^9 2}\end{aligned}$$

The integral converges which implies that the series $\sum_{j=2}^{\infty}\frac{1}{j\ln^{10}j}$ converges by the Integral Test.

10.8.84 We apply the Limit Comparison Test using the convergent series $\sum_{k=1}^{\infty}\frac{1}{k^3}$ as a comparison series.

$$L=\lim_{k\to\infty}\frac{\tan^{-1}\frac{1}{k^3}}{\frac{1}{k^3}}=\lim_{k\to\infty}\frac{\frac{1}{1+1/k^6}\cdot\left(-\frac{3}{k^4}\right)}{-\frac{3}{k^4}}=\lim_{k\to\infty}\frac{1}{1+1/k^6}=1$$

Because $0<L<\infty$, the series $\sum_{k=1}^{\infty}\tan^{-1}\frac{1}{k^3}$ converges by Limit Comparison Test.

10.8.85 The series $\frac{1}{2\cdot 3}+\frac{1}{4\cdot 5}+\frac{1}{6\cdot 7}+\frac{1}{8\cdot 9}+\cdots$ is expressed in summation notation as $\sum_{k=1}^{\infty}\frac{1}{2k(2k+1)}$.

We apply the Limit Comparison Test using the convergent p-series $\sum_{k=1}^{\infty}\frac{1}{k^2}$.

$$L = \lim_{k\to\infty} \frac{1/\big(2k(2k+1)\big)}{1/k^2} = \lim_{k\to\infty} \frac{k^2}{4k^2+2k} = \frac{1}{4}$$

So we have $0 < L < \infty$ and it follows that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k(2k+1)}$ converges by the Limit Comparison Test.

10.8.86 Applying the Ratio Test, we have

$$r = \lim_{k\to\infty} \frac{|a|^{k+1}/(k+1)!}{|a|^k/k!} = \lim_{k\to\infty} \frac{|a|}{k+1} = 0.$$

Because $r < 1$, the series $\displaystyle\sum_{k=1}^{\infty} \frac{|a|^k}{k!}$ converges and it follows that the series $\displaystyle\sum_{k=1}^{\infty} \frac{a^k}{k!}$ converges.

10.8.87

a. False. For example, the Limit Comparison Series can be used to show that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ converges using the comparison series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2}$, but this comparison series will not work with the Comparison Test.

b. True.

c. True.

d. False. The Alternating Series Test can only be used to show that a series converges.

10.8.88 Multiply the numerator and denominator by $\big(\sqrt{k^4+1}+k^2\big)$:

$$\sum_{k=1}^{\infty} \big(\sqrt{k^4+1}-k^2\big)\frac{\big(\sqrt{k^4+1}+k^2\big)}{\big(\sqrt{k^4+1}+k^2\big)} = \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k^4+1}+k^2}\right).$$

The inequalities $\sqrt{k^4+1}+k^2 > \sqrt{k^4+1} > \sqrt{k^4} = k^2$ imply that

$$\sqrt{k^4+1}-k^2 = \frac{1}{\sqrt{k^4+1}+k^2} < \frac{1}{k^2}.$$

Because $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series, it follows from the Comparison Test that $\displaystyle\sum_{k=1}^{\infty} \big(\sqrt{k^4+1}-k^2\big)$ converges.

10.8.89 Using the sum $1+2+\cdots+k = k(k+1)/2$ from Chapter 5, we have

$$\sum_{k=1}^{\infty} \frac{k}{1+2+\cdots+k} = \sum_{k=1}^{\infty} \frac{2k}{k(k+1)} = \sum_{k=1}^{\infty} \frac{2}{k+1}.$$

We apply the Limit Comparison Test using the divergent harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ as the comparison series.

$$L = \lim_{k\to\infty} \frac{2/(k+1)}{1/k} = \lim_{k\to\infty} \frac{2k}{k+1} = 2$$

Because $0 < L < \infty$, $\displaystyle\sum_{k=1}^{\infty} \frac{k}{1+2+\cdots+k}$ diverges by the Limit Comparison Test.

10.8.90 A Chapter 5 result states that

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$

This implies that $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1^3+2^3+3^3+\cdots+k^3}{k^5}\right)$ equals $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k^2(k+1)^2/4}{k^5}\right) = \sum_{k=1}^{\infty}\frac{(k+1)^2}{4k^3}$. Using the divergent harmonic series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ as a comparison series, we have

$$L = \lim_{k\to\infty}\frac{(k+1)^2/(4k^3)}{1/k} = \lim_{k\to\infty}\frac{k^3+2k^2+k}{4k^3} = \frac{1}{4}.$$

Because $0 < L < \infty$, the series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1^3+2^3+3^3+\cdots+k^3}{k^5}\right)$ diverges by the Limit Comparison Test.

10.8.91 The series $\displaystyle\sum_{k=0}^{\infty}\ln\left(\frac{2k+1}{2k+4}\right)$ consists of negative terms because $(2k+1)/(2k+4) < 1$. Notice that $\displaystyle\sum_{k=0}^{\infty}\ln\left(\frac{2k+1}{2k+4}\right) = -\sum_{k=0}^{\infty}\ln\left(\frac{2k+4}{2k+1}\right)$ where $\displaystyle\sum_{k=0}^{\infty}\ln\left(\frac{2k+4}{2k+1}\right)$ consists of positive terms. So we show that the series $\displaystyle\sum_{k=0}^{\infty}\ln\left(\frac{2k+1}{2k+4}\right)$ diverges by showing that $\displaystyle\sum_{k=0}^{\infty}\ln\left(\frac{2k+4}{2k+1}\right)$ diverges. To do this, we apply the Limit Comparison test using the divergent comparison series $\displaystyle\sum_{k=0}^{\infty}\frac{1}{k}$.

$$\begin{aligned}
L &= \lim_{k\to\infty}\frac{\ln\big((2k+4)/(2k+1)\big)}{1/k} && \text{Limit Comparison Test}\\
&= \lim_{k\to\infty}\frac{\frac{2}{2k+4}-\frac{2}{2k+1}}{-1/k^2} && \text{L'Hôpital's Rule}\\
&= \lim_{k\to\infty}\frac{2(2k+1)-2(2k+4)}{(2k+1)(2k+4)}\cdot\frac{-k^2}{1} && \text{Common denominators.}\\
&= \lim_{k\to\infty}\frac{6k^2}{4k^2+10k+4} = \frac{3}{2} && \text{Evaluate.}
\end{aligned}$$

Because $0 < L < \infty$, the series $\displaystyle\sum_{k=0}^{\infty}\ln\left(\frac{2k+4}{2k+1}\right)$ diverges and so $\displaystyle\sum_{k=0}^{\infty}\ln\left(\frac{2k+1}{2k+4}\right)$ diverges.

10.8.92 Observe that $10^{\ln k} = e^{\ln 10^{\ln k}} = e^{(\ln k)(\ln 10)} = (e^{\ln k})^{\ln 10} = k^{\ln 10}$. So $\displaystyle\sum_{k=1}^{\infty}\frac{1}{10^{\ln k}} = \sum_{k=1}^{\infty}\frac{1}{k^{\ln 10}}$ and because $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{\ln 10}}$ is a p-series with $p = \ln 10 > 1$, $\displaystyle\sum_{k=1}^{\infty}\frac{1}{10^{\ln k}}$ converges by the p-series test.

(Alternate solution): We apply the Limit Comparison Test with the comparison series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$:

$$L = \lim_{k\to\infty}\frac{1/10^{\ln k}}{1/k^2} = \lim_{k\to\infty}\frac{e^{2\ln k}}{10^{\ln k}} = \lim_{k\to\infty}\left(\frac{e^2}{10}\right)^{\ln k} = 0.$$

Because $L = 0$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (because it is a convergent p-series), it follows by part 2 of the Limit Comparison Test that $\sum_{k=1}^{\infty} \frac{1}{10^{\ln k}}$ converges.

10.8.93 We apply the Limit Comparison Test with the comparison series $\sum_{k=1}^{\infty} \frac{1}{k}$:

$$\lim_{k\to\infty} \frac{1/(2^{\ln k}+2)}{1/k} = \lim_{k\to\infty} \frac{e^{\ln k}}{2^{\ln k}+2} = \lim_{k\to\infty} \frac{(e/2)^{\ln k}}{1+2^{(1-\ln k)}} = \frac{\lim_{k\to\infty} \left(\frac{e}{2}\right)^{\ln k}}{1-0} = \infty.$$

Because $L = \infty$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (because it is the harmonic series), it follows by part 3 of the Limit Comparison Test that $\sum_{k=1}^{\infty} \frac{1}{2^{\ln k}+2}$ diverges.

Chapter Ten Review

1

a. False. Let $a_n = 1 - \frac{1}{n}$. This sequence has limit 1.

b. False. The terms of a sequence tending to zero is necessary but not sufficient for convergence of the series.

c. True. This is the definition of convergence of a series.

d. False. If a series converges absolutely, the definition says that it does not converge conditionally.

e. True. It has limit 1 as $n \to \infty$.

f. False. The subsequence of the even terms has limit 1 and the subsequence of odd terms has limit -1, so the sequence does not have a limit.

g. False. It diverges by the Divergence Test because $\lim_{k\to\infty} \frac{k^2}{k^2+1} = 1 \neq 0$.

h. True. The given series converges by the Limit Comparison Test with the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, and thus its sequence of partial sums converges.

2

a. $\lim_{n\to\infty} \frac{12n^3+4n}{6n^3+5} = \lim_{n\to\infty} \frac{12+4/n^2}{6+5/n^3} = 2$.

b. Because the sequence $\left\{\frac{12n^3+4n}{6n^3+5}\right\}$ does not converge to 0, the sequence $\left\{(-1)^n \frac{12n^3+4n}{6n^3+5}\right\}$ oscillates and diverges.

3 $\sum_{k=0}^{9} (0.2)^k = \frac{1-(0.2)^{10}}{1-0.2} \approx 1.25$, and $\sum_{k=2}^{9} (0.2)^k = (0.2)^2 \frac{1-(0.2)^8}{1-0.2} \approx 0.0499999$.

4

a. $B_n = 1.0025B_{n-1} + 100$ for $n \geq 1$, and $B_0 = 100$.

b. $B_1 = 100 + 1.0025(100)$.
$B_2 = 100 + 1.0025(B_1) = 100 + 1.0025(100 + 1.0025(100)) = 100 + 1.0025(100) + 1.0025(100^2)$.
$B_n = 100 + 100(1.0025) + \cdots + 100(1.0025)^n = 100\left(\dfrac{1 - 1.0025^{n+1}}{1 - 1.0025}\right)$.

c. $B_{30} = 100\left(\dfrac{1 - 1.0025^{31}}{1 - 1.0025}\right) \approx 3219.11$ dollars.

5 Because the series converges, we must have $\lim\limits_{k\to\infty} a_k = 0$. Because it converges to 8, the partial sums converge to 8, so that $\lim\limits_{k\to\infty} S_k = 8$.

6 $\displaystyle\sum_{k=0}^{\infty} 3r^k = \frac{3}{1-r} = 6$, so $\dfrac{1}{2} = 1 - r$, so $r = \dfrac{1}{2}$.

7 Consider the constant sequence with $a_k = 1$ for all k. The sequence $\{a_k\}$ converges to 1, but the corresponding series $\sum a_k$ diverges by the divergence test.

8 This is not possible. If the series $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, then we must have $\lim\limits_{k\to\infty} a_k = 0$.

9

a. For $|x| < 1$, $\lim\limits_{k\to\infty} x^k = 0$, so this limit is zero.

b. This is a geometric series with ratio $-4/5$, so the sum is $\dfrac{1}{1 + 4/5} = \dfrac{5}{9}$.

10

a. $\displaystyle\lim_{k\to\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{k\to\infty}\frac{1}{k(k+1)} = 0$.

b. This series telescopes, and $S_n = 1 - \dfrac{1}{n+1}$, so $\lim\limits_{n\to\infty} S_n = 1$, which is the sum of the series.

11

a. This sequence converges because $\displaystyle\lim_{k\to\infty}\frac{k}{k+1} = \lim_{k\to\infty}\frac{1}{1+\frac{1}{k}} = \frac{1}{1+0} = 1$.

b. Because the sequence of terms has limit 1 (which means its limit isn't zero) this series diverges by the Divergence Test.

12 $\displaystyle\lim_{n\to\infty}\frac{3n^3+4n}{6n^3+5} = \lim_{n\to\infty}\frac{3+4/n^2}{6+5/n^3} = \frac{3}{6} = \frac{1}{2}$.

13 Because $\displaystyle\lim_{n\to\infty}\frac{3n^3+4n}{6n^3+5} = \frac{1}{2} \neq 0$, the sequence $\left\{(-1)^n\dfrac{3n^3+4n}{6n^3+5}\right\}$ oscillates and doesn't converge.

14 $\displaystyle\lim_{n\to\infty}\frac{(2n+2)!}{(2n)!n^2} = \lim_{n\to\infty}\frac{(2n+2)(2n+1)(2n)!}{(2n)!n^2} = \lim_{n\to\infty}\frac{4n^2+6n+2}{n^2} = 4$.

15 $\displaystyle\lim_{n\to\infty}\frac{2^n+5^{n+1}}{5^n} = \lim_{n\to\infty}\left(\frac{2}{5}\right)^n + 5 = 0 + 5 = 5$.

16 $\lim_{n\to\infty} \frac{n^2+4}{\sqrt{4n^4+1}} = \lim_{n\to\infty} \frac{1+4n^{-2}}{\sqrt{4+n^{-4}}} = \frac{1}{2}$.

17 $\lim_{n\to\infty} \frac{8^n}{n!} = 0$ because exponentials grow more slowly than factorials.

18 After taking logs, we want to compute

$$\lim_{n\to\infty} 2n\ln(1+3/n) = \lim_{n\to\infty} \frac{\ln(1+3/n)}{1/(2n)}.$$

By L'Hôpital's rule, this is $\lim_{n\to\infty} \frac{6n}{n+3}$ (after some algebraic manipulations), which is 6. Thus the original limit is e^6.

19 Because $-1 \le \sin n \le 1$, we have

$$\frac{-(3n^2+2n+1)}{4n^3+n} \le \frac{(3n^2+2n+1)\sin n}{4n^3+n} \le \frac{(3n^2+2n+1)}{4n^3+n}.$$

Because $\lim_{n\to\infty} \frac{-(3n^2+2n+1)}{4n^3+n} = 0 = \lim_{n\to\infty} \frac{(3n^2+2n+1)}{4n^3+n}$, we must have the the original sequence has limit 0.

20 $\lim_{n\to\infty} (n-\sqrt{n^2-1}) = \lim_{n\to\infty} \frac{n-\sqrt{n^2-1}}{1} \cdot \frac{n+\sqrt{n^2-1}}{n+\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{1}{n+\sqrt{n^2+1}} = 0.$

21 Take logs, and then evaluate $\lim_{n\to\infty} \frac{1}{\ln n} \ln(1/n) = \lim_{n\to\infty} (-1) = -1$, so the original limit is e^{-1}.

22 This series oscillates among the values $\pm 1/2, \pm\sqrt{3}/2, \pm 1$, and 0, so it has no limit.

23 $a_n = (-1/0.9)^n = (-10/9)^n$. The terms grow without bound so the sequence does not converge.

24 $\lim_{n\to\infty} \left(\frac{\pi}{2} - \tan^{-1} n\right) = \frac{\pi}{2} - \frac{\pi}{2} = 0.$

25

a. 80, 48, 32, 24, 20

b. Assume that $L = \lim_{n\to\infty} a_n$. Then $L = \frac{L}{2} + 8$, so $2L = L + 16$, so $L = 16$.

26

a. 81, 7.81025, 6.05008, 6.00139, 6.00004

b. Assume that $L = \lim_{n\to\infty} a_n$. Then $L = \sqrt{L/3+34}$, so $L^2 = L/3 + 34$, so $3L^2 - L - 102 = 0$, so $(L-6)(3L+17) = 0$. So we either have $L = 6$ or $L = -17/3$, but because all the terms of a_n are positive, we must have $L = 6$.

27 $\sum_{k=1}^{\infty} 3(1.001)^k = 3\sum_{k=1}^{\infty} (1.001)^k$. This is a geometric series with ratio greater than 1, so it diverges.

28 This is a geometric series with ratio 9/10, so the sum is $\frac{9/10}{1-9/10} = 9$.

29 Because $\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k$ converges, but $\sum_{k=0}^{\infty} \left(\frac{4}{3}\right)^k$ diverges, the given series diverges.

30 $\displaystyle\sum_{k=1}^{\infty} \frac{3^k - 5^k}{10^k} = \sum_{k=1}^{\infty} \left(\left(\frac{3}{10}\right)^k - \left(\frac{5}{10}\right)^k \right) = \frac{3/10}{1-3/10} - \frac{5/10}{1-5/10} = \frac{3}{7} - 1 = -\frac{4}{7}$.

31 Because $\ln\left(\dfrac{2k+1}{2k-1}\right) = \ln(2k+1) - \ln(2k-1)$, this series telescopes. The formula for S_n is $S_n = (\ln 3 - \ln 1) + (\ln 5 - \ln 3) + \cdots + (\ln(2n+1) - \ln(2n-1)) = \ln(2n+1)$. The given series therefore diverges because $\lim_{n\to\infty} S_n = \infty$.

32 This is a geometric series with ratio $-\dfrac{1}{5}$, so it converges and the sum is $\dfrac{1}{1+1/5} = \dfrac{5}{6}$

33 This series telescopes, and the formula for S_n is

$$\begin{aligned} S_n &= (\tan^{-1} 2 - \tan^{-1} 0) + (\tan^{-1} 3 - \tan^{-1} 1) + (\tan^{-1} 4 - \tan^{-1} 2) + \cdots + (\tan^{-1}(n+2) - \tan^{-1} n) \\ &= \tan^{-1}(n+2) + \tan^{-1}(n+1) - \tan^{-1} 1 - \tan^{-1} 0. \end{aligned}$$

So

$$\lim_{n\to\infty} S_n = \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4} = \frac{3\pi}{4}.$$

34 This series clearly telescopes, and

$$S_n = \left(\frac{1}{\sqrt{2}} - 1\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}}\right) + \cdots \left(\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}\right) = \frac{1}{\sqrt{n+1}} - 1,$$

so $\lim_{n\to\infty} S_n = -1$.

35 Note that $\dfrac{9}{(3k-2)(3k+1)} = \dfrac{3}{3k-2} - \dfrac{3}{3k+1}$. This series telescopes.

$$S_n = \left(3 - \frac{3}{4}\right) + \left(\frac{3}{4} - \frac{3}{7}\right) + \cdots + \left(\frac{3}{3n-2} - \frac{3}{3n+1}\right) = 3 - \frac{3}{3n+1},$$

so that $\lim_{n\to\infty} S_n = 3$, which is the value of the series.

36 $\displaystyle\sum_{k=1}^{\infty} 4^{-3k} = \sum_{k=1}^{\infty} (1/64)^k$. This is a geometric series with ratio $1/64$, so its sum is $\dfrac{1/64}{1-1/64} = \dfrac{1}{63}$.

37 $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{3^{k+2}} = \frac{1}{9}\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{9} \cdot \frac{2/3}{1-2/3} = \frac{2}{9}$.

38 $0.29292929\ldots = 0.29 + 0.0029 + 0.000029 + \cdots = \dfrac{0.29}{1-0.01} = \dfrac{0.29}{0.99} = \dfrac{29}{99}$.

39 $0.314141414\ldots = 0.3 + (0.014 + 0.00014 + 0.00000014 + \cdots) = \dfrac{3}{10} + \dfrac{0.014}{1-0.01} = \dfrac{3}{10} + \dfrac{14}{990} = \dfrac{311}{990}$.

40

a. $d_{n+1} = 100 + \dfrac{1}{2} d_n$; $d_1 = 100$.

b. Assume $\lim_{n\to\infty} d_n = L$. Then $L = 100 + \dfrac{1}{2}L$, so $L = 200$. So the steady-state amount of medication in the blood is 200 mg.

41 Note that $d_2 = 100 + \frac{1}{2}(100)$, $d_3 = 100 + \frac{1}{2}\left(100 + \frac{1}{2}(100)\right) = 100 + \frac{1}{2}(100) + \frac{1}{2^2}(100)$, and

$$d_n = 100 + \frac{1}{2}(100) + \cdots + \frac{1}{2^{n-1}}(100).$$

The steady state is $\lim_{n\to\infty} d_n$, which is the sum of the geometric series which is given by $\frac{100}{1-(1/2)} = 200$ mg. So the steady-state amount of medication in the blood is 200 mg.

42 We have $\frac{k}{k^{5/2}+1} < \frac{k}{k^{5/2}} = \frac{1}{k^{3/2}}$, and because $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is a convergent p-series, the given series converges.

43 The series can be written $\sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$, which is a p-series with $p = 2/3 < 1$, so this series diverges.

44 Apply the Limit Comparison Test with the convergent p-series $\sum_{j=1}^{\infty} \frac{1}{j^2}$. We have

$$\lim_{j\to\infty} \frac{\frac{2j^4+j^3+1}{3j^6+4}}{\frac{1}{j^2}} = \lim_{j\to\infty} \frac{2j^6+j^5+j^2}{3k^6+4} = \frac{2}{3}.$$

So the given series converges by the Limit Comparison Test.

45 The kth term does not have limit 0, so the given series diverges by the Divergence Test. Note that

$$\lim_{k\to\infty} \frac{2k^4+k^3+1}{3k^4+4} = \frac{2}{3} \neq 0.$$

46 We use the Ratio Test. The ratio of successive terms is

$$\frac{2(k+1)^{(k+1)/2}}{k^{k/2}\sqrt{k+1}} = 2\frac{(k+1)^{k/2}}{k^{k/2}} = 2\left(\left(1+\frac{1}{k}\right)^k\right)^{1/2},$$

which as limit $2\sqrt{e}$ as $k \to \infty$. Because $2\sqrt{e} > 1$, the given series diverges by the Ratio Test.

47 We have $\frac{7+\sin k}{k^2} \leq \frac{8}{k^2}$ and $\sum_{k=1}^{\infty} \frac{8}{k^2}$ converges (it is a p-series with $p > 1$), so by the Comparison Test, the given series converges.

48 $\lim_{k\to\infty} \frac{2k^4+1}{\sqrt{9k^8+2}} = \lim_{k\to\infty} \frac{2+1/k^4}{\sqrt{9+2/k^8}} = \frac{2}{\sqrt{9}} = \frac{2}{3} \neq 0$, so the given series diverges by the Divergence Test.

49 Note that $\sqrt{9k^{12}+2} \geq \sqrt{9k^{12}} \geq \sqrt{k^{12}} = k^6$ for $k \geq 1$. It follows that

$$\frac{k^4}{\sqrt{9k^{12}+2}} \leq \frac{k^4}{k^6} = \frac{1}{k^2}.$$

Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the given series converges by the Comparison Test.

50 We use the Ratio Test. The ratio of successive terms is

$$\sqrt{\frac{(k+1)!k^k}{(k+1)^{k+1}k!}} = \sqrt{\frac{k^k}{(k+1)^k}} = \frac{1}{\sqrt{\left(1+\dfrac{1}{k}\right)^k}},$$

which has limit $\sqrt{\dfrac{1}{e}} < 1$ as $k \to \infty$. The given series therefore converges by the Ratio Test.

51 This is a geometric series with ratio $2/e < 1$, so the given series converges.

52 Note that $\dfrac{1}{a_k} = \left(\left(1+\dfrac{3}{k}\right)^k\right)^2$, so $\lim\limits_{k\to\infty} \dfrac{1}{a_k} = \lim\limits_{k\to\infty} \left(\left(1+\dfrac{3}{k}\right)^k\right)^2 = (e^3)^2$, so $\lim\limits_{k\to\infty} a_k = \dfrac{1}{e^6} \neq 0$, so the given series diverges by the Divergence Test.

53 We use the Root Test. The kth root of the absolute value of the kth term is $\dfrac{k}{2k+3}$ which has limit $\dfrac{1}{2}$ as $k \to \infty$, so the given series converges.

54 We use the Limit Comparison Test with the convergent p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}$. We have

$$L = \lim_{k\to\infty} \frac{\sin \frac{1}{k^4}}{\frac{1}{k^4}} = 1,$$

so the given series converges by the Limit Comparison Test.

55 Applying the Ratio Test:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{(k+1)!}{e^{k+1}(k+1)^{k+1}}}{\frac{k!}{e^k k^k}} = \lim_{k\to\infty} \frac{(k+1)k^k}{e(k+1)^{k+1}} = \lim_{k\to\infty} \frac{1}{e} \cdot \left(\frac{k}{k+1}\right)^k = \frac{1}{e^2},$$

so the given series converges.

56 Use the Limit Comparison Test with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$:

$$\frac{1}{\sqrt{k^2+k}} \Big/ \frac{1}{k} = \frac{k}{\sqrt{k^2+k}} = \sqrt{\frac{k^2}{k^2+k}},$$

which has limit 1 as $k \to \infty$. Because $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the original series does as well.

57 The series can be written as $2\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{4^k}$, and because $\displaystyle\sum_{k=1}^{\infty} \left(\frac{5}{4}\right)^k$ is a divergent geometric series, the given series is divergent.

58 By the Ratio Test: $\lim\limits_{k\to\infty} \dfrac{a_{k+1}}{a_k} = \lim\limits_{k\to\infty} \dfrac{k+1}{e^{k+1}} \cdot \dfrac{e^k}{k} = \lim\limits_{k\to\infty} \dfrac{1}{e} \cdot \dfrac{k+1}{k} = \dfrac{1}{e} < 1$. Thus the given series converges.

59 Use the Ratio Test. The ratio of successive terms is $\dfrac{2 \cdot 4^{j+1}}{(2j+3)!} \cdot \dfrac{(2j+1)!}{2 \cdot 4^j} = \dfrac{4}{(2j+3)(2j+2)}$. This has limit 0 as $j \to \infty$, so the given series converges.

60 Use the Ratio Test. The ratio of successive term is $\frac{9^{k+1}}{(2k+2)!} \cdot \frac{(2k)!}{9^k} = \frac{9}{(2k+2)(2k+1)}$. This has limit 0 as $k \to \infty$, so the given series converges.

61 We use the Integral Test. Let $f(x) = \frac{\ln x}{x^{3/2}}$. Then $f'(x) = \frac{2 - 3\ln x}{2x^{5/2}}$, which is less than zero for $x \geq 2$. So f is decreasing, positive and continuous for $x \geq 3$. Using Integration by Parts, it can be shown that

$$\int f(x)\, dx = -\frac{4}{\sqrt{x}} - \frac{2\ln x}{\sqrt{x}},$$

so $\lim_{b\to\infty} \int_3^b f(x)\, dx = \frac{4}{\sqrt{3}} + \frac{2\ln 3}{\sqrt{3}}$. Because this integral converges, the given series converges by the Integral Test.

62 We use the Ratio Test. The absolute value of the ratio of successive terms is

$$\frac{(k+1)^2(2k-1)!}{k^2(2k+1)!} = \frac{(k+1)^2}{k^2(2k+1)(2k)},$$

which has limit zero as $k \to \infty$. The given series is therefore convergent by the Ratio Test.

63 Use the Comparison Test: $\frac{3}{2+e^k} < \frac{3}{e^k}$, but $\sum_{k=1}^{\infty} \frac{3}{e^k}$ converges because it is a geometric series with ratio $\frac{1}{e} < 1$. Thus the original series converges as well.

64 $\lim_{k\to\infty} a_k = \lim_{k\to\infty} k\sin(1/k) = \lim_{k\to\infty} \frac{\sin(1/k)}{1/k} = 1$, so the given series diverges by the Divergence Test.

65 $a_k = \frac{k^{1/k}}{k^3} = \frac{1}{k^{3-1/k}}$. For $k \geq 2$, then, $a_k < \frac{1}{k^2}$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the given series also converges, by the Comparison Test.

66 Use the Comparison Test: $\frac{1}{1+\ln k} > \frac{1}{k}$ for $k > 1$. Because $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the given series does as well.

67 Use the Ratio Test: $\frac{a_{k+1}}{a_k} = \frac{(k+1)^5}{e^{k+1}} \cdot \frac{e^k}{k^5} = \frac{1}{e} \cdot \left(\frac{k+1}{k}\right)^5$, which has limit $1/e < 1$ as $k \to \infty$. Thus the given series converges.

68 For $k > 5$, we have $k^2 - 10 > (k-1)^2$, so that $a_k = \frac{2}{k^2 - 10} < \frac{2}{(k-1)^2}$. Because $\sum_{k=4}^{\infty} \frac{2}{(k-1)^2}$ converges, the original series does as well.

69 We use the Limit Comparison Test with $\sum_{k=1}^{\infty} \frac{1}{k^2}$. We have

$$L = \lim_{k\to\infty} \frac{1 - \cos\frac{1}{k}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{\left(\sin\frac{1}{k}\right)\frac{1}{k^2}}{\frac{2}{k^3}} = \frac{1}{2}\lim_{k\to\infty} \frac{\sin\frac{1}{k}}{\frac{1}{k}} = \frac{1}{2}.$$

Because $0 < L < \infty$, and because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, we have that the given series converges.

70 We use the Limit Comparison Test with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$L = \lim_{k\to\infty} \frac{\frac{\pi}{2} - \tan^{-1} k}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\frac{1}{1+k^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{1+k^2} = 1.$$

Because $0 < L < \infty$, and because the Harmonic Series diverges, the given series diverges.

71 We use the Limit Comparison Test with the convergent p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$. We have

$$\begin{aligned} L &= \lim_{k\to\infty} \frac{\left(1 - \cos\frac{1}{k}\right)^2}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{2\left(1-\cos\frac{1}{k}\right)\left(\sin\frac{1}{k}\right)\left(-\frac{1}{k^2}\right)}{-\frac{2}{k^3}} \\ &= \lim_{k\to\infty} k\left(1-\cos\frac{1}{k}\right)\left(\sin\frac{1}{k}\right) = \lim_{k\to\infty}\left(1-\cos\frac{1}{k}\right)\lim_{k\to\infty}\frac{\sin(1/k)}{(1/k)} = 0\cdot 1 = 0. \end{aligned}$$

Because $L = 0$ and the comparison series converges, by the Limit Comparison Test, the given series converges.

72 We use the Limit Comparison Test with the convergent p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$. We have

$$\begin{aligned} L &= \lim_{k\to\infty} \frac{\left(\frac{\pi}{2} - \tan^{-1} k\right)^2}{1/k} = \lim_{k\to\infty} \frac{2\left(\frac{\pi}{2} - \tan^{-1} k\right)\frac{1}{1+k^2}}{2/k^3} = \lim_{k\to\infty}\left(\frac{\pi}{2} - \tan^{-1} k\right)\frac{k^3}{1+k^2} \\ &= \lim_{k\to\infty} \frac{\frac{\pi}{2} - \tan^{-1} k}{1/k}\cdot\frac{k^2}{1+k^2} = \lim_{k\to\infty}\frac{1/(1+k^2)}{1/k^2} = \lim_{k\to\infty}\frac{k^2}{1+k^2} = 1. \end{aligned}$$

Thus the given integral converges by the Limit Comparison Test.

73 Use the Limit Comparison Test with the harmonic series. Note that $\displaystyle\lim_{k\to\infty} \frac{\coth k}{k}\cdot\frac{k}{1} = \lim_{k\to\infty} \coth k = 1$. Because the harmonic series diverges, the given series does as well.

74 Use the Limit Comparison Test with the convergent geometric series whose kth term is $\dfrac{1}{e^k}$. We have

$$\lim_{k\to\infty} \frac{1}{\sinh k}\cdot\frac{e^k}{1} = \lim_{k\to\infty}\frac{2e^k}{e^k - e^{-k}} = 2\lim_{k\to\infty}\frac{1}{1-e^{-2k}} = 2.$$

The given series is therefore convergent.

75 Use the Divergence Test. $\displaystyle\lim_{k\to\infty} \tanh k = \lim_{k\to\infty}\frac{e^k + e^{-k}}{e^k - e^{-k}} = 1 \neq 0$, so the given series diverges.

76 Use the Limit Comparison Test with the convergent geometric series whose kth term is $\dfrac{1}{e^k}$. We have

$$\lim_{k\to\infty} \frac{1}{\cosh k}\cdot\frac{e^k}{1} = \lim_{k\to\infty}\frac{2e^k}{e^k + e^{-k}} = 2\lim_{k\to\infty}\frac{1}{1+e^{-2k}} = 2.$$

The given series is therefore convergent.

77 The corresponding series of all positive terms is the divergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^{3/7}}$. However, the sequence $\left\{\frac{1}{k^{3/7}}\right\}$ is nonincreasing and has limit 0, so the given alternating series is convergent by the Alternating Series Test. Therefore, the given series is conditionally convergent.

78 The corresponding series of all positive terms is convergent, because it is $\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$ which is a convergent p-series. The given series is therefore absolutely convergent.

79 $|a_k| = \frac{1}{k^2-1}$. Use the Limit Comparison Test with the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Because

$$\lim_{k\to\infty} \frac{\frac{1}{k^2-1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{k^2-1} = 1,$$

the given series converges absolutely.

80 We have $\ln(x+3) < x+3$, or $\frac{1}{\ln(x+3)} > \frac{1}{x+3}$ for $x \geq 1$. The series $\sum_{j=1}^{\infty} \frac{1}{j+3} = \sum_{j=4}^{\infty} \frac{1}{j}$ diverges, so the series $\sum_{j=1}^{\infty} \frac{1}{\ln(j+3)}$ diverges. However, the sequence $\left\{\frac{1}{\ln(j+3)}\right\}$ is decreasing and has limit 0, so the given series is convergent by the Alternating Series Test. The given series is therefore conditionally convergent.

81 The sequence $\left\{\frac{k^2+4}{2k^2+1}\right\}$ has limit $\frac{1}{2}$, which implies that the sequence $\left\{(-1)^{k+1}\frac{k^2+4}{2k^2+1}\right\}$ oscillates and diverges. The given series therefore diverges, because by the Divergence Test, the kth term does not have limit 0.

82 Using the Limit Comparison Test with the harmonic series, we consider

$$\lim_{k\to\infty} \left|\frac{a_k}{\frac{1}{k}}\right| = \lim_{k\to\infty} \frac{k}{\sqrt{k^2+1}} = \lim_{k\to\infty} \sqrt{\frac{k^2}{k^2+1}} = 1;$$

because the comparison series diverges, so does the original series. Thus the series is not absolutely convergent. However, the terms are clearly decreasing to zero, so it is conditionally convergent.

83 Use the Root Test on the absolute values of the sequence of terms:

$$\lim_{k\to\infty} |\sqrt[k]{a_k}| = \lim_{k\to\infty} \sqrt[k]{ke^{-k}} = \lim_{k\to\infty} \frac{k^{1/k}}{e} = \frac{1}{e}.$$

Thus, the original series is absolutely convergent by the Root Test.

84 The series of absolute values converges by the Comparison Test. Note that $e^k + e^{-k} > e^k$, so $\frac{1}{e^k+e^{-k}} < \frac{1}{e^k}$ for $k \geq 1$. Because the series $\sum_{k=0}^{\infty} \frac{1}{e^k}$ is a convergent geometric series, the given series converges absolutely.

85 Use the Ratio Test on the absolute values of the sequence of terms: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{10}{k+1} = 0$, so the series converges absolutely.

86 $\sum \frac{1}{k \ln k}$ does not converge because $\int_2^\infty \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \ln(\ln x)\Big|_2^\infty = \infty$, so the improper integral diverges. Thus the given series does not converge absolutely. However, it does converge conditionally because the terms are decreasing and approach zero.

87 Because $k^2 \ll 2^k$, $\lim_{k\to\infty} \frac{-2\cdot(-2)^k}{k^2} \neq 0$. The given series thus diverges by the Divergence Test. Alternatively, this result can be derived from either the Ratio Test or the Root Test.

88 By the Integral Test, the series converges if and only if the following integral converges:

$$\int_2^\infty \frac{1}{x \ln^p(x)}\,dx = \lim_{b\to\infty} \left(\frac{1}{1-p} \ln^{(1-p)}(x) \Big|_2^b \right) = \lim_{b\to\infty} \frac{1}{1-p} \ln^{(1-p)}(b) - \left(\frac{1}{1-p}\right)\cdot \ln^{(1-p)}(2).$$

This limit exists only if $1-p<0$, i.e. $p>1$. Note that the above calculation is for the case $p \neq 1$. In the case $p=1$, the integral also diverges.

89

a. $\sum_{k=1}^{5} \frac{1}{k^5} \approx 1.03666$.

b. $R_5 < \int_5^\infty \frac{dx}{x^5} = \lim_{b\to\infty} \int_5^b \frac{dx}{x^5} = \lim_{b\to\infty} -\frac{1}{4}\left(\frac{1}{b^4} - \frac{1}{5^4}\right) = \frac{1}{4\cdot 5^4} = 0.0004$.

c. Note that $\int_6^\infty \frac{dx}{x^5} = \frac{1}{4\cdot 6^4} \approx 0.00019$. So $L_5 = S_5 + \int_6^\infty \frac{dx}{x^5} \approx 1.03685$, and $U_5 = S_5 + \int_5^\infty \frac{dx}{x^5} \approx 1.03706$.

90

a. $\sum_{k=1}^{20} \frac{1}{k^2+1} \approx 1.0279418$.

b. $R_{20} < \int_{20}^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_{20}^b \frac{dx}{x^2+1} = \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 20) = \frac{\pi}{2} - \tan^{-1} 20 \approx 0.0499584$.

c. Note that $\int_{21}^\infty \frac{dx}{x^2+1} = \frac{\pi}{2} - \tan^{-1} 21 \approx 0.0475831$. So $L_{20} = S_{20} + \int_{21}^\infty \frac{dx}{x^2+1} \approx 1.0755249$, and $U_{20} = S_{20} + \int_{20}^\infty \frac{dx}{x^2+1} \approx 1.0779002$.

91 We have $R_n = \int_n^\infty \frac{dx}{(2x+5)^3} = \frac{1}{4(2n+5)^2}$. We want $\frac{1}{4(2n+5)^2} < \frac{1}{10^4}$, which leads to $n \geq 23$. Therefore $\sum_{k=1}^{\infty} \frac{1}{(2k+5)^3} \approx \sum_{k=1}^{23} \frac{1}{(2k+5)^3} \approx 0.0067$.

92 $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k+5)^3} \approx \sum_{k=1}^{5} \frac{(-1)^{k+1}}{(2k+5)^3} \approx 0.00213615$. We have $|S - S_5| \leq \frac{1}{(2\cdot 6+5)^3} = \frac{1}{17^3} = 0.000203542$.

93 The maximum error is a_{n+1}, so we want $a_{n+1} = \frac{1}{(k+1)^4} < 10^{-8}$, or $(k+1)^4 > 10^8$, so $k = 100$.

94

a. $\sum_{k=0}^{\infty} e^{kx} = \sum_{k=0}^{\infty} (e^x)^k = \frac{1}{1-e^x} = 2$, so $1 - e^x = 1/2$. Thus $e^x = 1/2$ and $x = -\ln(2)$.

b. $\displaystyle\sum_{k=0}^{\infty}(3x)^k = \frac{1}{1-3x} = 4$, so that $1-3x = \frac{1}{4}$, $x = \frac{1}{4}$.

95

a. Let T_n be the amount of additional tunnel dug during week n. Then $T_0 = 100$ and $T_n = .95 \cdot T_{n-1} = (.95)^n T_0 = 100(0.95)^n$, so the total distance dug in N weeks is

$$S_N = 100\sum_{k=0}^{N-1}(0.95)^k = 100\left(\frac{1-(0.95)^N}{1-0.95}\right) = 2000(1-0.95^N).$$

Then $S_{10} \approx 802.5$ meters and $S_{20} \approx 1283.03$ meters.

b. The longest possible tunnel is $S_\infty = 100\displaystyle\sum_{k=0}^{\infty}(0.95)^k = \frac{100}{1-0.95} = 2000$ meters.

96 Let t_n be the time required to dig meters $(n-1)\cdot 100$ through $n \cdot 100$, so that $t_1 = 1$ week. Then $t_n = 1.1 \cdot t_{n-1} = (1.1)^{n-1}t_1 = (1.1)^{n-1}$ weeks. The time required to dig 1500 meters is then

$$\sum_{k=1}^{15} t_k = \sum_{k=1}^{15}(1.1)^{k-1} \approx 31.77 \text{ weeks.}$$

So it is not possible.

97

a. The area of a circle of radius r is πr^2. For $r = 2^{1-n}$, this is $2^{2-2n}\pi$. There are 2^{n-1} circles on the n^{th} page, so the total area of circles on the n^{th} page is $2^{n-1}\cdot\pi 2^{2-2n} = 2^{1-n}\pi$.

b. The sum of the areas on all pages is $\displaystyle\sum_{k=1}^{\infty}2^{1-k}\pi = 2\pi\sum_{k=1}^{\infty}2^{-k} = 2\pi\cdot\frac{1/2}{1/2} = 2\pi$.

98 Because the given sequence is non-decreasing and bounded above by 1, it must have a limit. A reasonable conjecture is that the limit is 1.

99

a. $T_1 = \dfrac{\sqrt{3}}{16}$ and $T_2 = \dfrac{7\sqrt{3}}{64}$.

b. At stage n, 3^{n-1} triangles of side length $1/2^n$ are removed. Each of those triangles has an area of $\dfrac{\sqrt{3}}{4\cdot 4^n} = \dfrac{\sqrt{3}}{4^{n+1}}$, so a total of

$$3^{n-1}\cdot\frac{\sqrt{3}}{4^{n+1}} = \frac{\sqrt{3}}{16}\cdot\left(\frac{3}{4}\right)^{n-1}$$

is removed at each stage. Thus

$$T_n = \frac{\sqrt{3}}{16}\sum_{k=1}^{n}\left(\frac{3}{4}\right)^{k-1} = \frac{\sqrt{3}}{16}\sum_{k=0}^{n-1}\left(\frac{3}{4}\right)^{k} = \frac{\sqrt{3}}{4}\left(1-\left(\frac{3}{4}\right)^{n}\right).$$

c. $\displaystyle\lim_{n\to\infty} T_n = \frac{\sqrt{3}}{4}$ because $\left(\dfrac{3}{4}\right)^n \to 0$ as $n \to \infty$.

d. The area of the triangle was originally $\dfrac{\sqrt{3}}{4}$, so none of the original area is left.

100

a. The fraction of available wealth spent each month is $1-p$, so the amount spent in the n^{th} month is $W(1-p)^n$ (so that all \$$W$ is spent during the first month). The total amount spent is then $\displaystyle\sum_{n=1}^{\infty} W(1-p)^n = \frac{W(1-p)}{1-(1-p)} = W\left(\frac{1-p}{p}\right)$ dollars.

b. As $p \to 1$, the total amount spent approaches 0. This makes sense, because in the limit, if everyone saves all of the money, none will be spent. As $p \to 0$, the total amount spent gets larger and larger. This also makes sense, because almost all of the available money is being respent each month.

101 Ignoring the initial drop for the moment, the height after the n^{th} bounce is $10p^n$, so the total time spent in that bounce is $2\cdot\sqrt{2\cdot 10p^n/g}$ seconds. The total time before the ball comes to rest (now including the time for the initial drop) is then $\displaystyle\sqrt{20/g}+\sum_{i=1}^{\infty} 2\cdot\sqrt{2\cdot 10p^n/g} = \sqrt{\frac{20}{g}}+2\sqrt{\frac{20}{g}}\sum_{i=1}^{\infty}(\sqrt{p})^n = \sqrt{\frac{20}{g}}+2\sqrt{\frac{20}{g}}\frac{\sqrt{p}}{1-\sqrt{p}} = \sqrt{\frac{20}{g}}\left(1+\frac{2\sqrt{p}}{1-\sqrt{p}}\right) = \sqrt{\frac{20}{g}}\left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right)$ seconds.

Chapter 11

Power Series

11.1 Approximating Functions With Polynomials

11.1.1 Let the polynomial be $p(x)$. Then $p(0) = f(0)$, $p'(0) = f'(0)$, and $p''(0) = f''(0)$.

11.1.2 It generally increases, because the more derivatives of f are taken into consideration, the better "fit" the polynomial will provide to f.

11.1.3 The approximations are $p_0(0.1) = 1$, $p_1(0.1) = 1+\frac{0.1}{2} = 1.05$, and $p_2(0.1) = 1+\frac{0.1}{2}-\frac{0.01}{8} = 1.04875$.

11.1.4 The quadratic approximating polynomial for f is $p_2(x) = f(0) + f'(0) + \frac{f''(0)}{2!}x^2 = 1 + 2x - \frac{x^2}{2}$. It follows that $f(0.1) \approx p_2(0.1) = 1 + 2(0.1) - \frac{(0.1)^2}{2} = 1.195$.

11.1.5 The third-order Taylor polynomial is $p_3(x) == f(0) + f'(0) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + x^2 + x^3$. $f(0.2) \approx 1 + (0.2)^2 + (0.3)^3 = 1.048$.

11.1.6 The remainder is the difference between the value of the Taylor polynomial at a point and the true value of the function at that point, $R_n(x) = f(x) - p_n(x)$.

11.1.7 The third-order Taylor polynomial is

$$p_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 = 1 + (x-2) + 2(x-2)^3$$

, so $f(1.9) \approx 1 - 0.1 + 2(-0.1)^3 = 0.898$.

11.1.8 Choose $a = 25$ because it is near 26, and f and its derivatives are easy to evaluate at 25.

11.1.9

a. Note that $f(1) = 8$, and $f'(x) = 12\sqrt{x}$, so $f'(1) = 12$. Thus, $p_1(x) = 8 + 12(x-1)$.

b. $f''(x) = 6/\sqrt{x}$, so $f''(1) = 6$. Thus $p_2(x) = 8 + 12(x-2) + 3(x-1)^2$.

c. $p_1(1.1) = 12(.1) + 8 = 9.2$. $p_2(1.1) = 3(.1)^2 + 12(.1) + 8 = 9.23$.

11.1.10

a. Note that $f(1) = 1$, and that $f'(x) = -1/x^2$, so $f'(1) = -1$. Thus, $p_1(x) = 1 - (x-1) = -x + 2$.

b. $f''(x) = 2/x^3$, so $f''(1) = 2$. Thus, $p_2(x) = 2 - x + (x-1)^2$.

c. $p_1(1.05) = 0.95$. $p_2(1.05) = (0.05)^2 - 0.05 + 2 = 0.9525$.

11.1.11

a. $f'(x) = -2e^{-2x}$, so $p_1(x) = f(0) + f'(0)(x) = 1 - 2x$.

b. $f''(x) = 4e^{-2x}$, so $p_2(x) = f(0) + f'(0)(x) + \frac{1}{2}f''(0)(x^2) = 1 - 2x + 2x^2$.

c. $p_1(0.2) = 0.6$, and $p_2(0.2) = 1 - 2(0.2) + 2(0.04) = 0.68$.

11.1.12

a. $f'(x) = \frac{1}{2}x^{-1/2}$, so $p_1(x) = f(4) + f'(4)(x-4) = 2 + \frac{1}{4}(x-4)$.

b. $f''(x) = -\frac{1}{4}x^{-3/2}$, so $p_2(x) = f(4) + f'(4)(x-4) + \frac{1}{2}f''(4)(x-4)^2 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$.

c. $p_1(3.9) = 2 + \frac{1}{4}(-0.1) = 2 - .025 = 1.975$, and $p_2(3.9) = 2 - 0.025 - \frac{1}{64}(0.001) = 1.97484375$.

11.1.13

a. $f'(x) = -\frac{1}{(x+1)^2}$, so $p_1(x) = f(0) + f'(0)(x) = 1 - x$.

b. $f''(x) = \frac{2}{(x+1)^3}$, so $p_2(x) = f(0) + f'(0)(x) + \frac{1}{2}f''(0)x^2 = 1 - x + x^2$.

c. $p_1(0.05) = 0.95$, and $p_2(0.05) = 1 - 0.05 + 0.0025 = 0.9525$.

11.1.14

a. $f'(x) = -\sin x$, so $p_1(x) = \cos(\pi/4) - \sin(\pi/4)(x - \pi/4) = \frac{\sqrt{2}}{2}(1 - (x - \pi/4))$.

b. $f''(x) = -\cos x$, so

$$\begin{aligned} p_2(x) &= \cos(\pi/4) - \sin(\pi/4)(x-\pi/4) - \frac{1}{2}\cos(\pi/4)(x-\pi/4)^2 \\ &= \frac{\sqrt{2}}{2}\left(1 - (x-\pi/4) - \frac{1}{2}(x-\pi/4)^2\right). \end{aligned}$$

c. $p_1(.24\pi) \approx 0.7293211959$, $p_2(.24\pi) \approx 0.7289722527$.

11.1.15

a. $f'(x) = (1/3)x^{-2/3}$, so $p_1(x) = f(8) + f'(8)(x-8) = 2 + \frac{1}{12}(x-8)$.

b. $f''(x) = (-2/9)x^{-5/3}$, so $p_2(x) = f(8) + f'(8)(x-8) + \frac{1}{2}f''(8)(x-8)^2 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$.

c. $p_1(7.5) \approx 1.958333333$, $p_2(7.5) \approx 1.957465278$.

11.1.16

a. $f'(x) = \frac{1}{1+x^2}$, so $p_1(x) = f(0) + f'(0)(x) = x$.

b. $f''(x) = \frac{-2x}{(1+x^2)^2}$, so $p_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = x$.

c. $p_1(0.1) = p_2(0.1) = 0.1$.

11.1.17 $f(0) = 1, f'(0) = -6\sin 0 = 0, f''(0) = -36\cos 0 = -36$, $f'''(0) = 6^3 \sin 0 = 0$, and $f^{(4)}(0) = 6^4\cos 0 = 1296$, so that $p_1(x) = 1$, $p_2(x) = p_3(x) = 1 - 18x^2$, $p_4(x) = 1 - 18x^2 + 54x^4$.

11.1.18 $f(0) = 1$, $f'(0) = -e^0 = -1$, $f''(0) = e^0 = 1$, $f'''(0) = -e^0 = -1$, $f^{(4)}(0) = e^0 = 1$, and $f^{(5)}(0) = -1$, so that $p_1(x) = 1-x$, $p_2(x) = 1-x+\frac{x^2}{2}$, $p_3(x) = 1-x+\frac{x^2}{2}-\frac{x^3}{6}$, $p_4(x) = 1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}$, $p_5(x) = 1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}$.

11.1.19 $f(0) = 1$, $f'(0) = -3(1+0)^{-4} = -3$, $f''(0) = 12(1+0)^{-5} = 12$, $f'''(0) = -60(1+0)^{-6} = -60$, and $f^{(4)}(0) = 360(1+0)^{-7} = 360$, so that $p_1(x) = 1-3x$, $p_2(x) = 1-3x+6x^2$, $p_3(x) = 1-3x+6x^2-10x^3$, $p_4(x) = 1-3x+6x^2-10x^3+15x^4$.

11.1.20 If $f(x) = \cos x$, then $f(\pi/6) = \sqrt{3}/2$, $f'(\pi/6) = -\sin(\pi/6) = -1/2$, $f''(\pi/6) = -\cos(\pi/6) = -\sqrt{3}/2$, $f'''(\pi/6) = \sin(\pi/6) = 1/2$, $f^{(4)}(\pi/6) = \cos(\pi/6) = \sqrt{3}/2$, and $f^{(5)}(\pi/6) = -\sin(\pi/6) = -1/2$. Then

$$p_4(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x-\frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x-\frac{\pi}{6}\right)^3 + \frac{\sqrt{3}}{48}\left(x-\frac{\pi}{6}\right)^4$$

and

$$p_5(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x-\frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x-\frac{\pi}{6}\right)^3 + \frac{\sqrt{3}}{48}\left(x-\frac{\pi}{6}\right)^4 - \frac{1}{240}\left(x-\frac{\pi}{6}\right)^5.$$

11.1.21 Note that $f(1) = 1$, $f'(1) = 3$, $f''(1) = 6$, and $f'''(1) = 6$. Thus, $p_1(x) = 1 + 3(x-1)$, and $p_2(x) = 1 + 3(x-1) + 3(x-1)^2$, and $p_3(x) = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$.

11.1.22 Note that $f(1) = 8$, $f'(1) = \frac{4}{\sqrt{1}} = 4$, $f''(1) = \frac{-2}{(1)^{3/2}} = -2$, $f'''(1) = \frac{3}{1^{5/2}} = 3$, and $f^{(4)}(1) = -\frac{15}{2}\cdot 1^{-7/2} = -\frac{15}{2}$, so

$$p_3(x) = 8 + 4(x-1) - (x-1)^2 + \frac{1}{2}(x-1)^3$$

and

$$p_4(x) = 8 + 4(x-1) - (x-1)^2 + \frac{1}{2}(x-1)^3 - \frac{5}{16}(x-1)^4.$$

11.1.23 $f(x) = \ln x$ so $f(e) = 1$, $f'(e) = \frac{1}{e}$, $f''(e) = -\frac{1}{e^2}$, and $f'''(e) = \frac{2}{e^3}$.

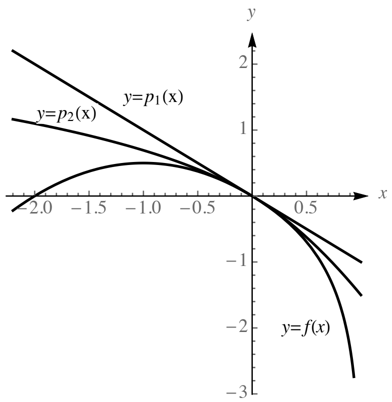
So,

$$p_3(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3.$$

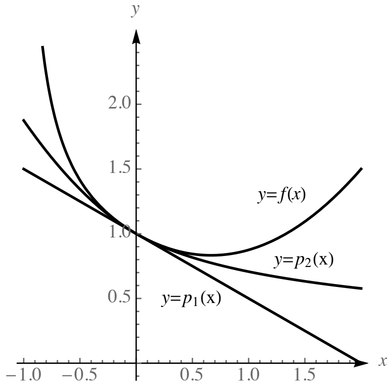
11.1.24 $f(x) = \sqrt[3]{x}$, so $f(8) = 2$, $f'(8) = \frac{1}{12}$, $f''(8) = -\frac{2}{9}(8)^{-5/3} = -\frac{1}{144}$ so

$$p_2(x) = x + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2.$$

11.1.25 $f(0) = 0$, $f'(0) = \frac{-1}{1-0} = -1$, $f''(0) = \frac{1}{(1-0)^2} = -1$, so that $p_1(x) = -x$, $p_2(x) = -x - \frac{1}{2}x^2$.



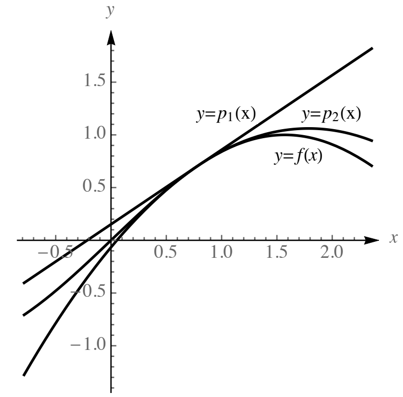
11.1.26 $f(0) = 1$, $f'(0) = (-1/2)(0+1)^{-3/2} = -1/2$, $f''(0) = (3/4)(0+1)^{-5/2} = 3/4$, so that $p_1(x) = 1 - \dfrac{x}{2}$, $p_2(x) = 1 - \dfrac{x}{2} + \dfrac{3}{8}x^2$.



11.1.27

a. $p_1(x) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\left(x - \dfrac{\pi}{4}\right)$, $p_2(x) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\left(x - \dfrac{\pi}{4}\right) - \dfrac{\sqrt{2}}{4}\left(x - \dfrac{\pi}{4}\right)^2$.

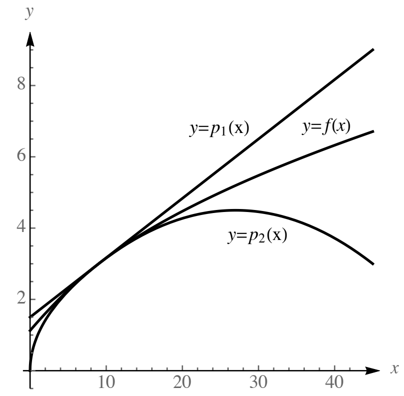
b.



11.1.28

a. $p_1(x) = 3 + \dfrac{1}{6}(x - 9)$, $p_2(x) = 3 + \dfrac{1}{6}(x - 9) - \dfrac{1}{216}(x - 9)^2$.

b.



11.1.29

a. $p_2(0.05) = 1.0246875$.

b. The absolute error is $\sqrt{1.05} - p_2(1.05) \approx 1.024695077 - 1.0246875 = 7.58 \times 10^{-6}$.

11.1.30

a. $p_2(0.08) = 0.9624$.

b. The absolute error is $p_2(0.08) - 1/\sqrt{1.08} \approx 0.9624 - 0.9622504482 \approx 1.50 \times 10^{-4}$.

11.1.31

a. $p_2(0.15) = 0.86125$.

b. The absolute error is $p_2(0.15) - e^{-0.15} \approx 0.86125 - 0.8607079764 \approx 5.42 \times 10^{-4}$.

11.1.32

a. $p_2(0.06) = 0.0582$.

b. The absolute error is $\ln(1.06) - p_2(.06) \approx 0.05826890812 - 0.0582 \approx 6.89 \times 10^{-5}$.

11.1.33

a. The 3rd-order Taylor polynomial of $f(x) = e^{-x}$ centered at $a = 0$ is

$$p(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3,$$

so the approximate value of $e^{0.12}$ is $p(0.12) = 1.127488$.

b. The absolute error is $\left|e^{0.12} - 1.127488\right| \approx 8.85 \times 10^{-6}$. (Answers may vary if intermediate calculations are rounded.)

11.1.34

a. The 3rd-order Taylor polynomial of $f(x) = \cos x$ centered at $a = 0$ is

$$p(x) = 1 - \frac{1}{2}x^2,$$

so the approximate value of $\cos(-0.2)$ is $p(-0.2) = 0.98$.

b. The absolute error is $|\cos(-0.2) - 0.98| \approx 6.66 \times 10^{-5}$. (Answers may vary if intermediate calculations are rounded.)

11.1.35

a. The 3rd-order Taylor polynomial of $f(x) = \tan x$ centered at $a = 0$ is

$$p(x) = x + \frac{1}{3}x^3$$

so the approximate value of $\tan(-0.1)$ is $p(-0.1) \approx -0.10033333$.

b. The absolute error is $|\tan(-0.1) - (-0.10033333)| \approx 1.34 \times 10^{-6}$. (Answers may vary if intermediate calculations are rounded.)

11.1.36

a. The 3rd-order Taylor polynomial of $f(x) = \ln x$ centered at $a = 1$ is

$$p(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3,$$

so the approximate value of $\ln(1.05)$ is $p(1.05) \approx 0.04879167$.

b. The absolute error is $|\ln(1.05) - 0.04879167| \approx 1.50 \times 10^{-6}$. (Answers may vary if intermediate calculations are rounded.)

11.1.37

a. The 3rd-order Taylor polynomial of $f(x) = \sqrt{x}$ centered at $a = 1$ is

$$p(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3,$$

so the approximate value of $\sqrt{1.06}$ is $p(1.06) \approx 1.0295635$.

b. The absolute error is $\left|\sqrt{1.06} - 1.0295635\right| \approx 4.86 \times 10^{-7}$. (Answers may vary if intermediate calculations are rounded.)

11.1.38

a. The 3rd-order Taylor polynomial of $f(x) = \sqrt[4]{x}$ centered at $a = 81$ is

$$p(x) = 3 + \frac{1}{108}(x-81) - \frac{1}{23328}(x-81)^2 + \frac{7}{22674816}(x-81)^3,$$

so the approximate value of $\sqrt[4]{79}$ is $p(79) \approx 2.98130754$.

b. The absolute error is $\left|\sqrt[4]{79} - 2.98130754\right| \approx 4.27 \times 10^{-8}$. (Answers may vary if intermediate calculations are rounded.)

11.1.39

a. The 3rd-order Taylor polynomial of $f(x) = \sinh x$ centered at $a = 0$ is

$$p(x) = x + \frac{1}{6}x^3,$$

so the approximate value of $\sinh 0.5$ is $p(0.5) \approx 0.52083333$.

b. The absolute error is $|\sinh(0.5) - 0.52083333| \approx 2.62 \times 10^{-4}$. (Answers may vary if intermediate calculations are rounded.)

11.1.40

a. The 3rd-order Taylor polynomial of $f(x) = \tanh x$ centered at $a = 0$ is

$$p(x) = x - \frac{1}{3}x^3,$$

so the approximate value of $\tanh 0.5$ is $p(0.5) \approx 0.45833333$.

b. The absolute error is $|\tanh(0.5) - 0.45833333| \approx 3.78 \times 10^{-3}$. (Answers may vary if intermediate calculations are rounded.)

11.1.41 $R_n(x) = \dfrac{\sin^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c between 0 and x.

11.1.42 $R_n(x) = \dfrac{\cos(2x)^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c between 0 and x.

11.1.43 $R_n(x) = \dfrac{(-1)^n e^{-c}}{(n+1)!}x^{n+1}$ for some c between 0 and x.

11.1.44 $R_n(x) = \dfrac{\cos^{(n+1)}(c)}{(n+1)!}\left(x - \dfrac{\pi}{2}\right)^{n+1}$ for some c between $\dfrac{\pi}{2}$ and x.

11.1.45 $R_n(x) = \dfrac{\sin^{(n+1)}(c)}{(n+1)!}\left(x - \dfrac{\pi}{2}\right)^{n+1}$ for some c between $\dfrac{\pi}{2}$ and x.

11.1.46 $R_n(x) = \dfrac{1}{(1-c)^{n+2}}\left(x^{n+1}\right)$ for some c between 0 and x.

11.1.47 $f(x) = \sin x$, so $f^{(4)}(x) = \cos x$. Because $\cos x$ is bounded in magnitude by 1, the remainder is bounded by $|R_4(x)| \le \dfrac{0.3^5}{5!} \approx 2.03 \times 10^{-5}$.

11.1.48 $f(x) = \cos(x)$, so $f^{(4)}(x) = \cos x$. Because $\cos x$ is bounded in magnitude by 1, the remainder is bounded by $|R_3(x)| \le \dfrac{0.45^4}{4!} \approx 1.7 \times 10^{-3}$.

11.1.49 $f(x) = e^x$, so $f^{(5)}(x) = e^x$. Because $e^{0.25}$ is bounded by 2, $|R_4(x)| \le 2 \cdot \dfrac{0.25^5}{5!} \approx 1.63 \times 10^{-5}$.

11.1.50 $f(x) = \tan(x)$, so $f^{(3)}(x) = 2\sec^2 x(\sec^2 x + 2\tan^2 x)$ and $f^{(3)}(x) < \dfrac{16}{3}$ on $[0, 0.3]$. Thus $|R_2(x)| \le \dfrac{16}{3} \cdot \dfrac{0.3^3}{3!} = 2.4 \times 10^{-2}$.

11.1.51 $f(x) = e^{-x}$, so $f^{(5)}(x) = -e^{-x}$. Because $f^{(5)}$ achieves its maximum magnitude in the range at $x = 0$, which has absolute value 1, $|R_4(x)| \le 1 \cdot \dfrac{0.5^5}{5!} \approx 2.6 \times 10^{-4}$.

11.1.52 $f(x) = \ln(1 + x)$, so $f^{(4)}(x) = \dfrac{-6}{(x+1)^4}$. On $[0, 0.4]$, the maximum magnitude is 6, so $|R_3(x)| \le 6 \cdot \dfrac{0.04^4}{4!} = 6.4 \times 10^{-3}$.

11.1.53 Here $n = 3$ or 4, so use $n = 4$, and $M = 1$ because $f^{(5)}(x) = \cos x$, so that $R_4(x) \le \dfrac{(\pi/4)^5}{5!} \approx 2.49 \times 10^{-3}$.

11.1.54 $n = 2$ or 3, so use $n = 3$, and $M = 1$ because $f^{(4)}(x) = \cos x$, so that $R_3(x) \le \dfrac{(\pi/4)^4}{4!} \approx 1.59 \times 10^{-2}$.

11.1.55 $n = 2$ and $M = e^{1/2} < 2$, so $R_2(x) \le 2 \cdot \dfrac{(1/2)^3}{3!} \approx 4.17 \times 10^{-2}$.

11.1.56 $n = 1$ or 2, so use 2, and $f^{(3)}(x) = 2\sec^2 x(\sec^2 x + 2\tan^2 x)$, which is bounded by $\dfrac{16}{3}$ on the range. Thus $R_2(x) \le \dfrac{16}{3} \cdot \dfrac{(\pi/6)^3}{3!} \approx 1.28 \times 10^{-1}$.

11.1.57 $n = 2$; $f^{(3)}(x) = \dfrac{2}{(1+x)^3}$, which achieves its maximum at $x = -0.2$: $|f^{(3)}(x)| = \dfrac{2}{0.8^3} < 4$. Then $R_2(x) \le 4 \cdot \dfrac{0.2^3}{3!} \approx 5.4 \times 10^{-3}$.

11.1.58 $n = 1$, $f'(x) = \dfrac{1}{2}(1+x)^{-1/2}$, which achieves its maximum magnitude at $x = -0.1$, where it is less than 1. Thus $R_1(x) \le \dfrac{0.1^2}{2!} = .005$.

11.1.59 Use the Taylor series for e^x at $x = 0$. The derivatives of e^x are e^x. On $[-0.5, 0]$, the maximum magnitude of any derivative is thus 1 and $x = 0$, so $|R_n(-0.5)| \le \dfrac{0.5^{n+1}}{(n+1)!}$, so for $R_n(-0.5) < 10^{-3}$ we need $n = 4$.

11.1.60 Use the Taylor series at $x = 0$ for $\sin x$. The magnitude of any derivative of $\sin x$ is bounded by 1, so $|R_n(0.2)| \le \dfrac{0.2^{n+1}}{(n+1)!}$, so for $R_n(0.2) < 10^{-3}$ we need $n = 3$.

11.1.61 Use the Taylor series for $\cos x$ at $x = 0$. The magnitude of any derivative of $\cos x$ is bounded by 1, so $|R_n(-0.25)| \le \dfrac{0.25^{n+1}}{(n+1)!}$, so for $|R_n(-0.25)| < 10^{-3}$ we need $n = 3$.

11.1.62 Use the Taylor series for $f(x) = \ln(1+x)$ at $x = 0$. Then $|f^{(n+1)}(x)| = \dfrac{n!}{(1+x)^{n+1}}$, which for $x \in [-0.15, 0]$ achieves its maximum at $x = -.15$. This maximum is less than $(1.2)^{n+1} \cdot n!$. Thus $|R_n(-0.15)| \le (1.2)^{n+1} \cdot n! \cdot \dfrac{.18^{n+1}}{(n+1)!} = \dfrac{1.2 \cdot (0.15)^{n+1}}{n}$, so for $|R_n(-0.15)| < 10^{-3}$ we need $n = 3$.

11.1.63 Use the Taylor series for $f(x) = \sqrt{x}$ at $x = 1$. Then $|f^{(n+1)}(x)| = \dfrac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2^{n+1}} x^{-(2n+1)/2}$, which achieves its maximum on $[1, 1.06]$ at $x = 1$. Then

$$|R_n(1.06)| \le \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2^{n+1}} \cdot \frac{(1.06-1)^{n+1}}{(n+1)!},$$

and for $|R_n(0.06)| < 10^{-3}$ we need $n = 1$.

11.1.64 Use the Taylor series for $f(x) = \sqrt{1/(1-x)}$ at $x = 0$. Then $|f^{(n+1)}(x)| = \dfrac{1 \cdot 3 \cdot \cdots \cdot (2n+1)}{2^{n+1}}(1-x)^{(-3-2n)/2}$, which achieves its maximum on $[0, 0.15]$ at $x = 0.15$. Thus

$$\begin{aligned} |R_n(0.15)| &\le \frac{1 \cdot 3 \cdot \cdots \cdot (2n+1)}{2^{n+1}} \cdot \left(\frac{1}{1-0.15}\right)^{(2n+3)/2} \cdot \frac{0.15^{n+1}}{(n+1)!} \\ &= \frac{1 \cdot 3 \cdot \cdots \cdot (2n+1)}{2^{n+1}(n+1)!} \cdot \left(\frac{0.15^{n+1}}{0.85^{(2n+3)/2}}\right), \end{aligned}$$

and for $|R_n(0.15)| < 10^{-3}$ we need $n = 3$.

11.1.65

a. False. If $f(x) = e^{-2x}$, then $f^{(n)}(x) = (-1)^n 2^n e^{-2x}$, so that $f^{(n)}(0) \neq 0$ and all powers of x are present in the Taylor series.

b. True. The constant term of the Taylor series is $f(0) = 1$. Higher-order terms all involve derivatives of $f(x) = x^5 - 1$ evaluated at $x = 0$; clearly for $n < 5$, $f^{(n)}(0) = 0$, and for $n > 5$, the derivative itself vanishes. Only for $n = 5$, where $f^{(5)}(x) = 5!$, is the derivative nonzero, so the coefficient of x^5 in the Taylor series is $f^{(5)}(0)/5! = 1$ and the Taylor polynomial of order 10 is in fact $x^5 - 1$.

c. True. The odd derivatives of $\sqrt{1+x^2}$ vanish at $x = 0$, while the even ones do not.

d. True. Because then $f''(a) = 0$.

11.1.66 Let $p(x) = \displaystyle\sum_{k=0}^{n} c_k(x-a)^k$ be the n^{th} polynomial for $f(x)$ at a. Because $f(a) = p(a)$, it follows that $c_0 = f(0)$. Now, the k^{th} derivative of $p(x)$, $1 \le k \le n$, is $p^{(k)}(x) = k!c_k +$ terms involving $(x-a)^i$, $i > 0$, so that $f^{(k)}(a) = p^{(k)}(a) = k! \cdot c_k$ so that $c_k = \dfrac{f^{(k)}(a)}{k!}$.

11.1.67

a. This matches (C) because for $f(x) = (1+2x)^{1/2}$, $f''(x) = -(1+2x)^{-3/2}$ so $\dfrac{f''(0)}{2!} = \dfrac{-1}{2}$.

b. This matches (E) because for $f(x) = (1+2x)^{-1/2}$, $f''(x) = 3(1+2x)^{-5/2}$, so $\dfrac{f''(0)}{2!} = \dfrac{3}{2}$.

c. This matches (A) because $f^{(n)}(x) = 2^n e^{2x}$, so that $f^{(n)}(0) = 2^n$, which is (A)'s pattern.

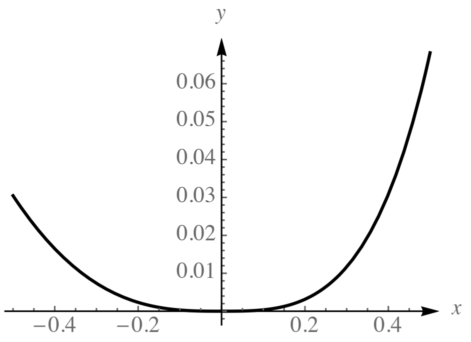
d. This matches (D) because $f''(x) = 8(1+2x)^{-3}$ and $f''(0) = 8$, so that $f''(0)/2! = 4$

e. This matches (B) because $f'(x) = -6(1+2x)^{-4}$ so that $f'(0) = -6$.

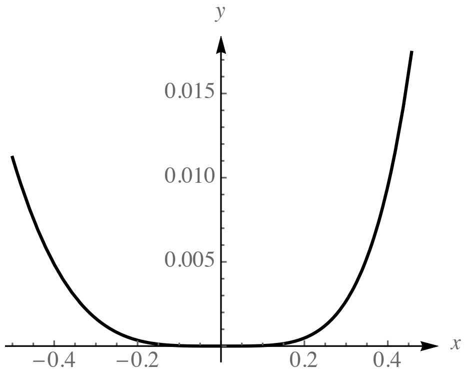
f. This matches (F) because $f^{(n)}(x) = (-2)^n e^{-2x}$, so $f^{(n)}(0) = (-2)^n$, which is (F)'s pattern.

11.1.68

a.



$|\ln(1-x) - p_2(x)|$



$|\ln(1-x) - p_3(x)|$

b. The error seems to be largest at $x = \dfrac{1}{2}$ and smallest at $x = 0$.

c. The error bound found in Example 7 for $|\ln(1-x) - p_3(x)|$ was 0.25. The actual error seems much less than that, about 0.02.

11.1.69

a. $p_2(0.1) = 0.1$. The maximum error in the approximation is $1 \cdot \dfrac{0.1^3}{3!} \approx 1.67 \times 10^{-4}$.

b. $p_2(0.2) = 0.2$. The maximum error in the approximation is $1 \cdot \dfrac{0.2^3}{3!} \approx 1.33 \times 10^{-3}$.

11.1.70

a. $p_1(0.1) = 0.1$. $f''(x) = 2\tan(x)(1 + \tan^2(x))$. Because $\tan(0.1) < 0.2$, $|f''(c)| \le 2(.2)(1 + .2^2) = 0.416$. Thus the maximum error is $\dfrac{0.416}{2!} \cdot 0.1^2 = 2.08 \times 10^{-3}$.

b. $p_1(0.2) = 0.2$. The maximum error is $\dfrac{0.416}{2} \cdot 0.2^2 = 8.32 \times 10^{-3}$.

11.1.71

a. $p_3(0.1) = 1 - .01/2 = 0.995$. The maximum error is $1 \cdot \dfrac{0.1^4}{4!} \approx 4.17 \times 10^{-6}$.

b. $p_3(0.2) = 1 - .04/2 = 0.98$. The maximum error is $1 \cdot \dfrac{0.2^4}{4!} \approx 6.67 \times 10^{-5}$.

11.1.72

a. $p_2(0.1) = 0.1$ (we can take $n = 2$ because the coefficient of x^2 is 0). $f^{(3)}(x) = -2 \cdot \dfrac{1 - 2x^2}{(1 + x^2)^2}$ has a maximum magnitude value of 2, the maximum error is $2 \cdot \dfrac{0.1^3}{3!} \approx 3.33 \times 10^{-4}$.

b. $p_2(0.2) = 0.2$. The maximum error is $2 \cdot \dfrac{0.2^3}{3!} \approx 2.67 \times 10^{-3}$.

11.1.73

a. $p_1(0.1) = 1.05$. Because $|f''(x)| = \dfrac{1}{4}(1 + x)^{-3/2}$ has a maximum value of $1/4$ at $x = 0$, the maximum error is $\dfrac{1}{4} \cdot \dfrac{0.1^2}{2} = 1.25 \times 10^{-3}$.

b. $p_1(0.2) = 1.1$. The maximum error is $\dfrac{1}{4} \cdot \dfrac{0.2^2}{2} = 5 \times 10^{-3}$.

11.1.74

a. $p_2(0.1) = 0.1 - 0.01/2 = 0.095$. Because $|f^{(3)}(x)| = \dfrac{2}{(x + 1)^3}$ achieves a maximum of 2 at $x = 0$, the maximum error is $2 \cdot \dfrac{0.1^3}{3!} \approx 3.33 \times 10^{-4}$.

b. $p_2(0.2) = 0.2 - 0.04/2 = 0.18$. The maximum error is $2 \cdot \dfrac{0.2^3}{3!} \approx 2.67 \times 10^{-3}$.

11.1.75

a. $p_1(0.1) = 1.1$. Because $f''(x) = e^x$ is less than 2 on $[0, 0.1]$, the maximum error is less than $2 \cdot \dfrac{0.1^2}{2!} = \dfrac{1}{100}$.

b. $p_1(0.2) = 1.2$. The maximum error is less than $2 \cdot \dfrac{0.2^2}{2!} = .04 = \dfrac{1}{25}$.

11.1.76

a. $p_1(0.1) = 0.1$. Because $f''(x) = \dfrac{x}{(1-x^2)^{3/2}}$ is less than 1 on $[0, 0.2]$, the maximum error is $1 \cdot \dfrac{0.1^3}{3!} \approx 1.67 \times 10^{-4}$.

b. $p_1(0.2) = 0.2$. The maximum error is $1 \cdot \dfrac{0.2^3}{3!} \approx 1.33 \times 10^{-3}$.

11.1.77

a.

	$\lvert\sec x - p_2(x)\rvert$	$\lvert\sec x - p_4(x)\rvert$
-0.2	3.39×10^{-4}	5.51×10^{-6}
-0.1	2.09×10^{-5}	8.51×10^{-8}
0.0	0	0
0.1	2.09×10^{-5}	8.51×10^{-8}
0.2	3.39×10^{-4}	5.51×10^{-6}

b. The errors are get smaller as $|x|$ decreases.

11.1.78

a.

	$\lvert\cos x - p_2(x)\rvert$	$\lvert\cos x - p_4(x)\rvert$
-0.2	6.66×10^{-5}	8.88×10^{-8}
-0.1	4.17×10^{-6}	1.39×10^{-9}
0.0	0	0
0.1	4.17×10^{-6}	1.39×10^{-9}
0.2	6.66×10^{-5}	8.88×10^{-8}

b. The errors are equal for positive and negative x. This makes sense, because $\cos(-x) = \cos(x)$ and $p_n(-x) = p_n(x)$ for $n = 2, 4$. The errors appear to get larger as x gets farther from zero.

11.1.79

a.

	$\lvert e^{-x} - p_1(x)\rvert$	$\lvert e^{-x} - p_2(x)\rvert$
-0.2	2.14×10^{-2}	1.4×10^{-3}
-0.1	5.17×10^{-3}	1.71×10^{-4}
0.0	0	0
0.1	4.84×10^{-3}	1.63×10^{-4}
0.2	1.87×10^{-2}	1.27×10^{-3}

b. The errors get smaller as $|x|$ decreases.

11.1.80 The true value of $\cos\dfrac{\pi}{12} = \dfrac{1+\sqrt{3}}{2\sqrt{2}} \approx 0.9659258263$. The 6th-order Taylor polynomial for $\cos x$ centered at $x = 0$ is

$$p_6(x) = 1 - \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720}.$$

Evaluating the polynomials at $x = \pi/12$ produces the following table:

n	$p_n\left(\frac{\pi}{12}\right)$	$\lvert p_n\left(\frac{\pi}{12}\right) - \cos\frac{\pi}{12}\rvert$
1	1.0000000000	3.41×10^{-2}
2	0.9657305403	1.95×10^{-4}
3	0.9657305403	1.95×10^{-4}
4	0.9659262729	4.47×10^{-7}
5	0.9659262729	4.47×10^{-7}
6	0.9659258257	5.47×10^{-10}

The 6th-order Taylor polynomial for $\cos x$ centered at $x = \pi/6$ is

$$p_6(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x - \frac{\pi}{6}\right)^3 + \frac{\sqrt{3}}{48}\left(x - \frac{\pi}{6}\right)^4 - \frac{1}{240}\left(x - \frac{\pi}{6}\right)^5 - \frac{\sqrt{3}}{1440}\left(x - \frac{\pi}{6}\right)^6.$$

Evaluating the polynomials at $x = \pi/12$ produces the following table:

n	$p_n\left(\frac{\pi}{12}\right)$	$\lvert p_n\left(\frac{\pi}{12}\right) - \cos\frac{\pi}{12}\rvert$
1	0.9969250977	3.10×10^{-2}
2	0.9672468750	1.32×10^{-3}
3	0.9657515877	1.74×10^{-4}
4	0.9659210972	4.73×10^{-6}
5	0.9659262214	3.95×10^{-7}
6	0.9659258342	7.88×10^{-9}

Comparing the tables shows that using the polynomial centered at $x = 0$ is more accurate when n is even while using the polynomial centered at $x = \pi/6$ is more accurate when n is odd. To see why, consider the remainder. Let $f(x) = \cos x$. By Theorem 9.2, the magnitude of the remainder when approximating $f(\pi/12)$ by the polynomial p_n centered at 0 is:

$$\left|R_n\left(\frac{\pi}{12}\right)\right| = \frac{|f^{(n+1)}(c)|}{(n+1)!}\left(\frac{\pi}{12}\right)^{n+1}$$

for some c with $0 < c < \frac{\pi}{12}$, while the magnitude of the remainder when approximating $f(\pi/12)$ by the polynomial p_n centered at $\pi/6$ is:

$$\left|R_n\left(\frac{\pi}{12}\right)\right| = \frac{|f^{(n+1)}(c)|}{(n+1)!}\left(\frac{\pi}{12}\right)^{n+1}$$

for some c with $\frac{\pi}{12} < c < \frac{\pi}{6}$. When n is odd, $|f^{(n+1)}(c)| = |\cos c|$. Because $\cos x$ is a positive and decreasing function over $[0, \pi/6]$, the magnitude of the remainder in using the polynomial centered at $\pi/6$ will be less than the remainder in using the polynomial centered at 0, and the former polynomial will be more accurate. When n is even, $|f^{(n+1)}(c)| = |\sin c|$. Because $\sin x$ is a positive and increasing function over $[0, \pi/6]$, the remainder in using the polynomial centered at 0 will be less than the remainder in using the polynomial centered at $\pi/6$, and the former polynomial will be more accurate.

11.1.81 The true value of $e^{0.35} \approx 1.419067549$. The 6th-order Taylor polynomial for e^x centered at $x = 0$ is

$$p_6(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}.$$

Evaluating the polynomials at $x = 0.35$ produces the following table:

n	$p_n(0.35)$	$\lvert p_n(0.35) - e^{0.35}\rvert$
1	1.350000000	6.91×10^{-2}
2	1.411250000	7.82×10^{-3}
3	1.418395833	6.72×10^{-4}
4	1.419021094	4.65×10^{-5}
5	1.419064862	2.69×10^{-6}
6	1.419067415	1.33×10^{-7}

The 6^{th}-order Taylor polynomial for e^x centered at $x = \ln 2$ is

$$p_6(x) = 2 + 2(x-\ln 2) + (x-\ln 2)^2 + \frac{1}{3}(x-\ln 2)^3 + \frac{1}{12}(x-\ln 2)^4 + \frac{1}{60}(x-\ln 2)^5 + \frac{1}{360}(x-\ln 2)^6.$$

Evaluating the polynomials at $x = 0.35$ produces the following table:

n	$p_n(0.35)$	$\lvert p_n(0.35) - e^{0.35}\rvert$
1	1.313705639	1.05×10^{-1}
2	1.431455626	1.24×10^{-2}
3	1.417987101	1.08×10^{-3}
4	1.419142523	7.50×10^{-5}
5	1.419063227	4.32×10^{-6}
6	1.419067762	2.13×10^{-7}

Comparing the tables shows that using the polynomial centered at $x = 0$ is more accurate for all n. To see why, consider the remainder. Let $f(x) = e^x$. By Theorem 9.2, the magnitude of the remainder when approximating $f(0.35)$ by the polynomial p_n centered at 0 is:

$$|R_n(0.35)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}(0.35)^{n+1} = \frac{e^c}{(n+1)!}(0.35)^{n+1}$$

for some c with $0 < c < 0.35$ while the magnitude of the remainder when approximating $f(0.35)$ by the polynomial p_n centered at $\ln 2$ is:

$$|R_n(0.35)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|0.35 - \ln 2|^{n+1} = \frac{e^c}{(n+1)!}(\ln 2 - 0.35)^{n+1}$$

for some c with $0.35 < c < \ln 2$. Because $\ln 2 - 0.35 \approx 0.35$, the relative size of the magnitudes of the remainders is determined by e^c in each remainder. Because e^x is an increasing function, the remainder in using the polynomial centered at 0 will be less than the remainder in using the polynomial centered at $\ln 2$, and the former polynomial will be more accurate.

11.1.82

a. Let x be a point in the interval on which the derivatives of f are assumed continuous. Then f' is continuous on $[a, x]$, and the Fundamental Theorem of Calculus implies that because f is an antiderivative of f', then $\int_a^x f'(t)\,dt = f(x) - f(a)$, or $f(x) = f(a) + \int_a^x f'(t)\,dt$.

b. Integrate the integral from part (a) by parts using $u = f'(t), dv = dt$ to obtain

$$\begin{aligned} f(x) &= f(a) + \int_a^x f'(t)\,dt = f(a) + [tf'(t)]\Big|_{t=a}^{x} - \int_a^x tf''(t)\,dt \\ &= f(a) + xf'(x) - af'(a) - \int_a^x tf''(t)\,dt \end{aligned}$$

Now, $\int_a^x xf''(t)\,dt = x\int_a^x f''(t)\,dt = xf'(x) - xf'(a)$. Substituting into the above, we obtain

$$\begin{aligned} f(x) &= f(a) + x\int_a^x f''(t)\,dt + xf'(a) - af'(a) - \int_a^x tf''(t)\,dt \\ &= f(a) + (x-a)f'(a) + \int_a^x (x-t)f''(t)\,dt \end{aligned}$$

c. We obtain (using $u = f''(t)$, $dv = (x-t)\,dt$),

$$\begin{aligned} f(x) &= f(a) + (x-a)f'(a) + \int_a^x (x-t)f''(t)\,dt \\ &= f(a) + (x-a)f'(a) - \frac{(x-t)^2}{2}f''(t)\bigg|_a^x + \frac{1}{2}\int_a^x (x-t)^2 f'''(t)\,dt \\ &= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(t) + \frac{1}{2}\int_a^x (x-t)^2 f'''(t)\,dt \end{aligned}$$

It is clear that continuing this process will give the desired result, because successive integrals of $x-t$ give $\dfrac{-1}{k!}(x-t)^k$.

d. **Lemma:** Let g and h be continuous functions on the interval $[a,b]$ with $g(t) \geq 0$. Then there is a number c in $[a,b]$ with

$$\int_a^b h(t)g(t)\,dt = h(c)\int_a^b g(t)\,dt$$

Proof: We note first that if $g(t) = 0$ for all t in $[a,b]$, then the result is clearly true. We can thus assume that there is some t in $[a,b]$ for which $g(t) > 0$. Because g is continuous, there must be an interval about this t on which g is strictly positive, so we may assume that

$$\int_a^b g(t)\,dt > 0$$

Because h is continuous on $[a,b]$, the Extreme Value Theorem shows that h has an absolute minimum value m and an absolute maximum value M on the interval $[a,b]$. Thus

$$m \leq h(t) \leq M$$

for all t in $[a,b]$, so

$$m\int_a^b g(t)\,dt \leq \int_a^b h(t)g(t)\,dt \leq M\int_a^b g(t)\,dt$$

Because $\int_a^b g(t)\,dt > 0$, we have

$$m \leq \frac{\int_a^b h(t)g(t)\,dt}{\int_a^b g(t)\,dt} \leq M.$$

Now there are points in $[a,b]$ at which $h(t)$ equals m and M, so the Intermediate Value Theorem shows that there is a point c in $[a,b]$ at which

$$h(c) = \frac{\int_a^b h(t)g(t)\,dt}{\int_a^b g(t)\,dt}$$

or

$$\int_a^b h(t)g(t)\,dt = h(c)\int_a^b g(t)\,dt.$$

Applying the lemma with $h(t) = \dfrac{f^{(n+1)}(t)}{n!}$, $g(t) = (x-t)^n$, we see that $R_n = \dfrac{f^{(n+1)}(c)}{n!}\displaystyle\int_a^x (x-t)^n\,dt = \dfrac{f^{(n+1)}(c)}{n!} \cdot \dfrac{1}{n+1}(x-a)^{n+1} = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some $c \in [a,b]$.

11.1.83

a. The slope of the tangent line to $f(x)$ at $x = a$ is by definition $f'(a)$; by the point-slope form for the equation of a line, we have $y - f(a) = f'(a)(x - a)$, or $y = f(a) + f'(a)(x - a)$.

b. The Taylor polynomial centered at a is $p_1(x) = f(a) + f'(a)(x - a)$, which is the tangent line at a.

11.1.84

a. $p_2(x) = f(a) + f'(a)(x - a) + \dfrac{f''(a)}{2}(x - a)^2$, so that $p_2'(x) = f'(a) + f''(a)(x - a)$ and $p_2''(x) = f''(a)$. If f has a local maximum at a, then $f'(a) = 0, f''(a) < 0$, but then $p_2'(a) = 0$ and $p_2''(a) < 0$ by the above, so that $p_2(x)$ also has a local maximum at a.

b. Similarly, if f has a local minimum at a, then $f'(a) = 0, f''(a) > 0$, but then $p_2'(a) = 0$ and $p_2''(a) > 0$ by the above, so that $p_2(x)$ also has a local minimum at a.

c. Recall that f has an inflection point at a if the second derivative of f changes sign at a. But $p_2''(x)$ is a constant, so p_2 does not have an inflection point at a (or anywhere else).

d. No. For example, let $f(x) = x^3$. Then $p_2(x) = 0$, so that the second-order Taylor polynomial has a local maximum at $x = 0$, but $f(x)$ does not. It also has a local minimum at $x = 0$, but $f(x)$ does not.

11.1.85

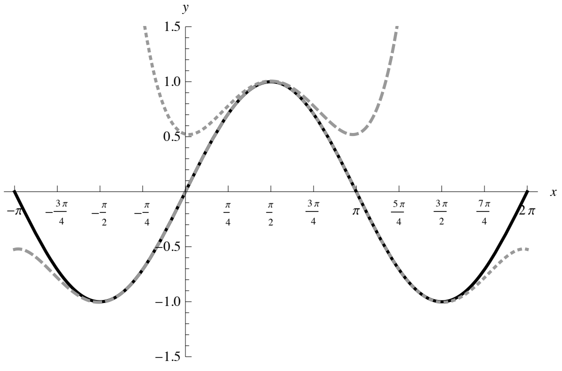
a. We have

$$\begin{aligned} f(0) &= f^{(4)}(0) = \sin 0 = 0 & f(\pi) &= f^{(4)}(\pi) = \sin\pi = 0 \\ f'(0) &= f^{(5)}(0) = \cos 0 = 1 & f'(\pi) &= f^{(5)}(0) = \cos\pi = -1 \\ f''(0) &= -\sin 0 = 0 & f''(\pi) &= -\sin\pi = 0 \\ f'''(0) &= -\cos 0 = -1 & f'''(\pi) &= -\cos\pi = 1. \end{aligned}$$

Thus

$$\begin{aligned} p_5(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \\ q_5(x) &= -(x - \pi) + \frac{1}{3!}(x - \pi)^3 - \frac{1}{5!}(x - \pi)^5. \end{aligned}$$

b. A plot of the three functions, with $\sin x$ the black solid line, $p_5(x)$ the dashed line, and $q_5(x)$ the dotted line is below.



$p_5(x)$ and $\sin x$ are almost indistinguishable on $[-\pi/2, \pi/2]$, after which $p_5(x)$ diverges pretty quickly from $\sin x$. $q_5(x)$ is reasonably close to $\sin x$ over the entire range, but the two are almost indistinguishable on $[\pi/2, 3\pi/2]$. $p_5(x)$ is a better approximation than $q_5(x)$ on about $[-\pi, \pi/2)$, while $q_5(x)$ is better on about $(\pi/2, 2\pi]$.

c. Evaluating the errors gives

x	$\lvert\sin x - p_5(x)\rvert$	$\lvert\sin x - q_5(x)\rvert$
$\frac{\pi}{4}$	3.6×10^{-5}	7.4×10^{-2}
$\frac{\pi}{2}$	4.5×10^{-3}	4.5×10^{-3}
$\frac{3\pi}{4}$	7.4×10^{-2}	3.6×10^{-5}
$\frac{5\pi}{4}$	2.3	3.6×10^{-5}
$\frac{7\pi}{4}$	20.4	7.4×10^{-2}

d. $p_5(x)$ is a better approximation than $q_5(x)$ only at $x = \frac{\pi}{4}$, in accordance with part (b). The two are equal at $x = \frac{\pi}{2}$, after which $q_5(x)$ is a substantially better approximation than $p_5(x)$.

11.1.86

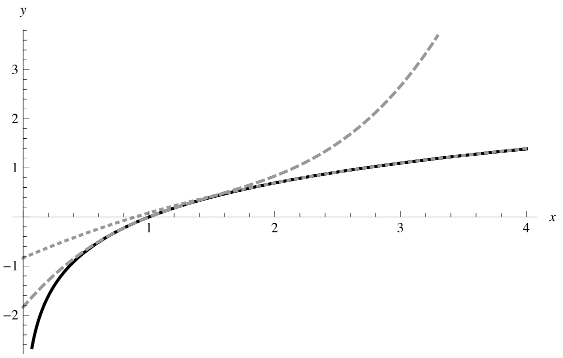
a. We have

$$\begin{aligned} f(1) &= \ln 1 = 0 & f(e) &= \ln e = 1 \\ f'(1) &= 1 & f'(e) &= \frac{1}{3} \\ f''(1) &= -1 & f''(e) &= -\frac{1}{e^2} \\ f'''(1) &= 2 & f'''(e) &= \frac{2}{e^3}. \end{aligned}$$

Thus

$$\begin{aligned} p_3(x) &= (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \\ q_3(x) &= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3. \end{aligned}$$

b. A plot of the three functions, with $\ln x$ the black solid line, $p_3(x)$ the dashed line, and $q_3(x)$ the dotted line is below.



c. Evaluating the errors gives

x	$\lvert\ln x - p_3(x)\rvert$	$\lvert\ln x - q_3(x)\rvert$
0.5	2.6×10^{-2}	3.6×10^{-1}
1.0	0	8.4×10^{-2}
1.5	1.1×10^{-2}	1.6×10^{-2}
2.0	1.4×10^{-1}	1.5×10^{-3}
2.5	5.8×10^{-1}	1.1×10^{-5}
3.0	1.6	2.7×10^{-5}
3.5	3.3	1.4×10^{-3}

d. $p_3(x)$ is a better approximation than $q_3(x)$ for $x = 0.5$, 1.0, and 1.5, and $q_3(x)$ is a better approximation for the other points. To see why this is true, note that on $[0.5, 4]$ that $f^{(4)}(x) = -\frac{6}{x^4}$ is bounded in magnitude by $\frac{6}{0.5^4} = 96$, so that (using P_3 for the error term for p_3 and Q_3 as the error term for q_3)

$$P_3(x) \le 96 \cdot \frac{|x-1|^4}{4!} = 4\,|x-1|^4\,, \qquad Q_3(x) \le 96 \cdot \frac{|x-e|^4}{4!} = 4\,|x-e|^4\,.$$

Thus the relative sizes of $P_3(x)$ and $Q_3(x)$ are governed by the distance of x from 1 and e. Looking at the different possibilities for x reveals why the results in part (c) hold.

11.1.87

a. We have

$$f(36) = \sqrt{36} = 6 \qquad f(49) = \sqrt{49} = 7$$
$$f'(36) = \frac{1}{2} \cdot \frac{1}{\sqrt{36}} = \frac{1}{12} \qquad f'(49) = \frac{1}{2} \cdot \frac{1}{\sqrt{49}} = \frac{1}{14}.$$

Thus

$$p_1(x) = 6 + \frac{1}{12}(x-36) \qquad q_1(x) = 7 + \frac{1}{14}(x-49).$$

b. Evaluating the errors gives

x	$\lvert\sqrt{x} - p_1(x)\rvert$	$\lvert\sqrt{x} - q_1(x)\rvert$
37	5.7×10^{-4}	6.0×10^{-2}
39	5.0×10^{-3}	4.1×10^{-2}
41	1.4×10^{-2}	2.5×10^{-2}
43	2.6×10^{-2}	1.4×10^{-2}
45	4.2×10^{-2}	6.1×10^{-3}
47	6.1×10^{-2}	1.5×10^{-3}

c. $p_1(x)$ is a better approximation than $q_1(x)$ for $x \le 41$, and $q_1(x)$ is a better approximation for $x \ge 43$. To see why this is true, note that $f''(x) = -\frac{1}{4}x^{-3/2}$, so that on $[36, 49]$ it is bounded in magnitude by $\frac{1}{4} \cdot 36^{-3/2} = \frac{1}{864}$. . Thus (using P_1 for the error term for p_1 and Q_1 for the error term for q_1)

$$P_1(x) \le \frac{1}{864} \cdot \frac{|x-36|^2}{2!} = \frac{1}{1728}(x-36)^2, \qquad Q_1(x) \le \frac{1}{864} \cdot \frac{|x-49|^2}{2!} = \frac{1}{1728}(x-49)^2.$$

It follows that the relative sizes of $P_1(x)$ and $Q_1(x)$ are governed by the distance of x from 36 and 49. Looking at the different possibilities for x reveals why the results in part (b) hold.

11.1.88

a. The quadratic Taylor polynomial for $\sin x$ centered at $\frac{\pi}{2}$ is

$$\begin{aligned} p_2(x) &= \sin\frac{\pi}{2} + \cos\frac{\pi}{2}\cdot\left(x-\frac{\pi}{2}\right) - \frac{1}{2}\sin\frac{\pi}{2}\cdot\left(x-\frac{\pi}{2}\right)^2 \\ &= 1 - \frac{1}{2}\left(x-\frac{\pi}{2}\right)^2 \\ &= -\frac{1}{2}x^2 + \frac{\pi}{2}x + 1 - \frac{\pi^2}{8}. \end{aligned}$$

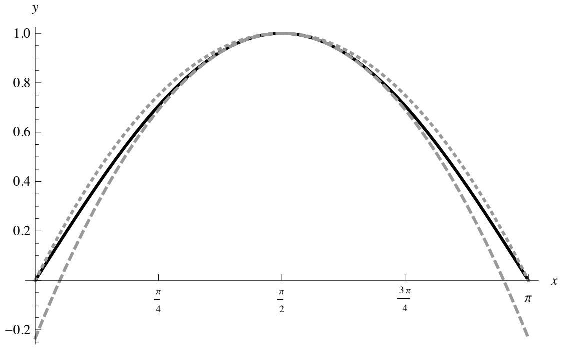
b. Let $q(x) = ax^2 + bx + c$. Because $q(0) = \sin 0 = 0$, we must have $c = 0$, so that $q(x) = ax^2 + bx$. Then the other two conditions give us a pair of linear equation in a and b:

$$\begin{aligned} \frac{\pi^2}{4}a + \frac{\pi}{2}b &= 1 \\ \pi^2 a + \pi b &= 0 \end{aligned}$$

where the first equation comes from the fact that $q(\pi/2) = \sin(\pi/2) = 1$ and the second from the fact that $q(\pi) = \sin\pi = 0$. Solving the linear system of equations gives $b = \frac{4}{\pi}$ and $a = -\frac{4}{\pi^2}$, so that

$$q(x) = -\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x.$$

c. A plot of the three function, with $\sin x$ the black solid line, $p_2(x)$ the dashed line, and $q(x)$ the dotted line is below.



d. Evaluating the errors gives

x	$\lvert\sin x - p_2(x)\rvert$	$\lvert\sin x - q(x)\rvert$
$\frac{\pi}{4}$	1.6×10^{-2}	4.3×10^{-2}
$\frac{\pi}{2}$	0	0
$\frac{3\pi}{4}$	1.6×10^{-2}	4.3×10^{-2}
π	2.3×10^{-1}	0

e. q is a better approximation than p at $x = \pi$, and the two are equal at $x = \frac{\pi}{2}$. At the other two points, however, $p_2(x)$ is a better approximation than $q(x)$. Clearly $q(x)$ will be exact at $x = 0$, $x = \frac{\pi}{2}$, and $x = \pi$, because it was chosen that way. Also clearly $p_2(x)$ will be exact at $x = \frac{\pi}{2}$ since it is the Taylor polynomial centered at $\frac{\pi}{2}$. The fact that $p_2(x)$ is a better approximation than $q(x)$ at the two intermediate points is a result of the way the polynomials were constructed: the goal of $p_2(x)$ was to be as good an approximation as possible near $x = \frac{\pi}{2}$, while the goal of $q(x)$ was to match $\sin x$ at three given points. Overall, it appears that $q(x)$ does a better job over the full range (the total area between $q(x)$ and $\sin x$ is certainly smaller than the total area between $p_2(x)$ and $\sin x$).

11.2 Properties of Power Series

11.2.1 $c_0 + c_1 x + c_2 x^2 + c_3 x^3$.

11.2.2 Yes. Note that we can write the given series as $\displaystyle\sum_{k=0}^{\infty} 5^k (x-4)^k$, which is a power series with $c_k = 5^k$ and $a = 4$.

11.2.3 Generally the Ratio Test or Root Test is used.

11.2.4 Yes. It is a power series where the center $a = 0$ and $c_k = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$

11.2.5 The radius of convergence does not change, but the interval of convergence may change at the endpoints.

11.2.6 The radius is 4 and the interval is $(-1, 7)$.

11.2.7 The series converges for $|x-2| \le 10$, so the radius of convergence is 10 and the interval is $[-8, 12]$.

11.2.8 The radius is $\dfrac{7-(-3)}{2} = 5$, and the center is then $7 - 5 = 2$.

11.2.9 Using the Root Test: $\displaystyle\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} |2x| = |2x|$. So the radius of convergence is $\dfrac{1}{2}$. At $x = 1/2$ the series is $\displaystyle\sum_{k=0}^{\infty} 1$ which diverges, and at $x = -1/2$ the series is $\displaystyle\sum_{k=0}^{\infty} (-1)^k$ which also diverges. So the interval of convergence is $(-1/2, 1/2)$

11.2.10 Using the Ratio Test: $\displaystyle\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(x-1)^{k+1}}{(k+1)!} \cdot \frac{k!}{(x-1)^k}\right| = \lim_{k\to\infty} \left|\frac{x-1}{k+1}\right| = 0$. Thus the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

11.2.11 Using the Ratio Test: $\displaystyle\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)^{k+1} x^{k+1}}{k^k x^k}\right| = \lim_{k\to\infty} (k+1) \left(\frac{k+1}{k}\right)^k |x| = \infty$ (for $x \ne 0$) because $\displaystyle\lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k = e$. Thus, the radius of convergence is 0 and the series only converges at $x = 0$.

11.2.12 Using the Ratio Test: $\displaystyle\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)!(x-10)^{k+1}}{k!(x-10)^k}\right| = \lim_{k\to\infty} (k+1)\,|x-10| = \infty$ (for $x \ne 10$). Thus, the radius of convergence is 0 and the series only converges at $x = 10$.

11.2.13 Using the Root Test: $\displaystyle\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sin(1/k)|x| = \sin(0)|x| = 0$. Thus, the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

11.2.14 Using the Root Test: $\displaystyle\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \frac{2\,|x-3|}{k^{1/k}} = 2\,|x-3|$. Thus, the radius of convergence is $1/2$. When $x = 7/2$, we have the harmonic series (which diverges), and when $x = 5/2$, we have the alternating harmonic series which converges. The interval of convergence is thus $[5/2, 7/2)$.

11.2.15 Using the Root Test: $\displaystyle\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \frac{|x|}{3} = \frac{|x|}{3}$, so the radius of convergence is 3. At $x = -3$, the series is $\sum (-1)^k$, which diverges. At $x = 3$, the series is $\sum 1$, which diverges. So the interval of convergence is $(-3, 3)$.

11.2.16 Using the Root Test: $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \frac{|x|}{5} = \frac{|x|}{5}$, so the radius of convergence is 5. At $x = 5$, we obtain $\sum (-1)^k$ which diverges. At $x = -5$, we have $\sum 1$, which also diverges. So the interval of convergence is $(-5, 5)$.

11.2.17 Using the Root Test: $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \frac{|x|}{k} = 0$, so the radius of convergence is infinite and the interval of convergence is $(-\infty, \infty)$.

11.2.18 We use the Ratio Test. The limit of the absolute value of the ratio of successive terms is

$$\lim_{k\to\infty} |x|\sqrt{\frac{k}{k+1}} = |x|.$$

This is less than 1 for $|x| < 1$, so the radius of convergence is 1. At $x = -1$ we have the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ which is a divergent p-series, and for $x = 1$ we have the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ which is a convergent alternating series. The interval of convergence is $(-1, 1]$.

11.2.19 We use the Ratio Test. The limit of the absolute value of the ratio of successive terms is

$$\lim_{k\to\infty} |x|\frac{2^k+1}{2^{k+1}+1} = |x| \lim_{k\to\infty} \frac{1+1/2^k}{2+1/2^k} = \frac{|x|}{2},$$

so the radius of convergence is 2. For $x = 2$, we have the series $\sum_{k=1}^{\infty} \frac{2^k}{2^k+1}$ which diverges by the Divergence Test because $\lim_{k\to\infty} \frac{2^k}{2^k+1} = 1 \neq 0$. Similarly, the series for $x = -2$ fails the divergence test because the kth term does not have limit 0 as $k \to \infty$. So the interval of convergence is $(-2, 2)$.

11.2.20 Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(2x)^{k+1}}{(k+1)!} \cdot \frac{k!}{(2x)^k}\right| = \lim_{k\to\infty} \left|\frac{2x}{k+1}\right| = 0$. So the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

11.2.21 The given series is $\sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{k!}$. Note that the limit of the absolute value of successive terms is

$$\lim_{k\to\infty} \frac{x^2}{k+1} = 0,$$

which is less than 1 for all x. So the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

11.2.22 The given series is $\sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{k^2}$. Note that the limit of the absolute value of successive terms is

$$\lim_{k\to\infty} \frac{x^2 k^2}{(k+1)^2} = x^2,$$

which is less than 1 for $-1 < x < 1$. So the radius of convergence is 1. For $x = 1$ we have the convergent alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ and for $x = -1$ we have its opposite (which also converges), so the interval of convergence is $[-1, 1]$.

11.2.23 We use the Root Test. The limit of the kth root of the absolute value of the kth term is

$$\lim_{k\to\infty} \frac{|x-1|}{k^{1/k}} = |x-1|.$$

This is less than 1 for $0 < x < 2$. The radius of convergence is 1. For $x = 0$, we have -1 times the Harmonic Series (which diverges) and for $x = 2$ we have the Alternating Harmonic series which converges by the Alternating Series Test. Therefore, the interval of convergence is $(0, 2]$.

11.2.24 Using the Ratio Test:

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\left(\frac{(k+1)(x-4)^{k+1}}{2^{k+1}} \cdot \frac{2^k}{k(x-4)^k}\right)\right| = \lim_{k\to\infty} \left(\frac{k+1}{k} \cdot \frac{|x-4|}{2}\right) = \frac{|x-4|}{2},$$

so that the radius of convergence is 2. The interval is $(2, 6)$, because at the left endpoint, the series becomes $\sum_{k=1}^{\infty} k$ (which diverges) and at the right endpoint, it becomes $\sum_{k=1}^{\infty} (-1)^k k$ (which diverges).

11.2.25 Using the Ratio Test:

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} |4x-1| \frac{k^2+4}{k^2+2k+5} = |4x-1|.$$

This is less than 1 for $0 < x < \frac{1}{2}$. The radius of convergence is $\frac{1}{4}$. For $x = \frac{1}{2}$ we obtain the convergent series $\sum_{k=0}^{\infty} \frac{1}{k^2+4}$, which converges because $\frac{1}{k^2+4} < \frac{1}{k^2}$, and $\sum_{k=0}^{\infty} \frac{1}{k^2}$ is a convergent p-series. For $x = 0$, we obtain the alternating version of that same series, which is clearly absolutely convergent and therefore convergent. The interval of convergence is $\left[0, \frac{1}{2}\right]$.

11.2.26 We use the Root Test. The limit of the kth root of the absolute value of the kth term is

$$\lim_{k\to\infty} |3x+2| \frac{1}{k^{1/k}} = |3x+2|.$$

This is less than 1 for $-1 < 3x + 2 < 1$, or $-1 < x < -\frac{1}{3}$. The radius of convergence is $\frac{1}{3}$. At $x = -\frac{1}{3}$, the series is the divergent Harmonic series, while at $x = -1$ the series is the convergent Alternating Harmonic series. The interval of convergence is therefore $\left[-1, -\frac{1}{3}\right)$.

11.2.27 We use the Root Test. The limit of the kth root of the absolute value of the kth term is

$$\lim_{k\to\infty} |2x-4| \frac{k^{10/k}}{10} = \frac{|2x-4|}{10}.$$

This is less than one for $-10 < 2x - 4 < 10$, or $-3 < x < 7$. The radius of convergence is therefore 5. For the endpoints, the series diverges by the Divergence Test. The interval of convergence is thus $(-3, 7)$.

11.2.28 We use the Ratio Test. The limit of the absolute value of the ratio of the $k+1$st term to the kth term is

$$\lim_{k\to\infty} |x+3| \frac{k}{k+1} \cdot \frac{\ln^2 k}{\ln^2(k+1)} = |x+3| \cdot 1 \cdot \left(\lim_{k\to\infty} \frac{\ln^2 k}{\ln^2(k+1)}\right)^2 = |x+3| \left(\lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{1}{k+1}}\right)^2 = |x+3|.$$

This is less than 1 for $-1 < x + 3 < 1$, or $-4 < x < -2$. The radius of convergence is 1. For $x = -2$, the resulting series is $\sum_{k=2}^{\infty} \frac{1}{k \ln^2 k}$ which can be shown to converge using the Integral Test. For $x = -4$, the series is the alternating version of the one for $x = -2$, so it converges absolutely. The interval of convergence is therefore $[-4, -2]$.

11.2.29 Using the Ratio Test: $\lim_{k\to\infty}\left|\frac{(k+1)^2x^{2k+2}}{(k+1)!}\cdot\frac{k!}{k^2x^{2k}}\right| = \lim_{k\to\infty}\frac{k+1}{k^2}x^2 = 0$, so the radius of convergence is infinite, and the interval of convergence is $(-\infty,\infty)$.

11.2.30 Using the Root Test: $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}k^{1/k}|x-1| = |x-1|$. The radius of convergence is therefore 1. At both $x=2$ and $x=0$ the series diverges by the Divergence Test. The interval of convergence is therefore $(0,2)$.

11.2.31 Using the Ratio Test: $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{2k+3}}{3^k}\cdot\frac{3^{k-1}}{x^{2k+1}}\right| = \frac{x^2}{3}$ so that the radius of convergence is $\sqrt{3}$. At $x=\sqrt{3}$, the series is $\sum_{k=1}^{\infty}3\sqrt{3}$, which diverges. At $x=-\sqrt{3}$, the series is $\sum_{k=1}^{\infty}(-3\sqrt{3})$, which also diverges, so the interval of convergence is $(-\sqrt{3},\sqrt{3})$.

11.2.32 $\sum_{k=0}^{\infty}\left(\frac{-x}{10}\right)^{2k} = \sum_{k=0}^{\infty}\left(\frac{x^2}{100}\right)^k$. Using the Root Test: $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\frac{x^2}{100} = \frac{x^2}{100}$, so that the radius of convergence is 10. At $x=\pm 10$, the series is then $\sum_{k=0}^{\infty}1$, which diverges, so the interval of convergence is $(-10,10)$.

11.2.33 Using the Root Test: $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\frac{(|x-1|)k}{k+1} = |x-1|$, so the series converges when $|x-1|<1$, so for $0<x<2$. The radius of convergence is 1. At $x=2$, the series diverges by the Divergence Test. At $x=0$, the series diverges as well by the Divergence Test. Thus the interval of convergence is $(0,2)$.

11.2.34 Using the Ratio Test:

$$\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|} = \left|\frac{(-2)^{k+1}(x+3)^{k+1}}{3^{k+2}}\cdot\frac{3^{k+1}}{(-2)^k(x+3)^k}\right| = \frac{2}{3}|x+3|.$$

Thus the series converges when $\frac{2}{3}|x+3|<1$, or $-\frac{9}{2}<x<-\frac{3}{2}$. At $x=-\frac{9}{2}$, the series diverges by the Divergence Test. At $x=-\frac{3}{2}$, the series diverges by the Divergence Test. Thus the interval of convergence is $\left(-\frac{9}{2},-\frac{3}{2}\right)$.

11.2.35 Using the Ratio Test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)^{20}x^{k+1}}{(2k+3)!}\cdot\frac{(2k+1)!}{x^kk^{20}}\right| = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^{20}\frac{|x|}{(2k+2)(2k+3)} = 0,$$

so the radius of convergence is infinite, and the interval of convergence is $(-\infty,\infty)$.

11.2.36 Using the Root Test: $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\frac{|x^3|}{27} = \frac{|x^3|}{27}$, so the radius of convergence is 3. The series is divergent by the Divergence Test for $x=\pm 3$, so the interval of convergence is $(-3,3)$.

11.2.37 Using the Ratio Test: $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)!\,x^{k+1}}{(k+1)^{k+1}}\cdot\frac{k^k}{k!\,x^k}\right| = \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k|x| = \frac{1}{e}|x|$. The radius of convergence is therefore e.

11.2.38 Using the Root Test: $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k|x| = ex$. Thus, the radius of convergence is $\frac{1}{e}$.

11.2.39 Using the Root Test:

$$\lim_{k\to\infty}\sqrt[k]{|a_k|}=|x|\lim_{k\to\infty}\left(\frac{k}{k+4}\right)^k=|x|\lim_{k\to\infty}\frac{1}{\left(1+\frac{4}{k}\right)^k}=\frac{|x|}{e^4}.$$

This is less than 1 for $|x|<e^4$, so the radius of convergence is e^4.

11.2.40 We use the Ratio Test. We have

$$\begin{aligned}\lim_{k\to\infty}\frac{a_{k+1}}{a_k}&=|x|\lim_{k\to\infty}\frac{1-\cos\frac{1}{2^{k+1}}}{1-\cos\frac{1}{2^k}}\\&=|x|\lim_{k\to\infty}\frac{(\sin 2^{-k-1})2^{-k-1}(-1)\ln 2}{(\sin 2^{-k})2^{-k}(-1)\ln 2}\\&=|x|\lim_{k\to\infty}\frac{\frac{\sin 2^{-k-1}}{2^{-k-1}}2^{-2k-2}}{\frac{\sin 2^{-k}}{2^{-k}}2^{-2k}}\\&=\frac{|x|}{4},\end{aligned}$$

where we used the fact that $\lim_{z\to 0}\frac{\sin z}{z}=1$. The expression $\frac{|x|}{4}$ is less than 1 for $|x|<4$, so the radius of convergence is 4.

11.2.41 $f(3x)=\frac{1}{1-3x}=\sum_{k=0}^{\infty}3^kx^k$, which converges for $|x|<1/3$, and diverges at the endpoints.

11.2.42 $g(x)=\frac{x^3}{1-x}=\sum_{k=0}^{\infty}x^{k+3}$, which converges for $|x|<1$ and is divergent at the endpoints.

11.2.43 $h(x)=\frac{2x^3}{1-x}=\sum_{k=0}^{\infty}2x^{k+3}$, which converges for $|x|<1$ and is divergent at the endpoints.

11.2.44 $f(x^3)=\frac{1}{1-x^3}=\sum_{k=0}^{\infty}x^{3k}$. By the Root Test, $\lim_{k\to\infty}\sqrt[k]{|a_k|}=\left|x^3\right|$, so this series also converges for $|x|<1$. It is divergent at the endpoints.

11.2.45 $p(x)=\frac{4x^{12}}{1-x}=\sum_{k=0}^{\infty}4x^{k+12}=4\sum_{k=0}^{\infty}x^{k+12}$, which converges for $|x|<1$. It is divergent at the endpoints.

11.2.46 $f(-4x)=\frac{1}{1+4x}=\sum_{k=0}^{\infty}(-4x)^k=\sum_{k=0}^{\infty}(-1)^k4^kx^k$, which converges for $|x|<1/4$ and is divergent at the endpoints.

11.2.47 $f(3x)=\ln(1-3x)=-\sum_{k=1}^{\infty}\frac{(3x)^k}{k}=-\sum_{k=1}^{\infty}\frac{3^k}{k}x^k$ for $-1\le 3x<1$, or $-\frac{1}{3}\le x<\frac{1}{3}$. The radius of convergence is $\frac{1}{3}$. The series diverges at $\frac{1}{3}$ (harmonic series), and converges at $-\frac{1}{3}$ (alternating harmonic series). The interval of convergence is $\left[-\frac{1}{3},\frac{1}{3}\right)$.

11.2.48 $g(x)=x^3\ln(1-x)=-\sum_{k=1}^{\infty}\frac{x^{k+3}}{k}$ for $-1\le x<1$, so the radius of convergence is 1. The series diverges at 1 and converges at -1. The interval of convergence is $[-1,1)$.

11.2.49 $p(x) = 2x^6\ln(1-x) = -2\sum_{k=1}^{\infty}\frac{x^{k+6}}{k}$ for $-1 \le x < 1$, so the radius of convergence is 1. The series diverges at 1 (harmonic series) but converges at -1 (alternating harmonic series), so the interval of convergence is $[-1, 1)$.

11.2.50 $f(x^3) = \ln(1-x^3) = -\sum_{k=1}^{\infty}\frac{x^{3k}}{k}$ for $-1 \le x^3 < 1$ or $-1 \le x < 1$, so the radius of convergence is 1. The series diverges at 1 (harmonic series) but converges at -1 (alternating harmonic series), so the interval of convergence is $[-1, 1)$.

11.2.51 The power series for $f(x)$ is $\sum_{k=0}^{\infty}(2x)^k$, which is convergent for $-1 < 2x < 1$, so for $-1/2 < x < 1/2$. The power series for $g(x) = f'(x)$ is $\sum_{k=1}^{\infty}k(2x)^{k-1}\cdot 2 = 2\sum_{k=1}^{\infty}k(2x)^{k-1}$, also convergent on $|x| < 1/2$. At the endpoints the series diverges by the Divergence Test. The interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

11.2.52 The power series for $f(x)$ is $\sum_{k=0}^{\infty}x^k$, which is convergent for $-1 < x < 1$, so the power series for $g(x) = \frac{1}{2}f''(x)$ is $\frac{1}{2}\sum_{k=2}^{\infty}k(k-1)x^{k-2} = \frac{1}{2}\sum_{k=0}^{\infty}(k+1)(k+2)x^k$, which is also convergent on $|x| < 1$.The series diverges at the endpoints by the Divergence Test. The interval of convergence is $(-1, 1)$.

11.2.53 $f(x) = \frac{1}{1+x} = \sum_{k=0}^{\infty}(-x)^k$ for $-1 < -x < 1$, or $-1 < x < 1$. So

$$g(x) = -\frac{1}{(1+x)^2} = \sum_{k=1}^{\infty}-k(-x)^{k-1} = \sum_{k=1}^{\infty}(-1)^k kx^{k-1},$$

for $-1 < x < 1$. At each endpoint $x = \pm 1$, the series diverges by the Divergence Test. So the interval of convergence is $(-1, 1)$.

11.2.54 The power series for $f(x)$ is $\sum_{k=0}^{\infty}(-1)^k x^{2k}$, which is convergent on $|x| < 1$. Because $g(x) = -\frac{1}{2}f'(x)$, the power series for g is $-\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k 2kx^{2k-1} = \sum_{k=1}^{\infty}(-1)^{k+1}kx^{2k-1}$, which is also convergent on $|x| < 1$. The series diverges at the endpoints by the Divergence Test, so the interval of convergence is $(-1, 1)$.

11.2.55 The power series for $\frac{1}{1-3x}$ is $\sum_{k=0}^{\infty}(3x)^k$, which is convergent on $|x| < 1/3$. Because $g(x) = \ln(1 - 3x) = -3\int\frac{1}{1-3x}\,dx$ and because $g(0) = 0$, the power series for $g(x)$ is $-3\sum_{k=0}^{\infty}3^k\frac{1}{k+1}x^{k+1} = -\sum_{k=1}^{\infty}\frac{3^k}{k}x^k$. The series converges at $x = -\frac{1}{3}$ (alternating harmonic) and diverges at $x = \frac{1}{3}$ (harmonic series). The interval of convergence is $\left[-\frac{1}{3}, \frac{1}{3}\right)$.

11.2.56 The power series for $\frac{x}{1+x^2}$ is $x\sum_{k=0}^{\infty}(-1)^k x^{2k} = \sum_{k=0}^{\infty}(-1)^k x^{2k+1}$, which is convergent on $|x| < 1$. Because $g(x) = 2\int f(x)\,dx$, and because $g(0) = 0$, the power series for $g(x)$ is $2\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+2}x^{2k+2} =$

$\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{1}{k+1}x^{2k+2}$. This can be written as $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}x^{2k}$. At both endpoints, we have the alternating harmonic series which converges, so the interval of convergence is $[-1,1]$.

11.2.57 Recall that $\dfrac{d}{dx}\left(\dfrac{1}{1+x^2}\right)=-\dfrac{2x}{(1+x^2)^2}$, and by Example 3c, $\dfrac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$. So

$$-\frac{2x}{(1+x^2)^2}=-\frac{d}{dx}(1-x^2+x^4-x^6+\cdots)=2x-4x^3+6x^5-\cdots=\sum_{k=1}^{\infty}(-1)^{k+1}2kx^{2k-1},$$

for $|x|<1$. Checking the endpoints for $x=\pm1$ reveals that the series diverges by the Divergence Test.

11.2.58 Start with $g(x)=\dfrac{1}{1-x}$. The power series for $g(x)$ is $\displaystyle\sum_{k=0}^{\infty}x^k$. Because $f(x)=g(x^4)$, its power series is $\displaystyle\sum_{k=0}^{\infty}x^{4k}$. The radius of convergence is still 1, and the series is divergent at both endpoints. The interval of convergence is $(-1,1)$.

11.2.59 Note that $f(x)=\dfrac{3}{3+x}=\dfrac{1}{1+(1/3)x}$. Let $g(x)=\dfrac{1}{1+x}$. The power series for $g(x)$ is $\displaystyle\sum_{k=0}^{\infty}(-1)^kx^k$, so the power series for $f(x)=g((1/3)x)$ is $\displaystyle\sum_{k=0}^{\infty}(-1)^k3^{-k}x^k=\sum_{k=0}^{\infty}\left(\frac{-x}{3}\right)^k$. Using the Ratio Test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\left|\frac{3^{-(k+1)}x^{k+1}}{3^{-k}x^k}\right|=\frac{|x|}{3},$$

so the radius of convergence is 3. The series diverges at both endpoints by the Divergence test. The interval of convergence is $(-3,3)$.

11.2.60 Note that $f(x)=\dfrac{1}{2}\ln(1-x^2)$. The power series for $g(x)=\ln(1-x)$ is $\displaystyle-\sum_{k=1}^{\infty}\frac{1}{k}x^k$, so the power series for $f(x)=\dfrac{1}{2}g(x^2)$ is $\displaystyle-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}x^{2k}$. The radius of convergence is still 1. The series diverges at both 1 and -1, its interval of convergence is $(-1,1)$.

11.2.61 Note that $f(x)=\ln\sqrt{4-x^2}=\dfrac{1}{2}\ln(4-x^2)=\dfrac{1}{2}\left(\ln 4+\ln\left(1-\dfrac{x^2}{4}\right)\right)=\ln 2+\dfrac{1}{2}\ln\left(1-\dfrac{x^2}{4}\right)$. Now, the power series for $g(x)=\ln(1-x)$ is $\displaystyle-\sum_{k=1}^{\infty}\frac{1}{k}x^k$, so the power series for $f(x)$ is $\displaystyle\ln 2-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\cdot\frac{x^{2k}}{4^k}=\ln 2-\sum_{k=1}^{\infty}\frac{x^{2k}}{k2^{2k+1}}$. Now, $\displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\left|\frac{x^{2k+2}}{(k+1)2^{2k+3}}\cdot\frac{k2^{2k+1}}{x^{2k}}\right|=\lim_{k\to\infty}\frac{k}{4(k+1)}x^2=\frac{x^2}{4}$, so that the radius of convergence is 2. The series diverges at both endpoints, so its interval of convergence is $(-2,2)$.

11.2.62 By Example 5, the Taylor series for $g(x)=\tan^{-1}x$ is $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k+1}}{2k+1}$, so that $f(x)=g((2x)^2)$ has Taylor series $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k(2x)^{4k+2}}{2k+1}=\sum_{k=0}^{\infty}\frac{(-1)^k4^{2k+1}}{2k+1}x^{4k+2}$. Using the Ratio Test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\left|\frac{4^{2k+3}x^{4k+6}}{2k+3}\cdot\frac{2k+1}{4^{2k+1}x^{4k+2}}\right|=\lim_{k\to\infty}\frac{16(2k+1)}{2k+3}x^4=16x^4,$$

so that the radius of convergence is 1/2. The interval of convergence is $(-1/2,1/2)$.

11.2.63

a. True. This power series is centered at $x = 3$, so its interval of convergence will be symmetric about 3.

b. True. Use the Root Test. At the endpoints, the series diverges by the Divergence Test.

c. True. Substitute x^2 for x in the series.

d. True. Because the power series is zero on the interval, all its derivatives are as well, which implies (differentiating the power series) that all the c_k are zero.

11.2.64 The power series for $f(ax)$ is $\sum c_k(ax)^k$. Then $\sum c_k(ax)^k$ converges if and only if $|ax| < R$ (because $\sum c_k x^k$ converges for $|x| < R$), which happens if and only if $|x| < \frac{R}{|a|}$.

11.2.65 The power series for $f(x - a)$ is $\sum c_k(x - a)^k$. Then $\sum c_k(x - a)^k$ converges if and only if $|x - a| < R$, which happens if and only if $a - R < x < a + R$, so the radius of convergence is the same.

11.2.66 Replacing x by $x - 1$ gives $\ln x = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k}$. Using the Ratio Test: $\displaystyle\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(x-1)^{k+1}}{k+1} \cdot \frac{k}{(x-1)^k}\right| = \lim_{k\to\infty} \frac{k}{k+1} |x-1| = |x-1|$, so that the series converges for $|x - 1| < 1$. Checking the endpoints, the interval of convergence is $(0, 2]$.

11.2.67 This is a geometric series with ratio $\sqrt{x} - 2$, so its sum is $\dfrac{1}{1-(\sqrt{x}-2)} = \dfrac{1}{3-\sqrt{x}}$, for $-1 < \sqrt{x} - 2 < 1$. Simplifying the inequality, we have $1 < \sqrt{x} < 3$, or $1 < x < 9$.

11.2.68 Note that $\displaystyle\sum_{k=1}^{\infty} \frac{x^{2k}}{4^k} = \sum_{k=1}^{\infty} \frac{(x^2)^k}{4^k} = \sum_{k=1}^{\infty} \left(\frac{x^2}{4}\right)^k$, which implies that $\displaystyle\sum_{k=1}^{\infty} \frac{x^{2k}}{4^k}$ is a geometric series with $a = \dfrac{x^2}{4}$ and $r = \dfrac{x^2}{4}$. It follows that

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{4^k} = \frac{a}{1-r} = \frac{x^2/4}{1-x^2/4} = \frac{x^2}{4-x^2},$$

for $\dfrac{x^2}{4} < 1$, or $-2 < x < 2$.

11.2.69 This is a geometric series with ratio e^{-x}, so its sum is $\dfrac{1}{1-e^{-x}}$, for $e^{-x} < 1$, or $e^x > 1$ or $x > 0$.

11.2.70 This is a geometric series with ratio $\dfrac{x-2}{9}$, so its sum is $\dfrac{(x-2)/9}{1-(x-2)/9} = \dfrac{x-2}{9-(x-2)} = \dfrac{x-2}{11-x}$, for $-1 < \dfrac{x-2}{9} < 1$, or $-7 < x < 11$. It diverges at both endpoints.

11.2.71 This is a geometric series with ratio $(x^2 - 1)/3$, so its sum is $\dfrac{1}{1-\dfrac{x^2-1}{3}} = \dfrac{3}{3-(x^2-1)} = \dfrac{3}{4-x^2}$, convergent for $\left|x^2 - 1\right| < 3$, so that $-2 < x^2 < 4$ or $-2 < x < 2$. It diverges at both endpoints.

11.2.72 Substitute $2x$ for x in the power series for e^x to get $e^{2x} = \displaystyle\sum_{k=0}^{\infty} \frac{(2x)^k}{k!} = \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k$. The series converges for all x.

11.2.73 Substitute $-3x$ for x in the power series for e^x to get $e^{-3x} = \sum_{k=0}^{\infty} \frac{(-3x)^k}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{3^k}{k!} x^k$. The series converges for all x.

11.2.74 Multiply the power series for e^x by x^2 to get $x^2 e^x = \sum_{k=0}^{\infty} \frac{x^{k+2}}{k!}$, which converges for all x.

11.2.75 The power series for $x^m f(x)$ is $\sum c_k x^{k+m}$. The radius of convergence of this power series is determined by the limit

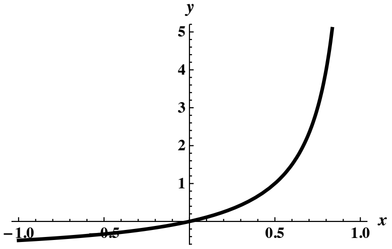
$$\lim_{k\to\infty} \left| \frac{c_{k+1}x^{k+1+m}}{c_k x^{k+m}} \right| = \lim_{k\to\infty} \left| \frac{c_{k+1}x^{k+1}}{c_k x^k} \right|,$$

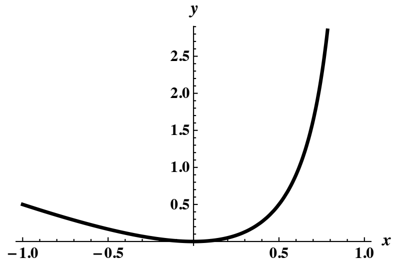
and the right-hand side is the limit used to determine the radius of convergence for the power series for $f(x)$. Thus the two have the same radius of convergence.

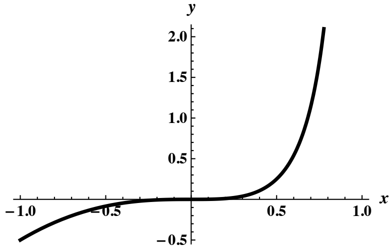
11.2.76

a. $R_n = f(x) - S_n(x) = \sum_{k=n}^{\infty} x^k$. This is a geometric series with ratio x. Its sum is then $R_n = \frac{x^n}{1-x}$ as desired.

b. $R_n(x)$ increases without bound as x approaches 1, and its absolute value smallest at $x = 0$ (where it is zero). In general, for $x > 0$, $R_n(x) < R_{n-1}(x)$, so the approximations get better the more terms of the series are included.

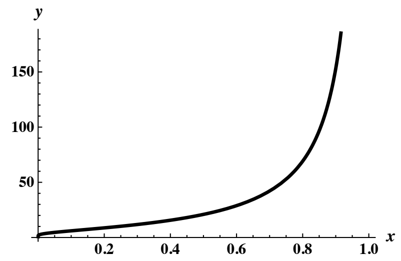






c. To minimize $|R_n(x)|$, set its derivative to zero. Assuming $n > 1$, we have $R_n'(x) = \frac{n(1-x)x^{n-1} + x^n}{(1-x)^2}$, which is zero for $x = 0$. There is a minimum at this critical point.

d. The following is a plot that shows, for each $x \in (0,1)$, the n required so that $R_n(x) < 10^{-6}$. The closer x gets to 1, the more terms are required in order for the estimate given by the power series to be accurate. The number of terms increases rapidly as $x \to 1$.



11.2.77

a. $f(x)g(x) = c_0d_0 + (c_0d_1 + c_1d_0)x + (c_0d_2 + c_1d_1 + c_2d_0)x^2 + \dots$

b. The coefficient of x^n in $f(x)g(x)$ is $\sum_{i=0}^{n} c_i d_{n-i}$.

11.2.78 The function $\frac{1}{\sqrt{1-x^2}}$ is the derivative of the inverse sine function, and $\sin^{-1}(0) = 0$, so the power series for $\sin^{-1} x$ is the integral of the given power series, or $x + \frac{1}{6}x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5}x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7}x^7 + \dots$. This can also be written $x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdots (2k-1)}{2 \cdot 4 \cdots 2k \cdot (2k+1)} x^{2k+1}$.

11.3 Taylor Series

11.3.1 The nth Taylor Polynomial is the nth sum of the corresponding Taylor Series.

11.3.2 In order to have a Taylor series centered at a, a function f must have derivatives of all orders on some interval containing a.

11.3.3 $\sum_{k=0}^{\infty} \frac{(x-2)^k}{k!}$.

11.3.4 $.\sum_{k=0}^{\infty} (k+1)x^k$.

11.3.5 Substitute x^2 for x in the Taylor series. By theorems proved in the previous section about power series, the interval of convergence does not change except perhaps at the endpoints of the interval.

11.3.6 The Taylor series terminates if $f^{(n)}(0) = 0$ for $n > N$ for some N. For $(1+x)^p$, this occurs if and only if p is an integer ≥ 0.

11.3.7 It means that the limit of the remainder term is zero.

11.3.8 For $f(x) = e^{-x}$, $f'(x) = -e^{-x}$, while $f''(x) = f(x) = e^{-x}$. So $f(0) = 1$, $f'(0) = -1$, $f''(0) = 1$, etc. Thus, $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$. Using Table 11.5, $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, so $e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$.

11.3.9

a. $f(x) = \frac{1}{x^2}$, so $f(1) = 1$. $f'(x) = -\frac{2}{x^3}$, so $f'(1) = -2$. $f''(x) = \frac{6}{x^4}$, so $f''(1) = 6$. $f'''(x) = -\frac{24}{x^5}$, so $f'''(1) = -24$. So the Taylor series is

$$1 - 2(x-1) + 6\frac{(x-1)^2}{2!} - 24\frac{(x-1)^3}{3!} + \cdots = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \cdots.$$

b. $\sum_{k=0}^{\infty} (-1)^k (k+1)(x-1)^k$.

c. $r = \lim_{k\to\infty} \left| \frac{(-1)^{k+1}(k+2)(x-1)^{k+1}}{(-1)^k(k+1)(x-1)^k} \right| = |x-1|$. This is less than 1 for $-1 < x-1 < 1$, or $0 < x < 2$. The series diverges at the endpoints by the Divergence Test, so the interval of convergence is $(0, 2)$.

11.3.10

a. $f(x) = \dfrac{1}{x^2}$, so $f(-1) = 1$. $f'(x) = -\dfrac{2}{x^3}$, so $f'(-1) = 2$. $f''(x) = \dfrac{6}{x^4}$, so $f''(-1) = 6$. $f'''(x) = -\dfrac{24}{x^5}$, so $f'''(-1) = 24$. So the Taylor series is

$$1 + 2(x+1) + 6\frac{(x+1)^2}{2!} + 24\frac{(x+1)^3}{3!} + \cdots = 1 + 2(x+1) + 3(x+1)^2 + 4(x+1)^3 + \cdots .$$

b. $\displaystyle\sum_{k=0}^{\infty}(k+1)(x+1)^k$.

c. $r = \displaystyle\lim_{k\to\infty}\left|\frac{(k+2)(x+1)^{k+1}}{(k+1)(x+1)^k}\right| = |x+1|$. This is less than 1 for $-1 < x+1 < 1$, or $-2 < x < 0$. The series diverges at the endpoints by the Divergence Test, so the interval of convergence is $(-2, 0)$.

11.3.11

a. Note that $f(0) = 1$, $f'(0) = -1$, $f''(0) = 1$, and $f'''(0) = -1$. So the Maclaurin series is $1 - x + x^2/2 - x^3/6 + \cdots$.

b. $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{k!}$.

c. The series converges on $(-\infty, \infty)$, as can be seen from the Ratio Test.

11.3.12

a. Note that $f(0) = 1$, $f'(0) = 0$, $f''(0) = -4$, $f'''(0) = 0$, $f^{(4)}(0) = 16$, ... Thus the Maclaurin series is $1 - 2x^2 + \dfrac{2x^4}{3} - \dfrac{4x^6}{45} + \cdots$.

b. $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{(2x)^{2k}}{(2k)!}$

c. The series converges on $(-\infty, \infty)$, as can be seen from the Ratio Test.

11.3.13

a. Note that $f(0) = 2$, $f'(x) = \dfrac{6}{(1-x)^4}$, so $f'(0) = 6$, $f''(x) = \dfrac{24}{(1-x)^5}$, so $f''(0) = 24$, and $f'''(x) = \dfrac{120}{(1-x)^6}$, so $f'''(0) = 120$. Thus the Maclaurin series is

$$2 + 6x + 12x^2 + 20x^3 + \cdots .$$

b. $\displaystyle\sum_{k=0}^{\infty}(k+1)(k+2)x^k$.

c. Using the Ratio Test, we see that the series converges on $(-1, 1)$. It does not converge at the endpoints by the Divergence Test.

11.3.14

a. $f(0) = 0$, $f'(x) = x\cos x + \sin x$, so $f'(0) = 0$. $f''(x) = -x\sin x + \cos x + \cos x$, so $f''(0) = 2$. $f'''(x) = -x\cos x - \sin x - 2\sin x$, so $f'''(0) = 0$. $f^{(4)}(x) = -4\cos x + x\sin x$, so $f^{(4)}(0) = -4$. Similarly, the 5th and 7th derivatives of f at 0 have value 0, while the 6th has value 6 and the 8th has value -8. So the series is

$$x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \frac{x^8}{5040} + \cdots .$$

b. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k}}{(2k-1)!}$.

c. The limit of the ratio of successive terms is 0, so the interval of convergence is $(-\infty,\infty)$.

11.3.15

a. For $f(x)=\dfrac{1}{1+x^2}$, we have $f(0)=1$, $f'(x)=-\dfrac{2x}{(1+x^2)^2}$ so $f'(0)=0$. $f''(x)=\dfrac{2(3x^2-1)}{(1+x^2)^3}$, so $f''(0)=-2$. Similarly, $f'''(0)=0$ and $f^{(4)}=24$, $f^{(5)}(0)=0$, and $f^{(6)}(0)=6!$. Thus the series is $1-x^2+x^4-x^6+\cdots$.

b. $\displaystyle\sum_{k=0}^{\infty}(-1)^k x^{2k}$.

c. The absolute value of the ratio of consecutive terms is x^2, so by the Ratio Test, the radius of convergence is 1. The series diverges at the endpoints by the Divergence Test, so the interval of convergence is $(-1,1)$.

11.3.16

a. Note that $f(0)=0$, $f'(0)=4$, $f''(0)=-16$, $f'''(0)=128$, and $f''''(0)=-1526$. Thus, the series is given by $4x-\dfrac{16x^2}{2}+\dfrac{128x^3}{6}-\dfrac{1536x^4}{24}+\cdots$.

b. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(k-1)!(4x)^k}{k!}=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(4x)^k}{k}$.

c. The absolute value of the ratio of consecutive terms is $\dfrac{4\,|x|\,k}{k+1}$, which has limit $4|x|$ as $k\to\infty$, so the interval of convergence is $(-1/4,1/4]$. Note that for $x=1/4$ we have the alternating harmonic series, while for $x=-1/4$ we have negative 1 times the harmonic series, which diverges.

11.3.17

a. Note that $f(0)=1$, and that $f^{(n)}(0)=2^n$. Thus, the series is given by $1+2x+\dfrac{4x^2}{2}+\dfrac{8x^3}{6}+\cdots$.

b. $\displaystyle\sum_{k=0}^{\infty}\frac{(2x)^k}{k!}$.

c. The absolute value of the ratio of consecutive terms is $\dfrac{2\,|x|}{n}$, which has limit 0 as $n\to\infty$. So by the Ratio Test, the interval of convergence is $(-\infty,\infty)$.

11.3.18

a. We have $f(0)=1$, while $f'(x)=-2(1+2x)^{-2}$, so $f'(0)=-2$. Then $f''(x)=8(1+2x)^{-3}$, so $f''(0)=8$. $f'''(x)=-48(1+2x)^{-4}$, so $f'''(0)=-48$. The series is given by $1-2x+4x^2-8x^3+\cdots$.

b. $\displaystyle\sum_{k=0}^{\infty}(-1)^k(2x)^k$.

c. The Root Test shows that the series converges absolutely for $|2x|<1$, or $|x|<1/2$. The interval of convergence is $(-1/2,1/2)$, because the series at both endpoints diverge by the Divergence Test.

11.3.19

a. We have $f(0) = 0$, while $f'(x) = \dfrac{2}{x^2+4}$, so $f'(0) = \dfrac{1}{2}$. $f''(x) = -\dfrac{4x}{(x^2+4)^2}$, so $f''(0) = 0$. Continuing, we have $f'''(x) = \dfrac{4(3x^2-4)}{(x^2+4)^3}$, so $f'''(0) = -\dfrac{1}{4}$. In a similar manner, $f^{(4)}(0) = 0$, $f^{(5)}(0) = \dfrac{3}{4}$, $f^{(6)}(0) = 0$, and $f^{(7)}(0) = -\dfrac{45}{8}$. So the series is $\dfrac{x}{2} - \dfrac{x^3}{24} + \dfrac{x^5}{160} - \dfrac{x^7}{896} + \cdots$.

b. $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)\cdot 2^{2k+1}} x^{2k+1}$.

c. The ratio of consecutive terms has limit $\dfrac{x^2}{4}$, so by the Ratio Test, the radius of convergence is $|x| < 2$. Also, at the endpoints we have convergence by the Alternating Series Test, so the interval of convergence is $[-2, 2]$.

11.3.20

a. $f(0) = 0$, $f'(x) = 3\cos 3x$, so $f'(0) = 3$. $f''(0) = 0$, but $f'''(x) = -27\cos 3x$, so $f'''(0) = -27$. Similarly, $f^{(4)}(0) = 0$, but $f^{(5)}(0) = 243$ and $f^{(7)}(0) = -3^7$. So the series is $3x - \dfrac{9x^3}{2} + \dfrac{81x^5}{40} - \dfrac{243x^7}{560} + \cdots$.

b. $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{3^{2k+1}}{(2k+1)!} x^{2k+1}$.

c. The ratio of successive terms is $\dfrac{9}{2n(2n+1)}x^2$, which has limit zero as $n \to \infty$, so the interval of convergence is $(-\infty, \infty)$.

11.3.21

a. Note that $f(0) = 1$, $f'(0) = \ln 3$, $f''(0) = \ln^2 3$, $f''(0) = \ln^3 3$. So the first four terms of the desired series are $1 + (\ln 3)x + \dfrac{\ln^2 3}{2}x^2 + \dfrac{\ln^3 3}{6}x^3 + \cdots$.

b. $\displaystyle\sum_{k=0}^{\infty} \frac{(\ln^k 3)x^k}{k!}$.

c. The ratio of successive terms is $\dfrac{(\ln^{k+1} 3)x^{k+1}}{(k+1)!} \cdot \dfrac{k!}{(\ln^k 3)x^k} = \dfrac{\ln 3}{k+1}x$, and the limit as $k \to \infty$ of this quantity is 0, so the interval of convergence is $(-\infty, \infty)$.

11.3.22

a. Note that $f(0) = 0$, $f'(0) = \dfrac{1}{\ln 3}$, $f''(0) = -\dfrac{1}{\ln 3}$, $f'''(0) = \dfrac{2}{ln 3}$, $f''''(0) = -\dfrac{6}{\ln 3}$. So the first terms of the desired series are $0 + \dfrac{x}{\ln 3} - \dfrac{x^2}{2\ln 3} + \dfrac{x^3}{3\ln 3} - \dfrac{x^4}{4\ln 3} + \cdots$.

b. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k\ln 3}$.

c. The absolute value of the ratio of successive terms is $\left|\dfrac{x^{k+1}}{(k+1)\ln 3} \cdot \dfrac{k\ln 3}{x^k}\right| = \dfrac{k}{k+1}|x|$, which has limit $|x|$ as $k \to \infty$. Thus the radius of convergence is 1. At $x = -1$ we have a multiple of the harmonic series (which diverges) and at $x = 1$ we have a multiple of the alternating harmonic series (which converges) so the interval of convergence is $(-1, 1]$.

11.3.23

a. Note that $f(0) = 1$, $f'(0) = 0$, $f''(0) = 9$, $f'''(0) = 0$, etc. The first terms of the series are $1 + 9x^2/2 + 81x^4/4! + 3^6x^6/6! + \cdots$.

b. $\displaystyle\sum_{k=0}^{\infty} \frac{(3x)^{2k}}{(2k)!}$.

c. The absolute value of the ratio of successive terms is $\left|\frac{(3x)^{2k+2}}{(2k+2)!} \cdot \frac{(2k)!}{(3x)^{2k}}\right| = \frac{1}{(2k+2)(2k+1)} \cdot 9x^2$, which has limit 0 as $x \to \infty$. The interval of convergence is therefore $(-\infty, \infty)$.

11.3.24

a. Note that $f(0) = 0$, $f'(0) = 2$, $f''(0) = 0$, $f'''(0) = 8$, etc. The first terms of the series are $2x + 8x^3/6 + 32x^5/5! + 128x^7/7! + \cdots$, or $2x + \frac{4x^3}{3} + \frac{4x^5}{15} + \frac{8x^7}{315} + \cdots$.

b. $\displaystyle\sum_{k=0}^{\infty} \frac{2^{2k+1}x^{2k+1}}{(2k+1)!}$.

c. The absolute value of the ratio of successive terms is $\left|\frac{2^{2k+3}x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{2^{2k+1}x^{2k+1}}\right| = \frac{4}{(2k+3)(2k+2)}x^2$, which has limit 0 as $x \to \infty$. The interval of convergence is therefore $(-\infty, \infty)$.

11.3.25

a. $f(3) = 0$, $f'(x) = \frac{1}{x-2}$, so $f'(3) = 1$. $f''(x) = -\frac{1}{(x-2)^2}$, so $f''(3) = -1$. $f'''(x) = \frac{2}{(x-2)^3}$, so $f'''(3) = 2$, and $f^{(4)}(x) = -\frac{6}{(x-2)^4}$, so $f^{(4)}(3) = -6$. The series is

$$(x-3) - \frac{1}{2}(x-3)^2 + \frac{1}{3}(x-3)^3 - \frac{1}{4}(x-3)^4 + \cdots .$$

b. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-3)^k}{k}$.

c. The limit of the absolute value of the ratio of successive terms is $x - 3$. So the series converges for $-1 < x - 3 < 1$, or $2 < x < 4$. At the left endpoint it diverges (it is the opposite of the harmonic series) and at the right endpoint it converges (it is the alternating harmonic series), so the interval of convergence is $(2, 4]$.

11.3.26

a. Because all the derivatives of e^x are e^x, we have $f^{(n)}(2) = e^2$ for all n. Thus the series is given by

$$e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3 + \cdots$$

b. $\displaystyle\sum_{k=0}^{\infty} \frac{e^2(x-2)^k}{k!}$.

c. The limit of the ratio of the absolute value of successive terms is 0, so the series converges on $(-\infty, \infty)$.

11.3.27

a. Note that $f(\pi/2) = 1$, $f'(\pi/2) = \cos(\pi/2) = 0$, $f''(\pi/2) = -\sin(\pi/2) = -1$, $f'''(\pi/2) = -\cos(\pi/2) = 0$, and so on. Thus the series is given by $1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x - \frac{\pi}{2}\right)^6 + \cdots$.

b. $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}\left(x - \frac{\pi}{2}\right)^{2k}$.

11.3.28

a. Note that $f(\pi) = -1$, $f'(\pi) = -\sin\pi = 0$, $f''(\pi) = -\cos\pi = 1$, $f'''(\pi) = -\sin\pi = 0$, and so on. Thus the series is given by $-1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4 + \frac{1}{720}(x-\pi)^6 + \cdots$.

b. $\displaystyle\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{(2k)!}(x-\pi)^{2k}$.

11.3.29

a. Note that $f^{(k)}(1) = (-1)^k \frac{k!}{1^{k+1}} = (-1)^k \cdot k!$. Thus the series is given by $1-(x-1)+(x-1)^2-(x-1)^3+\cdots$.

b. $\displaystyle\sum_{k=0}^{\infty}(-1)^k(x-1)^k$.

11.3.30

a. Note that $f^{(k)}(2) = (-1)^k \frac{k!}{2^{k+1}}$. Thus the series is given by $\frac{1}{2} - \frac{x-2}{4} + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4 + \cdots$.

b. $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{1}{2^{k+1}}(x-2)^k$.

11.3.31

a. Note that $f^{(k)}(3) = (-1)^{k-1}\frac{(k-1)!}{3^k}$. Thus the series is given by $\ln(3) + \frac{x-3}{3} - \frac{1}{18}(x-3)^2 + \frac{1}{81}(x-3)^3 + \cdots$.

b. $\displaystyle\ln 3 + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k\cdot 3^k}(x-3)^k$.

11.3.32

a. Note that $f^{(k)}(\ln 2) = 2$. Thus the series is given by $2 + 2(x-\ln(2)) + (x-\ln(2))^2 + \frac{1}{3}(x-\ln(2))^3 + \frac{1}{12}(x-\ln(2))^4 + \cdots$.

b. $\displaystyle\sum_{k=0}^{\infty}\frac{2}{k!}(x-\ln(2))^k$.

11.3.33

a. Note that $f(1) = 2$, $f'(1) = 2\ln 2$, $f''(1) = 2\ln^2 2$, $f'''(1) = 2\ln^3 2$. The first terms of the series are $2 + (2\ln 2)(x-1) + (\ln^2 2)(x-1)^2 + \frac{(\ln^3 2)(x-1)^3}{3} + \cdots$.

b. $\displaystyle\sum_{k=0}^{\infty}\frac{2(x-1)^k\ln^k 2}{k!}$.

11.3.34

a. Note that $f(1) = 0$, and that $f'(x) = \ln x$, so $f'(1) = 0$. Then $f''(x) = \dfrac{1}{x}$, so $f''(1) = 1$, $f'''(x) = -\dfrac{1}{x^2}$, so $f'''(1) = -1$. $f^{(4)}(x) = \dfrac{2}{x^3}$, so $f^{(4)}(1) = 2$, and $f^{(5)}(x) = -\dfrac{6}{x^4}$, so $f^{(5)}(1) = -6$. So the series is

$$\frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!} + \frac{2(x-1)^4}{4!} - \frac{6(x-1)^5}{5!} + \cdots .$$

b. $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k (x-1)^k}{k(k-1)}$.

11.3.35 Because the Taylor series for $\ln(1+x)$ is $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$, the first four terms of the Taylor series for $\ln(1+x^2)$ are $x^2 - \dfrac{x^4}{2} + \dfrac{x^6}{3} - \dfrac{x^8}{4} + \cdots$, obtained by substituting x^2 for x.

11.3.36 Because the Taylor series for $\sin x$ is $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$, the first four terms of the Taylor series for $\sin x^2$ are $x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \dfrac{x^{14}}{7!} + \cdots$, obtained by substituting x^2 for x.

11.3.37 Because the Taylor series for $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, the first four terms of the Taylor series for $\frac{1}{1-2x}$ are $1 + 2x + 4x^2 + 8x^3 + \cdots$ obtained by substituting $2x$ for x.

11.3.38 Because the Taylor series for $\ln(1+x)$ is $x - x^2/2 + x^3/3 - x^4/4 + \cdots$, the first four terms of the Taylor series for $2x - 2x^2 + 8x^3/3 - 4x^4 + \cdots$ obtained by substituting $2x$ for x.

11.3.39 The Taylor series for $e^x - 1$ is the Taylor series for e^x, less the constant term of 1, so it is $x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$. Thus, the first four terms of the Taylor series for $\frac{e^x-1}{x}$ are $1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \cdots$, obtained by dividing the terms of the first series by x.

11.3.40 Because the Taylor series for $\cos x$ is $1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$, the first four terms of the Taylor series for $\cos x^3$ are $1 - \dfrac{x^6}{2!} + \dfrac{x^{12}}{4!} - \dfrac{x^{18}}{6!} + \cdots$, obtained by substituting x^3 for x.

11.3.41 Because the Taylor series for $(1+x)^{-1}$ is $1 - x + x^2 - x^3 + \cdots$, if we substitute x^4 for x, we obtain $1 - x^4 + x^8 - x^{12} + \cdots$.

11.3.42 The Taylor series for $\tan^{-1} x$ is $x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} - \cdots$. Thus, the Taylor series for $\tan^{-1} x^2$ is $x^2 - \dfrac{x^6}{3} + \dfrac{x^{10}}{5} - \dfrac{x^{14}}{7} - \cdots$ and, multiplying by x, the Taylor series for $x \tan^{-1} x^2$ is $x^3 - \dfrac{x^7}{3} + \dfrac{x^{11}}{5} - \dfrac{x^{15}}{7} - \cdots$.

11.3.43 The Taylor series for $\sinh x$ is $x + \dfrac{x^3}{6} + \dfrac{x^5}{120} + \dfrac{x^7}{5040} + \cdots$. Thus, the Taylor series for $\sinh x^2$ is $x^2 + \dfrac{x^6}{6} + \dfrac{x^{10}}{120} + \dfrac{x^{14}}{5040} + \cdots$ obtained by substituting x^2 for x.

11.3.44 The Taylor series for $\cosh x$ is $1 + \dfrac{x^2}{2} + \dfrac{x^4}{24} + \dfrac{x^6}{720} + \cdots$. Thus, the Taylor series for $\cosh 3x$ is $1 + \dfrac{9x^2}{2} + \dfrac{81x^4}{24} + \dfrac{729x^6}{720} + \cdots$, obtained by substituting $3x$ for x.

11.3.45

a. The binomial coefficients are $\binom{-2}{0} = 1$, $\binom{-2}{1} = \frac{-2}{1!} = -2$, $\binom{-2}{2} = \frac{(-2)(-3)}{2!} = 3$, $\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4$.

   Thus the first four terms of the series are $1 - 2x + 3x^2 - 4x^3 + \cdots$.

b. $1 - 2 \cdot 0.1 + 3 \cdot 0.01 - 4 \cdot 0.001 = 0.826$

11.3.46

a. The binomial coefficients are $\binom{1/2}{0} = 1$, $\binom{1/2}{1} = \frac{1/2}{1!} = \frac{1}{2}$, $\binom{1/2}{2} = \frac{(1/2)(-1/2)}{2!} = -\frac{1}{8}$, $\binom{1/2}{3} = \frac{(1/2)(-1/2)(-3/2)}{3!} = \frac{1}{16}$, so the first four terms of the series are $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$.

b. $1 + \frac{1}{2} \cdot 0.06 - \frac{1}{8} \cdot 0.06^2 + \frac{1}{16} \cdot 0.06^3 \approx 1.030$

11.3.47

a. The binomial coefficients are $\binom{1/4}{0} = 1$, $\binom{1/4}{1} = \frac{1/4}{1} = \frac{1}{4}$, $\binom{1/4}{2} = \frac{(1/4)(-3/4)}{2!} = -\frac{3}{32}$, $\binom{1/4}{3} = \frac{(1/4)(-3/4)(-7/4)}{3!} = \frac{7}{128}$, so the first four terms of the series are $1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 + \cdots$.

b. Substitute $x = 0.12$ to get approximately 1.029.

11.3.48

a. The binomial coefficients are $\binom{-3}{0} = 1$, $\binom{-3}{1} = -3$, $\binom{-3}{2} = \frac{(-3)(-4)}{2!} = 6$, $\binom{-3}{3} = \frac{(-3)(-4)(-5)}{3!} = -10$, so the first four terms of the series are $1 - 3x + 6x^2 - 10x^3 + \cdots$.

b. Substitute $x = 0.1$ to get 0.750.

11.3.49

a. The binomial coefficients are $\binom{-2/3}{0} = 1$, $\binom{-2/3}{1} = -\frac{2}{3}$, $\binom{-2/3}{2} = \frac{(-2/3)(-5/3)}{2!} = \frac{5}{9}$, $\binom{-2/3}{3} = \frac{(-2/3)(-5/3)(-8/3)}{3!} = -\frac{40}{81}$, so the first four terms of the series are $1 - \frac{2}{3}x + \frac{5}{9}x^2 - \frac{40}{81}x^3 + \cdots$.

b. Substitute $x = 0.18$ to get 0.89512.

11.3.50

a. The binomial coefficients are $\binom{2/3}{0} = 1$, $\binom{2/3}{1} = \frac{2}{3}$, $\binom{2/3}{2} = \frac{(2/3)(-1/3)}{2!} = -\frac{1}{9}$, $\binom{2/3}{3} = \frac{(2/3)(-1/3)(-4/3)}{3!} = \frac{4}{81}$, so the first four terms of the series are $1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3 + \cdots$

b. Substitute $x = 0.02$ to get ≈ 1.013289284.

11.3.51 $\sqrt{1 + x^2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \cdots$. By the Ratio Test, the radius of convergence is 1. At the endpoints, the series obtained are convergent by the Alternating Series Test. Thus, the interval of convergence is $[-1, 1]$.

11.3.52 $\sqrt{4+x} = 2\sqrt{1+x/4} = 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + \cdots$. The interval of convergence is $(-4, 4]$.

11.3.53 $\sqrt{9-9x} = 3\sqrt{1-x} = 3 - \frac{3}{2}x - \frac{3}{8}x^2 - \frac{3}{16}x^3 - \cdots$. The interval of convergence is $[-1, 1]$.

11.3.54 $\sqrt{1-4x} = 1 - 2x - 2x^2 - 4x^3 - \cdots$, obtained by substituting $-4x$ for x in the original series. The interval of convergence of $[-1/4, 1/4)$.

11.3.55 $\sqrt{a^2+x^2} = a\sqrt{1+\frac{x^2}{a^2}} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \cdots$. The series converges when $\frac{x^2}{a^2}$ is less than 1 in magnitude, so the radius of convergence is a. The series given by the endpoints is convergent by the Alternating Series Test, so the interval of convergence is $[-a, a]$.

11.3.56 $\sqrt{4-16x^2} = 2\sqrt{1-(2x)^2} = 2 - 4x^2 - 4x^4 - 8x^6 - \cdots$. Because $2x$ was substituted for x to produce this series, this series converges when $-1 < 2x < 1$, or $-\frac{1}{2} < x < \frac{1}{2}$. Because only even powers of x appear in the series, the series at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ are identical, and are convergent. Thus the interval of convergence is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

11.3.57 $(1+4x)^{-2} = 1 - 2(4x) + 3(4x)^2 - 4(4x)^3 + \cdots = 1 - 8x + 48x^2 - 256x^3 + \cdots$.

11.3.58 $\frac{1}{(1-4x)^2} = (1-4x)^{-2} = 1 - 2(-4x) + 3(-4x)^2 - 4(-4x)^3 + \cdots = 1 + 8x + 48x^2 + 256x^3$.

11.3.59 $\frac{1}{(4+x^2)^2} = (4+x^2)^{-2} = \frac{1}{16}(1+(x^2/4))^{-2} = \frac{1}{16}\left(1 - 2\cdot\frac{x^2}{4} + 3\cdot\frac{x^4}{16} - 4\cdot\frac{x^6}{64} + \cdots\right) = \frac{1}{16} - \frac{1}{32}x^2 + \frac{3}{256}x^4 - \frac{1}{256}x^6 + \cdots$

11.3.60 Note that $x^2 - 4x + 5 = 1 + (x-2)^2$, so $(1+(x-2)^2)^{-2} = 1 - 2(x-2)^2 + 3(x-2)^4 - 4(x-2)^6 + \cdots$.

11.3.61 $(3+4x)^{-2} = \frac{1}{9}\left(1+\frac{4x}{3}\right)^{-2} = \frac{1}{9} - \frac{2}{9}\left(\frac{4x}{3}\right) + \frac{3}{9}\left(\frac{4x}{3}\right)^2 - \frac{4}{9}\left(\frac{4x}{3}\right)^3 + \cdots$.

11.3.62 $(1+4x^2)^{-2} = (1+(2x)^2)^{-2} = 1 - 2(2x)^2 + 3(2x)^4 - 4(2x)^6 + \cdots = 1 - 8x^2 + 48x^4 - 256x^6 + \cdots$.

11.3.63 The interval of convergence for the Taylor series for $f(x) = \sin x$ is $(-\infty, \infty)$. The remainder is $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c. Because $f^{(n+1)}(x)$ is $\pm \sin x$ or $\pm \cos x$, we have

$$\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \frac{1}{(n+1)!}\left|x^{n+1}\right| = 0$$

for any x.

11.3.64 The interval of convergence for the Taylor series for $f(x) = \cos 2x$ is $(-\infty, \infty)$. The remainder is $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c. The nth derivative of $\cos 2x$ is 2^n times either $\pm \sin x$ or $\pm \cos x$, so that $f^{(n+1)}$ is bounded by 2^{n+1} in magnitude. Thus $\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}\left|x^{n+1}\right| = \lim_{n\to\infty} \frac{(2|x|)^{n+1}}{(n+1)!} = 0$ for any x.

11.3.65 The interval of convergence for the Taylor series for e^{-x} is $(-\infty, \infty)$. The remainder is $R_n(x) = \frac{(-1)^{n+1}e^{-c}}{(n+1)!}x^{n+1}$ for some c. Thus $\lim_{n\to\infty} |R_n(x)| = 0$ for any x.

11.3.66 The interval of convergence for the Taylor series for $f(x) = \cos x$ is $(-\infty, \infty)$. The remainder is $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x - \pi/2)^{n+1}$ for some c. Because $f^{n+1}(x)$ is $\pm\cos x$ or $\pm\sin x$, we have

$$\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{1}{(n+1)!} \left|(x-\pi/2)^{n+1}\right| = 0$$

for any x.

11.3.67

a. False. Not all of its derivatives are defined at zero - in fact, none of them are.

b. True. The derivatives of $\csc x$ involve positive powers of $\csc x$ and $\cot x$, both of which are defined at $\pi/2$, so that $\csc x$ has continuous derivatives at $\pi/2$.

c. False. For example, the Taylor series for $f(x^2)$ doesn't converge at $x = 1.9$, because the Taylor series for $f(x)$ doesn't converge at $1.9^2 = 3.61$.

d. False. The Taylor series centered at 1 involves derivatives of f evaluated at 1, not at 0.

e. True. The follows because the Taylor series must itself be an even function.

11.3.68

a. The relevant Taylor series are: $\cos 2x = 1 - 2x^2 + \dfrac{2}{3}x^4 - \dfrac{4}{45}x^6 + \cdots$, and $2\sin x = 2x - \frac{1}{3}x^3 + \frac{1}{60}x^5 - \cdots$. Thus, the first four terms of the resulting series are $\cos 2x + 2\sin x = 1 + 2x - 2x^2 - \dfrac{1}{3}x^3 + \dfrac{2}{3}x^4 + \cdots$.

b. Because each series converges (absolutely) on $(-\infty, \infty)$, so does their sum. The radius of convergence is ∞.

11.3.69

a. The relevant Taylor series are: $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \dfrac{x^6}{6!} + \cdots$ and $e^{-x} = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \dfrac{x^5}{5!} + \dfrac{x^6}{6!} + \cdots$. Thus the first four terms of the resulting series are $\dfrac{1}{2}\left(e^x + e^{-x}\right) = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$.

b. Because each series converges (absolutely) on $(-\infty, \infty)$, so does their sum. The radius of convergence is ∞.

11.3.70

a. The first four terms of the Taylor series for $\sin x$ are $x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \dfrac{x^7}{5040}$, so the first four terms for $\dfrac{\sin x}{x}$ are $1 - \dfrac{x^2}{6} + \dfrac{x^4}{120} - \dfrac{x^6}{5040}$.

b. The radius of convergence is the same as that for $\sin x$, namely ∞.

11.3.71

a. Use the binomial theorem. The binomial coefficients are $\dbinom{-2/3}{0} = 1$, $\dbinom{-2/3}{1} = -\dfrac{2}{3}$, $\dbinom{-2/3}{2} = \dfrac{(-2/3)(-5/3)}{2!} = \dfrac{5}{9}$, $\dbinom{-2/3}{3} = \dfrac{(-2/3)(-5/3)(-8/3)}{3!} = -\dfrac{40}{81}$ and then, substituting x^2 for x, we obtain $1 - \dfrac{2}{3}x^2 + \dfrac{5}{9}x^4 - \dfrac{40}{81}x^6 + \cdots$.

b. The radius of convergence is determined from $\left|x^2\right| < 1$, so it is 1.

11.3.72

a. The first four terms of $\cos x$ are $1-\dfrac{x^2}{2}+\dfrac{x^4}{24}-\dfrac{x^6}{720}$, so the first four terms of $\cos x^2$ are $1-\dfrac{x^4}{2}+\dfrac{x^8}{24}-\dfrac{x^{12}}{720}$, and thus the first four terms of $x^2\cos x^2$ are $x^2-\dfrac{x^6}{2}+\dfrac{x^{10}}{24}-\dfrac{x^{14}}{720}$.

b. The radius of convergence is ∞.

11.3.73

a. From the binomial formula, the Taylor series for $(1-x)^p$ is $\sum\binom{p}{k}(-1)^k x^k$, so the Taylor series for $(1-x^2)^p$ is $\sum\binom{p}{k}(-1)^k x^{2k}$. Here $p=1/2$, and the binomial coefficients are $\binom{1/2}{0}=1$, $\binom{1/2}{1}=\dfrac{1/2}{1!}=\dfrac{1}{2}$, $\binom{1/2}{2}=\dfrac{(1/2)(-1/2)}{2!}=-\dfrac{1}{8}$, $\binom{1/2}{3}=\dfrac{(1/2)(-1/2)(-3/2)}{3!}=\dfrac{1}{16}$ so that $(1-x^2)^{1/2}=1-\dfrac{1}{2}x^2-\dfrac{1}{8}x^4-\dfrac{1}{16}x^6+\cdots$.

b. The radius of convergence is determined from $\left|x^2\right| < 1$, so it is 1.

11.3.74

a. Because $b^x=e^{x\ln b}$, the Taylor series is $1+x\ln b+\dfrac{1}{2!}(x\ln b)^2+\dfrac{1}{3!}(x\ln b)^3+\cdots$

b. Because the series for e^x converges on $(-\infty,\infty)$, the radius of convergence for the series in part a is ∞.

11.3.75

a. $f(x)=(1+x^2)^{-2}$; using the binomial series and substituting x^2 for x we obtain $1-2x^2+3x^4-4x^6+\cdots$.

b. The radius of convergence is determined from $\left|x^2\right| < 1$, so it is 1.

11.3.76 Because $f(36)=6$, and $f'(x)=\dfrac{1}{2}x^{-1/2}$, $f'(36)=\dfrac{1}{12}$, $f''(x)=-\dfrac{1}{4}x^{-3/2}$, $f''(36)=-\dfrac{1}{864}$, $f'''(x)=\dfrac{3}{8}x^{-5/2}$, and $f'''(36)=\dfrac{3}{62208}$, the first four terms of the Taylor series are $6+\dfrac{1}{12}(x-36)-\dfrac{1}{864\cdot 2!}(x-36)^2+\dfrac{3}{62208\cdot 3!}(x-36)^3$. Evaluating at $x=39$ we obtain 6.245008681.

11.3.77 Because $f(64)=4$, and $f'(x)=\dfrac{1}{3}x^{-2/3}$, $f'(64)=\dfrac{1}{48}$, $f''(x)=-\dfrac{2}{9}x^{-5/3}$, $f''(64)=-\dfrac{1}{4608}$, $f'''(x)=\dfrac{10}{27}x^{-8/3}$, and $f'''(64)=\dfrac{10}{1769472}=\dfrac{5}{884736}$, the first four terms of the Taylor series are $4+\dfrac{1}{48}(x-64)-\dfrac{1}{4608\cdot 2!}(x-64)^2+\dfrac{5}{884736\cdot 3!}(x-64)^3$. Evaluating at $x=60$, we obtain 3.914870274.

11.3.78 Because $f(4)=\dfrac{1}{2}$, and $f'(x)=-\dfrac{1}{2}x^{-3/2}$, $f'(4)=-\dfrac{1}{16}$, $f''(x)=\dfrac{3}{4}x^{-5/2}$, $f''(4)=\dfrac{3}{128}$, $f'''(x)=-\dfrac{15}{8}x^{-7/2}$, and $f'''(4)=-\dfrac{15}{1024}$, the first four terms of the Taylor series are $\dfrac{1}{2}-\dfrac{1}{16}(x-4)+\dfrac{3}{128\cdot 2!}(x-4)^2-\dfrac{15}{1024\cdot 3!}(x-4)^3$. Evaluating at $x=3$, we get 0.5766601563.

11.3.79 Because $f(16)=2$, and $f'(x)=\dfrac{1}{4}x^{-3/4}$, $f'(16)=\dfrac{1}{32}$, $f''(x)=-\dfrac{3}{16}x^{-7/4}$, $f''(16)=-\dfrac{3}{2048}$, $f'''(x)=\dfrac{21}{64}x^{-11/4}$, and $f'''(16)=\dfrac{21}{131072}$, the first four terms of the Taylor series are $2+\dfrac{1}{32}(x-16)-\dfrac{3}{2048\cdot 2!}(x-16)^2+\dfrac{21}{131072\cdot 3!}(x-16)^3$. Evaluating at $x=13$, we get 1.898937225.

11.3.80 Evaluate the binomial coefficient $\binom{-1}{k} = \dfrac{(-1)(-2)\cdots(-1-k+1)}{k!} = (-1)^k$, so that the binomial expansion for $(1+x)^{-1}$ is $\displaystyle\sum_{k=0}^{\infty}(-1)^k x^k$. Substituting $-x$ for x, we obtain $(1-x)^{-1} = \displaystyle\sum_{k=0}^{\infty}(-1)^k(-x)^k = \sum_{k=0}^{\infty} x^k$.

11.3.81 Evaluate the binomial coefficient

$$\begin{aligned}\binom{1/2}{k} &= \frac{(1/2)(-1/2)(-3/2)\cdots(1/2-k+1)}{k!} = \frac{(1/2)(-1/2)\cdots((3-2k)/2)}{k!} \\ &= (-1)^{k-1}2^{-k}\frac{1\cdot 3\cdots(2k-3)}{k!} = (-1)^{k-1}2^{-k}\frac{(2k-2)!}{2^{k-1}\cdot(k-1)!\cdot k!} = (-1)^{k-1}2^{1-2k}\cdot\frac{1}{k}\binom{2k-2}{k-1}.\end{aligned}$$

This is the coefficient of x^k in the Taylor series for $\sqrt{1+x}$. Substituting $4x$ for x, the Taylor series becomes

$$\sum_{k=0}^{\infty}(-1)^{k-1}2^{1-2k}\cdot\frac{1}{k}\binom{2k-2}{k-1}(4x)^k = \sum_{k=0}^{\infty}(-1)^{k-1}\frac{2}{k}\binom{2k-2}{k-1}x^k.$$

If we can show that k divides $\binom{2k-2}{k-1}$, we will be done, for then the coefficient of x^k will be an integer. But

$$\begin{aligned}\binom{2k-2}{k-1} - \binom{2k-2}{k-2} &= \frac{(2k-2)!}{(k-1)!(k-1)!} - \frac{(2k-2)!}{(k-2)!k!} = \frac{(2k-2)!}{(k-1)!(k-1)!} - \frac{(2k-2)!(k-1)}{(k-1)!(k-1)!k} \\ &= \frac{k(2k-2)! - (k-1)(2k-2)!}{k(k-1)!(k-1)!} = \frac{1}{k}\frac{(2k-2)!}{(k-1)!(k-1)!} = \frac{1}{k}\binom{2k-2}{k-1}\end{aligned}$$

and thus we have shown that k divides $\binom{2k-2}{k-1}$.

11.3.82 The two Taylor series are:

$$\begin{aligned}&8 + \frac{1}{16}(x-64) - \frac{1}{4096}(x-64)^2 + \frac{1}{524288}(x-64)^3 - \frac{5}{268435456}(x-64)^4 + \cdots \\ &9 + \frac{1}{18}(x-81) - \frac{1}{5832}(x-81)^2 + \frac{1}{944784}(x-81)^3 - \frac{5}{612220032}(x-81)^4 + \cdots.\end{aligned}$$

Evaluating these Taylor series at $n = 2, 3, 4$ (after the quadratic, cubic, and quartic terms) we obtain the errors:

n	64	81
2	9.064×10^{-4}	-8.297×10^{-4}
3	-7.019×10^{-5}	-5.813×10^{-5}
4	6.106×10^{-6}	-4.550×10^{-6}

The errors using the Taylor series centered at 81 are consistently smaller.

11.3.83

a. The Maclaurin series for $\sin x$ is $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$. Squaring the first four terms yields

$$\begin{aligned}&\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7\right)^2 \\ &= x^2 - \frac{2}{3!}x^4 + \left(\frac{2}{5!} + \frac{1}{3!3!}\right)x^6 + \left(-2\cdot\frac{1}{7!} - 2\cdot\frac{1}{3!5!}\right)x^8 \\ &= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8.\end{aligned}$$

b. The Maclaurin series for $\cos x$ is $1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$. Substituting $2x$ for x in the Maclaurin series for $\cos x$ and then computing $(1 - \cos 2x)/2$, we obtain

$$\begin{aligned}
&(1-(1-\frac{1}{2}(2x)^2+\frac{1}{4!}(2x)^4-\frac{1}{6!}(2x)^6)+\frac{1}{8!}(2x)^8)/2\\
&= (2x^2 - \frac{2}{3}x^4 + \frac{4}{45}x^6 - \frac{2}{315}x^8)/2\\
&= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8,
\end{aligned}$$

and the two are the same.

c. If $f(x) = \sin^2 x$, then $f(0) = 0$, $f'(x) = \sin 2x$, so $f'(0) = 0$. $f''(x) = 2\cos 2x$, so $f''(x) = 2$, $f'''(x) = -4\sin 2x$, so $f'''(0) = 0$. Note that from this point $f^{(n)}(0) = 0$ if n is odd and $f^{(n)}(0) = \pm 2^{n-1}$ if n is even, with the signs alternating for every other even n. Thus, the series for $\sin^2 x$ is

$$2x^2/2 - 8x^4/4! + 32x^6/6! - 128x^8/8! + \cdots = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8 + \cdots.$$

11.3.84

a. The Maclaurin series for $\cos x$ is $1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$. Squaring the first four terms yields

$$\begin{aligned}
&\left(1-\frac{1}{2}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6\right)^2\\
&= 1 - \left(\frac{1}{2}+\frac{1}{2}\right)x^2 + \left(\frac{1}{4!}+\frac{1}{4!}+\frac{1}{4}\right)x^4 + \left(-\frac{1}{6!}-\frac{1}{6!}-\frac{1}{2\cdot 4!}-\frac{1}{2\cdot 4!}\right)x^6\\
&= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6.
\end{aligned}$$

b. Substituting $2x$ for x in the Maclaurin series for $\cos x$ and then computing $(1 + \cos 2x)/2$, we obtain

$$\begin{aligned}
&\left(1+1-\frac{1}{2}(2x)^2+\frac{1}{4!}(2x)^4-\frac{1}{6!}(2x)^6\right)/2\\
&= \left(2 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right)/2\\
&= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6,
\end{aligned}$$

and the two are the same.

c. If $f(x) = \cos^2 x$, then $f(0) = 1$. Also, $f'(x) = -2\cos x \sin x = -\sin 2x$. So $f'(0) = 0$. $f''(x) = -2\cos 2x$, so $f''(0) = -2$. $f'''(x) = 8\sin 2x$, so $f'''(0) = 0$. Note that from this point on, $f^{(n)}(0) = 0$ if n is odd, and $f^{(n)}(0) = \pm 2^{n-1}$ if n is even, with the signs alternating for every other even n. Thus, the series for $\cos^2 x$ is

$$1 - 2x^2/2 + 8x^4/4! - 32x^6/6! + \cdots = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots.$$

11.3.85 There are many solutions. For example, first find a series that has $(-1, 1)$ as an interval of convergence, say $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$. Then the series $\frac{1}{1-x/2} = \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k$ has $(-2, 2)$ as its interval of convergence. Now shift the series up so that it is centered at 4. We have $\sum_{k=0}^{\infty}\left(\frac{x-4}{2}\right)^k$, which has interval of convergence $(2, 6)$.

11.3.86

a. The Maclaurin series in question are

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$
$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots ,$$

so substituting the series for $\sin x$ for x in the series for e^x (and considering only those terms that will give us an exponent at most 3), we obtain $e^{\sin x} = 1 + (x - \frac{1}{3!}x^3) + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = 1 + x + \frac{1}{2}x^2 + \cdots$.

b. The Maclaurin series in question are

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$
$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots ,$$

so substituting the series for $\tan x$ for x in the series for e^x (and considering only those terms that will give us an exponent at most 3), we obtain $e^{\tan x} = 1 + (x + \frac{1}{3}x^3) + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = 1 + x + \frac{1}{2}x^2 + \cdots$.

c. The Maclaurin series in question are

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$
$$\sqrt{1 + x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots ,$$

so substituting the series for $\sin x$ for x in the series for $\sqrt{1+x^2}$ (and considering only those terms that will give us an exponent at most 4), we obtain $\sqrt{1 + \sin^2 x} = 1 + \frac{1}{2}(x - \frac{1}{3!}x^3)^2 - \frac{1}{8}x^4 + \cdots = 1 + \frac{1}{2}x^2 - \frac{7}{24}x^4 + \cdots$.

11.3.87 Use the Taylor series for $\cos x$ centered at $\pi/4$:

$$\frac{\sqrt{2}}{2}(1 - (x - \pi/4) - \frac{1}{2}(x - \pi/4)^2 + \frac{1}{6}(x - \pi/4)^3 + \cdots).$$

The remainder after n terms (because the derivatives of $\cos x$ are bounded by 1 in magnitude) is

$$|R_n(x)| \le \frac{1}{(n+1)!} \cdot \left(\frac{\pi}{4} - \frac{2\pi}{9}\right)^{n+1}.$$

Solving for $|R_n(x)| < 10^{-4}$, we obtain $n = 3$. Evaluating the first four terms (through $n = 3$) of the series we get 0.7660427050. The true value is $\approx$ 0.7660444431.

11.3.88 Use the Taylor series for $\sin x$ centered at π:

$$-(x - \pi) + \frac{1}{6}(x - \pi)^3 - \frac{1}{120}(x - \pi)^5 + \cdots .$$

The remainder after n terms (because the derivatives of $\sin x$ are bounded by 1 in magnitude) is

$$|R_n(x)| \le \frac{1}{(n+1)!} \cdot (\pi - 0.98\pi)^{n+1}.$$

Solving for $|R_n(x)| < 10^{-4}$, we obtain $n = 2$. Evaluating the first term of the series gives 0.06283185307. The true value is $\approx$ 0.06279051953.

11.3.89

a. Use the Taylor series for $(125+x)^{1/3}$ centered at $x=0$. Using the first four terms and evaluating at $x=3$ gives a result (5.03968) accurate to within 10^{-4}.

b. Use the Taylor series for $x^{1/3}$ centered at $x=125$. Note that this gives the identical Taylor series except that the exponential terms are $(x-125)^n$ rather than x^n. Thus we need terms up through $(x-125)^3$, just as before, evaluated at $x=128$, and we obtain the identical result.

c. Because the two Taylor series are the same except for the shifting, the results are equivalent.

11.3.90

a. To show that $f'(0)=0$, we compute the limits of the left and right difference quotients and show that they are both zero:

$$\lim_{x\to 0^+}\frac{e^{-1/x^2}-0}{x}=\lim_{x\to 0^+}\frac{e^{-1/x^2}}{x} \qquad \text{and} \qquad \lim_{x\to 0^-}\frac{e^{-1/x^2}-0}{x}=\lim_{x\to 0^-}\frac{e^{-1/x^2}}{x}.$$

For the limit from the right, use the substitution $x=\frac{1}{\sqrt{y}}$; then $y=x^2$ and the limit becomes

$$\lim_{y\to\infty} e^{-y}\sqrt{y}=\lim_{y\to\infty}\frac{\sqrt{y}}{e^y}=0,$$

because exponentials dominate power functions. Similarly, for the limit from the left, use the substitution $x=-\frac{1}{\sqrt{y}}$; then again $y=x^2$ and the limit becomes

$$\lim_{y\to\infty}(-e^{-y}\sqrt{y})=-\lim_{y\to\infty}\frac{\sqrt{y}}{e^y}=0.$$

Since the left and right limits are both zero, it follows that f is differentiable at $x=0$, and its derivative is zero.

b. Because $f^{(k)}(0)=0$, the Taylor series centered at 0 has only one term:$f(x)=f(0)=0$, so the Taylor series is zero.

c. It does not converge to $f(x)$ because $f(x)\neq 0$ for all $x\neq 0$.

11.3.91 Consider the remainder after the first term of the Taylor series. Taylor's Theorem indicates that

$$R_1(x)=\frac{f''(c)}{2}(x-a)^2 \qquad \text{for some } c \text{ between } x \text{ and } a,$$

so that $f(x)=f(a)+f'(a)(x-a)+\frac{f''(c)}{2}(x-a)^2$. But $f'(a)=0$, so that for every x in an interval containing a, there is a c between x and a such that $f(x)=f(a)+\frac{f''(c)}{2}(x-a)^2$.

a. If $f''(x)>0$ on the interval containing a, then for every x in that interval, we have $f(x)=f(a)+\frac{f''(c)}{2}(x-a)^2$ for some c between x and a. But $f''(c)>0$ and $(x-a)^2>0$, so that $f(x)>f(a)$ and a is a local minimum.

b. If $f''(x)<0$ on the interval containing a, then for every x in that interval, we have $f(x)=f(a)+\frac{f''(c)}{2}(x-a)^2$ for some c between x and a. But $f''(c)<0$ and $(x-a)^2>0$, so that $f(x)<f(a)$ and a is a local maximum.

11.4 Working with Taylor Series

11.4.1 Replace f and g by their Taylor series centered at a, and evaluate the limit.

11.4.2 Integrate the Taylor series for $f(x)$ centered at a, and evaluate it at the endpoints.

11.4.3 Substitute -0.6 for x in the Taylor series for e^x centered at 0. Note that this series is an alternating series, so the error can easily be estimated by looking at the magnitude of the first neglected term.

11.4.4

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{1-\left(1-x^2/2!+x^4/4!-\cdots\right)}{x}=\lim_{x\to 0}\left(x/2!-x^3/4!+\cdots\right)=0.$$

11.4.5

$$\begin{aligned}\frac{d}{dx}\sinh x&=\frac{d}{dx}\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)\\&=1+\frac{3x^2}{3!}+\frac{5x^4}{5!}+\cdots=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\frac{x^{2k}}{(2k)!}+\cdots.\end{aligned}$$

11.4.6 It must have derivatives of all orders on some interval containing a.

11.4.7 Because $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$, we have $\frac{e^x-1}{x}=1+\frac{x}{2!}+\cdots$, so $\lim_{x\to 0}\frac{e^x-1}{x}=1$.

11.4.8 Because $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$, we have $\frac{\tan^{-1}x-x}{x^3}=-\frac{1}{3}+\frac{x^2}{5}-\cdots$. So $\lim_{x\to 0}\frac{\tan^{-1}x-x}{x^3}=-\frac{1}{3}$.

11.4.9 Because $-\ln(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\cdots$, we have $\frac{-x-\ln(1-x)}{x^2}=\frac{1}{2}+\frac{x}{3}+\frac{x^2}{4}+\cdots$, so $\lim_{x\to 0}\frac{-x-\ln(1-x)}{x^2}=\frac{1}{2}$.

11.4.10 Because $\sin 2x=2x-\frac{4x^3}{3}+\frac{4x^5}{15}+\cdots$, we have $\frac{\sin 2x}{x}=2-\frac{4x^2}{3}+\frac{4x^4}{15}+\cdots$, so $\lim_{x\to 0}\frac{\sin 2x}{x}=2$.

11.4.11 We compute that

$$\begin{aligned}\frac{e^x-e^{-x}}{x}&=\frac{1}{x}\left(\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)-\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\cdots\right)\right)\\&=\frac{1}{x}\left(2x+\frac{x^3}{3}+\cdots\right)=2+\frac{x^2}{3}+\cdots\end{aligned}$$

so the limit of $\frac{e^x-e^{-x}}{x}$ as $x\to 0$ is 2.

11.4.12 Because $-e^x=-1-x-\frac{x^2}{2}-\frac{x^3}{6}+\cdots$, we have $\frac{1+x-e^x}{4x^2}=-\frac{1}{8}-\frac{x}{24}+\cdots$, so $\lim_{x\to 0}\frac{1+x-e^x}{4x^2}=-\frac{1}{8}$.

11.4.13 We compute that

$$\begin{aligned}\frac{2\cos 2x-2+4x^2}{2x^4}&=\frac{1}{2x^4}\left(2(1-\frac{(2x)^2}{2}+\frac{(2x)^4}{24}-\frac{(2x)^6}{720}+\cdots)-2+4x^2\right)\\&=\frac{1}{2x^4}\left(\frac{(2x)^4}{12}-\frac{(2x)^6}{360}+\cdots\right)=\frac{2}{3}-\frac{4x^2}{45}+\cdots\end{aligned}$$

so the limit of $\frac{2\cos 2x-2+4x^2}{2x^4}$ as $x\to 0$ is $\frac{2}{3}$.

11.4.14 We substitute $t = \dfrac{1}{x}$ and find $\lim\limits_{t\to 0} \dfrac{\sin t}{t}$. We compute that

$$\frac{\sin t}{t} = \frac{1}{t}\left(t - \frac{t^3}{6} + \cdots\right) = 1 - \frac{t^2}{6} + \cdots$$

so the limit of $x \sin\left(\dfrac{1}{x}\right)$ as $x \to \infty$ is 1.

11.4.15 We have $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$, so that

$$\frac{\ln(1+x) - x + x^2/2}{x^3} = \frac{x^3/3 - x^4/4 + \cdots}{x^3} = \frac{1}{3} - \frac{x}{4} + \cdots$$

so that $\lim\limits_{x\to 0} \dfrac{\ln(1+x) - x + x^2/2}{x^3} = \dfrac{1}{3}$.

11.4.16 The Taylor series for $\ln(x-3)$ centered at $x = 4$ is

$$(x-4) - \frac{1}{2}(x-4)^2 + \cdots .$$

We compute that

$$\begin{aligned}\frac{x^2-16}{\ln(x-3)} &= \frac{x^2-16}{(x-4) - \frac{1}{2}(x-4)^2 + \cdots} = \frac{(x-4)(x+4)}{(x-4) - \frac{1}{2}(x-4)^2 + \cdots} \\ &= \frac{x+4}{1 - \frac{1}{2}(x-4) + \cdots}\end{aligned}$$

so the limit of $\dfrac{x^2-16}{\ln(x-3)}$ as $x \to 4$ is 8.

11.4.17 We compute that

$$\begin{aligned}\frac{3\tan^{-1}x - 3x + x^3}{x^5} &= \frac{1}{x^5}\left(3\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) - 3x + x^3\right) \\ &= \frac{1}{x^5}\left(\frac{3x^5}{5} - \frac{3x^7}{7} + \cdots\right) = \frac{3}{5} - \frac{3x^2}{7} + \cdots\end{aligned}$$

so the limit of $\dfrac{3\tan^{-1}x - 3x + x^3}{x^5}$ as $x \to 0$ is $\dfrac{3}{5}$.

11.4.18 The Taylor series for $\sqrt{1+x}$ centered at 0 is

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots .$$

We compute that

$$\begin{aligned}\frac{\sqrt{1+x} - 1 - (x/2)}{4x^2} &= \frac{1}{4x^2}\left(\left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots\right) - 1 - \frac{x}{2}\right) \\ &= \frac{1}{4x^2}\left(-\frac{x^2}{8} + \frac{x^3}{16} + \cdots\right) = -\frac{1}{32} + \frac{x}{64} + \cdots\end{aligned}$$

so the limit of $\dfrac{\sqrt{1+x} - 1 - (x/2)}{4x^2}$ as $x \to 0$ is $-\dfrac{1}{32}$.

11.4.19 The Taylor series for $\sin 2x$ centered at 0 is

$$\sin 2x = 2x - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 - \frac{1}{7!}(2x)^7 + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots .$$

Thus

$$\begin{aligned}\frac{12x - 8x^3 - 6\sin 2x}{x^5} &= \frac{12 - 8x^3 - (12x - 8x^3 + \frac{8}{5}x^5 - \frac{16}{105}x^7 + \cdots)}{x^5}\\ &= -\frac{8}{5} + \frac{16}{105}x^2 - \cdots ,\end{aligned}$$

so $\lim\limits_{x\to 0} \dfrac{12x - 8x^3 - 6\sin 2x}{x^5} = -\dfrac{8}{5}$.

11.4.20 The Taylor series for $\ln x$ centered at 1 is

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \cdots .$$

We compute that

$$\frac{x-1}{\ln x} = \frac{x-1}{(x-1) - \frac{1}{2}(x-1)^2 + \cdots} = \frac{1}{1 - \frac{1}{2}(x-1) + \cdots}$$

so the limit of $\dfrac{x-1}{\ln x}$ as $x \to 1$ is 1.

11.4.21 The Taylor series for $\ln(x-1)$ centered at 2 is

$$\ln(x-1) = (x-2) - \frac{1}{2}(x-2)^2 + \cdots .$$

We compute that

$$\frac{x-2}{\ln(x-1)} = \frac{x-2}{(x-2) - \frac{1}{2}(x-2)^2 + \cdots} = \frac{1}{1 - \frac{1}{2}(x-2) + \cdots}$$

so the limit of $\dfrac{x-2}{\ln(x-1)}$ as $x \to 2$ is 1.

11.4.22 Because $e^{1/x} = 1 + (1/x) + 1/(2x^2) + \cdots$, we have

$$x(e^{1/x} - 1) = 1 + 1/(2x) + \cdots .$$

Thus, $\lim\limits_{x\to\infty} x(e^{1/x} - 1) = 1$.

11.4.23 Computing Taylor series centers at 0 gives

$$e^{-2x} = 1 - 2x + \frac{1}{2!}(-2x)^2 + \frac{1}{3!}(-2x)^3 + \cdots = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \cdots$$

$$e^{-x/2} = 1 - \frac{x}{2} + \frac{1}{2!}\left(-\frac{x}{2}\right)^2 + \frac{1}{3!}\left(-\frac{x}{2}\right)^3 + \cdots = 1 - \frac{x}{2} + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \cdots .$$

Thus

$$\begin{aligned}\frac{e^{-2x} - 4e^{-x/2} + 3}{2x^2} &= \frac{1 - 2x + 2x^2 - \frac{4}{3}x^3 + \cdots - (4 - 2x + \frac{1}{2}x^2 - \frac{1}{12}x^3 + \cdots) + 3}{2x^2}\\ &= \frac{\frac{3}{2}x^2 - \frac{5}{4}x^3 + \cdots}{2x^2}\\ &= \frac{3}{4} - \frac{5}{8}x + \cdots\end{aligned}$$

so $\lim\limits_{x\to 0} \dfrac{e^{-2x} - 4e^{-x/2} + 3}{2x^2} = \dfrac{3}{4}$.

11.4.24 The Taylor series for $(1-2x)^{-1/2}$ centered at 0 is

$$(1-2x)^{-1/2} = 1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \cdots .$$

We compute that

$$\begin{aligned}\frac{(1-2x)^{-1/2} - e^x}{8x^2} &= \frac{1}{8x^2}\left(\left(1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \cdots\right) - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\right)\\ &= \frac{1}{8x^2}\left(x^2 + \frac{7x^3}{3} + \cdots\right) = \frac{1}{8} + \frac{7x}{24} + \cdots\end{aligned}$$

so the limit of $\frac{(1-2x)^{-1/2} - e^x}{8x^2}$ as $x \to 0$ is $\frac{1}{8}$.

11.4.25

a. $f'(x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) = \sum_{k=1}^{\infty} k\frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = f(x).$

b. $f'(x) = e^x$ as well.

c. The series converges on $(-\infty, \infty)$.

11.4.26

a. $f'(x) = \frac{d}{dx}(\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}) = \sum_{k=1}^{\infty}(-1)^k(2k)\frac{x^{2k-1}}{(2k)!} = \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k-1}}{(2k-1)!} = -\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}.$

b. $f'(x) = -\sin x.$

c. The series converges on $(-\infty, \infty)$, because the series for $\cos x$ does.

11.4.27

a. $f'(x) = \frac{d}{dx}(\ln(1+x)) = \frac{d}{dx}(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}x^k) = \sum_{k=1}^{\infty}(-1)^{k+1}x^{k-1} = \sum_{k=0}^{\infty}(-1)^k x^k.$

b. This is the power series for $\frac{1}{1+x}$.

c. The Taylor series for $\ln(1+x)$ converges on $(-1, 1)$, as does the Taylor series for $\frac{1}{1+x}$.

11.4.28

a. $f'(x) = \frac{d}{dx}(\sin x^2) = \frac{d}{dx}(\sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+2}}{(2k+1)!}) = \sum_{k=0}^{\infty}(-1)^k \cdot 2(2k+1)\frac{x^{4k+1}}{(2k+1)!} = 2\sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+1}}{(2k)!} = 2x\sum_{k=0}^{\infty}(-1)^k \frac{x^{4k}}{(2k)!}$.

b. This is the power series for $2x\cos x^2$.

c. Because the Taylor series for $\sin x^2$ converges everywhere, the Taylor series for $2x\cos x^2$ does as well.

11.4.29

a.

$$f'(x) = \frac{d}{dx}(e^{-2x}) = \frac{d}{dx}(\sum_{k=0}^{\infty}\frac{(-2x)^k}{k!}) = \frac{d}{dx}(\sum_{k=0}^{\infty}(-2)^k \frac{x^k}{k!}) = -2\sum_{k=1}^{\infty}(-2)^{k-1}\frac{x^{k-1}}{(k-1)!} = -2\sum_{k=0}^{\infty}\frac{(-2x)^k}{k!}.$$

b. This is the Taylor series for $-2e^{-2x}$.

c. Because the Taylor series for e^{-2x} converges on $(-\infty, \infty)$, so does this one.

11.4.30

a. We have

$$f'(x) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} x^k\right) = \frac{d}{dx}\left(1 + \sum_{k=1}^{\infty} x^k\right) = \sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty}(k+1)x^k.$$

b. From the formula for $(1+x)^p$ in the text, we see that the Taylor series for $\frac{1}{(1-x)^2}$ is

$$\sum_{k=0}^{\infty}\frac{(-2)(-3)\cdots(-2-k+1)}{k!}(-x)^k = \sum_{k=0}^{\infty}(-1)^k(-1)^k\frac{(k+1)!}{k!}x^k = \sum_{k=0}^{\infty}(k+1)x^k,$$

so that $f'(x)$ is simply $\frac{1}{(1-x)^2}$ as expected.

c. Since the Taylor series for $\frac{1}{1-x}$ converges on $(-1, 1)$, so does the series for $\frac{1}{(1-x)^2}$. Checking the endpoints, we see that the series diverges at both endpoints by the Divergence test, so that the interval of convergence for $f'(x)$ is also $(-1, 1)$.

11.4.31

a. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$, so $\frac{d}{dx}\tan^{-1} x^2 = 1 - x^2 + x^4 - x^6 + \cdots$.

b. This is the series for $\frac{1}{1+x^2}$.

c. Because the series for $\tan^{-1} x$ has a radius of convergence of 1, this series does too. Checking the endpoints shows that the interval of convergence is $(-1, 1)$.

11.4.32

a. $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots$, so $\frac{d}{dx}[-\ln(1-x)] = 1 + x + x^2 + x^3 + \cdots$.

b. This is the series for $\frac{1}{1-x}$.

c. The interval of convergence for $\frac{1}{1-x}$ is $(-1,1)$.

11.4.33

a. Because $y(0)=2$, we have $0=y'(0)-y(0)=y'(0)-2$ so that $y'(0)=2$. Differentiating the equation gives $y''(0)=y'(0)$, so that $y''(0)=2$. Successive derivatives also have the value 2 at 0, so the Taylor series is $2\sum_{k=0}^{\infty}\frac{t^k}{k!}$.

b. $2\sum_{k=0}^{\infty}\frac{t^k}{k!}=2e^t$.

11.4.34

a. Because $y(0)=0$, we see that $y'(0)=8$. Differentiating the equation gives $y''(0)+4y'(0)=0$, so $y''(0)+4\cdot 8=0$, $y''(0)=-4\cdot 8$. Continuing, $y'''(0)+4\cdot(-4\cdot 8)=0$, so $y'''(0)=4\cdot 4\cdot 8$, and in general $y^{(k)}(0)=(-1)^{k+1}2\cdot 4^k$ for $k\geq 1$, so the Taylor series is $2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(4t)^k}{k!}$.

b. $2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(4t)^k}{k!}=2(1-e^{-4t})$.

11.4.35

a. $y(0)=2$, so that $y'(0)=16$. Differentiating, $y''(t)-3y'(t)=0$, so that $y''(0)=48$, and in general $y^{(k)}(0)=3y^{(k-1)}(0)=3^{k-1}\cdot 16$. Thus the power series is $2+\frac{16}{3}\sum_{k=1}^{\infty}\frac{(3t)^k}{k!}=2+\sum_{k=1}^{\infty}\frac{3^{k-1}16}{k!}t^k$.

b. $2+\frac{16}{3}\sum_{k=1}^{\infty}\frac{(3t)^k}{k!}=2+\frac{16}{3}(e^{3t}-1)=\frac{16}{3}e^{3t}-\frac{10}{3}$.

11.4.36

a. $y(0)=2$, so $y'(0)=12+9=21$. Differentiating, $y^{(n)}(0)=6y^{(n-1)}(0)$ for $n>1$, so that $y^{(n)}(0)=6^{n-1}\cdot 21$ for $n\geq 1$. Thus the power series is $2+\sum_{k=1}^{\infty}21\cdot 6^{k-1}\frac{t^k}{k!}=2+\frac{7}{2}\sum_{k=1}^{\infty}\frac{(6t)^k}{k!}$.

b. $2+\frac{7}{2}\sum_{k=1}^{\infty}\frac{(6t)^k}{k!}=2+\frac{7}{2}(e^{6t}-1)=\frac{7}{2}e^{6t}-\frac{3}{2}$.

11.4.37 The Taylor series for e^{-x^2} is $\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{k!}$. Thus, the desired integral is

$$\int_0^{0.25}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{k!}\,dx=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)k!}\bigg|_0^{0.25}=\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)k!4^{2k+1}}.$$

Because this is an alternating series, to approximate it to within 10^{-4}, we must find n such that $a_{n+1}<10^{-4}$, or $\frac{1}{(2n+3)(n+1)!\cdot 4^{2n+3}}<10^{-4}$. This occurs for $n=1$, so $\sum_{k=0}^{1}(-1)^k\frac{1}{(2k+1)\cdot k!\cdot 4^{2k+1}}=\frac{1}{4}-\frac{1}{192}\approx 0.245$.

11.4.38 The Taylor series for $\sin x^2$ is $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{x^{4k+2}}{(2k+1)!}$. Thus the desired integral is

$$\int_0^{0.2}\sum_{k=0}^{\infty}(-1)^k\frac{x^{4k+2}}{(2k+1)!}\,dx=\sum_{k=0}^{\infty}(-1)^k\frac{x^{4k+3}}{(4k+3)(2k+1)!}\bigg|_0^{0.2}=\sum_{k=0}^{\infty}(-1)^k\frac{0.2^{4k+3}}{(4k+3)(2k+1)!}.$$

Because this is an alternating series, to approximate it to within 10^{-4}, we must find n such that $a_{n+1}<10^{-4}$, or $\dfrac{0.2^{4n+7}}{(4n+7)(2n+3)!}<10^{-4}$. This occurs first for $n=0$, so we obtain $\dfrac{0.2^3}{3\cdot 1!}\approx 2.67\times 10^{-3}$.

11.4.39 The Taylor series for $\cos 2x^2$ is $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{(2x^2)^{2k}}{(2k)!}=\sum_{k=0}^{\infty}(-1)^k\frac{4^kx^{4k}}{(2k)!}$. Note that $\cos x$ is an even function, so we compute the integral from 0 to 0.35 and double it:

$$2\int_0^{0.35}\sum_{k=0}^{\infty}(-1)^k\frac{4^kx^{4k}}{(2k)!}\,dx=2\left(\sum_{k=0}^{\infty}(-1)^k\frac{4^kx^{4k+1}}{(4k+1)(2k)!}\right)\bigg|_0^{0.35}=2\left(\sum_{k=0}^{\infty}(-1)^k\frac{4^k(0.35)^{4k+1}}{(4k+1)(2k)!}\right).$$

Because this is an alternating series, to approximate it to within $\frac{1}{2}\cdot 10^{-4}$, we must find n such that $a_{n+1}<\frac{1}{2}\cdot 10^{-4}$, or $\dfrac{4^{n+1}(0.35)^{4n+5}}{(4n+3)(2n+2)!}<\dfrac{1}{2}\cdot 10^{-4}$. This occurs first for $n=1$, and we have $2\left(0.35-\dfrac{4\cdot(0.35)^5}{5\cdot 2!}\right)\approx 0.696$.

11.4.40 The Taylor series for $(1+x^4)^{1/2}$ is $\displaystyle\sum_{k=0}^{\infty}\binom{1/2}{k}x^{4k}$, so the desired integral is

$$\int_0^{0.2}\sum_{k=0}^{\infty}\binom{1/2}{k}x^{4k}\,dx=\sum_{k=0}^{\infty}\frac{1}{4k+1}\binom{1/2}{k}x^{4k+1}\bigg|_0^{0.2}=\sum_{k=0}^{\infty}\frac{1}{4k+1}\binom{1/2}{k}(0.2)^{4k+1}.$$

This is an alternating series because the binomial coefficients alternate in sign, so to approximate it to within 10^{-4}, we must find n such that $a_{n+1}<10^{-4}$, or $\left|\dfrac{1}{4n+5}\dbinom{1/2}{n+1}(0.2)^{4n+5}\right|<10^{-4}$. This happens first for $n=0$, so the approximation is $\binom{1/2}{0}\cdot 0.2=0.2$.

11.4.41 $\tan^{-1}x=x-x^3/3+x^5/5-x^7/7+x^9/9-\cdots$, so $\displaystyle\int\tan^{-1}x\,dx=\int(x-x^3/3+x^5/5-x^7/7+x^9/9-\cdots)\,dx=C+\frac{x^2}{2}-\frac{x^4}{12}+\frac{x^6}{30}-\frac{x^8}{56}+\cdots$. Thus,

$$\int_0^{0.35}\tan^{-1}x\,dx=\frac{(0.35)^2}{2}-\frac{(0.35)^4}{12}+\frac{(0.35)^6}{30}-\frac{(0.35)^8}{56}+\cdots.$$

Note that this series is alternating, and $\dfrac{(0.35)^6}{30}<10^{-4}$, so we add the first two terms to approximate the integral to the desired accuracy. Calculating gives approximately 0.060.

11.4.42 $\ln(1+x^2)=x^2-\dfrac{x^4}{2}+\dfrac{x^6}{3}-\dfrac{x^8}{4}+\cdots$, so $\displaystyle\int\ln(1+x^2)\,dx=\int\left(x^2-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+\cdots\right)dx=C+\frac{x^3}{3}-\frac{x^5}{10}+\frac{x^7}{21}-\frac{x^9}{36}+\frac{x^{11}}{55}+\cdots$. Thus,

$$\int_0^{0.4}\ln(1+x^2)\,dx=\frac{(0.4)^3}{3}-\frac{(0.4)^5}{10}+\frac{(0.4)^7}{21}-\frac{(0.4)^9}{36}+\cdots.$$

Because $\dfrac{(0.4)^7}{21}<10^{-4}$, we add the first two terms to approximate the integral to the desired accuracy. Calculating gives approximately 0.020.

11.4.43 The Taylor series for $(1+x^6)^{-1/2}$ is $\displaystyle\sum_{k=0}^{\infty}\binom{-1/2}{k}x^{6k}$, so the desired integral is

$$\int_0^{0.5}\sum_{k=0}^{\infty}\binom{-1/2}{k}x^{6k}\,dx=\sum_{k=0}^{\infty}\frac{1}{6k+1}\binom{-1/2}{k}x^{6k+1}\bigg|_0^{0.5}=\sum_{k=0}^{\infty}\frac{1}{6k+1}\binom{-1/2}{k}(0.5)^{6k+1}.$$

This is an alternating series because the binomial coefficients alternate in sign, so to approximate it to within 10^{-4}, we must find n such that $a_{n+1}<10^{-4}$, or $\left|\frac{1}{6n+7}\binom{-1/2}{n+1}(0.5)^{6n+7}\right|<10^{-4}$. This occurs first for $n=1$, so we have $\binom{-1/2}{0}0.5+\frac{1}{7}\binom{-1/2}{1}(0.5)^7\approx 0.499$.

11.4.44 The Taylor series for $\frac{\ln(1+t)}{t}$ centered at 0 is $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{t^k}{k+1}$. The desired integral is thus $\displaystyle\int_0^{0.2}\sum_{k=0}^{\infty}(-1)^k\frac{t^k}{k+1}\,dt=\sum_{k=0}^{\infty}(-1)^k\frac{t^{k+1}}{(k+1)^2}\bigg|_0^{0.2}=\sum_{k=0}^{\infty}(-1)^k\frac{(0.2)^{k+1}}{(k+1)^2}$. This is an alternating series, so to approximate it to within 10^{-4}, we must find n such that $a_{n+1}<10^{-4}$, or $\frac{(0.2)^{n+2}}{(n+2)^2}<10^{-4}$. This occurs first for $n=3$, so we have $\displaystyle\sum_{k=0}^{3}(-1)^k\frac{(0.2)^{k+1}}{(k+1)^2}\approx 0.191$.

11.4.45 Use the Taylor series for e^x at 0: $1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}$.

11.4.46 Use the Taylor series for e^x at 0: $1+\frac{1/2}{1!}+\frac{(1/2)^2}{2!}+\frac{(1/2)^3}{3!}=1+\frac{1}{2}+\frac{1}{8}+\frac{1}{8\cdot 3!}$.

11.4.47 Use the Taylor series for $\cos x$ at 0: $1-\frac{2^2}{2!}+\frac{2^4}{4!}-\frac{2^6}{6!}$

11.4.48 Use the Taylor series for $\sin x$ at 0: $1-\frac{1^3}{3!}+\frac{1^5}{5!}-\frac{1^7}{7!}=1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}$.

11.4.49 Use the Taylor series for $\ln(1+x)$ evaluated at $x=1/2$: $\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{3}\cdot\frac{1}{8}-\frac{1}{4}\cdot\frac{1}{16}$.

11.4.50 Use the Taylor series for $\tan^{-1}x$ evaluated at 1/2: $\frac{1}{2}-\frac{1}{3}\cdot\frac{1}{8}+\frac{1}{5}\cdot\frac{1}{32}-\frac{1}{7}\cdot\frac{1}{128}$.

11.4.51 The Taylor series for f centered at 0 is $\frac{-1+\sum_{k=0}^{\infty}\frac{x^k}{k!}}{x}=\frac{\sum_{k=1}^{\infty}\frac{x^k}{k!}}{x}=\displaystyle\sum_{k=1}^{\infty}\frac{x^{k-1}}{k!}=\sum_{k=0}^{\infty}\frac{x^k}{(k+1)!}$. Evaluating both sides at $x=1$, we have $e-1=\displaystyle\sum_{k=0}^{\infty}\frac{1}{(k+1)!}$.

11.4.52 The Taylor series for f centered at 0 is

$$\frac{-1+\sum_{k=0}^{\infty}\frac{x^k}{k!}}{x}=\frac{\sum_{k=1}^{\infty}\frac{x^k}{k!}}{x}=\sum_{k=1}^{\infty}\frac{x^{k-1}}{k!}=\sum_{k=0}^{\infty}\frac{x^k}{(k+1)!}.$$

Differentiating, the Taylor series for $f'(x)$ is $f'(x)=\frac{(x-1)e^x+1}{x^2}=\displaystyle\sum_{k=1}^{\infty}\frac{kx^{k-1}}{(k+1)!}$. Evaluating both sides at 2 gives. $\frac{e^2+1}{4}=\displaystyle\sum_{k=1}^{\infty}\frac{k\cdot 2^{k-1}}{(k+1)!}$.

11.4.53 The Maclaurin series for $\ln(1+x)$ is

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}.$$

By the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}k}{x^k(k+1)}\right| = |x|$, so the radius of convergence is 1. The series diverges at -1 and converges at 1, so the interval of convergence is $(-1,1]$. Evaluating at 1 gives $\ln 2 = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

11.4.54 The Taylor series for $\ln(1+x)$ at 0 is $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$. By the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}k}{x^k(k+1)}\right| = |x|$, so the radius of convergence is 1. The series diverges at -1 and converges at 1, so the interval of convergence is $(-1,1]$. Evaluate both sides at $-1/2$ to get

$$f\left(-\frac{1}{2}\right) = \ln(1/2) = -\ln 2 = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(-1/2)^k}{k} = -\sum_{k=1}^{\infty}\frac{1}{k\cdot 2^k}.$$

so that $\ln 2 = \sum_{k=1}^{\infty}\frac{1}{k\cdot 2^k}$.

11.4.55 $\sum_{k=0}^{\infty}\frac{x^k}{2^k} = \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k = \frac{1}{1-\frac{x}{2}} = \frac{2}{2-x}$.

11.4.56 $\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{3^k} = \sum_{k=0}^{\infty}\left(\frac{-x}{3}\right)^k = \frac{1}{1+\frac{x}{3}} = \frac{3}{3+x}$.

11.4.57 $\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{4^k} = \sum_{k=0}^{\infty}\left(\frac{-x^2}{4}\right)^k = \frac{1}{1+\frac{x^2}{4}} = \frac{4}{4+x^2}$.

11.4.58 $\sum_{k=0}^{\infty}2^k x^{2k+1} = x\sum_{k=0}^{\infty}(2x^2)^k = \frac{x}{1-2x^2}$.

11.4.59 $\ln(1+x) = -\sum_{k=1}^{\infty}(-1)^k\frac{x^k}{k}$, so $\ln(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k}$, and finally $-\ln(1-x) = \sum_{k=1}^{\infty}\frac{x^k}{k}$.

11.4.60 $\sum_{k=0}^{\infty}\frac{(-1)^k x^{k+1}}{4^k} = -4\sum_{k=0}^{\infty}\left(\frac{-x}{4}\right)^{k+1} = -4(-1+\sum_{k=0}^{\infty}\left(\frac{-x}{4}\right)^k) = 4 - \frac{4}{1+\frac{x}{4}} = 4 - \frac{16}{4+x} = \frac{4x}{4+x}$

11.4.61

$$\begin{aligned}
\sum_{k=1}^{\infty}(-1)^k\frac{kx^{k+1}}{3^k} &= \sum_{k=1}^{\infty}(-1)^k\frac{k}{3^k}x^{k+1} = \sum_{k=1}^{\infty}k\left(-\frac{1}{3}\right)^k x^{k+1}\\
&= x^2\sum_{k=1}^{\infty}\left(-\frac{1}{3}\right)^k kx^{k-1} = x^2\sum_{k=1}^{\infty}\left(-\frac{1}{3}\right)^k\frac{d}{dx}(x^k)\\
&= x^2\frac{d}{dx}\left(\sum_{k=1}^{\infty}\left(-\frac{x}{3}\right)^k\right) = x^2\frac{d}{dx}\left(\frac{1}{1+\frac{x}{3}}\right) = -\frac{3x^2}{(x+3)^2}.
\end{aligned}$$

11.4.62 By Exercise 53, $\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x)$, so $\sum_{k=1}^{\infty} \frac{x^{2k}}{k} = \sum_{k=1}^{\infty} \frac{(x^2)^k}{k} = -\ln(1-x^2)$.

11.4.63 $\sum_{k=2}^{\infty} \frac{k(k-1)x^k}{3^k} = x^2 \sum_{k=2}^{\infty} \frac{k(k-1)x^{k-2}}{3^k} = x^2 \frac{d^2}{dx^2}\left(\sum_{k=2}^{\infty} \frac{x^k}{3^k}\right)$
$= x^2 \frac{d^2}{dx^2}\left(\sum_{k=2}^{\infty} \left(\frac{x}{3}\right)^k\right) = x^2 \frac{d^2}{dx^2}\left(\frac{x^2}{9} \cdot \frac{1}{1-\frac{x}{3}}\right) = x^2 \frac{d^2}{dx^2}\left(\frac{x^2}{9-3x}\right) = x^2 \frac{-6}{(x-3)^3} = \frac{-6x^2}{(x-3)^3}$.

11.4.64 $\sum_{k=2}^{\infty} \frac{x^k}{k(k-1)} = \sum_{k=2}^{\infty} \frac{x^k}{k-1} - \sum_{k=2}^{\infty} \frac{x^k}{k} = x\sum_{k=1}^{\infty} \frac{x^k}{k} - \sum_{k=1}^{\infty} \frac{x^k}{k} + x, = -x\ln(1-x) + \ln(1-x) + x = x + (1-x)\ln(1-x)$.

11.4.65

a. False. This is because $\frac{1}{1-x}$ is not continuous at 1, which is in the interval of integration.

b. False. The Ratio Test shows that the radius of convergence for the Taylor series for $\tan^{-1} x$ centered at 0 is 1.

c. True. $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$. Substitute $x = \ln 2$.

11.4.66 The Taylor series for e^{ax} centered at 0 is

$$e^{ax} = 1 + ax + \frac{(ax)^2}{2} + \frac{(ax)^3}{6} + \cdots .$$

We compute that

$$\begin{aligned} \frac{e^{ax}-1}{x} &= \frac{1}{x}\left(\left(1 + ax + \frac{(ax)^2}{2} + \frac{(ax)^3}{6} + \cdots\right) - 1\right) \\ &= \frac{1}{x}\left(ax + \frac{(ax)^2}{2} + \frac{(ax)^3}{6} + \cdots\right) = a + \frac{a^2x}{2} + \frac{a^3x^2}{6} + \cdots \end{aligned}$$

so the limit of $\frac{e^{ax}-1}{x}$ as $x \to 0$ is a.

11.4.67 The Taylor series for $\sin x$ centered at 0 is

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots .$$

We compute that

$$\begin{aligned} \frac{\sin ax}{\sin bx} &= \frac{ax - \frac{(ax)^3}{6} + \frac{(ax)^5}{120} - \cdots}{bx - \frac{(bx)^3}{6} + \frac{(bx)^5}{120} - \cdots} \\ &= \frac{a - \frac{a^3x^2}{6} + \frac{a^5x^4}{120} - \cdots}{b - \frac{b^3x^2}{6} + \frac{b^5x^4}{120} - \cdots} \end{aligned}$$

so the limit of $\frac{\sin ax}{\sin bx}$ as $x \to 0$ is $\frac{a}{b}$.

11.4.68 The Taylor series for $\sin ax$ centered at 0 is

$$\sin ax = ax - \frac{(ax)^3}{6} + \frac{(ax)^5}{120} - \cdots$$

and the Taylor series for $\tan^{-1} ax$ centered at 0 is

$$\tan^{-1} ax = ax - \frac{(ax)^3}{3} + \frac{(ax)^5}{5} - \cdots .$$

We compute that

$$\begin{aligned}\frac{\sin ax - \tan^{-1} ax}{bx^3} &= \frac{1}{bx^3}\left(\left(ax - \frac{(ax)^3}{6} + \frac{(ax)^5}{120} - \cdots\right) - \left(ax - \frac{(ax)^3}{3} + \frac{(ax)^5}{5} - \cdots\right)\right)\\ &= \frac{1}{bx^3}\left(\frac{(ax)^3}{6} - \frac{23(ax)^5}{120} + \cdots\right) = \frac{a^3}{6b} - \frac{23a^5}{120b}x^2 + \cdots\end{aligned}$$

so the limit of $\dfrac{\sin ax - \tan^{-1} ax}{bx^3}$ as $x \to 0$ is $\dfrac{a^3}{6b}$.

11.4.69 Compute instead the limit of the log of this expression, $\lim_{x\to 0} \frac{\ln(\sin x/x)}{x^2}$. If the Taylor expansion of $\ln(\sin x/x)$ is $\sum_{k=0}^{\infty} c_k x^k$, then $\lim_{x\to 0} \dfrac{\ln(\sin x/x)}{x^2} = \lim_{x\to 0} \sum_{k=0}^{\infty} c_k x^{k-2} = \lim_{x\to 0} c_0 x^{-2} + c_1 x^{-1} + c_2$, because the higher-order terms have positive powers of x and thus approach zero as x does. So compute the terms of the Taylor series of $\ln\left(\frac{\sin x}{x}\right)$ up through the quadratic term. The relevant Taylor series are: $\dfrac{\sin x}{x} = 1 - \dfrac{1}{6}x^2 + \dfrac{1}{120}x^4 - \cdots$, $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$ and we substitute the Taylor series for $\dfrac{\sin x}{x} - 1$ for x in the Taylor series for $\ln(1+x)$. Because the lowest power of x in the first Taylor series is 2, it follows that only the linear term in the series for $\ln(1+x)$ will give any powers of x that are at most quadratic. The only term that results is $-\dfrac{1}{6}x^2$. Thus $c_0 = c_1 = 0$ in the above, and $c_2 = -\dfrac{1}{6}$, so that $\lim_{x\to 0} \dfrac{\ln(\sin x/x)}{x^2} = -\dfrac{1}{6}$ and thus $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$.

11.4.70 We can find the Taylor series for $\ln(x + \sqrt{1+x^2})$ by substituting into $\ln(1+t)$ the Taylor series for $x + \sqrt{x^2+1} - 1$. The Taylor series in question are: $x + \sqrt{x^2+1} - 1 = x + \dfrac{1}{2}x^2 - \dfrac{1}{8}x^4 + \dfrac{1}{16}x^6 - \ldots$, $\ln(1+t) = t - \dfrac{1}{2}t^2 + \dfrac{1}{3}t^3 - \dfrac{1}{4}t^4 + \dfrac{1}{5}t^5 - \dfrac{1}{6}t^6 + \dfrac{1}{7}t^7 - \ldots$. Substituting the former into the latter and simplifying (not a simple task!), we obtain $\ln(x + \sqrt{x^2+1}) = x - \dfrac{1}{6}x^3 + \dfrac{3}{40}x^5 - \dfrac{5}{112}x^7 + \ldots$. Using the second definition, start with the Taylor series for $(1+t^2)^{-1/2}$, which is $1 - \dfrac{1}{2}t^2 + \dfrac{3}{8}t^4 - \dfrac{5}{16}t^6 + \ldots$, and integrate it: $\displaystyle\int_0^x \left(1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \ldots\right) dt = \left(t - \frac{1}{6}t^3 + \frac{3}{40}t^5 - \frac{5}{112}t^7 + \ldots\right)\Bigg|_0^x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \ldots$.

11.4.71 The Taylor series we need are $\cos x = 1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4 + \ldots$, and $e^t = 1 + t + \dfrac{1}{2!}t^2 + \dfrac{1}{3!}t^3 + \dfrac{1}{4!}t^4 + \ldots$. We are looking for powers of x^3 and x^4 that occur when the first series is substituted for t in the second series. Clearly there will be no odd powers of x, because $\cos x$ has only even powers. Thus the coefficient of x^3 is zero, so that $f^{(3)}(0) = 0$. The coefficient of x^4 comes from the expansion of $1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4$ in each term of e^t. Higher powers of x clearly cannot contribute to the coefficient of x^4. Thus consider $\left(1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4\right)^k$. The term $-\dfrac{1}{2}x^2$ generates $\dbinom{k}{2}$ terms of value $\dfrac{1}{4}x^4$ for $k \geq 2$, while the other term generates k terms of value

$\frac{1}{24}x^4$ for $k \geq 1$. These terms all have to be divided by the $k!$ appearing in the series for e^t. So the total coefficient of x^4 is $\frac{1}{24}\sum_{k=1}^{\infty}\frac{k}{k!} + \frac{1}{4}\sum_{k=2}^{\infty}\binom{k}{2}\frac{1}{k!}$, $= \frac{1}{24}\sum_{k=1}^{\infty}\frac{1}{(k-1)!} + \frac{1}{4}\sum_{k=2}^{\infty}\frac{1}{2\cdot(k-2)!}$, $= \frac{1}{24}\sum_{k=0}^{\infty}\frac{1}{k!} + \frac{1}{8}\sum_{k=0}^{\infty}\frac{1}{k!}$, $= \frac{1}{24}e + \frac{1}{8}e = \frac{e}{6}$ Thus $f^{(4)}(0) = \frac{e}{6}\cdot 4! = 4e$.

11.4.72 The Taylor series for $(1+x)^{-1/3}$ is $(1+x)^{-1/3} = 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \frac{35}{243}x^4 - \ldots$, so we want the coefficients of x^3 and x^4 in $(x^2+1)\left(1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \frac{35}{243}x^4\right)$. The coefficient of x^3 is $-\frac{1}{3} - \frac{14}{81} = -\frac{41}{81}$, and the coefficient of x^4 is $\frac{2}{9} + \frac{35}{243} = \frac{89}{243}$. Thus $f^{(3)}(0) = 6\cdot\frac{-41}{81} = \frac{-82}{27}$, and $f^{(4)}(0) = 24\cdot\frac{89}{243} = \frac{712}{81}$.

11.4.73 The Taylor series for $\sin t^2$ is $\sin t^2 = t^2 - \frac{1}{3!}t^6 + \frac{1}{5!}t^{10} - \ldots$, so that $\int_0^x \sin t^2\,dt = \frac{1}{3}t^3 - \frac{1}{7\cdot 3!}t^7 + \ldots\Big|_0^x = \frac{1}{3}x^3 - \frac{1}{7\cdot 3!}x^7 + \ldots$. Thus $f^{(3)}(0) = \frac{3!}{3} = 2$ and $f^{(4)}(0) = 0$.

11.4.74 $\frac{1}{1+t^4} = 1 - t^4 + t^8 + \ldots$, so that $\int_0^x \frac{1}{1+t^4}dt = t - \frac{1}{5}t^5 + \frac{1}{9}t^9 + \ldots\Big|_0^x = x - \frac{1}{5}x^5 + \ldots$. so that both $f^{(3)}(0)$ and $f^{(4)}(0)$ are zero.

11.4.75 Consider the series $\sum_{k=1}^{\infty}x^k = \frac{x}{1-x}$. Differentiating both sides gives $\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}kx^{k-1} = \frac{1}{x}\sum_{k=0}^{\infty}kx^k$ so that $\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty}kx^k$. Evaluate both sides at $x = 1/2$ to see that the sum of the series is $\frac{1/2}{(1-1/2)^2} = 2$. Thus the expected number of tosses is 2.

11.4.76

a. $\sum_{k=0}^{\infty}\frac{1}{6}\left(\frac{5}{6}\right)^{2k} = \frac{1}{6}\sum_{k=0}^{\infty}\left(\frac{25}{36}\right)^k = \frac{1}{6}\cdot\frac{1}{1-25/36} = \frac{6}{11}$.

b. Consider the series $\sum_{k=1}^{\infty}x^k = \frac{x}{1-x}$. Differentiating both sides gives $\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty}kx^{k-1}$ Evaluating at $x = 5/6$ and multiplying the result by $1/6$, we get $\frac{1}{6}\cdot\frac{1}{(1-5/6)^2} = 6$.

11.4.77 We look first for a Taylor series for $(1-k^2\sin^2\theta)^{-1/2}$. Because $(1-k^2x^2)^{-1/2} = (1-(kx)^2)^{-1/2} = \sum_{i=0}^{\infty}\binom{-1/2}{i}(kx)^{2i}$, and $\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \ldots$, substituting the second series into the first gives

$$\begin{aligned}\frac{1}{\sqrt{1-k^2\sin^2\theta}} &= 1 + \frac{1}{2}k^2\theta^2 + \left(-\frac{1}{6}k^2 + \frac{3}{8}k^4\right)\theta^4 \\ &+ \left(\frac{1}{45}k^2 - \frac{1}{4}k^4 + \frac{5}{16}k^6\right)\theta^6 + \left(\frac{-1}{630}k^2 + \frac{3}{40}k^4 - \frac{5}{16}k^6 + \frac{35}{128}k^8\right)\theta^8 + \ldots.\end{aligned}$$

Integrating with respect to θ and evaluating at $\pi/2$ (the value of the antiderivative is 0 at 0) gives

$$\begin{aligned}&\frac{1}{2}\pi + \frac{1}{48}k^2\pi^3 + \frac{1}{160}\left(-\frac{1}{6}k^2 + \frac{3}{8}k^4\right)\pi^5 + \frac{1}{896}\left(\frac{1}{45}k^2 - \frac{1}{4}k^4 + \frac{5}{16}k^6\right)\pi^7 \\ &+ \frac{1}{4608}\left(-\frac{1}{630}k^2 + \frac{3}{40}k^4 - \frac{5}{16}k^6 + \frac{35}{128}k^8\right)\pi^9.\end{aligned}$$

Evaluating these terms for $k = 0.1$ gives $F(0.1) \approx 1.574749680$. The true value is approximately 1.574745562.

11.4.78

a. $\dfrac{\sin t}{t} = \displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots.$

b. $\displaystyle\int_0^x \frac{\sin t}{t}\,dt = \sum_{k=0}^{\infty}\int_0^x (-1)^k \frac{t^{2k}}{(2k+1)!}\,dt = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)(2k+1)!}.$

c. This is an alternating series, so we want n such that $a_{n+1} < 10^{-3}$, or $\dfrac{0.5^{2n+3}}{(2n+3)(2n+3)!} < 10^{-3}$ (resp. $\dfrac{1^{2n+3}}{(2n+3)(2n+3)!} < 10^{-3}$), which gives $n = 1$ (resp. $n = 2$). Thus $\mathrm{Si}(0.5) \approx \dfrac{0.5}{1} - \dfrac{0.5^3}{3 \cdot 3!} \approx 0.4930555556$, $\mathrm{Si}(1.0) \approx 1 - \dfrac{1}{3 \cdot 3!} + \dfrac{1}{5 \cdot 5!} \approx 0.9461111111$.

11.4.79

a. By the Fundamental Theorem, $S'(x) = \sin x^2$, $C'(x) = \cos x^2$.

b. The relevant Taylor series are $\sin t^2 = t^2 - \frac{1}{3!}t^6 + \frac{1}{5!}t^{10} - \frac{1}{7!}t^{14} + \dots$, and $\cos t^2 = 1 - \frac{1}{2!}t^4 + \frac{1}{4!}t^8 - \frac{1}{6!}t^{12} + \dots$. Integrating, we have $S(x) = \frac{1}{3}x^3 - \frac{1}{7 \cdot 3!}x^7 + \frac{1}{11 \cdot 5!}x^{11} - \frac{1}{15 \cdot 7!}x^{15} + \dots$, and $C(x) = x - \frac{1}{5 \cdot 2!}x^5 + \frac{1}{9 \cdot 4!}x^9 - \frac{1}{13 \cdot 6!}x^{13} + \dots$.

c. $S(0.05) \approx \frac{1}{3}(0.05)^3 - \frac{1}{42}(0.05)^7 + \frac{1}{1320}(0.05)^{11} - \frac{1}{75600}(0.05)^{15} \approx 4.166664807 \times 10^{-5}$. $C(-0.25) \approx (-0.25) - \frac{1}{10}(-0.25)^5 + \frac{1}{216}(-0.25)^9 - \frac{1}{9360}(-0.25)^{13} \approx -.2499023616$.

d. The series is alternating. Because $a_{n+1} = \dfrac{1}{(4n+7)(2n+3)!}(0.05)^{4n+7}$, and this is less than 10^{-4} for $n = 0$, only one term is required.

e. The series is alternating. Because $a_{n+1} = \dfrac{1}{(4n+5)(2n+2)!}(0.25)^{4n+5}$, and this is less than 10^{-6} for $n = 1$, two terms are required.

11.4.80

a. $\dfrac{d}{dx}\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}}(e^{-x^2}).$

b. $e^{-t^2} = 1 - t^2 + \dfrac{t^4}{2!} - \dfrac{t^6}{3!} + \dots = \displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{t^{2k}}{k!}$, so that the Maclaurin series for the error function is $\mathrm{erf(x)} = \dfrac{2}{\sqrt{\pi}}\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5 \cdot 2!} - \dfrac{x^7}{7 \cdot 3!} + \dots\right)$.

c. $\mathrm{erf}(0.15) \approx \dfrac{2}{\sqrt{\pi}}\left(0.15 - \dfrac{0.15^3}{3} + \dfrac{0.15^5}{10} - \dfrac{0.15^7}{42}\right) \approx 0.1679959712.$

$\mathrm{erf}(-0.09) \approx \dfrac{2}{\sqrt{\pi}}\left(-0.09 + \dfrac{0.09^3}{3} - \dfrac{0.09^5}{10} + \dfrac{0.09^7}{42}\right) \approx -0.1012805939.$

d. The first omitted term in each case is $\dfrac{x^9}{9 \cdot 5!} = \dfrac{x^9}{1080}$. For $x = 0.15$, this is $\approx 3.56 \times 10^{-11}$. For $x = -0.09$, this is (in absolute value) $\approx 3.59 \times 10^{-13}$.

11.4.81

a. $J_0(x) = 1 - \frac{1}{4}x^2 + \frac{1}{16 \cdot 2!^2}x^4 - \frac{1}{2^6 \cdot 3!^2}x^6 + \dots$.

b. Using the Ratio Test: $\left|\frac{a_{k+1}}{a_k}\right| = \frac{x^{2k+2}}{2^{2k+2}((k+1)!)^2} \cdot \frac{2^{2k}(k!)^2}{x^{2k}} = \frac{x^2}{4(k+1)^2}$, which has limit 0 as $k \to \infty$ for any x. Thus the radius of convergence is infinite and the interval of convergence is $(-\infty, \infty)$.

c. Starting only with terms up through x^8, we have $J_0(x) = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \frac{1}{147456}x^8 + \dots$, $J_0'(x) = -\frac{1}{2}x + \frac{1}{16}x^3 - \frac{1}{384}x^5 + \frac{1}{18432}x^7 + \dots$, $J_0''(x) = -\frac{1}{2} + \frac{3}{16}x^2 - \frac{5}{384}x^4 + \frac{7}{18432}x^6 + \dots$ so that $x^2J_0(x) = x^2 - \frac{1}{4}x^4 + \frac{1}{64}x^6 - \frac{1}{2304}x^8 + \frac{1}{147456}x^{10} + \dots$, $xJ_0'(x) = -\frac{1}{2}x^2 + \frac{1}{16}x^4 - \frac{1}{384}x^6 + \frac{1}{18432}x^8 + \dots$, $x^2J_0''(x) = -\frac{1}{2}x^2 + \frac{3}{16}x^4 - \frac{5}{384}x^6 + \frac{7}{18432}x^8 + \dots$, and $x^2J_0''(x) + xJ_0'(x) + x^2J_0(x) = 0$.

11.4.82

a. Clearly $x = \sin s$ because BE, of length x, is the side opposite the angle measured by s in a right triangle with unit length hypotenuse.

b. In the formula $\frac{1}{2}r^2\theta$ for the formula for the area of a circular sector, we have $r = 1$, and $\theta = s$, so that the area is in fact $\frac{s}{2}$. But the area can also be expressed as an integral as follows: the area of the sector is the area under the circle between P and F (i.e. the area of the region $PAEF$), minus the area of the right triangle PEF. The area of the right triangle is $\frac{1}{2}x\sqrt{1-x^2}$ by the Pythagorean theorem and the formula for the area of a triangle. Equating these two formulae for the area of the sector, we have $\frac{s}{2} = \int_0^x \sqrt{1-t^2}\,dt - \frac{1}{2}x\sqrt{1-x^2}$, so $s = 2\int_0^x \sqrt{1-t^2}\,dt - x\sqrt{1-x^2}$.

c. The Taylor series for $\sqrt{1-t^2}$ is $1 - \frac{1}{2}t^2 - \frac{1}{8}t^4 - \frac{1}{16}t^6 - \frac{5}{128}t^8 - \dots$. Integrating and evaluating at x we have $s = \sin^{-1} x = 2\left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{1}{112}x^7 - \frac{5}{1152}x^9\right) - x\left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8\right) + \dots = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots$.

d. Suppose $x = \sin s = a_0 + a_1 s + a_2 s^2 + \dots$. Then $x = \sin(\sin^{-1}(x)) = a_0 + a_1\left(x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots\right) + a_2\left(x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots\right)^2 + \dots$. Equating coefficients yields $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = \frac{-1}{6}$, and so on.

11.4.83

a. Because $f(a) = g(a) = 0$, we use the Taylor series for $f(x)$ and $g(x)$ centered at a to compute that

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots}{g(a) + g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \cdots} \\
&= \lim_{x\to a}\frac{f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots}{g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \cdots} \\
&= \lim_{x\to a}\frac{f'(a) + \frac{1}{2}f''(a)(x-a) + \cdots}{g'(a) + \frac{1}{2}g''(a)(x-a) + \cdots} = \frac{f'(a)}{g'(a)}.
\end{aligned}$$

Because $f'(x)$ and $g'(x)$ are assumed to be continuous at a and $g'(a) \neq 0$,

$$\frac{f'(a)}{g'(a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

and we have that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

which is one form of L'Hôpital's Rule.

b. Because $f(a)=g(a)=f'(a)=g'(a)=0$, we use the Taylor series for $f(x)$ and $g(x)$ centered at a to compute that

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\frac{1}{6}f'''(a)(x-a)^3+\cdots}{g(a)+g'(a)(x-a)+\frac{1}{2}g''(a)(x-a)^2+\frac{1}{6}g'''(a)(x-a)^3+\cdots}\\ &= \lim_{x\to a}\frac{\frac{1}{2}f''(a)(x-a)^2+\frac{1}{6}f'''(a)(x-a)^3+\cdots}{\frac{1}{2}g''(a)(x-a)^2+\frac{1}{6}g'''(a)(x-a)^3+\cdots}\\ &= \lim_{x\to a}\frac{\frac{1}{2}f''(a)+\frac{1}{6}f'''(a)(x-a)+\cdots}{\frac{1}{2}g''(a)+\frac{1}{6}g'''(a)(x-a)+\cdots}=\frac{f''(a)}{g''(a)}.\end{aligned}$$

Because $f''(x)$ and $g''(x)$ are assumed to be continuous at a and $g''(a)\neq 0$,

$$\frac{f''(a)}{g''(a)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}$$

and we have that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}$$

which is consistent with two applications of L'Hôpital's Rule.

11.4.84

a. The power series for $\cos x$ has only even powers of x, so that the power series has the same value evaluated at $-x$ as it does at x.

b. The power series for $\sin x$ has only odd powers of x, so that evaluating it at $-x$ gives the opposite of its value at x.

Chapter Eleven Review

1

a. True. The approximations tend to get better as n increases in size, and also when the value being approximated is closer to the center of the series. Because 2.1 is closer to 2 than 2.2 is, and because $3>2$, we should have $|p_3(2.1)-f(2.1)|<|p_2(2.2)-f(2.2)|$.

b. False. The interval of convergence may or may not include the endpoints.

c. True. The interval of convergence is an interval centered at 0, and the endpoints may or may not be included.

d. True. Because $f(x)$ is a polynomial, all its derivatives vanish after a certain point (in this case, $f^{(12)}(x)$ is the last nonzero derivative).

e. True.

2 $p_3(x)=2x-\dfrac{(2x)^3}{3!}$.

3 $p_2(x)=1-\dfrac{3}{2}x^2$.

4 $p_2(x) = \frac{\pi}{3} - \frac{2((x-\frac{1}{2})}{\sqrt{3}} - \frac{2\left(x-\frac{1}{2}\right)^2}{3\sqrt{3}}$.

5 $p_2(x) = 1 - (x-1) + \frac{3}{2}(x-1)^2$.

6 $p_2(x) = 1 + x + \frac{x^2}{2}$.

7 $p_2(x) = 1 - \frac{1}{2}(x-1)^2$.

8 $\sinh x = x + \frac{x^3}{6} = \cdots$, so for $\sinh(-3x)$ we have $p_3(x) = -3x - \frac{9}{2}x^3$.

9 $p_3(x) = \frac{5}{4} + \frac{3(x-\ln 2)}{4} + \frac{5(x-\ln 2)^2}{8} + \frac{(x-\ln 2)^3}{8}$.

10

a. $p_1(x) = 1$, and $p_2(x) = 1 - \frac{x^2}{2}$.

b.

n	$p_n(-0.08)$	$\lvert p_n(-0.08) - \cos(-0.08)\rvert$
1	1	3.2×10^{-3}
2	0.997	1.7×10^{-6}

11

a. $p_1(x) = 1 + x$, and $p_2(x) = 1 + x + \frac{x^2}{2}$.

b.

n	$p_n(-0.08)$	$\left\lvert p_n(-0.08) - e^{-0.08}\right\rvert$
1	0.92	3.1×10^{-3}
2	0.923	8.4×10^{-5}

12

a. $p_1(x) = 1 + \frac{1}{2}x$, and $p_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$.

b.

n	$p_n(0.08)$	$\left\lvert p_n(0.08) - \sqrt{1+0.08}\right\rvert$
1	1.04	7.7×10^{-4}
2	1.039	3.0×10^{-5}

13

a. $p_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$, and $p_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$.

b.

n	$p_n(\pi/5)$	$\lvert p_n(\pi/5) - \sin(\pi/5)\rvert$
1	0.596	8.2×10^{-3}
2	0.587	4.7×10^{-4}

14 The bound is $|R_n(x)| \le M\dfrac{|x|^{n+1}}{(n+1)!}$, where M is a bound for $|e^x|$ (because e^x is its own derivative) on $[-1,1]$. Thus take $M = 3$ so that $|R_3(x)| \le \dfrac{3x^4}{4!} = \dfrac{x^4}{8}$. But $|x| < 1$, so this is at most $\dfrac{1}{8}$.

15 The derivatives of $\sin x$ are bounded in magnitude by 1, so $|R_n(x)| \le M\dfrac{|x|^{n+1}}{(n+1)!} \le \dfrac{|x|^{n+1}}{(n+1)!}$. But $|x| < \pi$, so $|R_3(x)| \le \dfrac{\pi^4}{24}$.

16 The third derivative of $\ln(1-x)$ is $\dfrac{-2}{(x-1)^3}$, which is bounded in magnitude by 16 on $|x| < 1/2$ (at $x = 1/2$). Thus $|R_3(x)| \le 16\dfrac{|x|^4}{4!} \le 16\dfrac{1}{2^4 4!} = \dfrac{1}{4!}$.

17 Using the Ratio Test, $\lim\limits_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\dfrac{(k+1)^2x^{k+1}}{(k+1)!}\cdot\dfrac{k!}{k^2x^k}\right| = \lim\limits_{k\to\infty}\left(\dfrac{k+1}{k}\right)^2\dfrac{|x|}{k+1} = 0$, so the radius of convergene is ∞ and the interval of convergence is $(-\infty,\infty)$.

18 Using the Ratio Test, $\lim\limits_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\dfrac{x^{4k+4}}{(k+1)^2}\cdot\dfrac{k^2}{x^{4k}}\right| = \lim\limits_{k\to\infty}\left(\dfrac{k}{k+1}\right)^2 x^4 = x^4$, so that the radius of convergence is 1. Because $\sum \dfrac{1}{k^2}$ converges, the given power series converges at both endpoints, so its interval of convergence is $[-1,1]$.

19 Using the Ratio Test, $\lim\limits_{k\to\infty}\dfrac{a_{k+1}}{a_k} = \lim\limits_{k\to\infty}\left|\dfrac{(x+1)^{2k+2}}{(k+1)!}\cdot\dfrac{k!}{(x+1)^{2k}}\right| = \lim\limits_{k\to\infty}\dfrac{1}{k+1}(x+1)^2 = 0$, so the radius of convergene is ∞ and the interval of convergence is $(-\infty,\infty)$.

20 Using the Ratio Test, $\lim\limits_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\dfrac{(x-1)^{k+1}}{(k+1)5^{k+1}}\cdot\dfrac{k5^k}{(x-1)^k}\right| = \lim\limits_{k\to\infty}\dfrac{k}{5k+5}|x-1| = \dfrac{1}{5}(|x-1|)$, so the series converges when $|1/5(x-1)| < 1$, or $-5 < x-1 < 5$, so that $-4 < x < 6$. The radius of convergence is 5. At $x = -4$, the series is the alternating harmonic series. At $x = 6$, it is the harmonic series, so the interval of convergence is $[-4,6)$.

21 By the Root Test, $\lim\limits_{k\to\infty}\sqrt[k]{|a_k|} = \lim\limits_{k\to\infty}\left(\dfrac{|x|}{9}\right)^3 = \dfrac{|x^3|}{729}$, so the series converges for $|x| < 9$. The radius of convergence is 9. The series given by letting $x = \pm 9$ are both divergent by the Divergence Test. Thus, $(-9,9)$ is the interval of convergence.

22 By the Ratio Test, $\lim\limits_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\dfrac{(x+2)^{k+1}}{\sqrt{k+1}}\cdot\dfrac{\sqrt{k}}{(x+2)^k}\right| = \lim\limits_{k\to\infty}\sqrt{\dfrac{k}{k+1}}(|x+2|) = |x+2|$, so that the series converges for $|x+2| < 1$, so $-3 < x < -1$. The radius of convergence is 1. At $x = -3$, we have a series which converges by the Alternating Series Test. At $x = -1$, we have the divergent $p-$series with $p = 1/2$. Thus, $[-3,-1)$ is the interval of convergence.

23 By the Ratio Test, $\lim_{k\to\infty}\left|\frac{(x+2)^{k+1}}{2^{k+1}\ln(k+1)}\cdot\frac{2^k\ln k}{(x+2)^k}\right| = \lim_{k\to\infty}\frac{\ln k}{2\ln(k+1)}\left|x+2\right| = \frac{|x+2|}{2}$. The radius of convergence is thus 2, and a check of the endpoints gives the divergent series $\sum\frac{1}{\ln k}$ at $x=0$ and the convergent alternating series $\sum\frac{(-1)^k}{\ln k}$ at $x=-4$. The interval of convergence is therefore $[-4,0)$.

24

$$r = \lim_{k\to\infty}\left|\frac{\frac{(2k+2)!(x-2)^{k+1}}{(k+1)!}}{\frac{(2k)!(x-2)^k}{k!}}\right| = \lim_{k\to\infty}\frac{(2k+2)(2k+1)\left|x-2\right|}{k+1} = \begin{cases}0 & \text{if } x=2\\ \infty & \text{if } x\neq 2\end{cases}.$$

So t the radius of convergence is 0 and the series only converges in the case that $x=2$, which implies that the interval of convergence is $\{2\}$.

25

$$r = \lim_{k\to\infty}\left|\frac{\frac{(-1)^{k+1}(2x+1)^{k+1}}{(k+1)^2 3^{k+1}}}{\frac{(-1)^k(2x+1)^k}{k^2 3^k}}\right| = \lim_{k\to\infty}\frac{k^2\left|2x+1\right|}{(k+1)^2 3} = \frac{|2x+1|}{3}.$$

This is less than 1 for $-3 < 2x+1 < 3$, so for $-2 < x < 1$. So the radius of convergence is $\frac{3}{2}$, and the interval of convergence is $[-2,1]$, because at the endpoints the series converges absolutely as the corresponding series of all positive terms is a convergent p-series.

26 We use the Ratio Test. $\lim_{k\to\infty}\left|\frac{x^{2k+3}}{2k+3}\cdot\frac{2k+1}{x^{2k+1}}\right| = x^2$. The radius of convergence is thus 1. At each endpoint we have a divergent series, so the interval of convergence is $(-1,1)$.

27

$$r = \lim_{k\to\infty}\left|\frac{\frac{(3k+3)!x^{k+1}}{((k+1)!)^3}}{\frac{(3k)!x^k}{(k!)^3}}\right| = \lim_{k\to\infty}\frac{(3k+3)(3k+2)(3k+1)\left|x\right|}{(k+1)^3} = 27\left|x\right|.$$

This is less than 1 for $|x| < \frac{1}{27}$. The radius of convergence is $R = \frac{1}{27}$.

28

$$r = \lim_{k\to\infty}\left|\frac{\frac{2^{k+1}(k+1)!(x-5)^{k+1}}{(k+1)^{k+1}}}{\frac{2^k k!(x-5)^k}{k^k}}\right| = \lim_{k\to\infty}\frac{2(k+1)k^k\left|x-5\right|}{(k+1)^{(k+1)}} = \lim_{k\to\infty}\frac{2k^k\left|x-5\right|}{(k+1)^k} = \lim_{k\to\infty}\frac{2\left|x-5\right|}{(1+1/k)^k} = \frac{2\left|x-5\right|}{e}.$$

This is less than 1 for $|x-5| < \frac{e}{2}$, so the radius of convergence is $R = \frac{e}{2}$.

29 The Maclaurin series for $f(x)$ is $\sum_{k=0}^{\infty} x^{2k}$. By the Root Test, this converges for $\left|x^2\right| < 1$, so $-1 < x < 1$. It diverges at both endpoints, so the interval of convergence is $(-1,1)$.

30 The Maclaurin series for $f(x)$ is determined by replacing x by $(-x)^3$ in the power series for $\frac{1}{1-x}$, so it is $\sum_{k=0}^{\infty}(-1)^k x^{3k}$. The radius of convergence is still 1. The series diverges at both endpoints, so the interval of convergence is $(-1,1)$.

31 The Maclaurin series for $f(x)$ is $\sum_{k=0}^{\infty}(-5x)^k = \sum_{k=0}^{\infty}(-5)^k x^k$. By the Root Test, this has radius of convergence 1/5. Checking the endpoints, we obtain an interval of convergence of $(-1/5, 1/5)$.

32 Replace x by $-x$ in the original power series, and multiply the result by $10x$, to get the Maclaurin series for $f(x)$, which is $\sum_{k=0}^{\infty}(-1)^k 10x^{k+1}$. By the Ratio Test, the radius of convergence is 1. Checking the endpoints, we obtain an interval of convergence of $(-1, 1)$.

33 Note that $\frac{1}{1-10x} = \sum_{k=0}^{\infty}(10x)^k$, so $\frac{1}{10}\cdot\frac{1}{1-10x} = \frac{1}{10}\sum_{k=0}^{\infty}(10x)^k$. Taking the derivative of $\frac{1}{10}\cdot\frac{1}{1-10x}$ gives $f(x)$. Thus, the Maclaurin series for $f(x)$ is $\frac{1}{10}\sum_{k=1}^{\infty}10k(10x)^{k-1} = \sum_{k=1}^{\infty}k(10x)^{k-1}$. Using the Ratio Test, we see that the radius of convergence is $1/10$, and checking endpoints we obtain an interval of convergence of $(-1/10, 1/10)$.

34 Integrating $\frac{1}{1-x}$ and then replacing x by $4x$ gives $-f(x)$, so the series for $f(x)$ is $-\sum_{k=0}^{\infty}\frac{1}{k+1}(4x)^{k+1}$. The Ratio Test shows that the series has a radius of convergence of $1/4$; checking the endpoints, we obtain an interval of convergence of $[-1/4, 1/4)$.

35 The first three terms are $1 + 3x + \frac{9x^2}{2}$. The series is $\sum_{k=0}^{\infty}\frac{(3x)^k}{k!}$.

36 The first three terms are $\frac{1}{3} - \frac{2(x-1)}{9} + \frac{4}{27}(x-1)^2$. The series is $\sum_{k=0}^{\infty}\frac{(-1)^k 2^k (x-1)^k}{3^{k+1}}$.

37 The first three terms are $-(x-\pi/2) + \frac{1}{6}(x-\pi/2)^3 - \frac{1}{120}(x-\pi/2)^5$. The series is

$$\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{(2k+1)!}\left(x-\frac{\pi}{2}\right)^{2k+1}.$$

38 The first three terms for $\frac{1}{1+x}$ are $1 - x + x^2$, so the first three terms of $x^2\cdot\frac{1}{1+x}$ are $x^2 - x^3 + x^4$. The series is $\sum_{k=0}^{\infty}(-1)^k x^{k+2}$.

39 The first three terms are $4x - \frac{1}{3}(4x)^3 + \frac{1}{5}(4x)^5$. The series is $\sum_{k=0}^{\infty}(-1)^k\frac{(4x)^{2k+1}}{2k+1}$.

40 The nth derivative of $f(x) = \sin(2x)$ is $\pm 2^n$ times either $\sin 2x$ or $\cos 2x$. Evaluated at $-\frac{\pi}{2}$, the even derivatives are therefore zero, and the $(2n+1)^{\text{st}}$ derivative is $(-1)^{n+1}2^{2n+1}$. The Taylor series for $\sin 2x$ around $x = -\frac{\pi}{2}$ is thus $-2\left(x+\frac{\pi}{2}\right) + \frac{2^3}{3!}\left(x+\frac{\pi}{2}\right)^3 - \frac{2^5}{5!}\left(x+\frac{\pi}{2}\right)^5 + \cdots$, and the general series is $\sum_{k=0}^{\infty}(-1)^{k+1}\frac{2^{2k+1}}{(2k+1)!}\left(x+\frac{\pi}{2}\right)^{2k+1}$.

41 The nth derivative of $\cosh(2x-2)$ at $x = 1$ is 0 if n is odd and is 2^n if n is even. The first 3 terms of the series are thus $1 + 2(x-1)^2 + \frac{2}{3}(x-1)^4$. The whole series can be written as $\sum_{k=0}^{\infty}\frac{2^{2k}}{(2k)!}(x-1)^{2k}$.

42 $f(0) = \frac{1}{4}$, $f'(x) = \frac{-2x}{(x^2+4)^2}$, so $f'(0) = 0$. $f''(x) = \frac{6x^2-8}{(x^2+4)^3}$, so $f''(0) = -\frac{1}{8}$. $f'''(0) = 0$, and $f''''(0) = \frac{3}{8}$. The first three terms are $\frac{1}{4} - \frac{x^2}{16} + \frac{x^4}{64}$. The series is given by $\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{4^{k+1}}$.

43 $f(x) = \binom{1/3}{0} + \binom{1/3}{1}x + \binom{1/3}{2}x^2 + \cdots = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \cdots$.

44 $f(x) = \binom{-1/2}{0} + \binom{-1/2}{1}x + \binom{-1/2}{2}x^2 + \cdots = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$.

45 $f(x) = \binom{-3}{0} + \binom{-3}{1}\frac{x}{2} + \binom{-3}{2}\frac{x^2}{4} + \cdots = 1 - \frac{3}{2}x + \frac{3}{2}x^2 + \cdots$.

46 $f(x) = \binom{-5}{0} + \binom{-5}{1}(2x) + \binom{-5}{2}(2x)^2 + \cdots = 1 - 10x + 60x^2 + \cdots$.

47 Note that $f(x) = \sinh x + \cosh x = e^x$. $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{e^c}{(n+1)!}x^{n+1}$ for some c. $\lim_{n\to\infty} \frac{e^c}{(n+1)!}x^{n+1} = 0$ for all x.

48 $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c in $(-1/2, 1/2)$. Now, $\left|f^{(n+1)}(c)\right| = \frac{n!}{(1+c)^{n+1}}$, so $\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} (2\,|x|)^{n+1} \cdot \frac{1}{n+1} \le \lim_{n\to\infty} 1^{n+1}\frac{1}{n+1} = 0$.

49 The Taylor series for $\cos x$ centered at 0 is

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots .$$

We compute that

$$\begin{aligned}\frac{x^2/2 - 1 + \cos x}{x^4} &= \frac{1}{x^4}\left(x^2/2 - 1 + \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)\right) \\ &= \frac{1}{x^4}\left(\frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) = \frac{1}{24} - \frac{x^2}{720} + \cdots\end{aligned}$$

so the limit of $\frac{x^2/2 - 1 + \cos x}{x^4}$ as $x \to 0$ is $\frac{1}{24}$.

50 The Taylor series for $\sin x$ centered at 0 is

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

and the Taylor series for $\tan^{-1} x$ centered at 0 is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

We compute that

$$\begin{aligned}&\frac{2\sin x - \tan^{-1} x - x}{2x^5} \\ &= \frac{1}{2x^5}\left(2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) - x\right) \\ &= \frac{1}{2x^5}\left(\frac{11x^5}{60} + \frac{359x^7}{2520} - \cdots\right) = -\frac{11}{120} + \frac{359x^2}{5040} - \cdots\end{aligned}$$

so the limit of $\frac{2\sin x - \tan^{-1} x - x}{2x^5}$ as $x \to 0$ is $-\frac{11}{120}$.

51 The Taylor series for $\ln(x-3)$ centered at 4 is

$$\ln(x-3) = (x-4) - \frac{1}{2}(x-4)^2 + \frac{1}{3}(x-4)^3 - \cdots .$$

We compute that

$$\begin{aligned}\frac{\ln(x-3)}{x^2-16} &= \frac{1}{(x-4)(x+4)}\left((x-4) - \frac{1}{2}(x-4)^2 + \frac{1}{3}(x-4)^3 - \cdots\right)\\ &= \frac{1}{(x-4)(x+4)}\left((x-4)\left(1 - \frac{1}{2}(x-4) + \frac{1}{3}(x-4)^2 - \cdots\right)\right)\\ &= \frac{1}{x+4}\left(1 - \frac{1}{2}(x-4) + \frac{1}{3}(x-4)^2 - \cdots\right)\end{aligned}$$

so the limit of $\dfrac{\ln(x-3)}{x^2-16}$ as $x \to 4$ is $\dfrac{1}{8}$.

52 The Taylor series for $\sqrt{1+2x}$ centered at 0 is

$$\sqrt{1+2x} = 1 + x - \frac{x^2}{2} + \frac{x^3}{2} - \cdots .$$

We compute that

$$\begin{aligned}\frac{\sqrt{1+2x}-1-x}{x^2} &= \frac{1}{x^2}\left(\left(1 + x - \frac{x^2}{2} + \frac{x^3}{2} - \cdots\right) - 1 - x\right)\\ &= \frac{1}{x^2}\left(-\frac{x^2}{2} + \frac{x^3}{2} - \cdots\right) = -\frac{1}{2} + \frac{x}{2} - \cdots\end{aligned}$$

so the limit of $\dfrac{\sqrt{1+2x}-1-x}{x^2}$ as $x \to 0$ is $-\dfrac{1}{2}$.

53 The Taylor series for $\sec x$ centered at 0 is

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots$$

and the Taylor series for $\cos x$ centered at 0 is

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots .$$

We compute that

$$\begin{aligned}&\frac{\sec x - \cos x - x^2}{x^4}\\ &= \frac{1}{x^4}\left(\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) - x^2\right)\\ &= \frac{1}{x^4}\left(\frac{x^4}{6} + \frac{31x^6}{360} + \cdots\right) = \frac{1}{6} + \frac{31x^2}{360} + \cdots\end{aligned}$$

so the limit of $\dfrac{\sec x - \cos x - x^2}{x^4}$ as $x \to 0$ is $\dfrac{1}{6}$.

54 The Taylor series for $(1+x)^{-2}$ centered at 0 is

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

and the Taylor series for $\sqrt[3]{1-6x}$ centered at 0 is

$$\sqrt[3]{1-6x} = 1 - 2x - 4x^2 - \frac{40x^3}{3} - \cdots .$$

We compute that

$$\begin{aligned}
&\frac{(1+x)^{-2} - \sqrt[3]{1-6x}}{2x^2} \\
&= \frac{1}{2x^2}\left(\left(1 - 2x + 3x^2 - 4x^3 + \cdots\right) - \left(1 - 2x - 4x^2 - \frac{40x^3}{3} - \cdots\right)\right) \\
&= \frac{1}{2x^2}\left(7x^2 + \frac{28x^3}{3} + \cdots\right) = \frac{7}{2} + \frac{14x}{3} + \cdots
\end{aligned}$$

so the limit of $\dfrac{(1+x)^{-2} - \sqrt[3]{1-6x}}{2x^2}$ as $x \to 0$ is $\dfrac{7}{2}$.

55 We have $e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \cdots$, so $\int e^{-x^2}\,dx = \int (1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \cdots)\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$. Thus, $\int_0^{1/2} e^{-x^2}\,dx = (0.5) - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{10} - \frac{(0.5)^7}{42} + \cdots$. Because $(0.5)^7/42 < 0.001$, we can calculate the approximation using the first three numbers shown, arriving at approximately 0.461.

56 $\tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - \cdots$, so $\int \tan^{-1}(x)\,dx = \int (x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - \cdots)\,dx = C + \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \frac{x^8}{56} + \cdots$. Thus, $\int_0^{0.5} \tan^{-1} x\,dx = \frac{(0.5)^2}{2} - \frac{(0.5)^4}{12} + \frac{(0.5)^6}{30} - \frac{(0.5)^8}{56} + \cdots$. Note that this series is alternating, and $\frac{(0.5)^6}{30} < 0.001$, so we add the first two terms showing to approximate the integral to the desired accuracy. Calculating gives approximately 0.120.

57 $x\cos x = x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + \cdots$, so $\int x\cos x\,dx = \int (x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + \cdots)\,dx = C + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{144} - \frac{x^8}{5760} + \frac{x^{10}}{403200} - \cdots$. Thus $\int_0^1 x\cos x\,dx = \frac{1}{2} - \frac{1}{8} + \frac{1}{144} - \frac{1}{5760} + \cdots$. Because $\frac{1}{5760} < 0.001$, we add the first three terms to approximate to the desired accuracy. Calculating gives $\int_0^1 x\cos x\,dx \approx 0.382$.

58 $x^2\tan^{-1} x = x^3 - x^5/3 + x^7/5 - x^9/7 + x^{11}/9 + \cdots$, so $\int x^2\tan^{-1}(x)\,dx = \int (x^3 - x^5/3 + x^7/5 - x^9/7 + x^{11}/9 - \cdots)\,dx = C + \frac{x^4}{4} - \frac{x^6}{18} + \frac{x^8}{40} - \frac{x^{10}}{70} + \cdots$. Thus, $\int_0^{0.5} x^2\tan^{-1} x\,dx = \frac{(0.5)^4}{4} - \frac{(0.5)^6}{18} + \frac{(0.5)^8}{40} - \frac{(0.5)^{10}}{70} + \cdots$. Note that this series is alternating, and $\frac{(0.5)^6}{18} < 0.001$, so we use the first term showing to approximate the integral to the desired accuracy. Calculating gives approximately 0.015.

59 The series for $f(x) = \sqrt{x}$ centered at $a = 121$ is $11 + \frac{x-121}{22} - \frac{(x-121)^2}{10648} + \frac{(x-121)^3}{2576816} + \cdots$. Letting $x = 119$ gives $\sqrt{119} \approx 11 - \frac{1}{11} - \frac{1}{2\cdot 11^3} - \frac{1}{2\cdot 11^5}$.

60 Because 20 degrees corresponds to $\frac{\pi}{9}$ radians, we consider the series for $\sin x$ centered at 0. We have $\sin x \approx x - x^3/3! + x^5/5! - x^7/7! + \cdots$, so $\sin \pi/9 \approx \frac{\pi}{9} - \frac{(\pi/9)^3}{3!} + \frac{(\pi/9)^5}{5!} - \frac{(\pi/9)^7}{7!}$.

61 $\tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + x^9/9 + \cdots$, so $\tan^{-1}(-1/3) \approx \dfrac{-1}{3} + \dfrac{1}{3\cdot 3^3} - \dfrac{1}{5\cdot 3^5} + \dfrac{1}{7\cdot 3^7}$.

62 $\sinh x = x + \dfrac{x^3}{6} + \dfrac{x^5}{120} + \dfrac{x^7}{5040} + \cdots$, so $\sinh(-1) \approx (-1) + \dfrac{(-1)^3}{6} + \dfrac{(-1)^5}{120} + \dfrac{(-1)^7}{5040}$.

63 Because $y(0) = 4$, we have $y'(0) - 16 + 12 = 0$, so $y'(0) = 4$. Differentiating the equation $n-1$ times and evaluating at 0 we obtain $y^{(n)}(0) = 4y^{(n-1)}(0)$, so that $y^{(n)}(0) = 4^n$. The Taylor series for $y(x)$ is thus $y(x) = 4 + 4x + \dfrac{4^2x^2}{2!} + \dfrac{4^3x^3}{3!} + \cdots$, or $y(x) = 3 + e^{4x}$.

64 We begin with $e^{-102x^2} = 1 - 102x^2 + \dfrac{102^2x^4}{2!} + \cdots$. For $n = 2$, we have $11.4\displaystyle\int_0^{0.14}(1 - 102x^2)\,dx = 11.4(x - 34x^3)\Big|_0^{0.14} = 0.5324256$.

65

a. The Taylor series for $\ln(1+x)$ is $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$. Evaluating at $x = 1$ gives $\ln 2 = \displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$.

b. The Taylor series for $\ln(1-x)$ is $-\displaystyle\sum_{k=1}^{\infty}\frac{x^k}{k}$. Evaluating at $x = 1/2$ gives $\ln(1/2) = -\displaystyle\sum_{k=1}^{\infty}\frac{1}{k2^k}$, so that $\ln 2 = \displaystyle\sum_{k=1}^{\infty}\frac{1}{k2^k}$.

c. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$. Using the two Taylor series above we have $f(x) = \displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} - \left(-\sum_{k=1}^{\infty}\frac{x^k}{k}\right) = \sum_{k=1}^{\infty}(1 + (-1)^{k+1})\frac{x^k}{k} = 2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$.

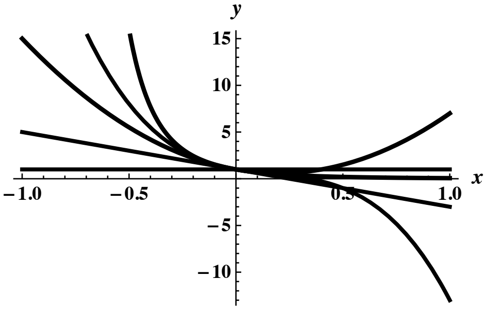
d. Because $\dfrac{1+x}{1-x} = 2$ when $x = \dfrac{1}{3}$, the resulting infinite series for $\ln 2$ is $2\displaystyle\sum_{k=0}^{\infty}\frac{1}{3^{2k+1}(2k+1)}$.

e. The first four terms of each series are: $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} \approx 0.5833333333$, $\dfrac{1}{2} + \dfrac{1}{8} + \dfrac{1}{24} + \dfrac{1}{64} \approx 0.6822916667$, $\dfrac{2}{3} + \dfrac{2}{81} + \dfrac{2}{1215} + \dfrac{2}{15309} \approx 0.6931347573$ The true value is $\ln 2 \approx 0.6931471806$. The third series converges the fastest, because it has 3^{k+1} in the denominator as opposed to 2^k, so its terms get small faster.

66

a. $p_3(x) = 1 - 4x + 10x^2 - 20x^3$.

b.



c. The constant polynomial looks like $f(x)$ only at 0. The linear polynomial looks like $f(x)$ on about $(-0.1, 0.1)$. The quadratic approximation looks like $f(x)$ on about $(-0.1, 0.1)$ as well, and the cubic approximation looks like $f(x)$ on about $(-0.2, 0.2)$.

Chapter 12

Parametric and Polar Curves

12.1 Parametric Equations

12.1.1 Given an input value of t, the point $(x(t), y(t))$ can be plotted in the xy-plane, generating a curve.

12.1.2 $x = 6\cos t$ and $y = 6\sin t$ for $0 \le t \le 2\pi$ generates the circle, because $x^2 + y^2 = 36\cos^2 t + 36\sin^2 t = 36$. Similarly, $x = 6\sin t$ and $y = 6\cos t$ for $0 \le t \le 2\pi$ generates the same curve.

12.1.3 Let $x = R\cos(\pi t/5)$ and $y = -R\sin(\pi t/5)$. Note that as t ranges from 0 to 10, $\pi t/5$ ranges from 0 to 2π. Because $x^2 + y^2 = R^2$, this curve represents a circle of radius R. Note also that for $t = 0$ the initial point is $(R, 0)$, and for small values of t the plotted points are in the third quadrant — so the curve is being traced with clockwise orientation.

12.1.4 Let $x = t$ and $y = -2t + 5$ for $t \in (-\infty, \infty)$.

12.1.5 Let $x = t^2$ and $y = t$ for $t \in (-\infty, \infty)$.

12.1.6 Both pairs of parametric equations describe the same segment of the parabola $x = y^2$, but the positive orientations of the curves are in opposite directions. The first curve starts at (0,0) and ends at (1,1); the second curve starts at (1,1) and ends at (0,0).

12.1.7 $\left.\dfrac{d}{dx}\right|_{t=2} = \left.\left(\dfrac{6t}{-6t^2}\right)\right|_{t=2} = \left.\left(\dfrac{-1}{t}\right)\right|_{t=2} = -\dfrac{1}{2}.$

12.1.8 With $t = 0$ the corresponding point on the curve is $(-2\sin 0, 2\cos 0) = (0, 2)$. As t increases from 0, the x-coordinate becomes negative, while the y coordinate decreases. Thus the curve is generated counterclockwise, running successively through $(0, 2)$, then $(-2, 0)$ for $t = \dfrac{\pi}{2}$, then $(0, -2)$ for $t = \pi$, then $(2, 0)$ for $t = \dfrac{3\pi}{2}$, and finally back to $(0, 2)$ for $t = 2\pi$. Then it repeats.

12.1.9 $x = t$, $y = t$, $0 \le t \le 6$; $x = 2t$, $y = 2t$, $0 \le t \le 3$; $x = 3t$, $y = 3t$, $0 \le t \le 2$. (Answers may vary).

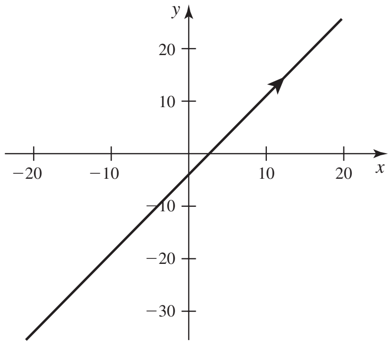
12.1.10 The arc length is given by $\displaystyle\int_0^1 \sqrt{25}\, dt = 5$. The distance between the endpoints $(1, 0)$ and $(4, 4)$. is $\sqrt{(4-1)^2 + (4-0)^2} = \sqrt{9 + 16} = 5$.

12.1.11

a.

t	x	y
-10	-20	-34
-5	-10	-19
0	0	-4
5	10	11
10	20	26

b.



c. Solving $x = 2t$ for t yields $t = x/2$, so $y = 3t - 4 = 3x/2 - 4$.

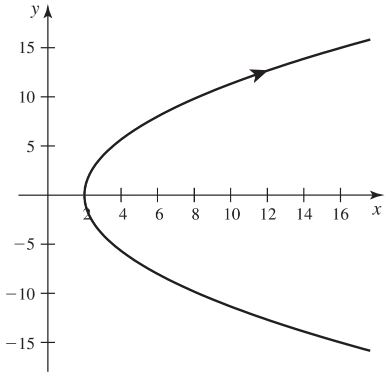
d. The curve is the line segment from $(-20, -34)$ to $(20, 26)$.

12.1.12

a.

t	x	y
-4	18	-16
-2	6	-8
0	2	0
2	6	8
4	18	16

b.



c. Solving $y = 4t$ for t yields $t = y/4$, so $x = t^2 + 2 = y^2/16 + 2$.

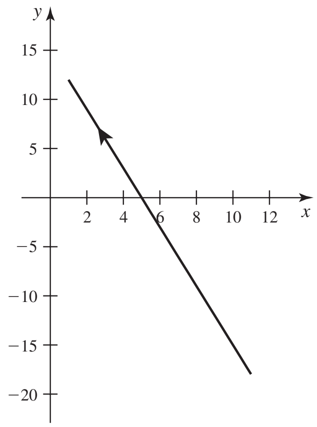
d. The curve is part of the parabola $x = y^2/16 + 2$ from $(18, -16)$ to $(18, 16)$.

12.1.13

a.

t	x	y
-5	11	-18
-3	9	-12
0	6	-3
3	3	6
5	1	12

b.



c. Solving $x = -t + 6$ for t yields $t = 6 - x$, so $y = 3t - 3 = 18 - 3x - 3 = 15 - 3x$.

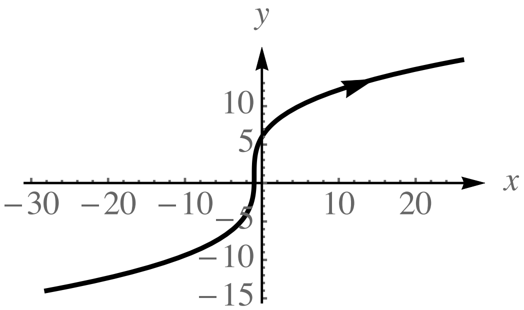
d. The curve is the line segment from $(11, -18)$ to $(1, 12)$.

12.1.14

a.

t	x	y
-3	-28	-14
-2	-9	-9
-1	-2	-4
0	-1	1
1	0	6
2	7	11
3	26	16

b.



c. Because $t = \sqrt[3]{x+1}$, we have $y = 5\sqrt[3]{x+1} + 1$.

d. The curve is a shifted and scaled version of the cube root function.

12.1.15

a. Because $t = x - 3$, we have $y = 1 - (x - 3) = 4 - x$.

b. This is a line segment starting at $(3, 1)$ and ending at $(4, 0)$.

12.1.16

a. Because $t = \dfrac{4-x}{3}$, we have $y = -2 + 6\left(\dfrac{4-x}{3}\right) = 6 - 2x$,

b. This is a line segment starting at $(7, -8)$ and ending at $(-5, 16)$.

12.1.17

a. Solving $x = \sqrt{t} + 4$ for t yields $t = (x - 4)^2$. Thus, $y = 3\sqrt{t} = 3(x - 4)$, where x ranges from 4 to 8. Note that all $t \geq 0$, $x > 0$, and $y > 0$.

b. The curve is the line segment from $(4, 0)$ to $(8, 12)$.

12.1.18

a. Solving $y = t+2$ for t yields $t = y-2$. Thus, $x = (t+1)^2 = (y-2+1)^2 = (y-1)^2$, where $-8 \le y \le 12$.

b. The curve is the part of the parabola $x = (y-1)^2$ from $(81, -8)$ to $(121, 12)$.

12.1.19

a. Note that $x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9$.

b. This represents an arc of the circle of radius 3 centered at the origin from $(-3, 0)$ to $(3, 0)$ traversed counterclockwise.

12.1.20

a. Note that $x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9$,

b. This represents an arc of the circle of radius 3 centered at the origin from $(3, 0)$ to $(0, 3)$ traversed counterclockwise.

12.1.21

a. Because $\cos^2 t + \sin^2 t = 1$, we have $x^2 + y = 1$, so $y = 1 - x^2$, $-1 \le x \le 1$.

b. This is a parabola opening downward with a vertex at $(0, 1)$, and starting at $(1, 0)$ and ending at $(-1, 0)$.

12.1.22

a. Note that $(1 - \sin^2 s) - \cos^2 s = 0$, so $x - y^2 = 0$, so $x = y^2$, $-1 \le y \le 1$.

b. This is a parabola opening to the right with a vertex at $(0, 0)$, starting at $(1, -1)$ and ending at $(1, 1)$.

12.1.23

a. Note that $x^2 + (y-1)^2 = \cos^2 t + \sin^2 t = 1$,

b. We have a circle of radius 1 centered at $(0, 1)$, traversed counterclockwise starting at $(1, 1)$.

12.1.24

a. Replacing $\sin t$ with x in the second equation, we have $y = 2x + 1$.

b. Because the sine function increases on $[0, \pi/2]$, x and y are both increasing on this interval also. When $t = 0$, $(x, y) = (0, 1)$ and when $t = \dfrac{\pi}{2}$, $(x, y) = (1, 3)$. So we have a line segment beginning at (0,1) and ending at (1,3).

12.1.25

a. Solving $x = r - 1$ for r yields $r = x + 1$. Thus, $y = r^3 = (x+1)^3$, where $-5 \le x \le 3$.

b. The curve is the part of the standard cubic curve, shifted one unit to the left, from $(-5, -64)$ to $(3, 64)$.

12.1.26

a. Solving $x = e^{2t}$ for t yields $t = \ln(\sqrt{x})$. Thus, $y = e^t + 1 = \sqrt{x} + 1$, where $1 \le x \le e^{50}$.

b. The curve is the part of the standard square root function, shifted one unit vertically, from the point $(1, 2)$ to $(e^{50}, e^{25} + 1)$.

12.1.27

a. Note that $x^2 + y^2 = 49\cos^2 2t + 49\sin^2 2t = 49$.

b. This represents an arc of the circle of radius 7 centered at the origin from $(-7, 0)$ to $(-7, 0)$ traversed counterclockwise. (So the whole circle is represented.)

12.1.28

a. Note that $(x - 1)^2 + (y - 2)^2 = 9\sin^2 4\pi t + 9\cos^2 4\pi t = 9$.

b. This represents the circle of radius 3 centered at $(1, 2)$ from $(1, 5)$ to $(1, 5)$ traversed counterclockwise.

12.1.29

a. $y = 1$, $-\infty < x < \infty$.

b. Because $y = 1$, this is a horizontal line with slope 0 and y-intercept 1.

12.1.30

a. $x = 5$, $-\infty < y < \infty$.

b. This is a vertical line with x-intercept 5.

12.1.31 Note that $x^2 + y^2 = 4\sin^2 8t + 4\cos^2 8t = 4$, so the curve is the circle $x^2 + y^2 = 4$.

12.1.32 Note that $4x^2 + y^2 = 4\sin^2 8t + 4\cos^2 8t = 4$, so the curve is the ellipse $4x^2 + y^2 = 4$.

12.1.33 Note that because $t = x$, we have $y = \sqrt{4 - t^2} = \sqrt{4 - x^2}$.

12.1.34 Note that $x^2 = t + 1$, so $y = \dfrac{1}{t + 1} = \dfrac{1}{x^2}$.

12.1.35 Because $\sec^2 t - 1 = \tan^2 t$, we have $y = x^2$.

12.1.36 Note that $(\sqrt[n]{x/a})^2 + (\sqrt[n]{y/b})^2 = \sin^2 t + \cos^2 t$, so $(\sqrt[n]{x/a})^2 + (\sqrt[n]{y/b})^2 = 1$.

12.1.37 Let $x = 4\cos t$ and $y = 4\sin t$ for $0 \le t \le 2\pi$. Then $x^2 + y^2 = 16\cos^2 t + 16\sin^2 t = 16$.

12.1.38 Let $x = 12\sin t$ and $y = 12\cos t$ for $0 \le t \le 2\pi$. Then $x^2 + y^2 = 144\cos^2 t + 144\sin^2 t = 144$, and for $t = 0$ the value of (x, y) is $(0, 12)$.

12.1.39 Let $x = \cos t + 2$ and $y = \sin t + 3$ for $0 \le t \le 2\pi$. Then $(x - 2)^2 + (y - 3)^2 = 1$, which is a circle with the desired center and radius and orientation.

12.1.40 Let $x = -2 + 8\sin t$ and $y = -3 + 8\cos t$ for $0 \le t \le 2\pi$. Then $(x+2)^2 + (y+3)^2 = 64\sin^2 t + 64\cos^2 t = 64$.

12.1.41 Let $x = x_0 + at$ and $y = y_0 + bt$. Letting $(x_0, y_0) = (0, 0)$, and then finding a and b so that the curve is at the point Q when $t = 1$ yields $x = 2t$, $y = 8t$ for $0 \le t \le 1$.

12.1.42 Let $x = x_0 + at$ and $y = y_0 + bt$. Letting $(x_0, y_0) = (-1, -3)$, and then finding a and b so that the curve is at the point Q when $t = 1$ yields $x = -1 + 7t$, $y = -3 - 13t$ for $0 \le t \le 1$.

12.1.43 Let $x = t$ and $y = 2t^2 - 4$, $-1 \le t \le 5$.

12.1.44 Let $x = t^3 - 3t$ and $y = t$, $-\infty < t < \infty$.

12.1.45 Let $x = 2$

12.1.46 Let $x = x_0 + at$ and $y = y_0 + bt$, and parametrize from $t = 0$ to $t = 1$. Because the point is at $(8, 2)$ when $t = 0$, we have $x_0 = 8$ and $y_0 = 2$. At $t = 1$, the point is at $(-2, 2)$, so that $-2 = 8 + a$ and $2 = 2 + b$. Thus $a = -10$ and $b = 0$, and our equations are $x = 8 - 10t$ and $y = 2$ for $0 \leq t \leq 1$.

12.1.47 Let $x = -2 + 4t$ and $y = 3 - 6t$, $0 \leq t \leq 1$, and $x = t + 1$, $y = 8t - 11$ for $1 \leq t \leq 2$.

12.1.48 Let $x = -4 + 4t$ and $y = 4 + 4t$, $0 \leq t \leq 1$, and $x = t - 1$, $y = 8 - 2(t - 1)^2$ for $1 \leq t \leq 3$.

12.1.49 Let $x = 1 + 2t$ and $y = 1 + 4t$, for $-\infty < t < \infty$. Note that $y = 2(1 + 2t) - 1$, so $y = 2x - 1$.

12.1.50 Let $x = -t$ and $y = t^2 + 1$, for $0 \leq t < \infty$.

12.1.51 Let $x = t^2$ and $y = t$, for $0 \leq t < \infty$. Note that $x = t^2 = y^2$, and that the starting point is $(0, 0)$.

12.1.52 Let $x = -2 - 6\cos t$ and $y = 2 - 6\sin t$, for $0 \leq t \leq \pi$. Then $(x + 2)^2 + (y - 2)^2 = 36$, so the curve represented is part of the circle of radius 6 centered at $(-2, 2)$. Note also that as t runs from 0 to π, the portion of the circle traversed is the lower portion, from $(-8, 2)$ to $(4, 2)$.

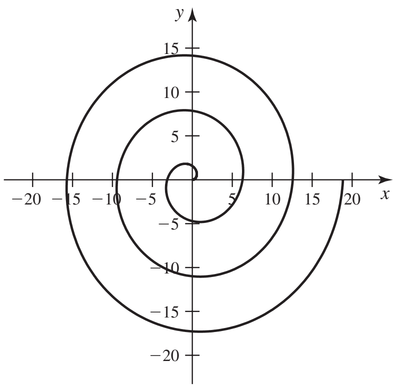
12.1.53 Let t be time in minutes, so $0 \leq t \leq 1.5$ Let $x = 400\cos(4\pi/3)t$ and $y = 400\sin(4\pi/3)t$. Then because $x^2 + y^2 = 400^2$, the path is a circle of radius 400. Note that the values of x and y are the same at $t = 0$ and $t = 1.5$, and that the circle is traversed counterclockwise.

12.1.54 Let t be time in seconds, so $0 \leq t \leq 60$ Let $x = 15\sin(\pi/30)t$ and $y = 15\cos(\pi/30)t$. Then because $x^2 + y^2 = 15^2$, the path is a circle of radius 15. Note that the values of x and y are the same at $t = 0$ and $t = 60$, and that the circle is traversed clockwise.

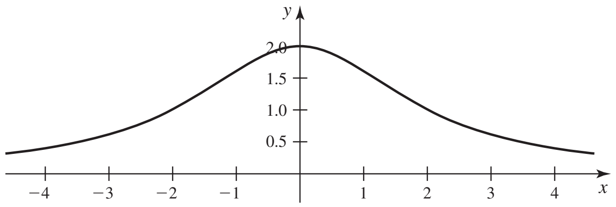
12.1.55 Let t be time in seconds, so $0 \leq t \leq 24$ Let $x = 50\cos(\pi/12)t$ and $y = 50\sin(\pi/12)t$. Then because $x^2 + y^2 = 50^2$, the path is a circle of radius 50. Note that the values of x and y are the same at $t = 0$ and $t = 24$, and that the circle is traversed counterclockwise.

12.1.56 Let t be time in minutes, so $0 \leq t \leq 3$. Because the low point is the origin, the circle we seek has its center at $(0, 20)$ and a radius of 20. Let $x = -20\sin(2\pi/3)t$ and $y = 20 - 20\cos(2\pi/3)t$. Then because $x^2 + (y - 20)^2 = 20^2$, the path is a circle of radius 20. Note that the values of x and y are the same for $t = 0$ and $t = 3$.

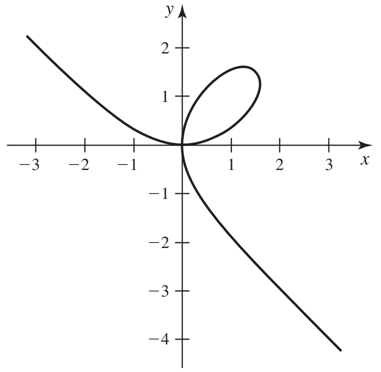
12.1.57



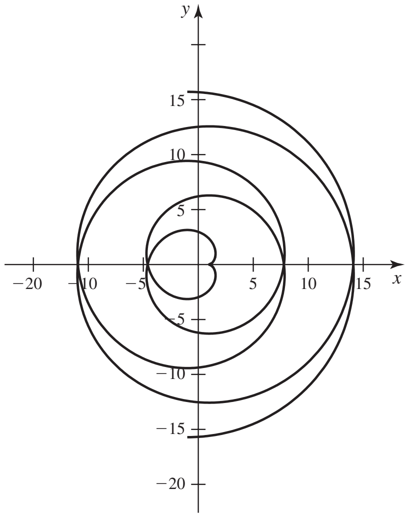
12.1.58



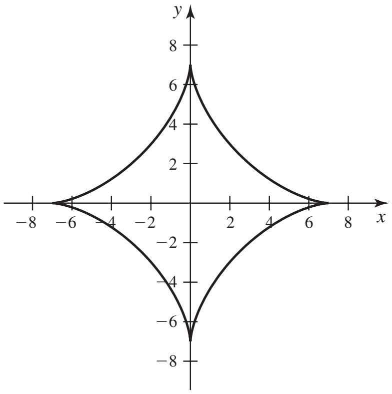
12.1.59



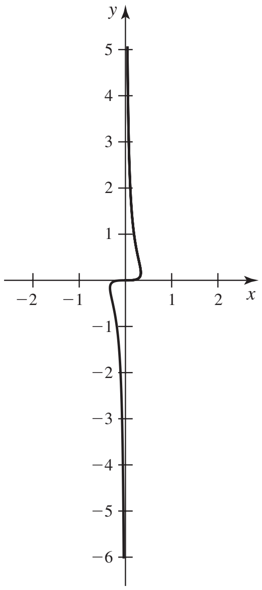
12.1.60



12.1.61

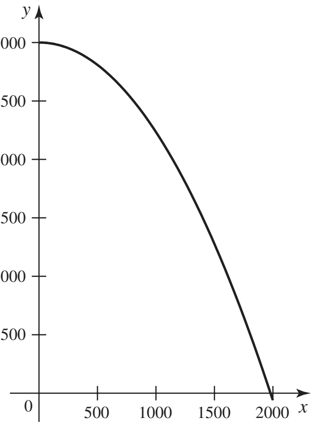


12.1.62



12.1.63 Let $x = 1 + \cos^2 t - \sin^2 t$ and $y = t$, for $-\infty < t < \infty$. Note that because $1 - \sin^2 t = \cos^2 t$, we have $x = 2\cos^2 t$, $y = t$.

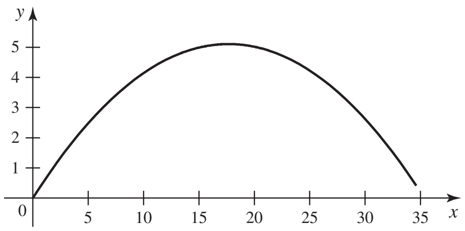
12.1.64 The packet lands when $y = 0$, so we seek a solution to $0 = -4.9t^2 + 3000$. So $t = \sqrt{\frac{3000}{4.9}} \approx 24.744$ seconds, at which point $x \approx 80 \cdot 24.744 \approx 1979.487$ meters.



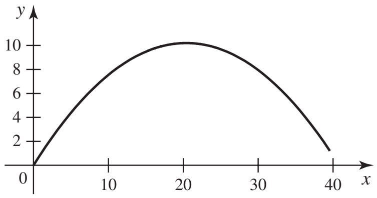
12.1.65 The packet lands when $y = 0$, so when $-4.9t^2 + 4000 = 0$ for $t > 0$. This occurs when $t = \sqrt{\frac{4000}{4.9}} \approx 28.571$ seconds. At that time, $x \approx 100 \cdot 28.57 = 2857$ meters.

12.1.66

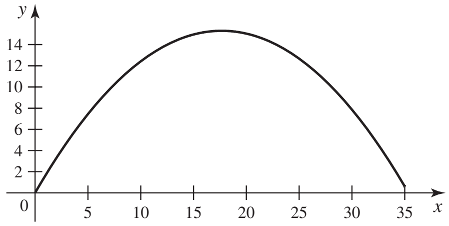
a.



$\theta = \pi/6$.



$\theta = \pi/4$.



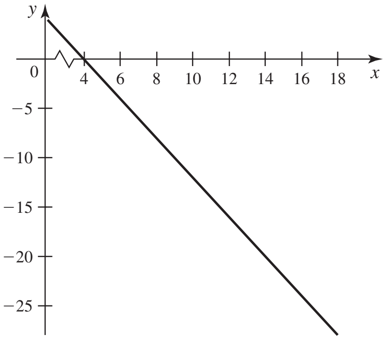
$\theta = \pi/3$.

b. The maximum appears to be reached when $\theta = \pi/4$.

12.1.67

a. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = -\dfrac{8}{4} = -2$ for all t. Because the curve is a line, the tangent line to the curve at the given point is the line itself.

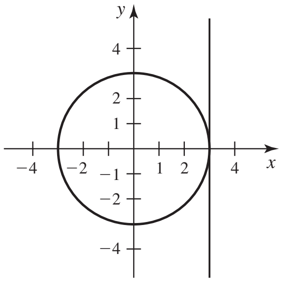
b.



12.1.68

a. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = -\dfrac{3\sin t}{3\cos t} = -\tan t$. At the given value of t, the value of $\dfrac{dy}{dx}$ doesn't exist, and the tangent line is the vertical line $x = 3$.

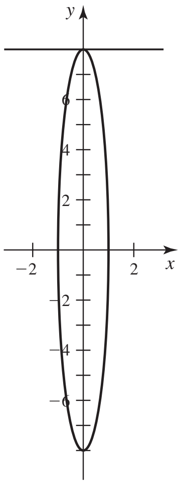
b.



12.1.69

a. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{8\cos t}{-\sin t} = -8\cot t$. At the given value of t, the value of $\dfrac{dy}{dx}$ is $-8\cot\pi/2 = 0$. The tangent line at the point $(0, 8)$ is thus the horizontal line $y = 8$.

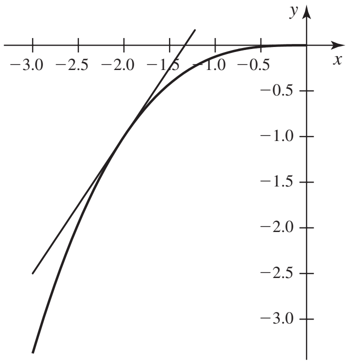
b.



12.1.70

a. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2}{2}$. At the given value of t, the value of $\dfrac{dy}{dx}$ is $\dfrac{3}{2}$, and the tangent line is $y = \dfrac{3}{2}x + 2$, tangent at the point $(-2, -1)$.

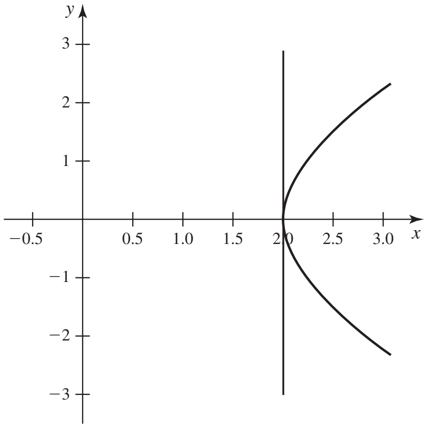
b.



12.1.71

a. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1+\dfrac{1}{t^2}}{1-\dfrac{1}{t^2}} = \dfrac{t^2+1}{t^2-1}$. At the given value of t, the derivative doesn't exist, and the tangent line is the vertical line $x = 2$, tangent at the point $(2, 0)$.

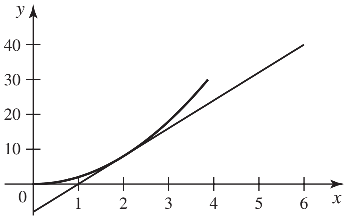
b.



12.1.72

a. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2}{1/(2\sqrt{t})} = 4\sqrt{t}$. At the given value of t, the value of $\dfrac{dy}{dx}$ is 8. The equation of the tangent line is $y = 8x - 8$, tangent at the point $(2, 8)$.

b.



12.1.73 The point corresponding to $t = 2$ is $(3, 10)$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2+1}{2t}$, so the slope at $t = 2$ is $\dfrac{13}{4}$. The equation of the tangent line is therefore $y - 10 = \dfrac{13}{4}(x-3)$, or $y = \dfrac{13}{4}x + \dfrac{1}{4}$.

12.1.74 The point corresponding to $t = \pi/4$ is $(\sqrt{2}/2, \sqrt{2}/2)$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-\sin t}{\cos t}$. At $t = \pi/4$, we have a slope of -1. The equation of the tangent line is thus $y - \sqrt{2}/2 = -1(x - \sqrt{2}/2)$, or $y = -x + \sqrt{2}$.

12.1.75 The point corresponding to $t = \pi/4$ is

$$\left(\frac{4\sqrt{2}+\pi\sqrt{2}}{8}, \frac{4\sqrt{2}-\pi\sqrt{2}}{8}\right).$$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t - (\cos t - t\sin t)}{-\sin t + (\sin t + t\cos t)} = \tan t$. At $t = \pi/4$, we have a slope of 1. The equation of the tangent line is thus $y - \dfrac{4\sqrt{2}-\pi\sqrt{2}}{8} = 1\left(x - \dfrac{4\sqrt{2}+\pi\sqrt{2}}{8}\right)$, or $y = x - \dfrac{\pi\sqrt{2}}{4}$.

12.1.76 The point corresponding to $t = 0$ is $(1, 0)$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1}{(t+1)e^t}$, so the slope at $t = 0$ is 1. The equation of the tangent line is thus $y = x - 1$.

12.1.77 $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4\cos t}{-4\sin t} = -\cot t$. We seek t so that $\cot t = -1/2$, so $t = \cot^{-1}(-1/2)$. The corresponding points on the curve are $\left(-\dfrac{4\sqrt{5}}{5}, \dfrac{8\sqrt{5}}{5}\right)$ and $\left(\dfrac{4\sqrt{5}}{5}, -\dfrac{8\sqrt{5}}{5}\right)$.

12.1.78 $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{8\cos t}{-2\sin t} = -4\cot t$. We seek t so that $\cot t = 1/4$, so $t = \cot^{-1}(1/4)$. The corresponding points on the curve are $\left(\dfrac{2\sqrt{17}}{17}, \dfrac{32\sqrt{17}}{17}\right)$ and $\left(-\dfrac{2\sqrt{17}}{17}, -\dfrac{32\sqrt{17}}{17}\right)$.

12.1.79 $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1+(1/t^2)}{1-(1/t^2)} = \dfrac{t^2+1}{t^2-1}$. We seek t so that $\dfrac{t^2+1}{t^2-1} = 1$, which never occurs. Thus, there are no points on this curve with slope 1.

12.1.80 $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = -\dfrac{4}{(1/2\sqrt{t})} = -8\sqrt{t}$ for $t \neq 0$. Note that this isn't 0 for t on the interval $(0,\infty)$, but it is the case that $\lim\limits_{t\to 0^+} \dfrac{dy}{dx} = 0$, so there is a flat tangent line at the point $(2,2)$, as long as the point is approached from the right.

12.1.81 The arc length is $\displaystyle\int_0^2 \sqrt{9+16}\, dt = 5t\Big|_0^2 = 10$.

12.1.82 The arc length is $\displaystyle\int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt = 3t\Big|_0^{2\pi} = 6\pi$.

12.1.83 The arc length is

$$\int_0^{\pi} \sqrt{(-\sin t - \cos t)^2 + (-\sin t + \cos t)^2}\, dt = \int_0^{\pi} \sqrt{2\sin^2 t + 2\cos^2 t}\, dt = \sqrt{2}t\Big|_0^{\pi} = \pi\sqrt{2}.$$

12.1.84 The arc length is

$$\int_0^{2\pi} \sqrt{(e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2}\, dt = \int_0^{2\pi} \sqrt{2e^{2t}\sin^2 t + 2e^{2t}\cos^2 t}\, dt$$
$$= \int_0^{2\pi} \sqrt{2}e^t\, dt = \sqrt{2}e^t\Big|_0^{2\pi} = \sqrt{2}(e^{2\pi} - 1).$$

12.1.85 The arc length is

$$\int_0^1 \sqrt{(4t^3)^2 + (2t^5)^2}\, dt = \int_0^1 \sqrt{16t^6 + 4t^{10}}\, dt = \int_0^1 t^3\sqrt{16+4t^4}\, dt = \frac{1}{16}\int_0^1 16t^3\sqrt{16+4t^4}\, dt.$$

Let $u = 16 + t^4$ so that $du = 16t^3\, dt$. Then we have

$$\frac{1}{16}\int_{16}^{20} u^{1/2}\, du = \frac{2}{48}u^{3/2}\Big|_{16}^{20} = \frac{1}{3}\left(5\sqrt{5} - 8\right).$$

12.1.86 The arc length is

$$\int_0^{\pi} \sqrt{(2\sin t + t^2 \sin t)^2 + (2\cos t + t^2\cos t)^2}\, dt$$
$$= \int_0^{\pi} \sqrt{4 + 4t^2\sin^2 t + t^4 \sin^2 t + 4t^2\cos^2 t + t^4\cos^2 t}\, dt$$
$$= \int_0^{\pi} \sqrt{4 + 4t^2 + t^4}\, dt = \int_0^{\pi} \sqrt{(t^2+2)^2}\, dt$$
$$= \int_0^{\pi} (t^2+2)\, dt = \left(\frac{t^3}{3} + 2t\right)\Big|_0^{\pi} = \frac{\pi^3}{3} + 2\pi.$$

12.1.87 The arc length is

$$\int_0^{\pi/4}\sqrt{((3\cos^2 2t)(-2\sin 2t))^2+((3\sin^2 2t)(2\cos 2t))^2}\,dt=\int_0^{\pi/4}\sqrt{36\cos^4 2t\sin^2 2t+36\sin^4 2t\cos^2 2t}\,dt$$
$$=6\int_0^{\pi/4}\cos 2t\sin 2t\sqrt{\cos^2 2t+\sin^2 2t}\,dt=6\int_0^{\pi/4}\cos 2t\sin 2t\,dt=3\int_0^{\pi/4}\sin 4t\,dt$$
$$=-\frac{3}{4}\cos 4t\Big|_0^{\pi/4}=-\frac{3}{4}(-1-1)=\frac{3}{2}.$$

12.1.88 The arc length is

$$\begin{aligned}\int_0^{\pi/2}\sqrt{\cos^2 t+(1+\sin t)^2}\,dt&=\sqrt{2}\int_0^{\pi/2}\sqrt{1+\sin t}\,dt\\&=\sqrt{2}\int_0^{\pi/2}\sqrt{1+\sin t}\cdot\frac{\sqrt{1-\sin t}}{\sqrt{1-\sin t}}\,dt=\sqrt{2}\int_0^{\pi/2}\frac{\cos t}{\sqrt{1-\sin t}}\,dt\\&=-\sqrt{2}\int_1^0 u^{-1/2}\,du\end{aligned}$$

Then let $u=1-\sin t$ so that $du=-\cos t\,dt$. We have

$$\sqrt{2}\int_0^1 u^{-1/2}\,du=\sqrt{2}\lim_{b\to 0^+}\int_b^1 u^{-1/2}\,du=\sqrt{2}\lim_{b\to 0^+}\left(2-2\sqrt{b}\right)=2\sqrt{2}.$$

12.1.89

a. False. This generates a circle in the counterclockwise direction.

b. True. Note that when t is increased by one, the value of $2\pi t$ is increased by 2π, which is the period of both the sine and the cosine functions.

c. False. This generates only the portion of the parabola in the first quadrant, omitting the portion in the second quadrant.

d. True. They describe the portion of the unit circle in the 4th and 1st quadrants.

e. True. This ellipse has vertical tangents at $t=0$ and $t=\pi$.

12.1.90

a. This corresponds to graph (D). Note that $t=0$ corresponds to the point $(-2,0)$ and as $t\to\infty$, both $x\to\infty$ and $y\to\infty$.

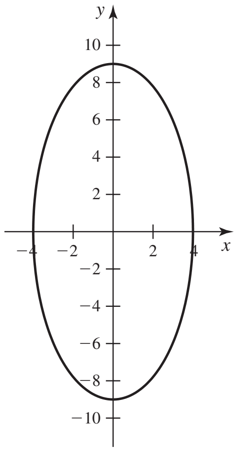
b. This corresponds to graph (B). Note that $-1\le x\le 1$ and $-1\le y\le 1$ for all values of t.

c. This corresponds to graph (A). Note that as $t\to-\infty$, we have $x\to-\infty$ and $y\to-\infty$.

d. This corresponds to graph (C). Note that $-3\le x\le 3$ and $-3\le y\le 3$ for all values of t.

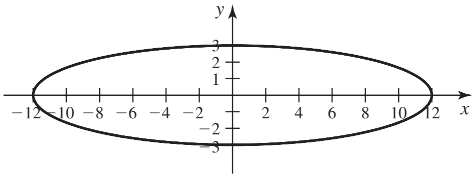
12.1.91

The entire curve is traversed for $0 \leq t \leq 2\pi$.



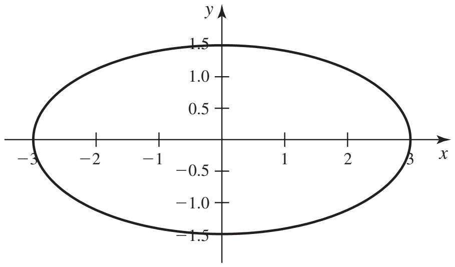
12.1.92

The entire curve is traversed for $0 \leq t \leq \pi$.



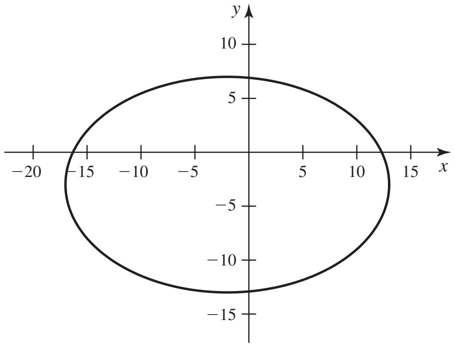
12.1.93

Let $x = 3\cos t$ and $y = \frac{3}{2}\sin t$ for $0 \leq t \leq 2\pi$. Then the major axis on the x-axis has length $2 \cdot 3 = 6$ and the minor axis on the y-axis has length $2 \cdot \frac{3}{2} = 3$. Note that $\left(\frac{x}{3}\right)^2 + \left(\frac{2y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$.



12.1.94

Let $x = 15\cos t - 2$ and $y = 10\sin t - 3$ for $0 \leq t \leq 2\pi$. Note that $\left(\frac{x+2}{15}\right)^2 + \left(\frac{y+3}{10}\right)^2 = \cos^2 t + \sin^2 t = 1$.
Then the major axis has length 30 and the minor axis has length 20.



12.1.95

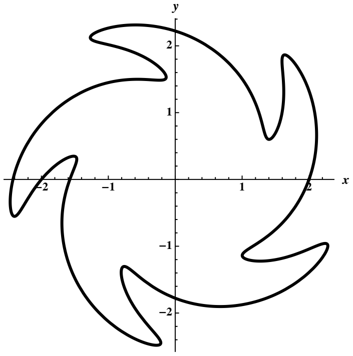
a. For $(1+s, 2s) = (1+2t, 3t)$, we must have$1+s = 1+2t$ and $2s = 3t$, so that $s = 2t$ and $2s = 3t$. The only solution to this pair of equations is $s = t = 0$, so these two lines intersect when $s = t = 0$, at the point $(1, 0)$.

b. For $(2+5s, 1+s) = (4+10t, 3+2t)$, we must have $2+5s = 4+10t$ and $1+s = 3+2t$, so that $s = 2+2t$ and $s = 2+2t$. This pair of equations has no solutions, so the lines are parallel.

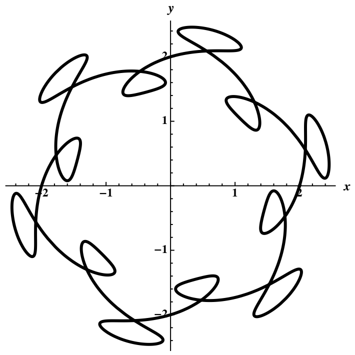
c. For $(1+3s, 4+2s) = (4-3t, 6+4t)$, we must have $1+3s = 4-3t$ and $4+2s = 6+4t$, so that $s = 1-t$ and $s = 1+2t$. The only solution to this pair of equations is $s = 1$ and $t = 0$, so these two lines intersect for these values of s and t, at the point $(4, 6)$.

12.1.96 All three represent portions of the parabola $x = 2\cdot(y-4)^2$ where x is between 0 and 32. However, the curve in part **b** only represents the portion of the parabola where $y \geq 4$, because for that curve, $y = 4+t^2 \geq 4$.

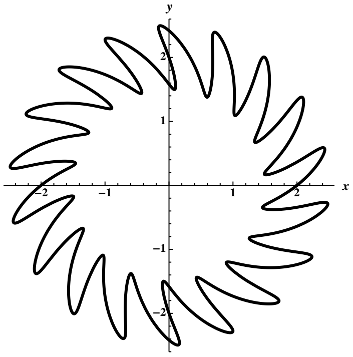
12.1.97



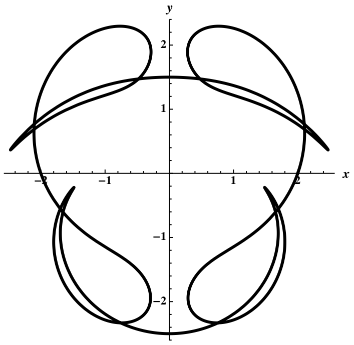
12.1.98



12.1.99



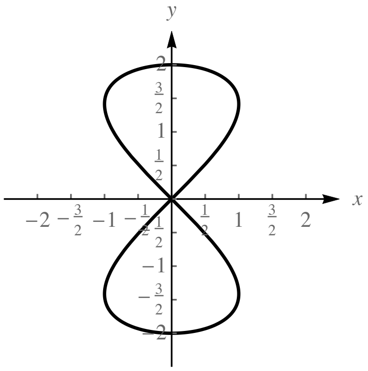
12.1.100



12.1.101

a. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2\cos t}{2\cos 2t}$. This is zero when $\cos t = 0$ but $\cos 2t \neq 0$, which occurs for $t = \pi/2$ and $t = 3\pi/2$. The corresponding points on the graph are $(0, 2)$ and $(0, -2)$.

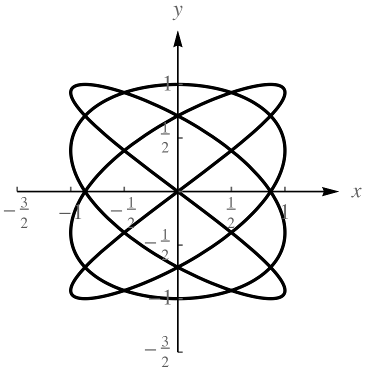
b. Using the derivative obtained above, we seek points where $\cos 2t = 0$ but $\cos t \neq 0$. This occurs for $t = \pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$. The corresponding points on the curve are $(1, \sqrt{2})$, $(-1, \sqrt{2})$, $(-1, -\sqrt{2})$, and $(1, -\sqrt{2})$.



12.1.102

a. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3\cos 3t}{4\cos 4t}$. This is zero when $\cos 3t = 0$ but $\cos 4t \neq 0$, which occurs for $t = \pi/6$, $\pi/2$, $5\pi/6$, $7\pi/6$, $3\pi/2$, and $t = 11\pi/6$. The corresponding points on the graph are the four points $(\pm\frac{\sqrt{3}}{2}, \pm 1)$ and the two points $(0, \pm 1)$.

b. Using the derivative obtained above, we seek points where $\cos 4t = 0$ but $\cos 3t \neq 0$. This occurs for $t = \dfrac{(2n+1)\pi}{8}$, $n = 0, 1, \ldots, 7$. The corresponding points on the curve are the four points $(\pm 1, \pm\sin(\pi/8))$ and $(\pm 1, \pm \sin(3\pi/8))$.



12.1.103 One arch is traced out by the parametric equations for $0 \le t \le 2\pi$. The base of the arch is in the interval $0 \le x \le 6\pi$. So the area is given by

$$\int_0^{6\pi} y\,dx = 9\int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt = 9\int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt$$
$$= 9\int_0^{2\pi} \left(1 - 2\cos t + \frac{1}{2} + \frac{1}{2}\cos 2t\right)\,dt = 9\left(\frac{3t}{2} - 2\sin t + \frac{1}{4}\sin 2t\right)\Big|_0^{2\pi} = 27\pi.$$

12.1.104 We find the area bounded by the ellipse in the first quadrant and multiply by 4. Let $f(t) = 3\cos t$ and $g(t) = 4\sin t$. Then we have $f'(t) = -3\sin t$, $\alpha = 0 = f(\pi/2)$ and $\beta = 3 = f(0)$. It follows that the area is $4\displaystyle\int_0^3 y\,dx = 4\int_{\pi/2}^0 4\sin t(-3\sin t)\,dt$. Evaluating the integral, we have

$$4\int_{\pi/2}^{0} 4\sin t(-3\sin t)\,dt = 48\int_0^{\pi/2} \sin^2 t\,dt = 24\int_0^{\pi/2} (1-\cos 2t)\,dt = 24\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^{\pi/2} = 12\pi.$$

12.1.105 The area is given by

$$\begin{aligned}4\int_{\pi/2}^{0}\sin^3 t(3\cos^2 t)(-\sin t)\,dt = 12\int_0^{\pi/2}\sin^4 t\cos^2 t\,dt &= \frac{3}{2}\int_0^{\pi/2}(1-\cos 2t)^2(1+\cos 2t)\,dt\\
&= \frac{3}{2}\int_0^{\pi/2}(1-\cos^2 2t)(1-\cos 2t)\,dt\\
&= \frac{3}{2}\int_0^{\pi/2}(1-\cos 2t-\cos^2 2t+\cos^3 2t)\,dt\\
&= \frac{3}{2}\int_0^{\pi/2}\left(1-\cos 2t-\frac{1+\cos 4t}{2}+(1-\sin^2 2t)(\cos 2t)\right)\,dt\\
&= \frac{3}{2}\int_0^{\pi/2}\left(\frac{1}{2}-\frac{\cos 4t}{2}-\sin^2 2t\cos 2t\right)\,dt\\
&= \frac{3}{2}\left(\frac{t}{2}-\frac{\sin 4t}{8}-\frac{\sin^3 2t}{6}\right)\Bigg|_0^{\pi/2} = \frac{3\pi}{8}.\end{aligned}$$

12.1.106 Revolving the parametric equations about the x-axis produces a sphere of radius 1. Therefore the area of this sphere is

$$\int_0^{\pi}2\pi\sin t\sqrt{(-\sin t)^2+(\cos t)^2}\,dt = 2\pi\int_0^{\pi}\sin t\,dt = 2\pi(-\cos t)\Big|_0^{\pi} = 2\pi(2) = 4\pi.$$

12.1.107

a. Observe that $x^2+(y-4)^2 = 9\cos^2 t+9\sin^2 t = 9$. So the curve is a circle of radius 3 centered at $(0,4)$.

b. Because the circle lies above the x-axis, the corresponding solid of revolution is a torus. The surface area is

$$\begin{aligned}S &= \int_0^{2\pi}2\pi g(t)\sqrt{(f'(t))^2+(g'(t))^2}\,dt = \int_0^{2\pi}2\pi(3\sin t+4)\sqrt{(-3\sin t)^2+(3\cos t)^2}\,dt\\
&= 6\pi\int_0^{2\pi}(3\sin t+4)\,dt = 6\pi(-3\cos t+4t)\Big|_0^{2\pi} = 6\pi(8\pi) = 48\pi^2.\end{aligned}$$

12.1.108 The area is given by

$$\begin{aligned}&\int_0^{\pi/2}2\pi\sin^3 t\sqrt{((3\cos^2 t)(-\sin t))^2+((3\sin^2 t)(\cos t))^2}\,dt = \int_0^{\pi/2}2\pi\sin^3 t\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}\,dt\\
&= \int_0^{\pi/2}6\pi\sin^3 t\sqrt{(\sin^2 t\cos^2 t)(\cos^2 t+\sin^2 t)}\,dt = 6\pi\int_0^{\pi/2}\sin^4 t\cos t\,dt = \frac{6}{5}\pi\sin^5 t\Big|_0^{\pi/2} = \frac{6\pi}{5}.\end{aligned}$$

12.1.109 The area is given by

$$\begin{aligned}\int_0^{2\pi}2\pi(1-\cos t)\sqrt{(1-\cos t)^2+\sin^2 t}\,dt &= \int_0^{2\pi}2\pi(1-\cos t)\sqrt{2-2\cos t}\,dt\\
&= 2\pi\sqrt{2}\int_0^{2\pi}(1-\cos t)^{3/2}\,dt = 2\pi\sqrt{2}\int_0^{2\pi}\left(2\sin^2\frac{t}{2}\right)^{3/2}\,dt\\
&= 8\pi\int_0^{2\pi}\sin^3\frac{t}{2}\,dt = 8\pi\int_0^{2\pi}\left(1-\cos^2\frac{t}{2}\right)\left(\sin\frac{t}{2}\right)\,dt.\end{aligned}$$

Now let $u = \cos\frac{t}{2}$ so that $du = -\frac{1}{2}\sin\frac{t}{2}$. Substituting gives

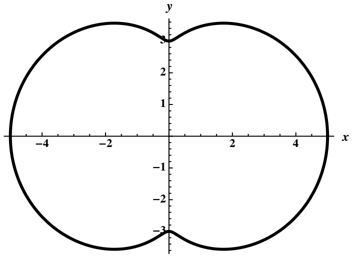
$$\begin{aligned}-16\pi\int_1^{-1}(1-u^2)\,du &= 16\pi\left(u-\frac{u^3}{3}\right)\Bigg|_{-1}^{1}\\
&= 16\pi\cdot\frac{4}{3} = \frac{64\pi}{3}.\end{aligned}$$

12.1.110 The area is given by

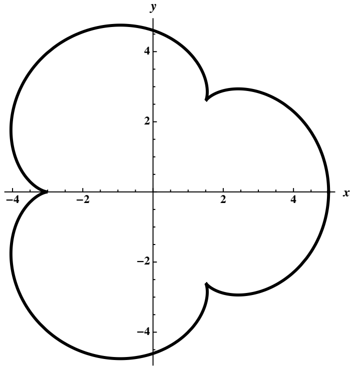
$$\int_1^5 2\pi\sqrt{t+1}\sqrt{\frac{1}{4(t+1)}+1}\,dt = \int_1^5 \pi\sqrt{4t+5}\,dt = \frac{\pi}{6}(4t+5)^{3/2}\bigg|_1^5 = \frac{\pi}{6}(125-27) = \frac{49\pi}{3}.$$

12.1.111 The area is given by $\int_0^1 2\pi(e^{3t}+1)\sqrt{4e^{4t}+9e^{6t}}\,dt$ which can be evaluated with a CAS to give about 1445.9.

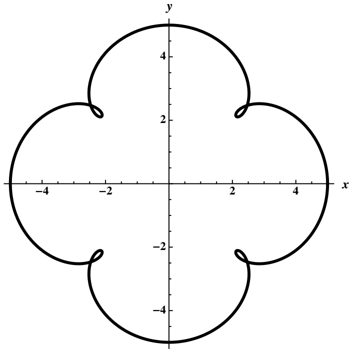
12.1.112



a.



b.

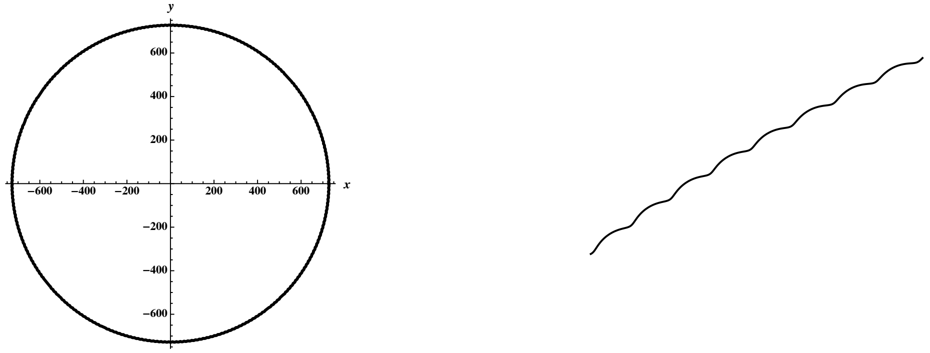


c.

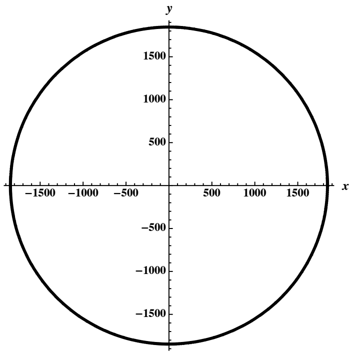
For a fixed a, there appear to be loops when $n > a$.

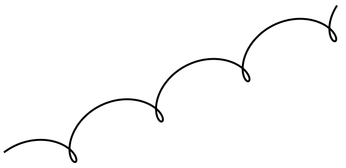
12.1.113

a.



b.





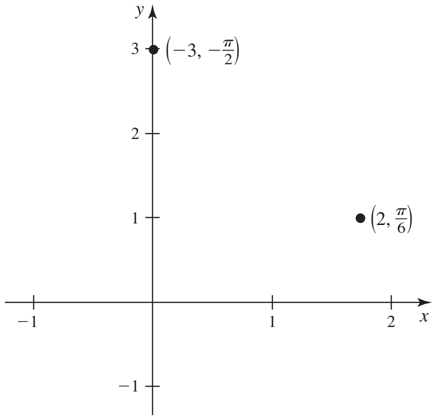
12.1.114 Note that $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, so $\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx} \cdot \frac{dx}{dt}\right) = \frac{dy}{dx}\frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{d}{dt}\left(\frac{dy}{dx}\right)$.
Also $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \cdot \frac{dx}{dt}$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. Thus,

$$\frac{d^2y}{dt^2} = \frac{dy}{dx} \cdot \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{d^2y}{dx^2} \cdot \frac{dx}{dt} = \frac{dy/dt}{dx/dt} \cdot \frac{d^2x}{dt^2} + \frac{d^2y}{dx^2} \cdot \left(\frac{dx}{dt}\right)^2.$$

Solving for $\frac{d^2y}{dx^2}$ yields $y'' = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t))^3} = \frac{f'(t)g''(t) - f''(t)g'(t)}{(f'(t))^3}$.

12.2 Polar Coordinates

12.2.1 The coordinates $(2, \pi/6)$, $(2, -11\pi/6)$, and $(-2, 7\pi/6)$ all give rise to the same point. Also, the coordinates $(-3, -\pi/2)$, $(3, \pi/2)$ and $(-3, 3\pi/2)$ give rise to the same point.



12.2.2 For a point with polar coordinates (r, θ), we have the Cartesian coordinates $x = r\cos\theta$ and $y = r\sin\theta$.

12.2.3 If a point has Cartesian coordinates (x, y) then $r^2 = x^2 + y^2$ and $\tan\theta = y/x$ for $x \neq 0$. If $x = 0$, then $\theta = \pi/2$ and $r = y$.

12.2.4 $r = 2a\cos\theta + 2b\sin\theta$.

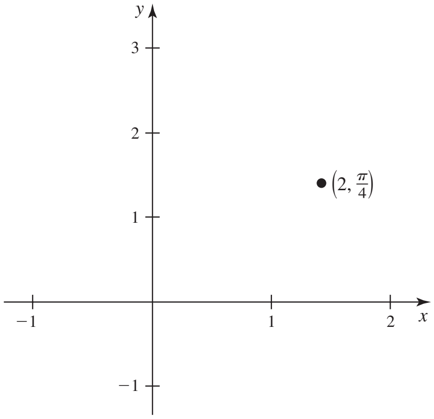
12.2.5 Because $x = r\cos\theta$, we have that the vertical line $x = 5$ has polar equation $r = 5\sec\theta$.

12.2.6 Because $y = r\sin\theta$, the horizontal line $y = 5$ has polar equation $r = 5\csc\theta$.

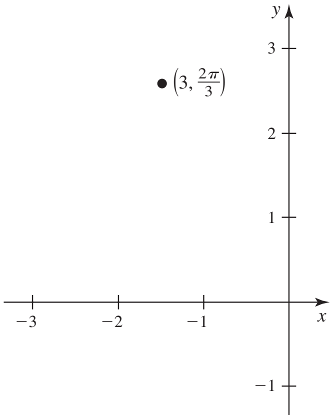
12.2.7 x-axis symmetry occurs if (r,θ) on the graph implies $(r,-\theta)$ is on the graph. y-axis symmetry occurs if (r,θ) on the graph implies $(r,\pi-\theta)=(-r,-\theta)$ is on the graph. Symmetry about the origin occurs if (r,θ) on the graph implies $(-r,\theta)=(r,\theta+\pi)$ is on the graph.

12.2.8 $(0,0)$, $(0,\pi)$, $(0,2\pi)$.

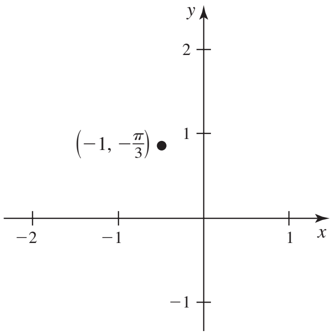
12.2.9 The coordinates $(2,\pi/4)$, $(-2,5\pi/4)$, and $(2,9\pi/4)$ represent the same point.



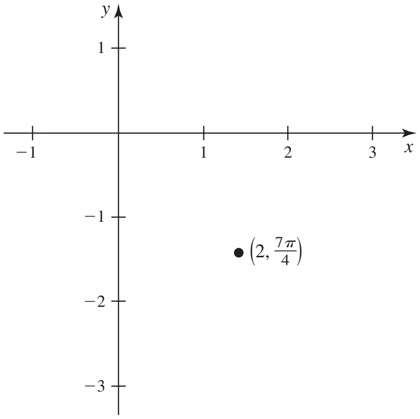
12.2.10 The coordinates $(3,2\pi/3)$, $(-3,5\pi/3)$ and $(3,8\pi/3)$ represent the same point.



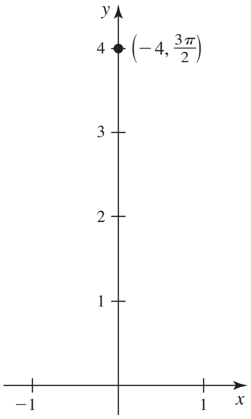
12.2.11 The coordinates $(-1,-\pi/3)$, $(1,2\pi/3)$ and $(1,-4\pi/3)$ represent the same point.



12.2.12 The coordinates $(2, 7\pi/4)$, $(-2, 3\pi/4)$ and $(2, -\pi/4)$ represent the same point.

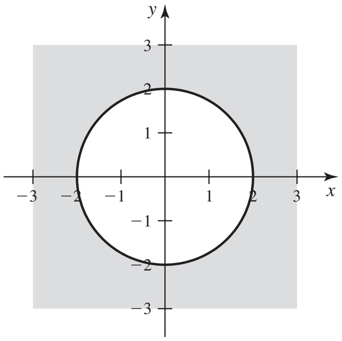


12.2.13 The coordinates $(-4, 3\pi/2)$, $(4, \pi/2)$ and $(-4, -\pi/2)$ represent the same point.



12.2.14 $A = (4, \pi/6) = (-4, 7\pi/6)$. $B = (3, \pi/4) = (-3, 5\pi/4)$. $C = (2, \pi/3) = (-2, 4\pi/3)$. $D = (4, \pi/2) = (-4, 3\pi/2)$. $E = (2, 4\pi/3) = (-2, \pi/3)$. $F = (4, -\pi/3) = (-4, 2\pi/3)$.

12.2.15



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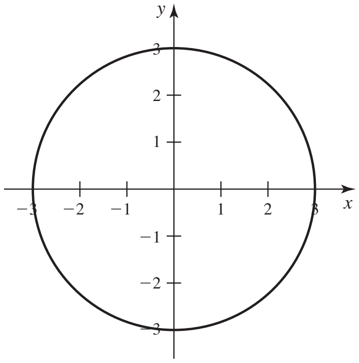
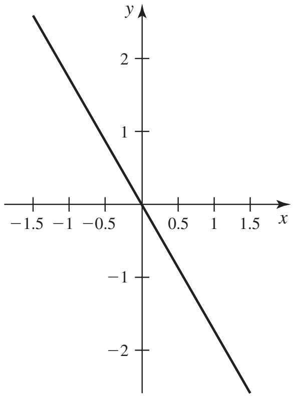
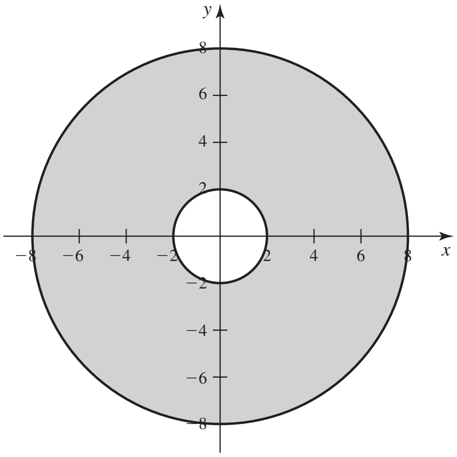


12.2.17



12.2.18



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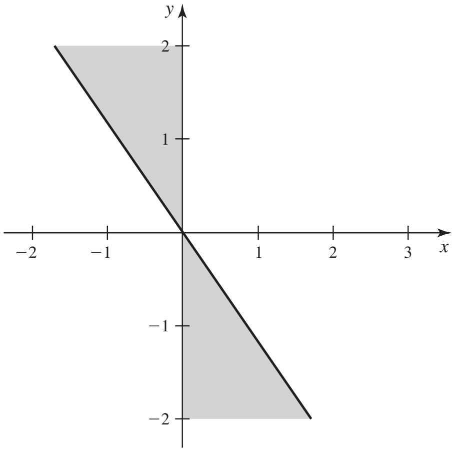
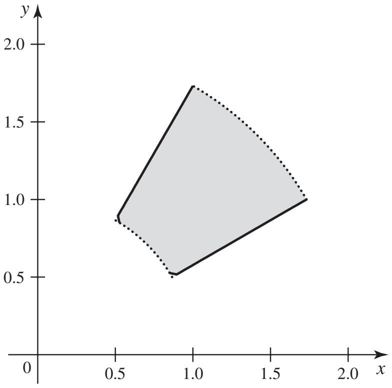
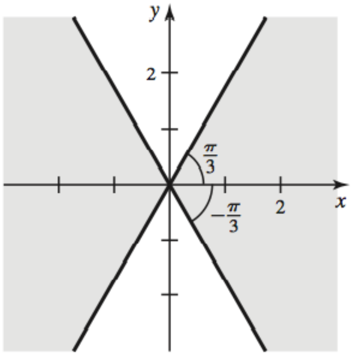


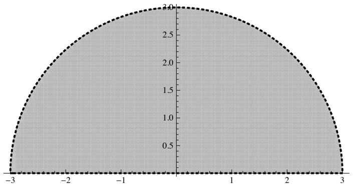
12.2.20



12.2.21



12.2.22



12.2.23 The angle (in radians) is $\frac{\pi}{2} - \frac{3\pi}{4}$, so the polar coordinates are $\left(100, \frac{\pi}{2} - \frac{3\pi}{4}\right) = \left(100, -\frac{\pi}{4}\right)$.

12.2.24 $\left(50, -\frac{2\pi}{3}\right)$.

12.2.25 $x = 3\cos(\pi/4) = \frac{3\sqrt{2}}{2}$. $y = 3\sin(\pi/4) = \frac{3\sqrt{2}}{2}$.

12.2.26 $x = \cos(2\pi/3) = -1/2$. $y = \sin(2\pi/3) = \dfrac{\sqrt{3}}{2}$.

12.2.27 $x = \cos(-\pi/3) = \dfrac{1}{2}$. $y = \sin(-\pi/3) = -\dfrac{\sqrt{3}}{2}$.

12.2.28 $x = 2\cos(7\pi/4) = 2 \cdot \dfrac{\sqrt{2}}{2} = \sqrt{2}$. $y = 2\sin(7\pi/4) = -\sqrt{2}$.

12.2.29 $x = -4\cos(3\pi/4) = 2\sqrt{2}$. $y = -4\sin(3\pi/4) = -2\sqrt{2}$.

12.2.30 $x = 4\cos(5\pi) = -4$. $y = 4\sin(5\pi) = 0$.

12.2.31 $r^2 = x^2 + y^2 = 4 + 4 = 8$, so $r = \sqrt{8}$. $\tan\theta = 1$, so $\theta = \pi/4$, so $(2\sqrt{2}, \pi/4)$ is one representation of this point, and $(-2\sqrt{2}, -3\pi/4)$ is another.

12.2.32 $r^2 = x^2 + y^2 = 1 + 0$, so $r = \pm 1$. $\tan\theta = 0$, so $\theta = 0, \pi$. $(-1, 0)$ is one representation of this point, and $(1, \pi)$ is another.

12.2.33 $r^2 = x^2 + y^2 = 1 + 3 = 4$, so $r = \pm 2$. $\tan\theta = \sqrt{3}$, so $\theta = \pi/3$, $4\pi/3$. $(2, \pi/3)$ is one representation of this point, and $(-2, -2\pi/3)$ is another.

12.2.34 $r^2 = 81$, so $r = \pm 9$. $\cos\theta = 0$, so $\theta = \pm\dfrac{\pi}{2}$. One representation of the given point is $(9, -\pi)/2$, and $(-9, \pi/2)$ is another.

12.2.35 $r^2 = 64$, so $r = \pm 8$. $\tan\theta = -\sqrt{3}$, so $\theta = -\pi/3$, $2\pi/3$. One representation of the given point is $(8, 2\pi/3)$, and $(-8, -\pi/3)$ is another.

12.2.36 $r^2 = 64 \cdot 3 + 64 = 256$, so $r = 16$. $\tan\theta = \dfrac{1}{\sqrt{3}}$, so $\theta = \pi/6, 7\pi/6$. One representation of the given point is $(16, 7\pi/6$ and another is $(16, -5\pi/6)$.

12.2.37 $x = -4$, so this is the vertical line through $(-4, 0)$.

12.2.38 Rewriting the equation in terms of sine and cosine, we have $r = \dfrac{\cos\theta}{\sin\theta}\dfrac{1}{\sin\theta}$, or $r\sin^2\theta = \cos\theta$. Multiplying both sides by r, we have $r^2\sin^2\theta = r\cos\theta$, or $y^2 = x$. This curve is a parabola with vertex at $(0, 0)$ which opens to the right.

12.2.39 Because $x^2 + y^2 = r^2 = 4$, this is a circle of radius 2 centered at the origin.

12.2.40 Because $y = r\sin\theta = 3\csc\theta\sin\theta = 3$, this is the horizontal line $y = 3$.

12.2.41 Note that $x^2 + y^2 = r^2 = 4\sin^2\theta + 8\sin\theta\cos\theta + 4\cos^2\theta = 4 + 8\sin\theta\cos\theta$. Also note that $x = r\cos\theta = 2\sin\theta\cos\theta + 2\cos^2\theta$ and $y = r\sin\theta = 2\sin^2\theta + 2\sin\theta\cos\theta$. Thus, $2x + 2y = 4 + 8\sin\theta\cos\theta$. If we combine these, we see that $x^2 + y^2 - (2x + 2y) = 0$. Thus $(x^2 - 2x + 1) + (y^2 - 2y + 1) = 2$, so we have the circle $(x - 1)^2 + (y - 1)^2 = 2$. This is a circle of radius $\sqrt{2}$ centered at $(1, 1)$.

12.2.42 Multiplying each side of the equation by r, we have $r^2 = -2r\cos\theta$, or $x^2 + 2x + y^2 = 0$. Completing the square, we have $(x + 1)^2 + y^2 = 1$, which is a circle of radius 1 centered at $(-1, 0)$. This result can also be obtained by using the summary box on circles in polar coordinates (we have $a = -1$, $b = 0$, and $r = \sqrt{(-1)^2 + 0} = 1$).

12.2.43 Multiplying each side of the equation by r, we have $r^2 = 6r\cos\theta + 8r\sin\theta$. Converting to Cartesian coordinates, we have $x^2 + y^2 = 6x + 8y$. Completing the square leads to $x^2 - 6x + 9 + y^2 - 8y + 16 = 25$ and then factoring we arrive at $(x - 3)^2 + (y - 4)^2 = 5^2$. So the curve is a circle of radius 5 centered at (3,4). This result can also be obtained by inspection using the summary box on circles in polar coordinates.

12.2.44 We have $r\sin\theta = \pm r\cos\theta$, so $y = \pm x$. These are lines through the origin with slopes ± 1.

12.2.45 $r\cos\theta = \sin 2\theta = 2\sin\theta\cos\theta$. Note that if $\cos\theta = 0$, then r can be any real number, and the equation is satisfied. For $\cos\theta \neq 0$, we have $x = r\cos\theta = 2\sin\theta\cos\theta$, so $r = 2\sin\theta$, and thus $y = r\sin\theta = 2\sin^2\theta$. Thus $x^2 + y^2 - 2y = 4\sin^2\theta\cos^2\theta + 4\sin^2\theta\sin^2\theta - 4\sin^2\theta = 4\sin^2\theta(\sin^2\theta + \cos^2\theta) - 4\sin^2\theta = 4\sin^2\theta - 4\sin^2\theta = 0$. Note also that $x^2 + y^2 - 2y = 0$ is equivalent to $x^2 + (y-1)^2 = 1$, so we have a circle of radius one centered at $(0, 1)$, as well as the line $x = 0$ which is the y-axis.

12.2.46 $r = \sin\theta\sec^2\theta$, so $x = r\cos\theta = \tan\theta = \dfrac{y}{x}$, so $y = x^2$, the standard parabola.

12.2.47 $r = 8\sin\theta$, so $r^2 = 8r\sin\theta$, so $x^2+y^2 = 8y$. This can be written $x^2+(y-4)^2 = 16$, which represents a circle of radius 4 centered at $(0, 4)$.

12.2.48 The given equation implies that $2r\cos\theta + 3r\sin\theta = 1$, so $2x + 3y = 1$. This is a line with slope $-\dfrac{2}{3}$ and y-intercept $\dfrac{1}{3}$.

12.2.49 We have $r\sin\theta = r^2\cos^2\theta$, so $r = \dfrac{\sin\theta}{\cos^2\theta} = \tan\theta\sec\theta$.

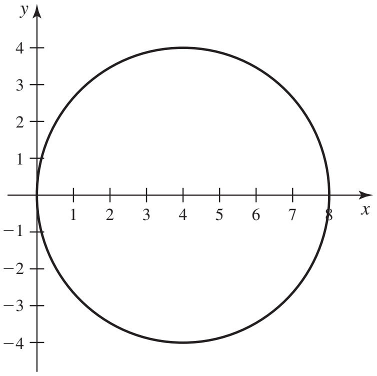
12.2.50 We have $y = r\sin\theta = 3$, so $r = \dfrac{3}{\sin\theta} = 3\csc\theta$.

12.2.51 We have $r\sin\theta = \dfrac{1}{r\cos\theta}$, so $r^2 = \sec\theta\csc\theta$.

12.2.52 We have $(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$, so $r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = 1$, and thus $2r\cos\theta = r^2$. Thus $r = 2\cos\theta$.

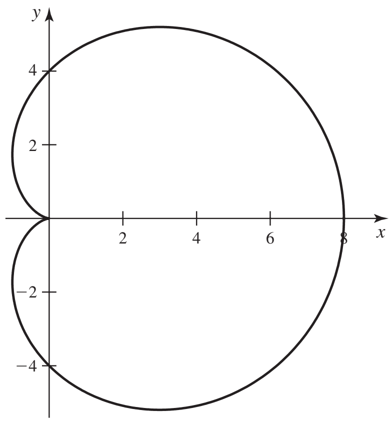
12.2.53

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
r	8	$4\sqrt{3}$	$4\sqrt{2}$	4	0	-4	$-4\sqrt{2}$	$-4\sqrt{3}$	-8

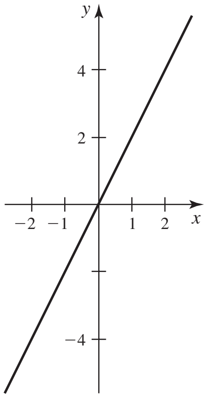


12.2.54

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	8	$4+2\sqrt{2}$	4	$4-2\sqrt{2}$	0	$4-2\sqrt{2}$	4	$4+2\sqrt{2}$	8

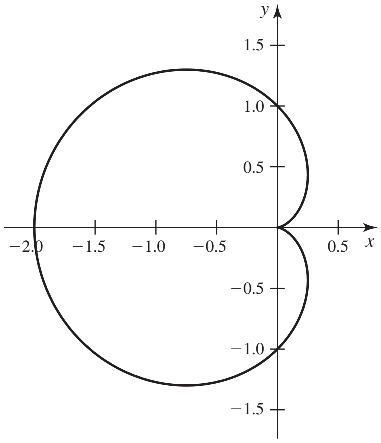


12.2.55 $r(\sin\theta - 2\cos\theta) = 0$ when $r = 0$ or when $\tan\theta = 2$, so the curve is a straight line through the origin of slope 2.

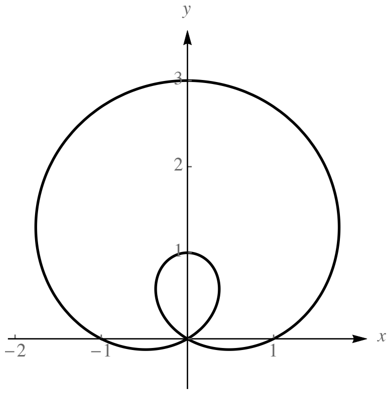


12.2.56

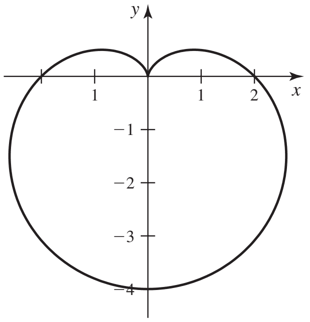
θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	0	$\frac{\sqrt{2}-1}{\sqrt{2}}$	1	$\frac{\sqrt{2}+1}{\sqrt{2}}$	2	$\frac{\sqrt{2}+1}{\sqrt{2}}$	1	$\frac{\sqrt{2}-1}{\sqrt{2}}$	0



12.2.57



12.2.58



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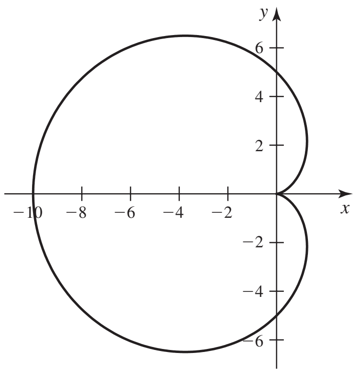

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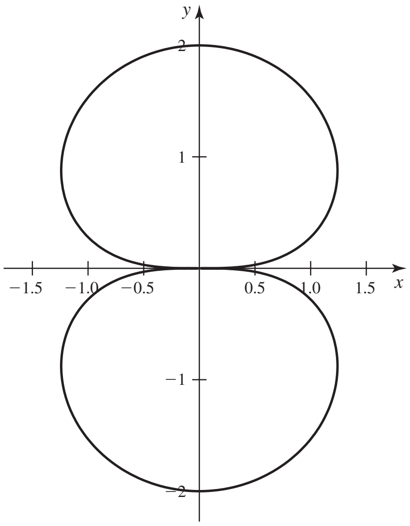
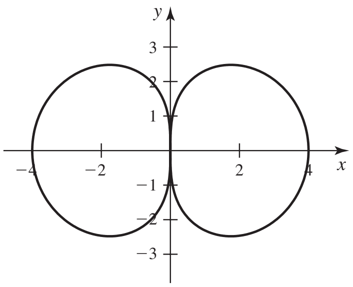
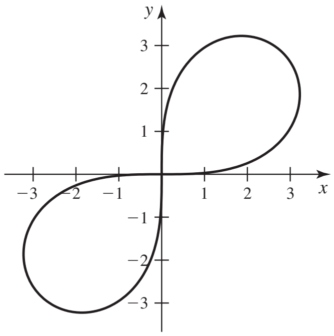


12.2.61



12.2.62



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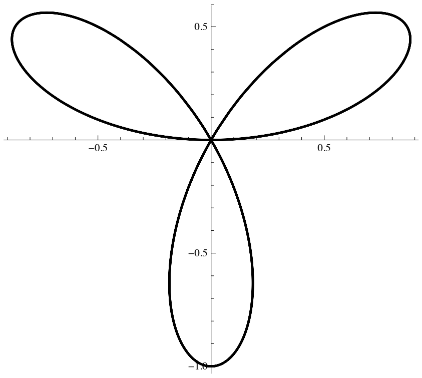
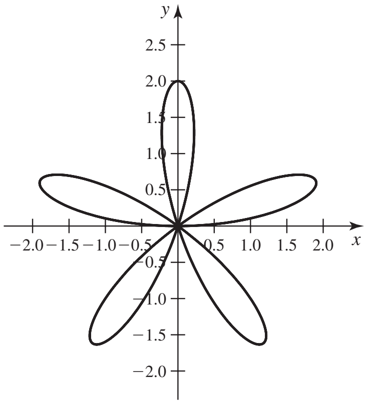
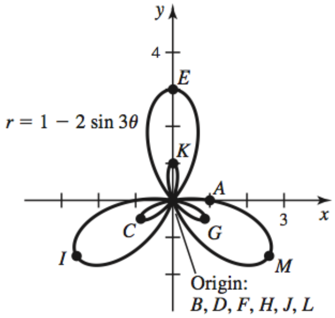


12.2.64

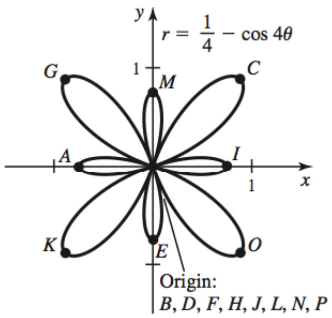


12.2.65 Points B, D, F, H, J and L have y-coordinate 0, so the graph is at the pole for each of these points. Points E, I, and M have maximal radius, so these correspond to the points at the tips of the outer loops. The points C, G and K correspond to the tips of the smaller loops. Point A corresponds to the polar point $(1, 0)$.



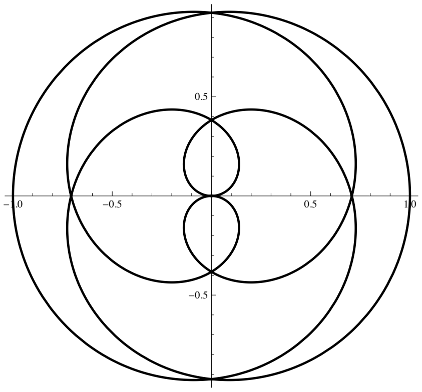
12.2.66 Points B, D, H and J have y-coordinate 0, so the graph is at the pole for each of these points. Points A and K lie where the graph intersects the negative x-axis. C and I are at the top of the two large loops, while F is is where the graph intersects the positive x-axis. E and G are the extreme points of the large wide loop.

12.2.67 Points B, D, F, H, J, L, N and P are at the origin. C, G, K and O are on the ends of the long loops, while A, E, I and M are at the ends of the smaller loops.

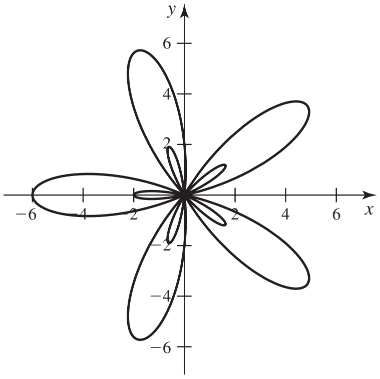


12.2.68 Points C, E, G and I are at the origin. B and D are at the ends of the two bigger loops, F and H are at ends of the two smaller loops. A and J are the points where the graph intersects the positive x-axis.

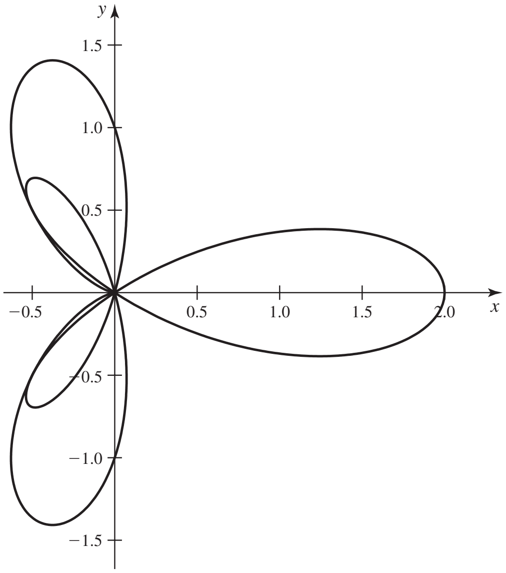
12.2.69 The interval $[0, 8\pi]$ generates the entire graph.



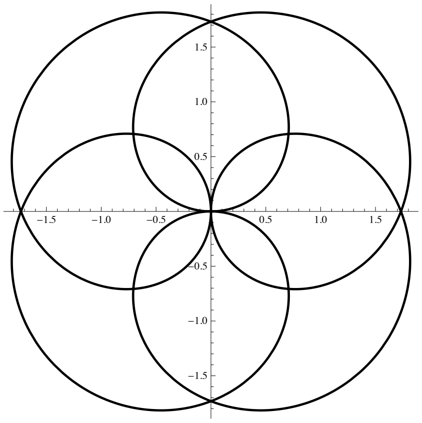
12.2.70 The interval $[0, 2\pi]$ generates the entire graph.



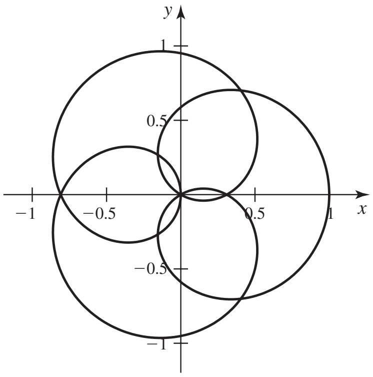
12.2.71 The interval $[0, 2\pi]$ generates the entire graph.



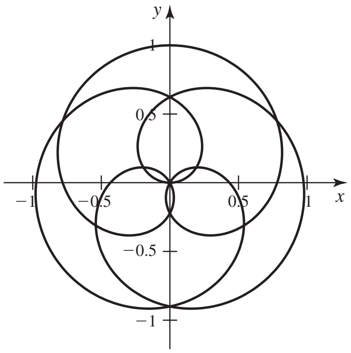
12.2.72 The interval $[0, 6\pi]$ generates the entire graph.



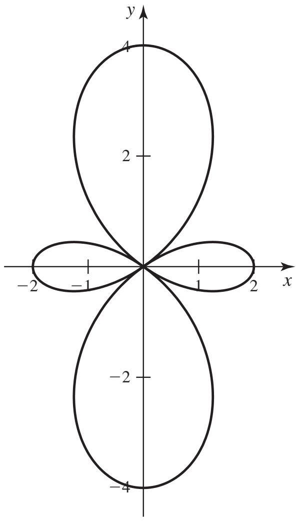
12.2.73 The interval $[0, 5\pi]$ generates the entire graph.



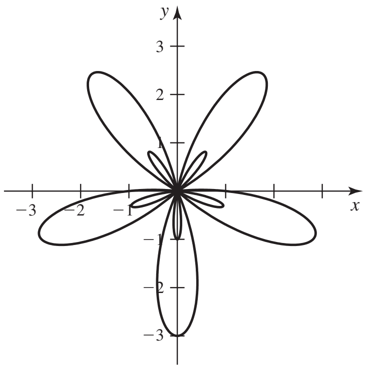
12.2.74 The interval $[0, 7\pi]$ generates the entire graph.



12.2.75 The interval $[0, 2\pi]$ generates the entire graph.



12.2.76 The interval $[0, 2\pi]$ generates the entire graph.



12.2.77

a. True. Note that $r^2 = 8$ and $\tan\theta = -1$.

b. True. Their intersection point (in Cartesian coordinates) is $(4, -2)$.

c. False. They intersect at the polar coordinates $(2, \pi/4)$ and $(2, 5\pi/4)$.

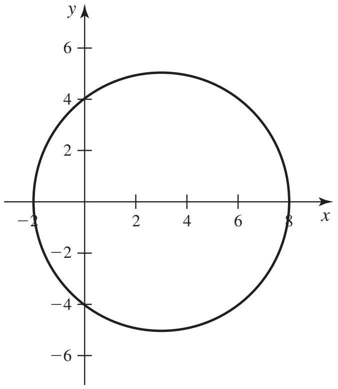
d. True. Note that for $\theta = \frac{3\pi}{2}$ we have $r = -3$. But the polar point $\left(-3, \dfrac{3\pi}{2}\right)$ is the same as the polar point $\left(3, \dfrac{\pi}{2}\right)$.

e. True. The first is the line $x = 2$ because $x = r\cos\theta = 2\sec\theta\cos\theta = 2$, and the second is $y = 3$ because $y = r\sin\theta = 3\csc\theta\sin\theta = 3$.

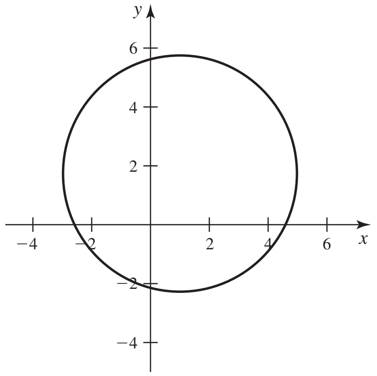
12.2.78 Let $x = r\cos\theta$ and $y = r\sin\theta$, so that $r^2 = x^2 + y^2$. Then the given equation can be written $(x^2 + y^2) - 2ax - 2by + (a^2 + b^2) = R^2$, which in turn can be written as $(x - a)^2 + (y - b)^2 = R^2$, which is the equation of a circle of radius R centered at (a, b).

12.2.79 Consider the circle with center $C(r_0, \theta_0)$, and let A be the origin and $B(r, \theta)$ be a point on the circle not collinear with A and C. Note that the length of side BC is R, and that the angle CAB has measure $\theta - \theta_0$. Applying the law of cosines to triangle CAB yields the equation $R^2 = r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)$, which is equivalent to the given equation.

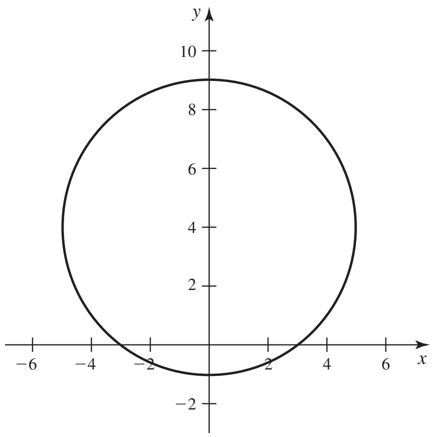
12.2.80 In relation to number 78, we have $2a = 6$, so $a = 3$ and $b = 0$. So $R^2 - a^2 - b^2 = R^2 - 9 = 16$, and thus $R^2 = 25$. Thus we have a circle centered at $(3, 0)$ with radius 5.



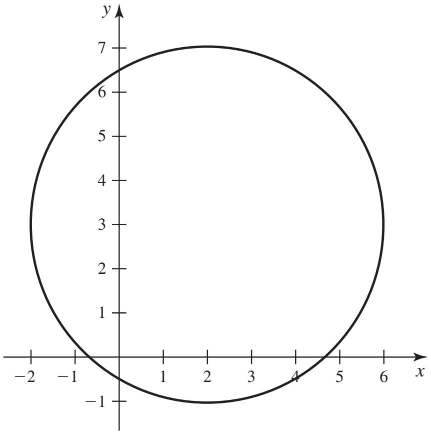
12.2.81 In relation to number 79, we have $r_0 = 2$ and $\theta_0 = \pi/3$, and $R^2 - 4 = 12$, so $R^2 = 16$. Thus this is a circle with polar center $(2, \pi/3)$ and radius 4.



12.2.82 In relation to number 79, we have $r_0 = 4$ and $\theta_0 = \pi/2$, and $R^2 - 16 = 9$, so $R^2 = 25$. Thus this is a circle with polar center $(4, \pi/2)$ and radius 5.



12.2.83 In relation to number 78, we have $a = 2$ and $b = 3$. So $R^2 - a^2 - b^2 = R^2 - 13 = 3$, and thus $R^2 = 16$. Thus we have a circle centered at $(2, 3)$ with radius 4.



12.2.84 The radius of a circle inscribed in a triangle with side lengths a, b, and c is $\frac{2A}{a+b+c}$ where A is the area of the triangle. So for the bigger circle, $R = r_0 = \frac{2}{2+2\sqrt{2}} = \frac{1}{1+\sqrt{2}}$. For each of the smaller circles, we have $R = \frac{1}{2+\sqrt{2}}$. The area inside the three circles is thus $2\pi \cdot \frac{1}{(2+\sqrt{2})^2} + \pi \cdot \frac{1}{(1+\sqrt{2})^2} \approx 1.078$. Because the area of the square is 2, there is more area inside the circles than outside the circles but inside the square. Using problem 75, the equation of the largest circle is $r^2 - 2r\left(\frac{1}{1+\sqrt{2}}\right)\cos(\theta - \pi/2) = 0$. The smaller circle in the 3rd quadrant has center with polar radius $r_0 = \frac{\sqrt{2}}{2} - \frac{1}{2+\sqrt{2}} = \sqrt{2} - 1$, so its equation is $r^2 - 2r\left(\sqrt{2} - 1\right)\cos(\theta - 5\pi/4) = R^2 - r_0^2 = \sqrt{2} - 3/2$, and the other circle has equation $r^2 - 2r\left(\sqrt{2} - 1\right)\cos(\theta + \pi/4) = \sqrt{2} - 3/2$.

12.2.85

a. During this 30-minute period, the plane moves on a straight line from the point (using Cartesian coordintes) $(0, 150)$, to $(100\cos\frac{\pi}{6}, 100\sin\frac{\pi}{6}) = (50\sqrt{3}, 50)$. So the distance travelled is

$$\sqrt{(50\sqrt{3})^2 + 100^2} \approx 132.3\,\text{miles}.$$

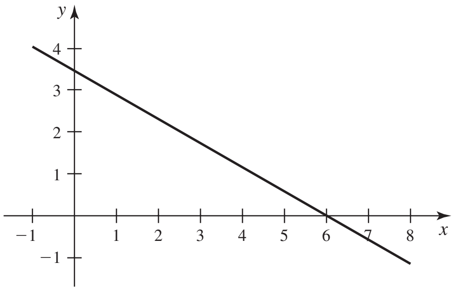
b. The average velocity is $\frac{132.3}{1/2} = 264.6$ miles per hour.

12.2.86

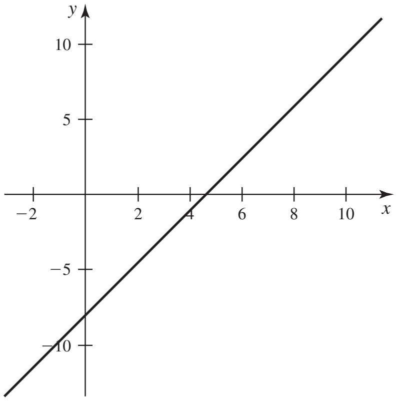
a. Given $y = mx + b$, let $x = r\cos\theta$ and $y = r\sin\theta$. Then $r\sin\theta = m(r\cos\theta) + b$, so $r\sin\theta - mr\cos\theta = b$, and thus $r(\sin\theta - m\cos\theta) = b$, and $r = \frac{b}{\sin\theta - m\cos\theta}$, provided $\sin\theta - m\cos\theta \neq 0$.

b. Using the right triangle shown, we see that $\frac{r_0}{r} = \cos(\theta_0 - \theta)$, so $r_0 = r\cos(\theta_0 - \theta)$.

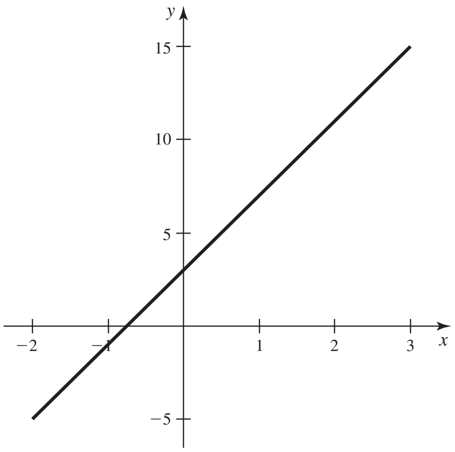
12.2.87 Using problem 86b, this is the line with $r_0 = 3$ and $\theta_0 = \frac{\pi}{3}$. So it is the line through the polar point $(3, \pi/3)$ in the direction of angle $\pi/3 + \pi/2 = 5\pi/6$. The Cartesian equation is $y = -\frac{x}{\sqrt{3}} + 2\sqrt{3}$.



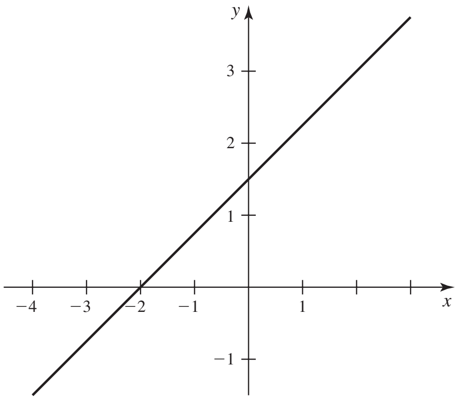
12.2.88 Using problem 86b, this is the line with $r_0 = 4$ and $\theta_0 = -\frac{\pi}{6}$. So it is the line through the polar point $(4, -\pi/6)$ in the direction of angle $-\pi/6 + \pi/2 = \pi/3$. The Cartesian equation is $y = \sqrt{3}x - 8$.



12.2.89 Using problem 86a, this is the line with $b = 3$ and $m = 4$, so $y = 4x + 3$.



12.2.90 Using problem 86a, this is the line with $b = 3/2$ and $m = 3/4$, so $y = \frac{3}{4}x + \frac{3}{2}$.



12.2.91

a. This matches (A), because we have $|a| = 1 = |b|$, and the graph is a cardioid.

b. This matches (C). This has an inner loop because $|a| = 1 < 2 = |b|$. Note that $r = 1$ when $\theta = 0$, so it can't be (D).

c. This matches (B). This has $|a| = 2 > 1 = |b|$, so it has an oval-like shape.

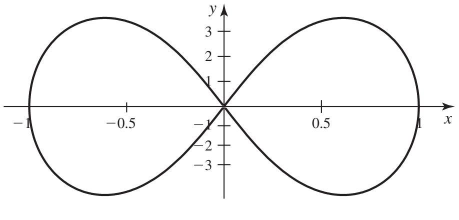
d. This matches (D). This has an inner loop because $|a| = 1 < 2 = |b|$. Note that $r = -1$ when $\theta = 0$, so this can't be (C).

e. This matches (E). Note that there is an inner loop because $|a| = 1 < 2 = |b|$, and that $r = 3$ when $\theta = \pi/2$.

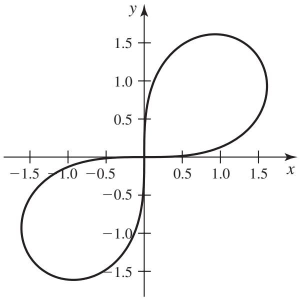
f. This matches (F).

12.2.92 As $b \to \infty$, the inner loop approaches the outer loop, so that for large b the graph appears to be a single circle with diameter b. Thus, there is no limiting curve as $b \to \infty$.

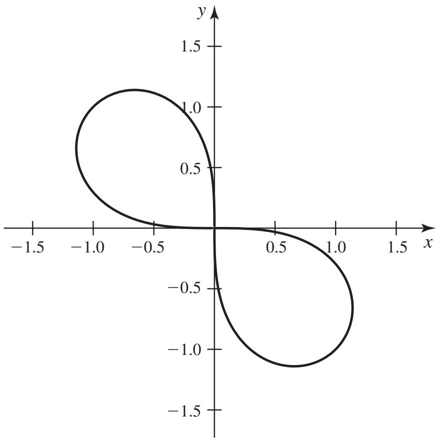
12.2.93



12.2.94



12.2.95



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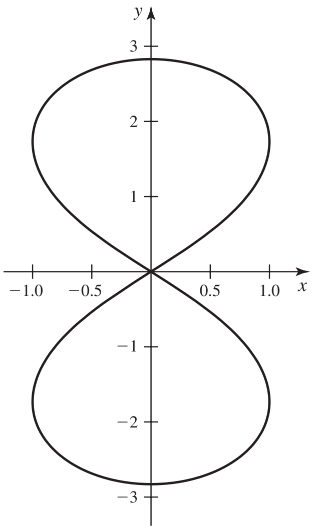

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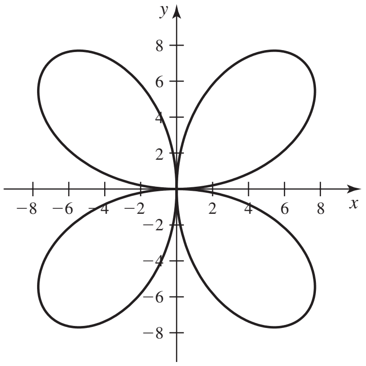

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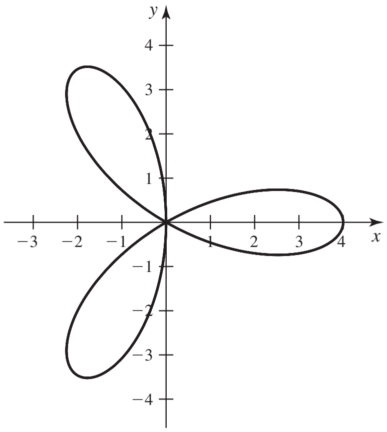

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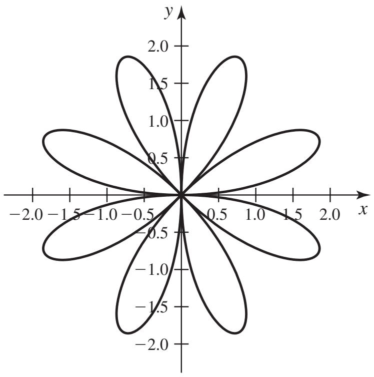
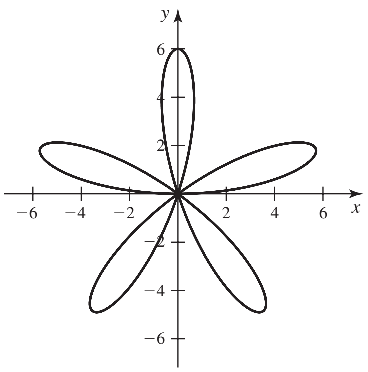
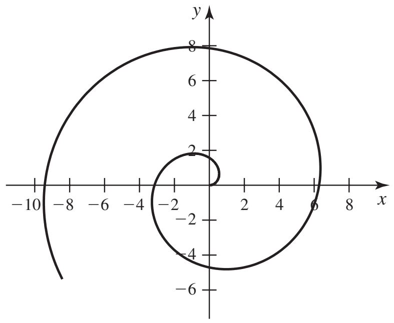


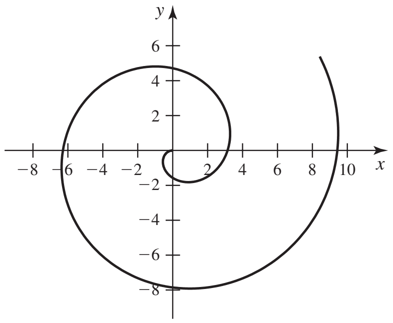
12.2.100



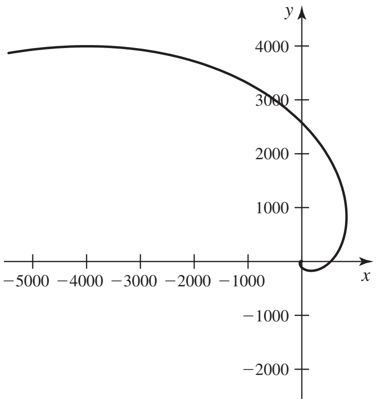
12.2.101 Note that $a \sin m\theta = 0$ for $\theta = \dfrac{k\pi}{m}$, $k = 1, 2, \ldots, 2m$. Thus the graph is back at the pole $r = 0$ for each of these values, and each of these gives rise to a distinct petal of the rose if m is odd. If m is even, then by symmetry, each petal for $k = 1, 2, \ldots \dfrac{m}{2}$ is equivalent to one for $k = \dfrac{m}{2} + 1, \dfrac{m}{2} + 2, \ldots, m$. (Note that this follows because the sine function is odd.) A similar result holds for the rose $r = a\cos\theta$.

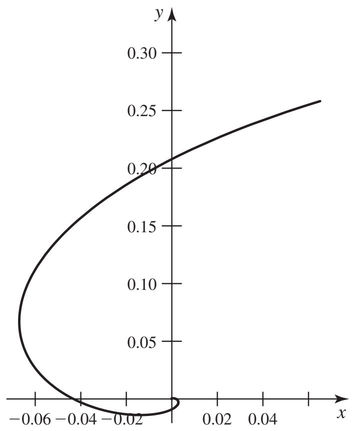
12.2.102 The spirals wind outward counterclockwise.



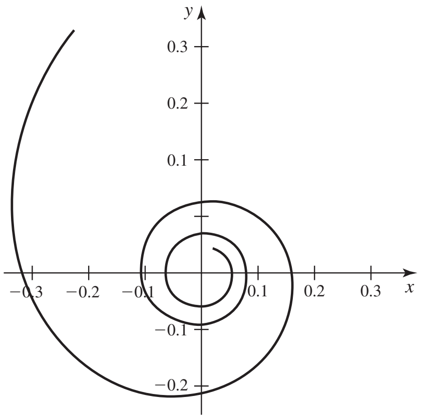


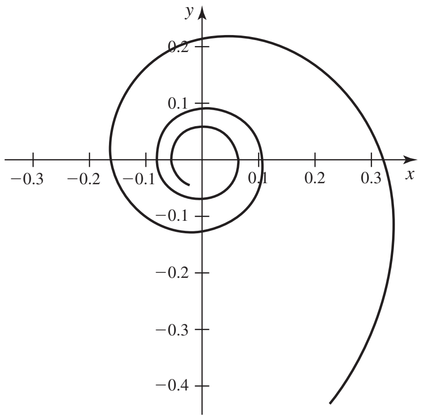
12.2.103 For $a = 1$, the spiral winds outward counterclockwise. For $a = -1$, the spiral winds inward counterclockwise.



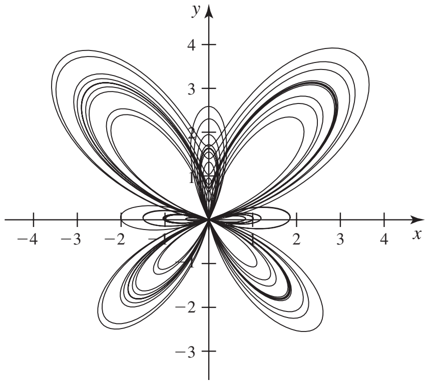


12.2.104 The spirals wind inward counterclockwise for $a = 1$ and outward clockwise for $a = -1$.





12.2.105



a.

b. It adds multiple layers of the same type of curve as $\sin^5 \theta/12$ oscillates between -1 and 1 for $0 \leq \theta \leq 24\pi$.

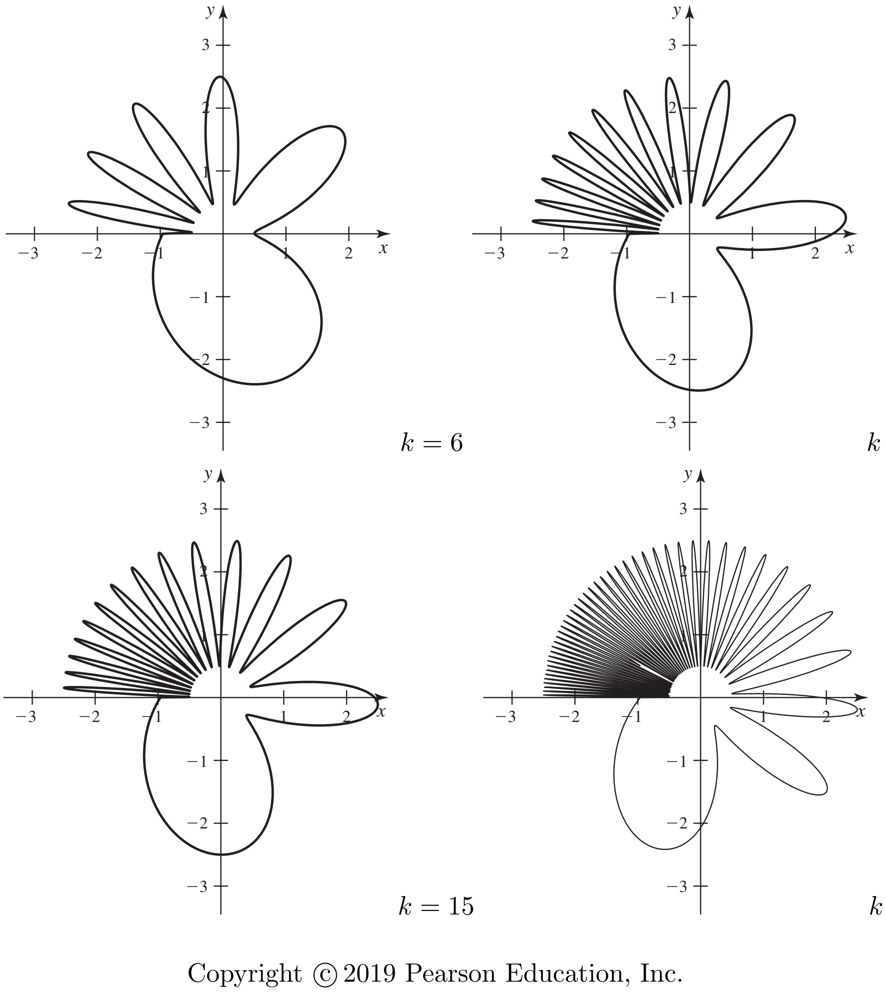
12.2.106

a. $f(0) = \cos(1) - 1.5$, and $f(2\pi) = \cos(((1+12\pi)^{1/2\pi})^{2\pi}) - 1.5 = \cos(1+12\pi) - 1.5 = \cos(1) - 1.5 = f(0)$. The points correspond to the polar points $(-0.960, 0)$.

b. No. The curve for $-\pi \leq \theta \leq 0$ has nowhere where the absolute value of the radius is equal to 1, whereas the curve for $\pi \leq \theta \leq 2\pi$ has numerous places where this is true, because a^x has a much bigger range on $[0, \pi]$ than on $[-\pi, 0]$.

c. Because $((1 + 2k\pi)^{1/2\pi})^0 = 1$ and $((1 + 2k\pi)^{1/2\pi})^{2\pi} = 1 + 2k\pi$, we have that $f(0) = \cos(1) - b = \cos(1 + 2k\pi) - b = \cos(1) - b = f(2\pi)$.

d.



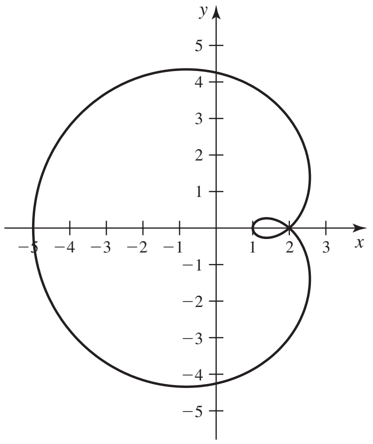
$k = 6$

$k = 12$

$k = 15$

$k = 45$

12.2.107



a.

b. $r = 3 - 4\cos \pi t$ is a limaçon, and $x - 2 = r\cos \pi t$ and $y = r\sin \pi t$ is a circle, and the composition of a limaçon and a circle is a limaçon.

12.2.108

a. The region is given by $\{(r, \theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$.

b. The inflow is given by $\{(r, \theta) : 1 \le r \le 2,\ \theta = 0\}$. The outflow is given by $\{(r, \theta) : 1 \le r \le 2,\ \theta = \pi\}$.

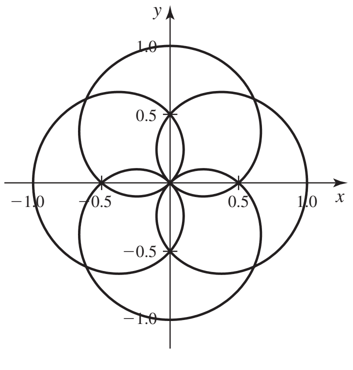
c. The tangential velocity at $(1.5, \pi/4)$ is $v(1.5) = 10 \cdot 1.5 = 15$ meters per second. At $(1.2, 3\pi/4)$ it is $v(1.2) = 10 \cdot 1.2 = 12$ meters per second, so it is greater at 1.5.

d. The velocity is greater at $r = 1.3$, because $\dfrac{20}{1.3} > \dfrac{20}{1.8}$.

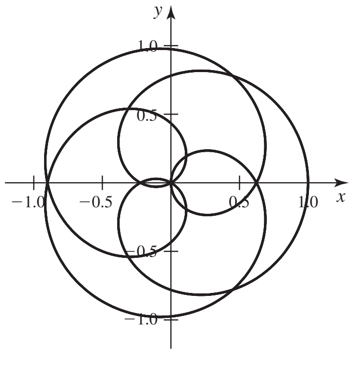
e. $\displaystyle\int_1^2 10r\,dr = 5r^2\Big|_1^2 = 15$, while $\int_1^2 \frac{20}{r}\,dr = 20\ln r\big|_1^2 \approx 13.86$, so the flow is greater in part c).

12.2.109 Because $\sin(\theta/2) = \sin(\pi - \theta/2) = \sin((2\pi - \theta)/2)$, we have that the graph is symmetric with respect to the x-axis.

12.2.110



a.



b.

c. If n is even, then the whole curve is generated for $0 \le \theta \le 2m\pi$. If n is odd, then the whole curve is generated for $0 \le \theta \le m\pi$.

12.2.111 Note that $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, so $r^2 = a^2(\cos^2\theta - \sin^2\theta)$, so $r^4 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$, so $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

12.3 Calculus in Polar Coordinates

12.3.1 Because $x = r\cos\theta$ and $y = r\sin\theta$, we have $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin(\theta)$.

12.3.2 This line is a vertical line passing through the origin and vertical lines have undefined slope.

12.3.3 Because slope is given relative to the horizontal and vertical coordinates, it is given by $\frac{dy}{dx}$, not by $\frac{dr}{d\theta}$.

12.3.4 This would be given by $\frac{1}{2}\int_\alpha^\beta (f(\theta)^2 - g(\theta)^2)\,d\theta$.

12.3.5 The slope is $\tan\frac{\pi}{3} = \sqrt{3}$.

12.3.6 $\theta = \frac{3\pi}{4}$.

12.3.7 The area is given by $\int_0^\pi \frac{1}{2}\theta\,d\theta = \left.\frac{\theta^2}{4}\right|_0^\pi = \frac{\pi^2}{4}$.

12.3.8 The curve passes through the origin for $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$. The corresponding slopes are $\tan\frac{\pi}{3} = \sqrt{3}$ and $\tan\frac{5\pi}{3} = -\sqrt{3}$.

12.3.9 Both curves pass through the origin, but for different values of θ.

12.3.10 The polar point $(-1, 3\pi/2)$ is equivalent to the polar point $(1, \pi/2)$ which does satisfy the equation.

12.3.11 $\frac{dy}{dx} = \frac{-\cos\theta\sin\theta + (1-\sin\theta)\cos\theta}{-\cos^2\theta - (1-\sin\theta)\sin\theta}$. At $(1/2, \pi/6)$, we have $\frac{dy}{dx} = \frac{0}{-1} = 0$.

12.3.12 $\frac{dy}{dx} = \frac{-4\sin^2\theta + 4\cos^2\theta}{-8\cos\theta\sin\theta}$. At $(2, \pi/3)$ we have $\frac{dy}{dx} = \frac{-2}{-2\sqrt{3}} = \frac{\sqrt{3}}{3}$.

12.3.13 $\frac{dy}{dx} = \frac{16\cos\theta\sin\theta}{-8\sin^2\theta + 8\cos^2\theta}$. At $(4, 5\pi/6)$ we have $\frac{dy}{dx} = \frac{-4\sqrt{3}}{4} = -\sqrt{3}$.

12.3.14 $\frac{dy}{dx} = \frac{\cos\theta\sin\theta + (4+\sin\theta)\cos\theta}{\cos^2\theta - (4+\sin\theta)\sin\theta}$. At $(4, 0)$ we have $\frac{dy}{dx} = \frac{4}{1} = 4$. At $(3, 3\pi/2)$ we have $\frac{dy}{dx} = \frac{0}{3} = 0$.

12.3.15 $\frac{dy}{dx} = \frac{-3\sin^2\theta + (6+3\cos\theta)\cos\theta}{-3\cos\theta\sin\theta - (6+3\cos\theta)\sin\theta}$. At both $(3, \pi)$ and $(9, 0)$, this doesn't exist.

12.3.16 $\frac{dy}{dx} = \frac{6\cos(3\theta)\sin\theta + 2\sin(3\theta)\cos\theta}{6\cos(3\theta)\cos\theta - 2\sin(3\theta)\sin\theta}$. The tips of the leaves occur at $\theta = \pi/6$, $\pi/2$ and $5\pi/6$. At $\pi/6$, we have $\frac{dy}{dx} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$. At $\pi/2$ we have $\frac{dy}{dx} = \frac{0}{2} = 0$. At $5\pi/6$ we have $\frac{dy}{dx} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$.

12.3.17 $\frac{dy}{dx} = \frac{-8\sin(2\theta)\sin\theta + 4\cos(2\theta)\cos\theta}{-8\sin(2\theta)\cos\theta - 4\cos(2\theta)\sin\theta}$. The tips of the leaves occur at $\theta = 0$, $\pi/2$, π and $3\pi/2$. At 0 and at π, we have that $\frac{dy}{dx}$ doesn't exist. At $\pi/2$ and $3\pi/2$ we have $\frac{dy}{dx} = 0$.

12.3.18 $\frac{dy}{dx} = \frac{0 + 3(\sqrt{2}/2)}{0 - 3(\sqrt{2}/2)} = -1$.

12.3.19 The slopes of the tangent lines are given by $\tan(\pi/4) = 1$ and $\tan(-\pi/4) = -1$.

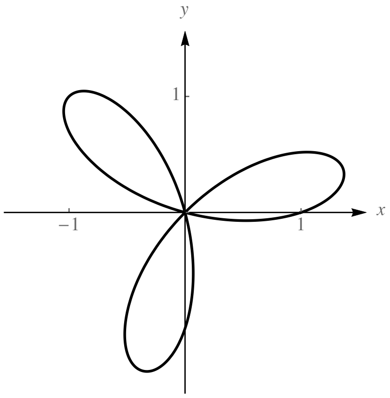
12.3.20 $\dfrac{dy}{dx} = \dfrac{2\sin\theta + 2\theta\cos\theta}{2\cos\theta - 2\theta\sin\theta}$. At $(\pi/2, \pi/4)$ this is $\dfrac{\sqrt{2} + \pi\sqrt{2}/4}{\sqrt{2} - \pi\sqrt{2}/4} \approx 8.32$.

12.3.21 $\cos\theta + \sin\theta = 0$ implies that $\cos\theta = -\sin\theta$, or $\tan\theta = -1$. So $\theta = \dfrac{3\pi}{4}$ or $\theta = \dfrac{7\pi}{4}$. Both of these represent the same line. So the line tangent to the curve at the origin is $\theta = \dfrac{3\pi}{4}$, which has a slope of -1.

12.3.22 The complete graph of this curve is generated for $0 \le \theta \le \pi$, and $4\cos\theta = 0$ implies that $\theta = \dfrac{\pi}{2}$. . So the equation of the line tangent to the curve at the origin is $\theta = \dfrac{\pi}{2}$, and because this is a vertical line, the slope of the tangent line is undefined.

12.3.23

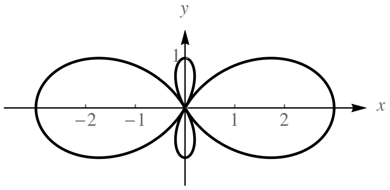
a. The curved is generated for $[0, \pi]$.



b. $\cos 3\theta + \sin 3\theta = 0$ implies that $\tan 3\theta = -1$, which means that $3\theta = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$, and $\dfrac{11\pi}{4}$. So $\theta = \dfrac{\pi}{4}, \dfrac{7\pi}{12}$, and $\dfrac{11\pi}{12}$. The slopes are $\tan\dfrac{\pi}{4} = 1$, $\tan\dfrac{7\pi}{12} \approx -3.73$, and $\tan\dfrac{11\pi}{12} \approx -0.27$.

12.3.24

a. The curved is generated for $[0, 2\pi]$.



b. $1 + 2\cos 2\theta = 0$ for $\cos 2\theta = -\dfrac{1}{2}$, which implies that $2\theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{8\pi}{3}$, and $\dfrac{10\pi}{3}$.
Thus $\theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$, and $\dfrac{5\pi}{3}$. The equations $\theta = \dfrac{\pi}{3}$ and $\theta = \dfrac{4\pi}{3}$ describe the same line, as do the lines $\theta = \dfrac{2\pi}{3}$ and $\theta = \dfrac{5\pi}{3}$. There are therefore two distinct tangent lines, with slopes $m = \tan\dfrac{\pi}{3} = \sqrt{3}$ and with slope $m = \tan\dfrac{2\pi}{3} = -\sqrt{3}$.

12.3.25 Note that the curve is at the origin at $\pi/2$, so there is vertical tangent at $(0, \pi/2)$. Also, $\dfrac{dy}{dx} = \dfrac{-4\sin^2\theta + 4\cos^2\theta}{-8\sin\theta\cos\theta} = \dfrac{1 - 2\sin^2\theta}{\sin(2\theta)}$. Thus, there are horizontal tangents at $\pi/4$ and $3\pi/4$ (at the polar points $(2\sqrt{2}, \pi/4)$ and $(-2\sqrt{2}, 3\pi/4)$). There is also a vertical tangent where $\theta = 0$, at the point $(4, 0)$.

12.3.26 Note that the curve is at the origin at $3\pi/2$, so there is vertical tangent at $(0, 3\pi/2)$. Also, $\dfrac{dy}{dx} = \dfrac{2\cos\theta\sin\theta + (2 + 2\sin\theta)\cos\theta}{2\cos^2\theta - (2 + 2\sin\theta)\sin\theta} = \dfrac{\cos\theta(2 + 4\sin\theta)}{(2 - 4\sin^2\theta) - 2\sin\theta}$. Thus, the are horizontal tangents where this expression is 0 at $\pi/2$ and $7\pi/6$ and $(11\pi/6)$ (at the polar points $(4, \pi/2)$ and $(1, 7\pi/6)$ and $(1, 11\pi/6)$). There are also vertical tangents where the denominator is 0 and the numerator isn't, which occurs at the point $(3, \pi/6)$ and at $(3, 5\pi/6)$.

12.3.27 Using the double angle identities somewhat liberally:

$$\frac{dy}{dx} = \frac{2\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta}{2\cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta} = \frac{\sin\theta(\cos 2\theta + \cos^2\theta)}{\cos\theta(\cos(2\theta) - \sin^2\theta)} = \frac{\sin\theta(3\cos^2\theta - 1)}{\cos\theta(1 - 3\sin^2\theta)} = \frac{\sin\theta(3\cos^2\theta - 1)}{\cos\theta(3\cos^2\theta - 2)}.$$

The numerator is 0 for $\theta = 0$ and for $\theta = \pm\cos^{-1}(\pm\sqrt{3}/3)$, so there are horizontal tangents at the corresponding points $(0, 0)$, $(0.943, 0.955)$, $(-0.943, 2.186)$, $(0.943, 4.097)$, and $(-0.943, 5.328)$. The denominator is 0 for $\theta = \pi/2$ and $3\pi/2$, and for $\theta = \pm\cos^{-1}(\pm\sqrt{6}/3)$, so there are vertical tangents at $(0, 0)$, $(0.943, 0.615)$, $(-0.943, 2.526)$, $(0.943, 3.757)$, and $(-0.943, 5.668)$.

12.3.28 The curve intersects the origin at $\theta = 7\pi/6$ and $\theta = 11\pi/6$, so those don't give rise to vertical or horizontal tangents. We have $\dfrac{dy}{dx} = \dfrac{6\cos\theta\sin\theta + (3 + 6\sin\theta)\cos\theta}{6\cos\theta\cos\theta - (3 + 6\sin\theta)\sin\theta} = \dfrac{\cos\theta(1 + 4\sin\theta)}{(2 - \sin\theta - 4\sin^2\theta)}$. Thus there are horizontal tangents for $\theta = \pi/2$ and $3\pi/2$, at the corresponding points $(9, \pi/2)$ and $(-3, 3\pi/2)$, and at the points where $\sin(\theta) = -1/4$, which are $(3/2, 3.394)$ and $(3/2, 6.031)$. There are vertical tangents where the denominator is 0, which occurs for $\theta = \sin^{-1}\left(-\dfrac{1}{8} \pm \dfrac{\sqrt{33}}{8}\right)$, so the corresponding points are $(-2.06, 5.28)$, and $(6.56, .634)$.

12.3.29 Suppose $2\cos\theta = 1 + \cos\theta$. Then $\cos\theta = 1$, so this occurs for $\theta = 0$ and $\theta = 2\pi$. At those values, $r = 2$, so the curves intersect at the polar point $(2, 0)$. The curves also intersect when $r = 0$, which occurs for $\theta = \pi/2$ and $\theta = 3\pi/2$ for the first curve and $\theta = \pi$ for the second.

12.3.30 Suppose $1 - \sin\theta = 1 + \cos\theta$, or $\tan\theta = -1$. Then $\theta = 3\pi/4$ or $\theta = 7\pi/4$. So the curves intersect at the polar points $(1 + \sqrt{2}/2, 7\pi/4)$ and $(1 - \sqrt{2}/2, 3\pi/4)$. They also intersect at the pole $(0, 0)$, which occurs for the first curve at $\pi/2$ and for the second curve at π.

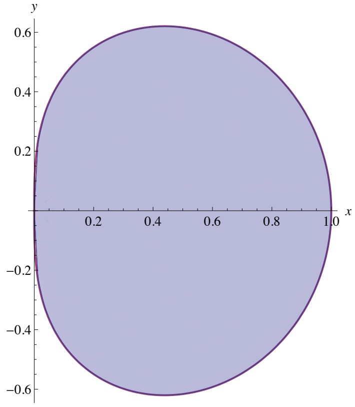
12.3.31 $2\sin 2\theta = 1$ implies that $\sin 2\theta = \dfrac{1}{2}$, so $2\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6}$, and $\dfrac{17\pi}{6}$, and therefore $\theta = \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{13\pi}{12}$, and $\dfrac{17\pi}{12}$. Using symmetry, we also see that they intersect at $\theta = \dfrac{7\pi}{12}, \dfrac{11\pi}{12}, \dfrac{19\pi}{12}$, and $\dfrac{23\pi}{12}$. So the curves intersect at the points

$$\left(1, \frac{\pi}{12}\right), \left(1, \frac{5\pi}{12}\right), \left(1, \frac{7\pi}{12}\right), \left(1, \frac{11\pi}{12}\right), \left(1, \frac{13\pi}{12}\right), \left(1, \frac{17\pi}{12}\right), \left(1, \frac{19\pi}{12}\right), \left(1, \frac{23\pi}{12}\right).$$

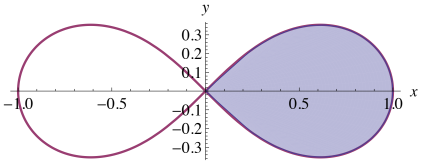
12.3.32 $\cos 2\theta = \sin 2\theta$ implies that $\tan 2\theta = 1$, so $2\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}$, so $\theta = \dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{8}, \dfrac{13\pi}{8}$. Using these and the symmetry of the curve, we find intersection points

$$\left(\frac{\sqrt{2}}{2}, \frac{\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{3\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{5\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{7\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{9\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{11\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{13\pi}{8}\right), \left(\frac{\sqrt{2}}{2}, \frac{15\pi}{8}\right).$$

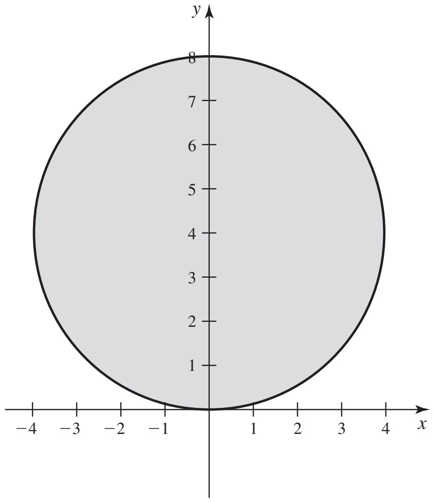
12.3.33 $A = 2 \cdot \frac{1}{2}\int_0^{\pi/2} \cos\theta \, d\theta = \sin\theta \Big|_0^{\pi/2} = 1.$



12.3.34 $A = 2 \cdot \frac{1}{2}\int_0^{\pi/4} \cos 2\theta \, d\theta = \frac{1}{2}\,(\sin 2\theta)\Big|_0^{\pi/4} = \frac{1}{2}.$



12.3.35 $A = \frac{1}{2}\int_0^{\pi} (8\sin\theta)^2 \, d\theta = 32\int_0^{\pi} \sin^2\theta \, d\theta = 32\int_0^{\pi} \frac{1-\cos 2\theta}{2} \, d\theta = 32\left(\frac{1}{2}\theta - \frac{\sin\theta\cos\theta}{2}\right)\Big|_0^{\pi} = 16\pi.$

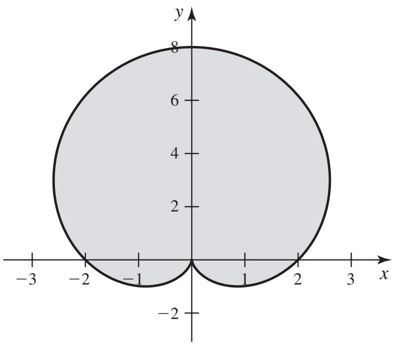


12.3.36

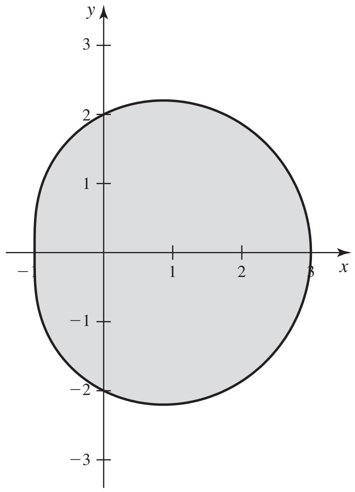
$$A = \frac{1}{2}\int_0^{2\pi} (4 + 4\sin\theta)^2 \, d\theta$$

$$= 8\int_0^{2\pi} (1 + 2\sin\theta + \sin^2\theta) \, d\theta$$

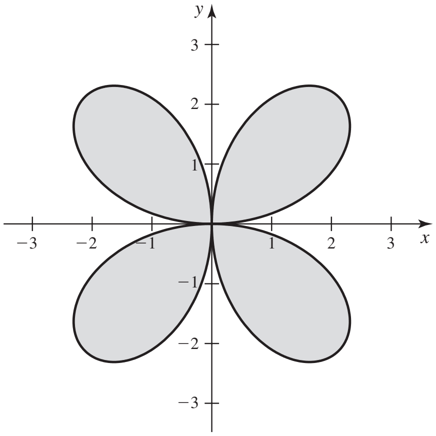
$$= 8\left(\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{\sin\theta\cos\theta}{2}\right)\Big|_0^{2\pi} = 24\pi.$$



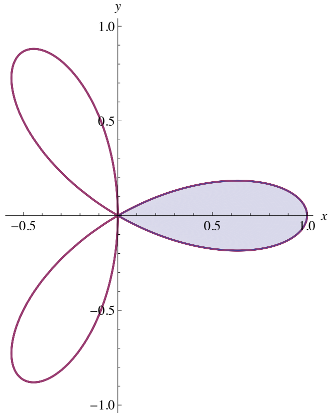
12.3.37 Using symmetry, we have $\frac{1}{2} \cdot 2\int_0^{\pi} (2 + \cos\theta)^2 \, d\theta = \int_0^{\pi} (4 + 4\cos\theta + \cos^2\theta) \, d\theta = \left(4\theta + 4\sin\theta + \frac{1}{2}\theta + \frac{\sin\theta\cos\theta}{2}\right)\Big|_0^{\pi} = \frac{9\pi}{2}$.



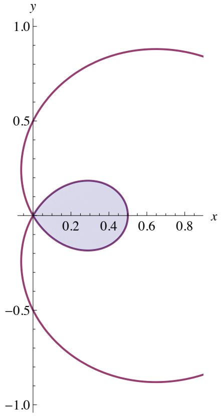
12.3.38 Because there are 4 symmetric leaves, we compute the area of 1/2 of one of the leaves, and then multiply by 8 to get the total area. We have $\frac{1}{2}\int_0^{\pi/4} 9\sin^2(2\theta) \, d\theta = \frac{9}{2}\int_0^{\pi/4} \sin^2(2\theta) \, d\theta = \frac{9}{4}\left(\theta - \frac{\sin 2\theta \cos 2\theta}{2}\right)\Big|_0^{\pi/4} = \frac{9\pi}{16}$. So the total area is $8 \cdot \frac{9\pi}{16} = \frac{9\pi}{2}$.



12.3.39 $2 \cdot \frac{1}{2}\int_0^{\pi/6} \cos^2 3\theta \, d\theta = \int_0^{\pi/6} \frac{1 + \cos 6\theta}{2} \, d\theta = \left(\theta/2 + \frac{\sin 6\theta}{6}\right)\Big|_0^{\pi/6} = \frac{\pi}{12}$.

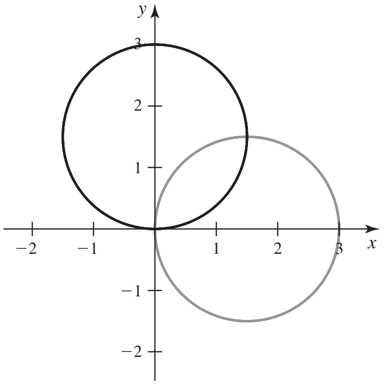


12.3.40 $2 \cdot \frac{1}{2}\int_0^{\pi/3} (\cos\theta - 1/2)^2 \, d\theta = \int_0^{\pi/3} (\cos^2\theta - \cos\theta + 1/4)\, d\theta = \int_0^{\pi/3} (\cos(2\theta)/2 - \cos\theta + 3/4)\, d\theta = (\sin(2\theta)/4 - \sin\theta + 3\theta/4)\Big|_0^{\pi/3} = \sqrt{3}/8 - \sqrt{3}/2 + \pi/4 = \pi/4 - \frac{3\sqrt{3}}{8}$.



12.3.41

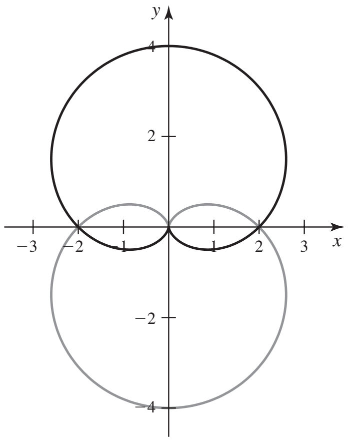
a. These curves intersect when $\sin\theta = \cos\theta$, which occurs at $\theta = \pi/4$ and $\theta = 5\pi/4$, and when $r = 0$ which occurs for $\theta = 0$ and $\theta = \pi$ for the first curve and $\theta = \pi/2$ and $\theta = 3\pi/2$ for the second curve. Only two of these intersection points are unique: the origin and the point $(3\sqrt{2}/2, \pi/4) = (-3\sqrt{2}/2, 5\pi/4)$.



b. By symmetry, we need to compute the area inside $r = 3\sin\theta$ between 0 and $\pi/4$ and then double that result. We have $2 \cdot \frac{1}{2}\int_0^{\pi/4} 9\sin^2\theta \, d\theta = \frac{9}{2}\int_0^{\pi/4} (1 - \cos(2\theta))\, d\theta = \frac{9}{2}\left(\theta - (1/2)\sin(2\theta)\right)\Big|_0^{\pi/4} = \frac{9}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{9}{8}(\pi - 2)$.

12.3.42

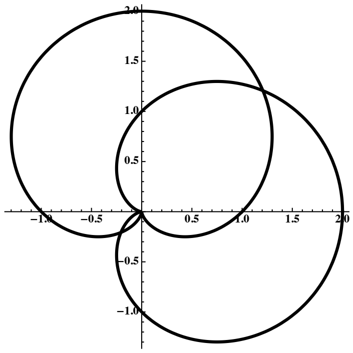
a. The curves intersect where $2 + 2\sin\theta = 2 - 2\sin\theta$, which occurs when $\sin\theta = 0$. The curves also intersect at the origin, which occurs for the first curve at $\theta = 3\pi/2$ and for the second curve at $\pi/2$. The only points of intersection are the origin, $(2, 0)$ and $(2, \pi)$.



b. By symmetry, we need to compute the area of the region inside $r = 2 - 2\sin\theta$ between 0 and $\pi/2$ and then quadruple it. We have $4 \cdot \dfrac{1}{2}\displaystyle\int_0^{\pi/2} (2-2\sin\theta)^2\,d\theta = 8\int_0^{\pi/2}(1-2\sin\theta+\sin^2\theta)\,d\theta = 8\int_0^{\pi/2}((3/2) - 2\sin\theta - (1/2)\cos 2\theta)\,d\theta = (12\theta + 16\cos\theta - 2\sin(2\theta))\Big|_0^{\pi/2} = 6\pi + 0 - 0 - (0 + 16 - 0) = 6\pi - 16.$

12.3.43

a. The curves intersect when $\sin\theta = \cos\theta$, which occurs for $\theta = \pi/4$ and $\theta = 5\pi/4$. The corresponding points are $\left(\frac{2+\sqrt{2}}{2}, \frac{\pi}{4}\right)$ and $\left(\frac{2-\sqrt{2}}{2}, \frac{5\pi}{4}\right)$. They also intersect at the pole: the first curve is at the pole at $(0, \pi)$ and the other at $(0, 3\pi/2)$.



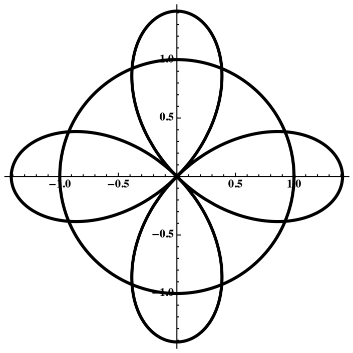
b. By symmetry, we can compute the area between $\pi/4$ and $5\pi/4$ inside $r = 1 + \cos\theta$ and then double it. This will include both the bigger and smaller enclosed regions. We have $2 \cdot \dfrac{1}{2}\displaystyle\int_{\pi/4}^{5\pi/4}(1 + \cos\theta)^2\,d\theta = \int_{\pi/4}^{5\pi/4}(1 + 2\cos\theta + (1/2)(1+\cos(2\theta)))\,d\theta = \int_{\pi/4}^{5\pi/4}((3/2) + 2\cos\theta + (1/2)(\cos 2\theta))\,d\theta =$

$(3\theta/2 + 2\sin\theta + (1/4)\sin 2\theta)\Big|_{\pi/4}^{5\pi/4}$

$= \left(\dfrac{15\pi}{8} - \sqrt{2} + \dfrac{1}{4}\right) - \left(\dfrac{3\pi}{8} + \sqrt{2} + \dfrac{1}{4}\right) = \dfrac{3\pi}{2} - 2\sqrt{2}.$

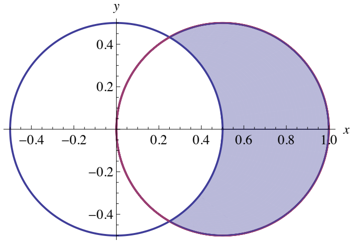
12.3.44

a. These curves intersect when $\cos(2\theta) = \dfrac{\sqrt{2}}{2}$, which occurs when $2\theta = \pi/4, 7\pi/4, 9\pi/4, 15\pi/4$, so for $\theta = \pi/8, 7\pi/8, 9\pi/8, 15\pi/8$. The intersection points are thus $(1, \pi/8)$, $(1, 3\pi/8)$, $(1, 5\pi/8)$, $(1, 7\pi/8)$, $(1, 9\pi/8)$, $(1, 11\pi/8)$, $(1, 13\pi/8)$, and $(1, 15\pi/8)$.

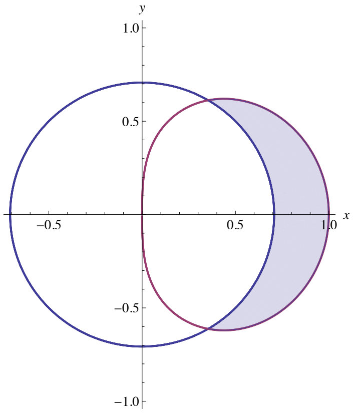


b. By symmetry, we can compute the area between 0 and $\pi/8$ within the circle $r = 1$ and add it to the area between $\pi/8$ and $\pi/4$ within the curve $\sqrt{2}\cos 2\theta$ and then multiply this by 8. The area within the circle between 0 and $\pi/8$ is (1/16)th the area of the circle, so this area is $\frac{\pi}{16}$. The area between $\pi/8$ and $\pi/4$ within $\sqrt{2}\cos 2\theta$ is given by $\dfrac{1}{2}\displaystyle\int_{\pi/8}^{\pi/4} 2\cos^2 2\theta\,d\theta = \dfrac{1}{2}\int_{\pi/8}^{\pi/4}(1 + \cos 4\theta)\,d\theta = \dfrac{1}{2}\,(\theta + (1/4)\sin 4\theta)\Big|_{\pi/8}^{\pi/4} = \dfrac{1}{2}\left(\dfrac{\pi}{4} + 0 - \left(\dfrac{\pi}{8} + \dfrac{1}{4}\right)\right) = \dfrac{\pi}{16} - \dfrac{1}{8}$. Adding this to the previously computed area gives that 1/8 of the total area is $\frac{\pi}{16} - \frac{1}{8} + \frac{\pi}{16} = \frac{\pi}{8} - \frac{1}{8}$. Thus the total area we are seeking is $\pi - 1$.

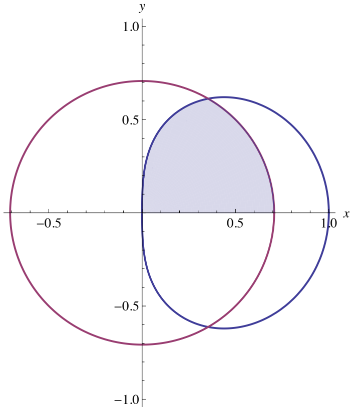
12.3.45 The area is given by $2 \cdot \frac{1}{2}\int_0^{\pi/3} (\cos^2\theta - (1/2)^2)\,d\theta = \int_0^{\pi/3} (\cos(2\theta)/2 + \frac{1}{4})\,d\theta = (\sin(2\theta)/4 + \theta/4)\Big|_0^{\pi/3} = \frac{\sqrt{3}}{8} + \frac{\pi}{12}$.



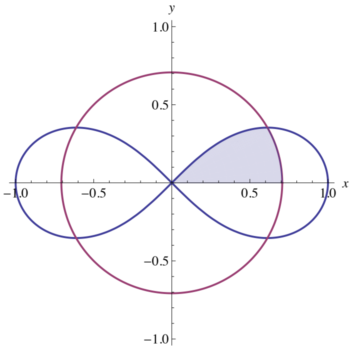
12.3.46 We have already computed the area inside $\sqrt{\cos\theta}$ to be 1. Now we must take away the portion of the circle with radius $1/\sqrt{2}$ between $\theta = -\pi/3$ and $\theta = \pi/3$. This is 1/3 of a circle, so the area being removed is $(1/3)\pi(1/2) = \pi/6$. We must also remove the area of the regions inside $\sqrt{\cos\theta}$ between $-\pi/2$ and $-\pi/3$ and $\pi/3$ and $\pi/2$. These have area $2 \cdot \frac{1}{2}\int_{\pi/3}^{\pi/2} \cos\theta\,d\theta = (\sin\theta)\Big|_{\pi/3}^{\pi/2} = 1 - \sqrt{3}/2$. So the area is $1 - (\pi/6 + (1 - \sqrt{3}/2)) = \sqrt{3}/2 - \pi/6$.



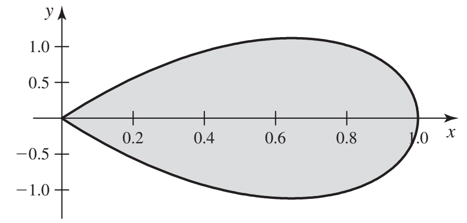
12.3.47 The region inside the circle between 0 and $\pi/3$ is 1/6 the area of a circle of radius $1/\sqrt{2}$ so it has area $(1/6)\pi(1/2) = \pi/12$. The rest of the area is represented by $\frac{1}{2}\int_{\pi/3}^{\pi} \cos\theta\,d\theta = \frac{1}{2}(\sin\theta)\Big|_{\pi/3}^{\pi} = \frac{1}{2}\left(1 - \sqrt{3}/2\right)$. The total area is therefore $\frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4}$.



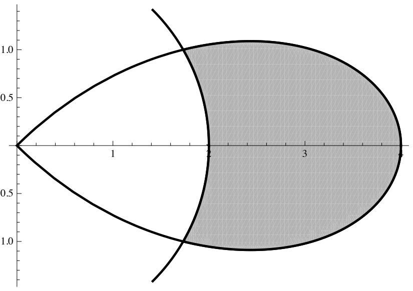
12.3.48 The region inside the circle between 0 and $\pi/6$ is 1/12 the area of a circle of radius $1/\sqrt{2}$ so it has area $(1/12)\pi(1/2) = \pi/24$. The rest of the area is represented by $\frac{1}{2}\int_{\pi/6}^{\pi/4} \cos 2\theta\,d\theta = \frac{1}{4}(\sin 2\theta)\Big|_{\pi/6}^{\pi/4} = \frac{1}{4}\left(1 - \sqrt{3}/2\right)$. The total area is thus $\frac{\pi}{24} + \frac{1}{4} - \frac{\sqrt{3}}{8}$.



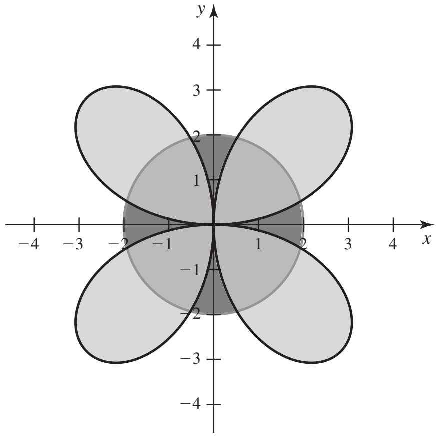
12.3.49 Using symmetry, we compute the area of 1/2 of one leaf, and then double it. We have $A = \frac{1}{2}\int_0^{\pi/10} \cos^2(5\theta)\, d\theta = \frac{1}{10}\int_0^{\pi/2} \cos^2 u\, du = \frac{1}{10}\left(\frac{1}{2}u + \frac{\cos u \sin u}{2}\right)\Big|_0^{\pi/2} = \frac{\pi}{40}$. So the area of one leaf is $2 \cdot \frac{\pi}{40} = \frac{\pi}{20}$.



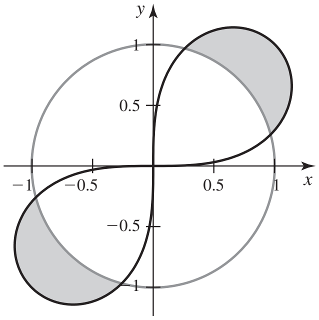
12.3.50 The curves intersect where $4\cos 2\theta = 2$, or $\theta = \pi/6$. By symmetry, we can compute the area of 1/2 of the tip of one leaf, and then multiply by 8. The area of 1/2 of the tip of one leaf is given by $\frac{1}{2}\int_0^{\pi/6} (4\cos(2\theta)^2 - 4)\, d\theta = \int_0^{\pi/6} (8\cos^2(2\theta) - 2)\, d\theta = (4\theta + \sin(4\theta) - 2\theta)\Big|_0^{\pi/6} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$. Thus the total area desired is $8\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) = \frac{8\pi}{3} + 4\sqrt{3}$.



12.3.51 Note that the area inside one leaf of the rose but outside the circle is given by $\frac{1}{2}\int_{\pi/12}^{5\pi/12} (16\sin^2(2\theta) - 4)\, d\theta = (2\theta - \sin(4\theta))\Big|_{\pi/12}^{5\pi/2} = \sqrt{3} + \frac{2\pi}{3}$. Also, the area inside one leaf of the rose is $\frac{1}{2}\int_0^{\pi/2} 16\sin^2(2\theta)\, d\theta = (4\theta - \sin(4\theta))\Big|_0^{\pi/2} = 2\pi$. Thus the area inside one leaf of the rose and inside the circle must be $2\pi - (\sqrt{3} + \frac{2\pi}{3}) = \frac{4\pi}{3} - \sqrt{3}$, and the total area inside the rose and inside the circle must be $4(\frac{4\pi}{3} - \sqrt{3}) = \frac{16\pi}{3} - 4\sqrt{3}$.



12.3.52 The curves intersect for $2\sin(2\theta) = 1$, which occurs in the first quadrant at $\theta = \pi/12$ and $\theta = 5\pi/12$. So one half of the total desired area is given by $\frac{1}{2}\int_{\pi/12}^{5\pi/12} (2\sin(2\theta) - 1)\, d\theta = \frac{1}{2}\left(-\cos(2\theta) - \theta\right)\Big|_{\pi/12}^{5\pi/12} = -\frac{1}{2}\left(-\sqrt{3} + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$. So the total desired area is $\sqrt{3} - \frac{\pi}{3}$.



12.3.53 The circles intersect for $\theta = \pi/6$ and $\theta = 5\pi/6$.

The area inside $r = 2\sin\theta$ but outside of $r = 1$ would be given by $\frac{1}{2}\int_{\pi/6}^{5\pi/6} (4\sin^2\theta - 1)\, d\theta = \frac{1}{2}\left(x - \sin(2x)\right)\Big|_{\pi/6}^{5\pi/6} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$. The total area of $r = 2\sin\theta$ is π. Thus, the area inside both circles is $\pi - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.

12.3.54 The inner loop is traced from $\theta = 2\pi/3$ to $\theta = 4\pi/3$. So the area is given by $\frac{1}{2}\int_{2\pi/3}^{4\pi/3} (2 + 4\cos\theta)^2\, d\theta = \int_{2\pi/3}^{4\pi/3} (2 + 8\cos\theta + 8\cos^2\theta)\, d\theta = \left(2\theta + 8\sin\theta + 4\theta + 2\sin(2\theta)\right)\Big|_{2\pi/3}^{4\pi/3} = 4\pi - 6\sqrt{3}$.

12.3.55 The inner loop is traced out between $\theta = \pi/6$ and $\theta = 5\pi/6$, so its area is given by $\frac{1}{2}\int_{\pi/6}^{5\pi/6} (3 - 6\sin\theta)^2\, d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (9 - 36\sin\theta + 36\sin^2\theta)\, d\theta = \frac{3}{2}\left(3\theta + 12\cos\theta + 6\theta - 3\sin(2\theta)\right)\Big|_{\pi/6}^{5\pi/6} = 9\pi - \frac{27\sqrt{3}}{2}$.

We can determine the area inside the outer loop by using symmetry and doubling the area of the region traced out between $5\pi/6$ and $3\pi/2$. Thus the area inside the outer region is $2 \cdot \frac{1}{2}\int_{5\pi/6}^{3\pi/2} (3 - 6\sin\theta)^2\, d\theta = 3\left(3\theta + 12\cos\theta + 6\theta - 3\sin(2\theta)\right)\Big|_{5\pi/6}^{3\pi/2} = 18\pi + \frac{27\sqrt{3}}{2}$. So the area outside the inner loop and inside the outer loop is $18\pi + \frac{27\sqrt{3}}{2} - \left(9\pi - \frac{27\sqrt{3}}{2}\right) = 9\pi + 27\sqrt{3}$.

12.3.56 The curves intersect at $\theta = \pi/3$, and using symmetry, the area we seek is $2 \cdot \frac{1}{2}\int_0^{\pi/3} (1 + \cos\theta)^2\, d\theta + 2 \cdot \frac{1}{2}\int_{\pi/3}^{\pi/2} (3\cos\theta)^2\, d\theta = \int_0^{\pi/3} (1 + 2\cos\theta + \cos^2\theta)\, d\theta + \int_{\pi/3}^{\pi/2} 9\cos^2\theta\, d\theta = \left(\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{\pi/3} + \left(\frac{9\theta}{2} + \frac{9\sin 2\theta}{4}\right)\Big|_{\pi/3}^{\pi/2} = \frac{\pi}{2} + 2 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8} + \frac{9\pi}{4} - \left(\frac{3\pi}{2} + \frac{9\sqrt{3}}{8}\right) = \frac{5\pi}{4}$.

12.3.57 One half of the area is given by $\frac{1}{2}\int_0^{\pi/2} 6\sin 2\theta\, d\theta = -\frac{3}{2}\cos 2\theta\Big|_0^{\pi/2} = 3$, so the total area is 6.

12.3.58 By symmetry, we can compute the area between $\theta = 5\pi/6$ and $\theta = 3\pi/2$ and double it. Thus, the total area we seek is given by $\int_{5\pi/6}^{3\pi/2} (2-4\sin\theta)^2\,d\theta = \int_{5\pi/6}^{3\pi/2} (4-16\sin\theta+16\sin^2\theta)\,d\theta =$ $(4\theta+16\cos\theta+8\theta-4\sin(2\theta))\Big|_{5\pi/6}^{3\pi/2} = 6\sqrt{3}+8\pi.$

12.3.59 The area is given by

$$\frac{1}{2}\int_0^{2\pi} (4-2\cos\theta)^2\,d\theta = \int_0^{2\pi} (8-8\cos\theta+2\cos^2\theta)\,d\theta = \left(8\theta-8\sin\theta+\theta+\frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} = 18\pi.$$

12.3.60 The area of one half of one leaf is $\frac{1}{2}\int_0^{\pi/6} 4\cdot\cos^2(3\theta)\,d\theta = \left(\theta+\frac{\sin(6\theta)}{6}\right)\Big|_0^{\pi/6} = \frac{\pi}{6}$. So the area of all 6 half-leaves is π.

12.3.61 Solving $4-2\cos\theta = 2+2\cos\theta$, we have $\cos\theta = \frac{1}{2}$, so $\theta = \pm\frac{\pi}{3}$.

The area of region A is

$$\int_0^{\pi/3} \frac{1}{2}((2+2\cos\theta)^2-(4-2\cos\theta)^2)\,d\theta = \frac{1}{2}\int_0^{\pi/3} (24\cos t-12)\,dt = \frac{1}{2}\,(24\sin t-12t)\Big|_0^{\pi/3} = 6\sqrt{3}-2\pi.$$

The area of region B is

$$\begin{aligned}
&\int_0^{\pi/3} \frac{1}{2}(4-2\cos\theta)^2\,d\theta + \int_{\pi/3}^{\pi} \frac{1}{2}(2+2\cos\theta)^2\,d\theta\\
&= \int_0^{\pi/3} (8-8\cos\theta+2\cos^2\theta)\,d\theta + \int_{\pi/3}^{\pi} (2+4\cos\theta+2\cos^2\theta)\,d\theta\\
&= \int_0^{\pi/3} (9-8\cos\theta+\cos 2\theta)\,d\theta + \int_{\pi/3}^{\pi} (3+4\cos\theta+\cos 2\theta)\,d\theta\\
&= \left(9\theta-8\sin\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/3} + \left(3\theta+4\sin\theta+\frac{1}{2}\sin 2\theta\right)\Big|_{\pi/3}^{\pi}\\
&= 3\pi-\frac{15\sqrt{3}}{4}+2\pi-\frac{9\sqrt{3}}{4}\\
&= 5\pi-6\sqrt{3}.
\end{aligned}$$

The area of region C is

$$\int_{\pi/3}^{\pi} \frac{1}{2}\left((4-2\cos\theta)^2-(2+2\cos\theta)^2\right)\,d\theta = \int_{\pi/3}^{\pi} (6-12\cos\theta)\,d\theta = (6\theta-12\sin\theta)\Big|_{\pi/3}^{\pi} = 4\pi+6\sqrt{3}.$$

12.3.62 Solving $4+\cos 2\theta = 3\cos 2\theta$ implies that $\cos 2\theta = 2$, which has no solutions. However, observe that if $\theta = \frac{\pi}{2}$, then $r = 4+\cos 2\theta = 3$, and if $\theta = \frac{\pi}{2}$, $r = 3\cos 2\theta = -3$. So the curves intersect at the points $\left(3, \pm\frac{\pi}{2}\right)$. Because the intersection points of the two curves correspond to different values of θ, we have to be careful when setting up the area integral. From the figure, we can see that the shaded area equals the area in the first quadrant that is inside the curve $r = 4+\cos 2\theta$ minus the area of one leaf of $r = 3\cos 2\theta$.

This equals

$$\begin{aligned}
&\int_0^{\pi/2} \frac{1}{2}(4+\cos 2\theta)^2\,d\theta - 2\int_0^{\pi/4} \frac{1}{2}(3\cos 2\theta)^2\,d\theta \\
&= .\int_0^{\pi/2} \left(8 + 4\cos 2\theta + \frac{1}{4} + \frac{\cos 4\theta}{4}\right)\,d\theta - 9\int_0^{\pi/4} \frac{1+\cos 4\theta}{2}\,d\theta \\
&= \left(\frac{33\theta}{4} + 2\sin 2\theta + \frac{\sin 4\theta}{16}\right)\Bigg|_0^{\pi/2} - 9\left(\frac{\theta}{2} + \frac{\sin 4\theta}{8}\right)\Bigg|_0^{\pi/4} \\
&= \frac{33\pi}{8} - \frac{9\pi}{8} = 3\pi.
\end{aligned}$$

12.3.63 Note that the diameter of the circle is a, and that the complete circle is traversed for $0 \le t \le \pi$. $L = \displaystyle\int_0^{\pi} \sqrt{(a\sin\theta)^2 + (a\cos\theta)^2}\,d\theta = \int_0^{\pi} a\,d\theta = \pi a.$

12.3.64 $L = \displaystyle\int_0^{2\pi} \sqrt{(2-2\sin\theta)^2 + 4\cos^2\theta}\,d\theta = \int_0^{2\pi} \sqrt{8 - 8\sin\theta}\,d\theta$. By symmetry, this is

$$2\sqrt{8}\int_{-\pi/2}^{\pi/2} \sqrt{1-\sin\theta}\cdot\frac{\sqrt{1+\sin\theta}}{\sqrt{1+\sin\theta}}\,d\theta = 2\sqrt{8}\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{1+\sin\theta}}\,d\theta.$$

Let $u = 1 + \sin\theta$ so that $du = \cos\theta\,d\theta$. Then we have $L = 2\sqrt{8}\displaystyle\int_0^2 u^{-1/2}\,du = 4\sqrt{8}\cdot\sqrt{u}\Big|_0^2 = 4\sqrt{8}\cdot\sqrt{2} = 16.$

12.3.65 $L = \displaystyle\int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\,d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2+4}\,d\theta$. Let $u = \theta^2 + 4$, so that $du = 2\theta\,d\theta$. Substituting gives $\displaystyle\frac{1}{2}\int_4^{4\pi^2+4} \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}\left(u^{3/2}\right)\Big|_4^{4\pi^2+4} = \frac{1}{3}\left(8(\pi^2+1)^{3/2} - 8\right) = \frac{8}{3}\left((\pi^2+1)^{3/2} - 1\right).$

12.3.66 $L = \displaystyle\int_0^{2\pi n} \sqrt{e^{2\theta} + e^{2\theta}}\,d\theta = \sqrt{2}\int_0^{2\pi n} e^{\theta}\,d\theta = \sqrt{2}\cdot e^{\theta}\Big|_0^{2\pi n} = \sqrt{2}(e^{2\pi n} - 1).$

12.3.67

$$\begin{aligned}
L &= 2\int_{-\pi/2}^{\pi/2} \sqrt{(4+4\sin\theta)^2 + (4\cos\theta)^2}\,d\theta = 8\int_{-\pi/2}^{\pi/2} \sqrt{2+2\sin\theta}\,d\theta = 8\sqrt{2}\int_{-\pi/2}^{\pi/2} \sqrt{1+\sin\theta}\cdot\frac{\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\,d\theta \\
&= 8\sqrt{2}\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin\theta}}\,d\theta.
\end{aligned}$$

Let $u = 1 - \sin\theta$, so that $du = -\cos\theta\,d\theta$. Then we have

$$L = 8\sqrt{2}\int_0^2 u^{-1/2}\,du = 16\sqrt{2}\left(\sqrt{2} - 0\right) = 32.$$

12.3.68 $L = \displaystyle\int_0^6 \sqrt{(4\theta^2)^2 + 64\theta^2}\,d\theta = \int_0^6 4\theta\sqrt{\theta^2+4}\,d\theta$. Let $u = \theta^2 + 4$ so that $du = 2\theta\,d\theta$. Then $L = \displaystyle 2\int_4^{40} u^{1/2}\,du = \frac{4}{3}\cdot u^{3/2}\Big|_4^{40} = \frac{4}{3}(40^{3/2} - 8) = \frac{32}{3}(10\sqrt{10} - 1).$

12.3.69 $L = \displaystyle\int_0^{\ln 8} \sqrt{4e^{4\theta} + 16e^{4\theta}}\,d\theta = 2\sqrt{5}\int_0^{\ln 8} e^{2\theta}\,d\theta = \sqrt{5}\cdot e^{2\theta}\Big|_0^{\ln 8} = \sqrt{5}(64-1) = 63\sqrt{5}.$

12.3.70

$$L=\int_0^{\pi/2}\sqrt{\frac{2}{(1+\cos\theta)^2}+\left(\frac{\sqrt{2}\sin\theta}{(1+\cos\theta)^2}\right)^2}\,d\theta=\int_0^{\pi/2}\sqrt{\frac{2(1+\cos\theta)^2+2\sin^2\theta}{(1+\cos\theta)^4}}\,d\theta$$
$$=\int_0^{\pi/2}\sqrt{\frac{4+4\cos\theta}{(1+\cos\theta)^4}}\,d\theta=2\int_0^{\pi/2}(1+\cos\theta)^{-3/2}\,d\theta.$$

Now note that $(\sqrt{1+\cos\theta})^{-3}=(\sqrt{2}\cos(\theta/2))^{-3}$, so $L=\frac{2}{2\sqrt{2}}\int_0^{\pi/2}\sec^3(\theta/2)\,d\theta=\frac{2}{\sqrt{2}}\int_0^{\pi/4}\sec^3 u\,du=\frac{1}{\sqrt{2}}\left(\sec u\tan u+\ln|\sec u+\tan u|\right)\Big|_0^{\pi/4}=\frac{\sqrt{2}}{2}\left(\sqrt{2}+\ln(\sqrt{2}+1)\right)=1+\frac{\sqrt{2}}{2}\ln(\sqrt{2}+1)$.

12.3.71 $L=\int_0^{\pi/2}\sqrt{\sin^6(\theta/3)+\sin^4(\theta/3)\cos^2(\theta/3)}\,d\theta=\int_0^{\pi/2}\sin^2(\theta/3)\,d\theta=\frac{1}{2}\int_0^{\pi/2}1-\cos(2\theta/3)\,d\theta=\frac{1}{2}\left(\theta-\frac{3}{2}\sin(2\theta/3)\right)\Big|_0^{\pi/2}=\frac{1}{2}\left(\frac{\pi}{2}-\frac{3\sqrt{3}}{4}\right)=\frac{2\pi-3\sqrt{3}}{8}$.

12.3.72 $\int_0^{\pi}\sqrt{4\cos^2(3\theta)+36\sin^2(3\theta)}\,d\theta=\int_0^{\pi}2\sqrt{1+8\sin^2\theta}\,d\theta\approx 13.36$.

12.3.73 $L=\int_0^{2\pi}\sqrt{(4-2\cos\theta)^2+4\sin^2\theta}\,d\theta=\int_0^{2\pi}\sqrt{20-16\cos\theta}\,d\theta=26.73$.

12.3.74 Using symmetry, we are seeking 4 times the curve traversed for $0\le\theta\le\pi/4$. Thus, $L=4\int_0^{\pi/4}\sqrt{6\sin 2\theta+6\cot 2\theta\cos 2\theta}\,d\theta=12.85$.

12.3.75

a. False. The area is given by $\frac{1}{2}\int_\alpha^\beta f(\theta)^2\,d\theta$.

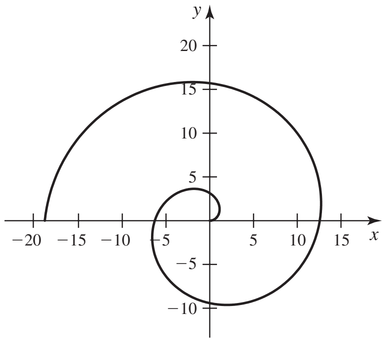
b. False. The slope is given by $\frac{dy}{dx}$, which can be computed using the formula

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{f'(\theta)\sin\theta+f(\theta)\cos\theta}{f'(\theta)\cos\theta-f(\theta)\sin\theta}.$$

c. True. For example, see number 23 above.

12.3.76 An equation of the unit circle in polar coordinates is $r=1$ and the equation the vertical line in polar coordinates is $r\cos\theta=\frac{\sqrt{2}}{2}$, or $r=\frac{\sqrt{2}}{2}\sec\theta$. So the area is $2\int_0^{\pi/4}\frac{1}{2}\left(1-\frac{1}{2}\sec^2\theta\right)\,d\theta=\left(\theta-\frac{1}{2}\tan\theta\right)\Big|_0^{\pi/4}=\frac{\pi}{4}-\frac{1}{2}$.

12.3.77 The first horizontal tangent line is at the origin. The next is at approximately $(4.0576, 2.0288)$, and the third at approximately $(9.8262, 4.9131)$. The first vertical tangent line is at approximately $(1.7206, 0.8603)$, the next is at about $(6.8512, 3.4256)$, and the next at approximately $(12.8746, 6.4373)$.



12.3.78

a. $L = \int_0^{\sqrt{8}} \sqrt{16\theta^2 + 16}\, d\theta = 4\int_0^{\sqrt{8}} \sqrt{\theta^2+1}\, d\theta \approx 20.5.$

b. $L(\theta) = 4\int_0^{\theta} \sqrt{\theta^2+1}\, d\theta = 2(\theta\sqrt{1+\theta^2} + \ln(\theta + \sqrt{\theta^2+1})).$

c. By the Fundamental Theorem of Calculus, $L'(\theta) = 4\sqrt{\theta^2+1} > 0$. $L''(\theta) = \dfrac{4\theta}{\sqrt{1+\theta^2}} \geq 0$ if $\theta \geq 0$. The arc length is increasing at an increasing rate.

12.3.79 $r'(\theta) = -ae^{-a\theta}$. Thus,

$$\begin{aligned} L = \int_0^{\infty} \sqrt{e^{-2a\theta} + a^2e^{-2a\theta}}\, d\theta &= \sqrt{1+a^2}\int_0^{\infty} e^{-a\theta}\, d\theta \\ &= \lim_{b\to\infty} \sqrt{1+a^2}\int_0^{b} e^{-a\theta}\, d\theta = \frac{\sqrt{1+a^2}}{-a} \lim_{b\to\infty} e^{-a\theta}\Big|_0^b \\ &= \frac{\sqrt{1+a^2}}{-a}(0-1) = \frac{\sqrt{1+a^2}}{a}. \end{aligned}$$

12.3.80

a. The area of one half of one leaf is $\frac{1}{2}\int_0^{\pi/(4m)} \cos^2(2m\theta)\, d\theta = \left(\frac{\theta}{4} + \frac{\sin(4m\theta)}{16m}\right)\Big|_0^{\pi/(4m)} = \frac{\pi}{16m}$. So the area of all $8m$ half-leaves is $\frac{\pi}{2}$.

b. The area of one half of one leaf is

$$\frac{1}{2}\int_0^{\pi/(4m+2)} \cos^2((2m+1)\theta)\, d\theta = \left(\frac{\theta}{4} + \frac{\sin(2(2m+1)\theta)}{4(4m+2)}\right)\Bigg|_0^{\pi/(4m+2)} = \frac{\pi}{8\cdot(2m+1)}.$$

So the area of all $2(2m+1)$ half-leaves is $\frac{\pi}{4}$.

12.3.81

a. $A_n = \frac{1}{2}\int_{(2n-2)\pi}^{(2n-1)\pi} e^{-2\theta}\, d\theta - \frac{1}{2}\int_{2n\pi}^{(2n+1)\pi} e^{-2\theta}\, d\theta = -\frac{1}{4}e^{-(4n-2)\pi} + \frac{1}{4}e^{-(4n-4)\pi} + \frac{1}{4}e^{-(4n+2)\pi} - \frac{1}{4}e^{-4n\pi}.$

b. Each term tends to 0 as $n \to \infty$ so $\lim_{n\to\infty} A_n = 0$.

c. $\dfrac{A_{n+1}}{A_n} = \dfrac{e^{-(4n+2)\pi} + e^{-(4n)\pi} + e^{-(4n+6)\pi} - e^{-(4n+4)\pi}}{e^{-(4n-2)\pi} + e^{-(4n-4)\pi} + e^{-(4n+2)\pi} - e^{-4n\pi}} = e^{-4\pi}$, so $\lim_{n\to\infty} \dfrac{A_{n+1}}{A_n} = e^{-4\pi}$.

12.3.82

a. The slope of the line tangent to $r = f(\theta)$ at P is $\left.\dfrac{dy}{dx}\right|_P$. Also, the slope of a line intersecting the x-axis at an angle α is $\tan\alpha$. (Note that in the picture, $\tan(\pi - \alpha) = -\tan(\alpha) = \dfrac{\text{rise}}{-\text{run}} =$ $-$slope of the tangent line.)

b. Draw a vertical line through P and let Q be the point where this line intersects the x-axis. Then in triangle OPQ we see $\tan\theta = \dfrac{y}{x}$.

c. Note that

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\tan\theta + \dfrac{f(\theta)}{f'(\theta)}}{1 - \dfrac{f(\theta)}{f'(\theta)}\tan\theta} = \tan\alpha.$$

Because $\alpha = \varphi + \theta$ and $\tan(\varphi + \theta) = \dfrac{\tan\theta + \tan\varphi}{1 - \tan\varphi\tan\theta}$, we see that $\tan\varphi = \dfrac{f(\theta)}{f'(\theta)}$.

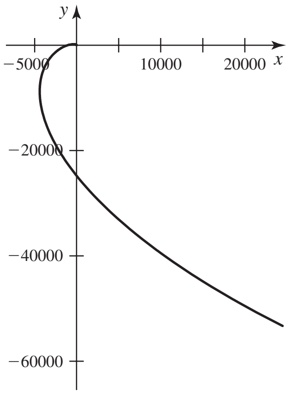
d. l is parallel to the x-axis when $\dfrac{dy}{dx} = 0$, or when $f'(\theta)\sin\theta + f(\theta)\cos\theta = 0$, hence if $\tan\theta = -\dfrac{f(\theta)}{f'(\theta)}$.

e. l is parallel to the y-axis when $\dfrac{dx}{dy} = 0$, which occurs when $f'(\theta)\cos\theta - f(\theta)\sin\theta = 0$, hence if $\tan\theta = \dfrac{f(\theta)}{f'(\theta)}$.

12.3.83

a. If $\cot\varphi = \dfrac{f'(\theta)}{f(\theta)}$ is constant for all θ, then $\varphi = \cot^{-1}\left(\dfrac{f'(\theta)}{f(\theta)}\right)$ is constant. Then $\dfrac{d}{d\theta}\ln(f(\theta)) = \dfrac{1}{f(\theta)} \cdot f'(\theta) = \cot\varphi$ is constant.

b. If $f(\theta) = Ce^{k\theta}$, then $\cot\varphi = \dfrac{f'(\theta)}{f(\theta)} = \dfrac{kCe^{k\theta}}{Ce^{k\theta}} = k$.



c.

12.3.84 If $r = f(\theta)$, then parametrically we have $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$. Note that $\dfrac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$ and $\dfrac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$. Thus

$$\sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2},$$

which can be written

$$\sqrt{(f'(\theta)\cos\theta)^2+(f'(\theta)\sin\theta)^2+(f(\theta)\sin\theta)^2+(f(\theta)\cos\theta)^2}=\sqrt{(f'(\theta))^2+(f(\theta))^2}.$$

Thus $L=\int_\alpha^\beta \sqrt{(f'(\theta))^2+(f(\theta))^2}\,d\theta$ as desired.

12.3.85 Suppose that the goat is tethered at the origin, and that the center of the corral is $(1,\pi)$. The circle that the goat can graze is $r=a$, and the corral is given by $r=-2\cos\theta$. The intersection occurs for $\theta=\cos^{-1}(-a/2)$.

The area grazed by the goat is twice the area of the sector of the circle $r=a$ between $\cos^{-1}(-a/2)$ and π, plus twice the area of the circle $r=-2\cos\theta$ between $\pi/2$ and $\cos^{-1}(-a/2)$. Thus we need to compute $A=\int_{\cos^{-1}(-a/2)}^{\pi} a^2\,d\theta+\int_{\pi/2}^{\cos^{-1}(-a/2)} 4\cos^2\theta\,d\theta = a^2\pi-a^2\cos^{-1}(-a/2)+\left(2\cos\theta\sin\theta+2\theta\right)\Big|_{\pi/2}^{\cos^{-1}(-a/2)} = a^2(\pi-\cos^{-1}(-a/2))-\pi-\frac{1}{2}a\sqrt{4-a^2}+2\cos^{-1}(-a/2)$. Note that $\pi-\cos^{-1}(-a/2)=\cos^{-1}(a/2)$, so this can be written as $(a^2-2)\cos^{-1}(a/2)+\pi-\frac{1}{2}a\sqrt{4-a^2}$. Note that for $a=0$ this is 0, and for $a=2$, this is π, as desired.

12.3.86 Imagine that the boundary of the concrete slab is the fence from the previous problem Then the area the goat could graze in the previous problem becomes the area it can't graze in this problem. If the slab weren't there, the goat could graze a region of area πa^2. Thus, the goat can graze a region of area $\pi a^2-\left((a^2-2)\cos^{-1}(a/2)+\pi-\frac{1}{2}a\sqrt{4-a^2}\right)=\pi(a^2-1)+\frac{1}{2}a\sqrt{4-a^2}+(2-a^2)\cos^{-1}(a/2)$. If $a=0$, this quantity is 0, while if $a=2$, this quantity is 3π.

12.3.87 Again, suppose that the goat is tethered at the origin, and that the center of the corral is $(1,\pi)$. The equation of the corral fence is given by $r=-2\cos\theta$. Note that to the right of the vertical line $\theta=\pi/2$, the goat can graze a half-circle of area $\pi a^2/2$. Also, there is a region in the 2nd quadrant and one in the 3rd quadrant of equal size that can also be grazed. Let this region have area A, so that the total area grazed will then be $\frac{\pi a^2}{2}+2A$.

Imagine that the goat is walking "west" from the polar point $(a,\pi/2)$, and is keeping the rope taut until his whole rope is along the fence in the third quadrant. Let φ be the central angle angle from the origin to the polar point $(1,\pi)$ to the point on the fence that the goat's rope is touching as he makes this walk. When the goat is at $(a,\pi/2)$, we have $\varphi=0$. When the goat is all the way to the fence, we have $\varphi=a$. Then length of the rope not along the fence is $a-\varphi$. Thus, the value of A is $\frac{1}{2}\int_0^a(a-\varphi)^2d\varphi=\frac{1}{2}\left(a^2\varphi-a\varphi^2+\frac{\varphi^3}{3}\right)\Big|_0^a=\frac{a^3}{6}$.

Thus, the goat can graze a region of area $\frac{\pi a^2}{2}+\frac{a^3}{3}$.

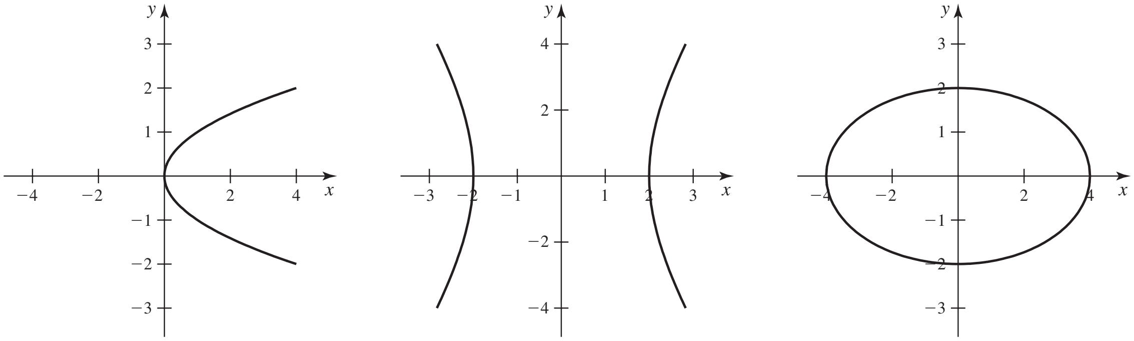
12.4 Conic Sections

12.4.1 A parabola is the set of points in the plane which are equidistant from a given fixed point and a given fixed line.

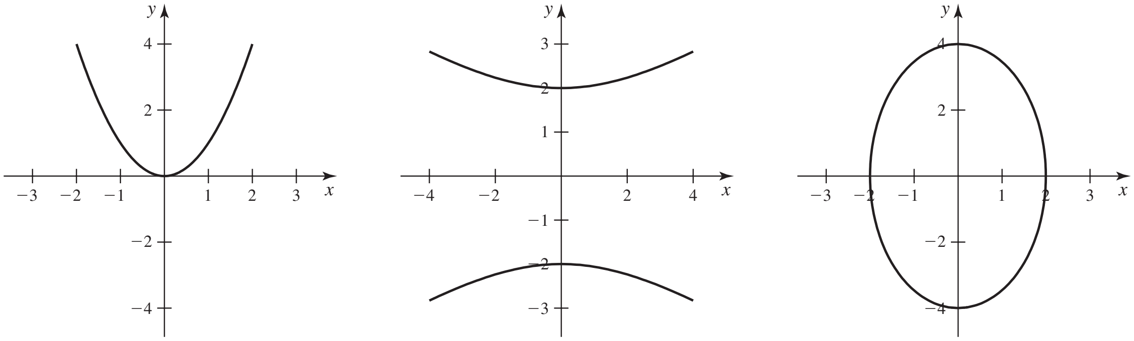
12.4.2 An ellipse is the set of points in the plane with the property that the sum of the distances from the point to two given fixed points is a given constant.

12.4.3 A hyperbola is the set of points in the plane with the property that the difference of the distances from the point to two given fixed points is a given constant.

12.4.4



12.4.5



12.4.6 $x^2 = 4py$, where $p < 0$.

12.4.7 $\left(\dfrac{x}{a}\right)^2 + \dfrac{y^2}{a^2 - c^2} = 1$.

12.4.8 $\left(\dfrac{y}{a}\right)^2 - \dfrac{x^2}{c^2 - a^2} = 1$.

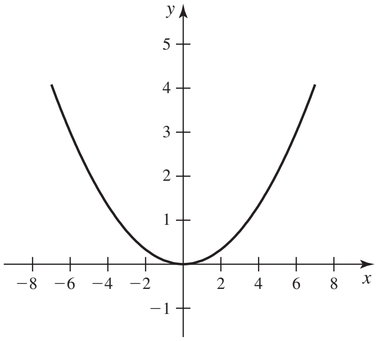
12.4.9 The foci for both are $(\pm ae, 0)$.

12.4.10 This is given by $r = \dfrac{ed}{1 + e\cos\theta}$, $-\pi < \theta < \pi$.

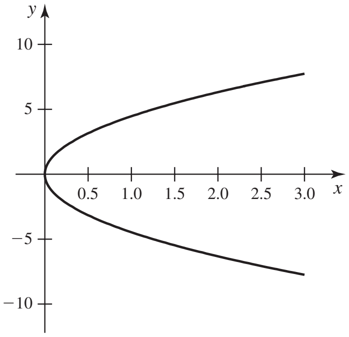
12.4.11 The asymptotes are $y = -\dfrac{b}{a} \cdot x$ and $y = \dfrac{b}{a} \cdot x$.

12.4.12 If $e = 1$, the conic section is a parabola. If $e > 1$, it is a hyperbola. If $0 < e < 1$, it is an ellipse.

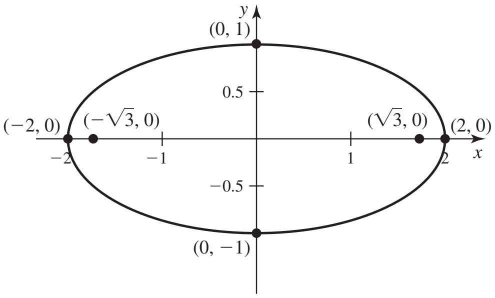
12.4.13 This is a parabola. Directrix: $y = -3$. Focus: $(0, 3)$.



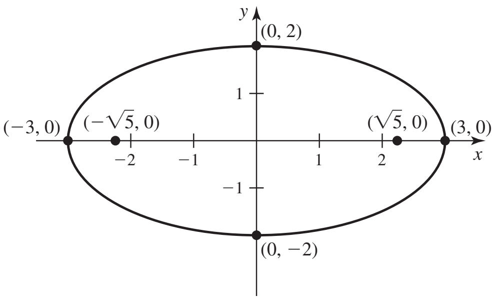
12.4.14 This is a parabola. Directrix: $x = -5$. Focus: $(5, 0)$.



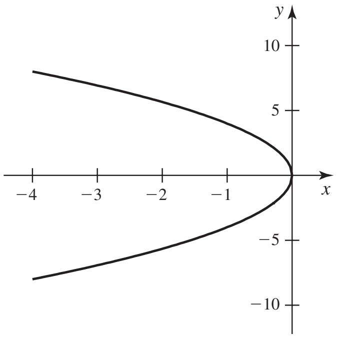
12.4.15 This is an ellipse. Vertices are $(\pm 2, 0)$, and the foci are $(\pm\sqrt{3}, 0)$. The major axis has length 4 and the minor axis has length 2.



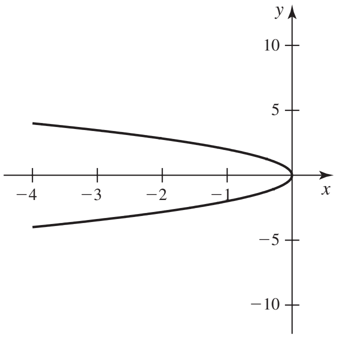
12.4.16 This is an ellipse. Vertices are $(\pm 3, 0)$, and the foci are $(\pm\sqrt{5}, 0)$. The major axis has length 6 and the minor axis has length 4.



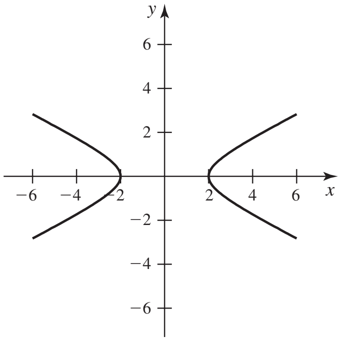
12.4.17 This is a parabola. Directrix: $x = 4$. Focus: $(-4, 0)$.



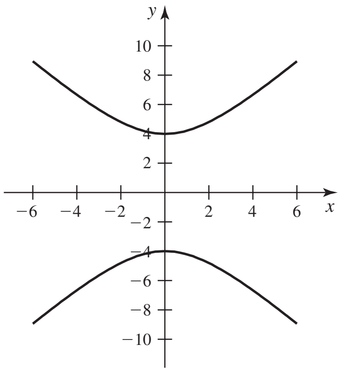
12.4.18 This is a parabola. Directrix: $x = 1$. Focus: $(-1, 0)$.



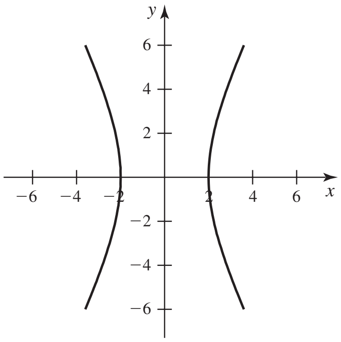
12.4.19 This is a hyperbola. The vertices are $(\pm 2, 0)$, and the foci are $(\pm\sqrt{5}, 0)$. The asymptotes are $y = \dfrac{\pm 1}{2} \cdot x$.



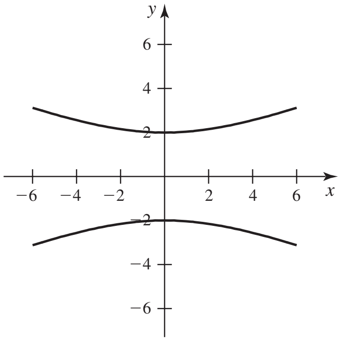
12.4.20 This is a hyperbola. The vertices are $(0, \pm 4)$, and the foci are $(0, \pm 5)$. The asymptotes are $y = \dfrac{\pm 4}{3} \cdot x$.



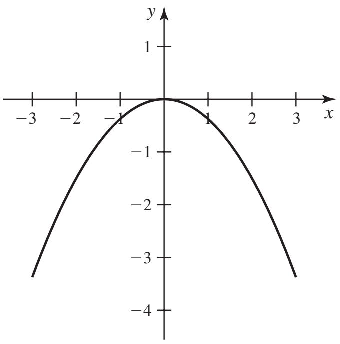
12.4.21 This is a hyperbola. The vertices are $(\pm 2, 0)$, and the foci are $(\pm 2\sqrt{5}, 0)$. The asymptotes are $y = \pm 2x$.



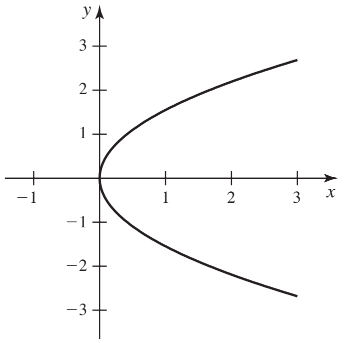
12.4.22 This is a hyperbola. The vertices are $(0, \pm 2)$, and the foci are $(0, \pm 29)$. The asymptotes are $y = \frac{\pm 2}{5} \cdot x$.



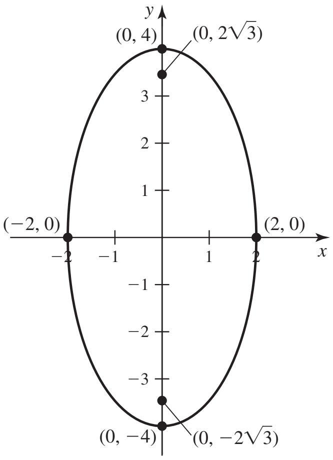
12.4.23 This is a parabola. Directrix: $y = \frac{2}{3}$. Focus: $\left(0, -\frac{2}{3}\right)$.



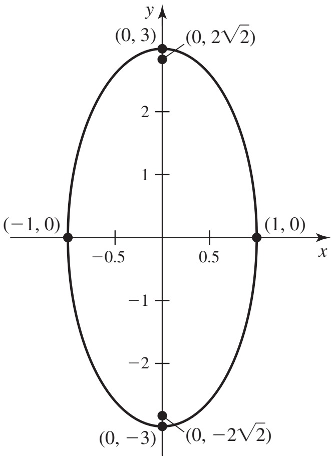
12.4.24 This is a parabola. Directrix: $x = -\dfrac{3}{5}$. Focus: $\left(\dfrac{3}{5}, 0\right)$.



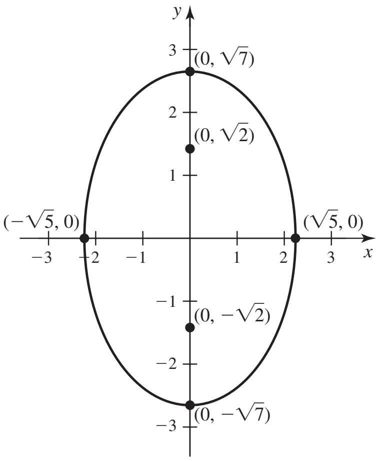
12.4.25 This is an ellipse. Vertices are $(0, \pm 4)$, and the foci are $(0, \pm 2\sqrt{3})$. The major axis has length 8 and the minor axis has length 4.



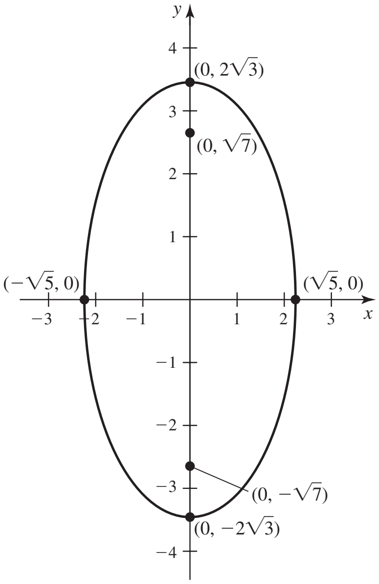
12.4.26 This is an ellipse. Vertices are $(0, \pm 3)$, and the foci are $(0, \pm 2\sqrt{2})$. The major axis has length 6 and the minor axis has length 2.



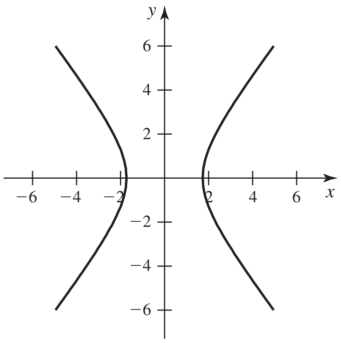
12.4.27 This is an ellipse. Vertices are $(0, \pm\sqrt{7})$, and the foci are $(0, \pm\sqrt{2})$. The major axis has length $2\sqrt{7}$ and the minor axis has length $2\sqrt{5}$.



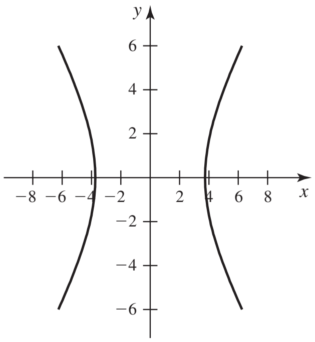
12.4.28 This is an ellipse. Vertices are $(0, \pm 2\sqrt{3})$, and the foci are $(0, \pm\sqrt{7})$. The major axis has length $4\sqrt{3}$ and the minor axis has length $2\sqrt{5}$.



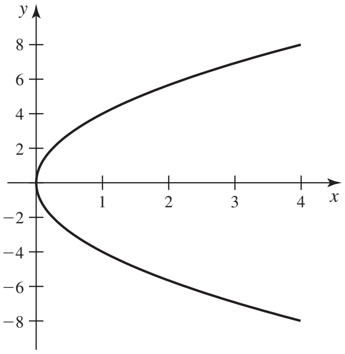
12.4.29 This is a hyperbola. The vertices are $(\pm\sqrt{3}, 0)$, and the foci are $(\pm 2\sqrt{2}, 0)$. The asymptotes are $y = \pm\sqrt{\dfrac{5}{3}} \cdot x$.



12.4.30 This is a hyperbola. The vertices are $(\pm\sqrt{14}, 0)$, and the foci are $(\pm\sqrt{34}, 0)$. The asymptotes are $y = \pm\sqrt{\dfrac{10}{7}} \cdot x$.

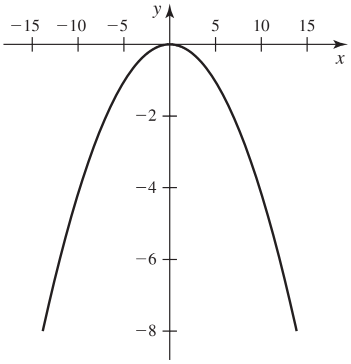


12.4.31



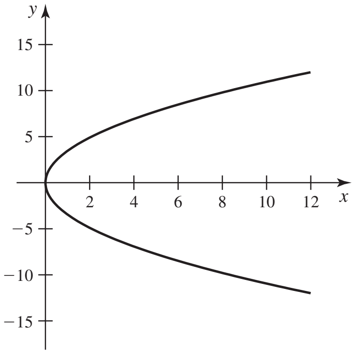
$y^2 = 16x.$

12.4.32



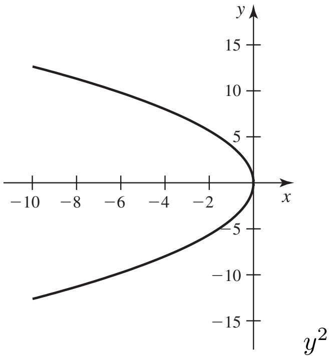
$x^2 = -24y.$

12.4.33



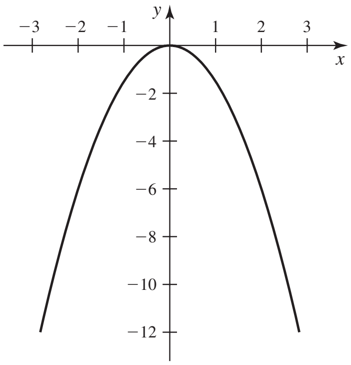
$y^2 = 12x.$

12.4.34

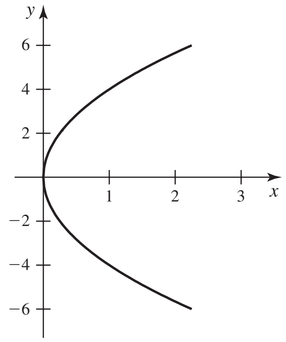


$y^2 = -16x.$

12.4.35 $x^2 = 4py$ and $4 = 4p(-6)$, so $p = -\frac{1}{6}$ and $x^2 = -\frac{2}{3} \cdot y$.



12.4.36 $y^2 = 4px$ and $(-4)^2 = 4p(1)$, so $p = 4$ and $y^2 = 16x$.



12.4.37 Because the vertex is $(-1, 0)$ and the parabola is symmetric about the x-axis, we have $y^2 = 4p(x+1)$ and because the directrix is one unit left of the vertex, we obtain $p = 1$ and $y^2 = 4(x + 1)$.

12.4.38 Because the vertex is $(0, 2)$ and the parabola is symmetric about the y-axis, we have $x^2 = 4p(y-2)$ and because the directrix is 2 units above the vertex, we obtain $p = -2$ and $x^2 = -8(y - 2)$.

12.4.39 $a = 4$, and $b = 3$, so the equation is $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$.

12.4.40 $a = 6$, and $a^2 = b^2 + c^2$ where $c = 4$, so $b^2 = 36 - 16 = 20$, and the equation is $\dfrac{x^2}{36} + \dfrac{y^2}{20} = 1$.

12.4.41 We have $a = 4$ and $c = 6$, so $b^2 = c^2 - a^2 = 20$, so the equation is $\dfrac{x^2}{16} - \dfrac{y^2}{20} = 1$.

12.4.42 We have $a = 1$, so the equation is of the form $x^2 - \dfrac{y^2}{b^2} = 1$. Because $(5/3, 8)$ is on the curve, we have $\dfrac{25}{9} - \dfrac{64}{b^2} = 1$, so $b = 6$. The equation is $x^2 - \dfrac{y^2}{36} = 1$.

12.4.43 $a = 5$, and the equation is of the form $\dfrac{x^2}{25} + \dfrac{y^2}{b^2} = 1$. Because $\left(4, \dfrac{3}{5}\right)$ is on the curve, we have $\dfrac{16}{25} + \dfrac{9}{25b^2} = 1$, so $b = 1$. The equation is $\dfrac{x^2}{25} + y^2 = 1$.

12.4.44 $a = 10$, and the equation is of the form $\dfrac{y^2}{100} + \dfrac{x^2}{b^2} = 1$, and because $(\sqrt{3}/2, 5)$ is on the curve, we have $\dfrac{1}{4} + \dfrac{3}{4b^2} = 1$, so $b = 1$. The equation is $x^2 + \dfrac{y^2}{100} = 1$.

12.4.45 We have $a = 2$, and because the asymptoes are $y = \dfrac{\pm bx}{a}$, we have that $b = 3$, so the equation is $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$.

12.4.46 We have $a = 2$ and because the asymptotes are $y = \dfrac{\pm a}{b} \cdot x$, we have $b = 1$, and the equation is $\dfrac{y^2}{4} - x^2 = 1$.

12.4.47 $a = 3$ and $b = 2$, so the equation is $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

12.4.48 $a = 10$ and $b = 8$, so $\dfrac{x^2}{100} + \dfrac{y^2}{64} = 1$.

12.4.49 We have $a = 4$ and $c = 5$, so $b^2 = 25 - 16 = 9$, so $b = 3$ and the equation is $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$.

12.4.50 We have $a = 6$ and $c = 10$, and $b^2 = 100 - 36 = 64$, so $b = 8$, and the equation is $\dfrac{y^2}{36} - \dfrac{x^2}{64} = 1$.

12.4.51

a. True. Note that if $x = 0$, the equation becomes $-y^2 = 9$, which has no solution.

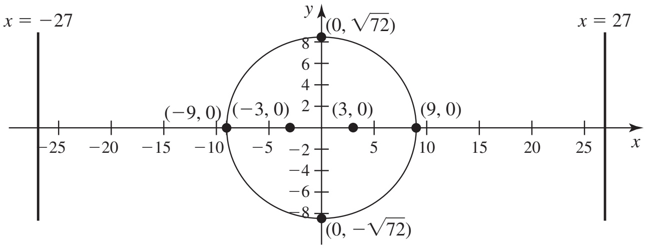
b. True. The slopes of the tangent lines range continuously from $-\infty$ to 0 to ∞ and then back through 0 to $-\infty$ again.

c. True. Given c and d, one can compute a, b, and e. See the summary after Theorem 10.3.

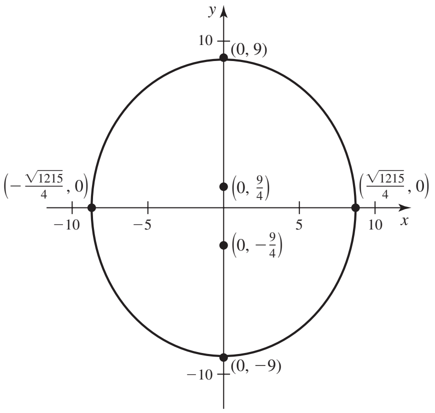
d. True. The vertex is exactly halfway between the focus and the directrix.

12.4.52 We have a vertex at $(0, 0)$ and the parabola passes through $(640, 152)$, so $152 = a(640)^2$, so $a = \dfrac{19}{51200} \approx 0.000371$. Thus, $y = \dfrac{19}{51200}x^2$, and the guy wire has length $L = \dfrac{19}{51200}(500^2) = \dfrac{11875}{128} \approx 92.77$ meters.

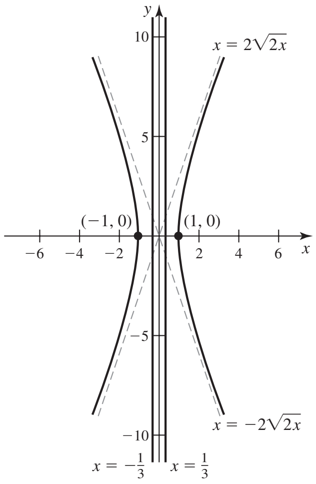
12.4.53 We have $a = 9$ and $e = \dfrac{1}{3}$, so $c = ae = 3$, and $b^2 = a^2 - c^2 = 72$, so the equation is $\dfrac{x^2}{81} + \dfrac{y^2}{72} = 1$.



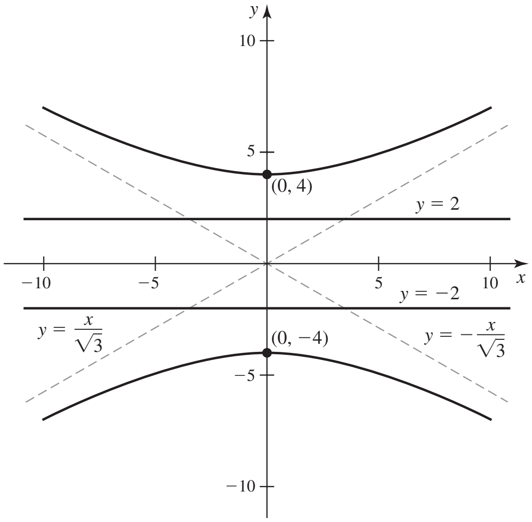
12.4.54 We have $a = 9$ and $e = \dfrac{1}{4}$, so $c = ae = \dfrac{9}{4}$, and $b^2 = a^2 - c^2 = 81 - \dfrac{81}{16} = \dfrac{1215}{16}$. Thus the equation is $\dfrac{16x^2}{1215} + \dfrac{y^2}{81} = 1$.



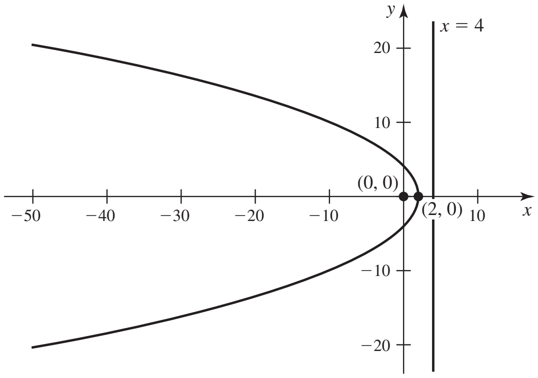
12.4.55 We have $a = 1$ and $e = 3$, so $c = ae = 3$ and $b^2 = c^2 - a^2 = 9 - 1 = 8$. Thus, the equation is $x^2 - \dfrac{y^2}{8} = 1$.



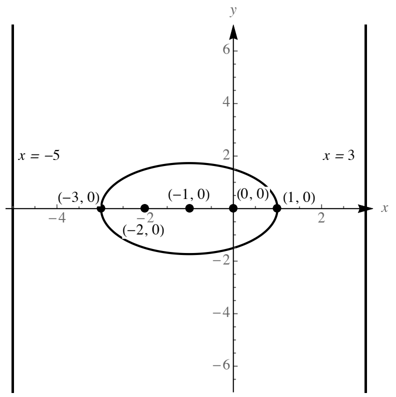
12.4.56 We have $a = 4$ and $e = 2$, so $c = ae = 8$ and $b^2 = c^2 - a^2 = 64 - 16 = 48$. Thus, the equation is $\dfrac{y^2}{16} - \dfrac{x^2}{48} = 1$.



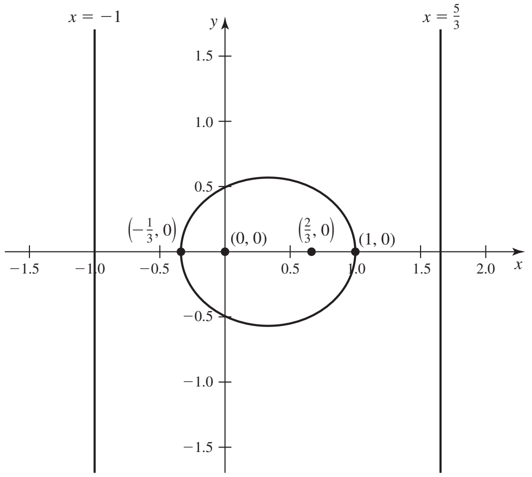
12.4.57 The vertex is $(2, 0)$. The focus is $(0, 0)$, and the directrix is the line $x = 4$.



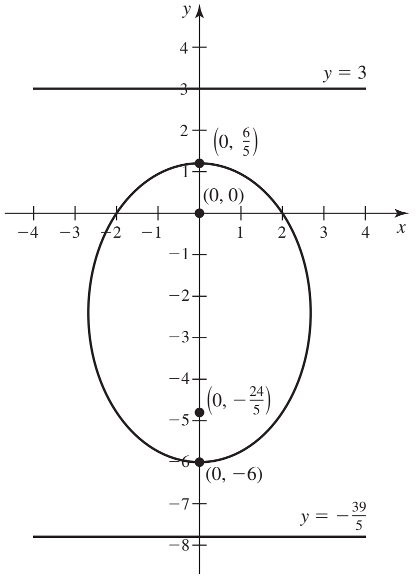
12.4.58 The vertices are $(1, 0)$ and $(-3, 0)$. The center is $(-1, 0)$. The foci are $(0, 0)$ and $(-2, 0)$.



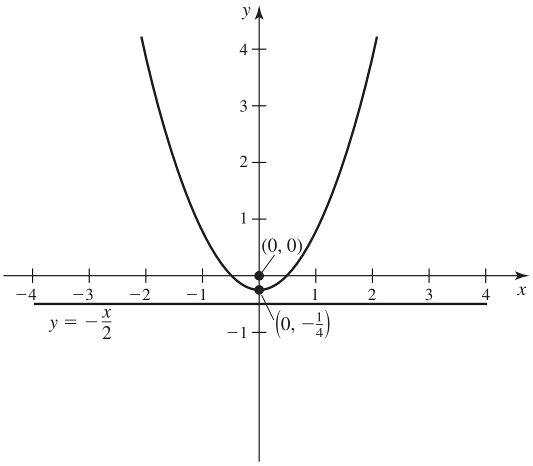
12.4.59 The vertices are $(1, 0)$ and $(-1/3, 0)$. The center is $(1/3, 0)$. The directrices are $x = -1$ and $x = 5/3$.



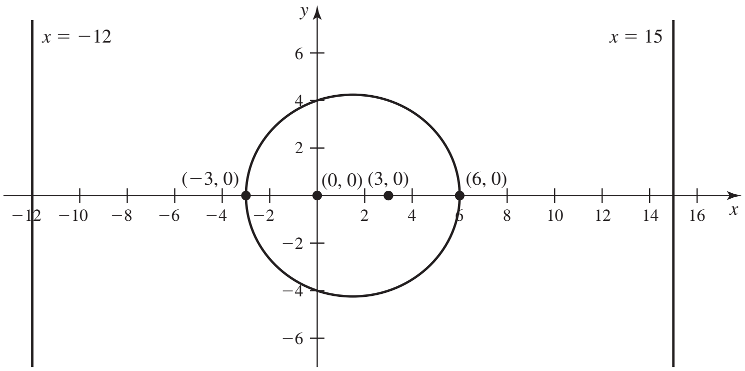
12.4.60 The vertices are $(0, 6/5)$ and $(0, -6)$. The center is $(0, -12/5)$. The foci are $(0, 0)$ and $(-24/5, 0)$.



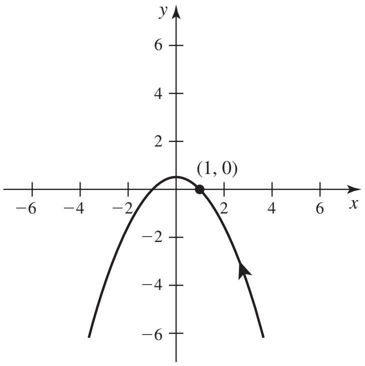
12.4.61 The vertex is $(0, -1/4)$, and the focus is $(0, 0)$. The directrix is the line $y = -\dfrac{1}{2}$.



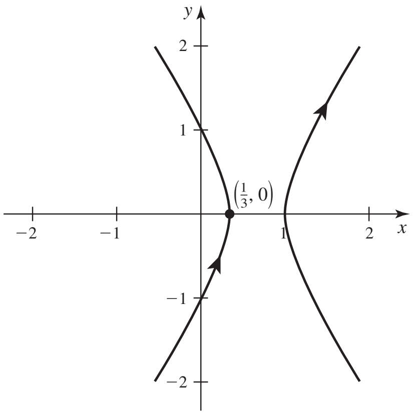
12.4.62 The vertices are $(6, 0)$ and $(-3, 0)$. The center is $(3/2, 0)$. The foci are $(0, 0)$ and $(3, 0)$.



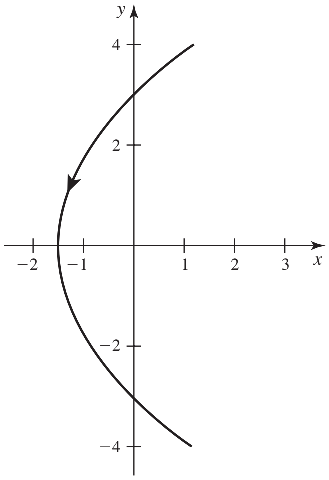
12.4.63 The parabola starts at $(1, 0)$ and goes through quadrants I, II, and III for $\theta \in [0, 3\pi/2]$. It then approaches $(1, 0)$ by traveling through quadrant IV for $\theta \in (3\pi/2, 2\pi)$.



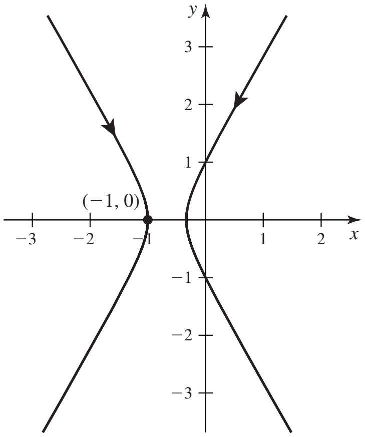
12.4.64 Note that the value of r for $\theta = 0$ is $1/3$. As θ proceeds to $\pi/2$, the curve is traced in the first quadrant and approaches the polar point $(1, \pi/2)$ From $\pi/2$ to π, the curve approaches the asymptote, and then appears along the asymptote in the fourth quadrant and heads toward the polar point $(-1, \pi)$. From π to 2π, the curve approaches the asymptote in the first quadrant, and then reappears in the third quadrant along the asymptote, and heads toward the point $(1/3, 2\pi)$.



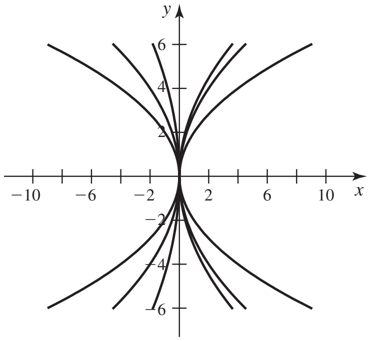
12.4.65 The parabola begins in the first quadrant and passes through the points $(0, 3)$ and then $(-3/2, 0)$ and $(0-3)$ as θ ranges from 0 to 2π.



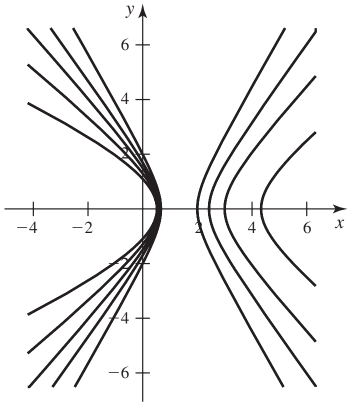
12.4.66 As θ ranges from 0 to $\pi/3$, the branch of the hyperbola in quadrant III starts at the point $(-1, 0)$, and approaches the asymptote (note that $r \to \infty$ as $\theta \to \pi/3^-$.) As θ takes on the values from $\pi/3$ to $\pi/2$, the portion of the parabola in quadrant I appears and heads toward the point $(0, 1)$. For θ ranging from $\pi/2$ to $3\pi/2$, the curve ranges from $(0, 1)$ to $(-1/3, 0)$ to $(0, -1)$. From $\theta = 3\pi/2$ to $\theta = 5\pi/3$, the curve approaches the asymptote in quadrant IV. From $5\pi/3$ to 2π, the curve reappears along the asymptote in quadrant II, and approaches the point $(-1, 0)$.



12.4.67 For negative p, the parabola opens to the left and for positive p it opens to the right. As p increases to 0, the parabola opens wider and as p decreases (for $p > 0$), it gets narrower.



12.4.68 As e gets larger, the vertices move closer to each other.



12.4.69 Differentiating gives $2x = -6y'$, so at $(-6,-6)$ we obtain $-12 = -6y'$, so $y' = 2$. Thus $y - (-6) = 2(x - (-6))$, or $y = 2x + 6$ is the equation of the tangent line.

12.4.70 Using implicit differentiation, we have $2yy' = 8$, and at the point $(8,-8)$, we have $y' = -\frac{1}{2}$. So $y - (-8) = -\frac{1}{2}(x - 8)$, or $y = -\frac{1}{2}x - 4$ is the equation of the tangent line.

12.4.71 Differentiating implicitly, we have $2yy' - \frac{x}{32} = 0$, so at $(6, -5/4)$ we have $-\frac{5}{2}y' - \frac{3}{16} = 0$, so $y' = -\frac{3}{40}$. The equation of the tangent line is $y + \frac{5}{4} = -\frac{3}{40}(x - 6)$, or $y = -\frac{3x}{40} - \frac{4}{5}$.

12.4.72 We have

$$\frac{dy}{dx} = \frac{-\dfrac{\cos\theta\sin\theta}{(1+\sin\theta)^2} + \dfrac{\cos\theta}{1+\sin\theta}}{-\dfrac{\cos^2\theta}{(1+\sin\theta)^2} - \dfrac{\sin\theta}{1+\sin\theta}} = \frac{\sin\theta - 1}{\cos\theta}.$$

At $\theta = \frac{\pi}{6}$ we have $y' = \frac{(1/2) - 1}{\sqrt{3}/2} = -\frac{\sqrt{3}}{3}$. The equation of the tangent line is therefore $y - \frac{1}{3} = -\frac{\sqrt{3}}{3}(x - \frac{\sqrt{3}}{3})$, or $y = -\frac{\sqrt{3}}{3}x + \frac{2}{3}$.

12.4.73 Using implicit differentiation, we have $\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$, or $y' = -\frac{b^2x}{a^2y}$. If (x_0, y_0) is the point of tangency, then $-\frac{b^2x_0}{a^2y_0} = \frac{y - y_0}{x - x_0}$, so $\frac{x_0(x - x_0)}{a^2} = -\frac{y_0(y - y_0)}{b^2}$, so $\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$.

12.4.74 Using implicit differentiation, we have $\frac{2x}{a^2}-\frac{2yy'}{b^2}=0$, so $y'=\frac{b^2x}{a^2y}$. At (x_0,y_0), we have $y'=\frac{b^2x_0}{a^2y_0}$, so the tangent line is given by $y-y_0=\frac{b^2x_0}{a^2y_0}(x-x_0)$, or $\frac{y(y-y_0)}{b^2}=\frac{x_0(x-x_0)}{a^2}$, or $\frac{y_0y}{b^2}-\frac{x_0x}{a^2}=\frac{y_0^2}{b^2}-\frac{x_0^2}{a^2}=-1$, so $\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$.

12.4.75 We have a hyperbola with focal point at the origin and directrix $y=-2$. Furthermore $P=(0,-4/3)$ is a vertex. Thus, $e=\frac{|PF|}{|PL|}=\frac{4/3}{2/3}=2$, and $r(\theta)=\frac{2(2)}{1-2\sin\theta}=\frac{4}{1-2\sin\theta}$.

12.4.76 We have an ellipse with focus at the origin and directrix $x=2$. Because $(2/3,0)$ is a vertex, $e=\frac{|PF|}{|PL|}=\frac{2/3}{4/3}=\frac{1}{2}$ and $r(\theta)=\frac{\frac{1}{2}\cdot 2}{1+\frac{1}{2}\cos\theta}=\frac{2}{2+\cos\theta}$.

12.4.77 The points on the intersection of the two circles are a distance of $2a+r$ from F_1 and a distance of r from F_2. So for P an intersection point, we have $|PF_1|-|PF_2|=2a$ for all r, and the set of all such points form a hyperbola with foci F_1 and F_2.

12.4.78

a. Making use of the substitution $\frac{x}{\sqrt{2}}=\sin t$, we have

$$\begin{aligned}A&=\int_0^1\left(\sqrt{1-\frac{x^2}{2}}-\frac{x^2}{\sqrt{2}}\right)dx=\int_0^{\pi/4}\sqrt{1-\sin^2 t}\sqrt{2}\cos t\,dt-\frac{1}{\sqrt{2}}\int_0^1 x^2\,dx\\&=\sqrt{2}\cdot\frac{1}{2}\left(\cos x\sin x+x\right)\Big|_0^{\pi/4}-\frac{1}{3\sqrt{2}}=\frac{\sqrt{2}}{2}(\sin(\pi/4)\cos(\pi/4)+\frac{\pi}{4})-\frac{1}{3\sqrt{2}}\\&=\frac{\sqrt{2}}{2}\cdot(\frac{1}{2}+\frac{\pi}{4})-\frac{\sqrt{2}}{6}=\frac{\sqrt{2}}{12}+\frac{\sqrt{2}\pi}{8}.\end{aligned}$$

b. About the x-axis, we obtain $\pi\int_0^1\left(1-\frac{x^2}{2}-\frac{x^4}{2}\right)dx=\pi\left(x-\frac{x^3}{6}-\frac{x^5}{10}\right)\Big|_0^1=\frac{11\pi}{15}\approx 2.304$.

About the y-axis, we obtain

$$\begin{aligned}&2\pi\int_0^1\left(x\sqrt{1-\frac{x^2}{2}}-\frac{x^2}{\sqrt{2}}\right)dx=2\pi\int_{1/2}^1 u^{1/2}\,du-\frac{1}{\sqrt{2}}\int_0^1 x^3\,dx\\&=2\pi\left(\frac{2}{3}u^{3/2}\right)\Big|_{1/2}^1-\left(\frac{1}{4\sqrt{2}}x^4\right)\Big|_0^1=2\pi\left(\frac{2}{3}-\frac{2}{6\sqrt{2}}-\frac{1}{4\sqrt{2}}\right)\approx 1.597.\end{aligned}$$

So the volume about the x-axis is greater.

12.4.79 $V_x=\pi\int_{-a}^a\left(b^2-\frac{b^2x^2}{a^2}\right)dx=\pi b^2\int_{-a}^a\left(1-\frac{x^2}{a^2}\right)dx=\pi b^2\left(x-\frac{x^3}{3a^2}\right)\Big|_{-a}^a=\frac{4\pi b^2a}{3}$.

$V_y=\pi\int_{-b}^b\left(a^2-\frac{a^2y^2}{b^2}\right)dy=\pi a^2\int_{-b}^b\left(1-\frac{y^2}{b^2}\right)dy=\pi a^2\left(y-\frac{y^3}{3b^2}\right)\Big|_{-b}^b=\frac{4\pi a^2b}{3}$.

These are different if $a\neq b$. In the case $a=b$, both volumes give $\frac{4\pi a^3}{3}$, the volume of a sphere.

12.4.80

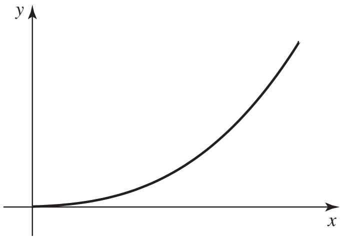
a. The focus is at $c = \sqrt{a^2+b^2}$. We have $A = 2b\int_a^c \sqrt{\frac{x^2}{a^2}-1}\,dx = \frac{2b}{a}\int_a^c \sqrt{x^2-a^2}\,dx$. Using either the substitution $x = a\sec\theta$ (or a table of integrals), we have

$$A = \frac{2b}{a}\cdot\left(\frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\ln\left(2\left(\sqrt{x^2-a^2}+x\right)\right)\right)\Big|_a^c$$

so

$$A = ab\ln(a) - ab\ln(\sqrt{a^2+b^2}+b) + \frac{\sqrt{a^2+b^2}}{a}\cdot b^2.$$

b.



12.4.81

a.

$$V_x = \pi\int_a^c\left(\sqrt{\frac{b^2x^2}{a^2}-b^2}\right)^2 dx = \pi\int_a^c\left(\frac{b^2x^2}{a^2}-b^2\right)dx = \pi b^2\left(\frac{x^3}{3a^2}-x\right)\Big|_a^c$$
$$= \pi b^2\left(\frac{c^3}{3a^2}-c-\frac{a}{3}+a\right) = \frac{\pi b^2}{3a^2}(c^3-3ca^2+2a^3) = \frac{\pi b^2}{3a^2}(a-c)^2(2a+c).$$

b. $V_y = 2\cdot 2\pi\int_a^c a^2b\sqrt{\frac{x^2}{a^2}-1}\,dx = 2\pi\int_0^{b^2/a^2} a^2b\sqrt{u}\,du = 2\pi a^2b\left(\frac{2}{3}u^{3/2}\right)\Big|_0^{b^2/a^2} = 2\pi a^2b\frac{2b^3}{3a^3} = \frac{4\pi b^4}{3a}$.

12.4.82

$$V_R = 2\pi\int_0^{\sqrt{h/a}} x(h-ax^2)\,dx = 2\pi\int_0^{\sqrt{h/a}}(xh-ax^3)\,dx = 2\pi\left(\frac{hx^2}{2}-\frac{ax^4}{4}\right)\Big|_0^{\sqrt{h/a}} = 2\pi\left(\frac{h^2}{2a}-\frac{h^2}{4a}\right) = \frac{\pi h^2}{2a}.$$

The cone has height h and radius $\sqrt{h/a}$, so $V_c = \frac{1}{3}\pi\left(\sqrt{\frac{h}{a}}\right)^2\cdot h = \frac{\pi h^2}{3a}$, and $V_R = \frac{3}{2}\cdot\frac{\pi h^2}{3a} = \frac{\pi h^2}{2a}$.

12.4.83

a. The slope of a line making an angle θ with the horizontal is $\tan\theta$. The slope of the tangent line at (x_0, y_0) is $y' = \frac{x}{2p}$, so $y' = \frac{x_0}{2p}$, so $\tan\theta = \frac{x_0}{2p}$.

b. The distance from $(0, y_0)$ to $(0, p)$ is $p - y_0$, and $\tan\varphi = \frac{\text{opposite}}{\text{adjacent}} = \frac{p-y_0}{x_0}$.

c. Because l is perpendicular to $y = y_0$, we have $\alpha + \theta = \pi/2$, or $\alpha = \frac{\pi}{2} - \theta$, so $\tan\alpha = \cot\theta = \frac{2p}{x_0}$.

d. $\tan\beta = \tan(\theta+\varphi) = \dfrac{\frac{x_0}{wp}+\frac{p-y_0}{x_0}}{1-\frac{p-y_0}{2p}} = \frac{x_0^2+2p^2-2py_0}{x_0(p+y_0)}$. Now because $x_0^2 = 4py_0$, we obtain $\tan\beta = \frac{4py_0+2p^2-2py_0}{x_0(p+y_0)}$
$= \frac{2p(p+y_0)}{x_0(p+y_0)} = \frac{2p}{x_0}$.

e. Because α and β are acute, we have that $\tan\alpha = \tan\beta$, so $\alpha = \beta$.

12.4.84

a. $e = \dfrac{|PF|}{|PL|}$, so $r = |PF| = e\,|PL| = e\,|-d - r\cos\theta|$, or $r = e(d + r\cos\theta)$. Solving for r yields $r = \dfrac{ed}{1 - e\cos\theta}$.

b. $r = |PF| = e\,|PL|$, or $r = e(d - r\sin\theta)$. Solving for r yields $r = \dfrac{ed}{1 + e\sin\theta}$.

c. $r = |PF| = e\,|PL|$, or $r = e(-d - r\sin\theta)$. So $r = e(d + r\sin\theta)$, and solving for r yields $r = \dfrac{ed}{1 - e\sin\theta}$.

12.4.85 Assume the two fixed points are at $(c, 0)$ and $(-c, 0)$. Let P be the point $(0, b)$, and note that P is equidistant from the two given points, so we must have $b^2 + c^2 = a^2$ by the Pythagorean theorem. Now let $Q = (u, 0)$ be on the ellipse for $u > c$. Then $u - c + (c + u) = 2a$, so $u = a$. Now let $R = (x, y)$ be an arbitrary point on the ellipse (assume $x > 0$ and $y > 0$ – the other cases are similar.) Using the triangles formed between the foci, R, and the projection of R onto the x-axis, we have $\sqrt{(x + c)^2 + y^2} = 2a - \sqrt{(c - x)^2 + y^2}$. Squaring both sides gives $(x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(c - x)^2 + y^2} + (c - x)^2 + y^2$. Isolating the root gives $\sqrt{(c - x)^2 + y^2} = \dfrac{1}{4a}\left((c - x)^2 + y^2 - (c + x)^2 - y^2 + 4a^2\right)$, so $\sqrt{(c - x)^2 + y^2} = a - \dfrac{c}{a}x$. Squaring again yields $(c - x)^2 + y^2 = a^2 - 2xc + \dfrac{c^2}{a^2}x^2$, so $c^2 - 2cx + x^2 + y^2 = a^2 - 2cx + \dfrac{c^2}{a^2}x^2$, or $x^2\left(1 - \dfrac{c^2}{a^2}\right) + y^2 = a^2 - c^2$. Thus $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$, which can be written $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, because $b^2 = a^2 - c^2$.

12.4.86 The intersection points of the branches with the x-axis are at $(-a, 0)$ and $(a, 0)$ because the distances to $(c, 0)$ and $(-c, 0)$ are $c + a$ and $c - a$, so the difference is $\pm 2a$. Consider one point on the right branch (the left branch will follow by a similar argument.) Let the distance from $(-c, 0)$ to (x, y) be u and the distance from $(c, 0)$ to (x, y) be v. Then $u = \sqrt{(c + x)^2 + y^2}$ and $v = \sqrt{(c - x)^2 + y^2}$, and because $u - v = 2a$, we have $\sqrt{(c + x)^2 + y^2} = 2a + \sqrt{(c - x)^2 + y^2}$. Squaring gives $(c + x)^2 + y^2 = 4a^2 + 4a\sqrt{(c - x)^2 + y^2} + (c - x)^2 + y^2$. Isolating the root gives $\sqrt{(c - x)^2 + y^2} = \dfrac{1}{4a}\left((c + x)^2 + y^2 - (c - x)^2 - y^2 - 4a^2\right)$, so $\sqrt{(c - x)^2 + y^2} = -a + \dfrac{c}{a}x$.

Squaring again yields $(c - x)^2 + y^2 = a^2 - 2xc + \dfrac{c^2}{a^2}x^2$, so $c^2 - 2cx + x^2 + y^2 = a^2 - 2cx + \dfrac{c^2}{a^2}x^2$, or $x^2\left(1 - \dfrac{c^2}{a^2}\right) + y^2 = a^2 - c^2$. Thus $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$, which can be written $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $b^2 = c^2 - a^2$.

12.4.87 Let the parabola be symmetric about the y-axis with vertex at the origin. Let the circle have radius r and be centered at $(r + a, 0)$, and let the line be $y = -a$. The distance form the point $P(x, y)$ to the line is $u = y + a$. The distance from the point P to the circle is $v = \sqrt{x^2 + (r + a - y)^2} - r$. Setting $u = v$ yields $y + a = \sqrt{x^2 + (r + a - y)^2} - r$, so $y + r + a = \sqrt{x^2 + (r + a - y)^2}$, and squaring gives $y^2 + 2(r + a)y + (r + a)^2 = x^2 + (r + a - y)^2$, so $y^2 + 2(r + a)y + (r + a)^2 = x^2 + (r + a)^2 - 2(r + a)y + y^2$, and thus $4(r + a)y = x^2$, so $y = \dfrac{1}{4(r + a)}x^2$, the equation of a parabola.

12.4.88 With focus at the origin, the cartesian equation of an ellipse with the second focus at $(-2c, 0)$ and major axis length $2a$, minor axis length $2b$ is $\dfrac{(x + c)^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Using $c = ae$ and polar coordinates yields $\dfrac{(r\cos\theta + ae)^2}{a^2} + \dfrac{r^2\sin^2\theta}{a^2(1 - e^2)} = 1$. Thus, $(1 - e^2)(r^2\cos^2\theta + 2aer\cos\theta + a^2e^2) + r^2\sin^2\theta = a^2(1 - e^2)$, so $r^2 - e^2r^2\cos^2\theta + 2ae(1 - e^2)r\cos\theta + (1 - e^2)a^2e^2 = a^2(1 - e^2)$. Gathering like terms gives $(1 - e^2\cos^2\theta)r^2 + 2ae(1 - e^2)\cos\theta \cdot r - a^2(1 - e^2)^2 = 0$. Using the quadratic formula, we have

$$r = \frac{-2ae(1 - e^2)\cos\theta + \sqrt{4a^2e^2(1 - e^2)^2\cos^2\theta + 4a^2(1 - e^2)^2(1 - e^2\cos^2\theta)}}{2(1 - e^2\cos^2\theta)}.$$

This can be written as

$$r = \frac{-2ae(1-e^2)\cos\theta + 2a(1-e^2)}{2(1-e^2\cos^2\theta)} = \frac{a(1-e^2)(-e\cos\theta+1)}{(1-e\cos\theta)(1+e\cos\theta)} = \frac{a(1-e^2)}{1+e\cos\theta}.$$

12.4.89 Let the hyperbolas be centered at the origin with equations $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $\dfrac{y^2}{B^2} - \dfrac{x^2}{A^2} = 1$ and eccentricities $e = \dfrac{c}{a}$ and $E = \dfrac{C}{B}$, respectively. Because the hyperbolas share a set of asymptotes $A = ra$ and $B = rb$ fro some $r > 0$, and

$$\begin{aligned} C^2 &= A^2 + B^2 = (ra)^2 + (rb)^2 \\ &= r^2(a^2+b^2) = r^2c^2. \end{aligned}$$

Then we have

$$\begin{aligned} e^{-2} + E^{-2} &= \left(\frac{c}{a}\right)^{-2} + \left(\frac{C}{B}\right)^{-2} = \frac{a^2}{c^2} + \frac{B^2}{C^2} \\ &= \frac{a^2}{c^2} + \frac{r^2b^2}{r^2c^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1. \end{aligned}$$

12.4.90 The focal chord of slope $m \neq 0$ has equation $y = m(x-p)$. Because $y^2 = 4px$, the focal chord and the parabola intersect for $(mx - mp)^2 = 4px$, which occurs (via the quadratic formula) at $x = \dfrac{(m^2 + 2 \pm 2\sqrt{m^2+1})p}{m^2}$. The corresponding y-values are $\dfrac{(m^2+2\pm 2\sqrt{m^2+1})p}{m} - mp$. Now $y' = \dfrac{2p}{y}$, so $y' = \dfrac{m}{1 \pm \sqrt{m^2+1}}$ at the two points. The product of these two values of y' is -1, so the two lines are perpendicular. If we call the intersection points found above (x_0, y_0) and (x_1, y_1), then the two lines intersect for

$$\frac{m}{1+\sqrt{m^2+1}}(x - x_0) + y_0 = \frac{m}{1-\sqrt{m^2+1}}(x - x_1) + y_1,$$

which when solved for x gives $x = -p$, so the two lines meet on the directrix.

Note that in the case of a vertical chord, we have $(x_0, y_0) = (p, 2p)$ and $(x_1, y_1) = (p, -2p)$, and thus the slopes of the tangent lines are 1 and -1, so their product is still -1 and thus they are perpendicular. Then the tangent lines meet when $1(x-p) + 2p = -1(x-p) - 2p$, which occurs when $x = -p$, so they still meet on the directrix.

12.4.91 The latus rectum L intersects the parabola at $x = p$, $y = \pm 2p$. The distance between any point $P(x, y)$ on the parabola to the left of L and L is $p - x$. The distance from F to P is $\sqrt{(x-p)^2 + y^2} = \sqrt{x^2 - 2px + p^2 + 4px} = \sqrt{x^2 + 2px + p^2} = x + p$ (because both x and p are positive.) Thus $D + |FP| = p - x + x + p = 2p$.

12.4.92 Because the latus rectum intersects the parabola at $(p, 2p)$ and $(p, -2p)$, its length is $4\,|p|$.

12.4.93 Let P be a point on the intersection of the latus rectum and the ellipse. The length of the latus rectum is twice the distance from P to the focus. Let l be the length from P to the focus, and let L be the distance from P to the other focal point. Then $l + L = 2a$, so $L^2 = 4c^2 + l^2$, and thus $(2a - l)^2 = 4c^2 + l^2$, and solving for l yields $l = a - \dfrac{c^2}{a}$. Because $c^2 = a^2 - b^2$, this can be written as $l = a - \dfrac{a^2 - b^2}{a} = a - (a - \dfrac{b^2}{a}) = \dfrac{b^2}{a}$. The length of the latus rectum is therefore $\dfrac{2b^2}{a}$. Now because $e = \dfrac{c}{a}$, we have $\sqrt{1-e^2} = \sqrt{1 - \dfrac{a^2-b^2}{a^2}} = \sqrt{\dfrac{b^2}{a^2}} = \dfrac{b}{a}$. The length of the latus rectum can thus also be written as $2b \cdot \dfrac{b}{a} = 2b\sqrt{1-e^2}$.

12.4.94 Let P be a point on the intersection of the latus rectum and the hyperbola. The length of the latus rectum is twice the distance from P to the focus. Let l be the length from P to the focus, and let L be the distance from P to the other focal point. Then $L-l=2a$, so $L^2=4c^2+l^2$, and thus $(2a+l)^2=4c^2+l^2$, and solving for l yields $l=\frac{c^2}{a}-a$. Because $c^2=a^2+b^2$, this can be written as $l=\frac{a^2+b^2}{a}-a=\frac{b^2}{a}$. The length of the latus rectum is therefore $\frac{2b^2}{a}$. Now because $e=\frac{c}{a}$, we have $\sqrt{e^2-1}=\sqrt{\frac{a^2+b^2}{a^2}-1}=\sqrt{\frac{b^2}{a^2}}=\frac{b}{a}$. The length of the latus rectum can thus also be written as $2b\cdot\frac{b}{a}=2b\sqrt{e^2-1}$.

12.4.95 Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$ and let the equation of the hyperbola be $\frac{x^2}{r^2}-\frac{y^2}{c^2-r^2}=1$. Let (x_0,y_0) be a point of intersection. By evaluating both equations at the point of intersection and subtracting, we obtain the result

$$\frac{x_0^2}{a^2}-\frac{x_0^2}{r^2}+\frac{y_0^2}{a^2-c^2}+\frac{y_0^2}{c^2-r^2}=0,$$

which can be written

$$\frac{r_0^2x_0^2-a^2x_0^2}{a^2r^2}+\frac{(c^2-r^2)y_0^2+(a^2-c^2)y_0^2}{(a^2-c^2)(c^2-r^2)}=0.$$

This equation can be rewritten in the form $\frac{x_0^2}{y_0^2}=\frac{a^2r^2}{(a^2-c^2)(c^2-r^2)}$, which we will use later.

Now implicitly differentiating the equation for the ellipse yields $\frac{2x}{a^2}+\frac{2yy'}{a^2-c^2}=0$, and thus the slope of the tangent line to the ellipse at (x_0,y_0) is $y_e'=-\frac{x_0}{y_o}\cdot\frac{a^2-c^2}{a^2}$. Differentiating the equation of the hyperbola gives $\frac{2x}{r^2}-\frac{2yy'}{c^2-r^2}=0$, so the slope of the tangent line to the hyperbola at the point of intersection is $y_h'=\frac{x_0}{y_0}\cdot\frac{c^2-r^2}{r^2}$.

Now consider the product

$$-1\cdot y_e'\cdot y_h'=\frac{x_0^2}{y_0^2}\cdot\frac{(a^2-c^2)(c^2-r^2)}{a^2r^2}.$$

By the result of the first paragraph, this is equal to 1, and thus the two curves are perpendicular at the point of intersection.

12.4.96 The vertical distance at x_0 is given by $d(x_0)=\frac{bx_0}{a}-\sqrt{\frac{x_0^2b^2}{a^2}-a^2}=\frac{b}{a}\left(x_0-\sqrt{x_0^2-\frac{a^4}{b^2}}\right)$. We have

$$\lim_{x_0\to\infty}d(x_0)=\frac{b}{a}\lim_{x_0\to\infty}\left(x_0-\sqrt{x_0^2-\frac{a^4}{b^2}}\right)=\frac{b}{a}\lim_{x_0\to\infty}\left(\frac{x_0^2-\left(x_0^2-\frac{a^4}{b^2}\right)}{x_0+\sqrt{x_0^2-\frac{a^4}{b^2}}}\right)=0.$$

12.4.97

a. The curve and the line intersect when $x^2-m^2(x^2-4x+4)-1=0$, which occurs for $\frac{2m^2\pm\sqrt{1+3m^2}}{m^2-1}$, assuming $m\neq\pm1$. So there are two solutions in this case – but if $-1<m<1$, one of the solutions is negative (the intersection lies on the other branch of the hyperbola.) If $m^2=1$, then the equation becomes $4x-5=0$, and there is only the solution $x=\frac{5}{4}$. So there are two intersection points on the right branch exactly for $|m|>1$. We have $v(m)=\frac{2m^2+\sqrt{1+3m^2}}{m^2-1}$ and $u(m)=\frac{2m^2-\sqrt{1+3m^2}}{m^2-1}$.

b. $\lim_{m\to 1^+} u(m) = \lim_{m\to 1^+} u(m)\cdot\dfrac{2m^2+\sqrt{1+3m^2}}{2m^2+\sqrt{1+3m^2}} = \lim_{m\to 1^+}\dfrac{4m^4-3m^2-1}{(m^2-1)(2m^2+\sqrt{1+3m^2})} =$
$\lim_{m\to 1^+}\dfrac{(m^2-1)(4m^2+1)}{(m^2-1)(2m^2+\sqrt{1+3m^2})} = \dfrac{5}{4}.$

$\lim_{m\to 1^+} v(m) = \lim_{m\to 1^+} v(m)\cdot\dfrac{2m^2-\sqrt{1+3m^2}}{2m^2-\sqrt{1+3m^2}} = \lim_{m\to 1^+}\dfrac{4m^4-3m^2-1}{(m^2-1)(2m^2-\sqrt{1+3m^2})} =$
$\lim_{m\to 1^+}\dfrac{(m^2-1)(4m^2+1)}{(m^2-1)(2m^2-\sqrt{1+3m^2})} = \lim_{m\to 1^+}\dfrac{(4m^2+1)}{(2m^2-\sqrt{1+3m^2})} = \infty.$

c. $\lim_{m\to\infty} u(m) = \lim_{m\to\infty}\dfrac{2-\sqrt{\dfrac{1}{m^4}+\dfrac{3}{m^2}}}{1-\dfrac{1}{m^2}} = 2.$

$\lim_{m\to\infty} v(m) = \lim_{m\to\infty}\dfrac{2+\sqrt{\dfrac{1}{m^4}+\dfrac{3}{m^2}}}{1-\dfrac{1}{m^2}} = 2.$

d. The expression $\lim_{m\to\infty} A(m)$ represents the area of the region bounded by the hyperbola and the line $x = 2$. It is given by $2\int_1^2 \sqrt{x^2-1}\,dx = 2\left(\dfrac{x}{2}\sqrt{x^2-1} - \dfrac{1}{2}\ln(x+\sqrt{x^2-1})\right)\Big|_1^2 = 2\sqrt{3} - \ln(2+\sqrt{3}).$

12.4.98

a. The area of the anvil is $A = 4\int_0^p \sqrt{1+y^2}\,dy = 4\int_0^{\tan^{-1}(p)} \sec^3 t\,dt = 2p\sqrt{1+p^2} + 2\ln(\sqrt{1+p^2}+p)$, where this last integral can be evaluated using the techniques of chapter 7 (or a table of integrals.)

The area of R is equal to the area of S when $2 = p\sqrt{1+p^2} + \ln(\sqrt{1+p^2}+p)$. Using a CAS, the result is $p \approx 0.8927$.

b. For R to have twice the area of S, we need $4 = p\sqrt{1+p^2} + \ln(\sqrt{1+p^2}+p)$, which occurs for $p \approx 1.5279$.

Chapter Twelve Review

1

a. False. For example, $x = r\cos t$, $y = r\sin t$ for $0 \le t \le 2\pi$ and $x = r\sin t$, $y = r\cos t$ for $0 \le t \le 2\pi$ generate the same circle.

b. False. Because $e^t > 0$ for all t, this only describes the portion of that line where $x > 0$.

c. True. They both describe the point whose cartesian coordinates are $(3\cos(-3\pi/4), 3\sin(-3\pi/4)) = (-3\cos(\pi/4), -3\sin(\pi/4)) = (-3/\sqrt{2}, -3/\sqrt{2})$.

d. False. The given integral counts the inner loop twice.

e. True. This follows because the equation $0 - x^2/4 = 1$ has no real solutions.

f. True. Note that the given equation can be written as $(x-1)^2 + 4y^2 = 4$, or $\dfrac{(x-1)^2}{4} + y^2 = 1$.

2 $y = 3 + 2\sin t = 3 + 2(x-1) = 2x + 1$, which implies that $y = 2x + 1$. This is a line segment starting at $(1, 3)$ and ending at $(2, 5)$. The curve $x = 1 + \sin t$, $y = 3 + 2\sin t$, for $\pi/2 \le t \le \pi$, has the opposite orientation, starting at $(2, 5)$ and ending at $(1, 3)$.

3 Note that $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$. This represents an ellipse generated counterclockwise.

4 Note that $(x+3)^2 + (y-6)^2 = 1$. This is the right half of a circle of radius 1 centered at $(-3, 6)$. It is generated clockwise.

5 $x = y^2$, so $y = \sqrt{x}$. This is a segment of a parabola opening to the right, starting at $(4, 2)$ and ending at $(9, 3)$.

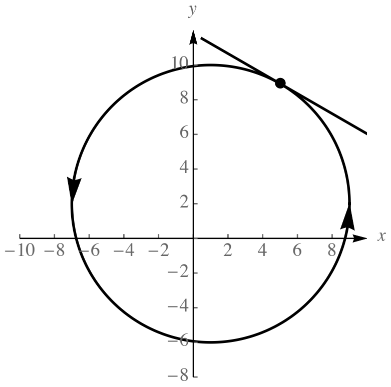
6 $1 + \cot^2 t = \csc^2 t$, so $1 - x = y - 1$, so $y = 2 - x$. This is a line segment starting at $(-1, 3)$ and ending at $(0, 2)$.

7

a. $(x-1)^2 + (y-2)^2 = 64\cos^2 t + 64\sin^2 t = 64$. This is the circle centered at $(1, 2)$ with radius 8.

b. $\left.\frac{dy}{dx}\right|_{t=\pi/3} = -\left.\frac{\cos t}{\sin t}\right|_{t=\pi/3} = -\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$.

c. Note that the point on the curve corresponding to $t = \pi/3$ has coordinates $(5, 2 + 4\sqrt{3}) \approx (5, 8.9)$.

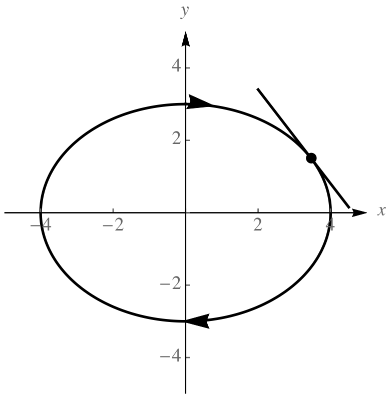


8

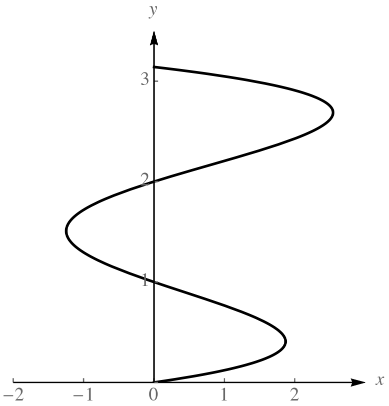
a. $\frac{x^2}{16} + \frac{y^2}{9} = \sin^2 2t + \cos^2 2t = 1$, so this is an ellipse.

b. $\left.\frac{dy}{dx}\right|_{t=\pi/6} = -\left.\frac{6\sin 2t}{8\cos 2t}\right|_{t=\pi/6} = -\frac{3\sqrt{3}}{4}$.

c. Note that the point on the curve corresponding to $t = \pi/6$ has coordinates $(2\sqrt{3}, 3/2) \approx (3.5, 1.5)$.



9 Let $y = t$, then $x = 5(t-1)(t-2)\sin t$.



10

a. $x = (-y)^2 + 4$, or $x = y^2 + 4$.

b. This is a segment of the upper half of a parabola opening to the right, starting at the point (8,2) and ending at (4,0).

c. The slope at $(5,1)$ is $\left.\dfrac{dy}{dx}\right|_{t=-1} = -\left.\dfrac{1}{2t}\right|_{t=-1} = \dfrac{1}{2}$.

11

a. $x^2 + (y+1)^2 = 9\cos^2(-t) + 9\sin^2(-t) = 9$, so $x^2 + (y+1)^2 = 9$.

b. This is the lower half of a circle of radius 3 centered at $(0,-1)$ which starts at $(3,-1)$ and ends at $(-3,-1)$.

c. The slope at $(0,-4)$ is $\left.\dfrac{dy}{dx}\right|_{t=\pi/2} = -\left.\dfrac{3\cos(-t)}{3\sin(-t)}\right|_{t=\pi/2} = 0$.

12

a. $y = 16\ln t = 16x$.

b. This is a line segment starting at (0,0) and ending at (2,32).

c. The slope at $(1,16)$ is the slope of the line segment, which is $\dfrac{32}{2} = 16$.

13 $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\sin t}{1-\cos t}$. At $t = \pi/6$, the slope of the tangent line is $\dfrac{1}{2-\sqrt{3}} = 2+\sqrt{3}$. So the equation of the tangent line is $y - (1-\sqrt{3}/2) = (2+\sqrt{3})(x - (\pi/6 - 1/2))$, or $y = (2+\sqrt{3})x + \left(2 - \dfrac{\pi}{3} - \dfrac{\pi\sqrt{3}}{6}\right)$.

At $t = 2\pi/3$, the slope of the tangent line is $\dfrac{\sqrt{3}}{3}$, so the equation of the tangent line is $y - \dfrac{3}{2} = \dfrac{\sqrt{3}}{3}\left(x - \left(\dfrac{2\pi}{3} - \dfrac{\sqrt{3}}{2}\right)\right)$, or $y = \dfrac{x}{\sqrt{3}} + 2 - \dfrac{2\pi}{3\sqrt{3}}$.

14 Let $y = t$, and $x = t^3 + t + 1$, for $0 \le t \le 2$.

15 From P to Q, we use $(x(t), y(t)) = tQ + (1-t)P = (t,t) + (t-1, 0) = (2t-1, t)$. So $x(t) = 2t-1$, $y(t) = t$, $0 \le t \le 1$.

From Q to P, we use $(x(t), y(t)) = tP + (1-t)Q = (-t, 0) + (1-t, 1-t) = (1-2t, 1-t)$, for $0 \le t \le 1$. Thus $x(t) = 1-2t$, $y(t) = 1-t$, $0 \le t \le 1$.

16 $x = t$, $y = t^3 + 2t$, $0 \le t \le 2$.

17 $x = 3\sin t$, $y = 3\cos t$, $0 \le t \le 2\pi$.

18 $x = 3\cos t$, $y = 2\sin t$, $-\pi/2 \le t \le \pi/2$.

19 The area is $\displaystyle\int_0^1 (t - t^3)(2t)\,dt = \int_0^1 (2t^2 - 2t^4)\,dt = \left(\frac{2t^3}{3} - \frac{2t^5}{5}\right)\Bigg|_0^1 = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$.
t

20 The area is $\displaystyle\int_{\pi/2}^0 (\sin 2t)(-\sin t)\,dt = \int_0^{\pi/2} 2\sin^2 t\cos t\,dt = \frac{2\sin^3 t}{3}\Bigg|_0^{\pi/2} = \frac{2}{3}$.

21 The volume is $\displaystyle 2\int_0^{\pi/4} 2\pi\cos t\sqrt{\cos^2 t + 4\sin^2 2t}\,dt \approx 9.1$.

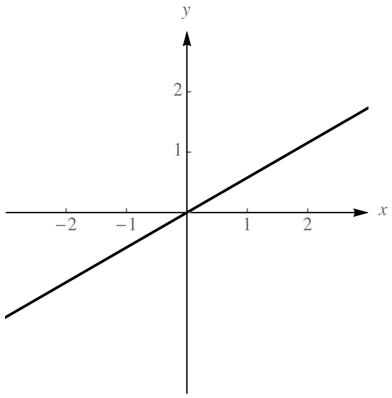
22 The arc length is

$$\begin{aligned}&\int_0^{\pi/3} \sqrt{(2e^{2t} + 3e^{2t}\cos 3t)^2 + (2e^{2t}\cos 3t - 3e^{2t}\sin 3t)^2}\,dt\\ &= \int_0^{\pi/3} \sqrt{13}e^{2t}\,dt = \frac{\sqrt{13}}{2}e^{2t}\Bigg|_0^{\pi/3} = \frac{\sqrt{13}}{2}(e^{2\pi/3} - 1).\end{aligned}$$

23 The arc length is

$$\begin{aligned}&\int_0^{\pi/4} \sqrt{(-2\sin 2t)^2 + (2 - 2\cos 2t)^2}\,dt = 2\sqrt{2}\int_0^{\pi/4} \sqrt{1 - \cos 2t}\,dt\\ &= 2\sqrt{2}\int_0^{\pi/4} \sqrt{1 - (1 - 2\sin^2 t)}\,dt = 4\int_0^{\pi/4} \sin t\,dt = -4\cos t\Bigg|_0^{\pi/4} = 4 - 2\sqrt{2}.\end{aligned}$$

24



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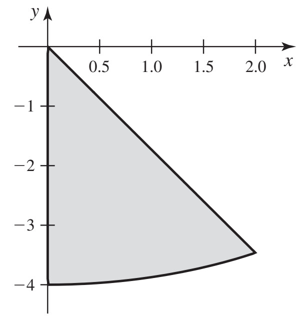
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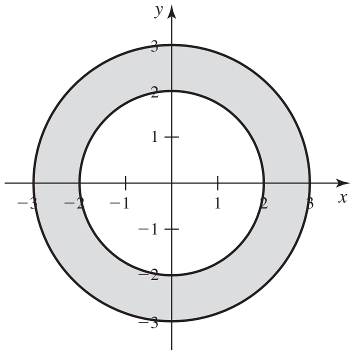
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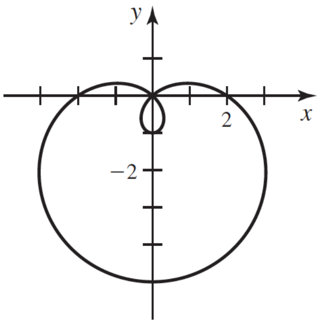

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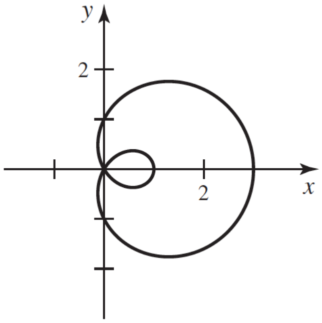

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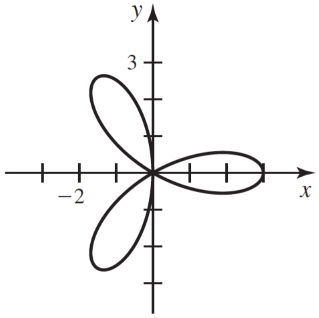

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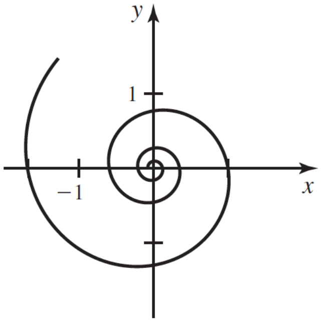

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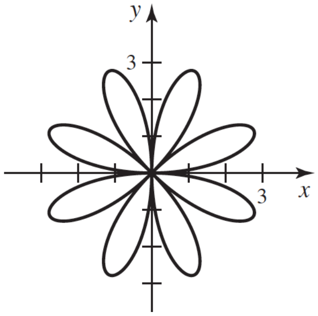

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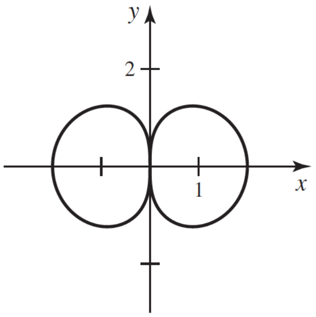


33 Liz should choose the cardioid, which is $r = 1 - \sin\theta$.

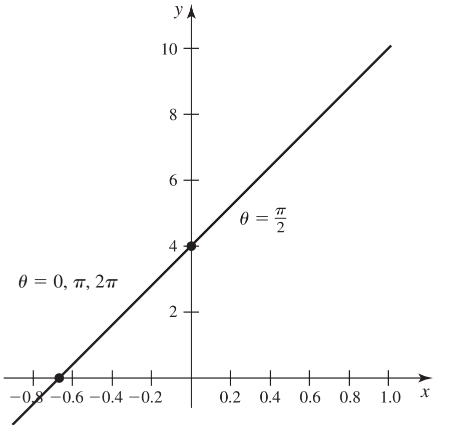
34 Jake should send $r^2 = \cos 2\theta$.

35 Letting $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$, we have $x^2 + y^2 + 2y - 6x = 0$, which can be written as $x^2 - 6x + 9 + y^2 + 2y + 1 = 10$, or $(x-3)^2 + (y+1)^2 = 10$, so this is a circle of radius $\sqrt{10}$ centered at $(3, -1)$.

36

a. We can write the equation as $r\sin\theta - 6r\cos\theta = 4$, or $y - 6x = 4$. This is a straight line with slope 6 and y-intercept 4.

b.



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c. Note that $\sin\theta - 6\cos\theta = 0$ for $\theta = \tan^{-1}(6)$. The whole curve can be generated for $\tan^{-1}(6) - \pi < \theta < \tan^{-1}(6) + \pi$.

37 If we let $r = 1 + \cos t$, then $x = r\cos t$ and $y = r\sin t$. The curve $r = 1 + \cos t$ is a cardioid.

38 We have $r\cos\theta = r^2\sin^2\theta$, so $r = \cot\theta\csc\theta$. The whole parabola is described by $0 < \theta < \pi$.

39 If $x = r\cos\theta$ and $y = r\sin\theta$, then $(r\cos\theta - 4)^2 + r^2\sin^2\theta = 16$, so $r^2\cos^2\theta - 8r\cos\theta + 16 + r^2\sin^2\theta = 16$, so $r^2 = 8r\cos\theta$, and thus $r = 8\cos\theta$. The complete circle can be described by $-\pi/2 \le \theta \le \pi/2$.

40

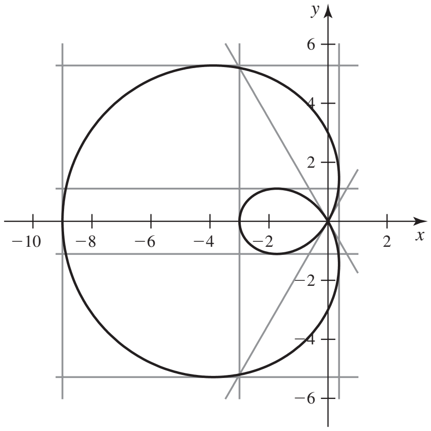
a. $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{6\sin\theta\cdot\sin\theta + (3 - 6\cos\theta)\cos\theta}{6\sin\theta\cos\theta - (3 - 6\cos\theta)\sin\theta} = \frac{6 - 12\cos^2\theta + 3\cos\theta}{12\sin\theta\cos\theta - 3\sin\theta}.$$

This is 0 when $\cos^2\theta - \frac{1}{4}\cos\theta - \frac{1}{2} = 0$, which (by the quadratic formula) occurs where $\cos\theta = \frac{1}{8} \pm \frac{\sqrt{33}}{8}$, so for $\theta \approx 0.568$, 2.206, 4.078, and 5.715.

The denominator is 0 when $\sin\theta = 0$ and when $12\cos\theta - 3 = 0$, or $\theta = \pm\cos^{-1}(1/4)$.

b. The curve is at the origin for $\theta = \pm\pi/3$, and because $\tan\pi/3 = \sqrt{3}$, the tangent lines have the equations $y = \pm\sqrt{3}x$.

c.



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41

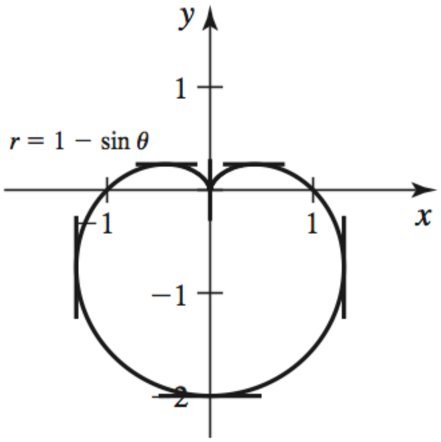
a. The curve intersects the origin at $\theta = \pi/2$, and there is a vertical tangent at $(0, \pi/2)$.

$$\frac{dy}{dx} = \frac{-\cos\theta\sin\theta + (1-\sin\theta)\cos\theta}{-\cos^2\theta - (1-\sin\theta)\sin\theta} = \frac{\cos\theta(1-2\sin\theta)}{\sin^2\theta - \cos^2\theta - \sin\theta} = \frac{\cos\theta(1-2\sin\theta)}{2\sin^2\theta - \sin\theta - 1}.$$

There are horizontal tangents when $\sin\theta = 1/2$, which occurs for $\theta = \pi/6, 5\pi/6$, and when $\cos\theta = 0$ (but not $\sin\theta = 1$) which occurs at $\theta = 3\pi/2$. So the horizontal tangents are at $(1/2, \pi/6)$, $((1/2, 5\pi/6)$, and $(2, 3\pi/2)$. There are vertical tangents when $2\sin^2\theta - \sin\theta - 1 = (2\sin\theta + 1)(\sin\theta - 1) = 0$, or $\theta = 7\pi/6$ and $\theta = 11\pi/6$. The vertical tangents are thus at $(3/2, 7\pi/6)$, $(3/2, 11\pi/6)$, and $(0, \pi/2)$, as well as the aforementioned $(0, \pi/2)$.

b. This is undefined.

c.



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42 $\sqrt{3} = 2\cos 2\theta$, so $\cos 2\theta = \frac{\sqrt{3}}{2}$, so $2\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$. Thus $\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$. The 4 intersection points are $\left(\sqrt{3}, \frac{\pi}{12}\right)$, $\left(\sqrt{3}, \frac{11\pi}{12}\right)$, $\left(\sqrt{3}, \frac{13\pi}{12}\right)$, $\left(\sqrt{3}, \frac{23\pi}{12}\right)$. Using symmetry, we see that there are also intersection points at $\left(\sqrt{3}, \frac{5\pi}{12}\right)$, $\left(\sqrt{3}, \frac{7\pi}{12}\right)$, $\left(\sqrt{3}, \frac{17\pi}{12}\right)$, $\left(\sqrt{3}, \frac{19\pi}{12}\right)$

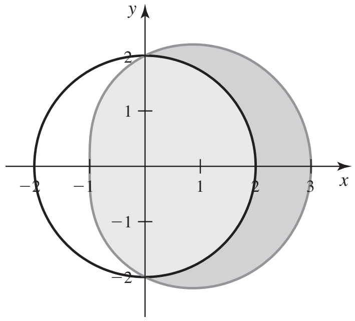
43 $\sqrt{\cos 3t} = \sqrt{\sin 3t}$, so for $t \neq 0$, $\tan 3t = 1$, so $3t = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}$. So $t = \frac{\pi}{12}, \frac{9\pi}{12}, \frac{17\pi}{12}$. The curves intersect at $\left(\frac{\pi}{12}, \frac{1}{\sqrt[4]{2}}\right)$, $\left(\frac{3\pi}{4}, \frac{1}{\sqrt[4]{2}}\right)$, $\left(\frac{17\pi}{12}, \frac{1}{\sqrt[4]{2}}\right)$. Also, they intersect at $(0, 0)$.

44 The area is given by $\int_0^{\pi/4} \frac{1}{2}\left(9\sin^2 4\theta\right)\, d\theta = \frac{9}{4}\int_0^{\pi/4} (1 - \cos 8\theta)\, d\theta = \frac{9}{4}\left(\theta - \frac{\sin 8\theta}{8}\right)\Big|_0^{\pi/4} = \frac{9\pi}{16}$.

45 The area is

$$\begin{aligned}
2\int_{7\pi/6}^{3\pi/2} \frac{1}{2}(1 + 2\sin t)^2\, dt &= \int_{7\pi/6}^{3\pi/2} (1 + 4\sin t + 4\sin^2 t)\, dt \\
&= \int_{7\pi/6}^{3\pi/2} (1 + 4\sin t + 2 - 2\cos 2t)\, dt = (3t - 4\cos t - \sin 2t)\Big|_{7\pi/4}^{3\pi/2} \\
&= \left(\frac{9\pi}{2} - 0 - 0\right) - \left(\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right) = \pi - \frac{3\sqrt{3}}{2}.
\end{aligned}$$

46 The curves intersect at $\theta = \pm\pi/2$. By symmetry, the area is twice the area outside the circle and inside the limaçon between 0 and $\pi/2$. We have $A = 2 \cdot \frac{1}{2}\int_0^{\pi/2} ((2 + \cos\theta)^2 - 2^2)\, d\theta = \int_0^{\pi/2} (4\cos\theta + \cos^2\theta)\, d\theta = \left(4\sin\theta + \frac{1}{2}(\cos\theta\sin\theta + \theta)\right)\Big|_0^{\pi/2} = 4 + \frac{\pi}{4}$.



47 $2 = 4\cos 2\theta$, so $\cos 2\theta = \frac{1}{2}$, so $\theta = \pm\frac{\pi}{6}$. The area is $4\int_0^{\pi/6} \frac{1}{2}(4\cos 2\theta - 2)\, d\theta = 4\int_0^{\pi/6} (2\cos 2\theta - 1)\, d\theta = 4(\sin 2\theta - \theta)\Big|_0^{\pi/6} = 4\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) = 2\sqrt{3} - \frac{2\pi}{3}$.

48 $\sqrt{2} + \sin\theta = 3\sin\theta$, so $\sin\theta = \frac{\sqrt{2}}{2}$, so $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$. The area is given by

$$\pi\left(\frac{3}{2}\right)^2 - 2\int_{\pi/4}^{\pi/2} \frac{1}{2}\left(9\sin^2\theta - (\sqrt{2} + \sin\theta)^2\right)\, d\theta.$$

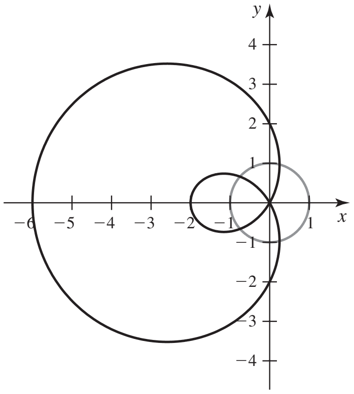
Note that $\left(9\sin^2\theta - (\sqrt{2}+\sin\theta)^2\right) = -2-2\sqrt{2}\sin\theta+8\sin^2\theta = 2-2\sqrt{2}\sin\theta-4\cos 2\theta$. So our area becomes

$$\begin{aligned}\frac{9\pi}{4} + \int_{\pi/4}^{\pi/2} (4\cos 2\theta + 2\sqrt{2}\sin\theta - 2)\,d\theta &= \frac{9\pi}{4} + \left(2\sin 2\theta - 2\sqrt{2}\cos\theta - 2\theta\right)\Big|_{\pi/4}^{\pi/2} \\ &= \frac{9\pi}{4} + (0-0-\pi) - \left(2-2-\frac{\pi}{2}\right) \\ &= \frac{7\pi}{4}.\end{aligned}$$

49 The area is given by $2\int_0^{\pi/2} \frac{1}{2}\left((1+\cos\theta)^2 - (1-\cos\theta)^2\right)\,d\theta = 4\int_0^{\pi/2} \cos\theta\,d\theta = 4\sin\theta\Big|_0^{\pi/2} = 4.$.

50

a.



There are 4 intersection points.

b. Note that $2-4\cos\theta = 1$ for $\theta = \cos^{-1}(1/4) \approx 1.32$, and $2-4\cos\theta = -1$ for $\theta = \cos^{-1}(3/4) \approx 0.73$. The points of intersection (in polar form) are approximately $(1, 1.32)$, $(1, 2\pi-1.32) \approx (1, 4.96)$, $(-1, 0.73)$, and $(-1, 2\pi-0.73) \approx (-1, 5.56)$.

51 $L = \int_0^{2\pi} \sqrt{(3-6\cos\theta)^2 + (6\sin\theta)^2}\,d\theta = \int_0^{2\pi} 3\sqrt{5-4\cos(\theta)}\,d\theta \approx 40.09.$

52 $L = \int_0^{2\pi} \sqrt{(3+2\cos\theta)^2 + (-2\sin\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{13+12\cos\theta}\,d\theta \approx 21.01.$

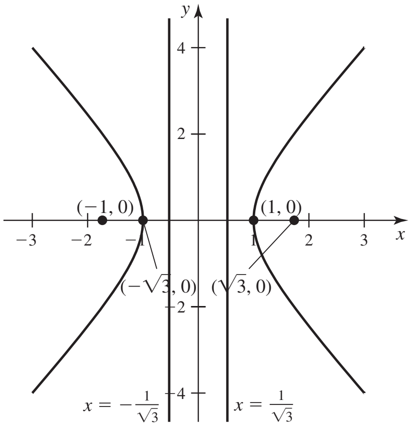
53

a. This represents a hyperbola with $a=1$ and $b=\sqrt{2}$.

b. The vertices are $(\pm 1, 0)$, the foci are $(\pm c, 0)$ where $c^2 = a^2+b^2 = 3$, so they are $(\pm\sqrt{3}, 0)$. The directrices are $x = \frac{\pm a^2}{c} = \frac{\pm 1}{\sqrt{3}}$.

c. The eccentricity is $e = \dfrac{c}{a} = \sqrt{3}$.

d.



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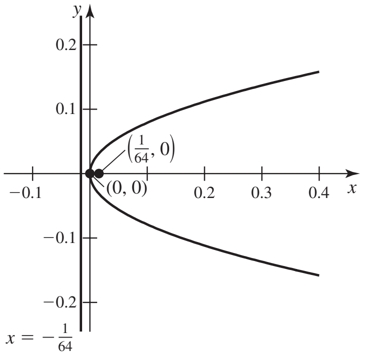
54

a. This represents a parabola.

b. We can write $y^2 = \dfrac{1}{16}x = 4 \cdot \dfrac{1}{64}x$, so $(p, 0) = (\dfrac{1}{64}, 0)$ is the focus, and the directrix is $x = -\dfrac{1}{64}$. The vertex is $(0, 0)$.

c. $e = 1$, because that is the case for all parabolas.

d.



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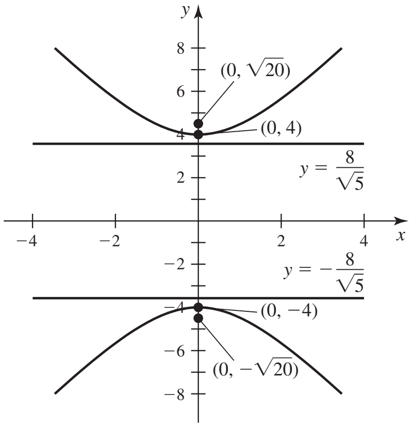
55

a. This can be written as $\dfrac{y^2}{16} - \dfrac{x^2}{4} = 1$. It is a hyperbola with $a = 4$ and $b = 2$.

b. The vertices are $(0, \pm 4)$. The foci are $(0, \pm c)$ where $c^2 = a^2 + b^2 = 16 + 4 = 20$, so they are $(0, \pm\sqrt{20})$. The directrices are $y = \dfrac{\pm a^2}{c} = \dfrac{\pm 16}{\sqrt{20}} = \dfrac{\pm 8}{\sqrt{5}}$.

c. The eccentricity is $e = \dfrac{c}{a} = \dfrac{\sqrt{20}}{4} = \dfrac{\sqrt{5}}{2}$.

d. 

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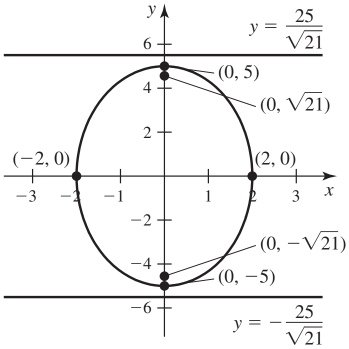
56

a. This represents an ellipse with $a = 5$ and $b = 2$.

b. The vertices are $(0, \pm 5)$. The foci are $(0, \pm c)$ where $c^2 = a^2 - b^2 = 25 - 4 = 21$, so they are $(0, \pm\sqrt{21})$. The directrices are $y = \dfrac{\pm a^2}{c} = \dfrac{\pm 25}{\sqrt{21}}$.

c. The eccentricity is $e = \dfrac{c}{a} = \dfrac{\sqrt{21}}{5}$.

d. 

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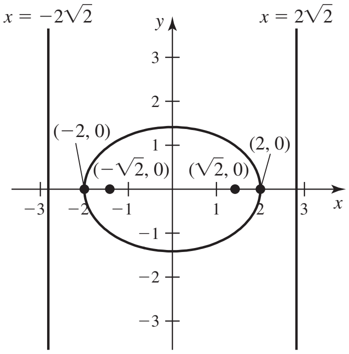
57

a. This can be written as $\dfrac{x^2}{4} + \dfrac{y^2}{2} = 1$, so it is an ellipse with $a = 2$ and $b = \sqrt{2}$.

b. The vertices are $(\pm 2, 0)$. The foci are $(\pm c, 0)$ where $c^2 = a^2 - b^2 = 4 - 2 = 2$, so they are $(\pm\sqrt{2}, 0)$. The directrices are $x = \dfrac{\pm a^2}{c} = \dfrac{\pm 4}{\sqrt{2}} = \pm 2\sqrt{2}$.

c. The eccentricity is $e = \dfrac{c}{a} = \dfrac{\sqrt{2}}{2}$.

d. 

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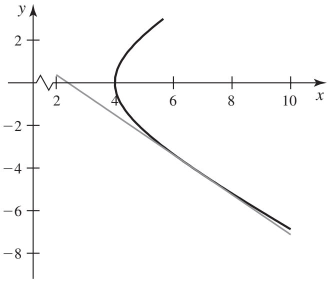
58 $\frac{x}{8} - \frac{2y}{9} \cdot \frac{dy}{dx} = 0$, so at the given point, $\frac{dy}{dx} = -\frac{15}{16}$. The equation of the tangent line is therefore $y + 4 = -\frac{15}{16}\left(x - \frac{20}{3}\right)$, or $y = -\frac{15}{16}x + \frac{9}{4}$.



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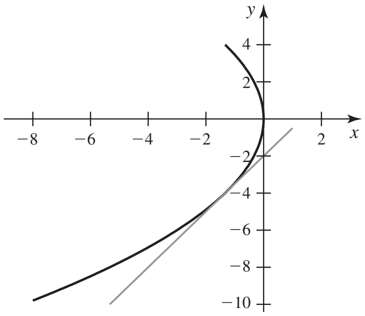
59 $2y\frac{dy}{dx} = -12$, so at the point in question, $\frac{dy}{dx} = 3/2$. So the equation of the tangent line is $y + 4 = \frac{3}{2}\left(x + \frac{4}{3}\right)$, or $y = \frac{3}{2}x - 2$.



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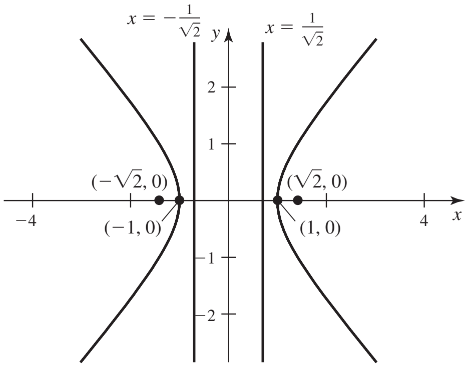
60

a. Recall that $\cos 2\theta = \cos^2\theta - \sin^2\theta$, so $r^2\cos(2\theta) = 1$ becomes $r^2(\cos^2\theta - \sin^2\theta) = x^2 - y^2 = 1$. The curve is a hyperbola.

b. With $a = b = 1$, we have $c^2 = 2$, so the vertices are $(\pm 1, 0)$ and the foci are $(\pm\sqrt{2}, 0)$. The directrices are $x = \pm\frac{a^2}{c} = \pm\frac{1}{\sqrt{2}}$. The eccentricity is $e = \frac{c}{a} = \sqrt{2}$.

c. It does not have the form as in Theorem 11.4 because it does not have a focus at the origin.



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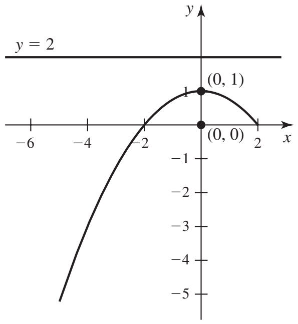
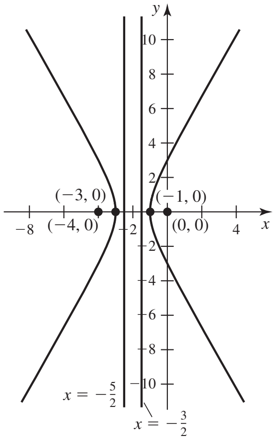
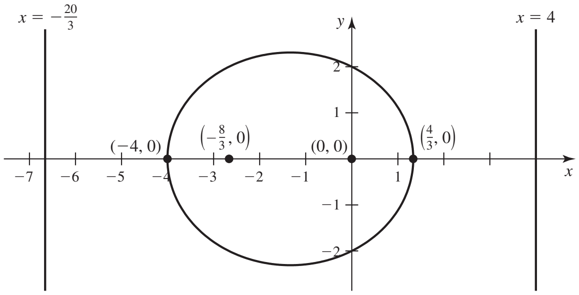
61 The eccentricity is 1, and the directrix is $y = 2$. The vertex is $(0,1)$ and the focus is $(0,0)$.



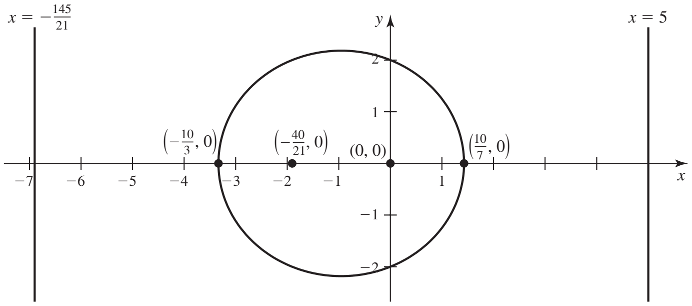
62 The eccentricity is 2, and the directrices are $x = -\dfrac{3}{2}$ and $x = -\dfrac{5}{2}$. The vertices are $(-1,0)$ and $(-3,0)$ and the foci are $(0,0)$ and $(-4,0)$.



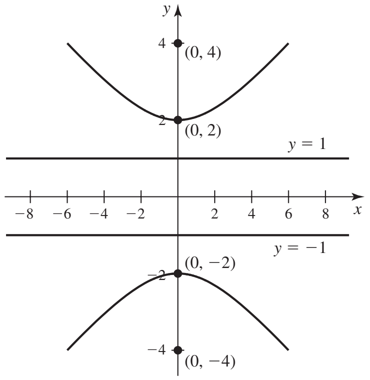
63 The eccentricity is $\dfrac{1}{2}$, and the directrices are $x = 4$ and $x = -\dfrac{20}{3}$. The vertices are $(\dfrac{4}{3},0)$ and $(-4,0)$ and the foci are $(0,0)$ and $(-\dfrac{8}{3},0)$.



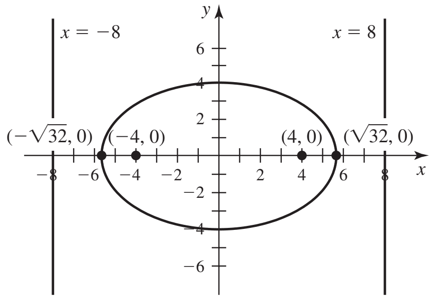
64 The eccentricity is $\dfrac{2}{5}$. The vertices are $(10/7,0)$ and $(-10/3,0)$, so the center is $(-20/21,0)$. The foci are $(0,0)$ and $(-40/21,0)$. The directrices are $x = -\dfrac{145}{21}$ and $x = 5$.



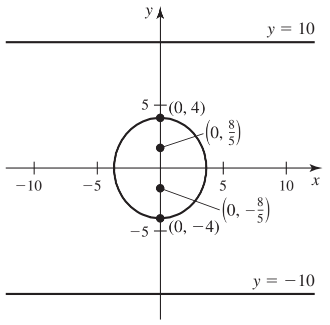
65 Because the center is halfway between the vertices, it is $(0,0)$. We must have $a = 2$ and because $\dfrac{a^2}{c} = d = 1$, we have $c = 4$. So $b^2 = 16-4 = 12$. The hyperbola has equation $\dfrac{y^2}{4} - \dfrac{x^2}{12} = 1$. The eccentricity is $\dfrac{c}{a} = \dfrac{4}{2} = 2$.



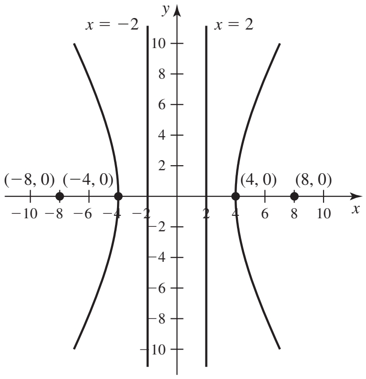
66 Because the center is halfway between the foci, it is $(0,0)$. We must have $c = 4$ and because $\dfrac{a^2}{c} = d = 8$, we have $a^2 = 32$. So $b^2 = a^2 - c^2 = 32 - 16 = 16$. The ellipse has equation $\dfrac{x^2}{32} + \dfrac{y^2}{16} = 1$. The eccentricity is $\dfrac{c}{a} = \dfrac{4}{4\sqrt{2}} = \dfrac{\sqrt{2}}{2}$.



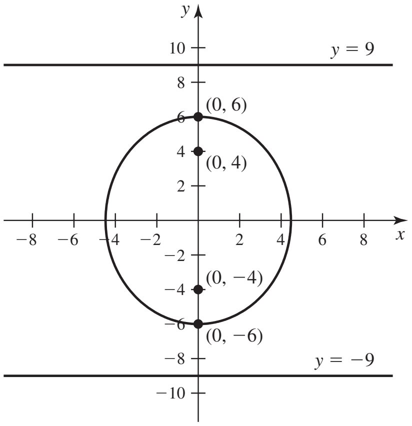
67 Because the center is halfway between the vertices, it is $(0,0)$. We must have $a = 4$ and because $\dfrac{a^2}{c} = d = 10$, we have $c = \dfrac{8}{5}$. So $b^2 = a^2 - c^2 = 16 - \dfrac{64}{25} = \dfrac{336}{25}$. The ellipse has equation $\dfrac{25x^2}{336} + \dfrac{y^2}{16} = 1$. The eccentricity is $\dfrac{c}{a} = \dfrac{8/5}{4} = \dfrac{2}{5}$.



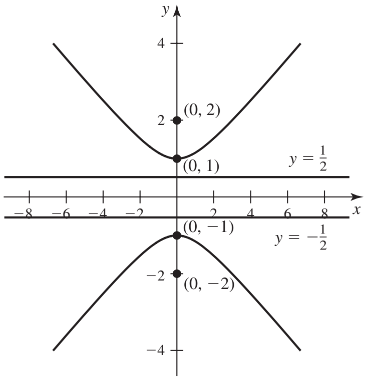
68 Because the center is halfway between the vertices, it is $(0,0)$. We must have $a = 4$ and because $\frac{a^2}{c} = d = 2$, we have $c = 8$. So $b^2 = 64 - 16 = 48$. The hyperbola has equation $\frac{x^2}{16} - \frac{y^2}{48} = 1$. The eccentricity is $\frac{c}{a} = \frac{8}{4} = 2$.



69 We have $a = 6$, $c = 4$ and $e = \frac{c}{a} = \frac{4}{6} = \frac{2}{3}$. Also, $b^2 = a^2 - c^2 = 36 - 16 = 20$, and the equation is $\frac{y^2}{36} + \frac{x^2}{20} = 1$. The vertices are $(\pm 2\sqrt{5}, 0)$. The directrices are $y = \pm\frac{a^2}{c} = \pm 9$.



70 We have $c = 2$, $e = \frac{c}{a} = 2$, so $a = 1$. Also, $b^2 = c^2 - a^2 = 3$, so the equation is $\frac{y^2}{1} - \frac{x^2}{3} = 1$. We have $d = \frac{a^2}{c} = \frac{1}{2}$. The vertices are $(0, \pm 1)$ and the directrices are $y = \pm\frac{1}{2}$.



71 The area of the ellipse in the first quadrant is $\frac{\pi ab}{4}$, so we are seeking θ_0 so that

$$\frac{\pi ab}{8} = \frac{1}{2}\int_0^{\theta_0} \frac{a^2 b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta}\, d\theta = \frac{a^2}{2}\int_0^{\theta_0} \frac{\sec^2\theta}{\frac{a^2}{b^2}\tan^2\theta + 1}\, d\theta.$$

Let $u = \frac{a}{b}\tan\theta$ so that $du = \frac{a}{b}\sec^2\theta\, d\theta$. Then we have

$$\frac{\pi ab}{8} = \frac{ab}{2}\int_0^{\frac{a}{b}\tan\theta_0} \frac{1}{1+u^2}\, du = \frac{ab}{2}\tan^{-1}(\frac{a}{b}\tan(\theta_0)).$$

Note that this equation is satisfied when $\tan(\theta_0) = \frac{b}{a}$, because then the expression on the right-hand side of that equation is $\frac{ab}{2}\cdot\frac{\pi}{4} = \frac{\pi ab}{8}$. So the desired value of m is $\tan(\theta_0) = \frac{b}{a}$.

72

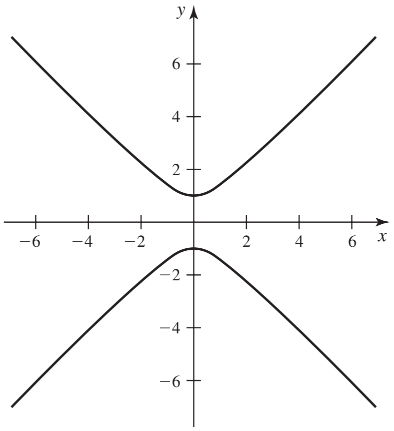
a. The curves are tangent when there is only one point of intersection in the first quadrant. This occurs when $x^2 - p^2x^4 = 1$ has only one solution. This quadratic-type equation $-p^2(x^2)^2 + (x^2) - 1 = 0$ has solution $x^2 = \frac{1 \pm \sqrt{1-4p^2}}{2p^2}$, and the discriminant $1 - 4p^2$ is 0 for $p = 1/2$.

b. The two curves intersect for $x^2 = \frac{1 \pm 0}{2p^2} = 2$, so for $x = \sqrt{2}$. The corresponding value for y is $\frac{1}{2}(\sqrt{2})^2 = 1$.

c. Using the same line of reasoning, we seek the value of p so that $\frac{x^2}{a^2} - \frac{p^2x^4}{b^2} = 1$, which yields the quadratic-type equation $p^2a^2(x^2)^2 - b^2(x^2) + a^2b^2 = 0$. The discriminant is 0 when $b^4 - 4p^2a^4b^2 = 0$, which occurs when $p = \frac{b}{2a^2}$. The point of intersection is $x = \sqrt{2}a$. The corresponding value of y is $px^2 = b$.

73 Note that $Q = (a\cos\theta, a\sin\theta)$ and $R = (b\cos\theta, b\sin\theta)$, where θ is the angle formed by l and the x-axis. Then $P = (a\sin\theta, b\cos\theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, because it satisfies that equation.

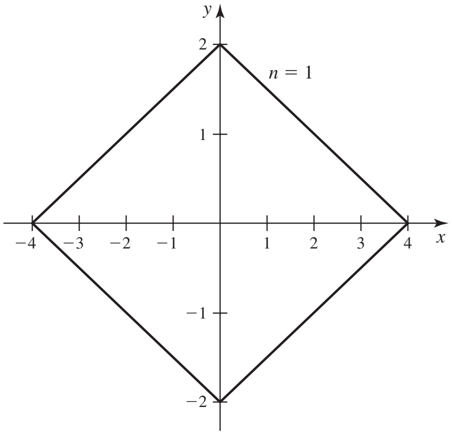
74

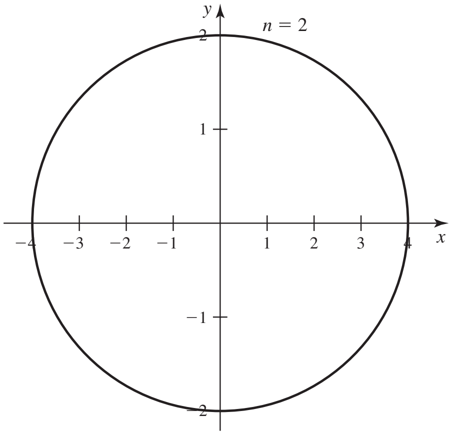


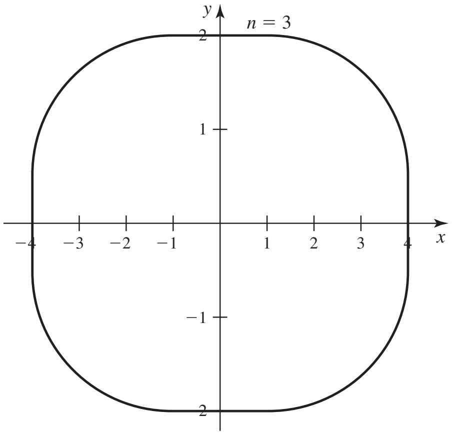
75

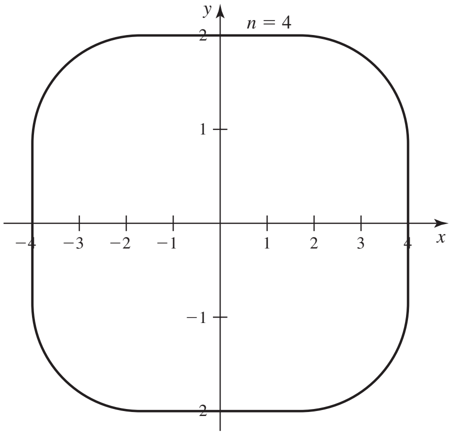
a. Let $\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$ Let $x = a \cdot \operatorname{sgn}(\cos t)\,|\cos(t)|^{2/n}$ and $y = b \cdot \operatorname{sgn}(\sin(t))\,|\sin(t)|^{2/n}$.

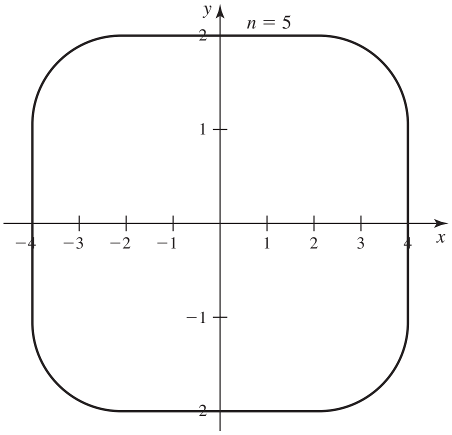
b.

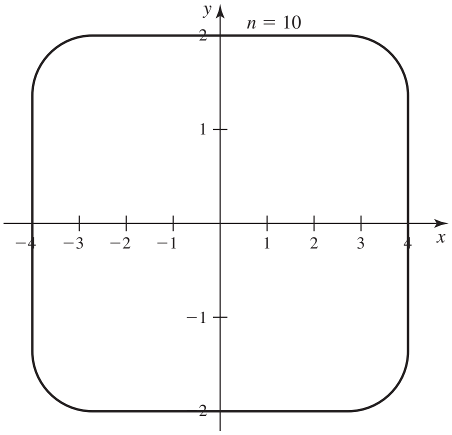












c. As n increases from near 0 to near 1, the curves change from star-shaped to a rectangular shape with corners at $(\pm a, 0)$ and $(0, \pm b)$. As n increases from 1 on, the curves become more rectangular with corners at $(\pm a, \pm b)$.

76 Suppose that $a^2 + c^2 = b^2 + d^2$, and that $ab + cd = 0$. Note that

$$x^2 + y^2 = a^2 \cos^2 t + 2ab \sin t \cos t + b^2 \sin^2 t + c^2 \cos^2 t + 2cd \sin t \cos t + d^2 \sin^2 t,$$

which can be rewritten as

$$(a^2 + c^2)\cos^2 t + (b^2 + d^2)\sin^2 t + (2ab + 2cd)\sin t \cos t.$$

Because $b^2 + d^2 = a^2 + c^2$ and because $2ab + 2cd = 0$, we can write this as

$$(a^2 + c^2)(\cos^2 t + \sin^2 t) = R^2,$$

so we have the circle $x^2 + y^2 = R^2$, as desired.

Chapter 13

Vectors and the Geometry of Space

13.1 Vectors in the Plane

13.1.1 The coordinates of a point determine its location, but a given point has no width or breadth, so it has no size or direction. A nonzero vector has size (magnitude) and direction, but it has no location in the sense that it can be translated to a different initial point and be considered the same vector.

13.1.2 A position vector is one with its tail at the origin.

13.1.3 Two vectors are equal if they have the same magnitude and direction. Given a position vector, any translation of that vector to a different initial point yields an equivalent vector. Because there are infinitely many such translations which don't change the given vector's direction or magnitude, there are infinitely many vectors equivalent to the given one.

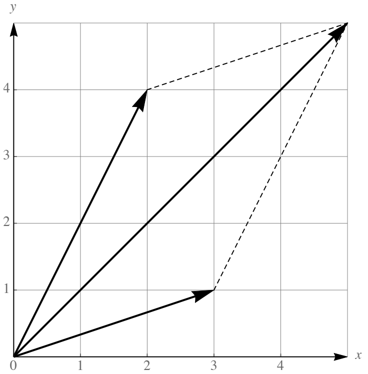
13.1.4 $\overrightarrow{PQ} = \langle 4, 0\rangle$ and $\overrightarrow{QP} = \langle -4, 0\rangle$.

13.1.5 $\mathbf{u} + \mathbf{v} = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle = \langle u_1 + v_1, u_2 + v_2\rangle$.

13.1.6 $\pm\dfrac{\langle 2, 3\rangle}{\sqrt{4+9}} = \pm\left\langle \dfrac{2}{\sqrt{13}}, \dfrac{3}{\sqrt{13}}\right\rangle$.

13.1.7 No. $|\langle 1, 1\rangle| = \sqrt{2} \neq 1$, so $\langle 1, 1\rangle$ is not a unit vector.

13.1.8 $\langle 3, 1\rangle + \langle 2, 4\rangle = \langle 5, 5\rangle$.



13.1.9 $|\mathbf{v}| = |\langle v_1, v_2\rangle| = \sqrt{v_1^2 + v_2^2}$.

13.1.10 $\mathbf{v} = \langle v_1, v_2\rangle = v_1\mathbf{i} + v_2\mathbf{j}$.

13.1.11 If $P(p_1, p_2)$ and $Q(q_1, q_2)$ are given, then $\left|\overrightarrow{PQ}\right| = |\langle q_1 - p_1, q_2 - p_2\rangle| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$.

13.1.12 The speed is $\left|\langle 2\sqrt{2}, 2\sqrt{2}\rangle\right| = \sqrt{8+8} = 4$ ft/s and the heading of the kayak is 45° north of east.

13.1.13 The vectors in choices a, c, and e are all equal to $\overrightarrow{CE}$.

13.1.14 The vectors in choices b, c, and e are equal to $\overrightarrow{BK}$.

13.1.15

a. $3\mathbf{v}$ b. $2\mathbf{u}$ c. $-3\mathbf{u}$ d. $-2\mathbf{u}$ e. $\mathbf{v}$

13.1.16

a. $2\mathbf{v}$ b. $-2\mathbf{v}$ c. $3\mathbf{u}$ d. $-5\mathbf{v}$ e. $-3\mathbf{u}$

13.1.17

a. $3\mathbf{u} + 3\mathbf{v}$ b. $\mathbf{u} + 2\mathbf{v}$ c. $2\mathbf{u} + 5\mathbf{v}$ d. $-2\mathbf{u} + 3\mathbf{v}$ e. $3\mathbf{u} + 2\mathbf{v}$

f. $-3\mathbf{u} - 2\mathbf{v}$ g. $-2\mathbf{u} - 4\mathbf{v}$ h. $\mathbf{u} - 4\mathbf{v}$ i. $-\mathbf{u} - 6\mathbf{v}$

13.1.18

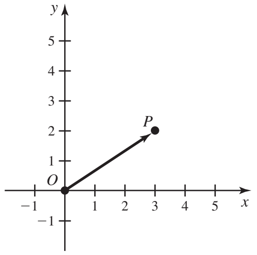
a. $\mathbf{u} + 3\mathbf{v}$ b. $\mathbf{u} + 3\mathbf{v}$ c. $2\mathbf{u} + 2\mathbf{v}$ d. $2\mathbf{u} - 3\mathbf{v}$ e. $-\mathbf{u} - 2\mathbf{v}$

f. $4\mathbf{u} + 4\mathbf{v}$ g. $-\mathbf{u} + 2\mathbf{v}$ h. $-\mathbf{u} + 2\mathbf{v}$ i. $2\mathbf{u} - 4\mathbf{v}$

13.1.19

a. $\overrightarrow{OP}$

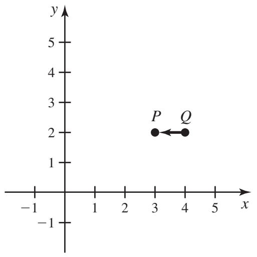
i.



ii. $|3\mathbf{i} + 2\mathbf{j}| = \sqrt{13}$.

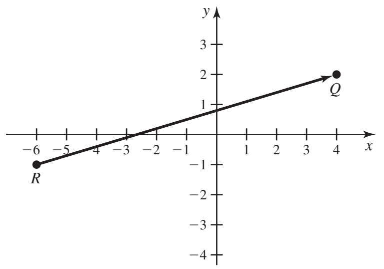
b. $\overrightarrow{QP}$

i.



ii. $|-\mathbf{i} + 0 \cdot \mathbf{j}| = 1$.

c. $\overrightarrow{RQ}$



i.

ii. $|10\mathbf{i} + 3\mathbf{j}| = \sqrt{109}$.

13.1.20

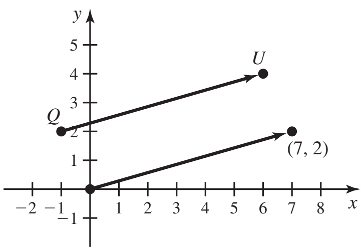
a. $\overrightarrow{AB} = \langle 6, 16\rangle - \langle -2, 0\rangle = \langle 8, 16\rangle$.

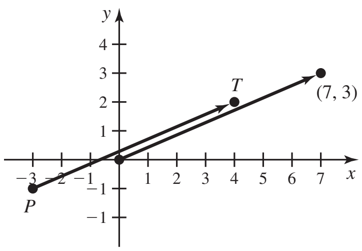
b. $\overrightarrow{AC} = \langle 1, 4\rangle - \langle -2, 0\rangle = \langle 3, 4\rangle$.

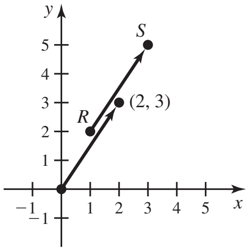
c. $\overrightarrow{EF} = \left\langle 3\sqrt{2}, -4\sqrt{2}\right\rangle - \left\langle \sqrt{2}, \sqrt{2}\right\rangle = \left\langle 2\sqrt{2}, -5\sqrt{2}\right\rangle$.

d. $\overrightarrow{CD} = \langle 5, 4\rangle - \langle 1, 4\rangle = \langle 4, 0\rangle$.

13.1.21 $\overrightarrow{QU} = \langle 7, 2\rangle$, $\overrightarrow{PT} = \langle 7, 3\rangle$, $\overrightarrow{RS} = \langle 2, 3\rangle$.







13.1.22 $\overrightarrow{PQ} = \langle 2, 3\rangle$, $\overrightarrow{RS} = \langle 2, 3\rangle$, and $\overrightarrow{TU} = \langle 2, 2\rangle$, so $\overrightarrow{PQ} = \overrightarrow{RS}$.

13.1.23 $\overrightarrow{QT} = \langle 5, 0\rangle$, while $\overrightarrow{SU} = \langle 3, -1\rangle$.

13.1.24 $\mathbf{u} + \mathbf{v} = \langle 4, -2\rangle + \langle -4, 6\rangle = \langle 0, 4\rangle$.

13.1.25 $\mathbf{w} - \mathbf{u} = \langle 0, 8\rangle - \langle 4, -2\rangle = \langle -4, 10\rangle$.

13.1.26 $2\mathbf{u} + 3\mathbf{v} = 2\langle 4, -2\rangle + 3\langle -4, 6\rangle = \langle -4, 14\rangle$.

13.1.27 $10\mathbf{u} - 3\mathbf{v} + \mathbf{w} = 10\langle 4, -2\rangle - 3\langle -4, 6\rangle + \langle 0, 8\rangle = \langle 52, -30\rangle$.

13.1.28 $|\mathbf{u} + \mathbf{v} + \mathbf{w}| = |\langle 3, -4\rangle + \langle 1, 1\rangle + \langle -1, 0\rangle| = |\langle 3, -3\rangle| = 3\sqrt{2}$.

13.1.29 $|-2\mathbf{v}| = |\langle -2, -2\rangle| = \sqrt{4+4} = 2\sqrt{2}$.

13.1.30 Two vectors parallel to $\mathbf{u}$ with magnitude four times that of $\mathbf{u}$ are the vectors $\pm 4\mathbf{u} = \pm 4\left\langle 3, -4\right\rangle = \pm\left\langle 12, -16\right\rangle$. So they are $\langle 12, -16\rangle$ and $\langle -12, 16\rangle$.

13.1.31 $|\mathbf{u} - \mathbf{v}| = |\langle 3, -4\rangle - \langle 1, 1\rangle| = |\langle 2, -5\rangle| = \sqrt{29}$. $|\mathbf{w} - \mathbf{u}| = |\langle -1, 0\rangle - \langle 3, -4\rangle| = |\langle -4, 4\rangle| = \sqrt{32} = 4\sqrt{2}$. $\mathbf{w} - \mathbf{u}$ has the greater magnitude.

13.1.32 $\dfrac{\langle -6, 8\rangle}{10} = \left\langle -\dfrac{3}{5}, \dfrac{4}{5}\right\rangle$.

13.1.33 $\langle -5, 12\rangle = 13\left\langle -\frac{5}{13}, \frac{12}{13}\right\rangle$.

13.1.34 $\overrightarrow{PQ} = \langle 4, -3\rangle = 5\left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle$.

13.1.35 $6\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle = \left\langle 3, 3\sqrt{3}\right\rangle$.

13.1.36 $-10\left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle = \langle -6, 8\rangle$.

13.1.37 $\mathbf{u} = 3\frac{\langle 5, -12\rangle}{\sqrt{25+144}} = \frac{3}{13}\langle 5, -12\rangle$.

13.1.38 $\mathbf{u} = -20\frac{\langle 6, -8\rangle}{10} = \langle -12, 16\rangle$.

13.1.39 $10 \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = 10 \cdot \frac{1}{\sqrt{9+4}} \cdot \langle 3, -2\rangle = \left\langle \frac{30}{\sqrt{13}}, -\frac{20}{\sqrt{13}}\right\rangle$.

13.1.40

a. $3\langle 8, 15\rangle = \langle 24, 45\rangle$.

b. $3\frac{\langle 8, 15\rangle}{\sqrt{64+225}} = 3\left\langle \frac{8}{17}, \frac{15}{17}\right\rangle = \left\langle \frac{24}{17}, \frac{45}{17}\right\rangle$.

13.1.41 $\overrightarrow{QR} = \langle 2, 6\rangle - \langle 3, -4\rangle = \langle -1, 10\rangle = -\mathbf{i} + 10\mathbf{j}$.

13.1.42 $\overrightarrow{PQ} = \langle 3, -4\rangle - \langle -4, 1\rangle = \langle 7, -5\rangle = 7\mathbf{i} - 5\mathbf{j}$.

13.1.43 $\mathbf{u} = \frac{\overrightarrow{PR}}{\left|\overrightarrow{PR}\right|} = \frac{\langle 2, 6\rangle - \langle -4, 1\rangle}{\left|\overrightarrow{PR}\right|} = \frac{\langle 6, 5\rangle}{\sqrt{36+25}} = \left\langle 6/\sqrt{61}, 5/\sqrt{61}\right\rangle$ is one unit vector, while the other is $-\mathbf{u} = \left\langle -6/\sqrt{61}, -5/\sqrt{61}\right\rangle$

13.1.44 $\mathbf{u} = \frac{\overrightarrow{QR}}{\left|\overrightarrow{QR}\right|} = \frac{\langle -1, 10\rangle}{\sqrt{101}} = -\frac{1}{\sqrt{101}}\mathbf{i} + \frac{10}{\sqrt{101}}\mathbf{j}$.

13.1.45 $\overrightarrow{QP} = \langle -4, 1\rangle - \langle 3, -4\rangle = \langle -7, 5\rangle$. A unit vector parallel to $\overrightarrow{QP}$ is $\frac{1}{\sqrt{74}}\langle -7, 5\rangle$. So the desired vectors are $\frac{4}{\sqrt{74}}\langle -7, 5\rangle$ and $-\frac{4}{\sqrt{74}}\langle -7, 5\rangle$.

13.1.46 Using the result of the previous problem, the two vectors are $4\left\langle 6/\sqrt{61}, 5/\sqrt{61}\right\rangle = \left\langle 24/\sqrt{61}, 20/\sqrt{61}\right\rangle$ and $-4\left\langle 6/\sqrt{61}, 5/\sqrt{61}\right\rangle = \left\langle -24/\sqrt{61}, -20/\sqrt{61}\right\rangle$.

13.1.47

a. Because the magnitude of $\mathbf{v}$ is $\sqrt{36+64} = 10$, the two desired vectors are $\langle 6/10, -8/10\rangle = \langle 3/5, -4/5\rangle$ and $\langle -3/5, 4/5\rangle$.

b. If the magnitude of $\mathbf{v}$ is 1, then $\sqrt{\frac{1}{9} + b^2} = 1$, so $b^2 = \frac{8}{9}$, so $b = \pm\frac{2\sqrt{2}}{3}$.

c. If the magnitude of $\mathbf{w}$ is 1, then $\sqrt{a^2 + \dfrac{a^2}{9}} = 1$, so $\dfrac{10a^2}{9} = 1$, so $a = \pm\dfrac{3}{\sqrt{10}}$.

13.1.48 The Cartesian coordinates of $P$are $(r\cos\theta, r\sin\theta)$ and therefore $\overrightarrow{OP} = \langle r\cos\theta, r\sin\theta\rangle$.

13.1.49 $\left\langle 8\cos\dfrac{5\pi}{6}, 8\sin\dfrac{5\pi}{6}\right\rangle = \left\langle -4\sqrt{3}, 4\right\rangle$.

13.1.50 $\left\langle 40\cos\dfrac{\pi}{4}, 40\sin\dfrac{\pi}{4}\right\rangle = \left\langle 20\sqrt{2}, 20\sqrt{2}\right\rangle$.

13.1.51 $\left\langle 30\cos\left(-\dfrac{\pi}{5}\right), 30\sin\left(-\dfrac{\pi}{5}\right)\right\rangle = \left\langle 15\sqrt{3}, -15\right\rangle$.

13.1.52 $\left\langle 100\cos\left(-\dfrac{\pi}{4}\right), 100\sin\left(-\dfrac{\pi}{4}\right)\right\rangle = \left\langle 50\sqrt{2}, -50\sqrt{2}\right\rangle$.

13.1.53

a. $\mathbf{v_a} = \langle -320, 0\rangle$ and $\mathbf{w} = \langle -40\cos 45°, -40\sin 45°\rangle = \left\langle -\dfrac{40}{\sqrt{2}}, -\dfrac{40}{\sqrt{2}}\right\rangle = \left\langle -20\sqrt{2}, -20\sqrt{2}\right\rangle$. Therefore the plane's vector is given by $\mathbf{v_g} = \mathbf{v_a} + \mathbf{w} = (-320 - 20\sqrt{2})\mathbf{i} - 20\sqrt{2}\mathbf{j}$.

b. The magnitude of $\mathbf{v_g}$ is $\sqrt{(-320 - 20\sqrt{2})^2 + (-20\sqrt{2})^2} \approx 349.43$ miles per hour.
$\theta = \tan^{-1}\left(\dfrac{20\sqrt{2}}{320 + 20\sqrt{2}}\right) \approx 0.0810$ radians, or about 4.64 degrees south of west.

13.1.54

a. $\mathbf{v_a} = \langle 400, 0\rangle$; $\mathbf{v_g} = \langle 420\cos 5°, 420\sin 5°\rangle \approx \langle 418.4, 36.6\rangle$; $\mathbf{w} = \mathbf{v_g} - \mathbf{v_a} \approx \langle 418.4, 36.6\rangle - \langle 400, 0\rangle = \langle 18.4, 36.6\rangle$.

b. Wind speed is $|\mathbf{w}| \approx |\langle 18.4, 36.6\rangle| \approx 41$ mi/hr. The heading is $\tan^{-1}(36.6/18.4) \approx 63.3$ degrees north of east.

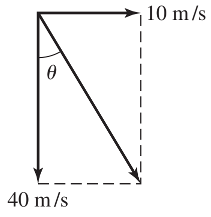
13.1.55 $\mathbf{v_g} = \langle 0, 480\rangle$ and $\mathbf{w} = \langle 20\cos(-60°), 20\sin(-60°)\rangle = \left\langle 10, -10\sqrt{3}\right\rangle$. Therefore $\mathbf{v_a} = \mathbf{v_g} - \mathbf{w} = \langle 0, 480\rangle - \left\langle 10, -10\sqrt{3}\right\rangle = \left\langle -10, 480 + 10\sqrt{3}\right\rangle$.

$|\mathbf{v_a}| = \left|\left\langle -10, 480 + 10\sqrt{3}\right\rangle\right| \approx 497.4$ mi/hr; The heading is $180° + \tan^{-1}\left(\dfrac{480 + 10\sqrt{3}}{-10}\right) \approx 91.15°$ which is about 1.2° west of north.

13.1.56 Let $\mathbf{w} = \langle 0, -10\rangle$ represent the water, and let $\mathbf{b} = \langle 20, 0\rangle$ represent the boat relative to the shore. Then the vector $\mathbf{v}$ which represents the boat relative to the water is given by $\mathbf{v} + \mathbf{w} = \mathbf{b}$, so $\mathbf{v} = \mathbf{b} - \mathbf{w} = \langle 20, 10\rangle$. Note that $|\langle 20, 10\rangle| = 10\sqrt{5} \approx 22.4$ miles per hour represents the speed of the boat. The direction is the direction of the vector $\langle 2, 1\rangle$ which is $\tan^{-1}(1/2) \cdot \dfrac{180}{\pi} \approx 26.6$ degrees north of east.

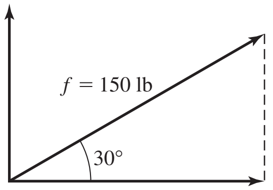
13.1.57 Let $\mathbf{w} = \langle 0, -5\rangle$ represent the water, and let $\mathbf{v_w} = \langle 40, 0\rangle$ represent the boat relative to the water. Then relative to the shore we have $\mathbf{v_s} = \mathbf{w} + \mathbf{v_w} = \langle 40, -5\rangle$, which has magnitude $\sqrt{1625} \approx 40.3$ km/hr.

13.1.58 Let $\mathbf{p}$ represent the vector's terminal velocity vector. $\mathbf{p} = \langle 10, -40\rangle$, and $|\mathbf{p}| = \sqrt{100 + 1600} = 10\sqrt{17}$. $\theta = \tan^{-1}(10/40) \approx 0.245$ radians, or 14.037 degrees. Thus, the speed is $10\sqrt{17}$ meters per second, and the direction is about 14.037 degrees east of vertical.



13.1.59 Let $\mathbf{u} = \mathbf{i}$ represent the current and $\mathbf{v} = \sqrt{3}\cos(\pi/6)\mathbf{i} + \sqrt{3}\sin(\pi/6)\mathbf{j} = \frac{3}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ represent the boat relative to land. If $\mathbf{w}$ represents the wind, then $\mathbf{u} + \mathbf{w} = \mathbf{v}$, so $\mathbf{w} = \mathbf{v} - \mathbf{u} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$. Then $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$, or 60 degrees. The speed of the wind is 1 meter per second in the direction 60 degrees north of east (or 30 degrees east of north).

13.1.60 $\mathbf{F} = 150\cos(\pi/6)\mathbf{i} + 150\sin(\pi/6)\mathbf{j} = 75\sqrt{3}\mathbf{i} + 75\mathbf{j}$. The horizontal component of the force is $75\sqrt{3}$ pounds, and the vertical component is 75 pounds.



13.1.61

a. $\mathbf{F} = 40\cos(\pi/3)\mathbf{i} + 40\sin(\pi/3)\mathbf{j} = 20\mathbf{i} + 20\sqrt{3}\mathbf{j}$, so the horizontal component is 20 and the vertical is $20\sqrt{3}$.

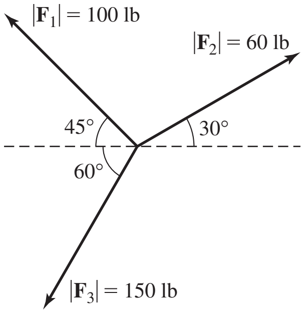
b. Yes. If it is 45 degrees, the horizontal component would be $40\cos(\pi/4) = 20\sqrt{2} > 20$.

c. No. If it is 45 degrees, the vertical component would be $40\sin(\pi/4) = 20\sqrt{2} < 20\sqrt{3}$.

13.1.62 Let $\mathbf{F_1} = 100\cos(\pi/3)\mathbf{i} + 100\sin(\pi/3)\mathbf{j} = 50\mathbf{i} + 50\sqrt{3}\mathbf{j}$, and let $\mathbf{F_2} = 60\cos(\pi/6)\mathbf{i} + 60\sin(\pi/6)\mathbf{j} = 30\sqrt{3}\mathbf{i} + 30\mathbf{j}$. Note that $\mathbf{F_2}$ has a greater horizontal component, because $30\sqrt{3} \approx 51.96 > 50$.

13.1.63 Let the magnitude of the force on the two chains be f. Let $\mathbf{F_1} = (-\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j})f$ and let $\mathbf{F_2} = (\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j})f$. Then $\mathbf{F_1} + \mathbf{F_2} - 500\mathbf{j} = 0$, and solving for f yields $f = 250\sqrt{2}$ pounds.

13.1.64 $\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} = \sqrt{2}(-50\mathbf{i} + 50\mathbf{j}) + (30\sqrt{3}\mathbf{i} + 30\mathbf{j}) + (-75\mathbf{i} - 75\sqrt{3}\mathbf{j}) = (-50\sqrt{2} + 30\sqrt{3} - 75)\mathbf{i} + (50\sqrt{2} + 30 - 75\sqrt{3})\mathbf{j}$. Thus $|\mathbf{F}|$ is about $\sqrt{(-50\sqrt{2} + 30\sqrt{3} - 75)^2 + (50\sqrt{2} + 30 - 75\sqrt{3})^2}$ which is about 98.19 pounds. $\theta = \tan^{-1}\left(\frac{50\sqrt{2} + 30 - 75\sqrt{3}}{-50\sqrt{2} + 30\sqrt{3} - 75}\right) \approx$ 17.3 degrees. The magnitude of the net force is about 98 pounds in the direction 17.3 degrees south of west.



13.1.65

a. True. This follows because $(\mathbf{u}+\mathbf{v})+\mathbf{w} = (\mathbf{w}+\mathbf{u})+\mathbf{v}$ (vector addition is commutative and associative.)

b. True. This is because $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

c. False. For example, if $\mathbf{u} = \langle 3, 4\rangle$ and $\mathbf{v} = \langle -3, -1\rangle$, then $|\mathbf{u} + \mathbf{v}| = |\langle 0, 3\rangle| = 3$, while $|\mathbf{u}| = 5$.

d. False. For example, if $\mathbf{u} = \langle 3, 4\rangle$ and $\mathbf{v} = \langle -1, -4\rangle$, then $|\mathbf{u} + \mathbf{v}| = |\langle 2, 0\rangle| = 2$, while $|\mathbf{u}| + |\mathbf{v}| = 5 + \sqrt{17}$.

e. False. For example, if $\mathbf{u} = \langle 3, 0\rangle$ and $\mathbf{v} = \langle 6, 0\rangle$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel, but have different lengths.

f. False. For example, given $A(0, 0)$, $B(3, 4)$, $C(1, 1)$ and $D(4, 5)$, we have $\overrightarrow{AB} = \langle 3, 4\rangle$ and $\overrightarrow{CD} = \langle 3, 4\rangle$, but $A \neq C$ and $B \neq D$.

g. False. For example, $\mathbf{u} = \langle 0, 1\rangle$ and $\mathbf{v} = \langle -1, 0\rangle$ are perpendicular, but $|\mathbf{u}+\mathbf{v}| = \sqrt{2}$, while $|\mathbf{u}|+|\mathbf{v}| = 2$.

h. True. Suppose $\mathbf{v} = k\mathbf{u}$ with $k > 0$. Then

$$|\mathbf{u}+\mathbf{v}| = |\mathbf{u}+k\mathbf{u}| = |(1+k)\mathbf{u}| = (1+k)\,|\mathbf{u}| = |\mathbf{u}| + k\,|\mathbf{u}| = |\mathbf{u}| + |k\mathbf{u}| = |\mathbf{u}| + |\mathbf{v}|\,.$$

13.1.66 $\overrightarrow{AB} = \langle 3, 6\rangle$ and $\overrightarrow{CD} = \langle b-a+2, a-b-7\rangle$. The system of linear equations $-a+b+2 = 3$, $a-b-7 = 6$ has no solution, so there are no values for a and b which will make these vectors equal.

13.1.67 $10\,\langle a, b\rangle = \langle 2, -3\rangle$, so $10a = 2$, and $a = 1/5$. Also, $10b = -3$, so $b = -3/10$. Thus $\mathbf{x} = \langle 1/5, -3/10\rangle$.

13.1.68 $2\,\langle a, b\rangle + \langle 2, -3\rangle = \langle -4, 1\rangle$, so $\langle a, b\rangle = \langle -3, 2\rangle = \mathbf{x}$.

13.1.69 $3\,\langle a, b\rangle - 4\,\langle 2, -3\rangle = \langle -4, 1\rangle$, so $\langle a, b\rangle = \dfrac{1}{3}\,\langle 4, -11\rangle = \mathbf{x}$.

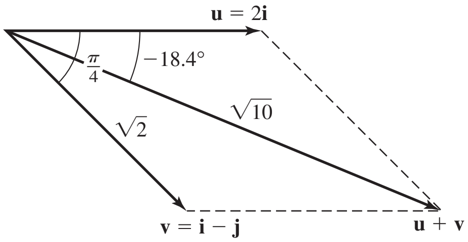
13.1.70 Because $2\mathbf{u}+3\mathbf{v} = \mathbf{i}$ and $-2(\mathbf{u}-\mathbf{v}) = -2\mathbf{j}$, we can conclude that $3\mathbf{v}+2\mathbf{v} = \mathbf{i}-2\mathbf{j}$ (by adding), so $\mathbf{v} = \dfrac{1}{5}\mathbf{i} - \dfrac{2}{5}\mathbf{j}$. It then follows that $\mathbf{u} = \mathbf{v}+\mathbf{j} = \dfrac{1}{5}\mathbf{i} - \dfrac{2}{5}\mathbf{j} + \mathbf{j} = \dfrac{1}{5}\mathbf{i} + \dfrac{3}{5}\mathbf{j}$.

13.1.71 $\langle 4, -8\rangle = 4\mathbf{i} + -8\mathbf{j}$.

13.1.72 Suppose $\langle 4, -8\rangle = c_1\,\langle 1, 1\rangle + c_2\,\langle -1, 1\rangle$. Then $c_1 - c_2 = 4$ and $c_1 + c_2 = -8$. Adding these two equations to each other yields $2c_1 = -4$, so $c_1 = -2$. And thus $c_2 = -6$. We have $\langle 4, -8\rangle = -2\mathbf{u} - 6\mathbf{v}$.

13.1.73 Let $\langle a, b\rangle = c_1\mathbf{u} + c_2\mathbf{v}$. Then $c_1 - c_2 = a$ and $c_1 + c_2 = b$. Adding these two equations to each other yields $2c_1 = a+b$, so $c_1 = \dfrac{a+b}{2}$. And thus $c_2 = \dfrac{b-a}{2}$. We have $\langle a, b\rangle = \dfrac{a+b}{2}\mathbf{u} + \dfrac{b-a}{2}\mathbf{v}$.

13.1.74 Let $\mathbf{u}$ represent the motion of the ant on the paper, and let $\mathbf{v}$ represent the motion of the paper. $\mathbf{u}+\mathbf{v} = 2\mathbf{i} + (\mathbf{i}-\mathbf{j}) = 3\mathbf{i} - \mathbf{j}$. $|\mathbf{u}+\mathbf{v}| = \sqrt{9+1} = \sqrt{10}$. $\theta = \tan^{-1}(-1/3) = -18.4$ degrees. The ant moves in the direction 18.4 degrees south of east with speed $\sqrt{10}$ miles per hour.



13.1.75

a. The sum is $\mathbf{0}$ because each vector has exactly one additive inverse in the set among the 12 vectors.

b. The 6:00 vector, because the others cancel in pairs, but this vector remains.

c. If we remove the 1:00 through 6:00 vectors, the sum is as large as possible, because all the vectors are pointing toward the left side of the clock. Removing any 6 consecutive vectors gives a sum whose magnitude is as large as possible.

d. Let $\mathbf{w}$ be the vector that points from 12:00 toward 6:00 but which has length r equal to the radius of the clock. The sum we are seeking is $12\mathbf{w}$. The sum of the vectors pointing to 1:00 and 11:00 add up to $(2-\sqrt{3})\mathbf{w}$, the sum of the vectors pointing to 2:00 and 10:00 is $\mathbf{w}$, the vectors pointing to 3:00 and 9:00 add up to $2\mathbf{w}$, the vectors pointing to 4:00 and 8:00 add up to $3\mathbf{w}$, and the vectors pointing to 5:00 and 7:00 add up to $(\sqrt{3}+2)\mathbf{w}$. Finally, the single vector pointing to 6:00 is $2\mathbf{w}$. The sum of all of these is $12\mathbf{w}$.

13.1.76 Because the ring doesn't move, $\mathbf{F}_3$ is the opposite of $\mathbf{F_1}+\mathbf{F}_2$, so $\mathbf{F_3} = \left\langle 50\sqrt{2} - 30\sqrt{3}, -50\sqrt{2} - 30\right\rangle$. Thus $|\mathbf{F}_3| \approx 102.44$ pounds, and the direction is given by $\alpha = \tan^{-1}\left(\dfrac{50\sqrt{2}+30}{50\sqrt{2}-30\sqrt{3}}\right) \approx 79.45$ degrees south of east.

13.1.77 $\mathbf{u}+\mathbf{v} = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle = \langle u_1+v_1, u_2+v_2\rangle = \langle v_1+u_1, v_2+u_2\rangle = \langle v_1, v_2\rangle + \langle u_1, u_2\rangle = \mathbf{v}+\mathbf{u}$.

13.1.78

$$\begin{aligned}(\mathbf{u}+\mathbf{v})+\mathbf{w} &= (\langle u_1, u_2\rangle + \langle v_1, v_2\rangle) + \langle w_1, w_2\rangle = \langle u_1+v_1, u_2+v_2\rangle + \langle w_1, w_2\rangle \\ &= \langle (u_1+v_1)+w_1, (u_2+v_2)+w_2\rangle = \langle u_1+(v_1+w_1), u_2+(v_2+w_2)\rangle \\ &= \langle u_1, u_2\rangle + \langle v_1+w_1, v_2+w_2\rangle = \mathbf{u}+(\mathbf{v}+\mathbf{w}).\end{aligned}$$

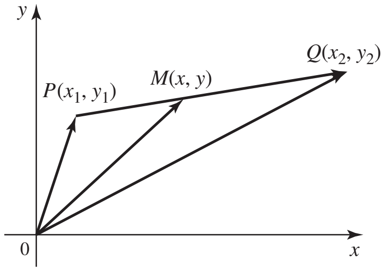
13.1.79 $a(c\mathbf{v}) = a(c\,\langle v_1, v_2\rangle) = a\,\langle cv_1, cv_2\rangle = \langle a(cv_1), a(cv_2)\rangle = \langle (ac)v_1, (ac)v_2\rangle = (ac)\,\langle v_1, v_2\rangle = (ac)\mathbf{v}$.

13.1.80

$$\begin{aligned}a(\mathbf{u}+\mathbf{v}) &= a(\langle u_1, u_2\rangle + \langle v_1, v_2\rangle) = a(\langle u_1+v_1, u_2+v_2\rangle) = \langle a(u_1+v_1), a(u_2+v_2)\rangle = \langle au_1+av_1, au_2+av_2\rangle \\ &= \langle au_1, au_2\rangle + \langle av_1, av_2\rangle = a\mathbf{u}+a\mathbf{v}.\end{aligned}$$

13.1.81 $(a+c)\mathbf{v} = (a+c)\,\langle v_1, v_2\rangle = \langle (a+c)v_1, (a+c)v_2\rangle = \langle av_1+cv_1, av_2+cv_2\rangle = \langle av_1, av_2\rangle + \langle cv_1, cv_2\rangle = a\mathbf{v}+c\mathbf{v}$.

13.1.82 Let $M(x,y)$ be the midpoint. Because $\overrightarrow{OM} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ}$, we have $\langle x, y\rangle = \langle x_1, y_1\rangle + \frac{1}{2}(\langle x_2, y_2\rangle - \langle x_1, y_1\rangle) = \langle x_1, y_1\rangle + \left\langle \frac{1}{2}x_2, \frac{1}{2}y_2\right\rangle + \left\langle -\frac{1}{2}x_1, -\frac{1}{2}y_1\right\rangle = \left\langle \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right\rangle$.



13.1.83 $|c\mathbf{v}| = |c\,\langle v_1, v_2\rangle| = |\langle cv_1, cv_2\rangle| = \sqrt{(cv_1)^2+(cv_2)^2} = \sqrt{c^2}\sqrt{v_1^2+v_2^2} = |c|\,|\mathbf{v}|$.

13.1.84 Yes. Because $\overrightarrow{PQ} = \overrightarrow{RS}$, we have that these two vectors are parallel and have the same magnitude. Thus the quadrilateral $RSQP$ is a parallelogram. Hence, $\overrightarrow{PR}$ is parallel to $\overrightarrow{QS}$ and they have the same magnitude, and are thus equal.

13.1.85

a. Note that $-6\mathbf{u} = \mathbf{v}$, so $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent. But there is no scalar c so that $c\mathbf{u} = \mathbf{w}$, nor any scalar d so that $d\mathbf{v} = \mathbf{w}$ so $\{\mathbf{u}, \mathbf{w}\}$ is linearly independent and $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent.

b. Two nonzero vectors are linearly independent when they are not parallel, and are linearly dependent when they are parallel.

c. Suppose $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Consider the equation $c_1\mathbf{u}+c_2\mathbf{v} = \mathbf{w}$ for a given vector $\mathbf{w}$. We are seeking a solution for the system of linear equatons $c_1u_1+c_2v_1 = w_1$ and $c_1u_2+c_2v_2 = w_2$. The solution for this system is given by $c_1 = \frac{1}{u_1v_2-u_2v_1}(v_2w_1-v_1w_2)$ and $c_2 = \frac{1}{u_1v_2-u_2v_1}(-u_2w_1+u_1w_2)$, provided $u_1v_2-u_2v_1 \neq 0$. This condition is equivalent to saying that $\mathbf{v}$ is not a multiple of $\mathbf{u}$. Thus a solution to the system of linear equations exists exactly when the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

13.1.86 Suppose that $u_1v_1+u_2v_2 = 0$. Then $\frac{u_2}{u_1}\cdot\frac{v_2}{v_1} = -1$. Let $m_1 = \frac{u_2}{u_1}$, and note that this is the slope of the line containing the vector $\mathbf{u}$. Likewise let $m_2 = \frac{v_2}{v_1}$, and note that this is the slope of the line containing the vector $\mathbf{v}$. Because $m_1\cdot m_2 = -1$, the vectors are perpendicular.

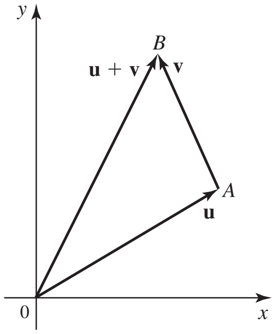
13.1.87

a. If **u** and **v** are parallel, we must have $\frac{a}{2} = \frac{5}{6}$, so $a = \frac{5}{3}$.

b. If **u** and **v** are perpendicular, we must have $2a + 30 = 0$, so $a = -15$.

13.1.88

a. The triangle rule states that in any triangle, the length of one side is less than or equal to the sum of the lengths of the other two sides. Suppose that $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{AB}$. Then $\mathbf{u} + \mathbf{v} = \overrightarrow{OB}$. The triangle rule applied to triangle OAB assures us that $\left|\overrightarrow{OB}\right| \leq \left|\overrightarrow{OA}\right| + \left|\overrightarrow{AB}\right|$, so $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$.



b. Equality occurs when **u** and **v** are parallel and in the same direction, so that $\mathbf{v} = k\mathbf{u}$ where $k > 0$. Then $|\mathbf{u} + \mathbf{v}| = |(1 + k)\mathbf{u}| = (1 + k)\,|\mathbf{u}| = |\mathbf{u}| + |k\mathbf{u}| = |\mathbf{u}| + |\mathbf{v}|$.

13.2 Vectors in Three Dimensions

13.2.1 Starting at the origin $(0, 0, 0)$, move 3 units in the positive x-direction, 2 units in the negative y-direction, and 1 unit in the positive z-direction, to arrive at the point $(3, -2, 1)$.

13.2.2 Every point in the xz-plane has a y-coordinate of 0.

13.2.3 The plane $x = 4$ is parallel to the yz-plane, but contains all of the points with x-coordinate 4. It is perpendicular to the x-axis.

13.2.4 $\mathbf{u} = \langle 0 - 3, -6 - 5, 3 - (-2)\rangle = \langle -3, -11, 5\rangle$.

13.2.5 $\mathbf{u} + \mathbf{v} = \langle 3 + 6, 5 + (-5), -7 + 1\rangle = \langle 9, 0, -6\rangle$. $3\mathbf{u} - \mathbf{v} = \langle 9, 15, -21\rangle - \langle 6, -5, 1\rangle = \langle 3, 20, -22\rangle$.

13.2.6 $\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

13.2.7 Because $\sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14} < \sqrt{0^2 + 0^2 + (-4)^2} = 4$, the point $(0, 0, -4)$ is further from the origin.

13.2.8 $\overrightarrow{PQ} = \langle 1 - (-1), 3 - (-4), -6 - (6)\rangle = \langle 2, 7, -12\rangle = 2\mathbf{i} + 7\mathbf{j} - 12\mathbf{k}$.

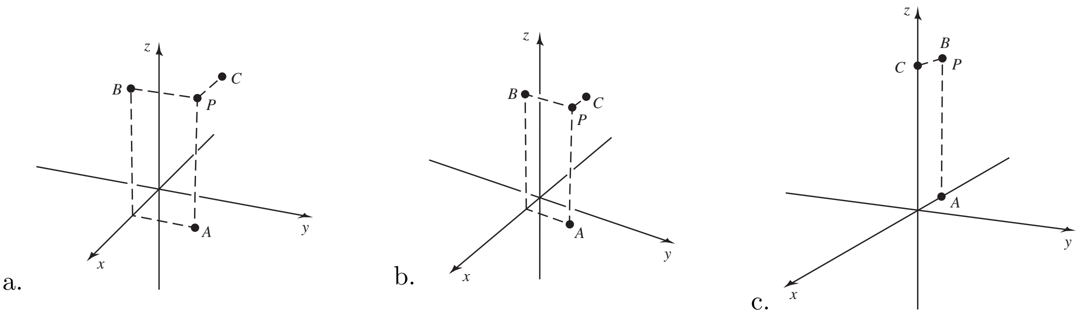
13.2.9 $A(3, 0, 5)$, $B(3, 4, 0)$, $C(0, 4, 5)$.

13.2.10 $A(5, 0, 10)$, $B(0, 8, 0)$, $C(5, 8, 10)$.

13.2.11 $A(3, -4, 5)$, $B(0, -4, 0)$, $C(0, -4, 5)$.

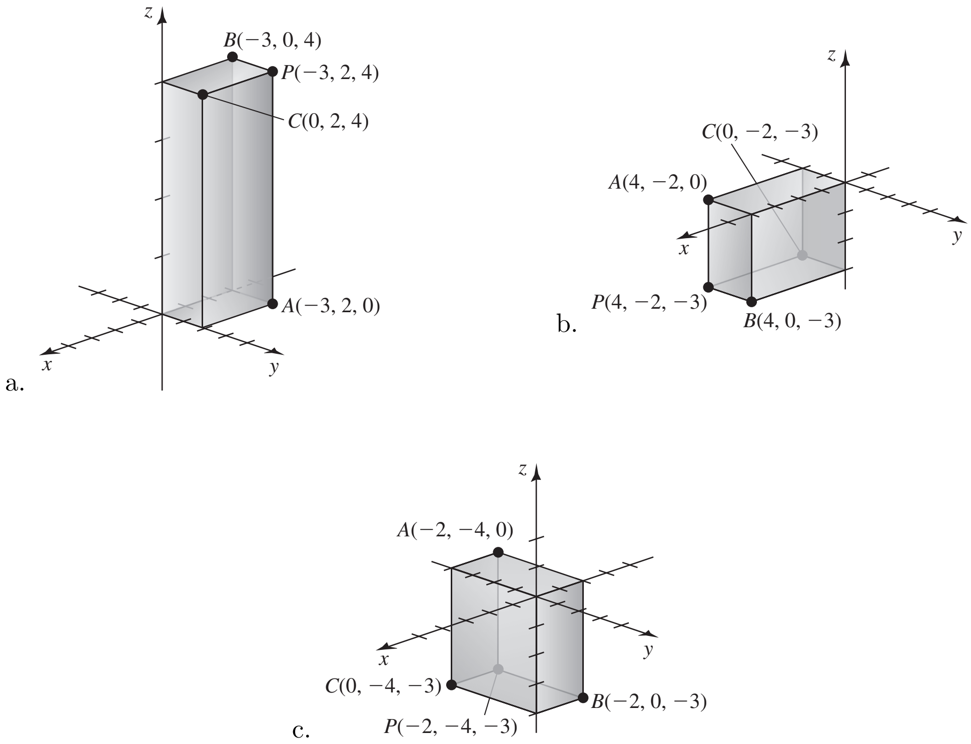
13.2.12 $A(-3, -3, 0)$, $B(0, 0, 3)$, $C(-3, -3, 3)$.

13.2.13



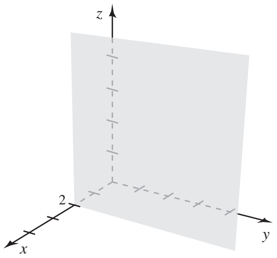
a. b. c.

13.2.14

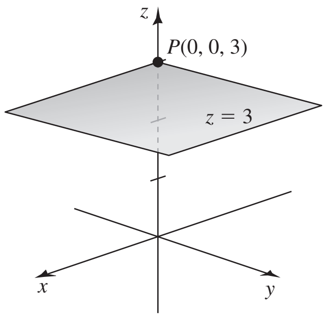

z
B(−3, 0, 4)
P(−3, 2, 4)
C(0, 2, 4)
A(−3, 2, 0)
x
y
C(0, −2, −3)
A(4, −2, 0)
P(4, −2, −3)
B(4, 0, −3)
A(−2, −4, 0)
C(0, −4, −3)
B(−2, 0, −3)
P(−2, −4, −3)

a. b. c.

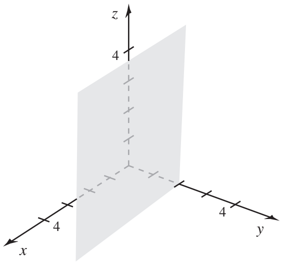
13.2.15



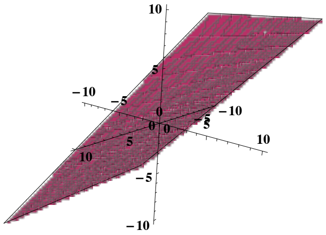
13.2.16



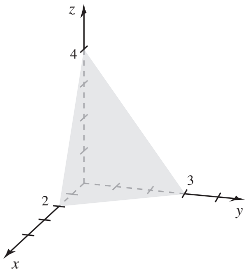
13.2.17



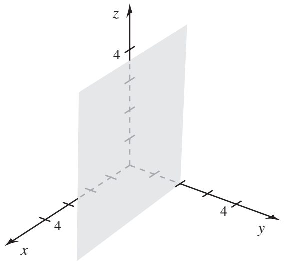
13.2.18



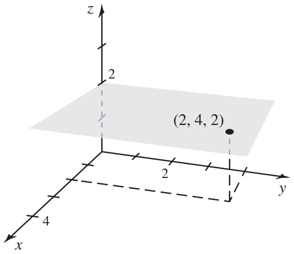
13.2.19



13.2.20

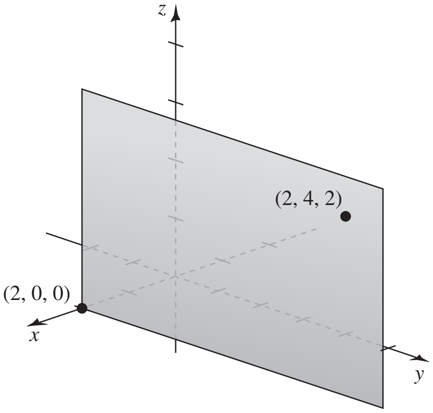


13.2.21



The plane $z = 2$.

13.2.22



The plane $x = 2$.

13.2.23 $(x-1)^2 + (y-2)^2 + (z-3)^2 = 16$.

13.2.24 Note that the radius r of this sphere would be $r = \sqrt{(3-1)^2 + (4-2)^2 + 5^2} = \sqrt{33}$. The equation of the sphere is thus $(x-1)^2 + (y-2)^2 + z^2 = 33$.

13.2.25 $(x+2)^2 + y^2 + (z-4)^2 \le 1$.

13.2.26 Note that the radius of the ball is given by $r = \sqrt{1^2 + (4-(-2))^2 + (8-6)^2} = \sqrt{41}$. The ball is thus described by the inequality $x^2 + (y+2)^2 + (z-6)^2 \le 41$.

13.2.27 The midpoint of the line segment $\overline{PQ}$ is $\left(\dfrac{1+2}{2}, \dfrac{0+3}{2}, \dfrac{5+9}{2}\right) = (3/2, 3/2, 7)$. The radius of the sphere is $r = \frac{1}{2}\sqrt{(2-1)^2+(3-0)^2+(9-5)^2} = \frac{\sqrt{26}}{2}$. The equation of the sphere is therefore $(x-3/2)^2 + (y-3/2)^2+(z-7)^2 = \frac{13}{2}$.

13.2.28 The midpoint of the line segment $\overline{PQ}$ is $\left(\frac{-4+0}{2}, \frac{2+2}{2}, \frac{3+7}{2}\right) = (-2, 2, 5)$. The radius of the sphere is $r = \frac{1}{2}\sqrt{(0-(-4))^2+(2-2)^2+(7-3)^2} = \sqrt{8}$. The equation of the sphere is therefore $(x+2)^2 + (y-2)^2+(z-5)^2 = 8$.

13.2.29 This is a sphere centered at $(1, 0, 0)$ of radius 3.

13.2.30 Completing the square, we have $(x+1)^2 + (y^2-2y+1)+z^2 = 25$, which can be written as $(x+1)^2+(y-1)^2+z^2 = 5^2$. This is a sphere centered at $(-1, 1, 0)$ with radius 5.

13.2.31 Completing the squares, we have $x^2 + (y^2-2y+1)+(z^2-4z+4) = 4+5$, so we have $x^2+(y-1)^2+(z-2)^2 = 3^2$, which describes a sphere of radius 3 centered at $(0, 1, 2)$.

13.2.32 Completing the squares, we have $(x^2-6x+9)+(y^2+6y+9)+(z^2-8z+16) = 2+9+9+16 = 36$, so we have $(x-3)^2+(y+3)^2+(z-4)^2 = 6^2$, which describes a sphere of radius 6 centered at $(3, -3, 4)$.

13.2.33 Completing the square, we have $x^2 + (y^2-14y+49)+z^2 \geq -13+49 = 36$, which can be written as $x^2+(y-7)^2+z^2 \geq 6^2$. This is the outside of a ball centered at $(0, 7, 0)$ with radius 6. (Including the sphere itself.)

13.2.34 Completing the square, we have $x^2 + (y^2-14y+49)+z^2 \leq -13+49 = 36$, which can be written as $x^2+(y-7)^2+z^2 \leq 6^2$. This is a ball centered at $(0, 7, 0)$ with radius 6.

13.2.35 Completing the squares, we have $(x^2-8x+16)+(y^2-14y+49)+(z^2-18z+81) \leq 79+16+49+81 = 225$, which can be written as $(x-4)^2+(y-7)^2+(z-9)^2 \leq 15^2$. This is a ball centered at $(4, 7, 9)$ with radius 15.

13.2.36 Completing the squares, we have $(x^2-8x+16)+(y^2+14y+49)+(z^2-18z+81) \geq 65+16+49+81 = 211$, which can be written as $(x-4)^2+(y+7)^2+(z-9)^2 \geq 211$. This is the outside of a ball centered at $(4, -7, 9)$ with radius $\sqrt{211}$. (Including the sphere itself.)

13.2.37 Completing the squares, we have $(x^2-2x+1)+(y^2+6y+9)+z^2 = -10+1+9 = 0$, or $(x-1)^2+(y+3)^2+z^2 = 0$. This is the single point $(1, -3, 0)$.

13.2.38 Completing the squares, we have $(x^2-4x+4)+(y^2+6y+9)+z^2 = -14+4+9 = -1$. This can be written as $(x-2)^2+(y+3)^2+z^2 = -1$, and no real numbers satisfy this equations.

13.2.39

a. $3\langle 4, -3, 0\rangle + 2\langle 0, 1, 1\rangle = \langle 12, -7, 2\rangle$.

b. $4\langle 4, -3, 0\rangle - \langle 0, 1, 1\rangle = \langle 16, -13, -1\rangle$.

c. $|\langle 4, -3, 0\rangle + 3\langle 0, 1, 1\rangle| = |\langle 4, 0, 3\rangle| = \sqrt{16+0+9} = 5$.

13.2.40

a. $3\langle -2, -3, 0\rangle + 2\langle 1, 2, 1\rangle = \langle -4, -5, 2\rangle$.

b. $4\langle -2, -3, 0\rangle - \langle 1, 2, 1\rangle = \langle -9, -14, -1\rangle$.

c. $|\langle -2, -3, 0\rangle + 3\langle 1, 2, 1\rangle| = |\langle 1, 3, 3\rangle| = \sqrt{1+9+9} = \sqrt{19}$.

13.2.41

a. $3\langle -2,1,-2\rangle + 2\langle 1,1,1\rangle = \langle -4,5,-4\rangle$.

b. $4\langle -2,1,-2\rangle - \langle 1,1,1\rangle = \langle -9,3,-9\rangle$.

c. $|\langle -2,1,-2\rangle + 3\langle 1,1,1\rangle| = |\langle 1,4,1\rangle| = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$.

13.2.42

a. $3\langle -5,0,2\rangle + 2\langle 3,1,1\rangle = \langle -9,2,8\rangle$.

b. $4\langle -5,0,2\rangle - \langle 3,1,1\rangle = \langle -23,-1,7\rangle$.

c. $|\langle -5,0,2\rangle + 3\langle 3,1,1\rangle| = |\langle 4,3,5\rangle| = \sqrt{16+9+25} = \sqrt{50} = 5\sqrt{2}$.

13.2.43

a. $3\langle -7,11,8\rangle + 2\langle 3,-5,-1\rangle = \langle -15,23,22\rangle$.

b. $4\langle -7,11,8\rangle - \langle 3,-5,-1\rangle = \langle -31,49,33\rangle$.

c. $|\langle -7,11,8\rangle + 3\langle 3,-5,-1\rangle| = |\langle 2,-4,5\rangle| = \sqrt{4+16+25} = \sqrt{45} = 3\sqrt{5}$.

13.2.44

a. $3\langle -4,-8\sqrt{3},2\sqrt{2}\rangle + 2\langle 2,3\sqrt{3},-\sqrt{2}\rangle = \langle -8,-18\sqrt{3},4\sqrt{2}\rangle$.

b. $4\langle -4,-8\sqrt{3},2\sqrt{2}\rangle - \langle 2,3\sqrt{3},-\sqrt{2}\rangle = \langle -18,-35\sqrt{3},9\sqrt{2}\rangle$.

c. $\left|\langle -4,-8\sqrt{3},2\sqrt{2}\rangle + 3\langle 2,3\sqrt{3},-\sqrt{2}\rangle\right| = \left|\langle 2,\sqrt{3},-\sqrt{2}\rangle\right| = \sqrt{4+3+2} = \sqrt{9} = 3$.

13.2.45

a. $\overrightarrow{PQ} = \langle 3-1,11-5,2-0\rangle = \langle 2,6,2\rangle = 2\mathbf{i}+6\mathbf{j}+2\mathbf{k}$.

b. $|\langle 2,6,2\rangle| = \sqrt{4+36+4} = \sqrt{44} = 2\sqrt{11}$.

c. $\left\langle 1/\sqrt{11},3/\sqrt{11},1/\sqrt{11}\right\rangle$ and $\left\langle -1/\sqrt{11},-3/\sqrt{11},-1/\sqrt{11}\right\rangle$.

13.2.46

a. $\overrightarrow{PQ} = \langle 1-5,14-11,13-12\rangle = \langle -4,3,1\rangle = -4\mathbf{i}+3\mathbf{j}+1\mathbf{k}$.

b. $|\langle -4,3,1\rangle| = \sqrt{16+9+1} = \sqrt{26}$.

c. $\left\langle -4/\sqrt{26},3/\sqrt{26},1/\sqrt{26}\right\rangle$ and $\left\langle 4/\sqrt{26},-3/\sqrt{26},-1/\sqrt{26}\right\rangle$.

13.2.47

a. $\overrightarrow{PQ} = \langle -3+3,-4-1,1-0\rangle = \langle 0,-5,1\rangle = -5\mathbf{j}+1\mathbf{k}$.

b. $|\langle 0,-5,1\rangle| = \sqrt{25+1} = \sqrt{26}$.

c. $\left\langle 0,-5/\sqrt{26},1/\sqrt{26}\right\rangle$ and $\left\langle 0,5/\sqrt{26},-1/\sqrt{26}\right\rangle$.

13.2.48

a. $\overrightarrow{PQ} = \langle 3-3,9-8,11-12\rangle = \langle 0,1,-1\rangle = \mathbf{j}-\mathbf{k}$.

b. $|\langle 0,1,-1\rangle| = \sqrt{1+1} = \sqrt{2}$.

c. $\left\langle 0,1/\sqrt{2},-1/\sqrt{2}\right\rangle$ and $\left\langle 0,-1/\sqrt{2},1/\sqrt{2}\right\rangle$.

13.2.49

a. $\overrightarrow{PQ} = \langle -2-0,4-0,0-2\rangle = \langle -2,4,-2\rangle = -2\mathbf{i}+4\mathbf{j}-2\mathbf{k}$.

b. $|\langle -2,4,-2\rangle| = \sqrt{4+16+4} = 2\sqrt{6}$.

c. $\left\langle -1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}\right\rangle$ and $\left\langle 1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6}\right\rangle$.

13.2.50

a. $\overrightarrow{PQ} = \langle 1-a, 1-b, -1-c\rangle = (1-a)\mathbf{i} + (1-b)\mathbf{j} + (-1-c)\mathbf{k}$.

b. $|\langle (1-a),(1-b),(-1-c)\rangle| = \sqrt{(1-a)^2+(1-b)^2+(-1-c)^2}$.

c. $\dfrac{\pm 1}{\sqrt{(1-a)^2+(1-b)^2+(-1-c)^2}}\langle 1-a, 1-b, -1-c\rangle$.

13.2.51

a. The airplane's velocity vector (without wind) is given by $\langle 0,20,0\rangle$, the wind's is given by $\langle 20,0,0\rangle$ and the downdraft's is $\langle 0,0,-10\rangle$. The sum of these is $\langle 20,20,-10\rangle$.

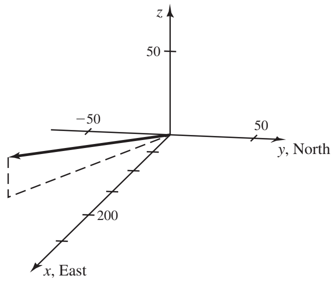
b. The speed is $|\langle 20,20,-10\rangle| = \sqrt{400+400+100} = 30$ mi/hr

13.2.52

a. The airplane's velocity vector (without wind) is given by $\langle 10,0,0\rangle$, the wind's is given by $\langle 0,-5,0\rangle$ and the updraft's is $\langle 0,0,5\rangle$. The sum of these is $\langle 10,-5,5\rangle$.

b. The speed is $|\langle 10,-5,5\rangle| = \sqrt{100+25+25} = 5\sqrt{6} \approx 12.2$ mi/hr

13.2.53 The airplane's velocity is $\mathbf{v_1} = 250\mathbf{i}$. The crosswind is blowing $\mathbf{v_2} = -25\sqrt{2}\mathbf{i} - 25\sqrt{2}\mathbf{j}$. The updraft is $\mathbf{v_3} = 30\mathbf{k}$. We have $|\mathbf{v_1}+\mathbf{v_2}+\mathbf{v_3}| = \left|\left\langle 250-25\sqrt{2}, -25\sqrt{2}, 30\right\rangle\right| \approx 219.596$ miles per hour. The direction is sketched in the diagram—it is slightly south of east and upward.



13.2.54 $\mathbf{F} = \mathbf{F_1}+\mathbf{F_2}+\mathbf{F_3} = \langle 60,20,30\rangle$. $|\mathbf{F}| = 10\sqrt{36+4+9} = 70$. The direction can be described using the angles α, β, and γ defined by $\alpha = \cos^{-1}(6/7) \approx 31$ degrees, $\beta = \cos^{-1}(20/70) \approx 73.4$ degrees, and $\gamma = \cos^{-1}(3/7) \approx 64.62$ degrees.

13.2.55 The component in the east direction is $(20\cos 30°)(\cos 45°) = 5\sqrt{6}$ knots. In the north direction, it is $(20\cos 30°)(\sin 45°) = 5\sqrt{6}$ knots. In the vertical direction, it is $20\sin 30° = 10$ knots.

13.2.56 $\mathbf{F_1}+\mathbf{F_2} = \langle 10,6,3\rangle + \langle 0,4,9\rangle = \langle 10,10,12\rangle$, so we need $\mathbf{F_3} = \langle -10,-10,-12\rangle$.

13.2.57

a. False. For example, let $\mathbf{u} = \langle 1,0,0\rangle$, $\mathbf{v} = \langle 0,1,0\rangle$ and $\mathbf{w} = \langle 1,1,0\rangle$. Then both $\mathbf{u}$ and $\mathbf{v}$ make a 45 degree angle with $\mathbf{w}$, but $\mathbf{u}+\mathbf{v} = \mathbf{w}$ makes a zero degree angle with $\mathbf{w}$.

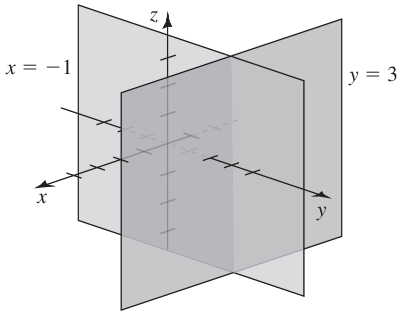
b. False. For example, $\mathbf{i}$ and $\mathbf{j}$ form a 90 degree angle with $\mathbf{k}$, as does $\mathbf{i}+\mathbf{j}$.

c. False. $\mathbf{i}+\mathbf{j}+\mathbf{k} = \langle 1,1,1\rangle \neq \langle 0,0,0\rangle$.

d. True. They intersect at the point $(1,1,1)$.

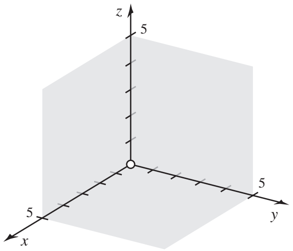
13.2.58

This is the set of all points which lie either on the plane $x = -1$ or the plane $y = 3$ or both.



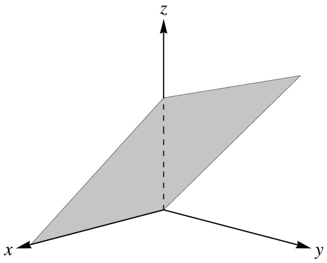
13.2.59

This represents all the points in 3-space, excluding the three axes.



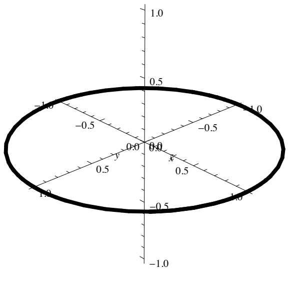
13.2.60

This represents the plane which is perpendicular to the yz-plane, and which intersects the yz-plane in the line $y = z$.



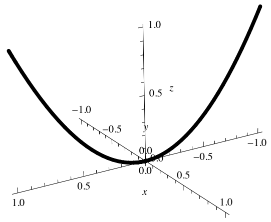
13.2.61

This represents a circle of radius 1 centered at $(0, 0, 0)$ in the xy-plane.



13.2.62

This represents a parabola in the xz-plane.



13.2.63 If $z = 1$, then $x^2 + y^2 + 1 = 5$, so $x^2 + y^2 = 4$. We have a circle of radius 2 centered at $(0, 0, 1)$ in the plane $z = 1$.

13.2.64 If $z = 6$ and $x^2 + y^2 + z^2 = 36$, then $x^2 + y^2 = 0$, so $x = 0$ and $y = 0$. This consists of the single point $(0, 0, 6)$.

13.2.65 Planes parallel to the xz-plane have the form $y = c$ for a constant c, so we must have $y = 4$. Thus, we must have $(x - 2)^2 + (z - 1)^2 = 9$ and $y = 4$.

13.2.66 The intersection of the planes $y = -5$ and $z = 1$ is a line parallel to the x-axis that contains the given point.

13.2.67 $\mathbf{v} = |\mathbf{v}| \dfrac{\mathbf{v}}{|\mathbf{v}|} = 6 \left\langle \dfrac{1}{3}, -\dfrac{2}{3}, \dfrac{2}{3} \right\rangle$.

13.2.68 $\mathbf{w} = 10 \dfrac{\mathbf{w}}{|\mathbf{w}|} = 10 \left\langle \dfrac{2}{3}, \dfrac{\sqrt{2}}{3}, \dfrac{\sqrt{3}}{3} \right\rangle$.

13.2.69 $-5 \dfrac{\mathbf{v}}{|\mathbf{v}|} = -5 \left\langle \dfrac{3}{4}, -\dfrac{1}{2}, \dfrac{\sqrt{3}}{4} \right\rangle = \left\langle -\dfrac{15}{4}, \dfrac{5}{2}, -\dfrac{5\sqrt{3}}{4} \right\rangle$.

13.2.70 Because the magnitude of $\mathbf{v}$ is $\sqrt{9 + 4 + 36} = 7$, the desired vectors are $\pm 10 \langle 3/7, -2/7, 6/7 \rangle = \pm \langle 30/7, -20/7, 60/7 \rangle$.

13.2.71 Because the magnitude of $\mathbf{v}$ is $\sqrt{36 + 64 + 0} = 10$, the desired vectors are $\pm 20 \langle .6, -.8, 0 \rangle = \pm \langle 12, -16, 0 \rangle$.

13.2.72 Because the magnitude of $\mathbf{v}$ is $\sqrt{1 + 1 + 0} = \sqrt{2}$, the desired vectors are $\pm 3 \left\langle 1/\sqrt{2}, -1/\sqrt{2}, 0 \right\rangle$.

13.2.73 Because the magnitude of $\mathbf{v}$ is $\sqrt{1 + 1 + 1} = \sqrt{3}$, the desired vectors are $\pm 3 \left\langle -1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3} \right\rangle = \pm \left\langle -\sqrt{3}, -\sqrt{3}, \sqrt{3} \right\rangle$.

13.2.74 In order for the points to be collinear, we would require $\langle 4 - 1, 7 - 2, 1 - 3 \rangle = k \langle x - 1, y - 2, 2 - 3 \rangle$. Thus we seek a solution to the system of equations $3 = k(x - 1)$, $5 = k(y - 2)$, $-2 = -k$. Thus $k = 2$, $x = \dfrac{5}{2}$ and $y = \dfrac{9}{2}$.

13.2.75

a. Because $\overrightarrow{PQ} = \langle 1, -1, 2 \rangle$ and $\overrightarrow{PR} = \langle 3, -3, 6 \rangle = 3 \langle 1, -1, 2 \rangle$ they are collinear. Q is between P and R.

b. Because $\overrightarrow{PQ} = \langle 4, 8, -8 \rangle$ and $\overrightarrow{PR} = \langle -1, -2, 2 \rangle = -\dfrac{1}{4} \langle 4, 8, -8 \rangle$ they are collinear. P is between Q and R.

c. Because $\overrightarrow{PQ} = \langle 1, -5, 3\rangle$ and $\overrightarrow{PR} = \langle 2, -3, 6\rangle$ are not parallel, the given points are not collinear.

d. Because $\overrightarrow{PQ} = \langle 2, 13, 3\rangle$ and $\overrightarrow{PR} = \langle -3, -2, -1\rangle$ are not parallel, the given points are not collinear.

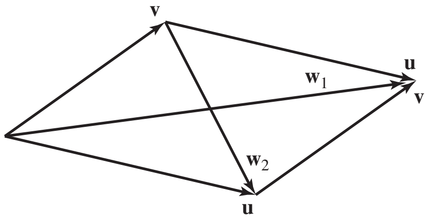
13.2.76 The diagonal of the box has magnitude $\sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$, so the longest rod that will fit in the box has length $\sqrt{29}$ feet.

13.2.77 Let $P(1, -\sqrt{3}, 0)$, $Q(1, \sqrt{3}, 0)$, $R(-2, 0, 0)$, and $S(0, 0, -2\sqrt{3})$ be the given points. Note that $\overrightarrow{PS} = \left\langle -1, \sqrt{3}, -2\sqrt{3}\right\rangle$, $\overrightarrow{QS} = \left\langle -1, -\sqrt{3}, -2\sqrt{3}\right\rangle$, $\overrightarrow{RS} = \left\langle 2, 0, -2\sqrt{3}\right\rangle$. Let $x(\overrightarrow{PS} + \overrightarrow{QS} + \overrightarrow{RS}) = -500\mathbf{k}$, then $-6\sqrt{3}x = -500$, so $x = \dfrac{250}{3\sqrt{3}}$. Then $x\overrightarrow{PS} = \dfrac{250}{3\sqrt{3}}\left\langle -1, \sqrt{3}, -2\sqrt{3}\right\rangle = \dfrac{250}{3}\left\langle -1/\sqrt{3}, 1, -2\right\rangle$. $x\overrightarrow{QS} = \dfrac{250}{3\sqrt{3}}\left\langle -1, -\sqrt{3}, -2\sqrt{3}\right\rangle = \dfrac{250}{3}\left\langle -1/\sqrt{3}, -1, -2\right\rangle$. $x\overrightarrow{RS} = \dfrac{250}{3\sqrt{3}}\left\langle 2, 0, -2\sqrt{3}\right\rangle = \dfrac{250}{3}\left\langle 2/\sqrt{3}, 0, -2\right\rangle$.

13.2.78 Let $A(2, 0, 0)$, $B(0, 2, 0)$, $C(-2, 0, 0)$, $D(0, -2, 0)$, and $E(0, 0, -4)$ be the given points. Note that $\overrightarrow{AE} = \langle -2, 0, -4\rangle$, $\overrightarrow{BE} = \langle 0, -2, -4\rangle$, $\overrightarrow{CE} = \langle 2, 0, -4\rangle$, $\overrightarrow{DE} = \langle 0, 2, -4\rangle$. We are seeking x so that $x(\overrightarrow{AE} + \overrightarrow{BE} + \overrightarrow{CE} + \overrightarrow{DE}) = -500\mathbf{k}$. Thus we require $-16x = -500$, so $x = \dfrac{125}{4}$. Then $x\overrightarrow{AE} = \dfrac{125}{4}\langle -2, 0, -4\rangle$. $x\overrightarrow{BE} = \dfrac{125}{4}\langle 0, -2, -4\rangle$. $x\overrightarrow{CE} = \dfrac{125}{4}\langle 2, 0, -4\rangle$, and $x\overrightarrow{DE} = \dfrac{125}{4}\langle 0, 2, -4\rangle$.

13.2.79 Let $R(x, y, z)$ be the fourth vertex. Then perhaps $\overrightarrow{OQ} = \overrightarrow{RP}$, so $\langle 2, 4, 3\rangle = \langle 1 - x, 4 - y, 6 - z\rangle$, so $x = -1$, $y = 0$, and $z = 3$, so $R(-1, 0, 3)$ is one possible desired vertex. We could also have $\overrightarrow{RP} = -\overrightarrow{OQ}$, in which case $\langle -2, -4, -3\rangle = \langle 1 - x, 4 - y, 6 - z\rangle$, so $R(3, 8, 9)$ is the other vertex. We could also have $\overrightarrow{OP} = \overrightarrow{RQ}$, so $\langle 1, 4, 6\rangle = \langle 2 - x, 4 - y, 3 - z\rangle$ and $R(1, 0, -3)$ is the desired point.

13.2.80 Suppose that the parallelogram has vertices $RTDS$, and $\mathbf{u} = \overrightarrow{RS}$ and $\mathbf{v} = \overrightarrow{RT}$ are two sides. Then we also have $\mathbf{u} = \overrightarrow{TD}$, and $\mathbf{v} = \overrightarrow{SD}$. But then $\mathbf{u} + \mathbf{v} = \overrightarrow{RS} + \overrightarrow{SD} = \overrightarrow{RD}$ is one diagonal and $\mathbf{u} - \mathbf{v} = \overrightarrow{TD} + \overrightarrow{DS} = \overrightarrow{TS}$ is the other diagonal.



13.2.81 Let $M(x, y, z)$ be the midpoint. Because $\overrightarrow{OM} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ}$, we have

$$\begin{aligned}\langle x, y, z\rangle &= \langle x_1, y_1, z_1\rangle + \frac{1}{2}(\langle x_2, y_2, z_2\rangle - \langle x_1, y_1, z_1\rangle)\\ &= \langle x_1, y_1, z_1\rangle + \left\langle \frac{1}{2}x_2, \frac{1}{2}y_2, \frac{1}{2}z_2\right\rangle + \left\langle -\frac{1}{2}x_1, -\frac{1}{2}y_1, -\frac{1}{2}z_1\right\rangle = \left\langle \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right\rangle.\end{aligned}$$

13.2.82 We complete the squares by adding $a^2 + b^2 + c^2$ to both sides of the given equation, giving

$$(x^2 - 2ax + a^2) + (y^2 - 2by + b^2) + (z^2 - 2cz + c^2) = d + a^2 + b^2 + c^2,$$

which can be written $(x - a)^2 + (y - b)^2 + (z - c)^2 = d + a^2 + b^2 + c^2$. This represents the set of points whose distance from (a, b, c) is $\sqrt{d + a^2 + b^2 + c^2}$, so it represents a sphere as long as $d + a^2 + b^2 + c^2 > 0$. The center is (a, b, c) and the radius is $\sqrt{d + a^2 + b^2 + c^2}$.

13.2.83

a. $\mathbf{u} + \mathbf{v} = -\mathbf{w}$ (by the geometric definition of vector addition), so $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$.

b. Let $\mathbf{M_1} = \overrightarrow{EB}$, $\mathbf{M_2} = \overrightarrow{FO}$, and $\mathbf{M_3} = \overrightarrow{GA}$. Consider triangle EAB. We have $\overrightarrow{EA} + \overrightarrow{AB} + \overrightarrow{BE} = \mathbf{0}$, so $\frac{1}{2}\mathbf{u} + \mathbf{v} = -\overrightarrow{BE} = \overrightarrow{EB} = \mathbf{M_1}$. Using similar arguments, we have $\mathbf{M_2} = \frac{1}{2}\mathbf{v} + \mathbf{w}$ and $\mathbf{M_3} = \frac{1}{2}\mathbf{w} + \mathbf{u}$.

c. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be the vectors from O to the points 1/3 of the way along $\mathbf{M_1}$, $\mathbf{M_2}$ and $\mathbf{M_3}$ respectively. Because $-\mathbf{w} = \mathbf{u} + \mathbf{v}$, we have $\frac{\mathbf{u} - \mathbf{w}}{3} = \frac{\mathbf{u}}{3} + \frac{\mathbf{u} + \mathbf{v}}{3} = \frac{2}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}$. Also, $\mathbf{a} = \frac{1}{2}\mathbf{u} + \frac{1}{3}\mathbf{M_1} = \frac{1}{2}\mathbf{u} + \frac{1}{3}\left(\frac{1}{2}\mathbf{u} + \mathbf{v}\right) = \frac{1}{2}\mathbf{u} + \frac{1}{6}\mathbf{u} + \frac{1}{3}\mathbf{v} = \frac{2}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}$. Thus $\mathbf{a} = \frac{\mathbf{u} - \mathbf{w}}{3}$. Also, $\mathbf{b} = -\frac{2}{3}\mathbf{M_2} = -\frac{2}{3}(\frac{1}{2}\mathbf{v} + (-\mathbf{u} - \mathbf{v})) = \frac{2}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}$. We also have $\mathbf{c} = -\frac{1}{2}\mathbf{w} + \frac{1}{3}\mathbf{M_3} = -\frac{1}{2}\mathbf{w} + \frac{1}{3}(\frac{1}{2}\mathbf{w} + \mathbf{u}) = \frac{1}{3}\mathbf{u} + -\frac{1}{3}(-\mathbf{u} - \mathbf{v}) = \frac{2}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}$. Thus $\mathbf{a} = \mathbf{b} = \mathbf{c}$.

d. Because $\mathbf{a} = \mathbf{b} = \mathbf{c}$, the medians all meet at a point that divides each median in a 2:1 ratio.

13.2.84

a. The coordinates of M_1 are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, 0\right)$, and thus

$$\overrightarrow{RM}_1 = \left\langle \frac{x_1 + x_2}{2} - x_3, \frac{y_1 + y_2}{2} - y_3, -z_3 \right\rangle.$$

b.

$$\begin{aligned}\overrightarrow{OZ}_1 = \overrightarrow{OR} + \frac{2}{3}\overrightarrow{RM}_1 &= \langle x_3, y_3, z_3 \rangle + \left\langle \frac{x_1 + x_2 - 2x_3}{3}, \frac{y_1 + y_2 - 2y_3}{3}, -\frac{2z_3}{3} \right\rangle \\ &= \left\langle \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_3}{3} \right\rangle.\end{aligned}$$

c. The coordinates of M_2 are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_3}{2}\right)$. We have

$$\overrightarrow{PM}_2 = \left\langle \frac{x_2 + x_3 - 2x_1}{2}, \frac{y_2 + y_3 - 2y_1}{2}, \frac{z_3}{2} \right\rangle,$$

and

$$\begin{aligned}\overrightarrow{OZ}_2 = \overrightarrow{OP} + \frac{2}{3}\overrightarrow{PM}_2 &= \langle x_1, y_1, 0 \rangle + \left\langle \frac{x_2 + x_3 - 2x_1}{3}, \frac{y_2 + y_3 - 2y_1}{3}, \frac{z_3}{3} \right\rangle \\ &= \left\langle \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_3}{3} \right\rangle.\end{aligned}$$

d. The coordinates of M_3 are $\left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}, \frac{z_3}{2}\right)$. We have

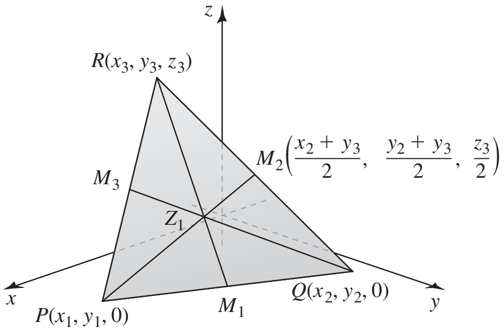
$$\overrightarrow{QM}_3 = \left\langle \frac{x_1 + x_3 - 2x_2}{2}, \frac{y_1 + y_3 - 2y_2}{2}, \frac{z_3}{2} \right\rangle,$$

and

$$\begin{aligned}\overrightarrow{OZ}_3 = \overrightarrow{OQ} + \frac{2}{3}\overrightarrow{QM}_3 &= \langle x_2, y_2, 0 \rangle + \left\langle \frac{x_1 + x_3 - 2x_2}{3}, \frac{y_1 + y_3 - 2y_2}{3}, \frac{z_3}{3} \right\rangle \\ &= \left\langle \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_3}{3} \right\rangle.\end{aligned}$$

e. The medians all intersect at the point $Z_1 = Z_2 = Z_3 = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_3}{3}\right)$.

f. The intersection point is $\left(\frac{2+4+6}{3}, \frac{4+1+3}{3}, \frac{4}{3}\right) = (4, 8/3, 4/3)$.



13.2.85

a. $\mathbf{u} + \mathbf{v} = \overrightarrow{PR}$ and $\mathbf{w} + \mathbf{x} = \mathbf{x} + \mathbf{w} = \overrightarrow{PR}$, so $\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{x}$.

b. $\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v} = \mathbf{m} = \frac{1}{2}(\mathbf{u} + \mathbf{v})$.

c. $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{w} = \mathbf{n} = \frac{1}{2}(\mathbf{x} + \mathbf{w})$.

d. We have $\mathbf{n} = \frac{1}{2}(\mathbf{x} + \mathbf{w}) = \frac{1}{2}(\mathbf{u} + \mathbf{v}) = \mathbf{m}$.

e. Because $\mathbf{m}$ and $\mathbf{n}$ are equal, they are parallel. A similar argument will show that the other two sides are parallel as well.

13.2.86 The midpoints are $A\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, 0\right)$, $B\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, 0\right)$, $C\left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}, \frac{z_4}{2}\right)$, and $D\left(\frac{x_4+x_1}{2}, \frac{y_4+y_1}{2}, \frac{z_4}{2}\right)$. So

$$\overrightarrow{AB} = \left\langle \frac{x_3-x_1}{2}, \frac{y_3-y_1}{2}, 0 \right\rangle = \overrightarrow{DC},$$

and

$$\overrightarrow{BC} = \left\langle \frac{x_4-x_2}{2}, \frac{y_4-y_2}{2}, \frac{z_4}{2} \right\rangle = \overrightarrow{AD},$$

so quadrilateral $ABCD$ is a parallelogram.

13.3 Dot Products

13.3.1 $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$, where θ is the angle between the two vectors.

13.3.2 If $\mathbf{u} = \langle a, b, c \rangle$ and $\mathbf{v} = \langle r, s, t \rangle$, then $\mathbf{u} \cdot \mathbf{v} = ar + bs + ct$.

13.3.3 $\langle 2, 3, -6 \rangle \cdot \langle 1, -8, 3 \rangle = 2 \cdot 1 + 3 \cdot (-8) + (-6) \cdot 3 = -40$.

13.3.4 Because the angle between a vector and itself is 0, we have $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|\,|\mathbf{v}| \cos\theta = |\mathbf{v}|\,|\mathbf{v}| = |\mathbf{v}|^2$.

13.3.5 Given non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, the angle between them is $\cos^{-1}(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|})$.

13.3.6 By definition, $\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta$, so we have $-2 = 4\cos\theta$, or $\cos\theta = -\frac{1}{2}$, and so $\theta = \frac{2\pi}{3}$.

13.3.7 $\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u}\left(\dfrac{\mathbf{v}}{|\mathbf{v}|}\right) = -2\left\langle \dfrac{2}{3}, -\dfrac{1}{3}, -\dfrac{2}{3}\right\rangle = \left\langle -\dfrac{4}{3}, \dfrac{2}{3}, \dfrac{4}{3}\right\rangle$.

13.3.8 $\mathbf{u}\cdot\mathbf{v} = 4 + 2 - 6 = 0$, so the vectors are orthogonal.

13.3.9 $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\pi = 1\cdot 1\cdot -1 = -1$.

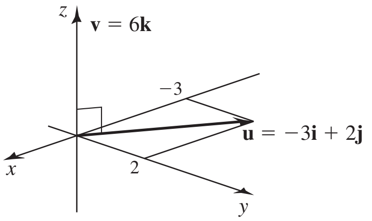
13.3.10 The work done by a force $\mathbf{F}$ in moving an object along a displacement vector $\mathbf{d}$ is $w = \mathbf{F}\cdot\mathbf{d}$.

13.3.11 There are 2.

13.3.12 There are infinitely many.

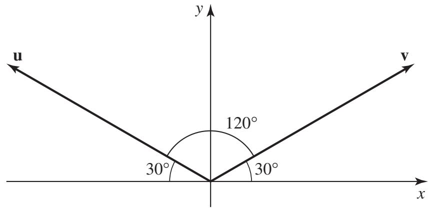
13.3.13 $\mathbf{u}\cdot\mathbf{v} = 4\cdot 6\cdot\cos(\pi/2) = 0$.

13.3.14 Because these vectors are perpendicular, we have $\cos\theta = 0$ where θ is the angle between them, so their dot product is zero.



13.3.15 The angle between these vectors is $\pi/4$. Thus, their dot product is $10\cdot 10\sqrt{2}\cdot\dfrac{\sqrt{2}}{2} = 100$.

13.3.16 The angle between these vectors is 120°, so their dot product is $|\mathbf{u}|\,|\mathbf{v}|\cos(120^\circ) = 2\cdot 2\cdot -\dfrac{1}{2} = -2$.



13.3.17 $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta = 1\cdot 1\cdot\cos(\pi/3) = 1/2$.

13.3.18 $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta = 1\cdot 2\cdot\cos(3\pi/4) = 2\cdot(-\sqrt{2}/2) = -\sqrt{2}$.

13.3.19 $\mathbf{u}\cdot\mathbf{v} = 1 - 1 = 0$, so $\theta = \cos^{-1}(0) = \pi/2$.

13.3.20 $\mathbf{u}\cdot\mathbf{v} = -50 + 0 = -50$, so $\theta = \cos^{-1}\left(\dfrac{-50}{10\sqrt{50}}\right) = \cos^{-1}(-\sqrt{2}/2) = 3\pi/4$.

13.3.21 $\mathbf{u}\cdot\mathbf{v} = 1 + 0 = 1$, so $\theta = \cos^{-1}\left(\dfrac{1}{2}\right) = \pi/3$.

13.3.22 $\mathbf{u}\cdot\mathbf{v} = -2 - 2 = -4$, so $\theta = \cos^{-1}\left(-\dfrac{4}{4}\right) = \pi$.

13.3.23 $\mathbf{u}\cdot\mathbf{v} = 4\cdot 4 + 3\cdot(-6) = -2$. The angle between the vectors is thus

$$\cos^{-1}\left(-\frac{2}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(-\frac{2}{5\cdot 2\sqrt{13}}\right) \approx 1.627\,\text{radians}.$$

13.3.24 $\mathbf{u}\cdot\mathbf{v} = 3\cdot 0 + 4\cdot 4 + 0\cdot 5 = 16$. The angle between the vectors is thus

$$\cos^{-1}\left(\frac{16}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{16}{5\cdot\sqrt{41}}\right) \approx 1.047\,\text{radians}.$$

13.3.25 $\mathbf{u}\cdot\mathbf{v} = -10 + 0 + 12 = 2$. The angle between the vectors is thus

$$\cos^{-1}\left(\frac{2}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{116}\cdot\sqrt{14}}\right) \approx 1.521\,\text{radians}.$$

13.3.26 $\mathbf{u}\cdot\mathbf{v} = -27 - 25 + 2 = -50$. The angle between the vectors is thus

$$\cos^{-1}\left(-\frac{50}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(-\frac{50}{\sqrt{38}\cdot\sqrt{107}}\right) \approx 2.472\,\text{radians}.$$

13.3.27 $\mathbf{u}\cdot\mathbf{v} = 2 + 0 - 6 = -4$. The angle between the vectors is thus

$$\cos^{-1}\left(-\frac{4}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(-\frac{4}{\sqrt{13}\cdot\sqrt{21}}\right) \approx 1.815\,\text{radians}.$$

13.3.28 $\mathbf{u}\cdot\mathbf{v} = 2 + 16 - 12 = 6$. The angle between the vectors is thus

$$\cos^{-1}\left(\frac{6}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{6}{\sqrt{53}\cdot\sqrt{24}}\right) \approx 1.402\,\text{radians}.$$

13.3.29 Note that $\overrightarrow{PQ} = \langle 2, 3, -2\rangle$, $\overrightarrow{QR} = \langle -4, 0, 3\rangle$, and $\overrightarrow{RP} = \langle 2, -3, -1\rangle$, and these have lengths $\sqrt{17}$, 5 and $\sqrt{14}$ respectively. The angle at P measures $\cos^{-1}\left(\frac{3}{\sqrt{17}\sqrt{14}}\right) \approx 78.8$ degrees. The angle at Q measures $\cos^{-1}\left(\frac{14}{5\sqrt{17}}\right) \approx 47.2$ degrees, and the angle at R measures $\cos^{-1}\left(\frac{11}{5\sqrt{14}}\right) \approx 54$ degrees.

13.3.30 Note that $\overrightarrow{PQ} = \langle 1, 11\rangle$, $\overrightarrow{QR} = \langle -4, -5\rangle$, and $\overrightarrow{RP} = \langle 3, -6\rangle$, and these vectors have lengths $\sqrt{122}$, $\sqrt{41}$ and $\sqrt{45}$, respectively. The angle at P measures $\cos^{-1}\left(\frac{63}{\sqrt{122}\sqrt{45}}\right) \approx 31.76$ degrees. The angle at Q measures $\cos^{-1}\left(\frac{59}{\sqrt{122}\sqrt{41}}\right) \approx 33.47$ degrees, and the angle at R measures $\cos^{-1}\left(-\frac{18}{\sqrt{45}\sqrt{41}}\right) \approx 114.78$ degrees.

13.3.31 $\text{proj}_{\mathbf{v}}\mathbf{u} = 3\mathbf{i}$. $\text{scal}_{\mathbf{v}}\mathbf{u} = 3$.

13.3.32 $\text{proj}_{\mathbf{v}}\mathbf{u} = -3\mathbf{i}$. $\text{scal}_{\mathbf{v}}\mathbf{u} = -3$.

13.3.33 $\text{proj}_{\mathbf{v}}\mathbf{u} = 3\mathbf{j}$. $\text{scal}_{\mathbf{v}}\mathbf{u} = 3$.

13.3.34 $\text{proj}_{\mathbf{v}}\mathbf{u} = -\mathbf{v} = \langle -2, -2\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = -2\sqrt{2}$.

13.3.35 $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{12}{20}\langle -4, 2\rangle = \langle -12/5, 6/5\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{12}{\sqrt{20}} = \frac{6}{\sqrt{5}}$.

13.3.36 $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{50}{40}\langle 2, 6\rangle = \langle 5/2, 15/2\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{50}{\sqrt{40}} = \frac{25}{\sqrt{10}}$.

13.3.37 $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = -\frac{14}{19}\langle 1, 3, -3\rangle = \langle -14/19, -42/19, 42/19\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = -\frac{14}{\sqrt{19}}$.

13.3.38 $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = -\frac{30}{20}\langle 0, 4, -2\rangle = \langle 0, -6, 3\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = -\frac{30}{\sqrt{20}} = -3\sqrt{5}$.

13.3.39 $\text{proj}_{\mathbf{v}}\mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \dfrac{6}{6}\langle -1,1,-2\rangle = \langle -1,1,-2\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \dfrac{6}{\sqrt{6}} = \sqrt{6}$.

13.3.40 $\text{proj}_{\mathbf{v}}\mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \dfrac{0}{24}\langle 2,-4,2\rangle = \langle 0,0,0\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \dfrac{0}{\sqrt{24}} = 0$.

13.3.41 $w = 30\cdot 50\cos\pi/6 = 750\sqrt{3}$ foot-pounds.

13.3.42 $w = 10\cdot 20\cdot\cos 15^\circ \approx 193.185$ J.

13.3.43 $w = 10\cdot 5\cdot\cos 45^\circ = 25\sqrt{2}$ J.

13.3.44 $w = \langle 4,3,2\rangle\cdot\langle 8,6,0\rangle = 50$ J.

13.3.45 $w = (40\mathbf{i}+30\mathbf{j})\cdot 10\mathbf{i} = 400$ J.

13.3.46 $w = \langle 2,4,1\rangle\cdot\langle 2,4,6\rangle = 4+16+6 = 26$.

13.3.47 Parallel to: use $\mathbf{v} = \langle 1/2, -\sqrt{3}/2\rangle$. $\text{proj}_{\mathbf{v}}\mathbf{F} = \dfrac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = 5\sqrt{3}\left\langle 1/2, -\sqrt{3}/2\right\rangle = \left\langle 5\sqrt{3}/2, -15/2\right\rangle$.
Normal to: $\mathbf{N} = \langle 0,-10\rangle - \langle 5\sqrt{3}/2, -15/2\rangle = \langle -5\sqrt{3}/2, -5/2\rangle$.

13.3.48 Parallel to: use $\mathbf{v} = \langle 5/\sqrt{41}, -4/\sqrt{41}\rangle$.

$$\text{proj}_{\mathbf{v}}\mathbf{F} = \frac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{40}{\sqrt{41}}\left\langle 5/\sqrt{41}, 4/\sqrt{41}\right\rangle = \langle 200/41, 160/41\rangle.$$

Normal to: $\mathbf{N} = \langle 0,-10\rangle - \langle 200/41, 160/41\rangle = \langle -200/41, -570/41\rangle$.

13.3.49 The direction down the plane is $\mathbf{v} = \left\langle \cos\left(-\frac{\pi}{4}\right), \sin\left(-\frac{\pi}{4}\right)\right\rangle = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$ and the gravitational force acting on the block is $\mathbf{F} = \langle 0, 100(9.8)\rangle = \langle 0,-980\rangle$. The component of the force parallel to the plane is $\text{proj}_{\mathbf{v}}\mathbf{F} = \left(\dfrac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = 490\sqrt{2}\left\langle \dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right\rangle = \langle 490,-490\rangle$ and the component of the force perpendicular to the plane is $\mathbf{F} - \text{proj}_{\mathbf{v}}\mathbf{F} = \langle 0,-980\rangle - \langle 490,-490\rangle = \langle -490,-490\rangle$.

13.3.50

a. We have $|\mathbf{W}_{\text{par}}| = 100\sin 20^\circ \approx 34.2$ lbs and $|\mathbf{W}_{\text{perp}}| = 100\cos 20^\circ \approx 94$ lbs.

b. The value of μ must satisfy $\mu \geq \dfrac{|\mathbf{W}_{\text{par}}|}{|\mathbf{W}_{\text{perp}}|} \approx 0.36$ in order for there to be no sliding. This condition is met here, so the block does not slide.

c. In general, we have $|\mathbf{W}_{\text{par}}| = 100\sin\theta$ and $|\mathbf{W}_{\text{perp}}| = 100\cos\theta$. Sliding occurs if $\mu = 0.65 < \dfrac{|\mathbf{W}_{\text{par}}|}{|\mathbf{W}_{\text{perp}}|} = \tan\theta$. Because $\tan^{-1} 0.65 \approx 33^\circ$, sliding will occur at angles greater than 33 degrees.

13.3.51

a. False. One is a vector in the same direction as $\mathbf{u}$ and the other is a vector in the direction of $\mathbf{v}$, so if these vectors aren't in the same direction, they can't be equal.

b. True. This follows because $\mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) = |\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{v}$ and $\mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{v}\cdot\mathbf{u} + |\mathbf{v}|^2$, and these are equal if $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude.

c. True. Let $\mathbf{u} = \langle a,b,c\rangle$. Then $(\mathbf{u}\cdot\mathbf{i})^2 + (\mathbf{u}\cdot\mathbf{j})^2 + (\mathbf{u}\cdot\mathbf{k})^2 = a^2+b^2+c^2 = |\mathbf{u}|^2$.

d. False. For example, consider $\mathbf{u} = \langle 1,0\rangle$, $\mathbf{v} = \langle 0,1\rangle$, and $\mathbf{w} = \langle 2,0\rangle$.

e. False. Consider $\langle 1,-1,0\rangle$, $\langle 2,-1,-1\rangle$ and $\langle 3,-2,-1\rangle$. These are all orthogonal to $\langle 1,1,1\rangle$, but don't all lie in the same line.

f. True. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then $\text{proj}_{\mathbf{v}}\mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$, and this can't be zero unless $\mathbf{u}\cdot\mathbf{v} = 0$.

13.3.52 $\mathbf{v}\cdot\mathbf{w} = 4a - 24 + 21 = 0$ implies that $a = \dfrac{3}{4}$.

13.3.53 $\mathbf{v}\cdot\mathbf{w} = 6 - 10 + 9c = 0$ implies that $c = \dfrac{4}{9}$.

13.3.54 The two other vectors could be $\langle 0,1,-1\rangle$ and $\langle 1,0,0\rangle$.

13.3.55 We must have $4 - 8a + 2b = 0$, so $b = 4a - 2$. These vectors have the form $\langle 1, a, 4a-2\rangle$ where a can be any real number.

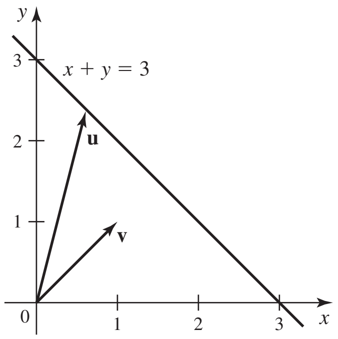
13.3.56 Let $\mathbf{u} = \pm\left\langle \sqrt{2}/2, \sqrt{2}/2, 0\right\rangle$, $\mathbf{v} = \pm\left\langle -\sqrt{2}/2, \sqrt{2}/2, 0\right\rangle$ and $\mathbf{w} = \pm\langle 0,0,1\rangle$.

13.3.57

a. $\text{proj}_{\mathbf{k}}\mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{k}}{\mathbf{k}\cdot\mathbf{k}}\mathbf{k} = \dfrac{|\mathbf{u}|\,|\mathbf{k}|\cos\theta}{\mathbf{k}\cdot\mathbf{k}}\mathbf{k} = \dfrac{1}{2}\mathbf{k}$, which is independent of $\mathbf{u}$.

b. Yes, because the scalar projection is the length of the vector projection. In fact, using the above result, it is equal to 1/2.

13.3.58 $\text{proj}_{\mathbf{v}}\mathbf{u} = \dfrac{3}{2}\langle 1,1\rangle$. Let $\mathbf{z} = \langle x,y\rangle$. Then $\text{proj}_{\mathbf{v}}\mathbf{z} = \dfrac{x+y}{2}\langle 1,1\rangle$, so we need $x+y=3$. All vectors from the origin whose terminal point is on the line $x+y=3$ will have the same projection onto $\mathbf{v}$. For example, $\langle 3,0\rangle$ will work.



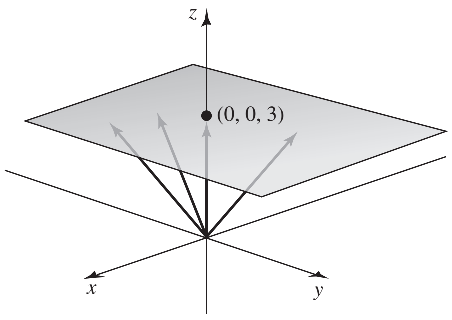
13.3.59 Using the idea from the last problem, any vector of the form $\langle x,y\rangle$ with $x+y=3$ will work, so any vector of the form $\langle x, 3-x\rangle$.

13.3.60 Note that

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle 1,2,3\rangle\cdot\langle 1,1,1\rangle}{3}\langle 1,1,1\rangle = \langle 2,2,2\rangle .$$

We are seeking $\langle x,y,z\rangle$ so that $\dfrac{x+y+z}{3}\langle 1,1,1\rangle = \langle 2,2,2\rangle$, so we require $x+y+z=6$. For example, the vector $\langle 3,2,1\rangle$ will work.

13.3.61 Note that $\text{proj}_{\mathbf{v}}\mathbf{u} = \dfrac{\langle 1,2,3\rangle\cdot\langle 0,0,1\rangle}{1}\langle 0,0,1\rangle = \langle 0,0,3\rangle$. We are seeking $\langle x,y,z\rangle$ so that $\dfrac{z}{1}\langle 0,0,1\rangle = \langle 0,0,3\rangle$, so we require $z=3$. Any vector of the form $\langle x,y,3\rangle$ will suffice.



13.3.62 Let $\mathbf{p} = \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{7}{2}\langle 1,1\rangle$. Then let $\mathbf{n} = \mathbf{u} - \mathbf{p} = \langle 4,3\rangle - \frac{7}{2}\langle 1,1\rangle = \langle 1/2, -1/2\rangle$.

13.3.63 Let $\mathbf{p} = \text{proj}_{\mathbf{v}}\mathbf{u} = -\frac{2}{5}\langle 2,1\rangle$. Then let $\mathbf{n} = \mathbf{u} - \mathbf{p} = \langle -2,2\rangle - -\frac{2}{5}\langle 2,1\rangle = \langle -6/5, 12/5\rangle$.

13.3.64 Let $\mathbf{p} = \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{7}{3}\langle 1,1,1\rangle$. Then let $\mathbf{n} = \mathbf{u} - \mathbf{p} = \langle 4,3,0\rangle - \frac{7}{3}\langle 1,1,1\rangle = \langle 5/3, 2/3, -7/3\rangle$.

13.3.65 Let $\mathbf{p} = \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{1}{2}\langle 2,1,1\rangle$. Then let $\mathbf{n} = \mathbf{u} - \mathbf{p} = \langle -1,2,3\rangle - \frac{1}{2}\langle 2,1,1\rangle = \langle -2, 3/2, 5/2\rangle$.

13.3.66 Let $P(x,y)$ be a point on ℓ. Then $\mathbf{n}$ is orthogonal to $\overrightarrow{P_0P} = \langle x - x_0, y - y_0\rangle$,, which implies that $\mathbf{n} \cdot \overrightarrow{P_0P} = \langle a,b\rangle \cdot \langle x - x_0, y - y_0\rangle = 0$, or $a(x - x_0) + b(y - y_0) = 0$.

13.3.67 We have $x_0 = 2$, $y_0 = 6$, and $\mathbf{n} = \langle a,b\rangle = \langle 3,-7\rangle$. So the equation of the line is $3(x-2)-7(y-6) = 0$, or $3x - 7y = -36$.

13.3.68 We have $x_0 = 1$, $y_0 = -3$, and $\mathbf{n} = \langle a,b\rangle = \langle 4,0\rangle$. So the equation of the line is $4(x-1)+0(y+3) = 0$, or $x = 1$.

13.3.69 The vector $\mathbf{n}$ has a slope of $\frac{3}{5}$ and because this vector is orthogonal to the line, the line has a slope of $-\frac{5}{3}$.

13.3.70 $\mathbf{I} \cdot \mathbf{J} = -\frac{1}{2} + \frac{1}{2} = 0$. Also, $|\mathbf{I}| = \sqrt{1/2 + 1/2} = 1 = |\mathbf{J}|$.

13.3.71 $\mathbf{I} = \left\langle 1/\sqrt{2}, 1/\sqrt{2}\right\rangle = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. $\mathbf{J} = \left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. $\mathbf{i} = \langle 1,0\rangle = \frac{\sqrt{2}}{2}(\mathbf{I} - \mathbf{J})$. $\mathbf{j} = \frac{\sqrt{2}}{2}(\mathbf{I} + \mathbf{J})$.

13.3.72 $\langle 2,-6\rangle = 2\mathbf{i} - 6\mathbf{j} = 2 \cdot \frac{\sqrt{2}}{2}(\mathbf{I} - \mathbf{J}) - 6 \cdot \frac{\sqrt{2}}{2}(\mathbf{I} + \mathbf{J}) = -2\sqrt{2}\mathbf{I} - 4\sqrt{2}\mathbf{J}$.

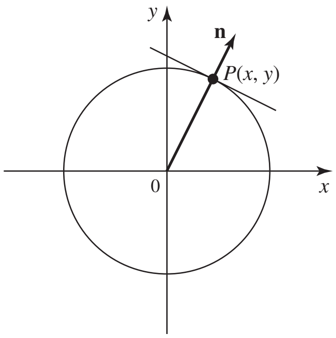
13.3.73

a. $|\mathbf{I}| = \sqrt{1/4 + 1/4 + 1/2} = 1$. $|\mathbf{J}| = \sqrt{1/2 + 1/2 + 0} = 1$. $|\mathbf{K}| = \sqrt{1/4 + 1/4 + 1/2} = 1$.

b. $\mathbf{I} \cdot \mathbf{J} = -1/2\sqrt{2} + 1/2\sqrt{2} = 0$. $\mathbf{I} \cdot \mathbf{K} = 1/4 + 1/4 - 1/2 = 0$, and $\mathbf{J} \cdot \mathbf{K} = -1/2\sqrt{2} + 1/2\sqrt{2} = 0$.

c. Let $\langle 1,0,0\rangle = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$. Then $\frac{1}{2}a - \frac{1}{\sqrt{2}}b + \frac{1}{2}c = 1$, $\frac{1}{2}a + \frac{1}{\sqrt{2}}b + \frac{1}{2}c = 0$, and $\frac{1}{\sqrt{2}}a - \frac{1}{\sqrt{2}}c = 0$. Solving this system of linear equations yields $a = \frac{1}{2}$, $b = -\frac{1}{\sqrt{2}}$, and $c = \frac{1}{2}$. Thus, $\langle 1,0,0\rangle = \frac{1}{2}\mathbf{I} + -\frac{1}{\sqrt{2}}\mathbf{J} + \frac{1}{2}\mathbf{K}$.

13.3.74



a. The equation of the circle is $x^2+y^2=1$, so $2x+2y\dfrac{dy}{dx}=0$, and we have $\dfrac{dy}{dx}=-\dfrac{x}{y}$. The slope of the line through the origin containing the vector $\langle x,y\rangle$ is $\dfrac{y}{x}$, and $\dfrac{y}{x}\cdot-\dfrac{x}{y}=-1$. Thus, the vector $\langle x,y\rangle$ is normal to the tangent vector.

b. Let $x=\cos\theta$ and $y=\sin\theta$. Then the previous part shows that $\mathbf{n}=\langle x,y\rangle=\langle\cos\theta,\sin\theta\rangle$.

c. The velocity is normal where $\langle x,y\rangle=k\langle 1,2\rangle$. Then $k^2+4k^2=1$, so $k=\dfrac{\pm1}{\sqrt{5}}$, and the points are $(\pm1/\sqrt{5},\pm2/\sqrt{5})$.

d. We need $\langle 1,2\rangle\cdot\langle x,y\rangle=0$, so $x+2y=0$. Because we also have $x^2+y^2=1$, we see that $4y^2+y^2=1$, so $y=\dfrac{\pm1}{\sqrt{5}}$, and $x=\dfrac{\mp2}{\sqrt{5}}$. The two points are $(2/\sqrt{5},-1/\sqrt{5})$ and $(-2/\sqrt{5},1/\sqrt{5})$.

e. The component of $\mathbf{v}=\langle 1,2\rangle$ normal to C is $\text{proj}_{\mathbf{n}}\mathbf{v}=\dfrac{x+2y}{x^2+y^2}\langle x,y\rangle=\left\langle x^2+2xy,xy+2y^2\right\rangle$. In terms of θ, this is $\left\langle\cos^2\theta+2\cos\theta\sin\theta,\sin\theta\cos\theta+2\sin^2\theta\right\rangle$.

f. The net flow is 0, so there is no accumulation inside the circle.

13.3.75

a. The faces on $y=0$ and $z=0$.

b. The faces on $y=1$ and $z=1$.

c. The faces on $x=0$ and $x=1$.

d. Because $\mathbf{Q}$ is tangential on this face, the scalar component of $\mathbf{Q}$ normal to the face is 0.

e. The scalar component of $\mathbf{Q}$ normal to $z=1$ is 1. Note that a vector normal to $z=1$ is $\langle 0,0,1\rangle$.

f. The scalar component of $\mathbf{Q}$ normal to $y=0$ is 2. Note that a vector normal to $y=0$ is $\langle 0,1,0\rangle$.

13.3.76

a. $\mathbf{r}_{0j}=\left\langle 2\cos(\dfrac{(j-1)\pi}{3}),2\sin(\dfrac{(j-1)\pi}{3})\right\rangle$ for $j=1,2,\ldots,6$.

b. $\mathbf{r}_{12}=\langle 2\cos(2\pi/3),2\sin(2\pi/3)\rangle=\left\langle -1,\sqrt{3}\right\rangle$.
$\mathbf{r}_{34}=\langle 2\cos(4\pi/3),2\sin(4\pi/3)\rangle=\left\langle -1,-\sqrt{3}\right\rangle$.
$\mathbf{r}_{61}=\langle 2\cos(\pi/3),2\sin(\pi/3)\rangle=\left\langle 1,\sqrt{3}\right\rangle$.

c. $\mathbf{r}_{07}=\mathbf{r}_{01}+\mathbf{r}_{02}=\left\langle 3,\sqrt{3}\right\rangle$.
$\mathbf{r}_{17}=\mathbf{r}_{02}=\langle 2\cos(\pi/3),2\sin(\pi/3)\rangle=\left\langle 1,\sqrt{3}\right\rangle$.
$\mathbf{r}_{47}=\langle 4,0\rangle+\mathbf{r}_{17}=\left\langle 5,\sqrt{3}\right\rangle$.
$\mathbf{r}_{75}=\mathbf{r}_{70}+\mathbf{r}_{05}=-2\left\langle 2,\sqrt{3}\right\rangle$.

13.3.77

a. Let the coordinates of R be (x,y,z). By symmetry, we have $y=0$. We must have $x^2+y^2+z^2=(x-\sqrt{3})^2+(y+1)^2+z^2=(x-\sqrt{3})^2+(y-1)^2+z^2=4$. The first equality gives $x^2+z^2=x^2-2\sqrt{3}x+3+1+z^2$, so $2\sqrt{3}x=4$, and $x=\dfrac{2}{\sqrt{3}}$. It then follows that $z=\dfrac{2\sqrt{2}}{\sqrt{3}}$.

b. We have $\mathbf{r}_{OP}=\left\langle\sqrt{3},-1,0\right\rangle$, $\mathbf{r}_{OQ}=\left\langle\sqrt{3},1,0\right\rangle$, $\mathbf{r}_{PQ}=\langle 0,2,0\rangle$, $\mathbf{r}_{OR}=\left\langle 2/\sqrt{3},0,2\sqrt{2}/\sqrt{3}\right\rangle$, and $\mathbf{r}_{PR}=\left\langle -\sqrt{3}/3,1,2\sqrt{2}/\sqrt{3}\right\rangle$.

13.3.78 $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$, so $|\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,|\cos\theta| \le |\mathbf{u}|\,|\mathbf{v}|$, because $|\cos\theta| \le 1$.

13.3.79 $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \mathbf{v}\cdot\mathbf{u}$.

13.3.80 $c(\mathbf{u}\cdot\mathbf{v}) = c(u_1v_1 + u_2v_2 + u_3v_3) = cu_1v_1 + cu_2v_2 + cu_3v_3 = (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3 = (c\mathbf{u})\cdot\mathbf{v}$. Using this result and the previous, we have $c(\mathbf{u}\cdot\mathbf{v}) = c(\mathbf{v}\cdot\mathbf{u}) = (c\mathbf{v})\cdot\mathbf{u} = \mathbf{u}\cdot(c\mathbf{v})$.

13.3.81 $\mathbf{u}(\mathbf{v}+\mathbf{w}) = \langle u_1, u_2, u_3\rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3\rangle = u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3) = u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + u_3v_3 + u_3w_3 = (u_1v_1 + u_2v_2 + u_3v_3) + (u_1w_1 + u_2w_2 + u_3w_3) = (\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{w})$.

13.3.82

a. Using the previous results, we have $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} + (\mathbf{u}+\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} = |\mathbf{u}|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2$.

b. If the two vectors are perpendicular, then $\mathbf{u}\cdot\mathbf{v} = 0$, so the above reduces to $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = |\mathbf{u}|^2 + |\mathbf{v}|^2$.

c. Using the previous results, we have $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} - (\mathbf{u}+\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) - \mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} = |\mathbf{u}|^2 - |\mathbf{v}|^2$.

13.3.83

a. $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2+c^2}}$, $\cos\beta = \dfrac{b}{\sqrt{a^2+b^2+c^2}}$, and $\cos\gamma = \dfrac{c}{\sqrt{a^2+b^2+c^2}}$. Thus, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{a^2}{a^2+b^2+c^2} + \dfrac{b^2}{a^2+b^2+c^2} + \dfrac{c^2}{a^2+b^2+c^2} = 1$.

b. We require $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{1}{2} + \dfrac{1}{2} + \cos^2\gamma = 1$, so $\gamma = 90°$. The vector could be $\langle 1, 1, 0\rangle$; it makes a 90 degree angle with $\mathbf{k}$.

c. We require $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{1}{4} + \dfrac{1}{4} + \cos^2\gamma = 1$, so $\gamma = 45°$. The vector could be $\langle 1, 1, \sqrt{2}\rangle$; it makes a 45 degree angle with $\mathbf{k}$.

d. No. If so, we would have $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{3}{4} + \dfrac{3}{4} + \cos^2\gamma = 1$, which would imply that $\cos^2\gamma = -\dfrac{1}{2}$, which can't occur.

e. If $\alpha = \beta = \gamma$, then $3\cos^2\alpha = 1$, and $\alpha = \cos^{-1}(\sqrt{3}/3) \approx 54.7356$ degrees. The vector could be $\langle 1, 1, 1\rangle$.

13.3.84 $|\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,|\cos\theta| = |\mathbf{u}|\,|\mathbf{v}|$ if and only if $\cos\theta = \pm 1$, which occurs only when $\mathbf{u}$ and $\mathbf{v}$ are parallel, or if either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

13.3.85 $\mathbf{u}\cdot\mathbf{v} = -24 - 15 + 6 = -33$. $|\mathbf{u}| = \sqrt{3^2 + (-5)^2 + 6^2} = \sqrt{70}$. $|\mathbf{v}| = \sqrt{(-8)^2 + 3^2 + 1^2} = \sqrt{74}$. Note that

$$33 < \sqrt{70}\sqrt{74},$$

so $|\mathbf{u}\cdot\mathbf{v}| < |\mathbf{u}|\,|\mathbf{v}|$.

13.3.86 We have $|\mathbf{u}\cdot\mathbf{v}| = \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab} \le |\mathbf{u}|\,|\mathbf{v}| = \sqrt{a+b}\sqrt{b+a} = a + b$. Thus,

$$\sqrt{ab} \le \frac{a+b}{2}.$$

13.3.87

a. We have $|\mathbf{u}+\mathbf{v}|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} + (\mathbf{u}+\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} = |\mathbf{u}|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2$

b. Note that $2(\mathbf{u}\cdot\mathbf{v}) \le 2\,|\mathbf{u}\cdot\mathbf{v}| \le 2\,|\mathbf{u}|\,|\mathbf{v}|$. Thus (using the previous part) we have,

$$|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 \le |\mathbf{u}|^2 + 2\,|\mathbf{u}|\,|\mathbf{v}| + |\mathbf{v}|^2 \le (|\mathbf{u}| + |\mathbf{v}|)^2.$$

c. Taking square roots of the previous result, and using the fact that the square root function is strictly increasing, we have $|\mathbf{u}+\mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$.

d. Because the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ form a triangle, we can interpret this as meaning that the sum of the lengths of any two sides of a triangle is greater than or equal to the length of the other side.

13.3.88 Let $\mathbf{v} = \langle u_2, u_3, u_1\rangle$. Then $\mathbf{u}\cdot\mathbf{v} = u_1u_2 + u_2u_3 + u_1u_3$ and $|\mathbf{u}|\,|\mathbf{v}| = u_1^2 + u_2^2 + u_3^2$. So the Cauchy-Schwarz inequality gives $u_1u_2 + u_2u_3 + u_3u_1 \le |\mathbf{u}|^2$. Thus,

$$(u_1+u_2+u_3)^2 = u_1^2+u_2^2+u_3^2 + 2(u_1u_2+u_2u_3+u_3u_1) \le u_1^2+u_2^2+u_3^2 + 2\,|\mathbf{u}|^2 = 3\,|\mathbf{u}|^2\,.$$

13.3.89

a. One diagonal consists of the sum of one side ($\mathbf{u}$) and the side opposite the side adjacent to $\mathbf{u}$, but because it is a parallelogram, the side opposite $\mathbf{v}$ is also $\mathbf{v}$. So the diagonal is $\mathbf{u}+\mathbf{v}$. The other diagonal is the difference of two adjacent sides, so it is $\mathbf{u}-\mathbf{v}$.

b. The two diagonals are equal when $|\mathbf{u}+\mathbf{v}| = |\mathbf{u}-\mathbf{v}|$. Squaring both sides, we see that this is equivalent to requiring $|\mathbf{u}|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 = |\mathbf{u}|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2$, which would imply that $2(\mathbf{u}\cdot\mathbf{v}) = -2(\mathbf{u}\cdot\mathbf{v})$, or $4(\mathbf{u}\cdot\mathbf{v}) = 0$. So if the diagonals are equal, the vectors are orthogonal. These steps are reversible, so the converse is also true.

c. $|\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 + |\mathbf{u}|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 = 2(|\mathbf{u}|^2 + |\mathbf{v}|^2)$.

13.4 Cross Products

13.4.1 Two parallel vectors have $\sin\theta = 0$ where θ is the angle between them. Thus, $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = 0$.

13.4.2 Two perpendicular vectors have $\sin\theta = 1$ where θ is the angle between them. Thus, $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = |\mathbf{u}|\,|\mathbf{v}|$.

13.4.3

a. This implies that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

b. This implies that $\mathbf{u}$ and $\mathbf{v}$ are parallel.

13.4.4 $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, so in particular, it is orthogonal to $\mathbf{u}$. So $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0$.

13.4.5 $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin(\pi/4) = \sqrt{2}/2$.

13.4.6 $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin(2\pi/3) = 12\cdot(\sqrt{3}/2) = 6\sqrt{3}$.

13.4.7 Because $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v})$, we have $\mathbf{v}\times\mathbf{u} = -3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$.

13.4.8 Because the angle between $\mathbf{v}$ and itself is 0 radians, we have $|\mathbf{v}\times\mathbf{v}| = |\mathbf{v}|\,|\mathbf{v}|\sin 0 = 0$. Thus $\mathbf{v}\times\mathbf{v}$ is the zero vector.

13.4.9 If $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, then $\mathbf{u}\times\mathbf{v}$ can be thought of as the determinant of the matrix

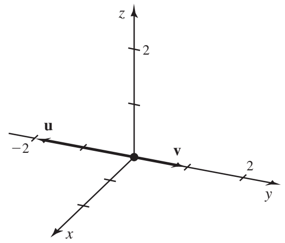
$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

13.4.10 The torque produced by the force $\mathbf{F}$ about the head of vector $\mathbf{r}$ is $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

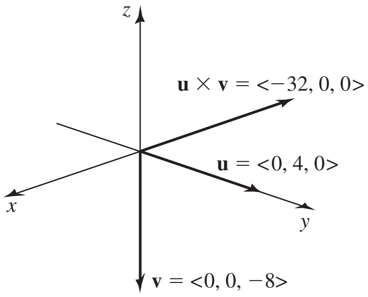
13.4.11 $\mathbf{u} \times \mathbf{v} = \langle 3,0,0\rangle \times \langle 0,5,0\rangle = \langle 0,0,15\rangle$.

13.4.12 $\mathbf{u} \times \mathbf{v} = \langle -4,0,0\rangle \times \langle 0,0,2\rangle = \langle 0,8,0\rangle$.

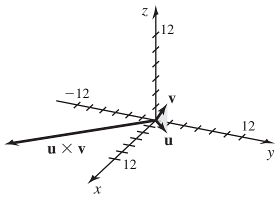
13.4.13 $\mathbf{u} \times \mathbf{v} = \langle 0,0,0\rangle$, so $|\mathbf{u} \times \mathbf{v}| = 0$.



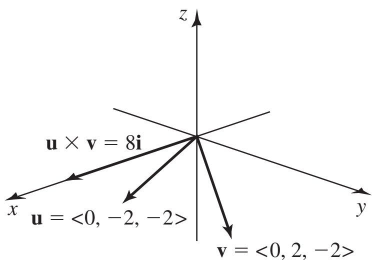
13.4.14 $\mathbf{u} \times \mathbf{v} = \langle -32,0,0\rangle$, so $|\mathbf{u} \times \mathbf{v}| = 32$.



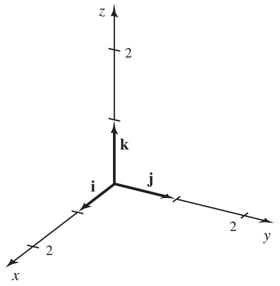
13.4.15 $\mathbf{u} \times \mathbf{v} = \left\langle 9\sqrt{2}, -9\sqrt{2}, 0\right\rangle$, so $|\mathbf{u} \times \mathbf{v}| = 18$.



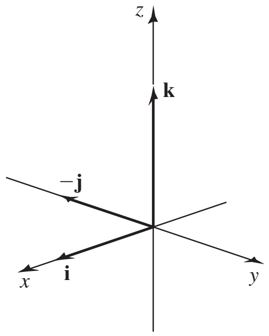
13.4.16 $\mathbf{u} \times \mathbf{v} = \langle 8,0,0\rangle$, so $|\mathbf{u} \times \mathbf{v}| = 8$.



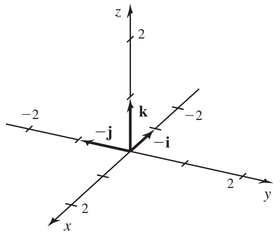
13.4.17 $\mathbf{j} \times \mathbf{k} = \mathbf{i}$.



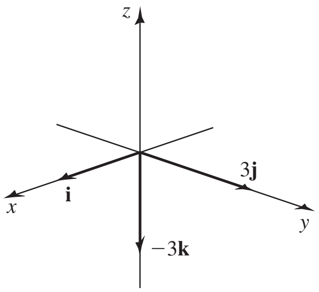
13.4.18 $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

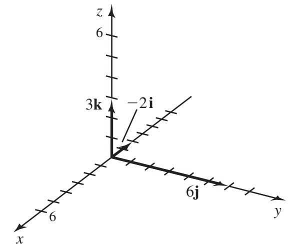


13.4.19 $-\mathbf{j} \times \mathbf{k} = -\mathbf{i}$.

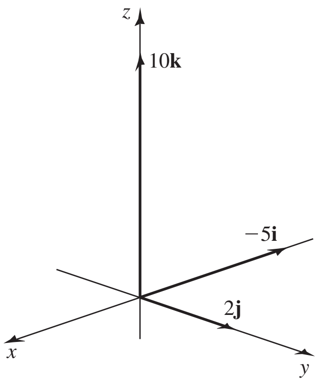

z
2
−2
k
−2
−j
−i
2
y
2
x

13.4.20 $3\mathbf{j} \times \mathbf{i} = -3\mathbf{k}$.





13.4.21 $-2\mathbf{i} \times 3\mathbf{k} = 6\mathbf{j}$.



13.4.22 $2\mathbf{j} \times -5\mathbf{i} = 10\mathbf{k}$.

13.4.23 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & 0 \\ 0 & 3 & -6 \end{vmatrix} = \langle -30, 18, 9 \rangle$. $\mathbf{v} \times \mathbf{u} = \langle 30, -18, -9 \rangle$.

13.4.24 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \langle -2, -4, -4 \rangle$. $\mathbf{v} \times \mathbf{u} = \langle 2, 4, 4 \rangle$.

13.4.25 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -9 \\ -1 & 1 & -1 \end{vmatrix} = \langle 6, 11, 5 \rangle$. $\mathbf{v} \times \mathbf{u} = \langle -6, -11, -5 \rangle$.

13.4.26 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 6 \\ 1 & 2 & -1 \end{vmatrix} = \langle -8, 9, 10 \rangle$. $\mathbf{v} \times \mathbf{u} = \langle 8, -9, -10 \rangle$.

13.4.27 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & -2 \\ 1 & 3 & -2 \end{vmatrix} = \langle 8, 4, 10 \rangle$. $\mathbf{v} \times \mathbf{u} = \langle -8, -4, -10 \rangle$.

13.4.28 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -10 & 15 \\ .5 & 1 & -.6 \end{vmatrix} = \langle -9, 8.7, 7 \rangle$. $\mathbf{v} \times \mathbf{u} = \langle 9, -8.7, -7 \rangle$.

13.4.29 $|\mathbf{u}\times\mathbf{v}| = |\langle -2,-6,9\rangle| = \sqrt{4+36+81} = 11.$

13.4.30 $|\mathbf{u}\times\mathbf{v}| = |\langle -2,5,-3\rangle| = \sqrt{4+9+25} = \sqrt{38}.$

13.4.31 $|\mathbf{u}\times\mathbf{v}| = |\langle 5,-4,7\rangle| = \sqrt{25+16+49} = 3\sqrt{10}.$

13.4.32 $|\mathbf{u}\times\mathbf{v}| = |\langle 4,26,28\rangle| = \sqrt{16+676+784} = \sqrt{1476} = 6\sqrt{41}.$

13.4.33 $\frac{1}{2}\cdot\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\cdot|\langle 3,0,1\rangle\times\langle 1,1,0\rangle| = \frac{1}{2}\cdot|\langle -1,1,3\rangle| = \frac{\sqrt{11}}{2}.$

13.4.34 Two of the sides are $\mathbf{u} = \langle 1,2,3\rangle$ and $\mathbf{v} = \langle 6,5,4\rangle$.

The area is $\frac{1}{2}|\mathbf{u}\times\mathbf{v}| = \frac{1}{2}\left\|\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 3\\ 6 & 5 & 4\end{vmatrix}\right\| = \frac{1}{2}|\langle -7,14,-7\rangle| = \frac{7\sqrt{6}}{2}.$

13.4.35 $\frac{1}{2}\cdot\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\cdot|\langle 2,10,2\rangle\times\langle 1,1,1\rangle| = \frac{1}{2}\cdot|\langle 8,0,-8\rangle| = 4\sqrt{2}.$

13.4.36 $\frac{1}{2}\cdot\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\cdot|\langle -2,3,2\rangle\times\langle 1,0,2\rangle| = \frac{1}{2}\cdot|\langle 6,6,-3\rangle| = \frac{9}{2}.$

13.4.37 The area is $\frac{1}{2}|\mathbf{u}\times\mathbf{v}| = \frac{1}{2}\left\|\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 3 & 3\\ 6 & 0 & 6\end{vmatrix}\right\| = \frac{1}{2}|\langle 18,0,-18\rangle| = 9\sqrt{2}.$

13.4.38 The area is $\frac{1}{2}|\mathbf{u}\times\mathbf{v}| = \frac{1}{2}\left\|\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 6 & 0\\ 4 & 4 & 4\end{vmatrix}\right\| = \frac{1}{2}|\langle 24,0,-24\rangle| = 12\sqrt{2}.$

13.4.39 If the points are collinear, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel, and $\overrightarrow{AB}\times\overrightarrow{AC} = \mathbf{0}$. On the other hand, if the points are not collinear, then the area of area of triangle ABC is not 0. So $\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| \neq 0$, and it follows that $\overrightarrow{AB}\times\overrightarrow{AC} \neq 0$.

13.4.40 $\overrightarrow{AB}\times\overrightarrow{AC} = \langle 2,2,6\rangle\times\langle 6,6,18\rangle = \langle 0,0,0\rangle$, so the points are collinear.

13.4.41 $\overrightarrow{AB}\times\overrightarrow{AC} = \langle 4,6,6\rangle\times\langle 7,12,13\rangle \neq \langle 0,0,0\rangle$, so the points are not collinear.

13.4.42 Let $\mathbf{u} = \langle 1,2,3\rangle$ and $\mathbf{v} = \langle -2,4,-1\rangle$. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 3\\ -2 & 4 & -1\end{vmatrix} = \langle -14,-5,8\rangle$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

13.4.43 Let $\mathbf{u} = \langle 0,1,2\rangle$ and $\mathbf{v} = \langle -2,0,3\rangle$. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 2\\ -2 & 0 & 3\end{vmatrix} = \langle 3,-4,2\rangle$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

13.4.44 Let $\mathbf{u} = \langle 8,0,4\rangle$ and $\mathbf{v} = \langle -8,2,1\rangle$. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 8 & 0 & 4\\ -8 & 2 & 1\end{vmatrix} = \langle -8,-40,16\rangle$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

13.4.45 $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 20 & 0 & 0 \end{vmatrix} = \langle 0, 20, -20 \rangle.$

13.4.46 $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 10 & 10 & 0 \end{vmatrix} = \langle -20, 20, 20 \rangle.$

13.4.47 $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 0 & 0 \\ 5 & 0 & -5 \end{vmatrix} = \langle 0, 50, 0 \rangle$ has magntitude 50, while $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 0 & 0 \\ 4 & -3 & 0 \end{vmatrix} = \langle 0, 0, -30 \rangle$ has magnitude 30, so the first force has greater magnitude.

13.4.48 $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & -5 \\ 1 & 0 & -10 \end{vmatrix} = \langle 0, 45, 0 \rangle$. The magnitude is 45 and the direction is in the positive y direction.

13.4.49 $|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = \frac{1}{4} \cdot 20 \cdot \frac{\sqrt{2}}{2} = \frac{5\sqrt{2}}{2}\,\mathrm{N} \cdot \mathrm{m}.$

13.4.50 $|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = \frac{5}{6} \cdot \frac{3}{2} \cdot \sin(\pi/2) = 1.25\,\mathrm{ft} \cdot \mathrm{lb}.$

13.4.51 Note that $\mathbf{r} = 0.66\mathbf{k}$, and $\mathbf{F} = 40\mathbf{j}$. $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.66 \\ 0 & 40 & 0 \end{vmatrix} = \langle -26.4, 0, 0 \rangle$. The magnitude of the torque is 26.4 Newton-meters and the direction is on the negative x axis.

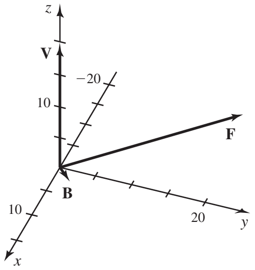
13.4.52

a. The torque of the shoulder has magnitude $2 \cdot 20 = 40$ ft lbs and the direction is perpendicular to $\mathbf{r}$ and $\mathbf{F}$, into the page.

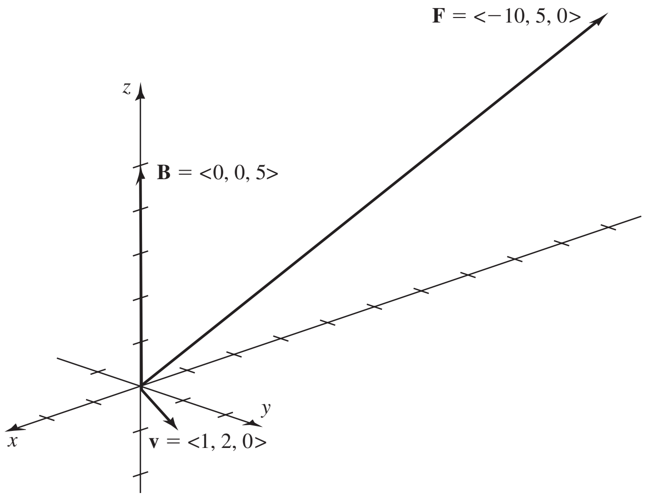
b. The torque of the elbow has magnitude $1 \cdot 20 = 20$ foot pounds, and the direction is the same as the torque of the shoulder, into the page.

13.4.53 $\mathbf{F} = 1 \cdot (\mathbf{v} \times \mathbf{B}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 20 \\ 1 & 1 & 0 \end{vmatrix} = \langle -20, 20, 0 \rangle.$

The magnitude of $\mathbf{F}$ is $20\sqrt{2}$ and the angle of the force is 135 degrees with the positive x axis in the xy-plane.



13.4.54 $\mathbf{F} = -1 \cdot (\mathbf{v} \times \mathbf{B}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 0 & 0 & 5 \end{vmatrix} = \langle -10, 5, 0 \rangle$. The magnitude of $\mathbf{F}$ is $5\sqrt{5}$ and the angle of the force is about 153.435 degrees with the positive x axis in the xy-plane.



13.4.55 $|\mathbf{F}| = |q(\mathbf{v}\times\mathbf{B})| = \left|-1.6\cdot 10^{-19}\right|\text{ C }\cdot 2\cdot 10^5\cdot 2\cdot\sin 45^\circ = 4.53\cdot 10^{-14}\,\text{kg}\cdot\text{m/s}^2$.

13.4.56 $\mathbf{F} = q(\mathbf{v}\times\mathbf{B}) = 5\cdot 10^{-12}\mathbf{k}$, $\mathbf{v} = 2\cdot 10^6\mathbf{j}$. We must have $\mathbf{B} = b\mathbf{i}$ for some scalar b. We have

$$1.6\cdot 10^{-19}(2\cdot 10^6\mathbf{j}\times b\mathbf{i}) = 5\cdot 10^{-12}\mathbf{k},$$

so $b = -\dfrac{5\cdot 10^{-12}}{1.6\cdot 10^{-19}\cdot 2\cdot 10^6} = -\dfrac{125}{8}$. So $\mathbf{B} = -15.625\mathbf{i}$.

13.4.57

a. False. For example $\mathbf{i}\times\mathbf{i} = \langle 0,0,0\rangle$, even though $\mathbf{i}\neq\langle 0,0,0\rangle$.

b. False. For example, $2\mathbf{i}\times 4\mathbf{j} = 8\mathbf{k}$ has magnitude 8, while $2\mathbf{i}$ has magnitude 2 and $4\mathbf{j}$ has magnitude 4.

c. False. If the compass directions are thought to lie in a plane, $\mathbf{u}\times\mathbf{v}$ doesn't lie in that plane, so it can't be a compass direction.

d. True. If both were nonzero, the first statement implies that the vectors are parallel, and the second that they are perpendicular, which can't both occur. So at least one of the vectors must be the zero vector.

e. False. $\mathbf{i}\times 2\mathbf{i} = \langle 0,0,0\rangle = \mathbf{i}\times 3\mathbf{i}$, but $2\mathbf{i}\neq 3\mathbf{i}$.

13.4.58 Note that $\langle a,a,2\rangle\times\langle 1,a,3\rangle = \left\langle a, 2-3a, -a+a^2\right\rangle$. So $a = 2$.

13.4.59 Let $\mathbf{u} = \langle u_1,u_2,u_3\rangle$. Then we have

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 1\\ u_1 & u_2 & u_3\end{vmatrix} = \langle -1,-1,2\rangle,$$

so $u_3-u_2 = -1$, $u_1-u_3 = -1$, and $u_2-u_1 = 2$. The solutions to this system of linear equation can be characterized by letting u_1 be arbitrary, and by letting $u_2 = u_1+2$ and $u_3 = u_1+1$. Thus, $\mathbf{u} = \langle u_1, u_1+2, u_1+1\rangle$ for any real number u_1.

13.4.60 Let $\mathbf{u} = \langle u_1,u_2,u_3\rangle$. Then we have

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 1\\ u_1 & u_2 & u_3\end{vmatrix} = \langle 0,0,1\rangle,$$

so $u_3-u_2 = 0$, $u_1-u_3 = 0$, and $u_2-u_1 = 1$. This system of linear equations has no solutions, so there are no vectors $\mathbf{u}$ which satisfy the given equation.

13.4.61 Two of the sides of the triangle are $\mathbf{u} = \langle -a, b, 0\rangle$ and $\mathbf{v} = \langle -a, 0, c\rangle$.

The area is $\frac{1}{2}|\mathbf{u}\times\mathbf{v}| = \frac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c\end{vmatrix}\right\| = \frac{1}{2}|\langle bc, ac, ab\rangle| = \frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}$.

13.4.62 $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \langle u_1, u_2, u_3\rangle \cdot \langle v_2w_3 - v_3w_2, w_1v_3 - v_1w_3, v_1w_2 - w_1v_2\rangle = u_1(v_2w_3 - v_3w_2) + u_2(w_1v_3 - v_1w_3) + u_3(v_1w_2 - w_1v_2)$. Note that this is exactly the expression for the determinant of the given matrix, as can be seen by expanding by cofactors across the top row.

13.4.63 $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |\mathbf{u}|\,|\mathbf{v}\times\mathbf{w}|\,|\cos\theta|$. Because $|\mathbf{v}\times\mathbf{w}|$ represents the area of the base, we just need to see that the height of the parallelepiped is $|\mathbf{u}|\,|\cos\theta|$. Note that the height is given by the scalar projection of $\mathbf{u}$ on $\mathbf{v}\times\mathbf{w}$, which has value $|\cos\theta|\,|\mathbf{u}|$. Thus the given expression represents the volume of the parallelepiped.

13.4.64 The volume is $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |\langle 3, 1, 0\rangle \cdot \langle 19, -9, -2\rangle| = 57 - 9 + 0 = 48$.

13.4.65 If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are coplanar, then the volume of the corresponding parallelepiped is 0 which implies that $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = 0$. If vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are not coplanar, then the volume of the corresponding parallelepiped is not 0 which implies that $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| \neq 0$.

13.4.66 Using one of the results above, we see that $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3\end{vmatrix}$ and similarly we have $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3\end{vmatrix}$. These two determinants are equal, as can be seen by fact that the underlying matrices differ by an even number of row transpositions.

13.4.67 Because $\mathbf{F} = q(\mathbf{v}\times\mathbf{B})$, we have $|\mathbf{F}| = |q|\,|\mathbf{v}|\,|\mathbf{B}|\sin\theta$. Thus, $\frac{m\,|\mathbf{v}|^2}{R} = |q|\,|\mathbf{v}|\,|\mathbf{B}|\sin\pi/2$. Therefore,

$$|\mathbf{v}| = \frac{R\,|q|\,|\mathbf{B}|}{m} = \frac{0.002\cdot 1.6\cdot 10^{-19}\cdot .05}{9\cdot 10^{-31}} \approx 1.758\cdot 10^7 \text{ m/s}.$$

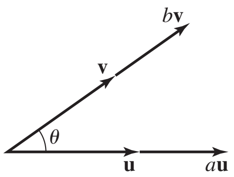
13.4.68

a. $|\mathbf{u}\times\mathbf{u}| = |\mathbf{u}|^2\sin 0 = 0$.

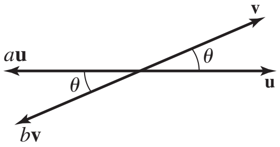
b. $\mathbf{u}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3\end{vmatrix} = \langle 0, 0, 0\rangle$.

c. Because $\mathbf{u}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{u})$, we must have $2(\mathbf{u}\times\mathbf{u}) = 0$, so $\mathbf{u}\times\mathbf{u} = 0$.

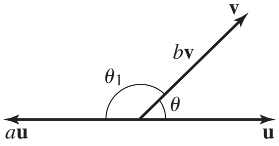
13.4.69 The result is trivial if either $a = 0$ or $b = 0$, so assume $ab \neq 0$. Note that the sine of the angle between $a\mathbf{u}$ and $b\mathbf{v}$ is the same as the sine of the angle between $\mathbf{u}$ and $\mathbf{v}$, as is demonstrated in the following diagrams.



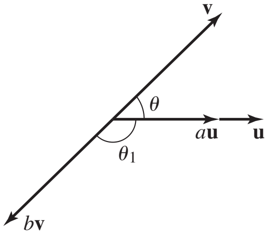
$a > 0,\ b > 0$



$a < 0,\ b < 0$



$a < 0,\, b > 0$



$a > 0,\, b < 0$

By the definition: $|(a\mathbf{u}) \times (b\mathbf{v})| = |a\mathbf{u}|\,|b\mathbf{v}| \sin\theta$, where θ is the angle between $a\mathbf{u}$ and $b\mathbf{v}$. But this is equal to $|a|\,|\mathbf{u}|\,|b|\,|\mathbf{v}|\sin\theta = |ab|\,(|\mathbf{u}|\,|\mathbf{v}|\sin\theta) = |ab|\,(|\mathbf{u}\times\mathbf{v}|)$. When a and b have the same sign, the directions are also the same, because they are determined by the right-hand rule (see diagrams above.) When a and b have opposite signs, the directions are opposite, but then $ab < 0$.

Using the determinant formula:

$$(a\mathbf{u}) \times (b\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ au_1 & au_2 & au_3 \\ bv_1 & bv_2 & bv_3 \end{vmatrix} = ab \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = ab(\mathbf{u}\times\mathbf{v}).$$

13.4.70 False. For example, $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j} \neq \mathbf{0}$.

13.4.71 True. $(\mathbf{u} - \mathbf{v}) \times (\mathbf{u} + \mathbf{v}) = \mathbf{u} \times \mathbf{u} + \mathbf{u} \times \mathbf{v} - (\mathbf{v} \times \mathbf{u}) - (\mathbf{v} \times \mathbf{v}) = 2(\mathbf{u} \times \mathbf{v}) = (2\mathbf{u} \times \mathbf{v})$.

13.4.72 This follows from number 66 above, together with the commutativity of the dot product.

13.4.73

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_2w_3 - v_3w_2 & v_3w_1 - v_1w_3 & v_1w_2 - v_2w_1 \end{vmatrix} \\
&= \langle u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3), u_3(v_2w_3 - v_3w_2) \\
&\quad - u_1(v_1w_2 - v_2w_1), u_1(v_3w_1 - v_1w_3) - u_2(v_2w_3 - v_3w_2)\rangle \\
&= \langle v_1(u_2w_2 + u_3w_3) - w_1(u_2v_2 + u_3v_3), v_2(u_1w_1 + u_3w_3) \\
&\quad - w_2(u_1v_1 + u_3v_3), v_3(u_1w_1 + u_2w_2) - w_3(u_1v_1 + u_2v_2)\rangle \\
&= \langle v_1(\mathbf{u}\cdot\mathbf{w}) - w_1(\mathbf{u}\cdot\mathbf{v}), v_2(\mathbf{u}\cdot\mathbf{w}) - w_2(\mathbf{u}\cdot\mathbf{v}), v_3(\mathbf{u}\cdot\mathbf{w}) - w_3(\mathbf{u}\cdot\mathbf{v})\rangle \\
&= (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}.
\end{aligned}$$

13.4.74 Consider the quantity $(\mathbf{w} \times \mathbf{x})$ as a single vector. Using the result of exercise 72, we have $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{x}) = \mathbf{u} \cdot (\mathbf{v} \times (\mathbf{w} \times \mathbf{x}))$. Now applying the result of exercise 73, this is equal to $\mathbf{u} \cdot ((\mathbf{v}\cdot\mathbf{x})\mathbf{w} - (\mathbf{v}\cdot\mathbf{w})\mathbf{x}) = (\mathbf{u}\cdot\mathbf{w})(\mathbf{v}\cdot\mathbf{x}) - (\mathbf{u}\cdot\mathbf{x})(\mathbf{v}\cdot\mathbf{w})$ as desired.

13.4.75

a. Suppose $\mathbf{u} \times \mathbf{z} = \mathbf{v}$. Then $\mathbf{v} \times (\mathbf{u} \times \mathbf{z}) = \mathbf{v} \times \mathbf{v} = \langle 0, 0, 0\rangle$. Now $\mathbf{v} \times (\mathbf{u} \times \mathbf{z}) = \mathbf{u}(\mathbf{v}\cdot\mathbf{z}) - \mathbf{z}(\mathbf{v}\cdot\mathbf{u})$ by exercise 73. If $\mathbf{u}\cdot\mathbf{v} = 0$, then we have $\mathbf{u}(\mathbf{v}\cdot\mathbf{z}) = \langle 0, 0, 0\rangle$. Any vector $\mathbf{z}$ which is perpendicular to $\mathbf{v}$ is a solution to this equation.

   Now suppose that the equation $\mathbf{u} \times \mathbf{z} = \mathbf{v}$ has a nonzero solution. Because the cross product of any two vectors is perpendicular to both of the vectors, we must have that $\mathbf{u} \times \mathbf{z} \cdot \mathbf{u} = 0$. But this means that $\mathbf{v} \cdot \mathbf{u} = 0$, as desired.

b. If there exists a vector $\mathbf{z}$ so that $\mathbf{u} \times \mathbf{z} = \mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ must be perpendicular. If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular nonzero vectors, then there must be a plane which contains $\mathbf{u}$ and a nonzero vector $\mathbf{z}$ so that $\mathbf{u} \times \mathbf{z} = \mathbf{v}$.

13.5 Lines and Planes in Space

13.5.1 $\langle 4, -8, 9\rangle$.

13.5.2 $x = 1 + 4t$, $y = 2$, $z = 3 - 6t$.

13.5.3 Subtract componentwise to obtain the vector $\mathbf{d} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$.

13.5.4 Let $\mathbf{d}$ be the direction vector as in the previous problem. Then $\mathbf{r}(t) = \langle x_0, y_0, z_0\rangle + t\mathbf{d}$.

13.5.5 The direction vector of the line is $\langle 1, 1, 1\rangle$, which is the direction normal to the plane. Thus, the line is perpendicular to the plane.

13.5.6 The direction vector of the line is $\langle 1, -2, 1\rangle$ which is perpendicular to the normal to the plane (because $\langle 1, -2, 1\rangle \cdot \langle 1, 1, 1\rangle = 0$), so the line is parallel to the plane.

13.5.7 One point and a normal vector determine a plane

13.5.8 The vector $\mathbf{n} = \langle -2, -3, 4\rangle$ is normal to this plane.

13.5.9 The point (x, y, z) where this plane intersects the x-axis has $y = z = 0$; substituting in the equation of the plane gives $x = -6$. Similarly, we see that the plane meets the y-axis at $y = -4$ and the z-axis at $z = 3$.

13.5.10 Substituting in the general equation for a plane gives $(x - 1) + (y - 0) + (z - 0) = 0$, which simplifies to $x + y + z = 1$.

13.5.11 The line is $\mathbf{r}(t) = \langle 0, 0, 1\rangle + t\langle 4, 7, 0\rangle$. The parametric equations are $x = 4t$, $y = 7t$, $z = 1$.

13.5.12 The line is $\mathbf{r}(t) = \langle -3, 2, -1\rangle + t\langle 1, -2, 0\rangle$. The parametric equations are $x = -3 + t$, $y = 2 - 2t$, $z = -1$.

13.5.13 The direction is $\langle 0, 1, 0\rangle$, so the line is $\mathbf{r}(t) = \langle 0, 0, 1\rangle + t\langle 0, 1, 0\rangle$. The parametric equations are $x = 0$, $y = t$, $z = 1$.

13.5.14 The direction is $\langle 1, 0, 0\rangle$ so the line is $\langle -2, 4, 3\rangle + t\langle 1, 0, 0\rangle$. The parametric equations are $x = -2 + t$, $y = 4$, $z = 3$.

13.5.15 The direction is $\langle 1, 2, 3\rangle$, so the line is $\mathbf{r}(t) = t\langle 1, 2, 3\rangle$. The parametric equations are $x = t$, $y = 2t$, $z = 3t$.

13.5.16 The direction is $\langle 8, -5, -6\rangle$, so the line is $\mathbf{r}(t) = \langle -3, 4, 6\rangle + t\langle 8, -5, -6\rangle$. The parametric equations are $x = -3 + 8t$, $y = 4 - 5t$, $z = 6 - 6t$.

13.5.17 The direction is $\langle -2, 8, -4\rangle$, so the line is $\mathbf{r}(t) = t\langle -2, 8, -4\rangle$. The parametric equations are $x = -2t$, $y = 8t$, $z = -4t$.

13.5.18 The direction is $\langle 4, -1, 0\rangle$, so the line is $\mathbf{r}(t) = \langle 1, -3, 4\rangle + t\langle 4, -1, 0\rangle$. The parametric equations are $x = 1 + 4t$, $y = -3 - t$, $z = 4$.

13.5.19 The direction is $\langle 1, 0, 2\rangle \times \langle 0, 1, 1\rangle = \langle -2, -1, 1\rangle$, so the line is $\mathbf{r}(t) = t\langle -2, -1, 1\rangle$. The parametric equations are $x = -2t$, $y = -t$, $z = t$.

13.5.20 The direction is $\langle 1, 1, -5\rangle \times \langle 0, 4, 0\rangle = \langle 20, 0, 4\rangle$, so the line is $\mathbf{r}(t) = \langle -3, 4, 2\rangle + t\langle 20, 0, 4\rangle$. The parametric equations are $x = -3 + 20t$, $y = 4$, $z = 2 + 4t$.

13.5.21 The direction is $\langle 1, 1, 2\rangle \times \langle 1, 0, 0\rangle = \langle 0, 2, -1\rangle$, so the line is $\mathbf{r}(t) = \langle -2, 5, 3\rangle + t\langle 0, 2, -1\rangle$. The parametric equations are $x = -2$, $y = 5 + 2t$, $z = 3 - t$.

13.5.22 The direction is $\langle 4,3,-5\rangle \times \langle 0,0,1\rangle = \langle 3,-4,0\rangle$, so the line is $\mathbf{r}(t) = \langle 0,2,1\rangle + t\langle 3,-4,0\rangle$. The parametric equations are $x = 3t$, $y = 2-4t$, $z = 1$.

13.5.23 The direction is $\langle -2,8,-4\rangle \times \langle -2,1,-1\rangle = \langle -4,6,14\rangle$, so the line is $\mathbf{r}(t) = \langle 1,2,3\rangle + t\langle -4,6,14\rangle$. The parametric equations are $x = 1-4t$, $y = 2+6t$, $z = 3+14t$.

13.5.24 The direction is $\langle 2,3,-4\rangle \times \langle 1,1,-1\rangle = \langle 1,-2,-1\rangle$, so the line is $\langle 1,0,-1\rangle + t\langle 1,-2,-1\rangle$. The parametric equations are $x = 1+t$, $y = -2t$, $z = -1-t$.

13.5.25 Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and solving the resulting system of linear equations gives $t = 1$ and $s = 5$, so the point of intersection of the lines occurs at $\mathbf{r}(1) = \mathbf{R}(5) = \langle 4,3,3\rangle$. The direction of the line perpendicular to both of these is $\langle 4,2,3\rangle \times \langle 1,2,3\rangle = \langle 0,-9,6\rangle$. The line we are seeking is therefore $\langle 4,3,3\rangle + t\langle 0,-9,6\rangle$. The parametric equations are $x = 4$, $y = 3-9t$, $z = 3+6t$.

13.5.26 Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and solving the resulting system of linear equations gives $t = 0$ and $s = 4$, so the point of intersection of the lines occurs at $\mathbf{r}(0) = \mathbf{R}(4) = \langle -2,0,0\rangle$. The direction of the line perpendicular to both of these is $\langle 3,2,3\rangle \times \langle 1,2,3\rangle = \langle 0,-6,4\rangle$. The line we are seeking is therefore $\langle -2,0,0\rangle + t\langle 0,-6,4\rangle$. The parametric equations are $x = -2$, $y = -6t$, $z = 4t$.

13.5.27 The line segment is $\mathbf{r}(t) = t\langle 1,2,3\rangle$, where $0 \le t \le 1$. The parametric equations are $x = t$, $y = 2t$, $z = 3t$, $0 \le t \le 1$.

13.5.28 The line segment is $\mathbf{r}(t) = \langle 1,0,1\rangle + t\langle -1,-2,0\rangle$, where $0 \le t \le 1$. The parametric equations are $x = 10t$, $y = -2t$, $z = 1$, $0 \le t \le 1$.

13.5.29 The line segment is $\mathbf{r}(t) = \langle 2,4,8\rangle + t\langle 5,1,-5\rangle$, where $0 \le t \le 1$. The parametric equations are $x = 2+5t$, $y = 4+t$, $z = 8-5t$, $0 \le t \le 1$.

13.5.30 The line segment is $\mathbf{r}(t) = \langle -1,-8,4\rangle + t\langle -8,13,-7\rangle$, where $0 \le t \le 1$. The parametric equations are $x = -1-8tt$, $y = -8+13t$, $z = 4-7t$, $0 \le t \le 1$.

13.5.31 The direction vectors are not parallel, so the lines either intersect in a single point or are skew. Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and solving the resulting system of linear equations gives $t = 0$ and $s = -3$, so the point of intersection of the lines occurs at $\mathbf{r}(0) = \mathbf{R}(-3) = \langle 1,3,2\rangle$

13.5.32 The direction vectors for the two lines are $\langle 2,1,3\rangle$ and $\langle 5,1,5\rangle$, which are not parallel, so the lines either intersect in a point or are skew. Setting $2t = 5s-2$, $t+2 = s+4$, and $3t-1 = 5s+1$ yields $t = 4$ and $s = 2$. Thus the point of intersection is $(8,6,11)$.

13.5.33 Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and attempting to solve the corresponding system of linear equations yields no solution. The lines aren't parallel since $\langle 0,-1,1\rangle$ is not a multiple of $\langle -7,4,-1\rangle$. Therefore the lines are skew.

13.5.34 Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and attempting to solve the corresponding system of linear equations yields no solution. The lines aren't parallel since $\langle 5,-2,3\rangle$ is not a multiple of $\langle 10,4,6\rangle$. Therefore the lines are skew.

13.5.35 These equations represent the same line. (So they are parallel and intersecting.) Note that $\mathbf{r}(3t-5) = \langle 1+2(3t-5), 7-3(3t-5), 6+(3t-5)\rangle = \langle -9+6t, 22-9t, 1+3t\rangle = \mathbf{R}(t)$.

13.5.36 Because the direction vectors are multiples of each other ($\langle 4,-6,4\rangle = -2\langle -2,3-2\rangle$), the lines are parallel. (Note also that the lines don't coincide.)

13.5.37 Because the direction vectors are multiples of each other ($\langle 1,-2,3\rangle = -\frac{1}{7}\langle -7,14,-21\rangle$), the lines are parallel. (Note also that the lines don't coincide.)

13.5.38

a. If they are to intersect, then we must have $2+2t = 6+s$, $8+t = 10-2s$, and $10+3t = 16-s$. This system of linear equations has the solution $t = 2$ and $s = 0$. So the lines intersect at $(6,10,16)$.

b. They do not collide, because they arrive at the intersection point at different times.

13.5.39 Let the point on the line be $P(7,2,4)$, so that $\mathbf{u} = \overrightarrow{PQ} = \langle -12,0,5\rangle$. Then

$$d = \frac{|\,\langle -12,0,5\rangle \times \langle 5,-1,12\rangle\,|}{|\,\langle 5,-1,12\rangle\,|} = \frac{|\,\langle 5,169,12\rangle\,|}{|\,\langle 5,-1,12\rangle\,|} = \frac{13\sqrt{170}}{\sqrt{170}} = 13.$$

13.5.40 Let the point on the line be $P(1,3,1)$, so that $\mathbf{u} = \overrightarrow{PQ} = \langle 4,3,0\rangle$. Then

$$d = \frac{|\,\langle 4,3,0\rangle \times \langle 3,-4,1\rangle\,|}{|\,\langle 3,-4,1\rangle\,|} = \frac{|\,\langle 3,-4,-25\rangle\,|}{|\,\langle 3,-4,1\rangle\,|} = \frac{5\sqrt{26}}{\sqrt{26}} = 5.$$

13.5.41

a. Yes. The distance between $A(5,45)$ and the line through (25, 16) parallel to $\mathbf{v} = \langle \cos 122°, \sin 122°\rangle$ is $\approx 1.3 < 2.25$.

b. No. The distance between $B(76,40)$ and the line through $P(25,16)$ and (50, 25) (i.e. $x = 25 + 25t$, $y = 16 + 9t$) is $\approx 5.3 > 2.25$.

c. $13.16° < \theta < 18.12°$. A vector parallel to the line of the shot is $\langle \cos\theta, \sin\theta\rangle$ and $\overline{PC} = \langle 50,14\rangle$. We need $d = |14\cos\theta - 50\sin\theta| < 2.25$ Using a root finder to solve $14\cos\theta - 50\sin\theta - 2.25 = 0$ and $14\cos\theta - 50\sin\theta + 2.25 = 0$, results in $\theta \approx 0.2297$ and $\theta \approx 0.3163$, respectively. Converting to degrees results in the interval given.

13.5.42 No, the balls will not collide. The cue ball bounces when its center reaches $(40,1)$ and it follows the line $x = 40 + \frac{\sqrt{2}}{2}t$, $y = 1 + \frac{\sqrt{2}}{2}t$, $t \geq 0$ after the bounce. The distance from $D(59,17)$ to this line is $d \approx 2.12$; d needs to ber less than 2 for a collision to occur.

13.5.43 Substituting in the general equation for a plane gives $(x-0) + (y-2) - (z-(-2)) = 0$, which simplifies to $x + y - z = 4$.

13.5.44 Substituting in the general equation for a plane gives $-1\,(x-2) + 2\,(y-3) - 3\,(z-0) = 0$, which simplifies to $-x + 2y - 3z = 4$.

13.5.45 A vector normal to the plane is given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \langle -2,-1,2\rangle$. Substituting in the general equation for a plane gives $-2(x-1) - 1(y-2) + 2(z-3) = 0$, which simplifies to $2x + y - 2z = -2$.

13.5.46 A vector normal to the plane is given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 1 \\ 4 & 2 & 0 \end{vmatrix} = \langle -2,4,14\rangle$. Substituting in the general equation for a plane gives $-2(x-3) + 4(y-0) + 2(z+2) = 0$, which simplifies to $x - 2y - 7z = 17$.

13.5.47 A normal to the plane is the direction vector of the line, which is $\langle 1,4,7\rangle$. Substituting in the general equation for a plane gives $1(x-0) + 4(y-0) + 7(z-0) = 0$, or $x + 4y + 7z = 0$.

13.5.48 A normal to the plane is the direction vector of the line, which is $\langle 2,3,4\rangle$. Substituting in the general equation for a plane gives $2(x-2) + 3(y-(-3)) + 4(z-5) = 0$, or $2x + 3y + 4z = 15$.

13.5.49 Let $P = (1,0,3)$, $Q = (0,4,2)$ and $R = (1,1,1)$. Then the vectors $\overrightarrow{PQ} = \langle -1,4,-1\rangle$ and $\overrightarrow{PR} = \langle 0,1,-2\rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 4 & -1 \\ 0 & 1 & -2 \end{vmatrix} = -7\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ is normal to the plane. The plane has equation $7\,(x-1) + 2\,(y-0) + 1\,(z-3) = 0$, which simplifies to $7x + 2y + z = 10$.

13.5.50 Let $P = (2, -1, 4)$, $Q = (1, 1, -1)$ and $R = (-4, 1, 1)$. Then the vectors $\overrightarrow{PQ} = \langle -1, 2, -5\rangle$ and $\overrightarrow{PR} = \langle -6, 2, -3\rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -5 \\ -6 & 2 & -3 \end{vmatrix} = 4\mathbf{i} + 27\mathbf{j} + 10\mathbf{k}$ is normal to the plane. The plane has equation $4(x - 2) + 27(y - (-1)) + 10(z - 4) = 0$, which simplifies to $4x + 27y + 10z = 21$.

13.5.51 The plane Q has normal vector $\langle -1, 2, -4\rangle$; therefore the parallel plane passing through the point $P_0(1, 0, 4)$ has equation $-1(x - 1) + 2(y - 0) - 4(z - 4) = 0$, which simplifies to $-x + 2y - 4z = -17$.

13.5.52 The plane Q has normal vector $\langle 2, 1, -1\rangle$; therefore the parallel plane passing through the point $P_0(0, 2, -2)$ has equation $2(x - 0) + 1(y - 2) - 1(z - (-2)) = 0$, which simplifies to $2x + y - z = 4$.

13.5.53 Observe that the vectors $\langle 1, 0, 0\rangle$ and $\langle 1, 2, 3\rangle$ lie in the plane, so $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} = -3\mathbf{j} + 2\mathbf{k}$ is normal to the plane. The plane has equation $0(x - 0) - 3(y - 0) + 2(z - 0) = 0$, which simplifies to $3y - 2z = 0$.

13.5.54 Observe that the vectors $\langle 0, 0, 1\rangle$ and $\langle 3, -1, 2\rangle$ lie in the plane, so $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 3 & -1 & 2 \end{vmatrix} = \mathbf{i} + 3\mathbf{j}$ is normal to the plane. The plane has equation $1(x-0)+3(y-0)+0(z-0) = 0$, which simplifies to $x+3y = 0$.

13.5.55 Observe that the vectors $\langle 1, 2, 3\rangle$ and $\langle -1, 0, 4\rangle$ lie in the plane, so $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 0 & 4 \end{vmatrix} = 8\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$ is normal to the plane. The plane has equation $8(x - 0) - 7(y - 0) + 2(z - 0) = 0$, which simplifies to $8x - 7y + 2z = 0$.

13.5.56 Observe that the points $P = (0, 0, 0)$ and $Q = (1, -1, 2)$ lie on the line ℓ. Therefore the vectors $\overrightarrow{PP_0} = \langle 1, -2, 3\rangle$ and $\overrightarrow{PQ} = \langle 1, -1, 2\rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PP_0} \times \overrightarrow{PQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 1 & -1 & 2 \end{vmatrix} = -\mathbf{i}+\mathbf{j}+\mathbf{k}$ is normal to the plane. The plane has equation $-1(x - 0) + 1(y - 0) + 1(z - 0) = 0$, which simplifies to $x - y - z = 0$.

13.5.57 Observe that the points $P = (-2, 0, 1)$ and $Q = (0, -2, -3)$ lie on the line ℓ. Therefore the vectors $\overrightarrow{PP_0} = \langle -2, 1, 1\rangle$ and $\overrightarrow{PQ} = \langle 2, -2, -4\rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PP_0} \times \overrightarrow{PQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 2 & -2 & -4 \end{vmatrix} = \langle -2, -6, 2\rangle = -2\langle 1, 3, -1\rangle$ is normal to the plane. The plane has equation $1(x - (-2)) + 3(y - 0) - 1(z - 1) = 0$, which simplifies to $x + 3y - z = -3$.

13.5.58 Observe that two points which lie in the intersection of the planes are $(4, 0, -2)$ and $(2, -2, 0)$. So the vectors $\langle 4, 0, -2\rangle$ and $\langle 2, -2, 0\rangle$ lie in the plane, so $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & -2 \\ 2 & -2 & 0 \end{vmatrix} = \langle -4, -4, -8\rangle$ is normal to the plane. Scaling gives the normal vector $\langle 1, 1, 2\rangle$. The plane has equation $x + y + 2z = 0$.

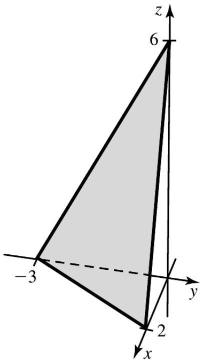
13.5.59 Yes. A direction vector for the line is $\langle 1, 2, 4\rangle$ which is perpendicular to the normal to the plane $\mathbf{n} = \langle 2, -1, 0\rangle$, so the line is parallel to the plane. A point on the line is $(1, 3, 5)$, so the plane $2x - y = -1$ is contains the line and is parallel to the given plane.

13.5.60 Yes. Solving the system of linear equations that arises from setting the coordinates equal to each other gives $t = 0$ and $s = 1$, which means that the point of intersection is $(0, 1, 4)$. To find $\mathbf{n}$ for the plane, we calculate $v \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 2 & 3 \end{vmatrix} = \langle 0, 3, -2 \rangle$. The equation of the plane is therefore

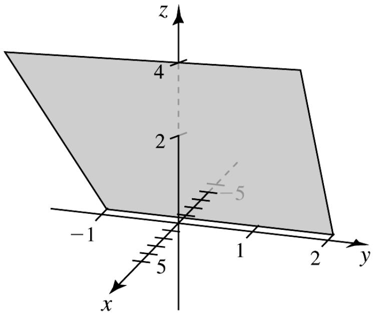
$$0(x - 0) + 3(y - 1) - 2(z - 4) = 0$$

or $3y - 2z = -5$.

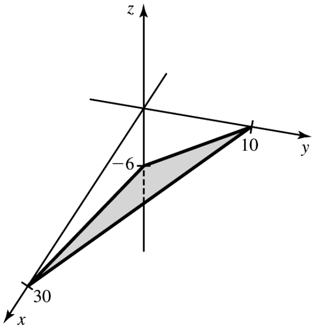
13.5.61 The x-intercept is found by setting $y = z = 0$ and solving $3x = 6$ to get $x = 2$. Similarly, we see that the y-intercept is -3 and the z-intercept is 6. The xy-trace is found by setting $z = 0$, which gives $3x - 2y = 6$. Similarly, the xz-trace is $3x + z = 6$ and the yz-trace is $-2y + z = 6$.



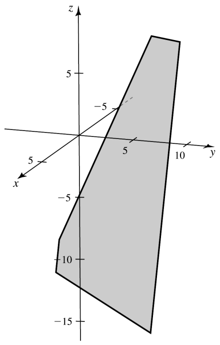
13.5.62 The x-intercept is found by setting $y = z = 0$ and solving $-4x = 16$ to get $x = -4$. Similarly, we see that the z-intercept is 2. Setting $x = z = 0$ gives $0 = 16$, so this plane does not intersect the y-axis. The xy-trace is found by setting $z = 0$, which gives $-4x = 16$ or $x = -4$. Similarly, the xz-trace is $-4x + 8z = 16$ or $x - 2z = -4$ and the yz-trace is $z = 2$.



13.5.63 The x-intercept is found by setting $y = z = 0$ which gives $x = 30$. Similarly, we see that the y-intercept is 10 and the z-intercept is -6. The xy-trace is found by setting z = 0, which gives $x + 3y = 30$. Similarly, the xz-trace is $x - 5z = 30$ and the yz-trace is $3y - 5z = 30$.



13.5.64 The x-intercept is found by setting $y = z = 0$ and solving $12x + 72 = 0$ to get $x = -6$. Similarly, we see that the y-intercept is 8 and the z-intercept is -18. The xy-trace is found by setting $z = 0$, which gives $12x - 9y = -72$ or $4x - 3y = -24$. Similarly, the xz-trace is $12x + 4z = -72$ or $3x + z = -18$ and the yz-trace is $9y - 4z = 72$.



13.5.65 The normal vectors to the planes are $\langle 1, 1, 4\rangle$ and $\langle -1, -3, 1\rangle$, and the dot product of these vectors is $-1 - 3 + 4 = 0$, so the planes are orthogonal.

13.5.66 The normal vectors to the planes are $\langle 2, 2, -3\rangle$ and $\langle -10, -10, 15\rangle$, and these are parallel because $\langle -10, -10, 15\rangle = -5\langle 2, 2, -3\rangle$. Thus, the planes are parallel.

13.5.67 The normal vectors to the planes are $\langle 3, 2, -3\rangle$ and $\langle -6, -10, 1\rangle$. These are neither parallel nor perpendicular, because one is not a multiple of the other and because their dot product is not 0. Thus, the planes are neither parallel nor perpendicular.

13.5.68 The normal vectors to the planes are $\langle 3, 2, 2\rangle$ and $\langle -6, -10, 19\rangle$, and the dot product of these vectors is $-18 - 20 + 38 = 0$, so the planes are orthogonal.

13.5.69 Rewrite R and T so we have $Q : 3x - 2y + z = 12$, $R : 3x - 2y + z = 0$, $T : 3x - 2y + z = 12$. This shows that Q and T are identical, and Q, R and T are parallel. Note that $\langle 3, -2, 1\rangle \cdot \langle -1, 2, 7\rangle = 0$ so S is orthogonal to Q, R and T.

13.5.70 The planes Q, R, S and T have normal vectors $\langle 1, 1, -1\rangle$, $\langle 0, 1, 1\rangle$, $\langle 1, -1, 0\rangle$ and $\langle 1, 1, 1\rangle$ respectively. None of these vectors are scalar multiples of each other, so no two of these planes are parallel (or identical). Observe that $\langle 1, 1, -1\rangle \cdot \langle 0, 1, 1\rangle = \langle 1, 1, -1\rangle \cdot \langle 1, -1, 0\rangle = 0$; therefore Q is orthogonal to R and S. We also have $\langle 1, -1, 0\rangle \cdot \langle 1, 1, 1\rangle = 0$, so S and T are orthogonal. All other pairs have non-zero dot products, so no other pairs are orthogonal.

13.5.71 The direction of the line is $\langle 2, -4, 1\rangle$, so the line is given by

$$\langle 2, 1, 3\rangle + t\,\langle 2, -4, 1\rangle = \langle 2 + 2t, 1 - 4t, 3 + t\rangle\,.$$

13.5.72 The direction of the line is $\langle 1, 0, 4\rangle$, so the line is given by $\langle 0, -10, -3\rangle + t\,\langle 1, 0, 4\rangle = \langle t, -10, -3 + 4t\rangle$.

13.5.73 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle -1, 2, 1\rangle$ and $\mathbf{n}_R = \langle 1, 1, 1\rangle$, are not multiples of each other; therefore these planes are not parallel and they intersect in a line ℓ. We need to find a point on ℓ and a vector in the direction of ℓ. Setting $x = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the yz-plane: $2y + z = 1$, $y + z = 0$. Solving these equations simultaneously gives $y = 1$ and $z = -1$, so $(0, 1, -1)$ is a point on ℓ. A vector in the direction of ℓ is $\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} = \langle 1, 2, -3\rangle$. Therefore ℓ has equation $\mathrm{r}\;(t) = \langle 0, 1, -1\rangle + t\langle 1, 2, -3\rangle = \langle t, 1 + 2t, -1 - 3t\rangle$, or $x = t$, $y = 1 + 2t$, $z = -1 - 3t$.

13.5.74 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle 1, 2, -1\rangle$ and $\mathbf{n}_R = \langle 1, 1, 1\rangle$, are not multiples of each other; therefore these planes are not parallel and they intersect in a line ℓ. We need to find a point on ℓ and a vector in the direction of ℓ. Setting $z = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the xy-plane: $x + 2y = 1$, $x + y = 1$. Solving these equations simultaneously gives $x = 1$ and $y = 0$, so $(1, 0, 0)$ is a point on ℓ. A vector in the direction of ℓ is $\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k} = \langle 3, -2, -1\rangle$. Therefore ℓ has equation $\mathbf{r}\,(t) = \langle 1, 0, 0\rangle + t\langle 3, -2, -1\rangle = \langle 1 + 3t, -2t, -t\rangle$, or $x = 1 + 3t$, $y = -2t$, $z = -t$.

13.5.75 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle 2, -1, 3\rangle$ and $\mathbf{n}_R = \langle -1, 3, 1\rangle$, are not multiples of each other; therefore these planes are not parallel and they intersect in a line ℓ. We need to find a point on ℓ and a vector in the direction of ℓ. Setting $z = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the xy-plane: $2x - y = 1$, $-x + 3y = 4$. Solving these equations simultaneously gives $x = \frac{7}{5}$ and $y = \frac{9}{5}$, so $\left(\frac{7}{5}, \frac{9}{5}, 0\right)$ is a point on ℓ. A vector in the direction of ℓ is $\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = -10\mathbf{i} - 5\mathbf{j} + 5\mathbf{k} = -5\langle 2, 1, -1\rangle$. Therefore ℓ has equation $\mathbf{r}\,(t) = \langle \frac{7}{5}, \frac{9}{5}, 0\rangle + t\langle 2, 1, -1\rangle = \langle \frac{7}{5} + 2t, \frac{9}{5} + t, -t\rangle$, or $x = \frac{7}{5} + 2t$, $y = \frac{9}{5} + t$, $z = -t$.

13.5.76 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle 1, -1, -2\rangle$ and $\mathbf{n}_R = \langle 1, 1, 1\rangle$, are not multiples of each other; therefore these planes are not parallel and they intersect in a line ℓ. We need to find a point on ℓ and a vector in the direction of ℓ. Setting $z = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the xy-plane: $x - y = 1$, $x + y = -1$. Solving these equations simultaneously gives $x = 0$ and $y = -1$, so $(0, -1, 0)$ is a point on ℓ. A vector in the direction of ℓ is $\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k} = \langle 1, -3, 2\rangle$. Therefore ℓ has equation $\mathbf{r}\,(t) = \langle 0, -1, 0\rangle + t\langle 1, -3, 2\rangle = \langle t, -1 - 3t, 2t\rangle$, or $x = t$, $y = -1 - 3t$, $z = 2t$.

13.5.77 The line and plane intersect for $x = t = 3$, so the point of intersection is $(3, 3, 3)$.

13.5.78 The intersection occurs for $y = -2 = -t + 4$, so for $t = 6$. The point of intersection is $(13, -2, 0)$.

13.5.79 The intersection occurs for $3(-2t + 5) + 2(3t - 5) - 4(4t - 6) = -3$, which occurs for $t = 2$. So the point is $(1, 1, 2)$.

13.5.80 The plane and line don't intersect. Notice that the direction of the line is $\langle 3, 1, -1\rangle$ which is perpendicular to the normal to the plane $\langle 2, -3, 3\rangle$. So the line is parallel to the plane and doesn't intersect it.

13.5.81

a. True. This curve passes through the origin at $t = -1/2$.

b. False. For example, the x-axis is not parallel to the line $\langle 0, 0, 1\rangle + t\,\langle 0, 1, 0\rangle$, but neither do they intersect.

c. False. The direction vector for the line is parallel to the normal to the plane, so the line is perpendicular to the plane, not parallel.

d. True. Because the direction vectors are multiples of each other, these represent the same line.

e. False. For example, the point $(0, 0, -1)$ is on the first plane but isn't on the second.

f. False. Two distinct lines determine a plane only if the lines are parallel or if they intersect.

g. True. Note that $\langle 5,-1,0\rangle \cdot \langle -1,-5,7\rangle = 0$ and $\langle 7,0,1\rangle \cdot \langle -1,-5,7\rangle = 0$.

13.5.82 The length of the orthogonal projection of $\overrightarrow{PQ}$ onto the normal vector $\mathbf{n}$ is the magnitude of the scalar component of $\overrightarrow{PQ}$ in the direction of $\mathbf{n}$ which is $\dfrac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{|\mathbf{n}|}$.

13.5.83 Let $P(2,0,0)$ be a point on the plane. Then $\overrightarrow{PQ} = \langle 4,-2,4\rangle$, and the distance from the point $Q(6,-2,4)$ to the plane is

$$\frac{|\langle 4,-2,4\rangle \cdot \langle 2,-1,2\rangle|}{\sqrt{4+1+4}} = \frac{18}{3} = 6.$$

13.5.84 Let $P(0,0,-1)$ be a point on the plane. Then $\overrightarrow{PQ} = \langle 1,2,-3\rangle$, and the distance form the point $Q(1,2,-4)$ to the plane is

$$\frac{|\langle 1,2,-3\rangle \cdot \langle 2,0,-5\rangle|}{\sqrt{4+0+25}} = \frac{17}{\sqrt{29}}.$$

13.5.85 We have $x = 1+4t$, $y = 2+7t$, and $z = 2t$. Solving for t and setting equal gives

$$\frac{x-1}{4} = \frac{y-2}{7} = \frac{z}{2}.$$

13.5.86 The line is given by $\langle 1,-2,3\rangle + t\,\langle 1,5,-4\rangle$, so $x = 1+t$, $y = -2+5t$, and $z = 3-4t$. Solving for t and setting equal gives

$$\frac{x-1}{1} = \frac{y+2}{5} = \frac{z-3}{-4}.$$

13.5.87 The angle θ between the vectors $\mathbf{n}_1 = \langle 5,2,-1\rangle$ and $\mathbf{n}_2 = \langle -3,1,2\rangle$ satisfies $\cos\theta = \dfrac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = -\dfrac{15}{\sqrt{30}\sqrt{14}} = -\dfrac{\sqrt{105}}{14}$, so $\theta = \cos^{-1}\left(-\dfrac{\sqrt{105}}{14}\right) \approx 2.392$ rad $\approx 137°$.

13.5.88 The x-intercept is found by setting $y = z = 0$ and solving for x, which gives the point $\left(\frac{d}{a},0,0\right)$, assuming $a \neq 0$. If $a = 0$, then all points on the x-axis lie on this plane when $d = 0$, and no points on the x-axis lie on the plane when $d \neq 0$. Similarly, the intersection of the plane with the y-axis is the point $\left(0,\frac{d}{b},0\right)$ when $b \neq 0$, the entire y-axis when $b = d = 0$ and empty when $b = 0$, $d \neq 0$; and the intersection of the plane with the z-axis is the point $\left(0,0,\frac{d}{c}\right)$ when $c \neq 0$, the entire z-axis when $c = d = 0$ and empty when $c = 0$, $d \neq 0$.

13.5.89 These planes have normal $\langle 2,3,0\rangle \times \langle -1,-1,2\rangle = \langle 6,-4,1\rangle$, so the planes all have an equation of the form $6x-4y+z = d$ for some real number d.

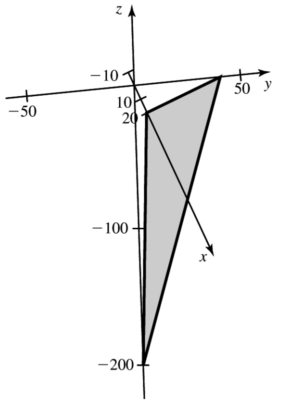
13.5.90 The normal to the plane we are seeking is $\langle 2,5,-3\rangle \times \langle -1,5,2\rangle = \langle 25,-1,15\rangle$. The equation of the plane has the form $25x - y + 15z = d$ for some d. Because the point $(0,-2,4)$ is on the plane, we have $0-(-2)+60 = d$, so the plane is given by $25x - y + 15z = 62$.

13.5.91 First we find the line of intersection of the first two planes. The direction of the line of intersection is $\langle 1,0,3\rangle \times \langle 0,1,4\rangle = \langle -3,-4,1\rangle$, and by inspection, a point on both planes is $(0,2,1)$. Thus the line of intersection is given by $\langle -3t, 2-4t, 1+t\rangle$. This intersects the plane $x+y+6z = 9$ when $-3t+(2-4t)+6(1+t) = 9$, or $8-t = 9$, so $t = -1$. Thus, the intersection is the single point $(3,6,0)$.

13.5.92 First we find the line of intersection of the first two planes. The direction of the line of intersection is $\langle 1,2,2\rangle \times \langle 0,1,4\rangle = \langle 6,-4,1\rangle$, and by inspection, a point on both planes is $(3,-2,2)$. Thus the line of intersection is given by $\langle 3+6t, -2-4t, 2+t\rangle$. This intersects the plane $x+2y+8z = 9$ when $3+6t+2(-2-4t)+8(2+t) = 9$, or $6t+15 = 9$, so $t = -1$. Thus, the intersection is the single point $(-3,2,1)$.

13.5.93

a.



b. The profit is $z = 10 \cdot 20 + 5 \cdot 10 - 200 = \$50$ which is positive.

c. The profit is 0 when x and y lie on the line $2x + y = 40$.

13.6 Cylinders and Quadric Surfaces

13.6.1 Because z is absent from the equation $x^2 + 2y^2 = 8$, this cylinder is parallel to the z-axis. Similarly, $z^2 + 2y^2 = 8$ is parallel to the x-axis and $x^2 + 2z^2 = 8$ is parallel to the y-axis.

13.6.2 This is a cylinder consisting of all lines parallel to the y-axis that pass through the parabola $x = z^2$ in the xz-plane.

13.6.3 The traces of a surface are the sets of points at which the surface intersects a plane that is parallel to one of the coordinate planes.

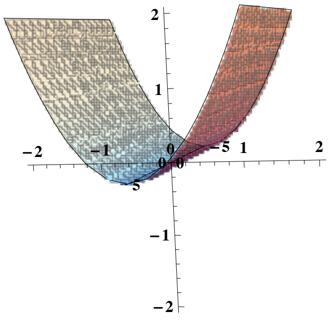
13.6.4 This is an elliptic paraboloid.

13.6.5 This is an ellipsoid.

13.6.6 This is a hyperboloid of two sheets.

13.6.7

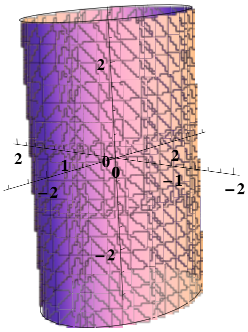
a. The cylinder is parallel to the x-axis.



b.

13.6.8

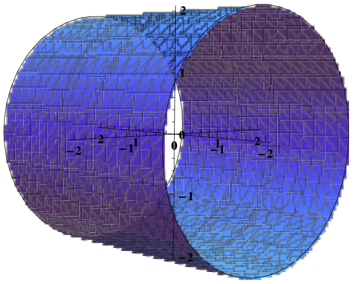
a. The cylinder is parallel to the z-axis.


2
2
1
0
0
−2
2
−1
−2
−2

b.

13.6.9

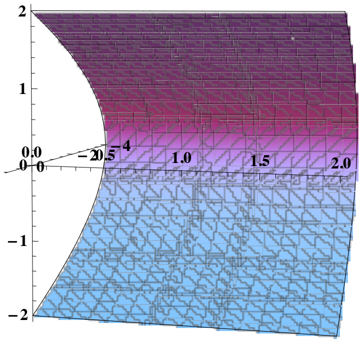
a. The cylinder is parallel to the y-axis.


2
1
−2
2
−1
1
0
0
−1
2
−2
−1
−2

b.

13.6.10

a. The cylinder is parallel to the y-axis.

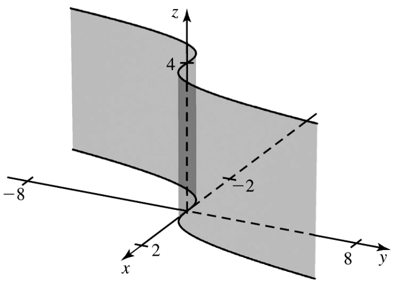

2
1
0.0
0
0
−2
0.5
4
1.0
1.5
2.0
−1
−2

b.

13.6.11

a. The cylinder is parallel to the z-axis.

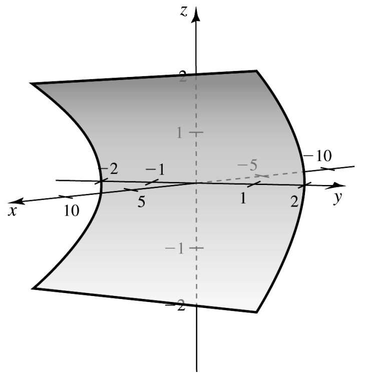
b.



13.6.12

a. The cylinder is parallel to the y-axis.

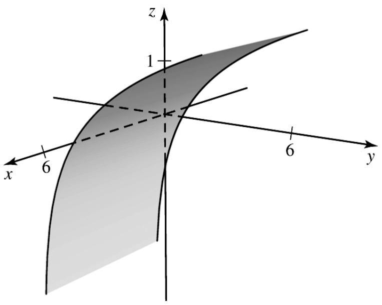
b.


z
2
1
−2
−1
−5
−10
x
10
5
1
2
y
−1
−2

13.6.13

a. The cylinder is parallel to the x-axis.

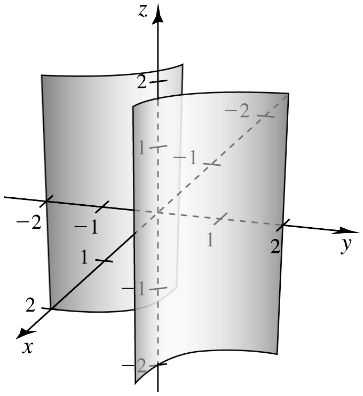
b.



13.6.14

a. The cylinder is parallel to the z-axis.

b.



13.6.15 This is an ellipsoid. The xy-trace is $x^2 + y^2 = 1$, which is a circle. The xz-trace is $x^2 + \dfrac{z^2}{25} = 1$ which is an ellipse. The yz-trace is $y^2 + \dfrac{z^2}{25} = 1$ which is also an ellipse.

13.6.16 This is a hyperboloid of one sheet. The xy-trace is $x^2 + y^2 = 1$, which is a circle. The xz-trace is $x^2 - \dfrac{z^2}{25} = 1$ which is a hyperbola. The yz-trace is $y^2 - \dfrac{z^2}{25} = 1$ which is also a hyperbola.

13.6.17 This is a paraboloid. The xy-trace is the single point $(0, 0, 0)$. The xz-trace is $z = 25x^2$ which is a parabola. The yz-trace is $z = -25y^2$ which is also a parabola.

13.6.18 This is a hyperbolic paraboloid. The xy-trace is $y = \pm x$ which are lines. The xz-trace is $z = 25x^2$ which is a parabola. The yz-trace is $z = -25y^2$ which is a parabola.

13.6.19 This is a hyperboloid of two sheets. The xz-trace is $z^2 - 25x^2 = 25$ which is a hyperbola. The yz-trace is $z^2 - 25y^2 = 25$ which is also a hyperbola.

13.6.20 This is an elliptic cone. The xy-trace is the single point $(0, 0, 0)$. The xz-trace is $z = \pm 5x$ which are lines. The yz-trace is $z = \pm 5y$ which are also lines.

13.6.21 This a hyperbolic paraboloid.

13.6.22 This is a hyperboloid of two sheets.

13.6.23 This is an elliptic paraboloid.

13.6.24 This is a hyperboloid of one sheet.

13.6.25 This is a hyperbolic cylinder.

13.6.26 This is an elliptic cylinder.

13.6.27 This is an elliptic paraboloid.

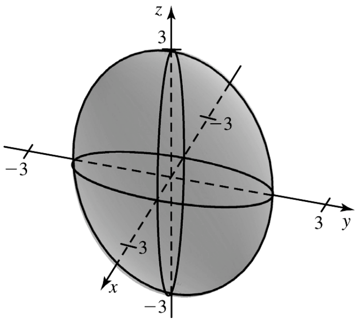
13.6.28 This is an elliptic cone.

13.6.29

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 1$, so the x-intercepts are $x = \pm 1$. Similarly we see that the y-intercepts are $y = \pm 2$ and the z-intercepts are $z = \pm 3$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x^2 + \frac{y^2}{4} = 1$, $x^2 + \frac{z^2}{9} = 1$, $\frac{y^2}{4} + \frac{z^2}{9} = 1$.

c. This is an ellipsoid.

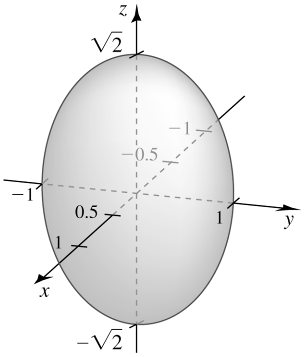


13.6.30

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $4x^2 = 1$, so the x-intercepts are $x = \pm\frac{1}{2}$. Similarly we see that the y-intercepts are $y = \pm 1$ and the z-intercepts are $z = \pm\sqrt{2}$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $4x^2 + y^2 = 1$, $4x^2 + \frac{z^2}{2} = 1$, $y^2 + \frac{z^2}{2} = 1$.

c. This is an ellipsoid.

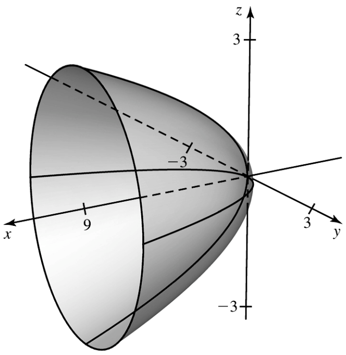


13.6.31

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x = y^2$, $x = z^2$, and $y^2 + z^2 = 0$ (which implies that $x = y = z = 0$).

c. This is an elliptic paraboloid.

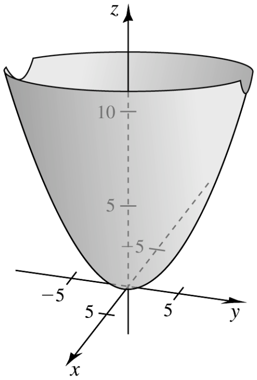


13.6.32

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $0 = \dfrac{x^2}{4} + \dfrac{y^2}{9}$ (which implies that $x = y = z = 0$), $z = \dfrac{x^2}{4}$, $z = \dfrac{y^2}{9}$.

c. This is an elliptic paraboloid.

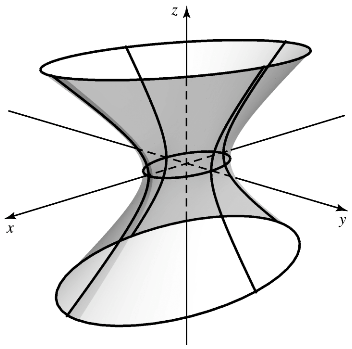


13.6.33

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 25$, so the x-intercepts are $x = \pm 5$. Similarly we see that the y-intercepts are $y = \pm 3$ and there are no z-intercepts.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$, $\dfrac{x^2}{25} - z^2 = 1$, $\dfrac{y^2}{9} - z^2 = 1$.

c. This is a hyperboloid of one sheet.

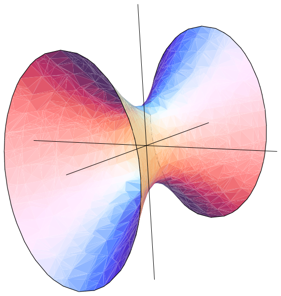


13.6.34

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which results in no intercepts. Similarly we see that the y-intercepts are $y = \pm 2$ and the z-intercepts are $z = \pm 3$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{y^2}{4} - \frac{x^2}{16} = 1$, $\frac{z^2}{9} - \frac{x^2}{16} = 1$, $\frac{y^2}{4} + \frac{z^2}{9} = 1$.

c. This is a hyperboloid of one sheet.

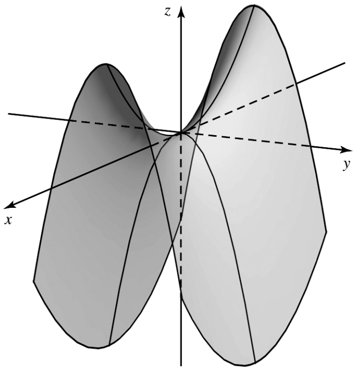


13.6.35

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{x^2}{9} - y^2 = 0$, $z = \frac{x^2}{9}$, $z = -y^2$.

c. This is a hyperbolic paraboloid.

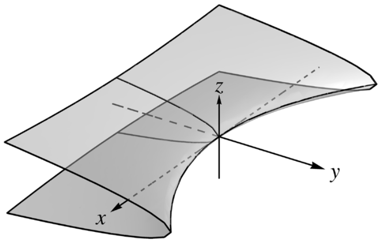


13.6.36

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $y = \frac{x^2}{16}$, $\frac{x^2}{16} - 4z^2 = 0$, $y = -4z^2$.

c. This is a hyperbolic paraboloid.

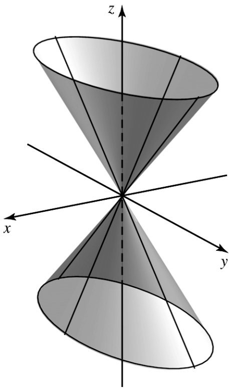


13.6.37

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x^2 + \frac{y^2}{4} = 0$ (which implies that $x = y = z = 0$), $x^2 = z^2$, $\frac{y^2}{4} = z^2$.

c. This is an elliptic cone.

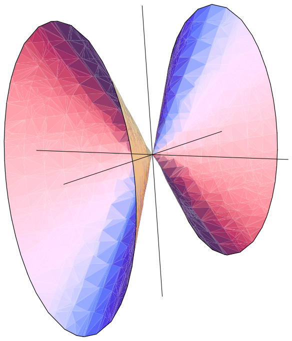


13.6.38

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $4y^2 = x^2$, $x^2 = z^2$, $4y^2 + z^2 = 0$ (which implies that $x = y = z = 0$).

c. This is an elliptic cone.

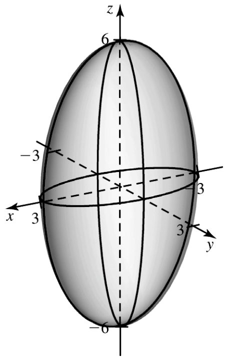


13.6.39

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 9$, so the x-intercepts are $x = \pm 3$. Similarly we see that the y-intercepts are $y = \pm 1$ and the z-intercepts are $z = \pm 6$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\dfrac{x^2}{3} + 3y^2 = 3$,, $\dfrac{x^2}{3} + \dfrac{z^2}{12} = 3$, $3y^2 + \dfrac{z^2}{12} = 3$.

c. This is an ellipsoid.

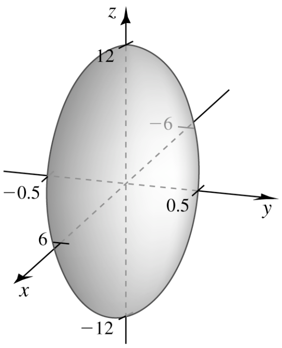


13.6.40

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 36$, so the x-intercepts are $x = \pm 6$. Similarly we see that the y-intercepts are $y = \pm\dfrac{1}{2}$ and the z-intercepts are $z = \pm 12$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\dfrac{x^2}{6} + 24y^2 = 6$, $\dfrac{x^2}{6} + \dfrac{z^2}{24} = 6$, $24y^2 + \dfrac{z^2}{24} = 6$.

c. This is an ellipsoid.

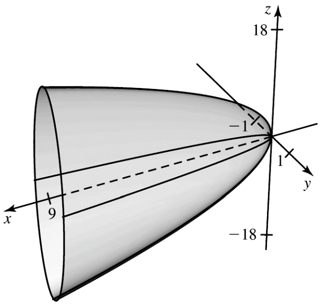


13.6.41

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x - 9y^2 = 0$, $9x - \dfrac{z^2}{4} = 0$, $81y^2 + \dfrac{z^2}{4} = 0$ (which implies that $x = y = z = 0$).

c. This is an elliptic paraboloid.

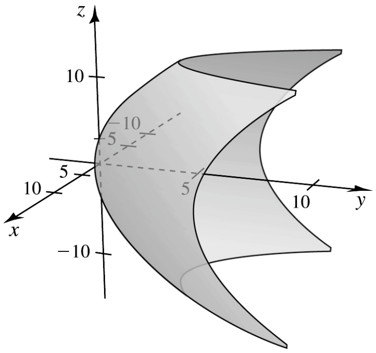


13.6.42

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $y - \dfrac{x^2}{16} = 0$, $\dfrac{x^2}{8} + \dfrac{z^2}{18} = 0$ (which implies that $x = y = z = 0$), $y - \dfrac{z^2}{36} = 0$.

c. This is an elliptic paraboloid.

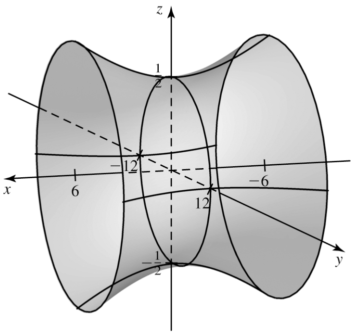


13.6.43

a. The y-intercept is found by setting $x = z = 0$ in the equation of this surface, which gives $y^2 = 144$, so the y-intercepts are $y = \pm 12$. Similarly we see that the z-intercepts are $z = \pm\frac{1}{2}$ and there are no x-intercepts.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $-\frac{x^2}{4} + \frac{y^2}{16} - 9 = 0$, $-\frac{x^2}{4} + 36z^2 - 9 = 0$, $\frac{y^2}{16} + 36z^2 - 9 = 0$.

c. This is an hyperboloid of one sheet.

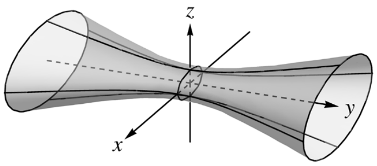


13.6.44

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 1$, so the x-intercepts are $x = \pm 1$. Similarly we see that the z-intercepts are $z = \pm\frac{1}{3}$ and there are no y-intercepts.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x^2 - \frac{y^2}{3} - 1 = 0$, $9z^2 + x^2 - 1 = 0$, $9z^2 - \frac{y^2}{3} - 1 = 0$.

c. This is a hyperboloid of one sheet.

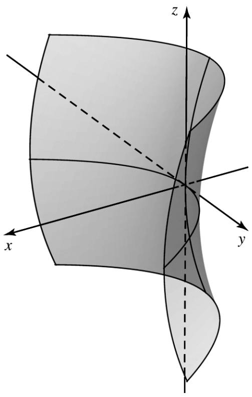


13.6.45

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $5x - \frac{y^2}{5} = 0$, $5x + \frac{z^2}{20} = 0$, $-\frac{y^2}{25} + \frac{z^2}{20} = 0$.

c. This is a hyperbolic paraboloid.

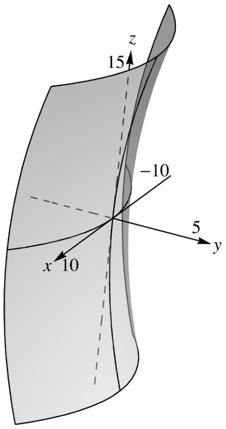


13.6.46

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $6y + \dfrac{x^2}{6} = 0$, $\dfrac{x^2}{6} - \dfrac{z^2}{24} = 0$, $6y - \dfrac{z^2}{24} = 0$.

c. This is a hyperbolic paraboloid.

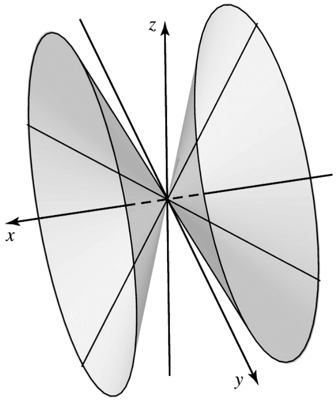


13.6.47

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\dfrac{y^2}{18} = 2x^2$, $\dfrac{z^2}{32} = 2x^2$, $\dfrac{z^2}{32} + \dfrac{y^2}{18} = 0$ (which implies that $x = y = z = 0$).

c. This is an elliptic cone.

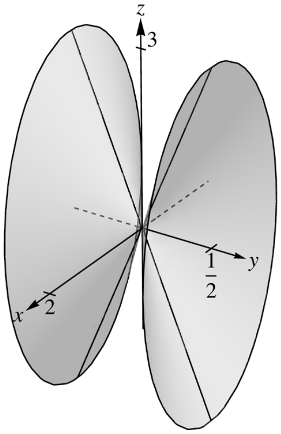


13.6.48

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{x^2}{3} = 3y^2$, $\frac{x^2}{3} + \frac{z^2}{12} = 0$, (which implies that $x = y = z = 0$), $\frac{z^2}{12} = 3y^2$.

c. This is an elliptic cone.

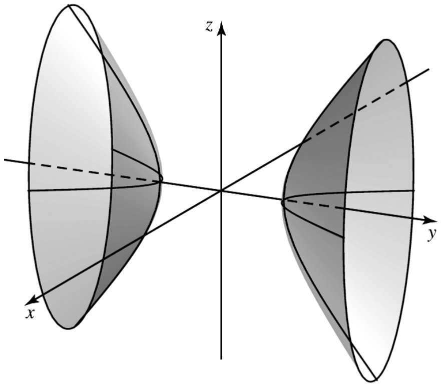


13.6.49

a. The y-intercept is found by setting $x = z = 0$ in the equation of this surface, which gives $y^2 = 4$, so the y-intercepts are $y = \pm 2$. There are no x or z-intercepts.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $-x^2 + \frac{y^2}{4} = 1$, $-x^2 - \frac{z^2}{9} = 1$ (no xz-trace), $\frac{y^2}{4} - \frac{z^2}{9} = 1$.

c. This is a hyperboloid of two sheets.

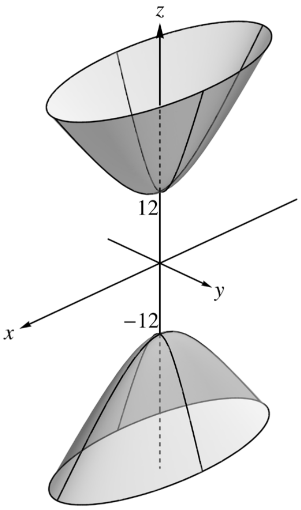


13.6.50

a. The z-intercept is found by setting $x = y = 0$ in the equation of this surface, which gives $z^2 = 24 \cdot 6$, so the z-intercepts are $x = \pm 12$. There are no x or y-intercepts.

b. b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $-\dfrac{x^2}{6} - 24y^2 - 6 = 0$ (no xy-trace), $-\dfrac{x^2}{6} + \dfrac{z^2}{24} - 6 = 0$, $-24y^2 + \dfrac{z^2}{24} - 6 = 0$.

c. This is a hyperboloid of two sheets.

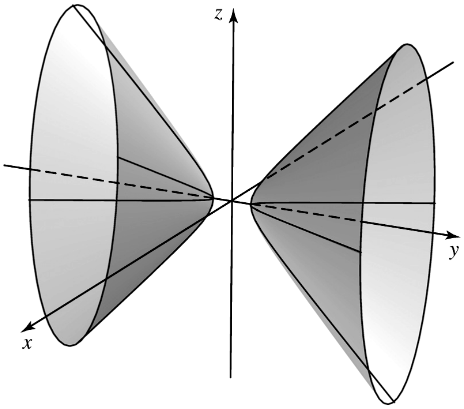


13.6.51

a. The y-intercept is found by setting $x = z = 0$ in the equation of this surface, which gives $3\ y^2 = 1$, so the y-intercepts are $y = \pm\dfrac{\sqrt{3}}{3}$. There are no x or z-intercepts.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $-\dfrac{x^2}{3} + 3y^2 = 1$, $-\dfrac{x^2}{3} - \dfrac{z^2}{12} = 1$ (no xz-trace), $3y^2 - \dfrac{z^2}{12} = 1$.

c. This is a hyperboloid of two sheets.



13.6.52 The equation $x^2 + y^2 - z^2 + 2z = 1$ can be written as $x^2 + y^2 - (z-1)^2 = 0$. This describes a cone shifted up one unit from the graph of $x^2 + y^2 - z^2 = 0$.

13.6.53 The equation $x^2 + 4y^2 + 9z^2 + 54z = 19$ can be written $x^2 + 4y^2 + 9(z^2 + 6z + 9) = 100$, or $x^2 + 4y^2 + 9(z+3)^2 = 100$. The graph of the corresponding ellipsoid is obtained by shifting the graph of the ellipsoid $x^2 + 4y^2 + 9z^2 = 100$ down 3 units.

13.6.54 Completing the square and rewriting the equation of the surface as $(x+1)^2 + y^2 + 4z^2 = 1$ shows that this surface is an ellipsoid centered at the point $(-1, 0, 0)$.

13.6.55 Completing the square and rewriting the equation of the surface as $9x^2 + (y+1)^2 - 4z^2 = 1$ shows that this surface is a hyperboloid of one sheet with axis the line ℓ: $\mathbf{r} = \langle 0, -1, t\rangle$.

13.6.56 Completing the square and rewriting the equation of the surface as $-(x-3)^2 - (y+4)^2 + \dfrac{z^2}{9} = 1$ shows that this surface is a hyperboloid of two sheets with axis the line ℓ: $\mathbf{r} = \langle 3, -4, t\rangle$.

13.6.57 Completing the square and rewriting the equation of the surface as $z^2 = \dfrac{(x-4)^2}{4} + (y-5)^2 + 12$ shows that this surface is a hyperboloid of two sheets.

13.6.58 Writing this as $z = -(x^2 + y^2)$, we see that it is a paraboloid opening downward

13.6.59

a. True.

b. True. Letting $z = 0$ in each equation gives the same result.

c. True. Letting z be a constant gives $y = 3x^2 - k$ for some constant k.

d. False. For example, letting $y = 0$ gives $3x^2 = z^2$ which does not represent a parabola.

e. False. Rather, up 4 units.

13.6.60

a. D. This surface is a cylinder parallel to the parabola $y = z^2$ in the yz-plane.

b. A. This surface is a plane.

c. E. This surface is an ellipsoid.

d. F. This surface is a hyperboloid of one sheet.

e. B. This surface is an elliptic cone.

f. C. This surface is a cylinder parallel to the graph $y = |x|$ in the xy-plane.

13.6.61 All of the quadric surfaces in Table 13.1 except the hyperbolic paraboloid can have circular cross-sections around a coordinate axis, and so can be generated by revolving a curve in one of the coordinate planes about a coordinate axis.

13.6.62

a. The ellipsoid has equal y- and z-intercepts, so the equation is $x^2 + 4y^2 + 4z^2 = 1$.

b. The ellipsoid has equal x- and z-intercepts, so the equation is $x^2 + 4y^2 + z^2 = 1$.

13.6.63 The volume is given by $3\int_{-1}^{1}(2 - x^2 - x^2)\,dx = 3\left(2x - \frac{2x^3}{3}\right)\Big|_{-1}^{1} = 3\left(2 - \frac{2}{3} - \left(-2 + \frac{2}{3}\right)\right) = 8$.

13.6.64

a. The light cone consists of all points (x, y, t) such that the distance from (x, y) to the origin is $c|t|$, where c is the speed of light; hence in these units an equation of the light cone is $x^2 + y^2 = t^2$.

b. The light cone consists of all points (x, y, t) such that the distance from (x, y) to the origin is $c|t|$, where c is the speed of light; hence in these units the equation of the light cone is $x^2 + y^2 = 9 \times 10^{16}t^2$.

13.6.65

a. Suppose that the y-axis is the major axis of an ellipse. When we rotate this ellipse about its major axis, we obtain an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The long axis therefore has a length of $2b$ and the other two axes have the same length $(c = a)$. So we obtain the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$, or $\frac{x^2 + z^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b > a$.

b. The length of the long axis is $2b = 11.1$, which implies that $b = 5.55$. The xz-trace is a circle of radius a with a circumference of 21.1 inches. So $2\pi a = 21.1$, which implies that $a = \frac{10.55}{\pi}$. So an equation for the football is $\frac{x^2 + z^2}{(10.55/\pi)^2} + \frac{y^2}{(5.55)^2} = 1$.

13.6.66 We are designing the upper half of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-3)^2}{c^2} = 1$ that equals $\frac{4x^2}{9} + 4y^2 = 1$ when $z = 3$. This implies that $a^2 = \frac{9}{4}$ and $b^2 = \frac{1}{4}$. Because the distance from the center of the base of the half-ellipsoid to the top of the ellipsoid is $\frac{1}{2}$, $c^2 = \frac{1}{4}$. So the desired equation is $\frac{4x^2}{9} + 4y^2 + 4(z-3)^2 = 1$.

13.6.67 The desired ellipsoid is obtained by shifting the standard equation of an ellipsoid up three units to obtain $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-3)^2}{c^2} = 1$. Now we need to find the constants a, b, and c. For $z = 3$, the equation of the cone reduces to $\frac{x^2}{2} + y^2 = \frac{9}{8}$, or $\frac{x^2}{(9/4)} + \frac{y^2}{(9/8)} = 1$ and the equation of the ellipsoid reduces to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Because the cone and ellipsoid coincide at $z = 3$, we let $a^2 = 9/4$ and $b^2 = 9/8$. Updating the equation of the ellipsoid, we have $\frac{4x^2}{9} + \frac{8y^2}{9} + \frac{(z-3)^2}{c^2} = 1$. Because the height of the truncated ellipsoid is $3/2$, we require the value of z to equal $9/2$ when$x = y = 0$. Substituting these values into the updated equation, we obtain $\frac{(9/2 - 3)^2}{c^2} = 1$, which implies that $c^2 = 9/4$. So the equation of the upper half of the ellipsoid is $\frac{4x^2}{9} + \frac{8y^2}{9} + \frac{4(z-3)^2}{9} = 1$, or $4x^2 + 8y^2 + 4(z-3)^2 = 9$, for $3 \le z \le 4.5$.

13.6.68 Because horizontal traces of the cup are circular, $a = b$. For $z = 0$, we have $x^2 + y^2 = 1$, which implies that $a^2 = b^2 = 1$. For $z = 6$, $x^2 + y^2 = 1 + \frac{36}{c^2}$, and also $x^2 + y^2 = 4$,which implies that $c^2 = 12$.

Chapter Thirteen Review

1

a. True. Addition of vectors is commutative.

b. False. For example, the vector in the direction of **i** with the length of **j** is **i**, but the vector in the direction of **j** with the length of **i** is **j**, and $\mathbf{i} \neq \mathbf{j}$.

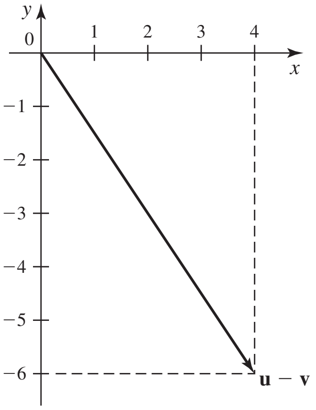
c. True, because it then follows that $\mathbf{u} = -\mathbf{v}$, so the two are parallel.

d. False. The direction vectors for the lines are not parallel, so the lines are not parallel.

e. True. The direction vector for the line is $\langle 1, 2, -1\rangle$ and the normal to the plane is $\langle 1, 2, 5\rangle$ and $\langle 1, 2, -1\rangle \cdot \langle 1, 2, 5\rangle = 1 + 4 - 5 = 0$, so the line and the plane are parallel.

f. True. The cross product of the normals to the two intersecting planes is the normal to the plane in question.

2



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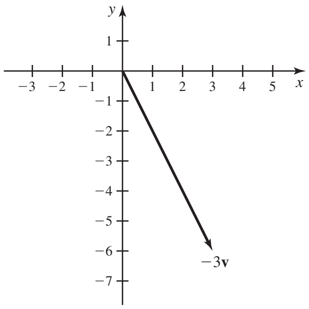

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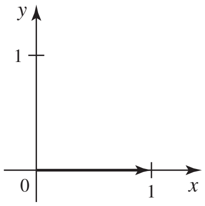

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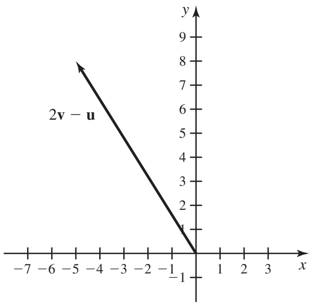


6 $\mathbf{u} - 3\mathbf{v} = \langle 2, 4, -5\rangle - \langle -18, 30, 6\rangle = \langle 20, -26, -11\rangle$.

7 $|\mathbf{u} + \mathbf{v}| = |\langle -4, 14, -3\rangle| = \sqrt{16 + 196 + 9} = \sqrt{221}$.

8 $\dfrac{\mathbf{u}}{|\mathbf{u}|} = \dfrac{1}{\sqrt{4 + 16 + 25}} \langle 2, 4, -5\rangle = \dfrac{1}{\sqrt{45}} \langle 2, 4, -5\rangle$.

9 $12\dfrac{\mathbf{w}}{|\mathbf{w}|} = 12\dfrac{\langle 4, -8, 8\rangle}{\sqrt{16 + 64 + 64}} = 12\left\langle \dfrac{1}{3}, -\dfrac{2}{3}, \dfrac{2}{3}\right\rangle$.

10 $10\mathbf{w} = 10\langle 4, -8, 8\rangle = \langle 40, -80, 80\rangle$.

11 $10\dfrac{\mathbf{w}}{|\mathbf{w}|} = \left\langle \dfrac{10}{3}, -\dfrac{20}{3}, \dfrac{20}{3} \right\rangle$.

12 $\mathbf{u} \cdot \mathbf{v} = 2(-6) + 4(10) + -5(2) = -12 + 40 - 10 = 18$.

13 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -5 \\ -6 & 10 & 2 \end{vmatrix} = \langle 58, 26, 44 \rangle$

14 We need $\langle -6, 10, 2 \rangle \cdot \langle a, 1, -3 \rangle = 0$, so $-6a + 10 - 6 = 0$, so $a = \dfrac{2}{3}$.

15 We need $\langle 4, -8, 8 \rangle = k \langle a, 6, -6 \rangle$ for some k. So $k = -\dfrac{4}{3}$ and $a = -3$.

16 We must have $a + c = 2$, $a + b = 2$, and $b + c = 2$. Thus $a = b = c = 1$.

17

a. $\mathbf{v} = 550 \left\langle -\sqrt{2}/2, \sqrt{2}/2 \right\rangle = \left\langle -275\sqrt{2}, 275\sqrt{2} \right\rangle$.

b. $\mathbf{v} = 550 \left\langle -\sqrt{2}/2, \sqrt{2}/2 \right\rangle + \langle 0, 40 \rangle = \left\langle -275\sqrt{2}, 275\sqrt{2} + 40 \right\rangle$.

18

a. This is equal to $\langle 2, -8, 5 \rangle - \langle 2, 0, 6 \rangle = \langle 0, -8, -1 \rangle$.

b. The midpoint is $\left(\dfrac{2+2}{2}, \dfrac{-8+0}{2}, \dfrac{5+6}{2} \right) = (2, -4, 5.5)$. The magnitude of $\overrightarrow{PM}$ is $\dfrac{1}{2} |\langle 0, -8, -1 \rangle| = \dfrac{1}{2}\sqrt{64+1} = \dfrac{1}{2}\sqrt{65}$.

c. $-\dfrac{8}{\sqrt{65}} \langle 0, -8, -1 \rangle = \dfrac{8}{\sqrt{65}} \langle 0, 8, 1 \rangle$.

19 $\{(x, y, z) \,|\, (x-1)^2 + y^2 + (z+1)^2 = 16\}$.

20 $\{(x, y, z) \,|\, (x-2)^2 + (y-4)^2 + (z+3)^2 < 100\}$.

21 $\{(x, y, z) \,|\, x^2 + (y-1)^2 + z^2 > 4\}$.

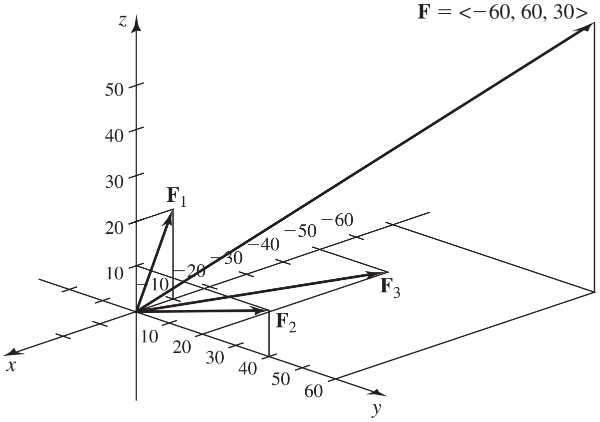
22 Completing the square gives $(x^2 - 6x + 9) + (y^2 + 8y + 16) + (z^2 - 2z + 1) = 23 + 9 + 16 + 1 = 49$, or $(x-3)^2 + (y+4)^2 + (z-1)^2 = 7^2$. This is a sphere of radius 7 centered at $(3, -4, 1)$.

23 Completing the square gives $(x^2 - x + 1/4) + (y^2 + 4y + 4) + (z^2 - 6z + 9) < -11 + 1/4 + 4 + 9 = \dfrac{9}{4}$, or $(x - 1/2)^2 + (y+2)^2 + (z-3)^2 < (3/2)^2$. This is a ball centered at $(1/2, -2, 3)$ of radius $3/2$.

24 Completing the square gives $x^2 + (y^2 - 10y + 25) + (z^2 - 6z + 9) = -34 + 25 + 9$, or $x^2 + (y-5)^2 + (z-3)^2 = 0$. This consists of the single point $(0, 5, 3)$.

25 Completing the square gives $(x^2 - 6x + 9) + y^2 + (z^2 - 20z + 100) > -9 + 9 + 100$, or $(x-3)^2 + y^2 + (z-10)^2 > 10^2$. These are the points outside of a sphere of radius 10 centered at $(3, 0, 10)$.

26 $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \langle -60, 60, 30 \rangle$.
$|\mathbf{F}| = 30\sqrt{4+4+1} = 90$.

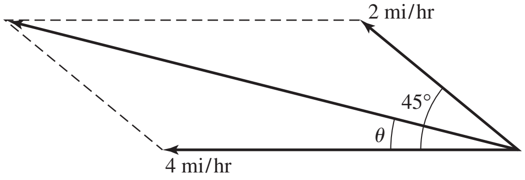


27 The magnitude of $\langle 0, 4, -50 \rangle$ is $\sqrt{2516} \approx 50.16$ meters per second. The direction is $\cos^{-1}(4/\sqrt{2516}) \approx 85.4$ degrees below the horizontal in the northerly horizontal direction.

28 The plane's original velocity vector is $250\,\langle 1, 0, 0 \rangle$, the crosswind's is $40\left\langle \sqrt{2}/2, \sqrt{2}/2, 0 \right\rangle$ and the downdraft's is $25\,\langle 0, 0, -1 \rangle$. The resulting velocity vector is $\left\langle 250 + 20\sqrt{2}, 20\sqrt{2}, -25 \right\rangle$. The speed is therefore $\sqrt{(250 + 20\sqrt{2})^2 + (20\sqrt{2})^2 + (-25)^2} \approx 280.8$ mph.

29 The magnitude of the net force is $|\mathbf{F}| = \sqrt{40^2 + 30^2} = 50$ pounds. $\alpha = \tan^{-1}\left(\frac{3}{4}\right) \approx 0.6435$ radians or 36.87 degrees. The net force has magnitude 50 pounds in the direction 36.9 degrees north of east.

30 The canoe's vector $\mathbf{u}$ is given by $\mathbf{u} = -4\mathbf{i} + (-\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}) = -(4 + \sqrt{2})\mathbf{i} + \sqrt{2}\mathbf{j}$. The magnitude of $\mathbf{u}$ is given by $\sqrt{(4 + \sqrt{2})^2 + (\sqrt{2})^2} \approx 5.5958$ miles per hour. $\theta = \tan^{-1}\left(\frac{\sqrt{2}}{(4 + \sqrt{2})}\right) \approx 0.2555$ radians, or about 14.64 degrees. The canoe has speed about 5.6 miles per hour in the direction 14.64 degrees north of west.



31 This is a circle of radius one centered at $(0, 2, 0)$ and sitting in the plane $y = 2$.

32

a. $\mathbf{u} \cdot \mathbf{v} = 0 - 3 + 20 = 17$. $|\mathbf{u}| = 5$ and $|\mathbf{v}| = \sqrt{42}$. Thus the angle between the vectors is $\cos^{-1}\left(\frac{17}{5\sqrt{42}}\right) \approx$ 1.02 radians.

b. $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{17}{42}\langle -4, 1, 5 \rangle$, and $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{17}{\sqrt{42}}$.

c. $\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{17}{25}\langle 0, -3, 4 \rangle$, and $\text{scal}_{\mathbf{u}}\mathbf{v} = \frac{17}{5}$.

33

a. $\mathbf{u} \cdot \mathbf{v} = -3 + 12 + 12 = 21$, $|\mathbf{u}| = 3$ and $|\mathbf{v}| = 9$, so $\theta = \cos^{-1}\left(\frac{21}{27}\right) \approx 0.68$ radian.

b. $\text{proj}_{\mathbf{v}}\mathbf{u} = \dfrac{21}{81}\langle 3,6,6\rangle = \langle 7/9, 14/9, 14/9\rangle$, and $\text{scal}_{\mathbf{v}}\mathbf{u} = \dfrac{21}{9} = \dfrac{7}{3}$.

c. $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{21}{9}\langle -1,2,2\rangle = \langle -7/3, 14/3, 14/3\rangle$, and $\text{scal}_{\mathbf{u}}\mathbf{v} = \dfrac{21}{3} = 7$.

34 We need to compute $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$. Note that

$$\mathbf{v}\cdot\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 10 & 2 \\ 4 & -8 & 8 \end{vmatrix} = \langle 96, 56, 8\rangle.$$

Then $\mathbf{u}\cdot\langle 96,56,8\rangle = 2(96) + 4(56) - 5(8) = 376$.

35 The work will be $25\cdot 20\cdot\cos 45^\circ = 250\sqrt{2}$ ft-lbs.

36 The work will be $\langle 2,3,4\rangle\cdot\langle 2,1,6\rangle = 31$J.

37 $|\mathbf{F}_{\text{perp}}| = 180\cos 30^\circ = 90\sqrt{3}$. $|\mathbf{F}_{\text{par}}| = 180\sin 30^\circ = 90$.

38 Work $= 90\cdot 10 = 900$ foot-lbs.

39 Two adjacent sides of the parallelogram are given by $\langle 0,-2,3\rangle$ and $\langle 3,0,1\rangle$, so the area is

$$|\langle 0,-2,3\rangle\times\langle 3,0,1\rangle| = |\langle -2,9,6\rangle| = 11.$$

40 Two adjacent sides of the triangle are given by $\langle 4,0,-4\rangle$ and $\langle -1,2,-5\rangle$, so the area is

$$\frac{1}{2}\left|\langle 4,0,-4\rangle\times\langle -1,2,-5\rangle\right| = \frac{1}{2}\left|\langle 8,24,8\rangle\right| = 4\sqrt{11}.$$

41 $\langle 2,-6,9\rangle\times\langle -1,0,6\rangle = \langle -36,-21,-6\rangle$. The length of this vector is $3\sqrt{12^2+7^2+2^2} = 3\sqrt{197}$. Thus the unit normals are $\dfrac{\pm 1}{\sqrt{197}}\langle 12,7,2\rangle$.

42

a. The angle is given by $\cos^{-1}\left(\dfrac{\langle 2,0,-2\rangle\cdot\langle 2,2,0\rangle}{2\sqrt{2}\cdot 2\sqrt{2}}\right) = \cos^{-1}(1/2) = \pi/3$.

b. The angle is also given by $\sin^{-1}\left(\dfrac{|\langle 2,0,-2\rangle\times\langle 2,2,0\rangle|}{2\sqrt{2}\cdot 2\sqrt{2}}\right) = \sin^{-1}\left(\dfrac{|\langle 4,-4,4\rangle|}{8}\right) = \sin^{-1}(4\sqrt{3}/8) = \pi/3$.

43 The torque is $\langle 3,2,1\rangle\times\langle 10,10,0\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 10 & 10 & 0 \end{vmatrix} = \langle -10,10,10\rangle$.

44 The magnitude of the torque is $50\cdot 0.25\sin 60^\circ = 6.25\sqrt{3}$ N-m.

45 $T(\theta) = (0.4)\cdot 98\cdot\sin\theta \approx 39.2\sin\theta$ Newton-meters. This has a maximum of 39.2 when $\sin\theta = 1$ (at $\theta = \pi/2$) and a minimum of 0 at $\theta = 0$. The direction of the torque does not change as the knee is lifted.

46 The direction is $\langle -6-2, 4-6, 0-(-1)\rangle = \langle -8,-2,1\rangle$. So the line is described by $\mathbf{r} = \langle 2,6,-1\rangle + t\langle 8,2,-1\rangle$, $\infty < t < \infty$.

47 The direction of the line segment is $\langle 2-0, -8-(-3), 1-9\rangle = \langle 2,-5,-8\rangle$. The line segment is described by $\mathbf{r} = \langle 0,-3,9\rangle + t\langle 2,-5,-8\rangle$, $0\le t\le 1$.

48 The direction is $\langle 2,-5,6\rangle$, so the line is given by $\mathbf{r} = \langle 0,1,1\rangle + t\,\langle 2,-5,6\rangle = \langle 2t, 1-5t, 1+6t\rangle$.

49 The direction is given by $\langle 0,-1,3\rangle \times \langle 2,-1,2\rangle = \langle 1,6,2\rangle$. Thus the line is given by $\mathbf{r} = \langle 0,1,1\rangle + t\,\langle 1,6,2\rangle = \langle t, 1+6t, 1+2t\rangle$.

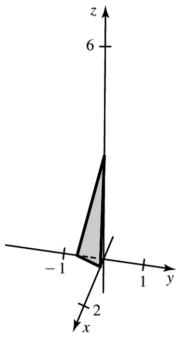
50 The direction is given by $\langle -2,1,7\rangle \times \langle 0,1,0\rangle = \langle -7,0,-2\rangle$. Thus the line is given by $\mathbf{r} = \langle 0,1,4\rangle + t\,\langle -7,0,-2\rangle = \langle -7t, 1, 4-2t\rangle$.

51

a. Let $P = (0,0,3)$, $Q = (1,0,-6)$ and $R = (1,2,3)$. Then the vectors $\overrightarrow{PQ} = \langle 1,0,-9\rangle$ and $\overrightarrow{PR} = \langle 1,2,0\rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -9 \\ 1 & 2 & 0 \end{vmatrix} = 18\mathbf{i} - 9\mathbf{j} + 2\mathbf{k}$ is normal to the plane. The plane has equation $18x - 9y + 2(z-3) = 0$, which simplifies to $18x - 9y + 2z = 6$.

b. The x-intercept is found by setting $y = z = 0$ and solving $18x = 6$ to obtain $x = \frac{1}{3}$. Similarly, the y and z-intercepts are $y = -\frac{2}{3}$ and $z = 3$.

c.



52 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle 2,1,-1\rangle$and $\mathbf{n}_R = \langle -1,1,1\rangle$, are not multiples of each other; therefore, these planes are not parallel and they intersect in a line ℓ. We need to find a point on ℓ and a vector in the direction of ℓ. Setting $x = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the yz plane: $y - z = 0$, $y + z = 1$. Solving these equations simultaneously gives $y = z = \frac{1}{2}$, so $\left(0, \frac{1}{2}, \frac{1}{2}\right)$ is a point on ℓ. A vector in the direction of ℓ is $\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} = \langle 2,-1,3\rangle$. Therefore ℓ has equation $\mathbf{r}(t) = \left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle + t\langle 2,-1,3\rangle = \left\langle 2t, \frac{1}{2} - t, \frac{1}{2} + 3t\right\rangle$, or $x = 2t$, $y = \frac{1}{2} - t$, $z = \frac{1}{2} + 3t$.

53 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle -3,1,2\rangle$ and $\mathbf{n}_R = \langle 3,3,4\rangle$ are not multiples of each other; therefore, these planes are not parallel and they intersect in a line ℓ. We need to find a point on ℓ and a vector in the direction of ℓ. Setting $x = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the yz plane: $y + 2z = 0$, $3y + 4z = 12$. Solving these equations simultaneously gives $y = 12$, $z = -6$, so $(0,12,-6)$ is a point on ℓ. A vector in the direction of ℓ is $\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 2 \\ 3 & 3 & 4 \end{vmatrix} = -2\mathbf{i} + 18\mathbf{j} - 12\mathbf{k} = -2\langle 1,-9,6\rangle$. Therefore, ℓ has equation $\mathbf{r}(t) = \langle 0,12,-6\rangle + t\langle 1,-9,6\rangle = \langle t, 12-9t, -6+6t\rangle$, or $x = t$, $y = 12 - 9t$, $z = -6 + 6t$.

54 A normal to the plane we are looking for is $\langle 2,1,-1\rangle$, because planes with the same normal are parallel. The desired plane is therefore $2(x-5)+1(y-0)-1(z-2)=0$, or $2x+y-z=8$.

55 The plane we are looking for has normal $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ 4 & 5 & -2 \end{vmatrix} = 4\mathbf{i}+2\mathbf{j}+13\mathbf{k}$. A point on both lines is $(4,5,1)$ which occurs for $s=1$ and $t=-1$. The desired plane is therefore $4(x-4)+2(y-5)+13(z-1)=0$, or $4x+2y+13z=39$.

56 The line has direction $\mathbf{v}=\langle 1,3,-3\rangle$, so the desired plane has equation $1(x-2)+3\,(y+3)-3\,(z-1)=0$, which simplifies to $x+3y-3z=-10$.

57 Let $P=(-2,3,1)$, $Q=(1,1,0)$ and $R=(-1,0,1)$. Then the vectors $\overrightarrow{PQ}=\langle 3,-2,-1\rangle$ and $\overrightarrow{PR}=\langle 1,-3,0\rangle$ lie in the plane, so $\mathbf{n}=\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -1 \\ 1 & -3 & 0 \end{vmatrix} = -\left(3\mathbf{i}+\mathbf{j}+7\mathbf{k}\right)$ is normal to the plane. The plane has equation $3\,(x+2)+1\,(y-3)+7\,(z-1)=0$, which simplifies to $3x+y+7z=4$.

58 Let the point on the line be $P(2,3,1)$, so that $\mathbf{u}=\overrightarrow{PQ}=\langle 1,1,-2\rangle$. Then

$$d=\frac{|\,\langle 1,1,-2\rangle\times\langle 1,0,-3\rangle\,|}{|\,\langle 1,0,-3\rangle\,|}=\frac{|\,\langle -3,1,-1\rangle\,|}{|\,\langle 1,0,-3\rangle\,|}=\frac{\sqrt{11}}{\sqrt{10}}.$$

59 Let $Q(2,2,2)$ be the given point and let $P(1,0,0)$ be a point on the plane. Then $\overrightarrow{PQ}=\langle 1,2,2\rangle$ is normal to the plane. So the distance from the point to the plane is $\left|\overrightarrow{PQ}\right|=|\langle 1,2,2\rangle|=\sqrt{9}=3$.

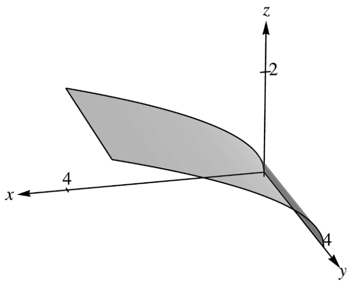
60

a. This surface is a cylinder consisting of lines parallel to the y-axis passing through the curve $z=\sqrt{x}$ in the xz-plane.

b. The xy-trace is found by setting $z=0$ in the equation $z=\sqrt{x}$, which gives all points $(0,y,0)$ (the y-axis). Similarly, we see that the xz-trace is the curve $z=\sqrt{x}$ and the yz-trace is the line $z=0$.

c. The x-intercept is found by setting $y=z=0$ in the equation $z=\sqrt{x}$, which gives the point $(0,0,0)$. The y-intercepts consist of all points on the y-axis, and the z-intercept is the point $(0,0,0)$.

d.



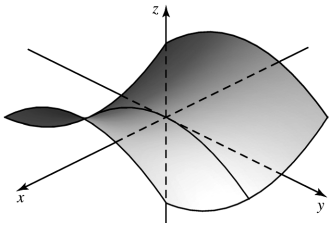
61

a. This surface is a hyperbolic paraboloid.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives $y = \pm 2x$ (two lines intersecting at the origin). Similarly, we see that the xz-trace is the parabola $z = \frac{x^2}{36}$ and the yz-trace is the parabola $z = -\frac{y^2}{144}$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y- and z-intercepts are also $(0, 0, 0)$.

d.



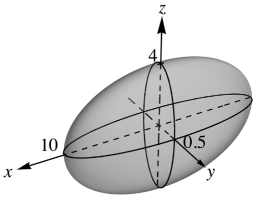
62

a. This surface is an ellipsoid.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the ellipse $\frac{x^2}{100} + 4y^2 = 1$. Similarly, we see that the xz-trace is the ellipse $\frac{x^2}{100} + \frac{z^2}{16} = 1$, and the yz-trace is the ellipse $4y^2 + \frac{z^2}{16} = 1$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the points $(\pm 10, 0, 0)$. Similarly, we see that the y-intercepts are the points $\left(0, \pm\frac{1}{2}, 0\right)$, and the z-intercepts are the points $(0, 0, \pm 4)$.

d.



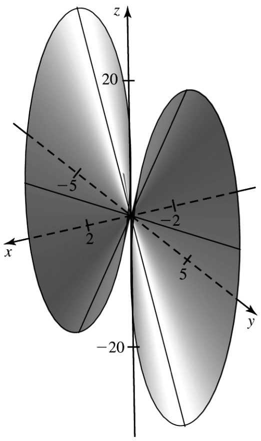
63

a. This surface is an elliptic cone.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives $y = \pm 2x$ (two lines intersecting at the origin). Similarly, we see that the xz-trace is $(0, 0, 0)$ and the yz-trace is $z = \pm 5y$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y- and z-intercepts are also $(0, 0, 0)$.

d.



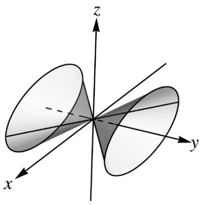
64

a. This surface is an elliptic cone.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives $y = \pm\dfrac{2x}{3}$ (two lines intersecting at the origin). Similarly, we see that the xz-trace is $(0, 0, 0)$, and the yz-trace is $y = \pm\dfrac{3z}{2}$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y- and z-intercepts are also $(0, 0, 0)$.

d.



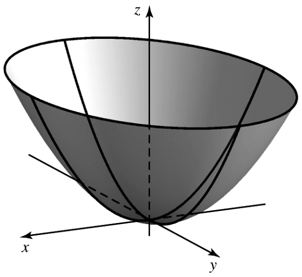
65

a. This surface is an elliptic paraboloid.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives $(0, 0, 0)$. Similarly, we see that the xz-trace is the parabola $z = \dfrac{x^2}{16}$, and the yz-trace is the parabola $z = \dfrac{y^2}{36}$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y- and z-intercepts are also $(0, 0, 0)$.

d.



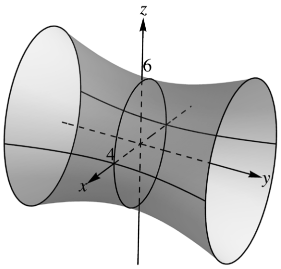
66

a. This surface is a hyperboloid of one sheet.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the hyperbola $\frac{x^2}{16} - \frac{y^2}{100} = 1$. Similarly, we see that the xz-trace is the ellipse $\frac{x^2}{16} + \frac{z^2}{36} = 1$, and the yz-trace is the hyperbola $\frac{z^2}{36} - \frac{y^2}{100} = 1$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the points $(\pm 4, 0, 0)$. Similarly, we see that there is no y-intercept, and the z-intercepts are $(0, 0, \pm 6)$.

d.



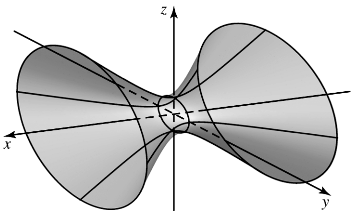
67

a. This surface is a hyperboloid of one sheet.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the hyperbola $y^2 - 2x^2 = 1$. Similarly, we see that the xz-trace is the hyperbola $4z^2 - 2x^2 = 1$, and the yz-trace is the ellipse $y^2 + 4z^2 = 1$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives no solutions. Similarly, we see that the y-intercepts are $(0, \pm 1, 0)$, and the z-intercepts are $\left(0, 0, \pm\frac{1}{2}\right)$.

d.



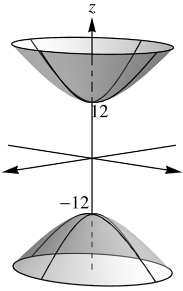
68

a. This surface is a hyperboloid of two sheets.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives no solutions. Similarly, we see that the xz-trace is the hyperbola $-\frac{x^2}{16} + \frac{z^2}{36} = 4$, and the yz-trace is the hyperbola $\frac{z^2}{36} - \frac{y^2}{25} = 4$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives no solutions. Similarly, we see that there is no y-intercept, and the z-intercepts are $(0, 0, \pm 12)$.

d.



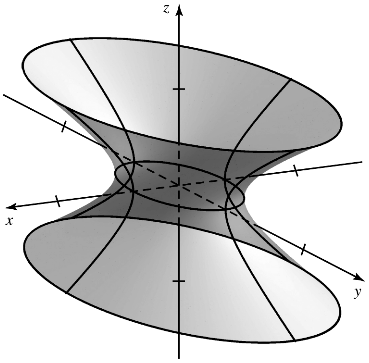
69

a. This surface is a hyperboloid of one sheet.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the ellipse $\frac{x^2}{4}+\frac{y^2}{16} = 4$. Similarly, we see that the xz-trace is the hyperbola $\frac{x^2}{4} - z^2 = 4$, and the yz-trace is the hyperbola $\frac{y^2}{16} - z^2 = 4$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the points $(\pm 4, 0, 0)$. Similarly, we see that the y-intercepts are $(0, \pm 8, 0)$, and there are no z-intercepts.

d.



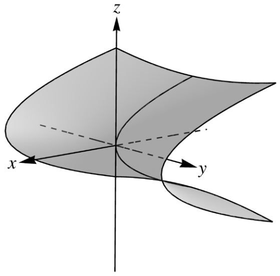
70

a. This surface is a hyperbolic paraboloid.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the parabola $x = \frac{y^2}{64}$. Similarly, we see that the xz-trace is the parabola $x = -\frac{z^2}{9}$ and the yz-trace is the $y = \pm\frac{8z}{3}$ (two lines intersecting at the origin).

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y- and z-intercepts are also $(0, 0, 0)$.

d.



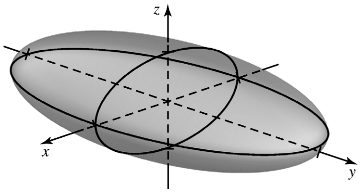
71

a. This surface is an ellipsoid.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the ellipse $\frac{x^2}{4}+\frac{y^2}{16} = 4$. Similarly, we see that the xz-trace is the ellipse $\frac{x^2}{4} + z^2 = 4$, and the yz-trace is the ellipse $\frac{y^2}{16} + z^2 = 4$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the points $(\pm 4, 0, 0)$. Similarly, we see that the y-intercepts are the points $(0, \pm 8, 0)$, and the z-intercepts are the points $(0, 0, \pm 2)$.

d.



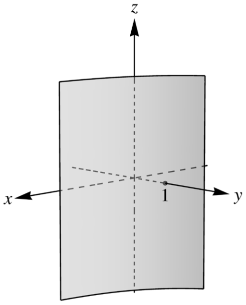
72

a. This surface is a cylinder consisting of lines parallel to the z-axis passing through the curve $y = e^{-x}$ in the xy-plane.

b. The xy-trace is found by setting $z = 0$ in the equation $y = e^{-x}$, which gives the curve $y = e^{-x}$. Similarly, we see that there is no xz-trace, and the yz-trace is the line consisting of all points $(0, 1, z)$.

c. The x-intercept is found by setting $y = z = 0$ in the equation $y = e^{-x}$, which has no solutions. Similarly, we see that the y-intercept is the point $(0, 1, 0)$, and there are no z-intercepts.

d.



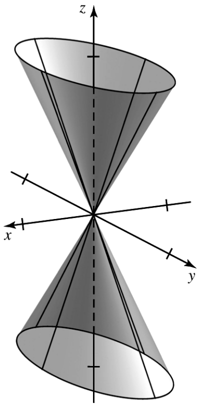
73

a. This surface is an elliptic cone.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the origin $(0, 0, 0)$. Similarly, we see that the xz-trace is $z = \pm\frac{8x}{3}$ (two lines intersecting at the origin), and the yz-trace is $y = \pm\frac{7z}{8}$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y and z-intercepts are also $(0, 0, 0)$.

d.



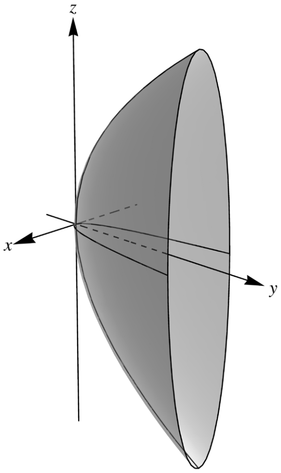
74

a. This surface is an elliptic paraboloid.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the parabola $y = 4x^2$. Similarly, we see that the xz-trace is the origin, and the yz-trace is the parabola $y = \frac{z^2}{9}$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y- and z-intercepts are also $(0, 0, 0)$.

d.



75

a. The graph of this function is part of a hyperboloid of two sheets and contains the origin, which matches A.

b. The graph of this function is a cylinder, which matches D.

c. The graph of this function is a hyperbolic paraboloid, which matches C.

d. The graph of this function is part of a hyperboloid of one sheet, which matches B.

76 The equation of the cylinder is $x^2+y^2=4$, $0\le z\le 6$. Perhaps the easiest approach to forming the top of the water bottle is to shift the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ up 8 units to obtain $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{(z-8)^2}{c^2}=1$ so that the top of the bottle corresponds to the narrowest portion of the hyperboloid before it is truncated. For $z=8$, the equation should simplify to $x^2+y^2=\frac{1}{4}$, which implies that $a^2=b^2=\frac{1}{4}$. So we have $4x^2+4y^2-\frac{(z-8)^2}{c^2}=1$ or $x^2+y^2-\frac{(z-8)^2}{4c^2}=\frac{1}{4}$. For $z=6$, the equation simplifies to $x^2+y^2=4$, which implies that $\frac{(6-8)^2}{4c^2}+\frac{1}{4}=4$, or $c^2=\frac{4}{15}$. Putting this all together, we have the equation $x^2+y^2-\frac{15(z-8)^2}{16}=\frac{1}{4}$ for the top part of the bottle. So the answer is $x^2+y^2=4$, $0\le z\le 6$ and $x^2+y^2-\frac{15(z-8)^2}{16}=\frac{1}{4}$, $6\le z\le 8$. Note: Answers vary depending upon which portion of a truncated hyperboloid is used to form the top part of the bottle.

Chapter 14

Vector-Valued Functions

14.1 Vector-Valued Functions

14.1.1 It has one, namely t.

14.1.2 It has three, namely $x = f(t)$, $y = g(t)$, and $z = h(t)$.

14.1.3 For every real number t that is put into the function, the output is a vector $\mathbf{r}(t)$.

14.1.4 It lies in the xz-plane.

14.1.5 Compute $\lim\limits_{t\to a} f(t) = L_1$, $\lim\limits_{t\to a} g(t) = L_2$, and $\lim\limits_{t\to a} h(t) = L_3$. Then $\lim\limits_{t\to a} \mathbf{r}(t) = \langle L_1, L_2, L_3\rangle$.

14.1.6 It is continuous at a exactly when the three component functions $x = f(t)$, $y = g(t)$ and $z = h(t)$ are continuous at a.

14.1.7 Note that $\langle 1, 2, 3\rangle$ is a direction vector for the line. The equation is $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + 3t\mathbf{k}$.

14.1.8 $x = \cos t$, $y = \sin t$, and $z = 10$, so $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 10\mathbf{k}$.

14.1.9 A direction vector for the line is $\overrightarrow{PQ} = \langle 2, 3, -4\rangle$, so the line is $\mathbf{r}(t) = \langle 2 + 2t, 3 + 3t, 7 - 4t\rangle$.

14.1.10 A direction vector for the line is $\langle 0, -1, 1\rangle$, so the line is $\mathbf{r}(t) = \langle 0, -3 - t, 2 + t\rangle$.

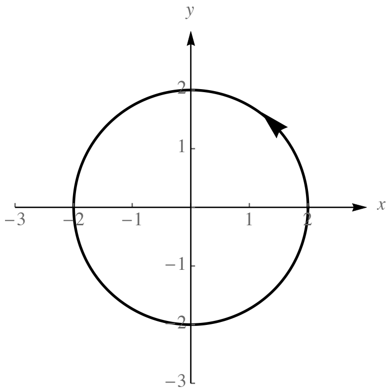
14.1.11 A direction vector for the line is a normal to the plane, which could be $\langle 2, 0, -1\rangle$, so the line is $\mathbf{r}(t) = \langle 3 + 2t, 4, 5 - t\rangle$.

14.1.12 A direction vector for the line of intersection is obtained by crossing the normals to the planes. So we calculate $\langle 2, 3, 4\rangle \times \langle 2, 3, 5\rangle = \langle 3, -2, 0\rangle$. So the line is given by $\mathbf{r}(t) = \langle 3t, 1 - 2t, 1\rangle$.

14.1.13 The line segment from P to Q is given by $(1 - t)\overrightarrow{OP} + t\overrightarrow{OQ} = (1 - t)\langle 1, 2, 1\rangle + t\langle 0, 2, 3\rangle = \langle 1 - t, 2, 1 + 2t\rangle$, $0 \le t \le 1$.

14.1.14 The line segment from P to Q is given by $(1 - t)\overrightarrow{OP} + t\overrightarrow{OQ} = (1 - t)\langle -4, -2, 1\rangle + t\langle -2, -2, 3\rangle = \langle -4 + 2t, -2, 1 + 2t\rangle$, $0 \le t \le 1$.

14.1.15



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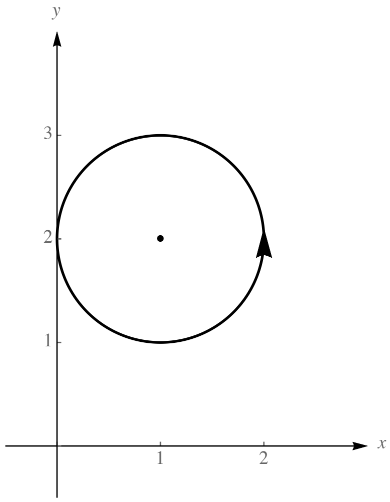

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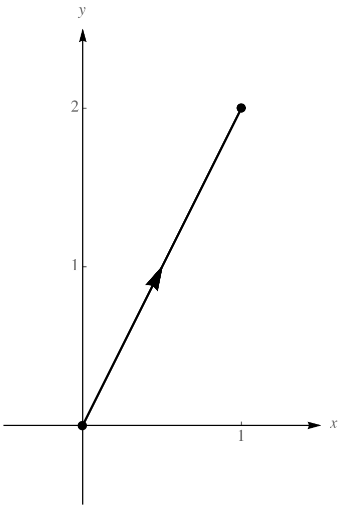

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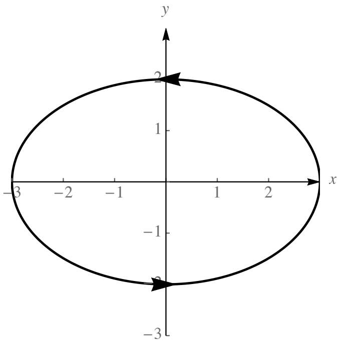

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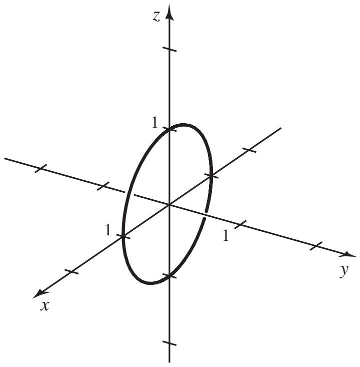

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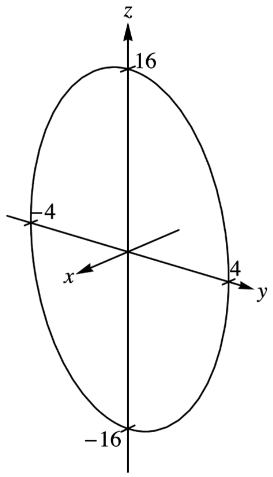

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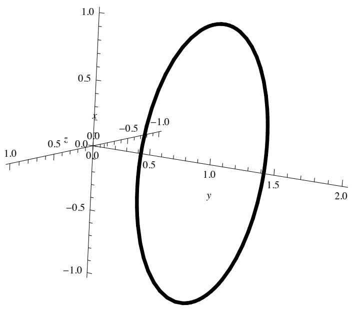

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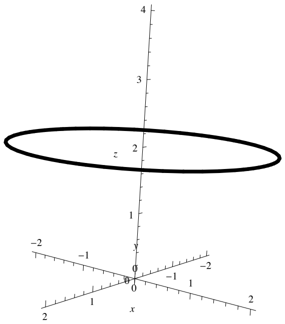

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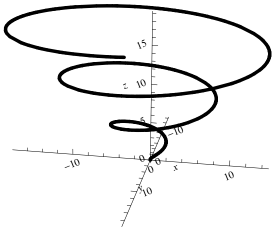

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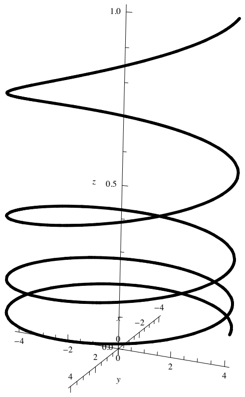
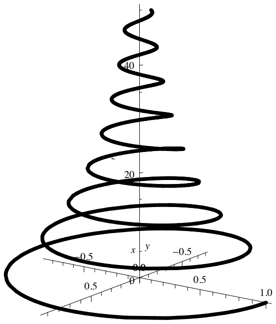
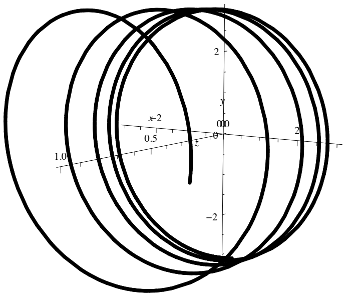


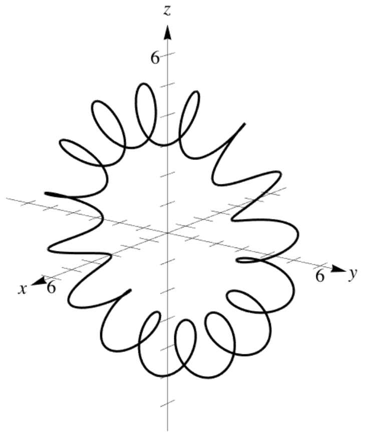
14.1.25



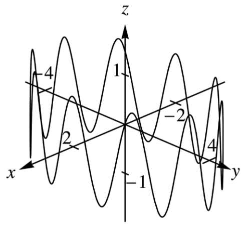
14.1.26



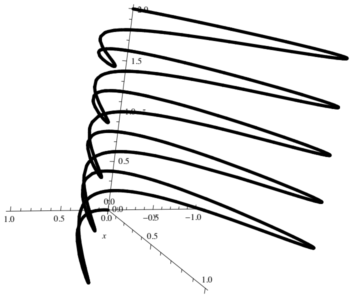
14.1.27 Note that the curve is closed (the initial point and the terminal point coincide), and is very "wavy."



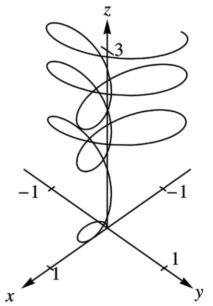
14.1.28 The projection onto the xy-plane is an ellipse elongate in the y direction. The curve oscillates in a sinusoidal wave in the z direction.



14.1.29 When viewed from the top, the curve looks parabolic.



14.1.30 When viewed from the top, the curve appears as a 3-petaled rose.



14.1.31 $\lim_{t\to\pi/2}\left\langle \cos 2t, -4\sin t, \frac{2t}{\pi}\right\rangle = \left\langle \cos\pi, -4\sin\pi/2, \frac{2\cdot\pi/2}{\pi}\right\rangle = \langle -1, -4, 1\rangle$.

14.1.32 $\lim_{t\to\ln 2}\left\langle 2e^t, 6e^{-t}, -4e^{-2t}\right\rangle = \left\langle 2e^{\ln 2}, 6e^{-\ln 2}, -4e^{-2\ln 2}\right\rangle = \langle 4, 3, -1\rangle$.

14.1.33 $\lim_{t\to\infty}\left\langle e^{-t}, -\frac{2t}{t+1}, \tan^{-1} t\right\rangle = \langle 0, -2, \pi/2\rangle$.

14.1.34 $\lim_{t\to 2}\left\langle \frac{t}{t^2+1}, -4e^{-t}\sin \pi t, \frac{1}{\sqrt{4t+1}}\right\rangle = \left\langle \frac{2}{5}, 0, \frac{1}{3}\right\rangle$.

14.1.35 Using l'Hôpital's rule (once in the first two components, twice in the third component):

$$\lim_{t\to 0}\left\langle \frac{\sin t}{t}, -\frac{e^t - t - 1}{t}, \frac{\cos t + t^2/2 - 1}{t^2}\right\rangle = \lim_{t\to 0}\left\langle \cos t, 1 - e^t, \frac{-\cos t + 1}{2}\right\rangle = \langle 1, 0, 0\rangle = \mathbf{i}.$$

14.1.36 Using l'Hôpital's rule (once in the first two components, but not in the 3rd component):

$$\lim_{t\to 0}\left\langle \frac{\tan t}{t}, \frac{-3t}{\sin t}, \sqrt{t+1}\right\rangle = \lim_{t\to 0}\left\langle \sec^2 t, \frac{-3}{\cos t}, \sqrt{t+1}\right\rangle = \langle 1, -3, 1\rangle = \mathbf{i} - 3\mathbf{j} + \mathbf{k}.$$

14.1.37

a. True. Letting $y = 0$ we see that $z = t^2 = x^2$, which is a parabola.

b. False. $x^2 + y^2 + z^2 = \sin^2 t + \cos^2 t + \sin^2 t = 1 + \sin^2 t \neq 1$.

c. True. The first component function approaches 0 as $t \to \infty$, while the others are periodic. The parametric equations $y = \sin t$ and $z = -\cos t$ form a circle in the yz-plane.

d. True. Both have limit $\langle 0, 0, 0\rangle$.

14.1.38 The first component function has domain $(-\infty, 1) \cup (1, \infty)$, and the second has domain $(-\infty, -2) \cup (-2, \infty)$, so the domain of $\mathbf{r}(t)$ is $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$.

14.1.39 The first component function has domain $[-2, \infty)$ and the second has domain $(-\infty, 2]$, so the domain of $\mathbf{r}(t)$ is $[-2, 2]$.

14.1.40 The first component function is defined everywhere, the second has domain $[0, \infty)$, and the third has domain $(-\infty, 0) \cup (0, \infty)$, so the domain of $\mathbf{r}(t)$ is $(0, \infty)$.

14.1.41 The first component function has domain $[-2, 2]$, the second has domain $[0, \infty)$, and the third has domain $(-1, \infty)$, so the domain of $\mathbf{r}(t)$ is $[0, 2]$.

14.1.42 The intersection occurs for $y = 1 = 2\sin t$, so for $t = \pi/6$ and $t = 5\pi/6$. The points of intersection are $(5\sqrt{3}, 1, 1)$ and $(-5\sqrt{3}, 1, 1)$.

14.1.43 The intersection occurs for $z = 16 = 4 + 3t$, so for $t = 4$. The point of intersection is $(4, 8, 16)$.

14.1.44 The intersection occurs for $x + y = \cos t + \sin t = 0$, so for $t = 3\pi/4$, $t = 7\pi/4$, $t = 11\pi/4$, and $t = 15\pi/4$. The points of intersection are $(-\sqrt{2}/2, \sqrt{2}/2, 3\pi/4)$, $(\sqrt{2}/2, -\sqrt{2}/2, 7\pi/4)$, $(-\sqrt{2}/2, \sqrt{2}/2, 11\pi/4)$, and $(\sqrt{2}/2, -\sqrt{2}/2, 15\pi/4)$.

14.1.45

a. This matches graph E. (It is a straight line.)

b. This matches graph D. (It is parabolic-like.)

c. This matches graph F. (It is a circle.)

d. This matches graph C. (It is a circular helix, elongated along the x-axis.)

e. This matches graph A. (It is a closed curve which isn't a circle.)

f. This matches graph B. (It is a circular helix, elongated along the y-axis.)

14.1.46

a. $\mathbf{r}(0) = \langle 50, 0, 0\rangle$.

b. $\lim_{t\to\infty} \frac{50\cos t}{e^t} = 0$ by the squeeze theorem, and likewise $\lim_{t\to\infty} \frac{50\sin t}{e^t} = 0$. Also, $\lim_{t\to\infty} \left(5 - \frac{5}{e^t}\right) = 5$. Thus we have $\lim_{t\to\infty} \mathbf{r}(t) = \langle 0, 0, 5\rangle$.

c. Let $x = 50e^{-t}\cos t$ and $y = 50e^{-t}\sin t$ and $z = 5 - 5e^{-t}$. Note that $x^2 + y^2 = 2500e^{-2t}$, so $r = 50e^{-t}$. We have $z = 5 - 5e^{-t} = 5 - \frac{r}{10}$.

14.1.47 We have $z = 4 = x^2 + y^2$, so we have a circle of radius 2 at a height of 4. This can be describe by $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle$.

14.1.48 We have $z = 3x^2 + y^2 + 1 = 5 - x^2 - 3y^2$, so $4x^2 + 4y^2 = 4$, or $x^2 + y^2 = 1$. Then $z = 3x^2 + y^2 + 1 = 2x^2 + x^2 + y^2 + 1 = 2x^2 + 2$. This can be described by $\mathbf{r}(t) = \left\langle \cos t, \sin t, 2 + 2\cos^2 t\right\rangle$.

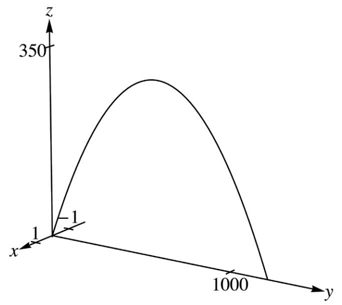
14.1.49 Letting $x = 5\cos t$ and $y = 5\sin t$ we have $x^2 + y^2 = 25$. So we have

$$\mathbf{r}(t) = \langle 5\cos t, 5\sin t, 10\cos t + 10\sin t\rangle .$$

14.1.50 If $z = y + 1 = x^2 + 1$, then $y = x^2$. If we let $x = t$, we have $y = t^2$ and $z = t^2 + 1$, so $\mathbf{r}(t) = \left\langle t, t^2, t^2 + 1\right\rangle$.

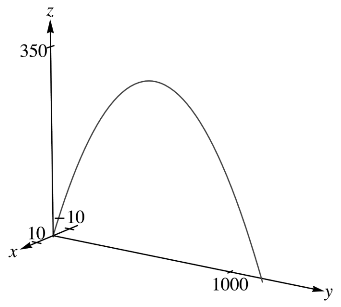
14.1.51

a.



With $a = 0$, we have $\mathbf{r}(t) = \left\langle 0, 75t, -5t^2 + 80t\right\rangle$. Note that $z = 0$ when $t = 0$ and when $-5t + 80 = 0$, or $t = 16$. At this time, $y = 75 \cdot 16 = 1200$ feet.

b.



With $a = 0.2$, we have $\mathbf{r}(t) = \left\langle .2t, (75 - .02)t, -5t^2 + 80t\right\rangle$. Note that $z = 0$ when $t = 0$ and when $-5t + 80 = 0$, or $t = 16$. At this time, $y = (75 - .02) \cdot 16 = 1199.7$ feet.

c. With $a = 2.5$, the ball travels $(75 - .025) \cdot 16 = 1196$ feet.

14.1.52 This is an elliptic cone. Note that

$$x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2(\cos^2 t + \sin^2 t) = t^2 = z^2.$$

14.1.53 This is a hyperboloid of one sheet. Note that

$$x^2+y^2-z^2=(t^2+1)\cos^2 t+(t^2+1)\sin^2 t-t^2=(t^2+1)(\cos^2 t+\sin^2 t)-t^2=(t^2+1)-t^2=1.$$

14.1.54 This is an elliptic paraboloid. Note that

$$x^2+y^2=t\cos^2 t+t\sin^2 t=t(\cos^2 t+\sin^2 t)=t=z.$$

14.1.55 This is an ellipsoid. Note that

$$x^2+\frac{y^2}{4}+\frac{z^2}{9}=0+\frac{4\cos^2 t}{4}+\frac{9\sin^2 t}{9}=\cos^2 t+\sin^2 t=1.$$

14.1.56 This is a torus. Note that

$$\begin{aligned}\left(3-\sqrt{x^2+y^2}\right)^2+z^2&=\left(3-\sqrt{(3+\cos 15t)^2\cos^2 t+(3+\cos 15t)^2\sin^2 t}\right)^2+\sin^2 15t\\&=\left(3-\sqrt{(3+\cos 15t)^2(\cos^2 t+\sin^2 t)}\right)^2+\sin^2 15t\\&=(3-(3+\cos 15t))^2+\sin^2 15t=\cos^2 15t+\sin^2 15t=1.\end{aligned}$$

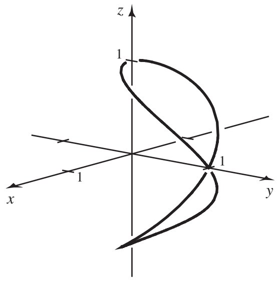
14.1.57 Consider $s(t)=\sqrt{(t^2-1)^2+(t-1)^2+(t-15)^2}=\sqrt{t^4-32t+227}$. Computing $s'(t)$ reveals $s'(t)=\dfrac{2(t^3-8)}{\sqrt{t^4-32t+227}}$. This is zero for $t=2$. So the point closest to P_0 is $(4,2,2)$. The distance between the two points is $\sqrt{179}$.

14.1.58 Consider

$$\begin{aligned}s(t)&=\sqrt{(\cos t-1)^2+(\sin t-1)^2+(t-3)^2}\\&=\sqrt{t^2-6t+\sin^2(t)-2\sin(t)+\cos^2(t)-2\cos(t)+11}\\&=\sqrt{t^2-6t-2\sin(t)-2\cos(t)+12}.\end{aligned}$$

Note that $s'(t)=\dfrac{t+\sin(t)-\cos(t)-3}{\sqrt{t^2-6t-2\sin(t)-2\cos(t)+12}}$ which is zero for approximately $t=1.8$. The closest point is approximately $(-0.23,0.97,1.8)$ and the distance is approximately 1.72.

14.1.59 First note that $x=\frac{1}{2}\sin 2t=\sin t\cos t$ and $y=\frac{1}{2}(1-\cos 2t)=\sin^2 t$. Then $x^2+y^2+z^2=\sin^2 t\cos^2 t+\sin^4 t+\cos^2 t=\sin^2 t(\cos^2 t+\sin^2 t)+\cos^2 t=\sin^2 t\cdot 1+\cos^2 t=1$. So all points on the curve are equidistant from the origin, so they lie on the sphere of radius one centered at the origin.

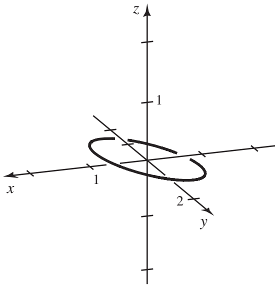


14.1.60 Suppose $a^2=b^2=c^2$. We have $x^2+y^2+z^2=a^2\sin^2 mt\cdot\cos^2 nt+b^2\sin^2 mt\cdot\sin^2 nt+c^2\cos^2 mt=a^2(\cos^2 nt+\sin^2 nt)\sin^2 mt+a^2\cos^2 mt=a^2\sin^2 mt+a^2\cos^2 mt=a^2$. Thus we have $x^2+y^2+z^2=a^2$. So the curve lies on a sphere.

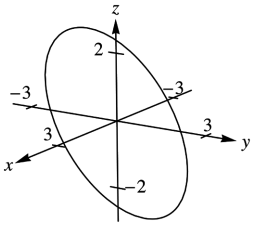
14.1.61 In order for $\sin(mt+mT)\cos(nt+nT)=\sin mt\cos nt$ and $\sin(mt+mT)\sin(nt+nT)=\sin mt\sin nt$ and $\cos(mt+mT)=\cos mt$ we would need $T=\dfrac{2\pi}{m}$ or a multiple of it, and then it would be necessary for $\sin(nt+nT)=\sin nt$, which would require $T=\dfrac{2\pi}{n}$, or a multiple of it. Thus, the smallest such T would be $\dfrac{2\pi}{(m,n)}$, where (m,n) represents the greatest common factor of m and n.

14.1.62 If $x = a\cos t + b\sin t$ and $y = c\cos t + d\sin t$ and $z = e\cos t + f\sin t$ then $x^2 + y^2 + z^2 = (a\cos t + b\sin t)^2 + (c\cos t + d\sin t)^2 + (e\cos t + f\sin t)^2 = (a^2+c^2+e^2)\cos^2 t + (b^2+d^2+f^2)\sin^2 t + 2(ab+cd+ef)\sin t\cos t$. If $ab + cd + ef = 0$ and if $a^2 + c^2 + e^2 = R^2 = b^2 + d^2 + f^2$, then we have $x^2 + y^2 + z^2 = R^2$, so all the points on the curve lie at a distance of R from the origin, so (because the curve lies in a plane) it is a circle of radius R centered at the origin.

14.1.63 This has the form mentioned in exercise 62, with $a = 1/\sqrt{2}$, $b = 1/\sqrt{3}$, $c = -1/\sqrt{2}$, $d = 1/\sqrt{3}$, $e = 0$, and $f = 1/\sqrt{3}$. Note that $a^2+c^2+e^2 = \frac{1}{2}+\frac{1}{2}+0 = 1 = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = b^2+d^2+f^2$, and $ab+cd+ef = 0$. So this is a circle of radius 1 centered at the origin.



14.1.64 Using the notation of exercise 62, we have $ab + cd + ef = 0$. However, we have $a^2 + c^2 + e^2 = 6$ and $b^2 + d^2 + e^2 = 12$. Thus $x^2 + y^2 + z^2 = 6\cos^2 t + 12\sin^2 t = 6 + 6\sin^2 t$, The curve is an oval-shaped closed curve.



14.1.65 Note that $\mathbf{r}(0) = \langle a, c, e\rangle$ and $\mathbf{r}(\pi/2) = \langle b, d, f\rangle$, and $\mathbf{r}(\pi) = \langle -a, -c, -e\rangle$ have their terminal points on the curve. So $\langle a, c, e\rangle - \langle -a, -c, -e\rangle = \langle 2a, 2c, 2e\rangle = 2\mathbf{r}(0)$ lies in the plane containing the curve, which implies that $\mathbf{r}(0)$ lies in the plane containing the curve, and that implies that the point $(0, 0, 0)$ is in the plane containing the curve. So a normal to the curve is $\mathbf{r}(0) \times \mathbf{r}(\pi/2) = \langle a, c, e\rangle \times \langle b, d, f\rangle = \langle cf - de, be - af, ad - bc\rangle$.

14.1.66

a. Assume that $\lim_{t\to a} \mathbf{r}(t) = \mathbf{L} = \langle L_1, L_2, L_3\rangle$. Then for any $\epsilon > 0$, there exists $\delta > 0$ so that $|\mathbf{r}(t) - \mathbf{L}| = \sqrt{(f(t) - L_1)^2 + (g(t) - L_2)^2 + (h(t) - L_3)^2} < \epsilon$ for $|t - a| < \delta$.

   Note that $|f(t) - L_1| \le \sqrt{(f(t) - L_1)^2 + (g(t) - L_2)^2 + (h(t) - L_3)^2}$, and likewise for $|g(t) - L_2|$ and $|h(t) - L_3|$. So given $\epsilon > 0$, choosing the δ mentioned in the paragraph above guarantees that the absolute values of the differences of the corresponding coordinate functions and the L_i are less than epsilon whenever $|t - a| < \delta$, as desired.

b. Assume $\lim_{t\to a} f(t) = L_1$, $\lim_{t\to a} g(t) = L_2$, and $\lim_{t\to a} h(t) = L_3$. Then for any $\epsilon > 0$, there exists $\delta_1 > 0$ so that $|f(t) - L_1| < \epsilon/\sqrt{3}$ whenever $|t - a| < \delta_1$, and there exists δ_2 so that $|g(t) - L_2| < \epsilon/\sqrt{3}$ whenever $|t - a| < \delta_2$, and there exists δ_3 so that $|h(t) - L_3| < \epsilon/\sqrt{3}$ whenever $|t - a| < \delta_3$.

   Now let $\epsilon > 0$ be given, and let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then

   $$|\mathbf{r}(t) - \langle L_1, L_2, L_3\rangle| = \sqrt{(f(t) - L_1)^2 + (g(t) - L_2)^2 + (h(t) - L_3)^2} \le \sqrt{3 \cdot \epsilon^2/3} = \sqrt{\epsilon^2} = \epsilon,$$

   as desired.

14.2 Calculus of Vector-Valued Functions

14.2.1 It is $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$.

14.2.2 $\mathbf{r}'(t)$ is a vector tangent to the curve $\mathbf{r}(t)$.

14.2.3 Divide the vector by its length, so if the vector is $\mathbf{r}'(t)$, form $\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

14.2.4 $\mathbf{r}'(t) = \left\langle 10t^9, 8, -\sin t\right\rangle$, so $\mathbf{r}''(t) = \left\langle 90t^8, 0, -\cos t\right\rangle$.

14.2.5 Compute the indefinite integral of each of the component functions, and then

$$\int \mathbf{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle.$$

14.2.6 $\displaystyle\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$.

14.2.7 $\mathbf{r}(0) = \langle 1, 3, 10\rangle + C$, so if $\mathbf{r}(0) = \langle 0, 0, 0\rangle$, then we must have $C = \langle -1, -3, -10\rangle$.

14.2.8 $\mathbf{r}'(0) = \langle 6, 2, 3\rangle$, so $\dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \left\langle \dfrac{6}{7}, \dfrac{2}{7}, \dfrac{3}{7}\right\rangle$.

14.2.9 $\mathbf{r}'(t) = \langle -\sin t, 2t, \cos t\rangle$.

14.2.10 $\mathbf{r}'(t) = \langle 4e^t, 0, 1/t\rangle$.

14.2.11 $\mathbf{r}'(t) = \left\langle 6t, 3/\sqrt{t}, -3/t^2\right\rangle$.

14.2.12 $\mathbf{r}'(t) = \langle 0, -6\sin 2t, 6\cos 3t\rangle$.

14.2.13 $\mathbf{r}'(t) = \left\langle e^t, -2e^{-t}, -8e^{2t}\right\rangle$.

14.2.14 $\mathbf{r}'(t) = \left\langle \sec^2 t, \sec t\tan t, -\sin 2t\right\rangle$.

14.2.15 $\mathbf{r}'(t) = \langle e^{-t}(1-t), 1+\ln t, \cos t - t\sin t\rangle$.

14.2.16 $\mathbf{r}'(t) = \left\langle -(t+1)^{-2}, (t^2+1)^{-1}, (t+1)^{-1}\right\rangle$.

14.2.17 $\mathbf{r}'(t) = \left\langle 1, 6t, 3t^2\right\rangle$, so $\mathbf{r}'(1) = \langle 1, 6, 3\rangle$.

14.2.18 $\mathbf{r}'(t) = \left\langle e^t, 3e^{3t}, 5e^{5t}\right\rangle$, so $\mathbf{r}'(0) = \langle 1, 3, 5\rangle$.

14.2.19 $\mathbf{r}'(t) = \langle 1, -2\sin 2t, 2\cos t\rangle$, so $\mathbf{r}'(\pi/2) = \langle 1, 0, 0\rangle$.

14.2.20 $\mathbf{r}'(t) = \left\langle 2\cos t, -3\sin t, \dfrac{1}{2}\cos(t/2)\right\rangle$, so $\mathbf{r}'(\pi) = \langle -2, 0, 0\rangle$.

14.2.21 $\mathbf{r}'(t) = \left\langle 8t^3, 9\sqrt{t}, -10/t^2\right\rangle$, so $\mathbf{r}'(1) = \langle 8, 9, -10\rangle$.

14.2.22 $\mathbf{r}'(t) = \left\langle 2e^t, -2e^{-2t}, 8e^{2t}\right\rangle$, so $\mathbf{r}'(\ln 3) = \langle 6, -2/9, 72\rangle$.

14.2.23 $\mathbf{r}'(t) = \langle 2, 2, 1\rangle$, so

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{3}\langle 2, 2, 1\rangle = \langle 2/3, 2/3, 1/3\rangle.$$

14.2.24 $\mathbf{r}'(t) = \langle -\sin t, \cos t, 0\rangle$, so

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{\sin^2 t + \cos^2 t + 0}} \langle -\sin t, \cos t, 0\rangle = \langle -\sin t, \cos t, 0\rangle .$$

14.2.25 $\mathbf{r}'(t) = \langle 0, -2\sin 2t, 4\cos 2t\rangle$, so

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{4\sin^2 2t + 16\cos^2 2t}} \langle 0, -2\sin 2t, 4\cos 2t\rangle = \frac{1}{\sqrt{1+3\cos^2 2t}} \langle 0, -\sin 2t, 2\cos 2t\rangle .$$

14.2.26 $\mathbf{r}'(t) = \langle \cos t, -\sin t, -\sin t\rangle$, so

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{\cos^2 t + \sin^2 t + \sin^2 t}} \langle \cos t, -\sin t, -\sin t\rangle = \frac{1}{\sqrt{1+\sin^2 t}} \langle \cos t, -\sin t, -\sin t\rangle .$$

14.2.27 $\mathbf{r}'(t) = \left\langle 1, 0, -2/t^2\right\rangle$, so

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1+(4/t^4)}} \left\langle 1, 0, -2/t^2\right\rangle = \frac{1}{\sqrt{t^4+4}} \left\langle t^2, 0, -2\right\rangle .$$

14.2.28 $\mathbf{r}'(t) = \left\langle 2e^{2t}, 4e^{2t}, -6e^{-3t}\right\rangle$, so

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{20e^{4t}+36e^{-6t}}} \left\langle 2e^{2t}, 4e^{2t}, -6e^{-3t}\right\rangle = \frac{1}{\sqrt{5e^{4t}+9e^{-6t}}} \left\langle e^{2t}, 2e^{2t}, -3e^{-3t}\right\rangle .$$

14.2.29 $\mathbf{r}'(t) = \langle -2\sin 2t, 0, 6\cos 2t\rangle$, so at $t = \pi/2$, we have $\mathbf{r}'(\pi/2) = \langle 0, 0, -6\rangle$. Thus, the unit tangent at $\pi/2$ is $\langle 0, 0, -1\rangle$.

14.2.30 $\mathbf{r}'(t) = \langle \cos t, -\sin t, -e^{-t}\rangle$, so at $t = 0$, we have $\mathbf{r}'(0) = \langle 1, 0, -1\rangle$. Thus, the unit tangent at 0 is $\left\langle 1/\sqrt{2}, 0, -1/\sqrt{2}\right\rangle$.

14.2.31 $\mathbf{r}'(t) = \left\langle 6, 0, -3/t^2\right\rangle$, so at $t = 1$, we have $\mathbf{r}'(1) = \langle 6, 0, -3\rangle$. Thus, the unit normal at 1 is $\left\langle 2/\sqrt{5}, 0, -1/\sqrt{5}\right\rangle$.

14.2.32 $\mathbf{r}'(t) = \left\langle \sqrt{7}e^t, 3e^t, 3e^t\right\rangle$, so at $t = \ln 2$, we have $\mathbf{r}'(\ln 2) = \left\langle 2\sqrt{7}, 6, 6\right\rangle$. Thus, the unit normal at $\ln 2$ is $\left\langle \sqrt{7}/5, 3/5, 3/5\right\rangle$.

14.2.33

$$\begin{aligned}(t^{12}+3t)\mathbf{u}'(t) + \mathbf{u}(t)(12t^{11}+3) &= (t^{12}+3t)\left\langle 6t^2, 2t, 0\right\rangle + \left\langle 2t^3, (t^2-1), -8\right\rangle (12t^{11}+3)\\ &= \left\langle 30t^{14}+24t^3, 14t^{13}-12t^{11}+9t^2-3, -96t^{11}-24\right\rangle .\end{aligned}$$

14.2.34 $(4t^8-6t^3)\mathbf{v}'(t) + \mathbf{v}(t)(32t^7-18t^2) = (4t^8-6t^3)\left\langle e^t, -2e^{-t}, -2e^{2t}\right\rangle + \left\langle e^t, 2e^{-t}, -e^{2t}\right\rangle (32t^7-18t^2) = \left\langle (4t^8+32t^7-6t^3-18t^2)e^t, (64t^7-36t^2-8t^8+12t^3)e^{-t}, (-8t^8-32t^7+12t^3+18t^2)e^{2t}\right\rangle$.

14.2.35

$$\begin{aligned}\mathbf{u}'(t^4-2t)\cdot(4t^3-2) &= \left\langle 6(t^4-2t)^2, 2(t^4-2t), 0\right\rangle (4t^3-2) = \left\langle 6(t^4-2t)^2(4t^3-2), 2(t^4-2t)(4t^3-2), 0\right\rangle\\ &= 4t(2t^3-1)(t^3-2)\left\langle 3t(t^3-2), 1, 0\right\rangle .\end{aligned}$$

14.2.36 $\mathbf{v}'(\sqrt{t})\cdot\dfrac{1}{2\sqrt{t}} = \left\langle e^{\sqrt{t}}, -2e^{-\sqrt{t}}, -2e^{2\sqrt{t}}\right\rangle\cdot\dfrac{1}{2\sqrt{t}} = \left\langle \dfrac{e^{\sqrt{t}}}{2\sqrt{t}}, -\dfrac{1}{\sqrt{t}e^{\sqrt{t}}}, -\dfrac{e^{2\sqrt{t}}}{\sqrt{t}}\right\rangle$.

14.2.37 $\mathbf{u}(t)\cdot\mathbf{v}'(t) + \mathbf{v}(t)\cdot\mathbf{u}'(t) = \left\langle 2t^3, (t^2-1), -8\right\rangle\cdot\left\langle e^t, -2e^{-t}, -2e^{2t}\right\rangle + \left\langle e^t, 2e^{-t}, -e^{2t}\right\rangle\cdot\left\langle 6t^2, 2t, 0\right\rangle = 2t^3e^t - 2(t^2-1)e^{-t} + 16e^{2t} + 6t^2e^t + 4te^{-t} + 0 = e^t(2t^3+6t^2) - 2e^{-t}(t^2-2t-1) + 16e^{2t}$.

14.2.38 $\mathbf{u}(t)\times\mathbf{v}'(t)+\mathbf{u}'(t)\times\mathbf{v}(t)=\left\langle 2t^3,(t^2-1),-8\right\rangle\times\left\langle e^t,-2e^{-t},-2e^{2t}\right\rangle+\left\langle 6t^2,2t,0\right\rangle\times\left\langle e^t,2e^{-t},-e^{2t}\right\rangle=$ $\left\langle -2e^{2t}t^2-16e^{-t}+2e^{2t},4e^{2t}t^3-8e^t,-4e^{-t}t^3-e^tt^2+e^t\right\rangle+\left\langle -2e^{2t}t,6e^{2t}t^2,12e^{-t}t^2-2e^tt\right\rangle=$ $\left\langle -2e^{2t}t^2-2e^{2t}t-16e^{-t}+2e^{2t},4e^{2t}t^3+6e^{2t}t^2-8e^t,-4e^{-t}t^3+12e^{-t}t^2-e^tt^2-2e^tt+e^t\right\rangle$.

14.2.39 This is equal to $\mathbf{u}(0)\cdot\mathbf{v}'(0)+\mathbf{u}'(0)\cdot\mathbf{v}(0)=\langle 0,1,1\rangle\cdot\langle 1,1,2\rangle+\langle 0,7,1\rangle\cdot\langle 0,1,1\rangle=1+2+7+1=11$.

14.2.40 This is equal to $\mathbf{u}(0)\times\mathbf{v}'(0)+\mathbf{u}'(0)\times\mathbf{v}(0)=\langle 0,1,1\rangle\times\langle 1,1,2\rangle+\langle 0,7,1\rangle\times\langle 0,1,1\rangle=\langle 1,1,-1\rangle+\langle 6,0,0\rangle=\langle 7,1,-1\rangle$.

14.2.41 This is equal to $-\sin(0)\cdot\mathbf{u}(0)+\cos(0)\cdot\mathbf{u}'(0)=\langle 0,7,1\rangle$.

14.2.42 This is equal to $\mathbf{u}'(0)\cos 0=\langle 0,7,1\rangle$.

14.2.43 $\mathbf{v}'(e^t)\cdot e^t=e^t\left\langle 2e^t,-2,0\right\rangle=\left\langle 2e^{2t},-2e^t,0\right\rangle$.

14.2.44 $\mathbf{u}'(t^3)\cdot 3t^2=3t^2\left\langle 0,1,2t^3\right\rangle=\left\langle 0,3t^2,6t^5\right\rangle$.

14.2.45 $\mathbf{v}'(g(t))g'(t)=\left\langle 4\sqrt{t},-2,0\right\rangle\left(\dfrac{1}{\sqrt{t}}\right)=\left\langle 4,-2/\sqrt{t},0\right\rangle$.

14.2.46 $g(t)\mathbf{v}'(t)+\mathbf{v}(t)g'(t)=(2\sqrt{t})\left\langle 2t,-2,0\right\rangle+\left\langle t^2,-2t,1\right\rangle\left(\dfrac{1}{\sqrt{t}}\right)=\left\langle 5t^{3/2},-6\sqrt{t},t^{-1/2}\right\rangle$.

14.2.47 $\mathbf{u}(t)\times\mathbf{v}'(t)+\mathbf{u}'(t)\times\mathbf{v}(t)=\left\langle 1,t,t^2\right\rangle\times\langle 2t,-2,0\rangle+\langle 0,1,2t\rangle\times\left\langle t^2,-2t,1\right\rangle=\left\langle 2t^2,2t^3,-2t^2-2\right\rangle+\left\langle 4t^2+1,2t^3,-t^2\right\rangle=\left\langle 6t^2+1,4t^3,-3t^2-2\right\rangle$

14.2.48 $\mathbf{u}(t)\cdot\mathbf{v}'(t)+\mathbf{u}'(t)\cdot\mathbf{v}(t)=\left\langle 1,t,t^2\right\rangle\cdot\langle 2t,-2,0\rangle+\langle 0,1,2t\rangle\cdot\left\langle t^2,-2t,1\right\rangle=(2t-2t+0)+0-2t+2t=0$.

14.2.49 $\left\langle t^2,2t^2,-2t^3\right\rangle\cdot\left\langle e^t,2e^t,3e^{-t}\right\rangle+\left\langle 2t,4t,-6t^2\right\rangle\cdot\left\langle e^t,2e^t,-3e^{-t}\right\rangle=t^2e^t+4t^2e^t-6t^3e^{-t}+2te^t+8te^t+18t^2e^{-t}=5t^2e^t+10te^t-6t^3e^{-t}+18t^2e^{-t}$.

14.2.50

$$\begin{aligned}&\left\langle t^3,-2t,-2\right\rangle\times\left\langle 1,-2t,-3t^2\right\rangle+\left\langle 3t^2,-2,0\right\rangle\times\left\langle t,-t^2,-t^3\right\rangle\\&=\left\langle 6t^3-4t,3t^5-2,2t-2t^4\right\rangle+\left\langle 2t^3,3t^5,2t-3t^4\right\rangle=\left\langle 8t^3-4t,6t^5-2,4t-5t^4\right\rangle.\end{aligned}$$

14.2.51 $\left\langle 3t^2,\sqrt{t},-2/t\right\rangle\cdot\langle -\sin t,2\cos 2t,-3\rangle+\langle 6t,1/(2\sqrt{t}),2/t^2\rangle\cdot\langle\cos t,\sin 2t,-3t\rangle=-3t^2\sin t+2\sqrt{t}\cos 2t+6t\cos t+\sin(2t)/(2\sqrt{t})$.

14.2.52 $\left\langle t^3,6,-2\sqrt{t}\right\rangle\times\left\langle 3,-24t,12t^{-3}\right\rangle+\left\langle 3t^2,0,-1/\sqrt{t}\right\rangle\times\left\langle 3t,-12t^2,-6t^{-2}\right\rangle=$ $\left\langle\dfrac{72}{t^3}-48t^{3/2},-6\sqrt{t}-12,-24t^4-18\right\rangle+\left\langle -12t^{3/2},18-3\sqrt{t},-36t^4\right\rangle=\left\langle\dfrac{72}{t^3}-60t^{3/2},6-9\sqrt{t},-60t^4-18\right\rangle$.

14.2.53 $\mathbf{r}'(t)=\langle 2t,1,0\rangle$. $\mathbf{r}''(t)=\langle 2,0,0\rangle$. $\mathbf{r}'''(t)=\langle 0,0,0\rangle$.

14.2.54 $\mathbf{r}'(t)=\left\langle 36t^{11}-2t,8t^7+3t^2,-4t^{-5}\right\rangle$. $\mathbf{r}''(t)=\left\langle 396t^{10}-2,56t^6+6t,20t^{-6}\right\rangle$.
$\mathbf{r}'''(t)=\left\langle 3960t^9,336t^5+6,-120t^{-7}\right\rangle$.

14.2.55 $\mathbf{r}'(t)=\langle -3\sin 3t,4\cos 4t,-6\sin 6t\rangle$. $\mathbf{r}''(t)=\langle -9\cos 3t,-16\sin 4t,-36\cos 6t\rangle$.
$\mathbf{r}'''(t)=\langle 27\sin 3t,-64\cos 4t,216\sin 6t\rangle$.

14.2.56 $\mathbf{r}'(t)=\left\langle 4e^{4t},-8e^{-4t},-2e^{-t}\right\rangle$. $\mathbf{r}''(t)=\left\langle 16e^{4t},32e^{-4t},2e^{-t}\right\rangle$. $\mathbf{r}'''(t)=\left\langle 64e^{4t},-128e^{-4t},-2e^{-t}\right\rangle$.

14.2.57 $\mathbf{r}'(t)=\left\langle\dfrac{1}{2\sqrt{t+4}},\dfrac{1}{(t+1)^2},2e^{-t^2}t\right\rangle$. $\mathbf{r}''(t)=\left\langle -\dfrac{1}{4(t+4)^{3/2}},-\dfrac{2}{(t+1)^3},e^{-t^2}\left(2-4t^2\right)\right\rangle$.
$\mathbf{r}'''(t)=\left\langle\dfrac{3}{8(t+4)^{5/2}},\dfrac{6}{(t+1)^4},4e^{-t^2}t\left(2t^2-3\right)\right\rangle$.

14.2.58 $\mathbf{r}'(t) = \left\langle \sec^2(t), 1 - \dfrac{1}{t^2}, -\dfrac{1}{t+1} \right\rangle$. $\mathbf{r}''(t) = \left\langle 2\tan(t)\sec^2(t), \dfrac{2}{t^3}, \dfrac{1}{(t+1)^2} \right\rangle$.
$\mathbf{r}'''(t) = \left\langle 2\sec^4(t) + 4\tan^2(t)\sec^2(t), -\dfrac{6}{t^4}, -\dfrac{2}{(t+1)^3} \right\rangle$.

14.2.59 $\displaystyle\int \left\langle t^4 - 3t, 2t - 1, 10 \right\rangle\, dt = \left\langle t^5/5 - 3t^2/2, t^2 - t, 10t \right\rangle + \mathbf{C}$.

14.2.60 $\displaystyle\int \left\langle 5t^{-4} - t^2, t^6 - 4t^3, 2/t \right\rangle\, dt = \left\langle (-5/3)t^{-3} - t^3/3, t^7/7 - t^4, 2\ln|t| \right\rangle + \mathbf{C}$.

14.2.61 $\displaystyle\int \langle 2\cos t, 2\sin 3t, 4\cos 8t \rangle\, dt = \langle 2\sin t, (-2/3)\cos 3t, (1/2)\sin 8t \rangle + \mathbf{C}$.

14.2.62 $\displaystyle\int \left\langle te^t, t\sin t^2, -\frac{2t}{\sqrt{t^2+4}} \right\rangle dt = \left\langle (t-1)e^t, -\frac{1}{2}\cos t^2, -2\sqrt{t^2+4} \right\rangle + \mathbf{C}$.

14.2.63 $\displaystyle\int \left\langle e^{3t}, \frac{1}{1+t^2}, \frac{-1}{\sqrt{2t}} \right\rangle dt = \left\langle e^{3t}/3, \tan^{-1} t, -\sqrt{2t} \right\rangle + \mathbf{C}$.

14.2.64 $\displaystyle\int \left\langle 2^t, \frac{1}{1+2t}, \ln t \right\rangle dt = \left\langle 2^t/(\ln 2), \frac{1}{2}\ln|1+2t|, t\ln t - t \right\rangle + \mathbf{C}$.

14.2.65 $\displaystyle\int \left\langle e^t, \sin t, \sec^2 t \right\rangle\, dt = \left\langle e^t, -\cos t, \tan t \right\rangle + \mathbf{C}$. Because $\mathbf{r}(0) = \langle 2, 2, 2 \rangle = \langle 1, -1, 0 \rangle + \mathbf{C}$, we have $\mathbf{C} = \langle 1, 3, 2 \rangle$. Thus, $\mathbf{r}(t) = \langle e^t, -\cos t, \tan t \rangle + \langle 1, 3, 2 \rangle = \langle 1 + e^t, 3 - \cos t, 2 + \tan t \rangle$.

14.2.66 $\displaystyle\int \langle 0, 2, 2t \rangle\, dt = \left\langle 0, 2t, t^2 \right\rangle + \mathbf{C}$. Because $\mathbf{r}(1) = \langle 4, 3, -5 \rangle$, we have $\mathbf{r}(1) = \langle 0, 2, 1 \rangle + C = \langle 4, 3, -5 \rangle$, so $\mathbf{C} = \langle 4, 1, -6 \rangle$. Thus, $\mathbf{r}(t) = \left\langle 4, 2t + 1, t^2 - 6 \right\rangle$.

14.2.67 $\displaystyle\int \left\langle 1, 2t, 3t^2 \right\rangle\, dt = \left\langle t, t^2, t^3 \right\rangle + \mathbf{C}$. Because $\mathbf{r}(1) = \langle 4, 3, -5 \rangle$, we have $\langle 1, 1, 1 \rangle + \mathbf{C} = \langle 4, 3, -5 \rangle$, so $\mathbf{C} = \langle 3, 2, -6 \rangle$, and $\mathbf{r}(t) = \left\langle t + 3, t^2 + 2, t^3 - 6 \right\rangle$.

14.2.68 $\displaystyle\int \left\langle \sqrt{t}, \cos \pi t, 4/t \right\rangle\, dt = \left\langle (2/3)t^{3/2}, (\sin \pi t)/\pi, 4\ln|t| \right\rangle + \mathbf{C}$. Because $\mathbf{r}(1) = \langle 2, 3, 4 \rangle$, we have $\langle 2/3, 0, 0 \rangle$
$+ \mathbf{C} = \langle 2, 3, 4 \rangle$, so $\mathbf{C} = \langle 4/3, 3, 4 \rangle$, and $\mathbf{r}(t) = \left\langle (2/3)t^{4/2} + 1/3, (\sin \pi t)/\pi + 3, 4\ln|t| + 4 \right\rangle$.

14.2.69 $\displaystyle\int \left\langle e^{2t}, 1 - 2e^{-t}, 1 - 2e^t \right\rangle\, dt = \left\langle e^{2t}/2, t + 2e^{-t}, t - 2e^t \right\rangle + \mathbf{C}$. Because $\mathbf{r}(0) = \langle 1, 1, 1 \rangle$, we have $\langle 1/2, 2, -2 \rangle + \mathbf{C} = \langle 1, 1, 1 \rangle$, so $\mathbf{C} = \langle 1/2, -1, 3 \rangle$, and $\mathbf{r}(t) = \left\langle e^{2t}/2 + 1/2, t + 2e^{-t} - 1, t - 2e^t + 3 \right\rangle$.

14.2.70 $\displaystyle\int \left\langle \frac{t}{(t^2+1)}, te^{-t^2}, -\frac{2t}{\sqrt{t^2+4}} \right\rangle dt = \left\langle \frac{1}{2}\ln(t^2+1), -\frac{e^{-t^2}}{2}, -2\sqrt{t^2+4} \right\rangle + \mathbf{C}$. Because
$\mathbf{r}(0) = \langle 1, 3/2, -3 \rangle$, we have $\langle 0, -1/2, -4 \rangle + \mathbf{C} = \langle 1, 3/2, -3 \rangle$, so $\mathbf{C} = \langle 1, 2, 1 \rangle$, and
$\mathbf{r}(t) = \left\langle \dfrac{1}{2}\ln(t^2+1) + 1, -\dfrac{e^{-t^2}}{2} + 2, -2\sqrt{t^2+4} + 1 \right\rangle$.

14.2.71 $\displaystyle\int_{-1}^{1} \left\langle 1, t, 3t^2 \right\rangle\, dt = \left\langle t, t^2/2, t^3 \right\rangle \Big|_{-1}^{1} = \langle 2, 0, 2 \rangle$.

14.2.72 $\displaystyle\int_{1}^{4} \left\langle 6t^2, 8t^3, 9t^2 \right\rangle\, dt = \left\langle 2t^3, 2t^4, 3t^3 \right\rangle \Big|_{1}^{4} = \langle 128, 512, 192 \rangle - \langle 2, 2, 3 \rangle = \langle 126, 510, 189 \rangle$.

14.2.73 $\displaystyle\int_0^{\ln 2} \left\langle e^t, e^t\cos \pi e^t, 0\right\rangle\,dt = \left\langle e^t, \frac{\sin \pi e^t}{\pi}, 0\right\rangle\Big|_0^{\ln 2} = \langle 2,0,0\rangle - \langle 1,0,0\rangle = \langle 1,0,0\rangle = \mathbf{i}$.

14.2.74

$$\int_{1/2}^{1}\left\langle \frac{3}{1+2t}, 0, -\pi\csc^2(\pi t/2)\right\rangle\,dt = \left\langle \frac{3}{2}\ln(1+2t), 0, 2\cot(\pi t/2)\right\rangle\Big|_{1/2}^{1}$$
$$= \langle (3/2)\ln 3, 0, 0\rangle - \langle (3/2)\ln 2, 0, 2\rangle = \langle (3/2)\ln(3/2), 0, -2\rangle\,.$$

14.2.75 $\displaystyle\int_{-\pi}^{\pi} \langle \sin t, \cos t, 2t\rangle\,dt = \left\langle -\cos t, \sin t, t^2\right\rangle\Big|_{-\pi}^{\pi} = \langle 0,0,0\rangle$.

14.2.76 $\displaystyle\int_0^{\ln 2} \left\langle e^{-t}, 2e^{2t}, -4e^t\right\rangle\,dt = \left\langle -e^{-t}, e^{2t}, -4e^t\right\rangle\Big|_0^{\ln 2} = \langle 1/2, 3, -4\rangle$.

14.2.77 $\displaystyle\int_0^{2} \left\langle te^t, 2te^t, -te^t\right\rangle\,dt = \left\langle (t-1)e^t, 2(t-1)e^t, -(t-1)e^t\right\rangle\Big|_0^{2} = \left\langle e^2+1, 2e^2+2, -e^2-1\right\rangle$ $= (e^2+1)\,\langle 1,2,-1\rangle$.

14.2.78 $\displaystyle\int_0^{\pi/4} \left\langle \sec^2 t, -2\cos t, -1\right\rangle\,dt = \langle \tan t, -2\sin t, -t\rangle\Big|_0^{\pi/4} = \left\langle 1, -\sqrt{2}, -\pi/4\right\rangle$.

14.2.79

a. False. For example, if $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, then $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$ is not parallel to $\mathbf{r}(t)$, and is in fact perpendicular to it.

b. True. $\mathbf{r}'(t) = \langle 1, 2t-2, -\pi\sin \pi t\rangle \neq \langle 0,0,0\rangle$. Each component function is differentiable, and the derivative is never $\langle 0,0,0\rangle$, so the function is smooth by definition.

c. True. This follows because $\displaystyle\int_{-a}^{a} o(x)\,dx = 0$ for any odd function $o(x)$.

14.2.80 $\mathbf{r}'(t) = \left\langle e^t, 2e^{2t}, 3e^{3t}\right\rangle$, so $\mathbf{r}'(0) = \langle 1,2,3\rangle$. We have $\mathbf{r}(0) = \langle 1,1,1\rangle$, so the tangent line is given by $\langle 1+t, 1+2t, 1+3t\rangle$.

14.2.81 $\mathbf{r}'(t) = \langle -\sin t, 2\cos 2t, 1\rangle$, so $\mathbf{r}'(\pi/2) = \langle -1,-2,1\rangle$. We have $\mathbf{r}(\pi/2) = \langle 2,3,\pi/2\rangle$, so the tangent line is given by $\langle 2-t, 3-2t, \pi/2+t\rangle$.

14.2.82 $\mathbf{r}'(t) = \left\langle \dfrac{1}{\sqrt{2t+1}}, \pi\cos \pi t, 0\right\rangle$, so $\mathbf{r}'(4) = \langle 1/3, \pi, 0\rangle$. We have $\mathbf{r}(4) = \langle 3,0,4\rangle$, so the tangent line is given by $\langle 3+(1/3)t, \pi t, 4\rangle$.

14.2.83 $\mathbf{r}'(t) = \langle 3,7,2t\rangle$, so $\mathbf{r}'(1) = \langle 3,7,2\rangle$. We have $\mathbf{r}(1) = \langle 2,9,1\rangle$, so the tangent line is given by $\langle 2+3t, 9+7t, 1+2t\rangle$.

14.2.84 $\mathbf{r}'(t) = \langle -a\sin t, a\cos t\rangle$, and $\mathbf{r}(t)\cdot\mathbf{r}'(t) = \langle a\cos t, a\sin t\rangle \cdot \langle -a\sin t, a\cos t\rangle = -a^2\cos t\sin t + a^2\sin t\cos t = 0$. So $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal for all t.

14.2.85 $\mathbf{r}'(t) = \langle 2at, 1\rangle$. We have $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$ when $(2at)(at^2+1) + t(1) = 0$, which occurs only for $t = 0$ because $2a^2t^2 + 2a + 1 > 0$ for all t. The corresponding point on the parabola is $(1,0)$.

14.2.86 $\mathbf{r}'(t) = \left\langle 1/(2\sqrt{t}), 0, 1\right\rangle$. We have $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$ when $1/2 + 0 + t = 0$, so for $t = -1/2$, which isn't in the domain. So the vectors $\mathbf{r}$ and $\mathbf{r}'$ are never orthogonal.

14.2.87 $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle$. We have $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$ when $-\sin t\cos t + \sin t\cos t + t = 0$, which occurs only for $t = 0$. So the only point on the helix where these vectors are orthogonal is at $t = 0$. This corresponds to the point $(1,0,0)$.

14.2.88 $\mathbf{r}'(t) = \langle -2\sin t, 8\cos t, 0\rangle$. We have $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$ when $-4\sin t\cos t + 64\sin t\cos t + 0 = 0$, which occurs when $60\sin t\cos t = 0$, or $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$, and $t = 2\pi$.

The corresponding points are $(2,0,0)$, $(0,8,0)$, $(-2,0,0)$, and $(0,-8,0)$.

14.2.89 Note that $\mathbf{r}(t) = \langle a_1t, a_2t, a_3t\rangle = t\langle a_1, a_2, a_3\rangle$ where the a_i's are real numbers has this property because $\mathbf{r}'(t) = \langle a_1, a_2, a_3\rangle$, and $\mathbf{r}(t)$ is a multiple of $\mathbf{r}'(t)$.

Also, $\mathbf{r}(t) = \left\langle a_1e^{kt}, a_2e^{kt}, a_3e^{kt}\right\rangle$ where k is a real number has this property, as its derivative is k times itself.

14.2.90 If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ lies on the sphere $x^2+y^2+z^2 = a^2$, then (differentiating with respect to t) we have $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} + 2z\dfrac{dz}{dt} = 0$, so $\langle x,y,z\rangle\cdot\langle x',y',z'\rangle = 0$, so $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$.

If $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$, then $x\dfrac{dx}{dt} + y\dfrac{dy}{dt} + z\dfrac{dz}{dt} = 0$, so $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} + 2z\dfrac{dz}{dt} = 0$, so by integrating both sides with respect to t, we have $x^2+y^2+z^2 = K$ for some constant $K > 0$, so the curve lies on a sphere.

14.2.91

a. $\mathbf{r}(t)\cdot\mathbf{r}'(t) = (a^2+b^2+c^2)t = |\mathbf{r}(t)|\,|\mathbf{r}'(t)|\cos\theta$, so $\cos\theta = \dfrac{(a^2+b^2+c^2)t}{\sqrt{a^2+b^2+c^2}t\cdot\sqrt{a^2+b^2+c^2}} = 1$, so $\theta = 0$.

b. $\mathbf{r}(t)\cdot\mathbf{r}'(t) = ax_0 + by_0 + cz_0 + (a^2+b^2+c^2)t = |\mathbf{r}(t)|\,|\mathbf{r}'(t)|\cos\theta$, so

$$\cos\theta = \frac{ax_0 + by_0 + cz_0 + (a^2+b^2+c^2)t}{\sqrt{(x_0+at)^2 + (y_0+bt)^2 + (z_0+ct)^2}\cdot\sqrt{a^2+b^2+c^2}}.$$

Because x_0, y_0, and z_0 are not all 0, $\cos\theta$ depends on t.

c. In part a, the curve is a straight line through the origin, so the position vector and the tangent vector are parallel for all t. In part b, the line is not through the origin, so the tangent vector (which is the direction vector for the line) is not parallel to the position vector.

14.2.92

$$\begin{aligned}\frac{d}{dt}(\mathbf{u}(t)+\mathbf{v}(t)) &= \frac{d}{dt}\langle u_1(t)+v_1(t), u_2(t)+v_2(t), u_3(t)+v_3(t)\rangle\\ &= \langle u_1'(t)+v_1'(t), u_2'(t)+v_2'(t), u_3'(t)+v_3'(t)\rangle = \mathbf{u}'(t)+\mathbf{v}'(t).\end{aligned}$$

14.2.93

$$\begin{aligned}\frac{d}{dt}(f(t)\mathbf{u}(t)) &= \frac{d}{dt}\langle f(t)u_1(t), f(t)u_2(t), f(t)u_3(t)\rangle\\ &= \langle f'(t)u_1(t) + f(t)u_1'(t)+, f'(t)u_2(t) + f(t)u_2'(t), f'(t)u_3(t) + f(t)u_3'(t)\rangle\\ &= f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t).\end{aligned}$$

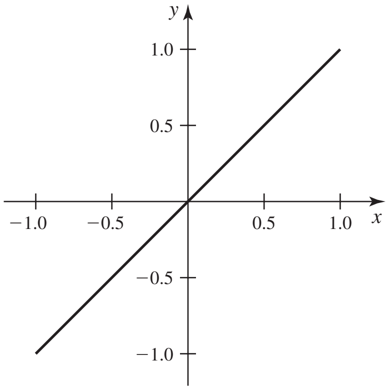
14.2.94

$$\begin{aligned}\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) &= \frac{d}{dt}\langle u_2v_3 - v_2u_3, u_3v_1 - v_3u_1, u_1v_2 - u_2v_1\rangle\\ &= \langle u_2'v_3 + u_2v_3' - (v_2'u_3 + v_2u_3'), u_3'v_1 + u_3v_1' - (u_1'v_3 + u_1v_3'), u_1'v_2 + u_1v_2' - (v_1'u_2 + v_1u_2')\rangle\\ &= \langle u_2'v_3 - v_2u_3' + u_2v_3' - v_2'u_3, u_3v_1' - u_1v_3' + u_3'v_1 - u_1'v_3, u_1'v_2 - v_1u_2' + u_1v_2' - v_1'u_2\rangle\\ &= \langle (u_2'v_3 - v_2u_3'), (u_3v_1' - u_1v_3'), (u_1'v_2 - v_1u_2')\rangle + \langle (u_2v_3' - v_2'u_3), (u_3'v_1 - u_1'v_3), (u_1v_2' - v_1'u_2\rangle\\ &= (\mathbf{u}'(t)\times\mathbf{v}(t)) + (\mathbf{u}(t)\times\mathbf{v}'(t))\end{aligned}$$

14.2.95

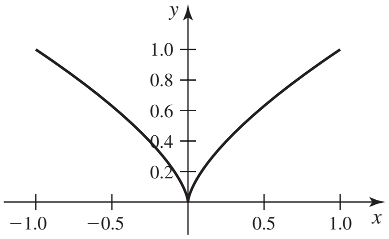
a.

$\mathbf{r}'(t) = \langle 3t^2, 3t^2 \rangle$, so $\mathbf{r}'(0) = \langle 0, 0 \rangle$. There is no cusp because $\lim_{t\to 0} \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2}{3t^2} = 1$ exists.



b.

$\mathbf{r}'(t) = \langle 3t^2, 2t \rangle$, so $\mathbf{r}'(0) = \langle 0, 0 \rangle$. There is a cusp because $\lim_{t\to 0} \dfrac{dy}{dx} = \lim\limits_{t\to 0} \dfrac{dy/dt}{dx/dt} = \lim\limits_{t\to 0} \dfrac{2t}{3t^2} = \lim\limits_{t\to 0} \dfrac{2}{3t}$ does not exist.



c. The curve $\mathbf{r}(t)$ for $-\infty < t < \infty$ traces out the whole curve $y = x^2$, while the curve $\mathbf{p}(t)$ only traces out the part in the first quadrant, because $x = t^2 > 0$ for all t.

d. $\mathbf{r}'(t) = \langle mt^{m-1}, nt^{n-1} \rangle$, so $\mathbf{r}'(0) = \langle 0, 0 \rangle$.

Assume $m > n$. There is a cusp because

$$\lim_{t\to 0} \frac{dy}{dx} = \lim_{t\to 0} \frac{dy/dt}{dx/dt} = \lim_{t\to 0} \frac{nt^{n-1}}{mt^{m-1}} = \lim_{t\to 0} (n/m) \frac{1}{t^{m-n}},$$

which does not exist.

Now assume $m < n$. There is a cusp because

$$\lim_{t\to 0} \frac{dx}{dy} = \lim_{t\to 0} \frac{dx/dt}{dy/dt} = \lim_{t\to 0} \frac{mt^{m-1}}{nt^{n-1}} = \lim_{t\to 0} (m/n) \frac{1}{t^{n-m}},$$

which does not exist.

14.3 Motion in Space

14.3.1 The velocity is the derivative of position, the speed is the magnitude of velocity, and the acceleration is the derivative of velocity.

14.3.2 For the circle $\mathbf{r}(t) = \langle a \cos t, a \sin t \rangle$ for $a > 0$, the two vectors are orthogonal, with the velocity vector tangent to the circle.

14.3.3 $m\mathbf{a}(t) = \mathbf{F}(t)$.

14.3.4 $m\mathbf{a}(t) = \langle mx''(t), my''(t), mz''(t)\rangle = \mathbf{F}(t) = \langle 0, 0, -mg\rangle$.

14.3.5 Integrate the acceleration to find an expression for the velocity plus a constant, and then use the initial velocity condition to find the constant.

14.3.6 Integrate the velocity to find an expression for the position plus a constant, and then use the initial position condition to find the constant.

14.3.7

a. When $96 - 32t = 0$, so $t = 3$.

b. $\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \langle 60, 96 - 32t\rangle\,dt = \left\langle 60t, 96t - 16t^2\right\rangle + \mathbf{C}$. Then because $\mathbf{r}(0) = \langle 0, 3\rangle$, we have $\mathbf{r}(t) = \left\langle 60t, -16t^2 + 96t + 3\right\rangle$.

14.3.8

a. The ball hits the ground when $y = 0$, so for $-16t^2 + 31t + 2 = 0$. This factors as $-(t-2)(16t+1) = 0$, so the ball hits the ground for $t = 2$.

b. We need to find x when $t = 2$. Because $x = 40t$, we have $x = 80$ when the ball hits the ground.

14.3.9

a. $\mathbf{v}(t) = \langle 6t, 8t\rangle$, so the speed is $\sqrt{36t^2 + 64t^2} = \sqrt{100t^2} = 10t$.

b. $\mathbf{a}(t) = \langle 6, 8\rangle$.

14.3.10

a. $\mathbf{v}(t) = \langle 5t, 12t\rangle$, so the speed is $\sqrt{25t^2 + 144t^2} = \sqrt{169t^2} = 13t$.

b. $\mathbf{a}(t) = \langle 5, 12\rangle$.

14.3.11

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2, -4\rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{20} = 2\sqrt{5}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 0\rangle$.

14.3.12

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle -2t, 6t^2\right\rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{4t^2 + 36t^4} = 2\,|t|\,\sqrt{1 + 9t^2}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -2, 12t\rangle$.

14.3.13

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 8\cos t, -8\sin t\rangle$, so the speed is $|\mathbf{r}'(t)| = 8$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -8\sin t, -8\cos t\rangle$.

14.3.14

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\sin t, 4\cos t\rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{9\sin^2 t + 16\cos^2 t} = \sqrt{9 + 7\cos^2 t}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -3\cos t, -4\sin t\rangle$.

14.3.15

a. $\mathbf{v}(t) = \langle 2t, 2t, t\rangle$, so the speed is $\sqrt{4t^2 + 4t^2 + t^2} = 3t$.

b. $\mathbf{a}(t) = \langle 2, 2, 1 \rangle$.

14.3.16

a. $\mathbf{v}(t) = \left\langle 4e^{2t}, 2e^{2t}, 4e^{2t} \right\rangle$, so the speed is $\sqrt{16e^{4t} + 4e^{4t} + 16e^{4t}} = 6e^{2t}$.

b. $\mathbf{a}(t) = \left\langle 8e^{2t}, 4e^{2t}, 8e^{2t} \right\rangle$.

14.3.17

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, -4, 6 \rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{1 + 16 + 36} = \sqrt{53}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 0, 0 \rangle$.

14.3.18

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 3\cos t, -5\sin t, 4\cos t \rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{25\cos^2 t + 25\sin^2 t} = 5$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -3\sin t, -5\cos t, -4\sin t \rangle$.

14.3.19

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 0, 2t, -e^{-t} \rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{4t^2 + e^{-2t}}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 2, e^{-t} \rangle$.

14.3.20

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -26\sin 2t, 24\cos 2t, 10\cos 2t \rangle$, so the speed is

$$|\mathbf{r}'(t)| = \sqrt{676\sin^2 2t + 576\cos^2 2t + 100\cos^2 2t} = 26.$$

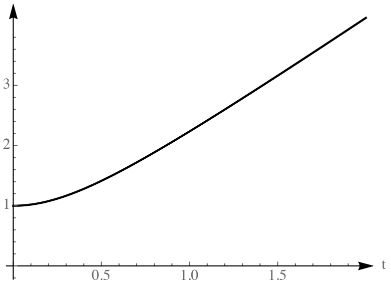
b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -52\cos 2t, -48\sin 2t, -20\sin 2t \rangle$.

14.3.21

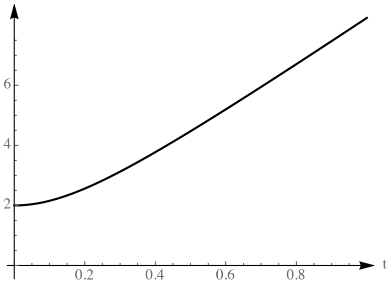
a. The interval must be shrunk by a factor of 2, so $[c, d] = [0, 1]$.

b. $\mathbf{r}'(t) = \langle 1, 2t \rangle$, and $\mathbf{R}'(t) = \langle 2, 8t \rangle$.

c. $|\mathbf{r}'(t)| = \sqrt{1 + 4t^2}$ and $|\mathbf{R}'(t)| = 2\sqrt{1 + 16t^2}$.

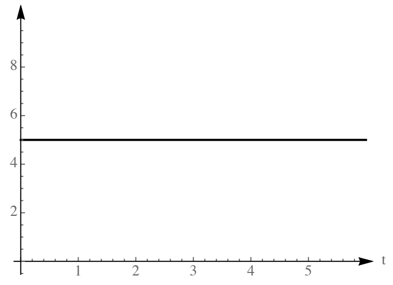


The speed of $\mathbf{r}(t)$.

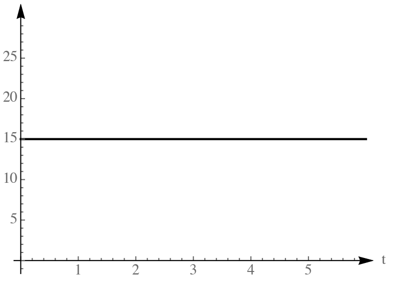


The speed of $\mathbf{R}(t)$.

14.3.22



The speed of $\mathbf{r}(t)$.



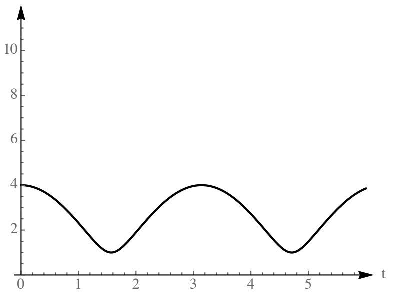
The speed of $\mathbf{R}(t)$.

a. The interval must be shrunk by a factor of 3, so $[c, d] = [0, 2]$.

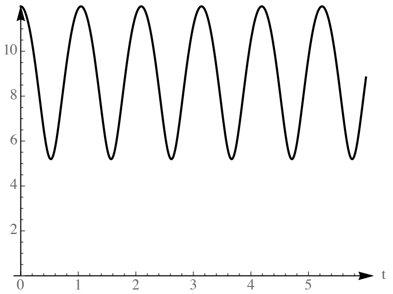
b. $\mathbf{r}'(t) = \langle 3, 4\rangle$, and $\mathbf{R}'(t) = \langle 9, 12\rangle$.

c. $|\mathbf{r}'(t)| = \sqrt{9+16} = 5$ and $|\mathbf{R}'(t)| = \sqrt{81+144} = 15$.

14.3.23



The speed of $\mathbf{r}(t)$.



The speed of $\mathbf{R}(t)$.

a. The interval must be shrunk by a factor of 1/3, so $[c, d] = [0, 2\pi/3]$.

b. $\mathbf{r}'(t) = \langle -\sin t, 4\cos t\rangle$, and $\mathbf{R}'(t) = \langle -3\sin 3t, 12\cos 3t\rangle$.

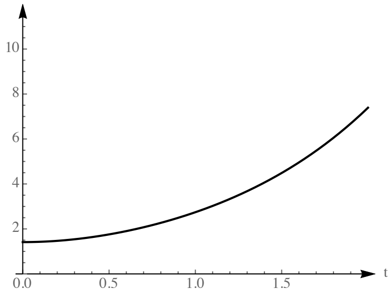
c. $|\mathbf{r}'(t)| = \sqrt{\sin^2 t + 16\cos^2 t}$ and $|\mathbf{R}'(t)| = 3\sqrt{\sin^2 3t + 16\cos^2 3t}$.

14.3.24

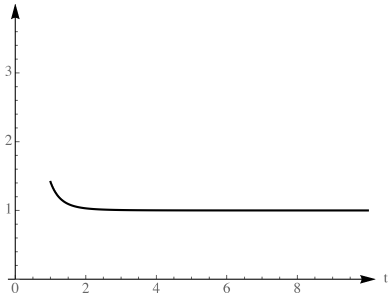
a. Because $e^0 = 1$ and $e^{\ln 10} = 10$, we have $[c, d] = [1, 10]$.

b. $\mathbf{r}'(t) = \langle -e^{-t}, e^{-t}\rangle$, and $\mathbf{R}'(t) = \left\langle -1, 1/t^2 \right\rangle$.

c. $|\mathbf{r}'(t)| = \sqrt{e^{2t} + e^{-2t}} = \sqrt{2\cosh(2t)}$ and $|\mathbf{R}'(t)| = \sqrt{1 + 1/t^4}$.



The speed of $\mathbf{r}(t)$.



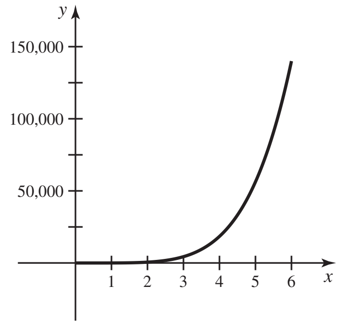
The speed of $\mathbf{R}(t)$.

14.3.25

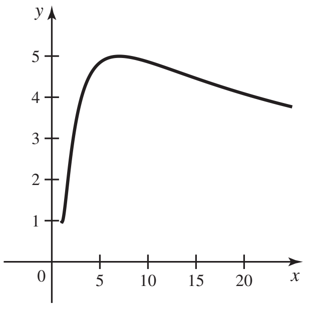
a. Because $e^{0^2} = 1$ and $e^{6^2} = e^{36}$, we have $[c, d] = [1, e^{36}]$.

b. $\mathbf{r}'(t) = \left\langle 2t, -8t^3, 18t^5 \right\rangle$, and $\mathbf{R}'(t) = \left\langle 1/t, (-4\ln t)/t, (9\ln^2 t)/t \right\rangle$.

c. $|\mathbf{r}'(t)| = 2t\sqrt{1 + 16t^4 + 81t^8}$ and $|\mathbf{R}'(t)| = \dfrac{1}{t}\sqrt{1 + 16\ln^2 t + 81\ln^4 t}$.



The speed of $\mathbf{r}(t)$.



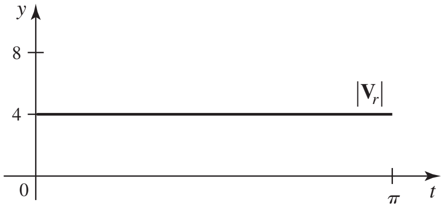
The speed of $\mathbf{R}(t)$.

14.3.26

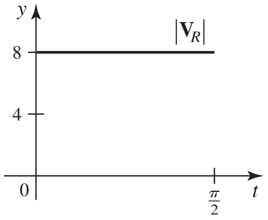
a. The interval must be shrunk by a factor of 1/2 so $[c, d] = [0, \pi/2]$.

b. $\mathbf{r}'(t) = \left\langle -4\sin 2t, 2\sqrt{2}\cos 2t, 2\sqrt{2}\cos 2t \right\rangle$, $\mathbf{R}'(t) = \left\langle -8\sin 4t, 4\sqrt{2}\cos 4t, 4\sqrt{2}\cos 4t \right\rangle$.

c. $|\mathbf{r}'(t)| = \sqrt{16\sin^2 2t + 8\cos^2 2t + 8\cos^2 2t} = \sqrt{16} = 4$, while we have $|\mathbf{R}'(t)| = \sqrt{64\sin^2 4t + 32\cos^2 4t + 32\cos^2 4t} = 8$.



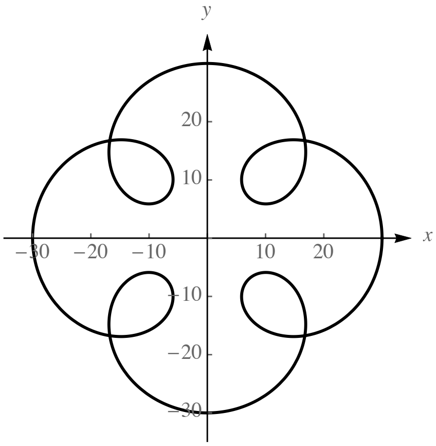
The speed of $\mathbf{r}(t)$.



The speed of $\mathbf{R}(t)$.

14.3.27

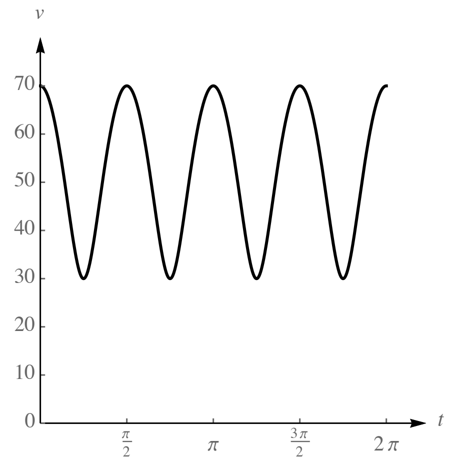
a.



b. $\mathbf{r}'(t) = \langle -20\sin t - 50\sin 5t, 20\cos t + 50\cos 5t \rangle$.

c.

$$\begin{aligned}
|\mathbf{v}(t)| &= \sqrt{\mathbf{r}'(t)\cdot\mathbf{r}'(t)} \\
&= \sqrt{400\sin^2 t + 2500\sin^2 5t + 2000\sin t\sin 5t + 400\cos^2 t + 2500\cos^2 5t + 2000\cos 5t\cos t} \\
&= \sqrt{400 + 2500 + 2000\sin t\sin 5t + 2000\cos t\cos 5t} \\
&= 10\sqrt{29 + 20(\sin 5t\sin t + \cos 5t\cos t)} = 10\sqrt{29 + 20\cos 4t}
\end{aligned}$$



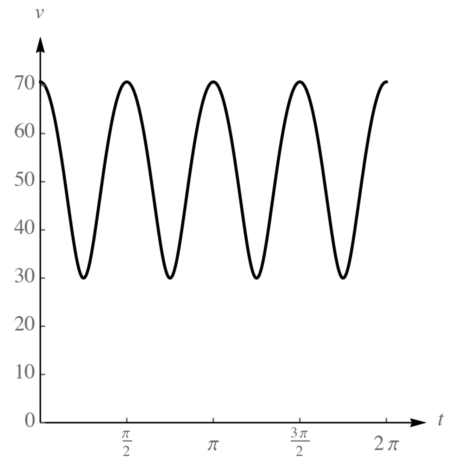
d. The maximum occurs when $\cos 4t = 1$, and the value is $10\sqrt{49} = 70$, while the minimum occurs when $\cos 4t = -1$ and the value. is $10\sqrt{9} = 30$.

14.3.28

a. Andrea is moving up and down.

b. $\mathbf{r}'(t) = \langle -20\sin t - 50\sin 5t, 20\cos t + 50\cos 5t, 10\cos 2t\rangle$. So, using the result from the previous problem,

$$\begin{aligned}|\mathbf{v}(t)| &= \sqrt{\mathbf{r}'(t)\cdot\mathbf{r}'(t)}\\ &= \sqrt{2900 + 2000\cos 4t + 100\cos^2 2t}\\ &= \sqrt{2900 + 2000\cos 4t + 50 + 50\cos 4t} = \sqrt{2950 + 2050\cos 4t} = 5\sqrt{118 + 82\cos 4t}.\end{aligned}$$



c. The maximum occurs when $\cos 4t = 1$ and the value is $5\sqrt{200} \approx 70.7$ and the minimum occurs when $\cos 4t = -1$ and the value is $5\sqrt{36} = 30$.

14.3.29 Note that $x^2 + y^2 = 64$, so the trajectory lies on a circle centered at the origin of radius 8. $\mathbf{r}(t)\cdot\mathbf{r}'(t) = \langle 8\cos 2t, 8\sin 2t\rangle \cdot \langle -16\sin 2t, 16\cos 2t\rangle = -128\sin 2t\cos 2t + 128\sin 2t\cos 2t = 0$.

14.3.30 Note that $x^2 + y^2 = 16\sin^2 t + 4\cos^2 t = 4 + 12\sin^2 t$ which is not a constant, so the trajectory does not lie on a circle centered at the origin.

14.3.31 Note that $x^2+y^2=(\sin t+\sqrt{3}\cos t)^2+(\sqrt{3}\sin t-\cos t)^2=(\sin^2 t+2\sin t\sqrt{3}\cos t+3\cos^2 t)+(3\sin^2 t-2\cos t\sqrt{3}\sin t+\cos^2 t)=4$, so the trajectory lies on a circle centered at the origin of radius 2. $\mathbf{r}(t)\cdot\mathbf{r}'(t)=\left\langle \sin t+\sqrt{3}\cos t, \sqrt{3}\sin t-\cos t\right\rangle\cdot\left\langle \cos t-\sqrt{3}\sin t, \sqrt{3}\cos t+\sin t\right\rangle=(\sin t\cos t-\sqrt{3}\sin^2 t+\sqrt{3}\cos^2 t-3\sin t\cos t)+(3\sin t\cos t+\sqrt{3}\sin^2 t-\sqrt{3}\cos^2 t-\sin t\cos t)=0$.

14.3.32 $x^2+y^2+z^2=9\sin^2 t+25\cos^2 t+16\sin^2 t=25$, so the trajectory lies on a sphere centered at the origin of radius 5. $\mathbf{r}(t)\cdot\mathbf{r}'(t)=\langle 3\sin t, 5\cos t, 4\sin t\rangle\cdot\langle 3\cos t, -5\sin t, 4\cos t\rangle=9\sin t\cos t-25\sin t\cos t+16\sin t\cos t=0$.

14.3.33 $\sqrt{\mathbf{r}(t)\cdot\mathbf{r}(t)}=\sqrt{\dfrac{25\sin^2 t+25\cos^2 t+25\sin^2 2t}{1+\sin^2 2t}}=\sqrt{\dfrac{25(1+\sin^2 2t)}{1+\sin^2 2t}}=\sqrt{25}=5.$

14.3.34 $\sqrt{\mathbf{r}(t)\cdot\mathbf{r}(t)}=\sqrt{\dfrac{16\cos^2 t+4t^2+16\sin^2 t}{4+t^2}}=\sqrt{\dfrac{4(4+t^2)}{4+t^2}}=\sqrt{4}=2.$

14.3.35 $\mathbf{v}(t)=\displaystyle\int \mathbf{a}(t)\,dt=\int \langle 0,1\rangle\ dt=\langle 0,t\rangle+\mathbf{C}$. Because $\mathbf{v}(0)=\langle 2,3\rangle$, we have $\mathbf{C}=\langle 2,3\rangle$. Thus, $\mathbf{v}(t)=\langle 2,t+3\rangle$.

$\mathbf{r}(t)=\displaystyle\int \mathbf{v}(t)\,dt=\int \langle 2,t+3\rangle\ dt=\left\langle 2t, t^2/2+3t\right\rangle+\mathbf{D}$. Because $\mathbf{r}(0)=\langle 0,0\rangle$, we have $\mathbf{D}=\langle 0,0\rangle$. Therefore, $\mathbf{r}(t)=\left\langle 2t, t^2/2+3t\right\rangle$.

14.3.36 $\mathbf{v}(t)=\displaystyle\int \mathbf{a}(t)\,dt=\int \langle 1,2\rangle\ dt=\langle t,2t\rangle+\mathbf{C}$. Because $\mathbf{v}(0)=\langle 1,1\rangle$, we have $\mathbf{C}=\langle 1,1\rangle$. Thus, $\mathbf{v}(t)=\langle t+1,2t+1\rangle$.

$\mathbf{r}(t)=\displaystyle\int \mathbf{v}(t)\,dt=\int \langle t+1,2t+1\rangle\ dt=\left\langle t^2/2+t, t^2+t\right\rangle+\mathbf{D}$. Because $\mathbf{r}(0)=\langle 2,3\rangle$, we have $\mathbf{D}=\langle 2,3\rangle$. Therefore, $\mathbf{r}(t)=\left\langle t^2/2+t+2, t^2+t+3\right\rangle$.

14.3.37 $\mathbf{v}(t)=\displaystyle\int \mathbf{a}(t)\,dt=\int \langle 0,10\rangle\ dt=\langle 0,10t\rangle+\mathbf{C}$. Because $\mathbf{v}(0)=\langle 0,5\rangle$, we have $\mathbf{v}(t)=\langle 0,10t+5\rangle$.

Also, $\mathbf{r}(t)=\displaystyle\int \mathbf{v}(t)\,dt=\int \langle 0,10t+5\rangle\ dt=\left\langle 0,5t^2+5t\right\rangle+\mathbf{D}$, and because $\mathbf{r}(0)=\langle 1,-1\rangle$, we have $\mathbf{r}(t)=\left\langle 1,5t^2+5t-1\right\rangle$.

14.3.38 $\mathbf{v}(t)=\displaystyle\int \mathbf{a}(t)\,dt=\int \langle 1,t\rangle\ dt=\left\langle t,\frac{t^2}{2}\right\rangle+\mathbf{C}$. Because $\mathbf{v}(0)=\langle 2,-1\rangle$, we obtain $\mathbf{v}(t)=\left\langle t+2,\dfrac{t^2}{2}-1\right\rangle$. Also, $\mathbf{r}(t)=\displaystyle\int \mathbf{v}(t)\,dt=\int \left\langle t+2,\frac{t^2}{2}-1\right\rangle\ dt=\left\langle \frac{t^2}{2}+2t,\frac{t^3}{6}-t\right\rangle+\mathbf{D}$, and because $\mathbf{r}(0)=\langle 0,8\rangle$, we have $\mathbf{r}(t)=\left\langle \dfrac{t^2}{2}+2t,\dfrac{t^3}{6}-t+8\right\rangle$.

14.3.39 $\mathbf{v}(t)=\displaystyle\int \mathbf{a}(t)\,dt=\int \langle \cos t,2\sin t\rangle\ dt=\langle \sin t,-2\cos t\rangle+\mathbf{C}$. Because $\mathbf{v}(0)=\langle 0,1\rangle$, we have $\mathbf{v}(t)=\langle \sin t,3-2\cos t\rangle$.

Also, $\mathbf{r}(t)=\displaystyle\int \mathbf{v}(t)\,dt=\int \langle \sin t,3-2\cos t\rangle\ dt=\langle -\cos t,3t-2\sin t\rangle+\mathbf{D}$, and because $\mathbf{r}(0)=\langle 1,0\rangle$, we have $\mathbf{r}(t)=\langle 2-\cos t,3t-2\sin t\rangle$.

14.3.40 $\mathbf{v}(t)=\displaystyle\int \mathbf{a}(t)\,dt=\int \left\langle e^{-t},1\right\rangle\ dt=\left\langle -e^{-t},t\right\rangle+\mathbf{C}$. Because $\mathbf{v}(0)=\langle 1,0\rangle$, we have $\mathbf{v}(t)=\langle 2-e^{-t},t\rangle$.

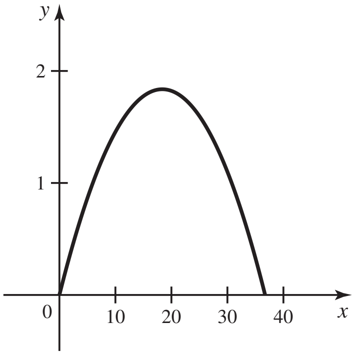
Also, $\mathbf{r}(t)=\displaystyle\int \mathbf{v}(t)\,dt=\int \left\langle 2-e^{-t},t\right\rangle\ dt=\left\langle 2t+e^{-t},t^2/2\right\rangle+\mathbf{D}$, and because $\mathbf{r}(0)=\langle 0,0\rangle$, we have $\mathbf{r}(t)=\left\langle 2t+e^{-t}-1,t^2/2\right\rangle$.

14.3.41

a. $\mathbf{v}(t) = \int \langle 0, -9.8\rangle\, dt = \langle 0, -9.8t\rangle + \mathbf{C}$, and because $\mathbf{v}(0) = \langle 30, 6\rangle$, we have $\mathbf{v}(t) = \langle 30, 6 - 9.8t\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int \langle 30, 6 - 9.8t\rangle\, dt = \left\langle 30t, 6t - 4.9t^2\right\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0\rangle$, we have $\mathbf{r}(t) = \left\langle 30t, 6t - 4.9t^2\right\rangle$.

b.



c. The ball hits the ground when $6t - 4.9t^2 = 0$, which occurs for $t = 6/4.9 \approx 1.22$ seconds. The range of the ball is approximately $30 \cdot 1.22 \approx 36.7$ meters.

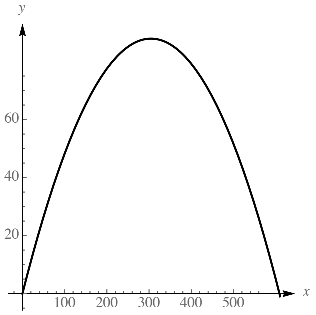
d. The maximum height occurs at time $T \approx 1.22/2 = .61$ seconds, and is $6T - 4.9T^2 \approx 1.84$ meters.

14.3.42

a. $\mathbf{v}(t) = \int \langle 0, -32\rangle\, dt = \langle 0, -32t\rangle + \mathbf{C}$, and because $\mathbf{v}(0) = 150\left\langle \sqrt{3}/2, 1/2\right\rangle = \left\langle 75\sqrt{3}, 75\right\rangle$, we have $\mathbf{v}(t) = \left\langle 75\sqrt{3}, -32t + 75\right\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int \left\langle 75\sqrt{3}, -32t + 75\right\rangle dt = \left\langle 75\sqrt{3}t, -16t^2 + 75t\right\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0\rangle$, we have $\mathbf{r}(t) = \left\langle 75\sqrt{3}t, -16t^2 + 75t\right\rangle$.

b.



c. The ball hits the ground when $-16t^2 + 75t = 0$, which occurs for $t = 75/16 = 4.6875$ seconds. The range of the ball is approximately $75\sqrt{3}(4.6875) \approx 608.9$ feet.

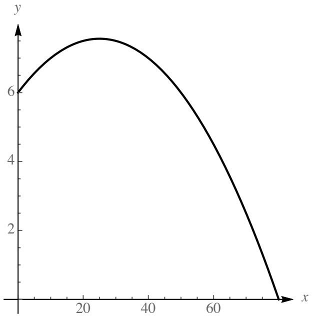
d. The maximum height occurs at time $T \approx 4.6875/2 \approx 2.344$, and is $-16(2.344)^2 + 75(2.344) \approx 87.89$ feet.

14.3.43

a. $\mathbf{v}(t) = \int \langle 0, -32\rangle\, dt = \langle 0, -32t\rangle + \mathbf{C}$, and because $\mathbf{v}(0) = \langle 80, 10\rangle$, we have $\mathbf{v}(t) = \langle 80, 10 - 32t\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int \langle 80, 10 - 32t\rangle\, dt = \left\langle 80t, 10t - 16t^2\right\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 6\rangle$, we have $\mathbf{r}(t) = \left\langle 80t, 6 + 10t - 16t^2\right\rangle$.

b.



c. The ball hits the ground when $-16t^2+10t+6=-2(t-1)(8t+3)=0$, which occurs for $t=1$ second. The range of the ball is $80\cdot 1=80$ feet.

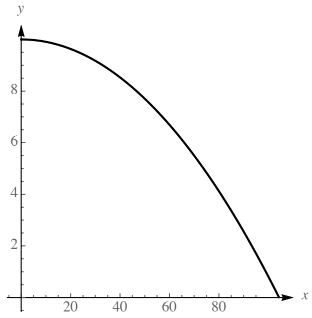
d. The maximum height occurs at time $T\approx 10/32\approx 0.3125$, and is $-16(.3125)^2+10(.3125)+6\approx 7.56$ feet.

14.3.44

a. $\mathbf{v}(t)=\int \langle 0,-32\rangle\,dt=\langle 0,-32t\rangle+\mathbf{C}$, and because $\mathbf{v}(0)=132\,\langle 1,0\rangle$, we have $\mathbf{v}(t)=\langle 132,-32t\rangle$.

Also, $\mathbf{r}(t)=\int \mathbf{v}(t)\,dt=\int \langle 132,-32t\rangle\,dt=\left\langle 132t,-16t^2\right\rangle+\mathbf{D}$, and because $\mathbf{r}(0)=\langle 0,10\rangle$, we have $\mathbf{r}(t)=\left\langle 132t,-16t^2+10\right\rangle$.

b.



c. The ball hits the ground when $-16t^2+10=0$, which occurs for $t=\sqrt{10/16}\approx 0.79$ second. The range of the ball is approximately $132(.79)\approx 104.36$ feet.

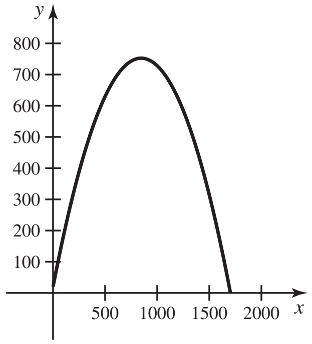
d. The maximum height occurs at time $T=0$, and is 10 feet.

14.3.45

a. $\mathbf{v}(t)=\int \langle 0,-32\rangle\,dt=\langle 0,-32t\rangle+\mathbf{C}$, and because $\mathbf{v}(0)=250\left\langle 1/2,\sqrt{3}/2\right\rangle=\left\langle 125,125\sqrt{3}\right\rangle$, we have $\mathbf{v}(t)=\left\langle 125,125\sqrt{3}-32t\right\rangle$.

Also, $\mathbf{r}(t)=\int \mathbf{v}(t)\,dt=\int \left\langle 125,125\sqrt{3}-32t\right\rangle\,dt=\left\langle 125t,125\sqrt{3}t-16t^2\right\rangle+\mathbf{D}$, and because $\mathbf{r}(0)=\langle 0,20\rangle$, we have $\mathbf{r}(t)=\left\langle 125t,20+125\sqrt{3}t-16t^2\right\rangle$.

b.



c. The ball hits the ground when $20+125\sqrt{3}t-16t^2=0$, which occurs for $t\approx 13.62$ seconds. The range of the ball is approximately $125\cdot 13.62\approx 1702$ feet.

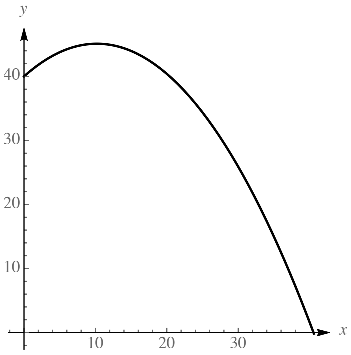
d. The maximum height occurs when $125\sqrt{3}-32t=0$, which is when $t\approx 6.77$ and it is about $20+125\sqrt{3}(6.77)-16(6.77)^2\approx 752.4$ feet.

14.3.46

a. $\mathbf{v}(t)=\int\langle 0,-9.8\rangle\,dt=\langle 0,-9.8t\rangle+\mathbf{C}$, and because $\mathbf{v}(0)=10\sqrt{2}\left\langle\sqrt{2}/2,\sqrt{2}/2\right\rangle=\langle 10,10\rangle$, we have $\mathbf{v}(t)=\langle 10,10-9.8t\rangle$.

Also, $\mathbf{r}(t)=\int\mathbf{v}(t)\,dt=\int\langle 10,10-9.8t\rangle\,dt=\left\langle 10t,10t-4.9t^2\right\rangle+\mathbf{D}$, and because $\mathbf{r}(0)=\langle 0,40\rangle$, we have $\mathbf{r}(t)=\left\langle 10t,40+10t-4.9t^2\right\rangle$.

b.



c. The ball hits the ground when $40+10t-4.9t^2=0$, which occurs when $t\approx 4.06$ seconds. The range of the ball is approximately $10\cdot 4.06=40.6$ meters.

d. The maximum height occurs at the time when $10-9.8t=0$, which is when $t=10/9.8\approx 1.02$. The height at this time is about 45.1 meters.

14.3.47 $\mathbf{v}(t)=\int\mathbf{a}(t)\,dt=\int\langle 0,0,10\rangle\,dt=\langle 0,0,10t\rangle+\mathbf{C}$. Because $\mathbf{v}(0)=\langle 1,5,0\rangle$, we have $\mathbf{v}(t)=\langle 1,5,10t\rangle$.

Also, $\mathbf{r}(t)=\int\mathbf{v}(t)\,dt=\int\langle 1,5,10t\rangle\,dt=\left\langle t,5t,5t^2\right\rangle+\mathbf{D}$, and because $\mathbf{r}(0)=\langle 0,5,0\rangle$, we have $\mathbf{r}(t)=\left\langle t,5t+5,5t^2\right\rangle$.

14.3.48 $\mathbf{v}(t)=\int\mathbf{a}(t)\,dt=\int\langle 1,t,4t\rangle\,dt=\left\langle t,t^2/2,2t^2\right\rangle+\mathbf{C}$. Because $\mathbf{v}(0)=\langle 20,0,0\rangle$, we have $\mathbf{v}(t)=\left\langle t+20,t^2/2,2t^2\right\rangle$.

Also, $\mathbf{r}(t)=\int\mathbf{v}(t)\,dt=\int\left\langle t+20,t^2/2,2t^2\right\rangle\,dt=\left\langle t^2/2+20t,t^3/6,2t^3/3\right\rangle+\mathbf{D}$, and because $\mathbf{r}(0)=\langle 0,0,0\rangle$, we have $\mathbf{r}(t)=\left\langle t^2/2+20t,t^3/6,2t^3/3\right\rangle$.

14.3.49 $\mathbf{v}(t)=\int\mathbf{a}(t)\,dt=\int\langle\sin t,\cos t,1\rangle\,dt=\langle-\cos t,\sin t,t\rangle+\mathbf{C}$. Because $\mathbf{v}(0)=\langle 0,2,0\rangle$, we have $\mathbf{v}(t)=\langle 1-\cos t,\sin t+2,t\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 1-\cos t, \sin t + 2, t\rangle\,dt = \left\langle t - \sin t, -\cos t + 2t, t^2/2\right\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0,0,0\rangle$, we have $\mathbf{r}(t) = \left\langle t-\sin t, 1-\cos t + 2t, t^2/2\right\rangle$.

14.3.50 $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \left\langle t, e^{-t}, 1\right\rangle\,dt = \left\langle t^2/2, -e^{-t}, t\right\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 0,0,1\rangle$, we have $\mathbf{v}(t) = \left\langle t^2/2, 1-e^{-t}, t+1\right\rangle$.

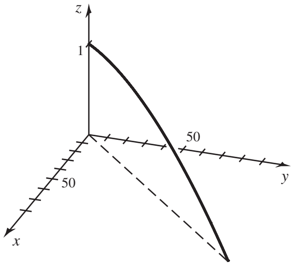
Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \left\langle t^2/2, 1-e^{-t}, t+1\right\rangle\,dt = \left\langle t^3/6, t+e^{-t}, t^2/2+t\right\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 4,0,0\rangle$, we have $\mathbf{r}(t) = \left\langle t^3/6+4, t+e^{-t}-1, t^2/2+t\right\rangle$.

14.3.51

a. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 0,0,-9.8\rangle\,dt = \langle 0,0,-9.8t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 200,200,0\rangle$, we have $\mathbf{v}(t) = \langle 200,200,-9.8t\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 200,200,-9.8t\rangle\,dt = \left\langle 200t, 200t, -4.9t^2\right\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0,0,1\rangle$, we have $\mathbf{r}(t) = \left\langle 200t, 200t, -4.9t^2+1\right\rangle$.

b.



c. The bullet hits the ground when $-4.9t^2+1=0$, which occurs for $t \approx .452$ seconds. At this time, the bullet is approximately at the point $(200\cdot 0.452, 200\cdot 0.452, 0) \approx (90.35, 90.35, 0)$. So its range is approximately $\sqrt{90.35^2+90.35^2} \approx 127.8$ meters.

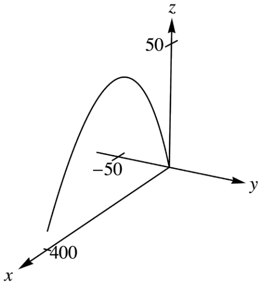
d. The maximum height of the bullet is its initial height of 1 meter.

14.3.52

a. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 0,-.8,-9.8\rangle\,dt = \langle 0,-.8t,-9.8t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 50,0,30\rangle$, we have $\mathbf{v}(t) = \langle 50,-.8t,-9.8t+30\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 50,-.8t,-9.8t+30\rangle\,dt = \left\langle 50t, -.4t^2, -4.9t^2+30t\right\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0,0,0\rangle$, we have $\mathbf{r}(t) = \left\langle 50t, -.4t^2, -4.9t^2+30t\right\rangle$.

b.



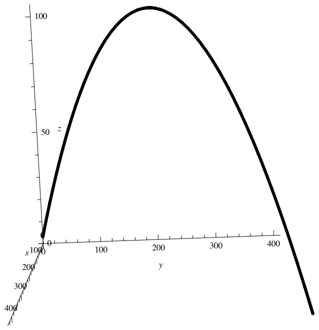
c. The ball hits the ground when $-4.9t^2+30t=0$, which occurs for $t \approx 30/4.9 \approx 6.12$ seconds. At this time, the ball is at the point $((50)(6.12), -.4(6.12)^2, 0) \approx (306, -14.98, 0)$. So its range is approximately $\sqrt{306^2+14.98^2} \approx 306.4$ meters.

d. The maximum height of the ball occurs when $-9.8t+30=0$, or when $t = 30/9.8 \approx 3.06$ seconds. At this time the ball's height is about $30(3.06) - 4.9(3.06)^2 \approx 45.92$ meters.

14.3.53

a. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 10, 0, -32\rangle\,dt = \langle 10t, 0, -32t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 60, 80, 80\rangle$, we have $\mathbf{v}(t) = \langle 10t + 60, 80, -32t + 80\rangle$.
Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 10t + 60, 80, -32t + 80\rangle\,dt = \left\langle 5t^2 + 60t, 80t, -16t^2 + 80t\right\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 3\rangle$, we have $\mathbf{r}(t) = \left\langle 5t^2 + 60t, 80t, -16t^2 + 80t + 3\right\rangle$.

b.



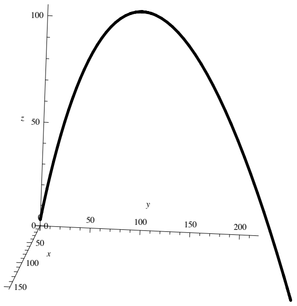
c. The ball hits the ground when $-16t^2 + 80t + 3 = 0$, which occurs for $t \approx 5.04$ seconds. At this time, the ball is at the point $(5(5.04)^2 + 60(5.04), 80(5.04), 0) \approx (429.4, 403.2, 0)$. So its range is approximately $\sqrt{429.4^2 + 403.2^2} \approx 589$ feet.

d. The maximum height of the ball occurs when $-32t + 80 = 0$, or when $t = 80/32 = 2.5$ seconds. At this time the ball's height is about $-16(2.5)^2 + 80(2.5) + 3 = 103$ feet.

14.3.54

a. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 0, 5, -32\rangle\,dt = \langle 0, 5t, -32t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 30, 30, 80\rangle$, we have $\mathbf{v}(t) = \langle 30, 5t + 30, -32t + 80\rangle$.
Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 30, 5t + 30, -32t + 80\rangle\,dt = \left\langle 30t, 5t^2/2 + 30t, -16t^2 + 80t\right\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 3\rangle$, we have $\mathbf{r}(t) = \left\langle 30t, 5t^2/2 + 30t, -16t^2 + 80t + 3\right\rangle$.

b.



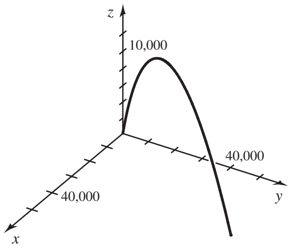
c. The ball hits the ground when $-16t^2 + 80t + 3 = 0$, which occurs for $t \approx 5.04$ seconds. At this time, the ball is at the point $(30(5.04), 5(5.04)^2/2 + 30(5.04), 0) \approx (151.2, 214.7, 0)$. So its range is approximately $\sqrt{151.2^2 + 214.7^2} \approx 263$ feet.

d. The maximum height of the ball occurs when $-32t + 80 = 0$, or when $t = 80/32 = 2.5$ seconds. At this time the ball's height is about $-16(2.5)^2 + 80(2.5) + 3 = 103$ feet.

14.3.55

a. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 0, 2.5, -9.8\rangle\;dt = \langle 0, 2.5t, -9.8t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 300, 400, 500\rangle$, we have $\mathbf{v}(t) = \langle 300, 2.5t + 400, 500 - 9.8t\rangle$.
Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 300, 2.5t + 400, 500 - 9.8t\rangle\;dt = \left\langle 300t, 1.25t^2 + 400t, 500t - 4.9t^2\right\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 10\rangle$, we have $\mathbf{r}(t) = \left\langle 300t, 1.25t^2 + 400t, 10 + 500t - 4.9t^2\right\rangle$.

b.



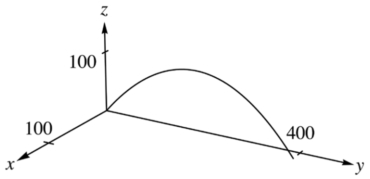
c. The rocket hits the ground when $-4.9t^2 + 500t + 10 = 0$, which occurs for $t \approx 102.1$ seconds. At this time, the rocket is at the point $(30630, 53870.5, 0)$. So its range is approximately $\sqrt{30630^2 + 53870.5^2} \approx 61969.6$ meters.

d. The maximum height of the rocket occurs when $-9.8t + 500 = 0$, or when $t = 500/9.8 \approx 51.02$ seconds. At this time the rocket's height is about $10 + 500(51.02) - 4.9(51.02)^2 \approx 12,765$ meters.

14.3.56

a. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 1.2, 0, -32\rangle\;dt = \langle 1.2t, 0, -32t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 0, 80, 80\rangle$, we have $\mathbf{v}(t) = \langle 1.2t, 80, 80 - 32t\rangle$.
Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 1.2t, 80, 80 - 32t\rangle\;dt = \left\langle .6t^2, 80t, 80t - 16t^2\right\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 0\rangle$, we have $\mathbf{r}(t) = \left\langle .6t^2, 80t, 80t - 16t^2\right\rangle$.

b.



c. The ball hits the ground when $-16t^2 + 80t = 0$, which occurs for $t = 80/16 = 5$ seconds. At this time, the ball is at the point $(15, 400, 0)$. So its range is approximately $\sqrt{15^2 + 400^2} \approx 400.28$ feet.

d. The maximum height of the ball occurs when $-32t + 80 = 0$, or when $t = 80/32 = 2.5$ seconds. At this time the ball's height is about $-16(2.5)^2 + 80 \cdot 2.5 = 100$ feet.

14.3.57

a. False. For example, if $\mathbf{v}(t) = \langle \cos t, \sin t\rangle$, then its speed is constantly 1 even though its components aren't constant.

b. True. They both generate $\{(x, y) \mid x^2 + y^2 = 1\}$.

c. False. For example, $\langle t, t, t\rangle$ has variable magnitude but constant direction.

d. True. If $\mathbf{a}(t) = \langle 0, 0, 0\rangle$, then $\mathbf{v}(t) = \langle 0, 0, 0\rangle + \mathbf{C}$ for a constant vector $\mathbf{C}$.

e. False. Recall that for two-dimensional motion the range is given by $\dfrac{|\mathbf{v}_0|^2 \sin 2\alpha}{g}$, so doubling the speed should quadruple the range.

f. True. The time of flight is given by $T = \dfrac{2\,|\mathbf{v}_0|\sin\alpha}{g}$, so doubling the speed doubles the time of flight.

g. True. For example, if $\mathbf{v}(t) = \langle e^t, e^t, e^t\rangle$, then $\mathbf{a}(t) = \langle e^t, e^t, e^t\rangle$ as well.

14.3.58 The time of flight is $T = \dfrac{2\,|\mathbf{v}_0|\sin\alpha}{g} = \dfrac{2\cdot 20}{32} = 1.25$ seconds.

The range of the flight is $\dfrac{|\mathbf{v}_0|^2\,2\sin\alpha\cos\alpha}{g} = \dfrac{400}{32} = 12.5$ feet.

The maximum height is given by $\dfrac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} = \dfrac{400}{64} = 6.25$ feet.

14.3.59 Note that $\langle u_0, v_0\rangle = 75\left\langle\sqrt{3}, 1\right\rangle$

The time of flight is $T = \dfrac{2\,|\mathbf{v}_0|\sin\alpha}{g} = \dfrac{2\cdot 75}{9.8} \approx 15.3$ seconds.

The range of the flight is $\dfrac{|\mathbf{v}_0|^2\,2\sin\alpha\cos\alpha}{g} = \dfrac{11250\sqrt{3}}{9.8} \approx 1988.3$ meters.

The maximum height is given by $\dfrac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} = \dfrac{75^2}{19.6} \approx 287$ meters.

14.3.60 The time of flight is $T = \dfrac{2\,|\mathbf{v}_0|\sin\alpha}{g} = \dfrac{2\cdot 80}{9.8} \approx 16.33$ seconds.

The range of the flight is $\dfrac{|\mathbf{v}_0|^2\,2\sin\alpha\cos\alpha}{g} = \dfrac{6400}{9.8} \approx 653$ meters.

The maximum height is given by $\dfrac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} = \dfrac{80^2}{19.6} \approx 326.5$ meters.

14.3.61 Note that $\langle u_0, v_0\rangle = 200\left\langle 1, \sqrt{3}\right\rangle$ The time of flight is $T = \dfrac{2\,|\mathbf{v}_0|\sin\alpha}{g} = \dfrac{2\cdot 200\sqrt{3}}{32} = 21.65$ seconds.

The range of the flight is $\dfrac{|\mathbf{v}_0|^2\,2\sin\alpha\cos\alpha}{g} = \dfrac{80,000\sqrt{3}}{32} = 4330.13$ feet.

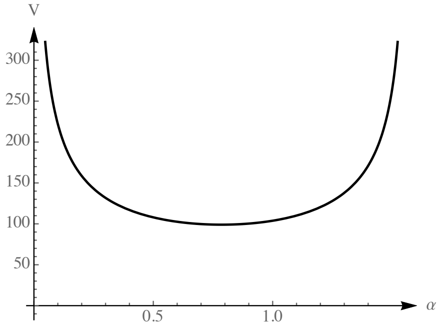
The maximum height is given by $\dfrac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} = \dfrac{(200\sqrt{3})^2}{64} = 1875$ feet.

14.3.62 Each of these values on the moon would be 6 times the corresponding value on the earth, because the factor of g in the denominator would result in an extra factor of 6 if g were replaced by $g/6$.

14.3.63 We desire $\dfrac{|\mathbf{v}_0|^2\,2\sin\alpha\cos\alpha}{g} = 300$ meters, so we require $\sin 2\alpha = 300\cdot\dfrac{9.8}{60^2} \approx .81666$. So $2\alpha = \sin^{-1}(.81666)$, and $\alpha \approx 27.4$ degrees or $\alpha \approx 62.62$ degrees.

14.3.64

a. Let V stand for the initial speed. The range is given by $\dfrac{V^2\sin 2\alpha}{g}$, so we require $\dfrac{9800}{\sin 2\alpha} = V^2$, so $V = \sqrt{9800\csc 2\alpha}$.

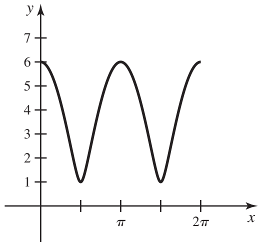


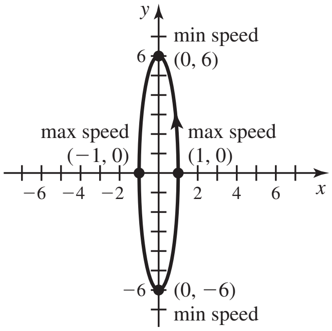
b. The speed is minimized when $\dfrac{dV}{d\alpha} = \dfrac{-9800\csc 2\alpha\cot 2\alpha}{\sqrt{9800\csc 2\alpha}} = 0$, which occurs when $\cos 2\alpha = 0$, or $\alpha = \pi/4$. At this value of α, the value of V is about 99 meters per second.

14.3.65

a. The velocity is $\mathbf{v}(t) = \langle -a\sin t, b\cos t\rangle$ and the speed is $\sqrt{a^2\sin^2 t + b^2\cos^2 t}$.

b.





c. Yes, as the diagram indicates.

d. Assume $a > b > 0$. Then the maximum speed occurs at $\pi/2$ and is equal to a, while the minimum speed occurs at π and is equal to b. So the ratio is $\dfrac{a}{b}$. In the case $b > a > 0$, the ratio is $\dfrac{b}{a}$.

14.3.66 Let the angle be α. Then $\mathbf{v}(t) = \langle 150\cos\alpha, 150\sin\alpha - 32t\rangle$, and

$$\mathbf{r}(t) = \left\langle 150t\cos\alpha, 150t\sin\alpha - 16t^2\right\rangle.$$

Because we require the ball to land in the hole, we need the point $(390, 40)$ to be on the curve. So $150t\cos\alpha = 390$ and $150t\sin\alpha - 16t^2 = 40$. Thus $t = \dfrac{390}{150\cos\alpha}$, and therefore $150\cdot\dfrac{390\cdot\sin\alpha}{150\cos\alpha} - 16\left(\dfrac{390}{150\cos\alpha}\right)^2 = 40$. This can be written $390\tan\alpha - \dfrac{2704}{25}\sec^2\alpha - 40 = 0$, or

$$390\tan\alpha - \frac{2704}{25}(1+\tan^2\alpha) - 40 = -\frac{2704}{25}\tan^2\alpha + 390\tan\alpha - \frac{3704}{25} = 0.$$

By the quadratic formula, we have

$$\tan\alpha = \frac{1}{208}\left(375 - \sqrt{81361}\right) \text{ and } \tan\alpha = \frac{1}{208}\left(375 + \sqrt{81361}\right).$$

Applying the inverse tangent and then writing the answer in degrees, we obtain $\alpha = 72.51$ degrees and $\alpha = 23.34$ degrees

14.3.67 Let the angle be α. Then $\mathbf{v}(t) = \langle 120\cos\alpha, 120\sin\alpha - 32t\rangle$, and

$$\mathbf{r}(t) = \left\langle 120t\cos\alpha, 120t\sin\alpha - 16t^2\right\rangle.$$

Because we require the ball to land in the hole, we need the point $(420, -50)$ to be on the curve. So $120t\cos\alpha = 420$ and $120t\sin\alpha - 16t^2 = -50$. Thus $t = \dfrac{420}{120\cos\alpha}$, and therefore $120\cdot\dfrac{420\cdot\sin\alpha}{120\cos\alpha} - 16\left(\dfrac{420}{120\cos\alpha}\right)^2 = -50$. This can be written $420\tan\alpha - 196\sec^2\alpha + 50 = -196\tan^2\alpha + 420\tan\alpha - 146 = 0$. By the quadratic formula, we have

$$\tan\alpha = \frac{1}{14}\left(15 - \sqrt{79}\right) \text{ and } \tan\alpha = \frac{1}{14}\left(15 + \sqrt{79}\right).$$

Applying the inverse tangent and then writing the answer in degrees, we obtain $\alpha = 59.63$ degrees and $\alpha = 23.58$ degrees.

14.3.68 Let s be the initial speed of the ball. Note that $\mathbf{v}(t) = \langle s\sqrt{2}/2, s\sqrt{2}/2 - 32t\rangle$ and

$$\mathbf{r}(t) = \left\langle st\sqrt{2}/2, st\sqrt{2}/2 - 16t^2\right\rangle.$$

Because we want the second coordinate to be 40 when the first coordinate is 390, we have $st\sqrt{2}/2 = 390$ and $st\sqrt{2}/2 - 16t^2 = 40$. Solving the first equation for t yields $t = \dfrac{780}{\sqrt{2}s}$. Putting this value into the second equation yields $s \cdot \dfrac{780}{\sqrt{2}s} \cdot \dfrac{\sqrt{2}}{2} - 16\left(\dfrac{780}{\sqrt{2}s}\right)^2 = 40$. Solving this last equation for s yields $s = \dfrac{\sqrt{8}\cdot 780}{\sqrt{350}} \approx 117.9$.

14.3.69 Let s be the initial speed of the ball. Note that $\mathbf{v}(t) = \langle s\sqrt{3}/2, s/2 - 32t\rangle$ and

$$\mathbf{r}(t) = \left\langle st\sqrt{3}/2, st/2 - 16t^2\right\rangle.$$

Because we want the second coordinate to be -50 when the first coordinate is 420, we have $st\sqrt{3}/2 = 420$ and $st/2 - 16t^2 = -50$. Solving the first equation for t yields $t = \dfrac{840}{\sqrt{3}s}$. Putting this value into the second equation yields $s \cdot \dfrac{840}{\sqrt{3}s} \cdot \dfrac{1}{2} - 16\left(\dfrac{840}{\sqrt{3}s}\right)^2 = -50$. Solving this last equation for s yields $s \approx 113.4$.

14.3.70

a. $\mathbf{v}(t) = \langle 40, -9.8t\rangle$ and $\mathbf{r}(t) = \left\langle 40t, 8 - 4.9t^2\right\rangle$. Let $x = 40t$ and $y = 8 - 4.9t^2$. Then the equation of trajectory is $y = 8 - 4.9\left(\dfrac{x}{40}\right)^2$. The equation of the outrun surface is $y = -\dfrac{1}{\sqrt{3}}x$. These curves intersect when $8 - 4.9\left(\dfrac{x}{40}\right)^2 = -\dfrac{1}{\sqrt{3}}x$, which when solved for x yields 201.487, and then $y = -116.329$. The distance from the origin to the landing points is $\sqrt{201.487^2 + (-116.329)^2} \approx 232.657$ meters.

b. In this scenario, $\mathbf{v}(t) = \langle 40 - .15t, -9.8t\rangle$ and $\mathbf{r}(t) = \left\langle 40t - .075t^2, 8 - 4.9t^2\right\rangle$. Let $x = 40t - .075t^2$ and $y = 8 - 4.9t^2$. Then $\dfrac{8-y}{4.9} = t^2$, so $x = 40\sqrt{\dfrac{8-y}{4.9}} - .075(\dfrac{8-y}{4.9})$. Because we also have $x = -\sqrt{3}y$, we are looking for the solution to the equation $-\sqrt{3}y = 40\sqrt{\dfrac{8-y}{4.9}} - .075(\dfrac{8-y}{4.9})$. This results in $y = -114.29$ and so $x = 197.95$. Then the length of the jump is $\sqrt{(-114.29)^2 + 197.95^2} \approx 228.575$ meters.

c. $\mathbf{v} = \langle 40\cos\theta, 40\sin\theta - 9.8t\rangle$, and $\mathbf{r}(t) = \left\langle 40t\cos\theta, 8 + 40t\sin\theta - 4.9t^2\right\rangle$. Let $x = 40t\cos\theta$ and $y = 8 + 40t\sin\theta - 4.9t^2$. Then the equation of trajectory is $y = 8 + x\tan\theta - 4.9\left(\dfrac{x}{40\cos\theta}\right)^2$. The equation of the outrun surface is $y = -\dfrac{1}{\sqrt{3}}x$. These intersect when $8 + x\tan\theta - 4.9\left(\dfrac{x}{40\cos\theta}\right)^2 = -\dfrac{1}{\sqrt{3}}x$. Solving for x in terms of θ (using a computer algebra system) gives

$$x = \frac{0.5\left(1.04785\sqrt{\tan^2\theta + 1.05164\tan\theta + 0.392835} + \tan\theta + 0.57735\right)}{0.0030625\tan^2\theta + 0.0030625}.$$

This is maximized for $\theta \approx 29.41$ degrees.

14.3.71

a. $\mathbf{v} = \langle 130, 0, -3 - 32t\rangle$ and $\mathbf{r}(t) = \left\langle 130t, 0, 6 - 3t - 16t^2\right\rangle$. When $x = 60$, $t = 6/13$, so $z = 6 - 3(6/13) - 16(6/13)^2 \approx 1.207$ feet. The flight lasts $t = 6/13$ seconds.

b. Suppose that the initial velocity is $\langle 130, 0, b\rangle$, so that $\mathbf{v}(t) = \left\langle 130t, 0, 6 + bt - 16t^2\right\rangle$. So $z = 3 = 6 + b(6/13) - 16(6/13)^2$, which when solved for b gives $b = .8846$.

c. In this scenario, we have $\mathbf{v}(t) = \langle 130, 8t, -3 - 32t\rangle$ and $\mathbf{r}(t) \left\langle 130t, 4t^2, 6 - 3t - 16t^2 \right\rangle$. As before, $x = 60$ when $t = 6/13$, and at that time $y = 4(6/13)^2 \approx 0.8521$ feet.

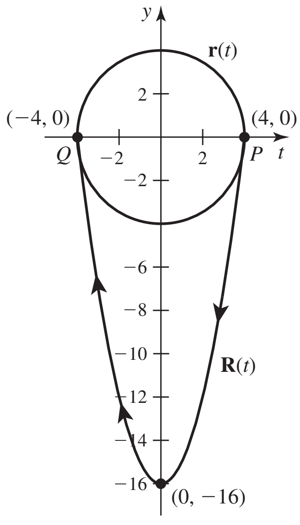
d. It moves more in the second half, because of the factor of t^2. This makes life more difficult for the batter!

e. In this case $\mathbf{v}(t) = \langle 130, ct, -3 - 32t\rangle$ and $\mathbf{r}(t) = \left\langle 130t, -3 + ct^2/2, 6 - 3t - 16t^2 \right\rangle$. Again, we have $t = 6/13$, and so we require $-3 + c\left(\dfrac{6}{13}\right)^2 \cdot \dfrac{1}{2} = 0$, so $c \approx 28.17$.

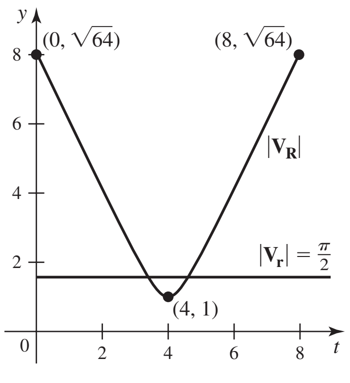
14.3.72 We have $x = u_0 t + x_0$, and $y = -\dfrac{gt^2}{2} + v_0 t + y_0$. Eliminating t gives $y = -\dfrac{g}{2}((x - x_0)/u_0)^2 + v_0((x - x_0)/u_0) + y_0$, which is a segment of a parabola. If $y(T) = 0$, then we have $-\dfrac{g}{2}T^2 + v_0 T + y_0 = 0$, so $T^2 - \dfrac{2v_0}{g}T - \dfrac{2y_0}{g} = 0$, so by the quadratic formula we have $T = \dfrac{v_0}{g} + \sqrt{\left(\dfrac{v_0}{g}\right)^2 + \dfrac{2y_0}{g}} = \dfrac{v_0 + \sqrt{v_0^2 + 2gy_0}}{g}$.

14.3.73 We have $\mathbf{v}(t) = \langle v_0 \cos\alpha, v_0 \sin\alpha - gt\rangle$, and $\mathbf{r}(t) = \left\langle v_0 t \cos\alpha, y_0 + v_0 t \sin\alpha - \dfrac{1}{2}gt^2 \right\rangle$. Suppose that object hits the ground at $(a, 0)$. Then $a = v_0 T \cos\alpha$ and $0 = y_0 + v_0 T \sin\alpha - \dfrac{1}{2}gT^2$ where T is the time of the flight. So by the quadratic formula, $T = \dfrac{v_0 \sin\alpha + \sqrt{v_0^2 \sin^2\alpha + 2gy_0}}{g}$. Thus $a = v_0 T \cos\alpha = v_0(\cos\alpha)\left(\dfrac{v_0 \sin\alpha + \sqrt{v_0^2 \sin^2\alpha + 2gy_0}}{g}\right)$ is the range. Because the maximum height when $y_0 = 0$ is $\dfrac{v_0^2 \sin^2\alpha}{2g}$, the maximum height in this scenario is $y_0 + \dfrac{v_0^2 \sin^2\alpha}{2g}$.

14.3.74



a.



b.

c. Both travelers arrive at $t = 8$.

14.3.75

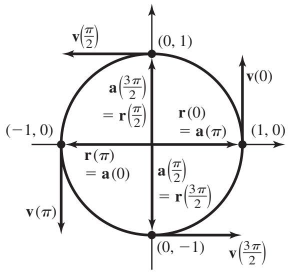
a. The object traverses the circle once over the interval $[0, 2\pi/\omega]$.

b. The velocity is $\mathbf{v}(t) = \langle -A\omega\sin(\omega t), A\omega\cos(\omega t)\rangle$. The velocity is not constant in direction, but it is constant in speed, because the speed is $|A\omega|$.

c. The acceleration is $\mathbf{a}(t) = -A\omega^2\langle\cos\omega t, \sin\omega t\rangle$.

d. The position and velocity are orthogonal. The position and acceleration are in opposite directions.

e.


$\mathbf{v}\left(\frac{\pi}{2}\right)$
(0, 1)
$\mathbf{v}(0)$
$\mathbf{a}\left(\frac{3\pi}{2}\right)$
$= \mathbf{r}\left(\frac{\pi}{2}\right)$
$\mathbf{r}(0)$
$= \mathbf{a}(\pi)$
(−1, 0)
(1, 0)
$\mathbf{r}(\pi)$
$= \mathbf{a}(0)$
$\mathbf{a}\left(\frac{\pi}{2}\right)$
$= \mathbf{r}\left(\frac{3\pi}{2}\right)$
$\mathbf{v}(\pi)$
(0, −1)
$\mathbf{v}\left(\frac{3\pi}{2}\right)$

14.3.76

a. Note that $\mathbf{r}(t) = \langle(-7/5)t+1, (6/5)t+2, (6/5)t+4\rangle$ has $\mathbf{r}(0) = \langle 1,2,4\rangle$ and $\mathbf{r}(5) = \langle -6,8,10\rangle$.

b. Consider $\mathbf{r}(t) = e^t\left\langle -\frac{7}{11}, \frac{6}{11}, \frac{6}{11}\right\rangle + \left\langle \frac{13}{6}, 1, 3\right\rangle$ for $\ln(11/6) \le t \le \ln(77/6)$. Note that

$$\mathbf{r}(\ln(11/6)) = \frac{11}{6}\langle -7/11, 6/11, 6/11\rangle + \langle 13/6, 1, 3\rangle = \langle -7/6, 1, 1\rangle + \langle 13/6, 1, 3\rangle = \langle 1, 2, 4\rangle$$

and

$$\mathbf{r}(\ln(77/6)) = \frac{77}{6}\langle -7/11, 6/11, 6/11\rangle + \langle 13/6, 1, 3\rangle = \langle -49/6, 7, 7\rangle + \langle 13/6, 1, 3\rangle = \langle -6, 8, 10\rangle.$$

Also $|\mathbf{r}'(t)| = |\langle e^t(-7/11), e^t(6/11), e^t(6/11)\rangle| = e^t\sqrt{\frac{49+36+36}{11^2}} = e^t$.

14.3.77

a. Consider $\mathbf{r}(t) = \langle 5\sin(\pi t/6), 5\cos(\pi t/6)\rangle$ for $0 \le t \le 12$. Note that the speed is the constant $5\pi/6$, and that $\mathbf{r}(0) = (0,5) = \mathbf{r}(12)$.

b. Consider $\mathbf{r}(t) = \langle 5\sin((1-e^{-t})/5), 5\cos((1-e^{-t})/5)\rangle$ for $-\ln(10\pi+1) \le t \le 0$. Note that

$$\mathbf{r}(-\ln(10\pi+1)) = (0,5) = \mathbf{r}(0).$$

Also note that the speed is $|\mathbf{r}'(t)| = |e^{-t}\langle\cos((1-e^{-t})/5), -\sin((1-e^{-t})/5)\rangle| = e^{-t}$.

14.3.78

a. Let $\mathbf{r}(t) = \langle\cos 6t, \sin 6t, 8t\rangle$. Then $\mathbf{r}$ moves along the circular helix, but the speed is

$$|\langle -6\sin 6t, 6\cos 6t, 8\rangle| = \sqrt{36\sin^2 6t + 36\cos^2 6t + 64} = 10.$$

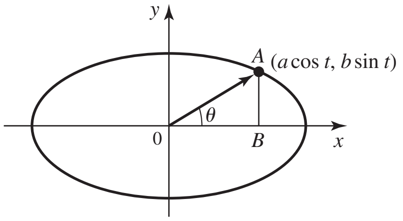
b. Let $\mathbf{r}(t) = \left\langle\cos(t^2/(2\sqrt{2})), \sin(t^2/(2\sqrt{2})), t^2/(2\sqrt{2})\right\rangle$. Then $\mathbf{r}$ moves along the circular helix, but the speed is

$$\left|\left\langle -\frac{t}{\sqrt{2}}\sin(t^2/(2\sqrt{2})), \frac{t}{\sqrt{2}}\cos(t^2/(2\sqrt{2})), t/\sqrt{2}\right\rangle\right| = \sqrt{\frac{t^2}{2}\sin^2(t^2/(2\sqrt{2})) + \frac{t^2}{2}\cos^2(t^2/(2\sqrt{2})) + \frac{t^2}{2}} = t$$

for $t \ge 0$.

14.3.79 Note that $x^2+y^2 = \cos^2 t + \sin^2 t = 1$, and $z = cy$, so this curve is the intersection of the cylinder $x^2+y^2 = 1$ with the plane $z = cy$, which results in an ellipse in that plane.

14.3.80



a. In right triangle ABO, we have $\tan\theta = \dfrac{AB}{OB} = \dfrac{b\sin t}{a\cos t} = \dfrac{b}{a}\tan t$.

b. $\theta = \tan^{-1}((b/a)\tan t)$, so $\theta'(t) = \dfrac{1}{1+((b/a)\tan t)^2}\cdot\dfrac{b}{a}\sec^2 t = \dfrac{ab}{a^2\cos^2 t + b^2\sin^2 t}$.

c. $\dfrac{dA}{dt} = \dfrac{dA}{d\theta}\cdot\dfrac{d\theta}{dt} = \dfrac{1}{2}\left|\mathbf{r}(\theta(t))\right|^2\cdot\dfrac{ab}{a^2\cos^2 t + b^2\sin^2 t} = \dfrac{1}{2}ab$.

d. Because $\dfrac{dA}{dt}$ is a constant, the object sweeps out equal areas in equal times as it moves about the ellipse.

14.3.81 $\mathbf{r}(t)$ can be written $\langle R\cos\phi\cos t, R\sin\phi\cos t, R\sin t\rangle$ where R is the radius of the sphere and ϕ is a real constant. Note that

$$\mathbf{v}(t) = \langle -R\cos\phi\sin t, -R\sin\phi\sin t, R\cos t\rangle$$

and $\mathbf{a}(t) = \langle -R\cos\phi\cos t, -R\sin\phi\cos t, -R\sin t\rangle$, and that $\mathbf{r}(t)\cdot\mathbf{a}(t) = -R^2\cos^2\phi\cos^2 t - R^2\sin^2\phi\cos^2 t - R^2\sin^2 t = -R^2(\cos^2\phi + \sin^2\phi)\cos^2 t - R^2\sin^2 t = -R^2(\cos^2 t + \sin^2 t) = -R^2 = -\left|\mathbf{v}(t)\right|^2$.

14.3.82

a. $\left|\mathbf{r}(t)\right|^2 = (a\cos t + b\sin t)^2 + (c\cos t + d\sin t)^2 = (a^2+c^2)\cos^2 t + (2ab+2cd)\sin t\cos t + (b^2+d^2)\sin^2 t$. In order for the path to be a circle, it would be sufficient that $a^2+c^2 = b^2+d^2$ and that $ab+cd = 0$.

b. In order for the path to be an ellipse, it would be sufficient that $ab+cd=0$.

14.3.83

a. If $t_1 > t_0$ are two values of t, we have $\mathbf{r}(t_1) - \mathbf{r}(t_0) = (f(t_1) - f(t_0))\langle a, b, c\rangle$, which is always a vector in the same direction, regardless of the values of t_1 and t_0.

b. $\mathbf{r}'(t) = f'(t)\langle a, b, c\rangle$ is a multiple of $\langle a, b, c\rangle$, so the tangent vector is always a multiple of the vector $\langle a, b, c\rangle$, so the motion of the object doesn't vary in direction, although it might vary in speed.

14.4 Length of Curves

14.4.1 $L = \displaystyle\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_a^b \sqrt{1+4}\,dt = \sqrt{5}(b-a)$.

14.4.2 To compute the length of the curve, compute $L = \displaystyle\int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt$.

14.4.3 The arc length is $L = \displaystyle\int_a^b \left|\mathbf{r}'(t)\right|\,dt = \int_a^b \left|\mathbf{v}(t)\right|\,dt$.

14.4.4 It travels a distance equal to the length of the path it traces in space: $\displaystyle\int_a^b \left|\mathbf{r}'(t)\right|\,dt = \int_a^b \left|\mathbf{v}(t)\right|\,dt$.

14.4.5 $L = \int_0^{\pi} |\langle -20\sin(2t), 20\cos(2t)\rangle|\, dt = 20\pi.$

14.4.6 This is given by $\int_0^2 \sqrt{\langle 1,-8,4\rangle \cdot \langle 1,-8,4\rangle}\, dt = \int_0^2 \sqrt{1+64+16}\, dt = 18.$

14.4.7 If the parameter t used to describe a trajectory also measures the arc length s of the curve that is generated, then we say that the curve is parametrized by its arc length. This occurs when $|\mathbf{v}(t)| = 1$.

14.4.8 Note that $|\mathbf{v}(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$, so it is parametrized by arc length.

The arc length is $s = \int_0^t |\mathbf{v}(t)|\, dt = \int_0^t 1\, dt = t.$

14.4.9 $L = \int_0^1 \sqrt{36t^2 + 64t^2}\, dt = \int_0^1 10t\, dt = 5t^2\Big|_0^1 = 5.$

14.4.10 $L = \int_0^1 \sqrt{9+16+1}\, dt = \sqrt{26}t\,\Big|_0^1 = \sqrt{26}.$

14.4.11 $L = \int_0^{\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt = \int_0^{\pi} 3\, dt = 3\pi.$

14.4.12 $L = \int_0^{2\pi/3} \sqrt{(-12\sin(3t))^2 + (12\cos(3t))^2}\, dt = 12(2\pi/3) = 8\pi.$

14.4.13 Note that $(\cos t + t\sin t)' = -\sin t + \sin t + t\cos t = t\cos t$, and $(\sin t - t\cos t)' = \cos t - (\cos t - t\sin t) = t\sin t$. $L = \int_0^{\pi/2} \sqrt{(t\cos t)^2 + (t\sin t)^2}\, dt = \int_0^{\pi/2} t\, dt = \frac{t^2}{2}\Big|_0^{\pi/2} = \frac{\pi^2}{8}.$

14.4.14 Note that $(\cos t + \sin t)' = -\sin t + \cos t$, and $(\cos t - \sin t)' = -\sin t - \cos t$.

$$L = \int_0^{2\pi} \sqrt{(-\sin t + \cos t)^2 + (-\sin t - \cos t)^2}\, dt = \int_0^{2\pi} \sqrt{2 - 2\cos t\sin t + 2\cos t\sin t}\, dt = \sqrt{2}(2\pi).$$

14.4.15 $L = \int_1^6 \sqrt{3^2 + (-4)^2 + 3^2}\, dt = \sqrt{34}(6-1) = 5\sqrt{34}.$

14.4.16 $L = \int_0^{6\pi} \sqrt{16\sin^2 t + 16\cos^2 t + 9}\, dt = 5(6\pi) = 30\pi.$

14.4.17 $L = \int_0^{4\pi} \sqrt{1 + 64\cos^2 t + 64\sin^2 t}\, dt = \sqrt{65}(4\pi).$

14.4.18 $L = \int_0^2 \sqrt{t^2 + (2t+1)}\, dt = \int_0^2 (t+1)\, dt = \left(\frac{t^2}{2} + t\right)\Big|_0^2 = 4.$

14.4.19 $L = \int_0^{\ln 2} \sqrt{4e^{4t} + 16e^{4t} + 16e^{4t}}\, dt = \int_0^{\ln 2} 6e^{2t}\, dt = 3e^{2t}\Big|_0^{\ln 2} = 12 - 3 = 9.$

14.4.20 $L = \int_0^4 \sqrt{4t^2 + 9t^4}\, dt = \int_0^4 t\sqrt{4+9t^2}\, dt$. Let $u = 4 + 9t^2$ so that $du = 18t\, dt$. Then $L = \frac{1}{18}\int_4^{148} \sqrt{u}\, du = \frac{1}{27}\, u^{3/2}\Big|_4^{148} = \frac{8}{27}(37\sqrt{37} - 1).$

14.4.21 $L = \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\, dt = \int_0^{\pi/2} 3\sin t \cos t\, dt = \frac{3}{2} \sin^2 t \Big|_0^{\pi/2} = \frac{3}{2}.$

14.4.22 $L = \int_0^{2\pi} \sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t}\, dt = 5(2\pi) = 10\pi.$

14.4.23 The speed is $\sqrt{36t^4 + 9t^4 + 225t^4} = \sqrt{270}\, t^2$. The length is thus $L = \int_0^4 \sqrt{270}\, t^2\, dt = \sqrt{270}\, \frac{t^3}{3} \Big|_0^4 = \sqrt{30}(64 - 0) = 64\sqrt{30}.$

14.4.24 The speed is $\sqrt{100t^2 \sin^2 t^2 + 100t^2 \cos^2 t^2 + 24^2 t^2}\, dt = \sqrt{100t^2 + 576t^2} = 26t$. The length is thus $L = \int_0^2 26t\, dt = 13t^2 \Big|_0^2 = 52.$

14.4.25 The speed is $2\sqrt{(13\cos 2t)^2 + (-12\sin 2t)^2 + (-5\sin 2t)^2} = 2 \cdot \sqrt{169} = 26$. The length is thus $L = \int_0^{\pi} 26\, dt = 26\pi.$

14.4.26 The speed is $e^t \sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2 + 1^2} = e^t \sqrt{3}$. The length is thus $L = \int_0^{\ln 2} \sqrt{3} e^t\, dt = \sqrt{3}\, e^t \Big|_0^{\ln 2} = \sqrt{3}(2 - 1) = \sqrt{3}.$

14.4.27 The length of the orbit is approximately $4 \int_0^{\pi/2} \sqrt{\sin^2 t + 0.999\cos^2 t}\, dt$ which can be verified with a computer algebra system to be about 6.28. Then the speed is approximately $\frac{6.28 \cdot 93,000,000}{365.25 \cdot 24} \approx 66,626$ miles per hour.

14.4.28 The length of the orbit is approximately

$$4 \int_0^{\pi/2} 5.203 \sqrt{\sin^2 t + \left(\frac{1.517}{1.524}\right)^2 \cos^2 t}\, dt \approx 32.616.$$

Then the speed is approximately $\frac{32.616 \cdot 93,000,000}{11.8618 \cdot 365.25 \cdot 24} \approx 29,171.7.$

14.4.29 $L = \int_0^{2\pi} \sqrt{4\sin^2 t + 16\cos^2 t}\, dt = \int_0^{2\pi} \sqrt{4 + 12\cos^2 t}\, dt = 2\int_0^{2\pi} \sqrt{1 + 3\cos^2 t}\, dt \approx 19.38.$

14.4.30 $L = \int_0^{2\pi} \sqrt{4\sin^2 t + 16\cos^2 t + 36\sin^2 t}\, dt = \int_0^{2\pi} \sqrt{16\sin^2 t + 16\cos^2 t + 24\sin^2 t}\, dt = 2\int_0^{2\pi} \sqrt{4 + 6\sin^2 t}\, dt \approx 32.85.$

14.4.31 $L = \int_{-2}^{2} \sqrt{1 + 64t^2}\, dt \approx 32.5.$

14.4.32 $L = \int_0^{\ln 3} \sqrt{e^{2t} + 4e^{-2t} + 1}\, dt \approx 2.73.$

14.4.33 Note that $|\mathbf{v}| = \sqrt{0 + \cos^2 t + \sin^2 t} = 1$, so it does use arc length as its parameter.

14.4.34 Note that $|\mathbf{v}| = \sqrt{1/3 + 1/3 + 1/3} = 1$, so it does use arc length as its parameter.

14.4.35 Note that $|\mathbf{v}| = \sqrt{1+4} \neq 1$, so it doesn't use arc length as a parameter.

Consider $\mathbf{r}(s) = \left\langle s/\sqrt{5}, 2s/\sqrt{5} \right\rangle$ for $0 \leq s \leq 3\sqrt{5}$. This has $|\mathbf{r}'(s)| = \dfrac{1}{\sqrt{5}}\sqrt{1+4} = 1$, so it does use arc length as its parameter.

14.4.36 Note that $|\mathbf{v}| = \sqrt{1+4+36} \neq 1$, so it doesn't use arc length as a parameter.

Consider $\mathbf{r}(s) = \left\langle (s/\sqrt{41}) + 1, (2s/\sqrt{41}) - 3, 6s/\sqrt{41} \right\rangle$ for $0 \leq s \leq 10\sqrt{41}$. This has $|\mathbf{r}'(s)| = \dfrac{1}{\sqrt{41}}\sqrt{1+4+36} = 1$, so it does use arc length as its parameter.

14.4.37 Note that $|\mathbf{v}| = \sqrt{4\sin^2 t + 4\cos^2 t} \neq 1$, so it doesn't use arc length as a parameter.

Consider $\mathbf{r}(s) = \langle 2\cos(s/2), 2\sin(s/2) \rangle$ for $0 \leq s \leq 4\pi$. This has $|\mathbf{r}'(s)| = \sqrt{\sin^2(s/2) + \cos^2(s/2)} = 1$, so it does use arc length as its parameter.

14.4.38 Note that $|\mathbf{v}| = \sqrt{17^2\sin^2 t + 15^2\cos^2 t + 8^2\cos^2 t} = \sqrt{289} = 17 \neq 1$, so it doesn't use arc length as a parameter.

Consider $\mathbf{r}(s) = \langle 17\cos(s/17), 15\sin(s/17), 8\sin(s/17) \rangle$ for $0 \leq s \leq 17\pi$. This has

$$|\mathbf{r}'(s)| = \sqrt{\sin^2(s/17) + (15/17)^2\cos^2(s/17) + (8/17)^2\cos^2(s/17)} = 1,$$

so it does use arc length as its parameter.

14.4.39 Note that $|\mathbf{v}| = \sqrt{4t^2\sin^2(t^2) + 4t^2\cos^2(t^2)} = 2t \neq 1$, so it doesn't use arc length as a parameter.

Consider $\mathbf{r}(s) = \langle \cos s, \sin s \rangle$ for $0 \leq s \leq \pi$. This has $|\mathbf{r}'(s)| = \sqrt{\sin^2 s + \cos^2 s} = 1$, so it does use arc length as its parameter.

14.4.40 Note that $|\mathbf{v}| = \sqrt{4t^2 + 16t^2 + 64t^2} = \sqrt{84}t = 2\sqrt{21}t \neq 1$, so it doesn't use arc length as a parameter.

Consider $\mathbf{r}(s) = \left\langle (1/\sqrt{21})s, (2/\sqrt{21})s, (4/\sqrt{21})s \right\rangle$ for $\sqrt{21} \leq s \leq 16\sqrt{21}$. This has $|\mathbf{r}'(s)| = \dfrac{1}{\sqrt{21}}\sqrt{1+4+16} = 1$, so it does use arc length as its parameter.

14.4.41 Note that $|\mathbf{v}| = \sqrt{e^{2t} + e^{2t} + e^{2t}} = \sqrt{3}e^t \neq 1$, so it doesn't use arc length as its parameter.

Consider $\mathbf{r}(s) = \left\langle \dfrac{s}{\sqrt{3}} + 1, \dfrac{s}{\sqrt{3}} + 1, \dfrac{s}{\sqrt{3}} + 1 \right\rangle$ for $s \geq 0$. This has $|\mathbf{r}'(s)| = 1$, so it does use arc length as its parameter.

14.4.42 Note that $|\mathbf{v}| = \sqrt{(1/2)\sin^2 t + (1/2)\sin^2/2 + \cos^2 t} = 1$, so it does use arc length as its parameter.

14.4.43

a. True. $L = \displaystyle\int_a^b S\,dt = S(b-a)$.

b. True. Both have length $L = \displaystyle\int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt = \int_a^b \sqrt{g'(t)^2 + f'(t)^2}\,dt$.

c. True. Both have length $L = \displaystyle\int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{(f'(u^2)(2u))^2 + (g'(u^2)(2u))^2}\,du$. The equality can be seen via the substitution $u^2 = t$.

d. False. It is not the case that $|\mathbf{r}'(t)| = 1$ for all t, because $|\mathbf{r}'(t)| = \sqrt{1+40t^2}$.

14.4.44

a. $\mathbf{r}(t) = \langle x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0), z_0 + t(z_1 - z_0) \rangle$, where $0 \leq t \leq 1$.

b. $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 \sqrt{(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2}\,dt$. This is equal to
$$\sqrt{(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2}.$$

c. The distance formula also gives the length of this line segment to be
$$\sqrt{(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2}.$$

14.4.45

a. Let $x = a\cos t$, $y = b\sin t$ and $z = c\sin t$. Then $x^2+y^2+z^2 = a^2\cos^2 t + b^2\sin^2 t + c^2\sin^2 t = a^2\cos^2 t + (b^2+c^2)\sin^2 t = a^2$, assuming $a^2 = b^2+c^2$. So the curve lies on a sphere, but also note that $cy = bz$, so the curve also lies in the plane $cy - bz = 0$. So the curve is a circle centered at the origin.

b. The circle has arc length
$$L = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t + c^2\cos^2 t}\,dt = \sqrt{a^2}\int_0^{2\pi}\sqrt{\sin^2 t+\cos^2 t}\,dt = 2a\pi.$$

c. As in an exercise from the previous section, the curve describes a circle when $a^2+c^2+e^2 = b^2+d^2+f^2 = R^2$ and $ab+cd+ef=0$. Note that $|\mathbf{r}(t)|^2 = (a\cos t + b\sin t)^2 + (c\cos t + d\sin t)^2 + (e\cos t + f\sin t)^2 = (a^2+c^2+e^2)\cos^2 t + (2ab+2cd+2ef)\sin t\cos t + (b^2+d^2+f^2)\sin^2 t$, so if the conditions are met, then the curve describes a circle of radius R and $L = \int_0^{2\pi}\sqrt{R^2}\,dt = 2\pi R$.

14.4.46 Assume $m \neq 0$. $\mathbf{r}'(t) = \left\langle mt^{m-1}, mt^{m-1}, (3m/2)t^{(3m/2-1)}\right\rangle$, so $|\mathbf{r}'(t)| = \dfrac{3\,|m|\,t^{m-1}}{2}\sqrt{\dfrac{4}{9}+\dfrac{4}{9}+t^m}$. So
$$L = \frac{3\,|m|}{2}\int_a^b t^{m-1}\sqrt{(8/9)+t^m}\,dt.$$
Let $u = \dfrac{8}{9}+t^m$, so that $\pm du = |m|\,t^{m-1}\,dt$. Then if $m > 0$ we have $L = \dfrac{3}{2}\displaystyle\int_{(8/9)+a^m}^{(8/9)+b^m}\sqrt{u}\,du = (8/9+b^m)^{3/2} - (8/9+a^m)^{3/2}$, and if $m < 0$, we have $L = \dfrac{3}{2}\displaystyle\int_{(8/9)+b^m}^{(8/9)+a^m}\sqrt{u}\,du = (8/9+a^m)^{3/2} - (8/9+b^m)^{3/2}$. Note that in the case $m = 0$, the curve is the constant $\mathbf{r}(t) = \langle 1,1,1\rangle$, so $L = 0$.

14.4.47

a. $\mathbf{r}'(t) = h'(t)\,\langle A, B\rangle$, so $|\mathbf{r}'(t)| = |h'(t)|\sqrt{A^2+B^2}$. Thus, $L = \sqrt{A^2+B^2}\displaystyle\int_a^b |h'(t)|\,dt$

b. $L = \sqrt{2^2+5^2}\displaystyle\int_0^4 3t^2\,dt = \sqrt{29}\,t^3\Big|_0^4 = 64\sqrt{29}$.

c. $L = \sqrt{4^2+10^2}\displaystyle\int_1^8 \left|-1/t^2\right|\,dt = \sqrt{116}\,\frac{1}{t}\Big|_8^1 = \frac{7\sqrt{29}}{4}$.

14.4.48 With $A = 4$, $a = 1$, and $k = 35$, we have
$$\mathbf{r}(t) = \langle (4+\cos 35t)\cos t, (4+\cos 35t)\sin t, \sin 35t\rangle = \langle 4\cos t + \cos t\cos 35t, 4\sin t + \sin t\cos 35t, \sin 35t\rangle.$$
Then $\mathbf{r}'(t) = \langle -4\sin t - 35\cos t\sin 35t - \cos 35t\sin t, 4\cos t - 35\sin t\sin 35t + \cos 35t\cos t, 35\cos 35t\rangle$. Using a computer algebra system, we find that $\displaystyle\int_0^{2\pi}|\mathbf{r}'(t)|\,dt \approx 221.4$ in.

14.4.49

a. Let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Then $y(t) = -4.9t^2+25t$ is 0 for $t > 0$ when $-4.9t+25 = 0$, or $t = 25/4.9 \approx 5.102$ seconds.

b. $L \approx \displaystyle\int_0^{5.102} \sqrt{400 + (25 - 9.8t)^2}\, dt$.

c. Let $u = -9.8t + 25$ so that $du = -9.8dt$. Then $L \approx \dfrac{1}{-9.8} \displaystyle\int_{25}^{-24.9996} \sqrt{400 + u^2}\, du \approx 124.43$ meters.

d. $x = u_0(5.102) = 20(5.102) = 102.04$ meters.

14.4.50

a. Let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Because $x^2 + y^2 = 1$, the path is the unit circle. The particle is at $(0, 1)$ at $t = 0$, and returns there at $t = \sqrt{2\pi}$.

b. The length of the path is $2\pi = \displaystyle\int_0^{\sqrt{2\pi}} \sqrt{(2t\cos t^2)^2 + (2t\sin t^2)^2}\, dt = \int_0^{\sqrt{2\pi}} 2t\, dt.$

c. This particle traces out the same circle as that one, but it does it faster, completing the circle in $\sqrt{2\pi}$ time units rather than 2π time units. Note that the speed of this particle is $2t$ rather than the constant 1 which is the speed of the other particle.

d. This also traces out the same circle, over the time interval $[0, \sqrt[n]{2\pi}]$.

e. It is also 2π because the arc is the same circle.

f. The second runner would win the race. The runners occupy the same position at $t = 1$ (namely the point $(\sin(1), \cos(1))$), so that is when one passes the other.

14.4.51 Recall that a curve is parametrized by arc length exactly when $|\mathbf{v}| = 1$. We have $\mathbf{v} = \langle a, b, c\rangle$, so $|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}$, which is equal to one if and only if $a^2 + b^2 + c^2 = 1$.

14.4.52 Suppose $a^2 = b^2 + c^2$. Recall that a curve is parametrized by arc length exactly when $|\mathbf{v}| = 1$. We have $\mathbf{v} = \langle -a\sin t, b\cos t, c\cos t\rangle$, so

$$|\mathbf{v}| = \sqrt{a^2\sin^2 t + b^2\cos^2 t + c^2\cos^2 t} = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2},$$

which is equal to one if and only if $a^2 = b^2 + c^2 = 1$.

14.4.53 $\displaystyle\int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b \sqrt{(cf'(t))^2 + (cg'(t))^2}\, dt = |c| \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt = |c|\, L.$

14.4.54

a. Let t_0 be a number on $[a, b]$, so that $\mathbf{r}(t_0) = \langle f(t_0), g(t_0), h(t_0)\rangle$ is a point on $\mathbf{r}$. Then $u^{-1}(t_0) = u_0$ is a number between $u^{-1}(a)$ and $u^{-1}(b)$. Note that $\mathbf{R}(u_0) = \left\langle f(u(u^{-1}(t_0))), g(u(u^{-1}(t_0))), h(u(u^{-1}(t_0)))\right\rangle$ $= \langle f(t_0), g(t_0), h(t_0)\rangle = \mathbf{r}(t_0)$. So for every point on $\mathbf{r}$ there is a corresponding point on $\mathbf{R}$.

Now suppose that z_0 is a number between $u^{-1}(a)$ and $u^{-1}(b)$, so that $\mathbf{R}(z_0) =$ $\langle f(u(z_0)), g(u(z_0)), h(u(z_0))\rangle$ is a point on $\mathbf{R}$. Then $u(z_0) = s_0$ is a number between a and b, and $\mathbf{r}(s_0) = \langle f(s_0), g(s_0), h(s_0)\rangle = \langle f(u(z_0)), g(u(z_0)), h(u(z_0))\rangle = \mathbf{R}(z_0)$. So for every point on $\mathbf{R}$ there is a corresponding point on $\mathbf{r}$.

Thus the two curves represent the same sets of points.

b. Assume $u'(t) > 0$. The case $u'(t) < 0$ is similar. The length of $\mathbf{R}$ is

$$L = \int_{u^{-1}(a)}^{u^{-1}(b)} \sqrt{(f'(u)u'(t))^2 + (g'(u)u'(t))^2 + (h'(u)u'(t))^2}\, dt.$$

Let $s = u(t)$ so that $ds = u'(t)\, dt$. Then we have $L = \displaystyle\int_a^b \sqrt{f'(s)^2 + g'(s)^2 + h'(s)^2}\, ds$, which is the length of $\mathbf{r}$.

14.5 Curvature and Normal Vectors

14.5.1 A straight line has zero curvature.

14.5.2 The curvature is a measure of the magnitude of the rate of change of the unit tangent vector with respect to arc length. It is a scalar function.

14.5.3 $\kappa = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right|$ or $\kappa = \dfrac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3}$.

14.5.4 The principal unit normal vector of a curve is a vector function whose value for any point on the curve is the vector perpendicular to the tangent to the curve, having unit length, and pointing to the inside of the curve.

14.5.5 $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$.

14.5.6 $\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$, where $a_N = \kappa\,|\mathbf{v}|^2 = \dfrac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|}$ and $a_T = \dfrac{d^2s}{dt^2} = \dfrac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|}$.

14.5.7 $\mathbf{B}$ is a length one vector mutually perpendicular to $\mathbf{T}$ and $\mathbf{N}$. The three vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ form a right-handed coordinate system.

14.5.8 $\mathbf{B} = \mathbf{T}\times\mathbf{N}$.

14.5.9 The torsion is the rate at which the curve moves out of the osculating lane.

14.5.10 $\tau = \dfrac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{|\mathbf{v}\times\mathbf{a}|\,r}$.

14.5.11 $\mathbf{r}'(t) = \langle 2,4,6\rangle$, so $\mathbf{T} = \dfrac{1}{2\sqrt{14}}\langle 2,4,6\rangle = \dfrac{1}{\sqrt{14}}\langle 1,2,3\rangle$.
So $\dfrac{d\mathbf{T}}{dt} = \langle 0,0,0\rangle$, and $\kappa = 0$.

14.5.12 $\mathbf{r}'(t) = \langle -2\sin t, -2\cos t\rangle$, so $\mathbf{T}(t) = \langle -\sin t, -\cos t\rangle$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{8}\left|\langle -2\cos t, 2\sin t, 0\rangle \times \langle -2\sin t, -2\cos t, 0\rangle\right| = \frac{1}{2}.$$

14.5.13 $\mathbf{r}'(t) = \langle 2, 4\cos t, -4\sin t\rangle$, so $\mathbf{T}(t) = \dfrac{1}{\sqrt{5}}\langle 1, 2\cos t, -2\sin t\rangle$. So

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{2\sqrt{5}}\left|\frac{1}{\sqrt{5}}\langle 0, -2\sin t, -2\cos t\rangle\right| = \frac{1}{5}.$$

14.5.14 $\mathbf{r}'(t) = \left\langle -2t\sin t^2, 2t\cos t^2\right\rangle$, so $\mathbf{T}(t) = \left\langle -\sin t^2, \cos t^2\right\rangle$. So

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{2t}\left|\left\langle -2t\cos(t^2), -2t\sin(t^2)\right\rangle\right| = 1.$$

14.5.15 $\mathbf{r}'(t) = \left\langle \sqrt{3}\cos t, \cos t, -2\sin t\right\rangle$, so $|\mathbf{v}(t)| = \sqrt{3\cos^2 t + \cos^2 t + 4\sin^2 t} = 2$.
Thus, $\mathbf{T}(t) = \dfrac{1}{2}\left\langle \sqrt{3}\cos t, \cos t, -2\sin t\right\rangle$. So

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{2}\left|\frac{1}{2}\left\langle -\sqrt{3}\sin t, -\sin t, -2\cos t\right\rangle\right| = \frac{1}{2}\cdot\frac{1}{2}\cdot 2 = \frac{1}{2}.$$

14.5.16 $\mathbf{r}'(t) = \langle 1, -\tan t\rangle$, so $|\mathbf{v}(t)| = \sqrt{1+\tan^2 t} = \sec t$. So $\mathbf{T}(t) = \langle \cos t, -\sin t\rangle$.

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \cos t\,|\langle -\sin t, -\cos t\rangle| = \cos t.$$

14.5.17 $\mathbf{r}'(t) = \langle 1, 4t\rangle$, so $|\mathbf{v}(t)| = \sqrt{16t^2+1}$. Thus, $\mathbf{T}(t) = \dfrac{1}{\sqrt{16t^2+1}}\langle 1, 4t\rangle$.

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{(16t^2+1)^{3/2}}\,|\langle 0,4,0\rangle \times \langle 1,4t,0\rangle| = \frac{1}{(16t^2+1)^{3/2}}\,|\langle 0,0,-4\rangle| = \frac{4}{(16t^2+1)^{3/2}}.$$

14.5.18 $\mathbf{r}'(t) = \left\langle -3\cos^2 t\sin t, 3\sin^2 t\cos t\right\rangle$, so $|\mathbf{v}(t)| = 3\sqrt{\cos^4 t\sin^2 t + \sin^4 t\cos^2 t} = 3\sqrt{\cos^2 t\sin^2 t} = 3\,|\cos t\sin t|$. Thus $\mathbf{T}(t) = \langle -\cos t, \sin t\rangle$.

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{3\,|\sin t\cos t|}\,|\langle \sin t, \cos t\rangle| = \frac{1}{3}\,|\sec t\csc t|$$

14.5.19 $\mathbf{r}'(t) = \left\langle \cos(\pi t^2/2), \sin(\pi t^2/2)\right\rangle$, so $|\mathbf{v}(t)| = 1$. So $\mathbf{T}(t) = \mathbf{v}(t) = \left\langle \cos(\pi t^2/2), \sin(\pi t^2/2)\right\rangle$.

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = 1\cdot\left|\left\langle -\pi t\sin(\pi t/^{2}2), \pi t\cos(\pi t^2/2)\right\rangle\right| = \pi t.$$

14.5.20 $\mathbf{r}'(t) = \left\langle \cos(t^2), \sin(t^2)\right\rangle$, so $|\mathbf{v}(t)| = 1$. So $\mathbf{T}(t) = \mathbf{v}(t) = \left\langle \cos(t^2), \sin(t^2)\right\rangle$.

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = 1\cdot\left|\left\langle -2t\sin(t^2), 2t\cos(t^2)\right\rangle\right| = 2t.$$

14.5.21 $\mathbf{r}'(t) = \langle 3\sin t, 3\cos t\rangle$ and $\mathbf{r}''(t) = \langle 3\cos t, -3\sin t\rangle$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{27}\,|\langle 3\cos t, -3\sin t, 0\rangle \times \langle 3\sin t, 3\cos t, 0\rangle| = \frac{1}{27}\,|\langle 0,0,9\rangle| = \frac{1}{3}.$$

14.5.22 $\mathbf{r}'(t) = \langle 4, 3\cos t, -3\sin t\rangle$ and $\mathbf{r}''(t) = \langle 0, -3\sin t, -3\cos t\rangle$.

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{125}\,|\langle 0, -3\sin t, -3\cos t\rangle \times \langle 4, 3\cos t, -3\sin t\rangle| = \frac{1}{125}\,|\langle -9, 12\cos t, 12\sin t\rangle| = \frac{15}{125} = \frac{3}{25}.$$

14.5.23 $\mathbf{r}'(t) = \langle 2t, 1, 0\rangle$ and $\mathbf{r}''(t) = \langle 2, 0, 0\rangle$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{(4t^2+1)^{3/2}}\,|\langle 2,0,0\rangle \times \langle 2t,1,0\rangle| = \frac{1}{(4t^2+1)^{3/2}}\,|\langle 0,0,2\rangle| = \frac{2}{(4t^2+1)^{3/2}}.$$

14.5.24 $\mathbf{r}'(t) = \left\langle \sqrt{3}\cos t, \cos t, -2\sin t\right\rangle$ and $\mathbf{r}''(t) = \left\langle -\sqrt{3}\sin t, -\sin t, -2\cos t\right\rangle$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{8}\left|\left\langle -\sqrt{3}\sin t, -\sin t, -2\cos t\right\rangle \times \left\langle \sqrt{3}\cos t, \cos t, -2\sin t\right\rangle\right| = \frac{1}{8}\left|\left\langle 2, -2\sqrt{3}, 0\right\rangle\right| = \frac{4}{8} = \frac{1}{2}.$$

14.5.25 $\mathbf{r}'(t) = \langle -4\sin t, \cos t, -2\sin t\rangle$ and $\mathbf{r}''(t) = \langle -4\cos t, -\sin t, -2\cos t\rangle$.

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{|\langle 2,0,-4\rangle|}{\left(\sqrt{20\sin^2(t)+\cos^2(t)}\right)^3} = \frac{2\sqrt{5}}{\left(20\sin^2(t)+\cos^2(t)\right)^{3/2}}.$$

14.5.26 $\mathbf{r}'(t) = \langle e^t(\cos t - \sin t), e^t(\cos t + \sin t), e^t\rangle$ and $\mathbf{r}''(t) = \langle -2e^t \sin t, 2e^t \cos t, e^t\rangle$. $|\mathbf{v}(t)| = \sqrt{3}e^t$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{3\sqrt{3}e^{3t}}\left|\left\langle -2e^t\sin t, 2e^t\cos t, e^t\right\rangle \times \left\langle e^t(\cos t - \sin t), e^t(\cos t + \sin t), e^t\right\rangle\right|$$

$$= \frac{1}{3\sqrt{3}e^{3t}}\left|e^{2t}\left\langle \cos t - \sin t, \cos t + \sin t, -2\right\rangle\right| = \frac{\sqrt{6}e^{2t}}{3\sqrt{3}e^{3t}} = \frac{\sqrt{2}}{3e^t}.$$

14.5.27 $\mathbf{r}'(t) = \langle 2\cos t, -2\sin t\rangle$, so $\mathbf{T} = \langle \cos t, -\sin t\rangle$ and $|\mathbf{T}| = 1$. $\mathbf{N}(t) = \langle -\sin t, -\cos t\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = -\sin t\cos t + \sin t \cos t = 0$.

14.5.28 $\mathbf{r}'(t) = \langle 4\cos t, 4\sin t, 10\rangle$, and $|\mathbf{r}'(t)| = 2\sqrt{29}$, so $\mathbf{T} = \frac{1}{\sqrt{29}}\langle 2\cos t, -2\sin t, 5\rangle$ and $|\mathbf{T}| = 1$. $\mathbf{N}(t) = \langle -\sin t, -\cos t, 0\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

14.5.29 $\mathbf{r}'(t) = \langle t, -3, 0\rangle$, and $|\mathbf{r}'(t)| = \sqrt{t^2+9}$, so $\mathbf{T} = \frac{1}{\sqrt{t^2+9}}\langle t, -3, 0\rangle$ and $|\mathbf{T}| = 1$. We have $\mathbf{T}'(t) = \left\langle \frac{9}{(\sqrt{t^2+9})^3}, \frac{3t}{(\sqrt{t^2+9})^3}, 0\right\rangle$, so $\mathbf{N}(t) = \frac{1}{\sqrt{t^2+9}}\langle 3, t, 0\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

14.5.30 $\mathbf{r}'(t) = \left\langle t, t^2\right\rangle$, and $|\mathbf{r}'(t)| = |t|\sqrt{t^2+1}$, so $\mathbf{T} = \frac{1}{\sqrt{t^2+1}}\langle 1, t\rangle$ and $|\mathbf{T}| = 1$.
$\mathbf{T}'(t) = \left\langle -\frac{t}{(\sqrt{t^2+1})^3}, \frac{1}{(\sqrt{t^2+1})^3}\right\rangle$, so $\mathbf{N}(t) = \frac{1}{\sqrt{t^2+1}}\langle -t, 1\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

14.5.31 $\mathbf{r}'(t) = \left\langle -2t\sin t^2, 2t\cos t^2\right\rangle$, and $|\mathbf{r}'(t)| = 2t$, so $\mathbf{T} = \left\langle -\sin t^2, \cos t^2\right\rangle$ and $|\mathbf{T}| = 1$. $\mathbf{T}'(t) = \left\langle -2t\cos t^2, -2t\sin t^2\right\rangle$, so $\mathbf{N}(t) = \left\langle -\cos t^2, -\sin t^2\right\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

14.5.32 $\mathbf{r}'(t) = \left\langle -3\cos^2 t\sin t, 3\sin^2 t\cos t\right\rangle$, and $|\mathbf{r}'(t)| = 3\left|\sin t\cos t\right|$, so $\mathbf{T} = \langle -\cos t, \sin t\rangle$ and $|\mathbf{T}| = 1$. $\mathbf{T}'(t) = \langle \sin t, \cos t\rangle$, so $\mathbf{N}(t) = \langle \sin t, \cos t\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

14.5.33 $\mathbf{r}'(t) = \langle 2t, 1\rangle$, and $|\mathbf{r}'(t)| = \sqrt{4t^2+1}$, so $\mathbf{T} = \frac{1}{\sqrt{4t^2+1}}\langle 2t, 1\rangle$ and $|\mathbf{T}| = 1$.
$\mathbf{T}'(t) = \left\langle \frac{2}{(\sqrt{4t^2+1})^3}, -\frac{4t}{(\sqrt{4t^2+1})^3}\right\rangle$, so $\mathbf{N}(t) = \frac{1}{\sqrt{4t^2+1}}\langle 1, -2t\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

14.5.34 $\mathbf{r}'(t) = \langle 1, -\tan t\rangle$, and $|\mathbf{r}'(t)| = \sec t$, so $\mathbf{T} = \langle \cos t, -\sin t\rangle$ and $|\mathbf{T}| = 1$. $\mathbf{T}'(t) = \langle -\sin t, -\cos t\rangle$, so $\mathbf{N}(t) = \langle -\sin t, -\cos t\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

14.5.35 $\mathbf{r}'(t) = \langle 1, 4, -6\rangle$ and $\mathbf{r}''(t) = \langle 0, 0, 0\rangle$. So $\kappa = 0$ and thus $a_N = 0$. Also, $a_T = 0$. We have $\mathbf{a} = 0\cdot\mathbf{T} + 0\cdot\mathbf{N} = \langle 0, 0, 0\rangle$.

14.5.36 $\mathbf{r}'(t) = \langle -10\sin t, -10\cos t\rangle$ and $\mathbf{r}''(t) = \langle -10\cos t, 10\sin t\rangle$. Note that $\mathbf{a}\cdot\mathbf{v} = 0$, so $a_T = 0$. $\mathbf{a}\times\mathbf{v} = \langle 0, 0, 100\rangle$, so $a_N = \frac{100}{10} = 10$. We have $\mathbf{a} = \langle -10\cos t, 10\sin t\rangle = 0\cdot\mathbf{T} + 10\mathbf{N}$.

14.5.37 In problem 26 above, we computed $|\mathbf{v}(t)| = \sqrt{3}e^t$ and $\kappa = \frac{\sqrt{2}}{3e^t}$. Also, $\mathbf{v}(t)\cdot\mathbf{a}(t) = 3e^{2t}$. Thus $a_T = \frac{3e^{2t}}{\sqrt{3}e^t} = \sqrt{3}e^t$ and $a_N = \frac{\sqrt{2}}{3e^t}\cdot 3e^{2t} = \sqrt{2}e^t$.

14.5.38 $\mathbf{r}'(t) = \langle 1, 2t\rangle$ and $\mathbf{r}''(t) = \langle 0, 2\rangle$. Note that $\mathbf{a}\cdot\mathbf{v} = 4t$, so $a_T = \frac{4t}{\sqrt{4t^2+1}}$. $\mathbf{a}\times\mathbf{v} = \langle 0, 0, -2\rangle$, so $a_N = \frac{2}{\sqrt{4t^2+1}}$. We have $\mathbf{a} = \langle 0, 2\rangle = \frac{4t}{\sqrt{4t^2+1}}\cdot\mathbf{T} + \frac{2}{\sqrt{4t^2+1}}\cdot\mathbf{N}$.

14.5.39 $\mathbf{r}'(t) = \left\langle 3t^2, 2t\right\rangle$ and $\mathbf{r}''(t) = \langle 6t, 2\rangle$. Note that $\mathbf{a}\cdot\mathbf{v} = 18t^3 + 4t$, so $a_T = \frac{18t^3+4t}{\sqrt{9t^4+4t^2}} = \frac{18t^2+4}{\sqrt{9t^2+4}}$. $\mathbf{a}\times\mathbf{v} = \left\langle 0, 0, 6t^2\right\rangle$, so $a_N = \frac{6t^2}{t\sqrt{9t^2+4}} = \frac{6t}{\sqrt{9t^2+4}}$. We have $\mathbf{a} = \langle 6t, 2\rangle = \frac{18t^2+4}{\sqrt{9t^2+4}}\cdot\mathbf{T} + \frac{6t}{\sqrt{9t^2+4}}\cdot\mathbf{N}$.

14.5.40 $\mathbf{r}'(t) = \langle -20\sin t, 20\cos t, 30\rangle$ and $\mathbf{r}''(t) = \langle -20\cos t, -20\sin t, 0\rangle$. Note that $\mathbf{a}\cdot\mathbf{v} = 0$,so $a_T = 0$. $\mathbf{a}\times\mathbf{v} = \langle -600\sin t, 600\cos t, -400\rangle$, so $a_N = \dfrac{200\sqrt{13}}{10\sqrt{13}} = 20$. We have $\mathbf{a} = \langle -20\cos t, -20\sin t, 0\rangle = 0\cdot\mathbf{T} + 20\mathbf{N}$.

14.5.41 $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \langle \cos t, -\sin t, 0\rangle \times \langle -\sin t, -\cos t, 0\rangle = \langle 0, 0, -1\rangle$. Because $\mathbf{B}$ is constant, $\tau = 0$.

14.5.42 $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \left\langle \dfrac{2\cos(t)}{\sqrt{29}}, \dfrac{-2\sin(t)}{\sqrt{29}}, \dfrac{5}{\sqrt{29}}\right\rangle \times \langle -\sin t, -\cos t, 0\rangle = \frac{1}{\sqrt{29}}\langle 5\cos t, -5\sin t, -2\rangle$. Also, $\tau = \dfrac{-d\mathbf{B}/dt}{ds/dt}\cdot\mathbf{N} = \left\langle \dfrac{5\sin(t)}{58}, \dfrac{5\cos(t)}{58}, 0\right\rangle \cdot \langle -\sin t, -\cos t, 0\rangle = -\dfrac{5}{58}$.

14.5.43 $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \left\langle \dfrac{t}{\sqrt{t^2+9}}, -\dfrac{3}{\sqrt{t^2+9}}, 0\right\rangle \times \left\langle \dfrac{3}{\sqrt{t^2+9}}, \dfrac{t}{\sqrt{t^2+9}}, 0\right\rangle = \langle 0, 0, 1\rangle$. Because $\mathbf{B}$ is constant, $\tau = 0$.

14.5.44 $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \left\langle \dfrac{1}{\sqrt{t^2+1}}, \dfrac{t}{\sqrt{t^2+1}}, 0\right\rangle \times \left\langle -\dfrac{t}{\sqrt{t^2+1}}, \dfrac{1}{\sqrt{t^2+1}}, 0\right\rangle = \langle 0, 0, 1\rangle$. Because $\mathbf{B}$ is constant, $\tau = 0$.

14.5.45 $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, -1\rangle$, so $\mathbf{T} = \left\langle (-2/\sqrt{5})\sin t, (2/\sqrt{5})\cos t, -1/\sqrt{5}\right\rangle$.

$\mathbf{r}''(t) = \langle -2\cos t, -2\sin t, 0\rangle$, so $\mathbf{N} = \langle -\cos t, -\sin t, 0\rangle$. Thus, $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \dfrac{1}{\sqrt{5}}\langle -\sin t, \cos t, 2\rangle$. Also, $\tau = \dfrac{-d\mathbf{B}/dt}{ds/dt}\cdot\mathbf{N} = -\dfrac{1}{\sqrt{5}}\cdot\dfrac{1}{\sqrt{5}}(\langle -\cos t, -\sin t, 0\rangle \cdot \langle -\cos t, -\sin t, 0\rangle = -\dfrac{1}{5}$.

14.5.46 $\mathbf{r}'(t) = \langle 1, \sinh t, -\cosh t\rangle$, so $\mathbf{r}''(t) = \langle 0, \cosh t, -\sinh t\rangle$. $\mathbf{r}'(t)\times\mathbf{r}''(t) = \langle 1, \sinh t, \cosh t\rangle$. Note that $|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{1+\sinh^2 t+\cosh^2 t} = \sqrt{2}\cosh t$. Therefore, $\mathbf{B} = \dfrac{1}{\sqrt{2}}\langle \operatorname{sech} t, \tanh t, 1\rangle$.

Now $\mathbf{r}'''(t) = \langle 0, \sinh t, -\cosh t\rangle$ so $(\mathbf{r}'(t)\times\mathbf{r}''(t))\cdot\mathbf{r}'''(t) = \langle 1, \sinh t, \cosh t\rangle \cdot \langle 0, \sinh t, -\cosh t\rangle = -1$. Thus $\tau = \dfrac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2} = -\dfrac{1}{2\cosh^2 t} = -\dfrac{1}{2}\operatorname{sech}^2 t$.

14.5.47 $\mathbf{r}'(t) = \langle 12, -5\sin t, 5\cos t\rangle$, so $\mathbf{r}''(t) = \langle 0, -5\cos t, -5\sin t\rangle$. Thus,

$$\mathbf{T} = \langle 12/13, (-5/13)\sin t, (5/13)\cos t\rangle$$

and

$$\mathbf{N} = \langle 0, -\cos t, -\sin t\rangle .$$

So

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \langle (5/13), (12/13)\sin t, (-12/13)\cos t\rangle .$$

Also, $\tau = -\dfrac{d\mathbf{B}/dt}{ds/dt}\cdot\mathbf{N} = -\dfrac{1}{13}\cdot\dfrac{1}{13}(\langle 0, 12\cos t, 12\sin t\rangle \cdot \langle 0, -\cos t, -\sin t\rangle = -\dfrac{1}{169}\cdot(-12) = \dfrac{12}{169}$.

14.5.48 $\mathbf{r}'(t) = \langle t\sin t, t\cos t, 1\rangle$, and $\mathbf{r}''(t) = \langle \sin t + t\cos t, \cos t - t\sin t, 0\rangle$. Then

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \left\langle t\sin t - \cos t, \sin t + t\cos t, -t^2\right\rangle .$$

We then have $\mathbf{B} = \dfrac{1}{\sqrt{1+t^2+t^4}}\left\langle t\sin t - \cos t, \sin t + t\cos t, -t^2\right\rangle$. Note that

$$\mathbf{r}'''(t) = \langle 2\cos t - t\sin t, -2\sin t - t\cos t, 0\rangle .$$

Then we have $\tau = \dfrac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2} = -\dfrac{(2+t^2)}{1+t^2+t^4}$.

14.5.49

a. False. For example, consider $\mathbf{r}(t) = \langle \cos t, \sin t, 1\rangle$. Then $\mathbf{T} = \langle -\sin t, \cos t, 0\rangle$ and also $\mathbf{N} = \langle -\cos t, -\sin t, 0\rangle$. Note that $\mathbf{T}$ and $\mathbf{N}$ lie in the xy-plane, but $\mathbf{r}$ doesn't.

b. False. $\mathbf{T}$ does depend on the orientation, but $\mathbf{N}$ doesn't. Reversing the orientation changes $\mathbf{T}$ to $-\mathbf{T}$, but leaves $\mathbf{N}$ alone.

c. False. Note that $|\mathbf{T}|$ is independent of orientation, so $\left|\dfrac{d\mathbf{T}}{ds}\right|$ is too.

d. True. As we have already seen for the circle, $\mathbf{v}\cdot\mathbf{a} = 0$. Thus $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} = 0\cdot\mathbf{T} + \kappa\,|\mathbf{v}|^2\,\mathbf{N} = \dfrac{1}{R}\mathbf{N}$.

e. False. For example, if the car's motion is given by $\mathbf{r}(t) = \langle 60\cos t, 60\sin t\rangle$, then the speed is a constant 60, but $\mathbf{a} = \langle -60\cos t, -60\sin t\rangle \neq \langle 0, 0\rangle$.

f. False. If it lies in the xy-plane, it will have zero torsion.

g. False. If it lies in the xy-plane, it might have very large curvature but zero torsion.

14.5.50 Let $\mathbf{r}(t) = \langle t, f(t), 0\rangle$. Then $\mathbf{r}'(t) = \langle 1, f'(t), 0\rangle$ and $\mathbf{r}''(t) = \langle 0, f''(t), 0\rangle$. Note that $|\mathbf{v}(t)| = \sqrt{1+f'(t)^2}$, and $\mathbf{a}\times\mathbf{v} = \langle 0, 0, -f''(t)\rangle$, so $\kappa = \dfrac{|f''(t)|}{(1+f'(t)^2)^{3/2}}$.

14.5.51 $f'(x) = 2x$ and $f''(x) = 2$, so

$$\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}$$

14.5.52 $f'(x) = -\dfrac{x}{\sqrt{a^2-x^2}}$ and $f''(x) = -\dfrac{a^2}{(a^2-x^2)^{3/2}}$, so

$$\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{\left|-\dfrac{a^2}{(a^2-x^2)^{3/2}}\right|}{\left(1+\left(-\dfrac{x}{\sqrt{a^2-x^2}}\right)^2\right)^{3/2}} = \frac{a^2}{a^3} = \frac{1}{a}.$$

14.5.53 $f'(x) = 1/x$ and $f''(x) = -1/x^2$, so

$$\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{1/x^2}{(1+(1/x)^2)^{3/2}} = \frac{x}{(x^2+1)^{3/2}}.$$

14.5.54 $f'(x) = -\tan x$ and $f''(x) = -\sec^2 x$, so

$$\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{\sec^2 x}{(1+\tan^2 x)^{3/2}} = \frac{\sec^2 x}{\sec^3 x} = \cos x.$$

14.5.55 $\mathbf{r}'(t) = \langle f'(t), g'(t)\rangle$ and $\mathbf{r}''(t) = \langle f''(t), g''(t)\rangle$. So

$$\mathbf{a}\times\mathbf{v} = \langle f''(t), g''(t), 0\rangle \times \langle f'(t), g'(t), 0\rangle = \langle 0, 0, g'(t)f''(t) - f'(t)g''(t)\rangle\,.$$

Because $|\mathbf{v}| = \sqrt{(f'(t))^2 + (g'(t))^2}$, we have

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{|f'g'' - g'f''|}{((f')^2 + (g')^2)^{3/2}}.$$

14.5.56 $f'(t) = a\cos t$ and $f''(t) = -a\sin t$, while $g'(t) = -a\sin t$ and $g''(t) = -a\cos t$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{|f'g'' - g'f''|}{((f')^2 + (g')^2)^{3/2}} = \frac{\left|-a^2\cos^2 t - a^2\sin^2 t\right|}{(a^2\cos^2 t + a^2\sin^2 t)^{3/2}} = \left|\frac{a^2}{a^3}\right| = \frac{1}{|a|}.$$

14.5.57 $f'(t) = a\cos t$ and $f''(t) = -a\sin t$, while $g'(t) = -b\sin t$ and $g''(t) = -b\cos t$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{|f'g'' - g'f''|}{((f')^2+(g')^2)^{3/2}} = \frac{\left|-ab\cos^2 t - ab\sin^2 t\right|}{(a^2\cos^2 t + b^2\sin^2 t)^{3/2}} = \frac{|ab|}{(a^2\cos^2 t + b^2\sin^2 t)^{3/2}}.$$

14.5.58 $f'(t) = -3a\cos^2 t\sin t$ and $f''(t) = 3a\cos t(3\sin^2 t - 1)$, while $g'(t) = 3a\sin^2 t\cos t$ and $g''(t) = 3a\sin t(3\cos^2 t - 1)$. So

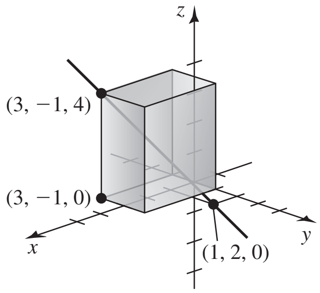
$$\kappa = \frac{|f'g'' - g'f''|}{((f')^2+(g')^2)^{3/2}} = \frac{\left|9a^2\cos^2 t\sin^2 t(3\cos^2 t - 1 + 3\sin^2 t - 1)\right|}{27a^3\left|\cos^3 t\sin^3 t\right|} = \frac{1}{3a\left|\cos t\sin t\right|}.$$

14.5.59 $f'(t) = 1$ and $f''(t) = 0$, while $g'(t) = 2at$ and $g''(t) = 2a$. So

$$\kappa = \frac{|f'g'' - g'f''|}{((f')^2+(g')^2)^{3/2}} = \frac{|2a - 0|}{(1+4a^2t^2)^{3/2}} = \frac{2\,|a|}{(1+4a^2t^2)^{3/2}}.$$

14.5.60

a.

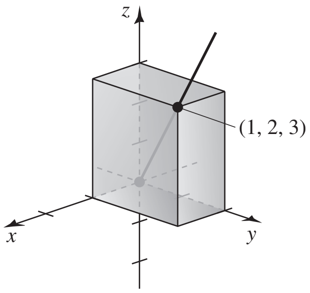


b. For line A: $\mathbf{v}(t) = \langle 2,-3,4\rangle$ and $\mathbf{a}(t) = \langle 0,0,0\rangle$. For line B: $\mathbf{v}(t) = \langle 6,-9,12\rangle$ and $\mathbf{a}(t) = \langle 0,0,0\rangle$. B has 3 times the velocity of A, but the same acceleration.

c. For both lines, we have $a_N = 0$ and $a_T = 0$. $\mathbf{a} = \langle 0,0,0\rangle = 0\cdot\mathbf{T} = 0\cdot\mathbf{N}$.

14.5.61

a.

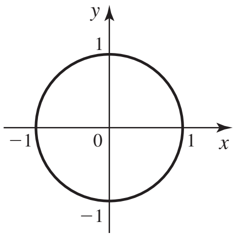


b. For line A: $\mathbf{v}(t) = \langle 1,2,3\rangle$ and $\mathbf{a}(t) = \langle 0,0,0\rangle$. For line B: $\mathbf{v}(t) = \langle 2t,4t,6t\rangle$ and $\mathbf{a}(t) = \langle 2,4,6\rangle$. A has constant velocity and zero acceleration, while B has linearly increasing velocity and constant acceleration.

c. For A, we have $a_T = a_N = 0$. For B, we have $a_T = \dfrac{56t}{\sqrt{56}t} = 2\sqrt{14}$ and $a_N = 0$ (because $\mathbf{v}$ and $\mathbf{a}$ are in the same direction).

14.5.62

a.

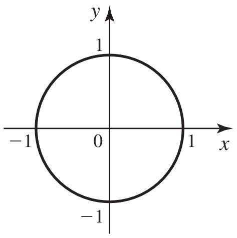


b. For curve A: $\mathbf{v}(t) = \langle -\sin t, \cos t\rangle$ and $\mathbf{a}(t) = \langle -\cos t, -\sin t\rangle$. For curve B: $\mathbf{v}(t) = \langle -3\sin 3t, 3\cos 3t\rangle$ and $\mathbf{a}(t) = \langle -9\cos 3t, -9\sin 3t\rangle$. A has constant velocity 1 while B has constant velocity 3. The magnitude of the acceleration on B is 9 times that on A.

c. For A, we have $\mathbf{a} = \mathbf{N}$, so $a_T = 0$ and $a_N = 1$. For B, we have $\mathbf{a} = 9\mathbf{N}$, so $a_T = 0$ and $a_N = 9$.

14.5.63

a.

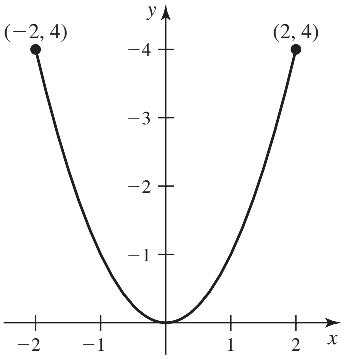


b. For curve A: $\mathbf{v}(t) = \langle -\sin t, \cos t\rangle$ and $\mathbf{a}(t) = \langle -\cos t, -\sin t\rangle$.
For curve B: $\mathbf{v}(t) = 2t\left\langle -\sin t^2, \cos t^2\right\rangle$ and $\mathbf{a}(t) = \left\langle -4t^2\cos t^2 - 2\sin t^2, -4t^2\sin t^2 + 2\cos t^2\right\rangle$.
A has constant velocity 1 while B does not have constant velocity.

c. For A, we have $\mathbf{a} = \mathbf{N}$, so $a_T = 0$ and $a_N = 1$. For B, we have $a_T = 2$ and $a_N = 4t^2$.

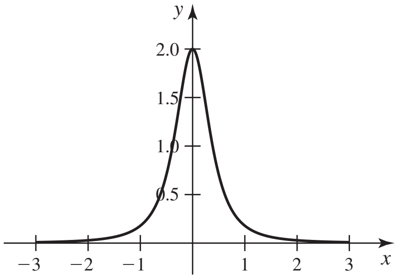
14.5.64

a.



b. $\kappa = \dfrac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \dfrac{2}{(1+4x^2)^{3/2}}$.

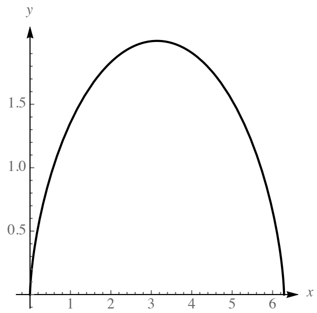
c.



d. $\kappa'(x) = -\dfrac{24x}{(1+4x^2)^{5/2}}$. κ has a maximum at 0. $\kappa''(x) = \dfrac{24(16x^2-1)}{(1+4x^2)^{7/2}}$. κ has inflection points at $x = \pm 1/4$.

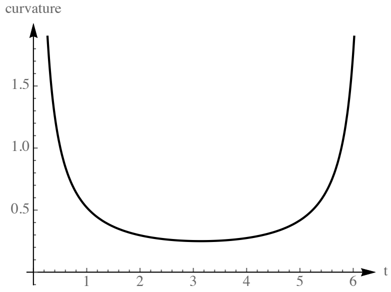
e. Symmetry of $y = x^2$ implies symmetry in κ, which does occur. The parabola appears to have greater curvature near 0 and less near the endpoints, which is consistent with the graph of κ.

14.5.65



a.

b. $\kappa = \dfrac{|f'g'' - f''g'|}{(f'(x)^2 + g'(x)^2)^{3/2}} = \dfrac{|(1-\cos t)\cos t - \sin t \sin t|}{((1-\cos t)^2 + \sin^2 t)^{3/2}} = \dfrac{1-\cos t}{2\sqrt{2}(1-\cos t)^{3/2}} = \dfrac{1}{2\sqrt{2}\sqrt{1-\cos t}}$.

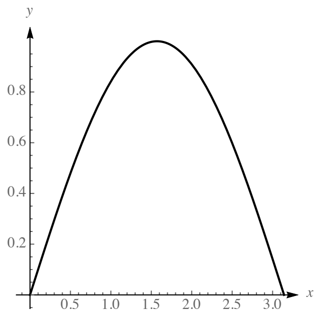


c.

d. $\kappa'(t) = -\dfrac{\sin t}{4\sqrt{2}(1-\cos t)^{3/2}}$. κ has a minimum at π. $\kappa''(t) = \dfrac{\sin^2\left(\dfrac{t}{2}\right)(\cos t + 3)}{4\sqrt{2}(1-\cos(t))^{5/2}}$. κ has no inflection points on the given interval.

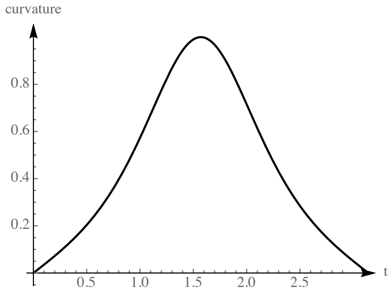
e. Symmetry of the given curve about π (on the interval $(0, 2\pi)$ implies symmetry in κ, which does occur. The curve is flatter near π and more curved near 0 and 2π.

14.5.66



a.

b. $\kappa = \dfrac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} = \dfrac{\sin x}{(1+\cos^2 x)^{3/2}}$.



c.

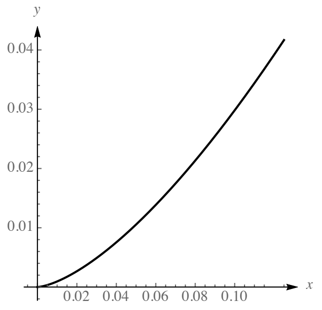
d. $\kappa'(x) = -\dfrac{\cos(x)(\cos(2x) - 3)}{\left(\cos^2(x) + 1\right)^{5/2}}$. κ has a maximum at $\pi/2$.

$$\kappa''(x) = -\frac{2\sqrt{2}(-4\sin(x) - 19\sin(3x) + \sin(5x))}{(\cos(2x)+3)^{7/2}}.$$

κ has inflection points at approximately 1.122 and $\pi - 1.122$.

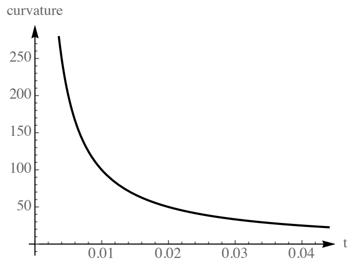
e. Symmetry of the given curve about $\pi/2$ (on the interval $(0, \pi)$ implies symmetry in κ, which does occur. The curve is flatter near 0 and π and more curved near $\pi/2$.

14.5.67



a.

b. $\kappa = \dfrac{|f'g'' - f''g'|}{(f'(x)^2 + g'(x)^2)^{3/2}} = \dfrac{\left|t \cdot 2t - 1 \cdot t^2\right|}{(t^2 + t^4)^{3/2}} = \dfrac{1}{t(1+t^2)^{3/2}}$.



c.

d. $\kappa'(t) = \dfrac{-4t^2 - 1}{t^2\left(t^2+1\right)^{5/2}}$. κ has no extrema. $\kappa''(t) = \dfrac{20t^4 + 7t^2 + 2}{t^3\left(t^2+1\right)^{7/2}}$. κ has no inflection points.

e. The curve gets flatter as $t \to \infty$.

14.5.68 $y'(x) = 1/x$ and $y''(x) = -1/x^2$. So $\kappa = \dfrac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \dfrac{1/x^2}{(1+(1/x^2))^{3/2}} = \dfrac{x}{(x^2+1)^{3/2}}$.

$\kappa'(x) = \dfrac{1-2x^2}{\left(x^2+1\right)^{5/2}}$, which is 0 for $x > 0$ only for $x = \sqrt{2}/2$. The first derivative test shows that this is where the maximum curvature exists, and the value of the maximum curvature is $\kappa(\sqrt{2}/2) = 2\sqrt{3}/9$.

14.5.69 $y'(x) = e^x$ and $y''(x) = e^x$. So $\kappa = \dfrac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \dfrac{e^x}{(1+(e^{2x}))^{3/2}}$.

$\kappa'(x) = \dfrac{e^x - 2e^{3x}}{\left(e^{2x}+1\right)^{5/2}}$, which is 0 for $x = -\dfrac{\ln 2}{2}$. The first derivative test shows that this is where the maximum curvature exists, and the value of the maximum curvature is $\kappa((-\ln 2)/2) = 2\sqrt{3}/9$.

14.5.70 By example 3, the curvature of a circle centered at the origin is the reciprocal of the radius of the circle. Because curvature is independent of the coordinate system, the curvature of the circle of curvature is the reciprocal of its radius, but by definition, its curvature is κ, so its radius is $\dfrac{1}{\kappa}$.

14.5.71 $\mathbf{r}'(t) = \langle 1, 2t\rangle$ and $\mathbf{r}''(t) = \langle 0, 2\rangle$. Thus $\kappa = \dfrac{2}{(1+4t^2)^{3/2}}$. At $t = 0$, we have $\kappa = 2$. So we are seeking a circle of radius $1/2$. The center of the osculating circle is the point along the normal line to the curve at $(0,0)$ which is $1/2$ unit from $(0,0)$ and is on the inside of the curve, so it is $(0, 1/2)$. The equation is $x^2 + (y - (1/2))^2 = \dfrac{1}{4}$.

14.5.72 $y' = 1/x$ and $y'' = -1/x^2$, so $\kappa(x) = \dfrac{\left|1/x^2\right|}{(1+(1/x)^2)^{3/2}} = \dfrac{x}{(x^2+1)^{3/2}}$. At $x = 1$ we have $\kappa = \dfrac{1}{2\sqrt{2}}$, so the radius of the osculating circle is $2\sqrt{2}$. The center of the osculating circle is the point along the the normal line to the curve at $(1,0)$ which is $2\sqrt{2}$ units from $(1,0)$ and is on the inside of the curve, so it is $(3,-2)$. The equation of the osculating circle is $(x-3)^2 + (y+2)^2 = 8$.

14.5.73 $\mathbf{r}'(t) = \langle 1-\cos t, \sin t\rangle$ and $\mathbf{r}''(t) = \langle \sin t, -\cos t\rangle$. Thus $\kappa(\pi) = \dfrac{(2-0)}{(2^2+0^2)^{3/2}} = \dfrac{2}{8} = \dfrac{1}{4}$, so the radius of the osculating circle is 4. The center of the osculating circle is the point along the normal line to the curve at $(\pi, 2)$ which is 4 units from $(\pi, 2)$ and is on the inside of the curve, so it is $(\pi, -2)$. The equation of the osculating circle is $(x-\pi)^2 + (y+2)^2 = 16$.

14.5.74

a. There is an instantaneous change in curvature from 0 to $\dfrac{1}{a}$ at the origin.

b. $\mathbf{r}'(t) = \left\langle \cos\dfrac{t^2}{2}, \sin\dfrac{t^2}{2}\right\rangle$ by the Fundamental Theorem of Calculus. So $|\mathbf{v}(t)| = 1$. Then $\kappa = |T'(t)| = \left|\left\langle -t\sin\dfrac{t^2}{2}, t\cos\dfrac{t^2}{2}\right\rangle\right| = t$. The curvature of the easement curve is 0 at the origin and gradually increases.

c. We would need the radius to be $\dfrac{1}{\kappa} = \dfrac{1}{t} = \dfrac{1}{1.2} \approx 0.83$.

14.5.75 $y' = n\cos nx$ and $y'' = -n^2 \sin nx$, so $\kappa(\pi/2n) = \dfrac{\left|-n^2\sin(\pi/2)\right|}{(1+(n^2\cos^2(\pi/2)))^{3/2}} = \dfrac{n^2}{(1+0)^{3/2}} = n^2$. This increases as n increases.

14.5.76 Note that $\mathbf{r}'(t) = \langle 1, 2t\rangle$, so $\mathbf{T} = \dfrac{1}{\sqrt{1+4t^2}}\langle 1, 2t\rangle$. So $\mathbf{T}'(t) = \left\langle -\dfrac{4t}{(4t^2+1)^{3/2}}, \dfrac{2}{(4t^2+1)^{3/2}}\right\rangle$, so $\mathbf{N} = \dfrac{1}{\sqrt{4t^2+1}}\langle -2t, 1\rangle$. Then

$$\mathbf{a} = \frac{2}{\sqrt{1+4t^2}}\left(\left\langle -\frac{2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}}\right\rangle + 2t\left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}\right\rangle\right) = \langle 0, 2\rangle.$$

14.5.77

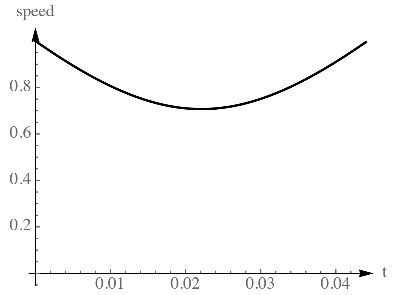
a. $\mathbf{r}'(t) = \langle V_0\cos\alpha, V_0\sin\alpha - gt\rangle$, so the speed is

$$\sqrt{V_0^2\cos^2\alpha + V_0^2\sin^2\alpha - 2gtV_0\sin\alpha + g^2t^2},$$

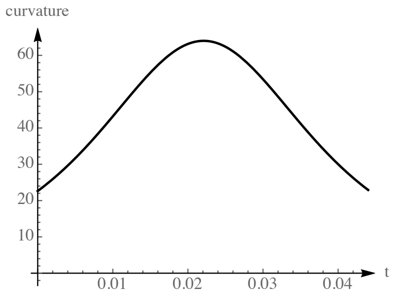
which can be written

$$\sqrt{V_0^2 - 2gtV_0\sin\alpha + g^2t^2}.$$

The graph shown is for $V_0 = 1$, $g = 32$, and $\alpha = 45$ degrees.



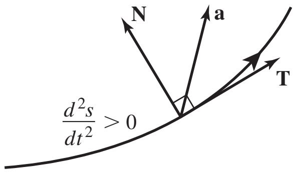
b. $\mathbf{a} = \langle 0, -g\rangle$. We have $\mathbf{a}\times\mathbf{v} = \langle 0, 0, gV_0\cos\alpha\rangle$. So $\kappa = \dfrac{|gV_0\cos\alpha|}{(V_0^2 - 2gtV_0\sin\alpha + g^2t^2)^{3/2}}$. The graph shown is for $V_0 = 1$, $g = 32$, and $\alpha = 45$ degrees.



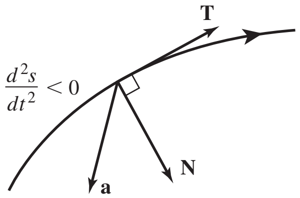
c. The speed has a minimum and the curvature has a maximum at the halfway time of the flight, namely at $\frac{V_0 \sin\alpha}{g}$.

14.5.78

Suppose $\frac{d^2s}{dt^2} > 0$. Recall that $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, and that $a_T = \frac{d^2s}{dt^2} > 0$. Also $a_N = \kappa\left|\mathbf{v}\right|^2 > 0$. So $\mathbf{a}$ is a linear combination of $\vec{T}$ and $\mathbf{N}$ with positive coefficients, so it lies between the two of them.



Suppose $\frac{d^2s}{dt^2} < 0$. Recall that $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, and that $a_T = \frac{d^2s}{dt^2} < 0$. Also $a_N = \kappa\left|\mathbf{v}\right|^2 > 0$. So $\mathbf{a}$ is a linear combination of $\mathbf{T}$ and $\mathbf{N}$ with a negative coefficient on $\mathbf{T}$ and a positive coefficient on $\mathbf{N}$, so it does not lie between the two of them.



14.5.79 $\mathbf{r}'(t) = \left\langle pbt^{p-1}, pdt^{p-1}, pft^{p-1}\right\rangle = pt^{p-1}\langle b,d,f\rangle$. $\mathbf{r}''(t) = p(p-1)t^{p-2}\langle b,d,f\rangle$. Because for any t, $\mathbf{r}''(t)$ and $\mathbf{r}'(t)$ are multiples of $\langle b,d,f\rangle$, their cross product is $\langle 0,0,0\rangle$. Thus $\kappa = \frac{0}{\left|\mathbf{v}\right|^3} = 0$. The given curve represents a straight line, which has zero curvature.

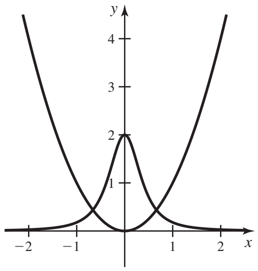
14.5.80 $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \frac{d\mathbf{T}}{ds}\left|\mathbf{r}'(t)\right|$. So

$$\frac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{\frac{d\mathbf{T}}{ds}\left|\mathbf{r}'(t)\right|}{\left|\frac{d\mathbf{T}}{ds}\left|\mathbf{r}'(t)\right|\right|} = \mathbf{N}.$$

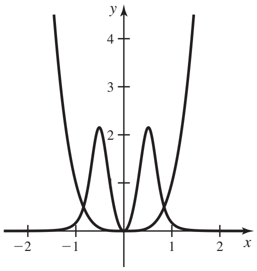
14.5.81

a. $f_n'(x) = 2nx^{2n-1}$ and $f_n''(x) = (2n)(2n-1)x^{2n-2}$. $\kappa = \frac{\left|2n(2n-1)x^{2n-2}\right|}{(1+4n^2x^{4n-2})^{3/2}}$. So $\kappa_1(x) = \frac{2}{(1+4x^2)^{3/2}}$, $\kappa_2(x) = \frac{12x^2}{(1+16x^6)^{3/2}}$, and $\kappa_3(x) = \frac{30x^4}{(1+36x^{10})^{3/2}}$.

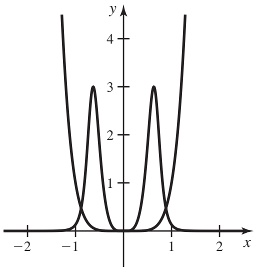
b.



$\kappa_1(x)$



$\kappa_2(x)$



$\kappa_3(x)$

Note that the curves are symmetric about the y-axis.

c. $\kappa'(x) = -\dfrac{4n(2n-1)x^{2n-1}\left(2n^2(4n-1)x^{4n}-(n-1)x^2\right)}{\left(4n^2x^{4n}+x^2\right)^2\sqrt{4n^2x^{4n-2}+1}}$. By symmetry, we can concentrate on the critical points for $x > 0$. We have $\kappa'(x) = 0$ for $x > 0$ and $n > 1$, when

$$x = 2^{\frac{1}{2-4n}}\left(\frac{\sqrt{n^2(4n-1)}}{\sqrt{n-1}}\right)^{\frac{1}{1-2n}}.$$

For $n = 1$, the maximum occurs at 0. For $n = 2$, it occurs at $\dfrac{1}{\sqrt{2}\cdot 7^{1/6}}$. For $n = 3$, it occurs at $\dfrac{1}{3^{1/5}11^{1/10}}$.

d. If the maximum curvature for f_n occurs at $\pm z_n$, then $\lim_{n\to\infty} z_n = 1$.

14.5.82

a. Write $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$. Then $\mathbf{T}\times\mathbf{a} = \mathbf{T}\times(a_T\mathbf{T}+a_N\mathbf{N}) = a_t(\mathbf{T}\times\mathbf{T}) + a_N(\mathbf{T}\times\mathbf{N}) = 0 + a_N\mathbf{B}$. Then we have $\dfrac{\mathbf{v}}{|\mathbf{v}|}\times\mathbf{a} = \kappa\,|\mathbf{v}|^2\,\mathbf{B}$, so $\mathbf{v}\times\mathbf{a} = \kappa\,|\mathbf{v}|^3\,\mathbf{B}$.

b. Taking norms of the last equation in part a, we have $|\mathbf{v}\times\mathbf{a}| = \kappa\,|\mathbf{v}|^3\cdot 1$, so $\kappa = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$.

14.5.83 We have that $\mathbf{B} = \dfrac{\mathbf{v}\times\mathbf{a}}{\kappa\,|\mathbf{v}|^3}$ and $\kappa = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$, so $\mathbf{B} = \dfrac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$.

14.5.84 $\tau = -\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{N} = -\dfrac{d\mathbf{B}/dt}{ds/dt}\cdot\mathbf{N} = -\dfrac{d\mathbf{B}/dt}{|\mathbf{v}|}\cdot\mathbf{N}$.

Chapter Fourteen Review

1

a. False. $\mathbf{r}(0) = \langle 1,1,0\rangle + \mathbf{C} = \langle 0,0,0\rangle$, so $\mathbf{C} = \langle -1,-1,0\rangle$.

b. True. The curvature of a circle is always the reciprocal of the radius.

c. True. Note that $\dfrac{x^2}{9} + \dfrac{z^2}{36} = 1$.

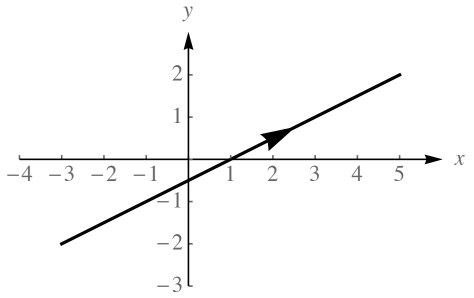
d. True. This follows because $\displaystyle\int \mathbf{r}'(t)\,dt = \int \langle 0,0,0\rangle\,dt = \langle 0,0,0\rangle + \mathbf{C} = \langle a,b,c\rangle$ for some real numbers a, b, and c.

e. False. Its length is $\sqrt{169\sin^2 t + 169\cos^2 t} = 13 \neq 1$.

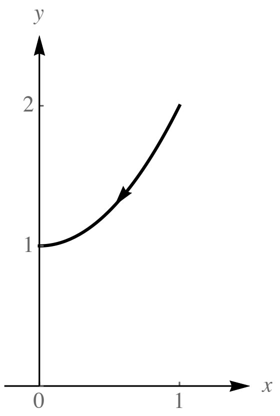
f. False. For example, for the curve $\mathbf{r}(t) = \left\langle t^2, t\right\rangle$, we have $\mathbf{N} = \dfrac{1}{\sqrt{1+4t^2}}\langle 1,-2t\rangle$. So if, for example, $t = 2$, we have $\mathbf{r}(2) = \langle 4,2\rangle$ and $\mathbf{N} = \dfrac{1}{\sqrt{17}}\langle 1,-4\rangle$, which aren't parallel.

2 This is a circle of radius one centered at $(0,2,0)$ and sitting in the plane $y = 2$.

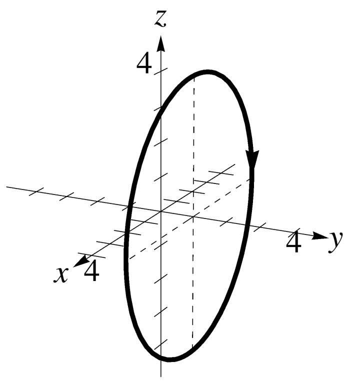
3 This is a straight line.



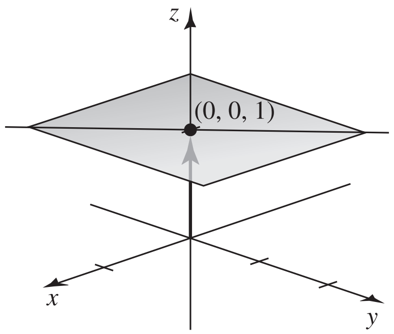
4 Note that if $x = \cos t$, then $y = 1 + x^2$. This is a portion of a parabola, from $(1, 2)$ to $(0, 1)$.



5 Note that $x^2 + z^2 = 16$. This is a circle of radius 4 centered at $(0, 1, 0)$ and sitting in the plane $y = 1$.



6 Note that if $\langle x, y, z\rangle = \langle e^t, 2e^t, 1\rangle$, then $y = 2x$ and $z = 1$. So this is a line in the plane $z = 1$.



7 Note that $x^2 + y^2 + z^2 = 2$, so this curve lies on a sphere of radius 2. Also, every point satisfies $z = y$, so it is a tilted circle centered at the origin of radius $\sqrt{2}$, sitting in the plane $x = z$.

8 $\mathbf{r}(t) = (1-t)\langle 2, -3, 0\rangle + t\langle 1, 4, 9\rangle = \langle 2-t, -3+7t, 9t\rangle$, $0 \le t \le 1$.

9 A direction vector for the line is $\langle 1, 5, 2\rangle \times \langle 1, 0, 3\rangle = \langle 15, -1, -5\rangle$. So the line is given by $\mathbf{r}(t) = \langle 4+15t, -2-t, 3-5t\rangle$.

10 Let $x = 3\cos t + 2$ and $y = 1$ and $z = 3\sin t$. Then $(x-2)^2 + z^2 = 9$, as desired. So $\mathbf{r}(t) = \langle 3\cos t + 2, 1, 3\sin t\rangle$, $0 \le t \le 2\pi$.

11 Let $x = 2$, $y = 3\cos t$, and $z = 4\sin t$. Then $\dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$, as desired. So $\mathbf{r}(t) = \langle 2, 3\cos t, 4\sin t\rangle$, $0 \le t \le 2\pi$.

12 Let $x = t$. Then we must have $y = x^2 = t^2$ and $z = x = t$. So $\mathbf{r}(t) = \langle t, t^2, t\rangle$.

13 Let $x = \cos t$ and $y = \sin t$. Then $x^2 + y^2 = 1$. Because $z = y$, we must also have $z = \sin t$. Thus $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t\rangle$, $0 \le t \le 2\pi$.

14 We must have $x^2 - 5y^2 = 10x^2 + 4y^2 - 36$, so $9x^2 + 9y^2 = 36$ and $x^2 + y^2 = 4$. Let $x = 2\cos t$ and $y = 2\sin t$. Then $z = 4\cos^2 t - 20\sin^2 t = 4\cos^2 t - 20(1-\cos^2 t) = 24\cos^2 t - 20$. So $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 24\cos^2 t - 20\rangle$, $0 \le t \le 2\pi$.

15 We must have $x^2 + 7y^2 + 2y^2 = 9$, so $x^2 + 9y^2 = 9$. Let $x = 3\cos t$ and $y = \sin t$ and $z = \sin t$. So $\mathbf{r}(t) = \langle 3\cos t, \sin t, \sin t\rangle$, $0 \le t \le 2\pi$.

16

a. $\lim_{t\to 0} \mathbf{r}(t) = \langle 1, -3\rangle$ and $\lim_{t\to\infty} \mathbf{r}(t)$ does not exist.

b. $\mathbf{r}'(t) = \langle 1, 2t\rangle$ so $\mathbf{r}'(0) = \langle 1, 0\rangle$.

c. $\mathbf{r}''(t) = \langle 0, 2\rangle$.

d. $\displaystyle\int \mathbf{r}(t)\,dt = \langle t^2/2 + t, t^3 - 3t\rangle + \langle C_1, C_2\rangle$.

17

a. $\lim_{t\to 0} \mathbf{r}(t) = \langle 1, 0\rangle$ and $\lim_{t\to\infty} \mathbf{r}(t) = \langle 0, 1\rangle$.

b. $\mathbf{r}'(t) = \left\langle \dfrac{-2}{(2t+1)^2}, \dfrac{1}{(1+t)^2}\right\rangle$ so $\mathbf{r}'(0) = \langle -2, 1\rangle$.

c. $\mathbf{r}''(t) = \left\langle \dfrac{8}{(2t+1)^3}, \dfrac{-2}{(1+t)^3}\right\rangle$.

d. $\int \mathbf{r}(t)\,dt = \left\langle \frac{1}{2}\ln|2t+1|, t-\ln|t+1| \right\rangle + \langle C_1, C_2\rangle$.

18

a. $\lim_{t\to 0}\mathbf{r}(t) = \langle 1,0,0\rangle$ and $\lim_{t\to\infty}\mathbf{r}(t) = \left\langle 0,0,\frac{\pi}{2}\right\rangle$.

b. $\mathbf{r}'(t) = \left\langle -2e^{-2t}, e^{-t}-te^{-t}, \frac{1}{1+t^2}\right\rangle$ so $\mathbf{r}'(0) = \langle -2,1,1\rangle$.

c. $\mathbf{r}''(t) = \left\langle 4e^{-2t}, te^{-t}-2e^{-t}, \frac{2t}{(1+t^2)^2}\right\rangle$.

d. $\int \mathbf{r}(t)\,dt = \left\langle -\frac{1}{2}e^{-2t}, (1+t)e^{-t}, t\tan^{-1}(t) - \frac{1}{2}\ln\left(t^2+1\right)\right\rangle + \langle C_1, C_2\rangle$.

19

a. $\lim_{t\to 0}\mathbf{r}(t) = \langle 0,3,0\rangle$ and $\lim_{t\to\infty}\mathbf{r}(t)$ does not exist.

b. $\mathbf{r}'(t) = \langle 2\cos 2t, -12\sin 4t, 1\rangle$ so $\mathbf{r}'(0) = \langle 2,0,1\rangle$.

c. $\mathbf{r}''(t) = \langle -4\sin 2t, -48\cos 4t, 0\rangle$.

d. $\int \mathbf{r}(t)\,dt = \left\langle -\frac{1}{2}\cos 2t, \frac{3}{4}\sin 4t, \frac{1}{2}t^2\right\rangle + \langle C_1, C_2\rangle$.

20 $\int_1^3 \left\langle 6t^2, 4t, 1/t\right\rangle\,dt = \left\langle 2t^3, 2t^2, \ln|t|\right\rangle\Big|_1^3 = \langle 54, 18, \ln 3\rangle - \langle 2,2,0\rangle = \langle 52, 16, \ln 3\rangle$.

21 $\int_{-1}^1 \left\langle \sin \pi t, 1, \frac{2}{(1+t^2)}\right\rangle\,dt = \left\langle \frac{-\cos\pi t}{\pi}, t, 2\tan^{-1}t\right\rangle\Big|_{-1}^1 = \langle 0,2,\pi\rangle$.

22 $\mathbf{u}(0)\cdot\mathbf{v}'(0) + \mathbf{u}'(0)\cdot\mathbf{v}(0) = \langle 2,7,0\rangle\cdot\langle 5,0,3\rangle + \langle 3,1,2\rangle\cdot\langle 3,-1,0\rangle = 10+0+0+9-1+0 = 18$.

23 $\mathbf{u}(0)\times\mathbf{v}'(0) + \mathbf{u}'(0)\times\mathbf{v}(0) = \langle 2,7,0\rangle\times\langle 5,0,3\rangle + \langle 3,1,2\rangle\times\langle 3,-1,0\rangle = \langle 21,-6,-35\rangle. + \langle 2,6,-6\rangle = \langle 23,0,-41\rangle$.

24 $\mathbf{u}'(0)\cdot 5e^{5\cdot 0} = \langle 3,1,2\rangle\cdot 5 = \langle 15,5,10\rangle$.

25 $\int \left\langle 1, \sin 2t, \sec^2 t\right\rangle\,dt = \left\langle t, -\frac{1}{2}\cos 2t, \tan t\right\rangle + \langle C_1, C_2, C_3\rangle$. Because $\mathbf{r}(0) = \langle 2,2,2\rangle$, we have $C_1 = 2$, $C_2 = \frac{5}{2}$, and $C_3 = 2$. So $\mathbf{r}(t) = \left\langle t+2, -\frac{1}{2}\cos 2t + \frac{5}{2}, \tan t + 2\right\rangle$.

26 $\int \left\langle e^t, 2e^{2t}, 6e^{3t}\right\rangle\,dt = \left\langle e^t, e^{2t}, 2e^{3t}\right\rangle + \langle C_1, C_2, C_3\rangle$. Because $\mathbf{r}(0) = \langle 1,3,-1\rangle$, we have $C_1 = 0$, $C_2 = 2$, and $C_3 = -3$. So $\mathbf{r}(t) = \left\langle e^t, e^{2t}+2, 2e^{3t}-3\right\rangle$.

27 $\int \left\langle \frac{4}{1+t^2}, 2t+1, 3t^2\right\rangle\,dt = \left\langle 4\tan^{-1}t, t^2+t, t^3\right\rangle + \langle C_1, C_2, C_3\rangle$. Because $\mathbf{r}(1) = \langle 0,0,0\rangle$, we have $C_1 = -\pi$, $C_2 = -2$, and $C_3 = -1$. So $\mathbf{r}(t) = \left\langle 4\tan^{-1}t - \pi, t^2+t-2, t^3-1\right\rangle$.

28 $\mathbf{r}'(t) = \langle 0, 6\cos 2t, -6\sin 2t\rangle$. $|\mathbf{r}'(t)| = 6$. So $\mathbf{T}(t) = \langle 0, \cos 2t, -\sin 2t\rangle$. At $t = \pi/4$ we have $T(\pi/4) = \langle 0,0,-1\rangle$.

29 $\mathbf{r}'(t) = \left\langle 2e^t, 2e^{2t}, 1\right\rangle$. $|\mathbf{r}'(t)| = \sqrt{(2e^{2t}+1)^2} = 2e^{2t}+1$. So $\mathbf{T}(t) = \left\langle \frac{2e^t}{2e^{2t}+1}, \frac{2e^{2t}}{2e^{2t}+1}, \frac{1}{2e^{2t}+1}\right\rangle$. At $t = 0$ we have $T(0) = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right\rangle$.

30

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 5t^2, 2t+10\right\rangle$, so the speed is $\sqrt{25t^4+4t^2+40t+100}$.

b. $\mathbf{a}(t) = \langle 10t, 2\rangle$.

31

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 4e^{4t}, 4e^{4t}, 2e^{4t}\right\rangle$, so the speed is $\sqrt{16e^{8t}+16e^{8t}+4e^{8t}} = \sqrt{36e^{8t}} = 6e^{4t}$.

b. $\mathbf{a}(t) = \left\langle 16e^{4t}, 16e^{4t}, 8e^{4t}\right\rangle$.

32 $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 1, 4\rangle\; dt = \langle t, 4t\rangle + \langle C_1, C_2\rangle$. Because $\mathbf{v}(0) = \langle 4, 3\rangle$, we have $\mathbf{v}(t). = \langle t+4, 4t+3\rangle$.

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle t+4, 4t+3\rangle\; dt = \left\langle \frac{t^2}{2}+4t, 2t^2+3t\right\rangle + \langle D_1, D_2\rangle$. Because the initial position is $\langle 0, 2\rangle$, we have $\mathbf{r}(t) = \left\langle \frac{t^2}{2}+4t, 2t^2+3t+2\right\rangle$.

33 $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle \cos t, 2\sin t\rangle\; dt = \langle \sin t, -2\cos t\rangle + \langle C_1, C_2\rangle$. Because $\mathbf{v}(0) = \langle 2, 1\rangle$, we have $\mathbf{v}(t). = \langle 2+\sin t, 3-2\cos t\rangle$.

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 2+\sin t, 3-2\cos t\rangle\; dt = \langle 2t-\cos t, 3t-2\sin t\rangle + \langle D_1, D_2\rangle$. Because the initial position is $\langle 1, 2\rangle$, we have $\mathbf{r}(t) = \langle 2t+2-\cos t, 3t+2-2\sin t\rangle$.

34 $\mathbf{r}'(t) = \langle 0, 8\cos t, -\sin t\rangle$. The vectors $\mathbf{r}'(t)$ and $\mathbf{r}(t)$ are orthogonal when $\mathbf{r}\cdot\mathbf{r}' = 0 + 64\sin t\cos t - \sin t\cos t = 63\sin t\cos t$ is zero. This occurs when either $\sin t = 0$ or $\cos t = 0$, so at $t = 0$, $t = \pi/2$, $t = 3\pi/2$ and $t = 2\pi$, which correspond to the points $(1, 0, 1)$, $(1, 8, 0)$, $(1, 0, -1)$ and $(1, -8, 0)$.

35

a. We have $\mathbf{a}(t) = \langle 0, -32\rangle$ and $\mathbf{v}(0) = 80\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle = \left\langle 40, 40\sqrt{3}\right\rangle$. So $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \langle 0, -32t\rangle + \left\langle 40, 40\sqrt{3}\right\rangle = \left\langle 40, -32t+40\sqrt{3}\right\rangle$. Then

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\,dt = \int \left\langle 40, -32t+40\sqrt{3}\right\rangle\; dt \\ &= \left\langle 40t + C_1, -16t^2+40\sqrt{3}t + C_2\right\rangle = \left\langle 40t, -16t^2+40\sqrt{3}t+3\right\rangle\end{aligned}$$

because $\mathbf{r}(0) = \langle 0, 3\rangle$.

b. The ball hits the ground when $-16t^2+40\sqrt{3}t+3 = 0$, which occurs for $t \approx 4.37$ s. At this time the ball's x-position is about $40(4.37) \approx 174.9$ ft.

c. The ball's maximum height occurs when $-32t+40\sqrt{3} = 0$, or $t = \frac{40\sqrt{3}}{32} \approx 2.17$ and is about $-16(2.17)^2 + 40\sqrt{3}(2.17) + 3 \approx 78$ ft.

36

a. We have $\mathbf{a}(t) = \langle 0, -9.8\rangle$ and $\mathbf{v}(0) = 6\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle = \left\langle 3\sqrt{3}, 3\right\rangle$. So $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \langle 0, -9.8t\rangle + \left\langle 3\sqrt{3}, 3\right\rangle = \left\langle 3\sqrt{3}, -9.8t + 3\right\rangle$. Then

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\,dt = \int \left\langle 3\sqrt{3}, -9.8t+3\right\rangle\,dt \\ &= \left\langle 3\sqrt{3}t + C_1, -4.9t^2 + 3t + C_2\right\rangle = \left\langle 3\sqrt{3}t, -4.9t^2 + 3t + 2\right\rangle\end{aligned}$$

because $\mathbf{r}(0) = \langle 0, 2\rangle$.

b. The rock hits the ground when $-4.9t^2 + 3t + 2 = 0$, which occurs for $t \approx 1.01$ s. At this time the rock's x-position is about $3\sqrt{3}(1.01) \approx 5.25$ ft.

c. The rock's maximum height occurs when $-9.8t + 3 = 0$, or $t = \frac{3}{9.8} \approx 0.306$ and is about $-4.9(0.306)^2 + 3(0.306) + 2 \approx 2.46$ ft.

37

a. We have $\mathbf{a}(t) = \langle 4, 0, -32\rangle$ and $\mathbf{v}(0) = \langle 40, 20, 40\rangle$. So $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \langle 4t, 0, -32t\rangle + \langle 40, 20, 40\rangle = \langle 4t + 40, 20, 40 - 32t\rangle$. Then

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 4t+40, 20, 40-32t\rangle\,dt = \left\langle 2t^2 + 40t, 20t, -16t^2 + 40t + 2\right\rangle$$

because $\mathbf{r}(0) = \langle 0, 0, 2\rangle$.

b. The ball hits the ground when $z = 0$, or $-16t^2 + 40t + 2 = 0$, or $t \approx 2.549$ s.

c. The ball's position at $t = 2.549$ is approximately $\langle 115, 51, 0\rangle$, so its distance from home plate is about $\sqrt{115^2 + 51^2} \approx 126$ ft.

38 We need $\frac{40^2 \sin 2\alpha}{9.8} = 100$, so $\sin 2\alpha = \frac{980}{1600}$. The two solutions to this equation are approximately $\alpha = 18.9°$, $71.1°$.

39

a. The trajectory is given by $\mathbf{r}(t) = \left\langle 50t, 50t - 16t^2\right\rangle$. The projectile is at $y = 30$ when $-16t^2 + 50t - 30 = 0$, which occurs at $t = \frac{1}{16}\left(25 \pm \sqrt{145}\right) \approx .81$ and 2.32. At these times, $x = 50t \approx 40.5$ and 116. The first time represents when the projectile has not yet reached the cliff, while the second time represents when the projectile lands on the cliff, so the coordinates of the landing spot are approximately $(116, 30)$.

b. The maximum height occurs where $y' = 0$, which occurs for $50 - 32t = 0$, or $t = 25/16$. The maximum height is $50 \cdot \frac{25}{16} - 16\left(\frac{25}{16}\right)^2 = \frac{625}{16} \approx 39.06$ feet.

c. As mentioned above, the flight ends at $t \approx 2.32$ seconds.

d. The length of the trajectory is $\int_0^{2.32} \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^{2.32} \sqrt{2500 + (50 - 32t)^2}\,dt$.

e. $L \approx 129$ feet.

f. Suppose the launch angle is α. Then $\mathbf{r}(t) = \left\langle 50\sqrt{2}t\cos\alpha, 50\sqrt{2}t\sin\alpha - 16t^2\right\rangle$. We want $y \geq 30$ when $x = 50$. We know that $x = 50$ when $t = \frac{\sec\alpha}{\sqrt{2}}$. At this time, we have $y = 50\tan\alpha - 8\sec^2\alpha$. This expression is greater than or equal to 30 for approximately $41.5 \leq \alpha \leq 79.4$.

40 The initial velocity of the ball is given by $\left\langle s\sqrt{3}/2, s/2\right\rangle$ where s is the initial speed of the ball. We have $\mathbf{r}(t) = \left\langle (s\sqrt{3}/2)t, -16t^2 + (s/2)t + 2\right\rangle$. We know that $(s\sqrt{3}/2)t = 10$ when $-16t^2 + (s/2)t + 2 = 0$. Solving the first equation for t gives $t = 20/(s\sqrt{3})$. Putting this into the second equation gives $-16(20/(s\sqrt{3}))^2 + (s/2)(20/(s\sqrt{3})) = -2$. Solving for s gives $s \approx 16.6$ feet per second.

41 The square of the distance from $P(3, 1, 6)$ to the curve is $s(t) = (t-3)^2 + t^4 + (3t-6)^2$. Note that $s'(t) = 2(t-3)+4t^3+6(3t-6) = 4t^3+20t-42$. This is zero for $t \approx 1.47$. So the closest point is approximately $(1, 47, 3.15, 4.4)$.

42

a. Let t_0 be the time until the ball is at the point $(18, 10)$. Using the theory from Section 14.3, the path of the ball is $\mathbf{r}(t) = \left\langle |v_0| \frac{\sqrt{2}}{2} t, -16t^2 + |v_0| \frac{\sqrt{2}}{2} t + 8 \right\rangle$. Thus $|v_0| \frac{\sqrt{2}}{2} t_0 = 18$. Thus $t_0 = \frac{18\sqrt{2}}{|v_0|}$. Because the second component of $\mathbf{r}$ is 10 at time t_0, we have

$$-16\left(\frac{18\sqrt{2}}{|v_0|}\right)^2 + 18 + 8 = 10.$$

Solving for $|v_0|$ gives $|v_0| = 18\sqrt{2}$ ft/s.

b. $\mathbf{v}(0) = 18\sqrt{2}\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle = \langle 18, 18 \rangle$.

c. We have $\mathbf{r}(t) = \langle 18t, -16t^2 + 18t + 8 \rangle$.

d. The front of the hoop is at the point $(17.25, 10)$. The distance is

$$s(t) = \sqrt{(18t - 17.25)^2 + (-16t^2 + 18t - 2)^2}.$$

e. We will minimize the square of the distance: $(18t - 17.25)^2 + (-16t^2 + 18t - 2)^2$. The derivative of this is $36(18t - 17.25) + 2(-16t^2 + 18t - 2)(-32t + 18)$, which is 0 for $t \approx 0.973$. $s(0.973) \approx 5.42$ in.

f. Yes, because the radius of the ball is less than 5.42 inches.

43 $L = \int_1^3 \sqrt{4t^2 + 8t + 4}\, dt = 2\int_1^3 \sqrt{(t+1)^2}\, dt = 2\int_1^3 (t+1)\, dt = 2\left(t^2/2 + t\right)\Big|_1^3 = 2(9/2+3-(1/2+1)) = 9 + 6 - 1 - 2 = 12$.

44 $L = \int_0^2 \sqrt{81t^7 + 9t^4}\, dt = \int_0^2 3t^2\sqrt{9t^3 + 1}\, dt$. Let $u = 9t^3 + 1$ so that $du = 27t^2\, dt$. We have $L = \frac{1}{9}\int_1^{73} u^{1/2}\, du = \frac{2}{27}\left(u^{3/2}\right)\Big|_1^{73} = \frac{2}{27}(73\sqrt{73} - 1)$.

45 $L = \int_0^{\pi/2} \sqrt{\cos^2 t + (1 - \sin t)^2 + 16}\, dt = \int_0^{\pi/2} \sqrt{18 - 2\sin t}\, dt \approx 6.42$.

46 $L = \int_0^{\pi/4} \sqrt{1 + \tan^2 t + \sec^2 t}\, dt = \int_0^{\pi/4} \sqrt{2}\sec t\, dt = \sqrt{2}\left(\ln|\sec t + \tan t|\right)\Big|_0^{\pi/4} = \sqrt{2}(\ln(\sqrt{2} + 1) - \ln(1 + 0)) = \sqrt{2}\ln(\sqrt{2} + 1)$.

47

a. $\mathbf{v}(t) = \int \left\langle 0, \sqrt{2}, 2t \right\rangle dt = \left\langle 0, \sqrt{2}t, t^2 \right\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 1, 0, 0 \rangle$, we have $\mathbf{C} = \langle 1, 0, 0 \rangle$, so $\mathbf{v}(t) = \langle 1, \sqrt{2}t, t^2 \rangle$.

b. $L = \int_0^3 \sqrt{1 + 2t^2 + t^4}\, dt = \int_0^3 (t^2 + 1)\, dt = \left(t^3/3 + t\right)\Big|_0^3 = 9 + 3 = 12$.

48 $|\mathbf{r}'(t)| = |\langle 4, -3 \rangle| = 5$. Let $s = \int_1^t 5\, du = 5t - 5$. Solving for t gives $t = \frac{s + 5}{5}$. Thus, $\mathbf{r}(s) = \langle 1 + 4(s + 5)/5, -3(s + 5)/5 \rangle = \left\langle 5 + \frac{4s}{5}, -3 - \frac{3s}{5} \right\rangle$ for $s \geq 0$.

49 $|\mathbf{r}'(t)| = \left|\left\langle 2t, 2\sqrt{2}t^{1/2}, 2\right\rangle\right| = \sqrt{4t^2+8t+4} = 2\sqrt{(t+1)^2} = 2(t+1)$. Let $s = \int_0^t 2(u+1)\,du = \left(u^2+2u\right)\Big|_0^t = t^2+2t$. Then $s+1=(t+1)^2$, so $t=\sqrt{s+1}-1$. Thus,

$$\mathbf{r}(s) = \left\langle (\sqrt{s+1}-1)^2, \frac{4\sqrt{2}}{3}(\sqrt{s+1}-1)^{3/2}, 2(\sqrt{s+1}-1)\right\rangle$$

for $s \geq 0$.

50

a. $\mathbf{r}'(t) = \langle -3\sin t, 4\cos t\rangle$, so $\mathbf{T} = \frac{1}{\sqrt{9+7\cos^2 t}}\langle -3\sin t, 4\cos t\rangle$. Because $\mathbf{N}$ is a unit vector perpendicular to $\mathbf{T}$ pointing toward the inside of the ellipse, it is $\mathbf{N} = \frac{1}{\sqrt{9+7\cos^2 t}}\langle -4\cos t, -3\sin t\rangle$. All of these are valid for $0 \leq t \leq 2\pi$.

b. $|\mathbf{r}'(t)| = \sqrt{9+7\cos^2 t}$. The derivative of this is $-\frac{7\cos t\sin t}{\sqrt{9+7\cos^2 t}}$, which is 0 at $t=0$, $t=\pi/2$, $t=\pi$, and $t=3\pi/2$. The speed is maximal at $t=0$ and $t=\pi$ and minimal at $t=\pi/2$ and $t=3\pi/2$.

c. $\kappa = \frac{|\mathbf{r}''(t)\times\mathbf{r}'(t)|}{(9+7\cos^2 t)^{3/2}} = \frac{|\langle 0,0,-12\rangle|}{(9+7\cos^2 t)^{3/2}} = \frac{12}{(9+7\cos^2 t)^{3/2}}$. The curvature is maximal at $t=\pi/2$ and $t=3\pi/2$ where the denominator is minimized, and is minimal at $t=0$ and $t=\pi$ where the denominator is maximized. Note that the velocity is maximized where the curvature is minimal, and vice versa.

d. In order for $\mathbf{r}$ and $\mathbf{N}$ to be parallel , we require $3\cos t = m\cdot -\frac{4\cos t}{\sqrt{9+7\cos^2 t}}$ and $4\sin t = m\cdot -\frac{3\sin t}{\sqrt{9+7\cos^2 t}}$ for some constant m. This only occurs when either $\sin t=0$ or $\cos t=0$. (If $\sin t=0$, then $m=-3$ and if $\cos t=0$ then $m=-4$.) So the corresponding points on $\mathbf{r}$ are $(3,0)$, $(0,4)$, $(-3,0)$, and $(0,-4)$.

51

a. $\mathbf{r}'(t) = \langle -6\sin t, 3\cos t\rangle$, so $\mathbf{T} = \frac{1}{\sqrt{1+3\sin^2 t}}\langle -2\sin t, \cos t\rangle$.

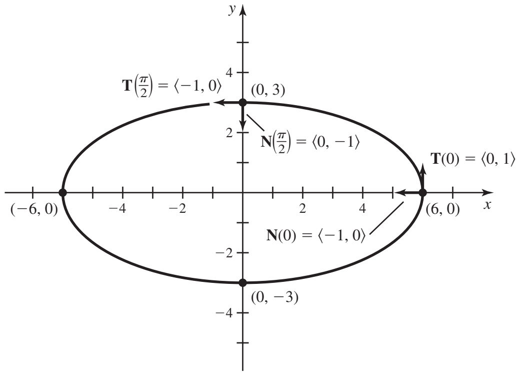
b. $\kappa = \frac{|\mathbf{r}''(t)\times\mathbf{r}'(t)|}{(3\sqrt{1+3\sin^2 t})^3} = \frac{|\langle 0,0,-18\rangle|}{(3\sqrt{1+3\sin^2 t})^3} = \frac{2}{3(\sqrt{1+3\sin^2 t})^3}$.

c. Note that $\frac{1}{\sqrt{1+3\sin^2 t}}\langle -\cos t, -2\sin t\rangle$ has length one, and is perpendicular to $\mathbf{T}$ (see part [d]), and points to the inside of the curve, so it is $\mathbf{N}$.

d. $\left|\frac{1}{\sqrt{1+3\sin^2 t}}\langle -\cos t, -2\sin t\rangle\right| = \frac{\sqrt{1+3\sin^2 t}}{\sqrt{1+3\sin^2 t}} = 1$ and

$$\mathbf{T}\cdot\mathbf{N} = \frac{1}{\sqrt{1+3\sin^2 t}}(2\sin t\cos t - 2\sin t\cos t) = 0.$$

e.


y
x
4
2
−2
−4
−4
−2
2
4
$\mathbf{T}\left(\frac{\pi}{2}\right) = \langle -1, 0\rangle$
(0, 3)
$\mathbf{N}\left(\frac{\pi}{2}\right) = \langle 0, -1\rangle$
$\mathbf{T}(0) = \langle 0, 1\rangle$
(−6, 0)
(6, 0)
$\mathbf{N}(0) = \langle -1, 0\rangle$
(0, −3)

52

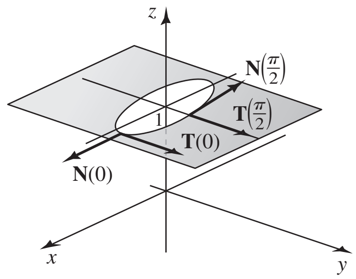
a. $\mathbf{r}'(t) = \langle -\sin t, 2\cos t, 0\rangle$, so $\mathbf{T} = \dfrac{1}{\sqrt{1+3\cos^2 t}}\langle -\sin t, 2\cos t, 0\rangle$.

b. $\kappa = \dfrac{|\mathbf{r}''(t)\times\mathbf{r}'(t)|}{(\sqrt{1+3\cos^2 t})^3} = \dfrac{|\langle 0,0,-2\rangle|}{(\sqrt{1+3\cos^2 t})^3} = \dfrac{2}{(\sqrt{1+3\cos^2 t})^3}$.

c. Note that $\dfrac{1}{\sqrt{1+3\cos^2 t}}\langle -2\cos t, -\sin t, 0\rangle$ has length one, and is perpendicular to **T** (see part [d]), and points to the inside of the curve, so it is **N**.

d. $\left|\dfrac{1}{\sqrt{1+3\cos^2 t}}\langle -2\cos t, -\sin t\rangle\right| = \dfrac{\sqrt{1+3\cos^2 t}}{\sqrt{1+3\cos^2 t}} = 1$ and $\mathbf{T}\cdot\mathbf{N} = \dfrac{1}{\sqrt{1+3\cos^2 t}}(2\sin t\cos t - 2\sin t\cos t) = 0$.

e.


z
x
y
1
$\mathbf{N}\left(\frac{\pi}{2}\right)$
$\mathbf{T}\left(\frac{\pi}{2}\right)$
T(0)
N(0)

53

a. $\mathbf{r}'(t) = \left\langle -\sin t, -2\sin t, \sqrt{5}\cos t\right\rangle$, so

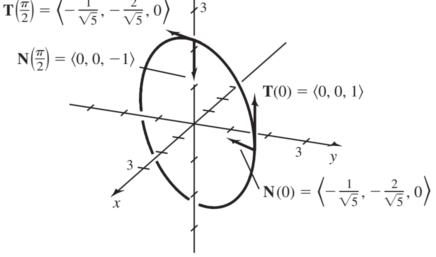
$$\mathbf{T} = \frac{1}{\sqrt{5}}\left\langle -\sin t, -2\sin t, \sqrt{5}\cos t\right\rangle.$$

b. $\kappa = \dfrac{|\mathbf{r}''(t)\times\mathbf{r}'(t)|}{(\sqrt{5})^3} = \dfrac{\left|\left\langle -2\sqrt{5}, \sqrt{5}, 0\right\rangle\right|}{(\sqrt{5})^3} = \dfrac{1}{\sqrt{5}}$.

c. $\dfrac{dT}{dt} = \dfrac{1}{\sqrt{5}}\left\langle -\cos t, -2\cos t, -\sqrt{5}\sin t\right\rangle = \mathbf{N}$.

d. $\left|\dfrac{1}{\sqrt{5}}\left\langle -\cos t, -2\cos t, -\sqrt{5}\sin t\right\rangle\right| = \dfrac{\sqrt{5}}{\sqrt{5}} = 1$ and $\mathbf{T}\cdot\mathbf{N} = \dfrac{1}{\sqrt{5}}(\sin t\cos t + 4\sin t\cos t - 5\sin t\cos t) = 0$.

e.


z
3
$\mathbf{T}\left(\frac{\pi}{2}\right) = \left\langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right\rangle$
$\mathbf{N}\left(\frac{\pi}{2}\right) = \langle 0, 0, -1\rangle$
$\mathbf{T}(0) = \langle 0, 0, 1\rangle$
3
y
3
x
$\mathbf{N}(0) = \left\langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right\rangle$

54

a. $\mathbf{r}'(t) = \langle 1, -2\sin t, 2\cos t\rangle$, so

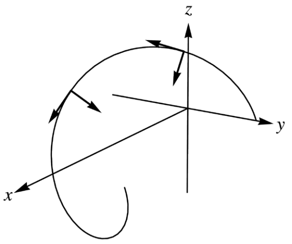
$$\mathbf{T} = \frac{1}{\sqrt{5}}\langle 1, -2\sin t, 2\cos t\rangle .$$

b. $\kappa = \dfrac{|\mathbf{r}''(t) \times \mathbf{r}'(t)|}{(\sqrt{5})^3} = \dfrac{|\langle -4, -2\sin(t), 2\cos(t)\rangle|}{(\sqrt{5})^3} = \dfrac{2}{5}$.

c. $\dfrac{dT}{dt} = \dfrac{1}{\sqrt{5}}\langle 0, -2\cos t, -2\sin t\rangle$, so

$$\mathbf{N} = \langle 0, -\cos t, -\sin t\rangle .$$

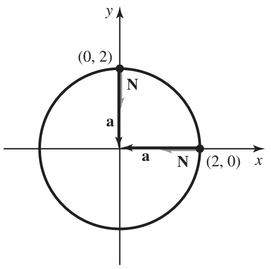
d. $|\mathbf{N}| = \sqrt{\cos^2 t + \sin^2 t} = 1$ and $\mathbf{T} \cdot \mathbf{N} = \dfrac{1}{\sqrt{5}}(0 + 2\sin t\cos t - 2\sin t\cos t) = 0$.

e. 

55

a. $\mathbf{v}(t) = \langle -2\sin t, 2\cos t\rangle$, and $\mathbf{a}(t) = \langle -2\cos t, -2\sin t\rangle$. Because $\mathbf{v} \cdot \mathbf{a} = 0$, we have $a_T = 0$. Note that $\mathbf{a} \times \mathbf{v} = \langle -2\cos t, -2\sin t, 0\rangle \times \langle -2\sin t, 2\cos t, 0\rangle = \langle 0, 0, -4\rangle$, so $a_N = \dfrac{4}{2} = 2$. We have $\mathbf{a} = \langle -2\cos t, -2\sin t\rangle = 2\mathbf{N} + 0 \cdot \mathbf{T}$.

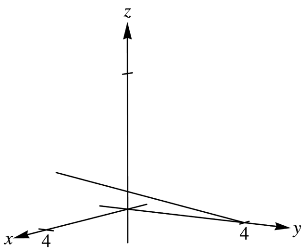
b. Note that at $t = 0$ we have $\mathbf{a} = \langle -2, 0\rangle = 2\langle -1, 0\rangle = 2\mathbf{N}$, and at $t = \pi/2$ we have $\mathbf{a} = \langle 0, -2\rangle = 2\langle 0, -1\rangle = 2\mathbf{N}$.



56

a. $\mathbf{v}(t) = \langle 3, -1, 1\rangle$, and $\mathbf{a}(t) = \langle 0, 0, 0\rangle$. Because $\mathbf{v} \cdot \mathbf{a} = 0$, we have $a_T = 0$. Note that $\mathbf{a} \times \mathbf{v} = \langle 0, 0, 0\rangle$, so $a_N = 0$. We have $\mathbf{a} = \langle 0, 0, 0\rangle = 0 \cdot \mathbf{N} + 0 \cdot \mathbf{T}$.

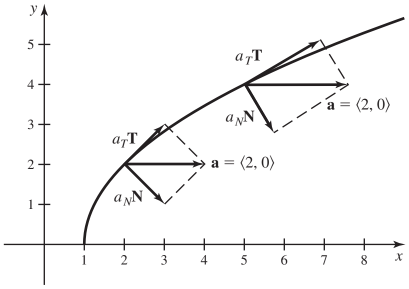
b. For all t, we have $\mathbf{a} = \langle 0, 0, 0\rangle = 0 \cdot \mathbf{N} + 0 \cdot \mathbf{T}$.



57

a. $\mathbf{v}(t) = \langle 2t, 2\rangle$, and $\mathbf{a}(t) = \langle 2, 0\rangle$. Because $\mathbf{v}\cdot\mathbf{a} = 4t$, we have $a_T = \dfrac{4t}{2\sqrt{t^2+1}} = \dfrac{2t}{\sqrt{t^2+1}}$. Note that $\mathbf{a}\times\mathbf{v} = \langle 0, 0, 4\rangle$, so $a_N = \dfrac{4}{2\sqrt{t^2+1}} = \dfrac{2}{\sqrt{t^2+1}}$. We have that $\mathbf{a} = \langle 2, 0\rangle = \dfrac{2}{\sqrt{t^2+1}}\mathbf{N} + \dfrac{2t}{\sqrt{t^2+1}}\mathbf{T}$.

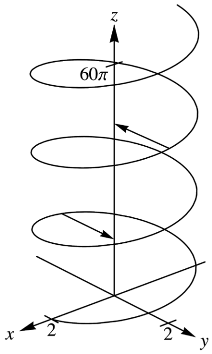
b. At $t = 1$, we have $\mathbf{a} = \langle 2, 0\rangle = \dfrac{2}{\sqrt{2}}\mathbf{T} + \dfrac{2}{\sqrt{2}}\mathbf{N} = \dfrac{2}{\sqrt{2}}\left\langle \sqrt{2}/2, \sqrt{2}/2\right\rangle + \dfrac{2}{\sqrt{2}}\left\langle \sqrt{2}/2, -\sqrt{2}/2\right\rangle$.
At $t = 2$, we have $\mathbf{a} = \langle 2, 0\rangle = \dfrac{4}{\sqrt{5}}\left\langle 2/\sqrt{5}, 1/\sqrt{5}\right\rangle + \dfrac{2}{\sqrt{5}}\left\langle 1/\sqrt{5}, -2/\sqrt{5}\right\rangle$.



58

a. $\mathbf{v}(t) = \langle -2\sin t, 2\cos t, 10\rangle$, and $\mathbf{a}(t) = \langle -2\cos t, -2\sin t, 0\rangle$. Because $\mathbf{v}\cdot\mathbf{a} = 0$, we have $a_T = 0$. Note that $\mathbf{a}\times\mathbf{v} = \langle -20\sin(t), 20\cos(t), -4\rangle$, so $a_N = \dfrac{4\sqrt{26}}{2\sqrt{26}} = 2$.

b. At $t = \pi/2$, we have $\mathbf{a} = \langle 0, -2, 0\rangle = 0\cdot\mathbf{T} + 2\mathbf{N} = 2\,\langle 0, -1, 0\rangle$.
At $t = 3\pi/2$, we have $\mathbf{a} = \langle 0, 2, 0\rangle = 0\cdot\mathbf{T} + 2\mathbf{N} = 2\,\langle 0, 1, 0\rangle$.



59 We have $\mathbf{r}'(t) = \left\langle 1, 2t, 3t^2\right\rangle$, so $\mathbf{r}'(1) = \langle 1, 2, 3\rangle$. Thus, $\mathbf{T}(1) = \dfrac{1}{\sqrt{3}}\,\langle 1, 1, 1\rangle$.

Then $\mathbf{N}(1) = \left\langle -\dfrac{11}{\sqrt{266}}, -4\sqrt{\dfrac{2}{133}}, \dfrac{9}{\sqrt{266}}\right\rangle$, so $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \left\langle \dfrac{3}{\sqrt{19}}, -\dfrac{3}{\sqrt{19}}, \dfrac{1}{\sqrt{19}}\right\rangle$. Also,

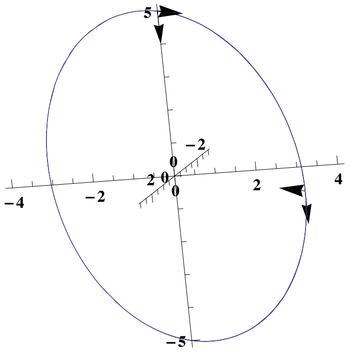
$$\tau = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2} = \frac{3}{19}.$$

60

a. $\mathbf{r}'(t) = \langle 3\cos t, 4\cos t, -5\sin t\rangle$, and $|\mathbf{r}'(t)| = 5$, so $\mathbf{T}(t) = \dfrac{1}{5}\,\langle 3\cos t, 4\cos t, -5\sin t\rangle$.

b. $\mathbf{T}'(t) = \dfrac{1}{5}\,\langle -3\sin t, -4\sin t, -5\cos t\rangle$, which has length one, so $\mathbf{N}(t) = \dfrac{1}{5}\,\langle -3\sin t, -4\sin t, -5\cos t\rangle$.

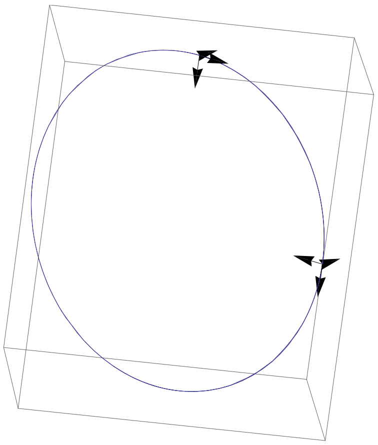
c. At $t = 0$ we have $\mathbf{T} = \langle 3/5, 4/5, 0\rangle$ and $\mathbf{N} = \langle 0, 0, -1/5\rangle$. At $t = \pi/2$ we have $\mathbf{T} = \langle 0, 0, -1\rangle$ and $\mathbf{N} = \langle -3/5, -4/5, 0\rangle$.



d. Yes.

e. $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \left\langle -\frac{4}{5}, \frac{3}{5}, 0 \right\rangle$.

f.



g. One should check that **T**, **N**, and **B** are all of unit length and are mutually orthogonal.

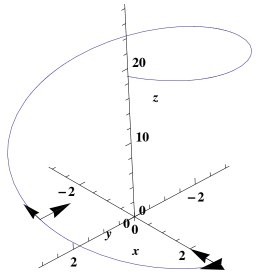
h. Because **B** is constant, $\tau = 0$.

61

a. $\mathbf{r}'(t) = \langle 3\cos t, -3\sin t, 4\rangle$, and $|\mathbf{r}'(t)| = 5$, so $\mathbf{T}(t) = \frac{1}{5}\langle 3\cos t, -3\sin t, 4\rangle$.

b. $\mathbf{T}'(t) = \frac{1}{5}\langle -3\sin t, -3\cos t, 0\rangle$, so $\mathbf{N}(t) = \langle -\sin t, -\cos t, 0\rangle$.

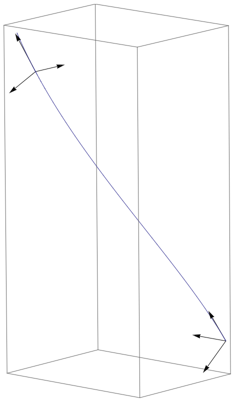
c. At $t = 0$ we have $\mathbf{T} = \langle 3/5, 0, 4/5\rangle$ and $\mathbf{N} = \langle 0, -1, 0\rangle$. At $t = \pi/2$ we have $\mathbf{T} = \langle 0, -3/5, 4/5\rangle$ and $\mathbf{N} = \langle -1, 0, 0\rangle$.



d. Yes.

e. $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{5}\langle 4\cos t, -4\sin t, -3\rangle$.

f.



g. One should check that **T**, **N**, and **B** are all of unit length and are mutually orthogonal.

h. $\tau = -\frac{d\mathbf{B}/dt}{ds/dt} \cdot \mathbf{N} = \frac{1}{25}\langle -4\sin t, -4\cos t, 0\rangle \cdot \langle -\sin t, -\cos t, 0\rangle = -\frac{4}{25}$.

62

a. $\mathbf{v}(t) = \langle 2a_1t + b_1, 2a_2t + b_2, 2a_3t + b_3\rangle$ and $\mathbf{a}(t) = \langle 2a_1, 2a_2, 2a_3\rangle$. Thus $\mathbf{v}\times\mathbf{a} = 2\langle a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2\rangle$, which is constant. Thus $\mathbf{B} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$ is a constant, so $\tau = 0$.

b. $\mathbf{a}'(t) = \langle 0, 0, 0\rangle$, so $\tau = \frac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{|\mathbf{v}\times\mathbf{a}|^2} = 0$.

63

a. First consider the case where $a_3 = b_3 = c_3 = 0$. Let $t \neq s$ be in the interval I, and consider $\mathbf{r}(t) \times \mathbf{r}(s)$. We will show that this vector is always a multiple of the same constant vector. We have

$$\begin{aligned}
&\mathbf{r}(t) \times \mathbf{r}(s)\\
&= \langle a_1f(t) + a_2g(t), b_1f(t) + b_2g(t), c_1f(t) + c_2g(t)\rangle \times \langle a_1f(s) + a_2g(s), b_1f(s) + b_2g(s), c_1f(s) + c_2g(s)\rangle\\
&= (f(t)g(s) - f(s)g(t))\,\langle c_2b_1 - b_2c_1, a_2c_1 - a_1c_2, a_1b_2 - a_2b_1\rangle,
\end{aligned}$$

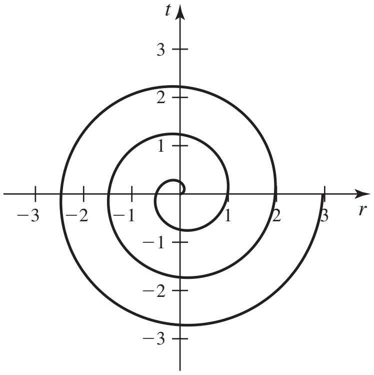
where this last computation admittedly requires patience. Because $\mathbf{r}(t) \times \mathbf{r}(s)$ is always orthogonal to the same vector, the vectors $\mathbf{r}(t)$ must all lie in the same plane.

Now consider the case where a_3, b_3, and c_3 are not necessarily 0, and consider $\mathbf{p}(t) = \mathbf{r}(t) - \langle a_3, b_3, c_3 \rangle$. Note that $\mathbf{p}(t)$ has the form required in the argument in the previous paragraph. Using the result above, the curve $\mathbf{p}(t)$ lies in a plane, which implies that $\mathbf{r}(t) = \mathbf{p}(t) + \langle a_3, b_3, c_3 \rangle$ lies in a plane as well, because we are just translating all the vectors $\mathbf{p}(t)$ by the same constant vector.

b. If the curve lies in a plane, the $\mathbf{B}$ is always normal to the plane with length 1. Hence $\mathbf{B}$ is constant, so $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = 0$.

64

a. The curve makes "one loop" for every 2π radians, so for $0 \le \theta \le 2\pi N$, it makes N loops. When $\theta = 2\pi N$, we have that the radius of the whole spiral is $R = tN$.



b. $L = \displaystyle\int_0^{2\pi N} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^{2\pi N} \sqrt{\left(\frac{t\theta}{2\pi}\right)^2 + \left(\frac{t}{2\pi}\right)^2}\, d\theta = \frac{t}{2\pi} \int_0^{2\pi N} \sqrt{\theta^2 + 1}\, d\theta.$

c. Note that $\lim_{\theta \to \infty} \dfrac{\theta^2 + 1}{\theta^2} = 1$, so when θ is large $\theta^2 \approx \theta^2 + 1$. So $L \approx \dfrac{t}{2\pi} \displaystyle\int_0^{2\pi N} \theta\, d\theta = \frac{t}{2\pi} \cdot (2\pi N)^2 \cdot \frac{1}{2} = t\pi N^2$. Because $N = \dfrac{R}{t}$, we have $L \approx \dfrac{\pi R^2}{t}$.

d. Let $t\theta_1/(2\pi) = r$ and $t\theta_2/(2\pi) = R$. We have $\theta_1 = \dfrac{2\pi r}{t} = \dfrac{2\pi \cdot .025}{1.5 \cdot 10^{-6}} = \dfrac{\pi}{3} \cdot 10^5$, and $\theta_2 = \dfrac{2\pi R}{t} = \dfrac{2\pi \cdot 0.059}{1.5 \cdot 10^{-6}} \approx 2.36 \cdot \dfrac{\pi}{3} \cdot 10^5$.

e. $L \approx \dfrac{1.5 \cdot 10^{-6}}{2\pi} \displaystyle\int_{(\pi/3)\cdot 10^5}^{(2.36\pi/3)\cdot 10^5} \theta\, d\theta \approx 5981.6$ meters. This is about 598,160 centimeters, and about 3.7 miles.

Chapter 15

Functions of Several Variables

15.1 Graphs and Level Curves

15.1.1 The independent variables are x and y and the dependent variable is z.

15.1.2 The domain of f is $\mathbb{R}^2$.

15.1.3 The domain of g is $\{(x,\, y) : x \neq 0 \text{ or } y \neq 0\}$.

15.1.4 The domain of h is $\{(x,\, y) : x - y \geq 0\}$.

15.1.5 We need three dimensions to plot points $(x,\, y,\, f\,(x,\, y))$.

15.1.6 Sketch the curves $f\,(x,\, y) = z_0$ in $\mathbb{R}^2$ for several values of z_0.

15.1.7 $f(2, 1) = \sqrt{10 - 2 + 1} = \sqrt{9} = 3$. $f(-9, -3) = \sqrt{10 + 9 - 3} = \sqrt{16} = 4$.

15.1.8 $g(1, 5, 3) = \dfrac{1 + 5}{3} = 2$. $g(3, 7, 2) = \dfrac{3 + 7}{2} = 5$.

15.1.9

a. 1300 ft.

b. Katie did. She is 100 ft higher than Zeke.

15.1.10 Katie remains at a constant elevation of 1500 during her hike and there isn't enough information to determine Zeke's elevation during his hike, although he started and ended at an elevation of 1500 feet.

15.1.11 The level curves $x^2 + y^2 = z_0$ are circles centered at (0, 0) in $\mathbb{R}^2$.

15.1.12 We need three dimensions to graph the level surfaces $f\,(x,\, y,\, z) = w_0$.

15.1.13 The function f has 6 independent variables, so $n = 6$.

15.1.14 We can sketch level surfaces in $\mathbb{R}^3$, or use colors to code the values of the function at points in $\mathbb{R}^3$.

15.1.15 The domain of f is $\mathbb{R}^2$.

15.1.16 The domain of f is $\mathbb{R}^2$

15.1.17 The domain of f is $\{(x,\, y) : x^2 + y^2 \leq 25\}$, which is the set of all points on or within the circle of radius 5 centered at the origin.

15.1.18 The domain of f is $\{(x,\, y) : x^2 + y^2 > 25\}$, which is the set of all points outside the circle of radius 5 centered at the origin.

15.1.19 The domain of f is $\{(x,\, y) : y \neq 0\}$.

15.1.20 The domain of f is $\{(x,\, y) : x \neq \pm y\}$.

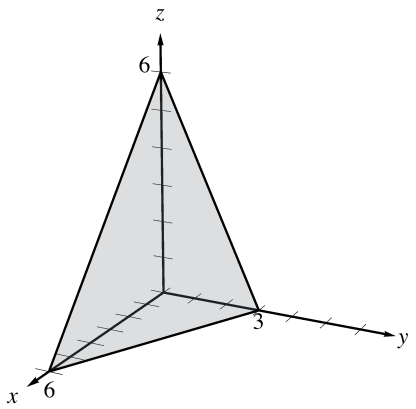
15.1.21 The domain of g is $\{(x,\, y) :\ y < x^2\}$.

15.1.22 The domain of f is $\{(x,\, y) :\ -1 \leq y - x^2 \leq 1\}$; which is the set of points lying between or on the parabolas $y = x^2 - 1$ and $y = x^2 + 1$.

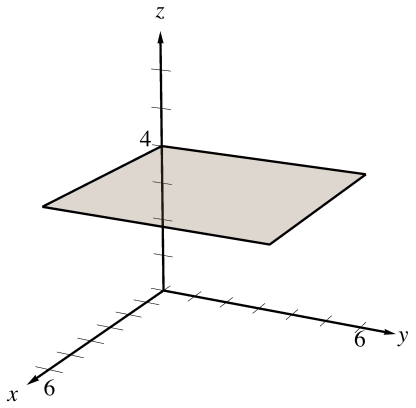
15.1.23 The domain of g is $\{(x,\, y) : x\,y \geq 0,\ (x,\, y) \neq (0,\, 0)\}$.

15.1.24 The domain of h is $\{(x,\, y) : x - 2y + 4 \geq 0\}$.

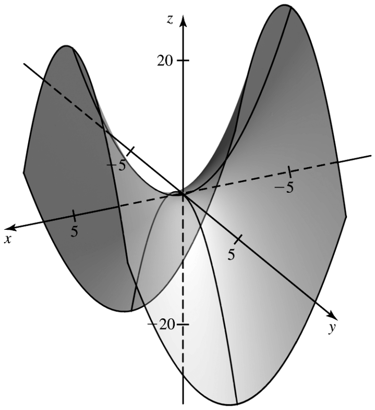
15.1.25 This surface is a plane; the function's domain is $\mathbb{R}^2$ and its range is $\mathbb{R}$.



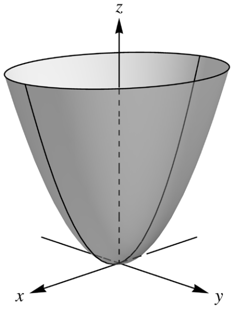
15.1.26 This surface is a plane; the function's domain is $\mathbb{R}^2$ and its range is $\{4\}$.



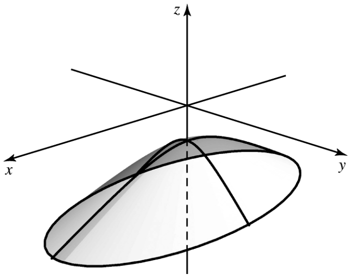
15.1.27 This surface is a hyperbolic paraboloid; the function's domain is $\mathbb{R}^2$ and its range is $\mathbb{R}$.



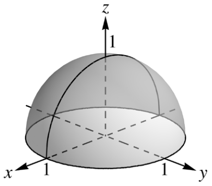
15.1.28 This surface is an elliptic paraboloid; the function's domain is $\mathbb{R}^2$ and its range is the interval $[0, \infty)$.



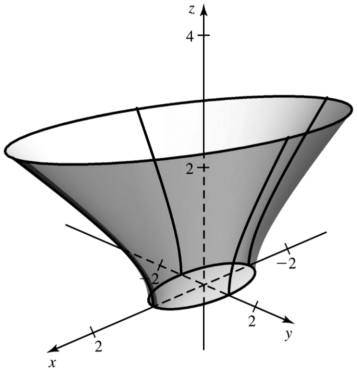
15.1.29 This surface is the lower part of a hyperboloid of two sheets; the function's domain is $\mathbb{R}^2$ and its range is the interval $(-\infty, -1]$.



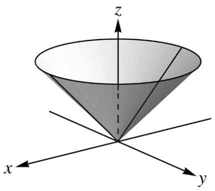
15.1.30 This surface is a hemisphere; the function's domain is $\{(x, y) : x^2 + y^2 \leq 1\}$ and its range is the interval $[0, 1]$.



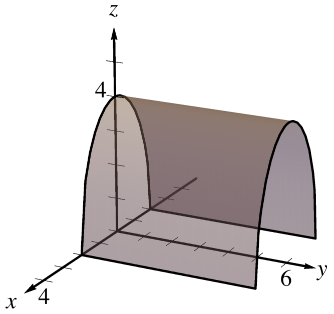
15.1.31 This surface is the upper part of a hyperboloid of one sheet; the function's domain is $\{(x, y) : x^2 + y^2 \geq 1\}$ and its range is the interval $[0, \infty)$.



15.1.32 This surface is the upper part of a circular cone; the function's domain is $\mathbb{R}^2$ and its range is the interval $[0, \infty)$.



15.1.33 This is the upper half of an elliptical cylinder. The domain is $\{(x, y) : -2 \leq x \leq 2, -\infty < y < \infty\}$, and the range is $[0, 4]$.



15.1.34

a. B. Notice that the level curves for the function consist of lines parallel to the x-axis.

b. E. Notice that the level curves for the function are hyperbolas.

c. C. Notice that the level curves are oval and are elongated along the y-axis; therefore the level sets match (C).

d. D. Notice that the level curves for the function are circles.

e. A. Notice that the level curves are oval and are elongated along the x-axis ; therefore the level sets match (A).

f. F. Notice that the level curves for the function are ellipses.

15.1.35

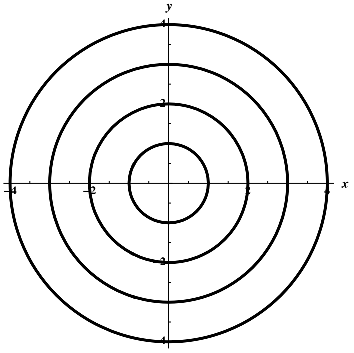
a. A. Notice that the range of the function in (A) is $[-1, 1]$.

b. D. Notice that the function in (D) becomes large and negative for$(x,\, y)$ near $(0,\, 0)$.

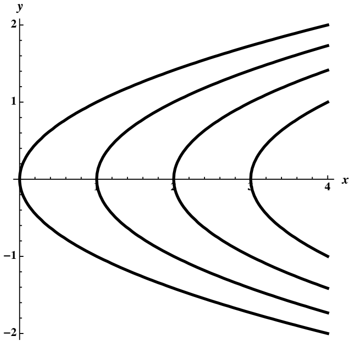
c. B. Notice that the function in (B) becomes large as you get close to $y = x$.

d. C. Notice that the function in (C) is everywhere positive.

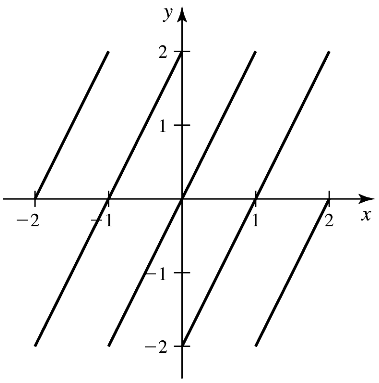
15.1.36



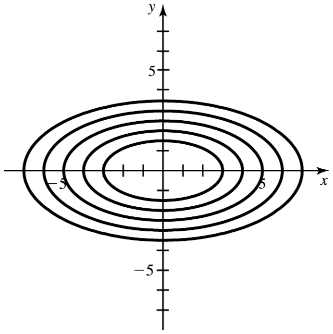
15.1.37



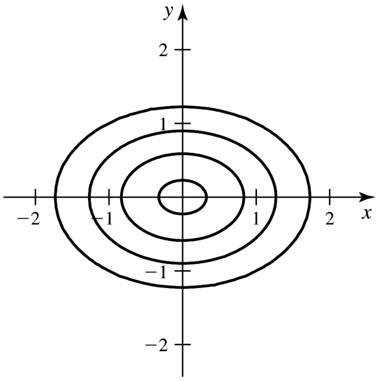
15.1.38



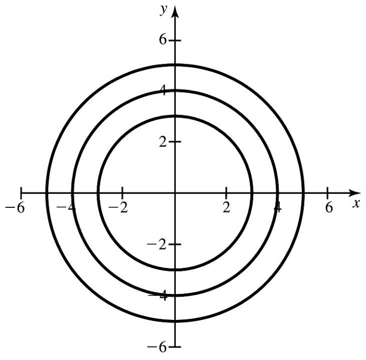
15.1.39



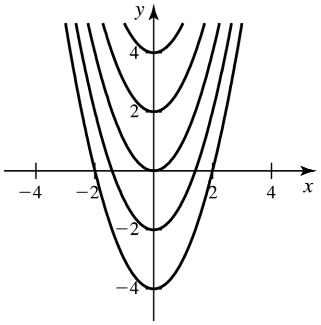
15.1.40



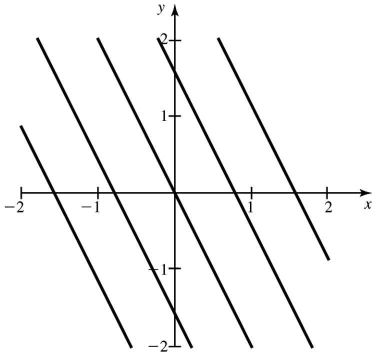
15.1.41



15.1.42



15.1.43

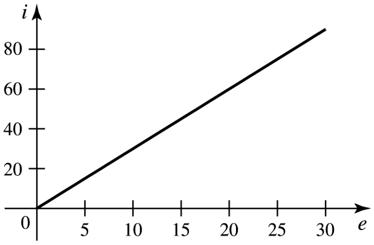


15.1.44

a. His ERA was $A(24,\, 224) = \dfrac{9 \cdot 24}{224} = 0.9643.$

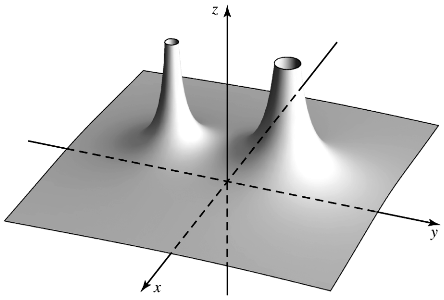
b. His ERA is $A\left(4,\, \dfrac{1}{3}\right) = \dfrac{9 \cdot 4}{1/3} = 108.$

c. The relationship is $e = \dfrac{i}{3}$, so a pitcher with an ERA of 3 gives up one run every three innings.



15.1.45

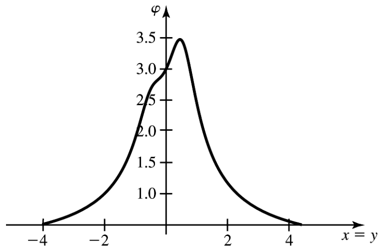
a.



b. The potential function is defined for all $(x,\, y)$ in $\mathbb{R}^2$ except $(0,\, 1)$ and $(0,\, -1)$.

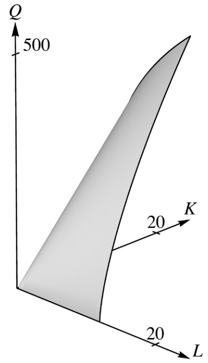
c. We have $\varphi(2,\, 3) \approx 0.93 > \varphi(3,\, 2) \approx 0.87$.

d.



15.1.46

a.

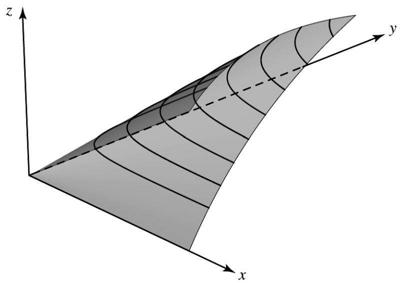


b. We have $Q = cL^aK^b = 40 \cdot (10)^{1/3}\, K^{2/3}$.

c. We have $Q = cL^aK^b = 40 \cdot (15)^{2/3}\, L^{1/3}$.

15.1.47

a.



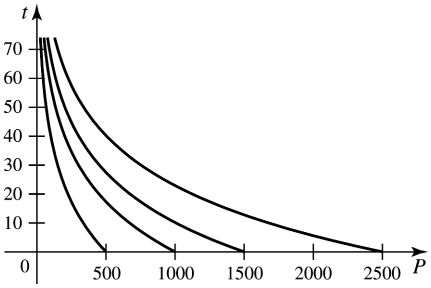
b. The maximum resistance is $R\,(10,\,10) = 5$ ohms.

c. This means $R\,(x,\,y) = R\,(y,\,x)$.

15.1.48

a. The set of points $(P,\,t)$ with $B = 2000$ is given by $Pe^{0.04t} = 2000$; solving for P gives $P = 2000e^{-0.04t}$.

b. The level curves are given by $P = Be^{0.04t}$ with $B = 500, 1000, 1500$ and 2000.

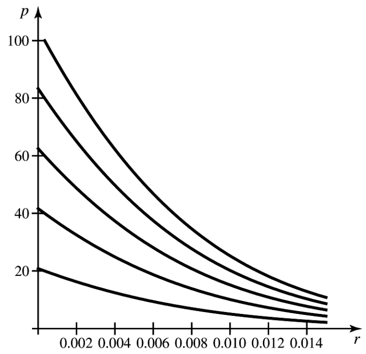


c. As t increases along a level curve, P decreases and vice versa.

15.1.49

a. Solving for P in the equation $B(P, r, t) = 20000$ with $t = 20$ years gives $P = \dfrac{20000r}{(1+r)^{240} - 1}$.

b. The level curves are given by $P = \dfrac{Br}{(1+r)^{240} - 1}$,with $B = 5000, 10000, 15000$ and 25000.



15.1.50 The domain of f is $\mathbb{R}^3$.

15.1.51 The domain of g is $\{(x, y, z) : x \neq z\}$, which is all points in $\mathbb{R}^3$ not on the plane given by $x = z$.

15.1.52 The domain of p is $\{(x, y, z) : x^2 + y^2 + z^2 \geq 9\}$, which is all points in $\mathbb{R}^3$ on or outside the sphere of radius 3 centered at the origin.

15.1.53 The domain of f is $\{(x, y, z) : y \geq z\}$, which is all points in $\mathbb{R}^3$ on or below the plane given by $z = y$.

15.1.54 The domain of Q is $\mathbb{R}^3$.

15.1.55 The domain of F is $\{(x, y, z) : x^2 \leq y\}$, which is all points on the side of the vertical cylinder $y = x^2$ that contains the positive y-axis.

15.1.56 The domain of f is $\{(w, x, y, z) : w^2 + x^2 + y^2 + z^2 \leq 1\}$, which is all points in R^4 on or inside the hypersphere of radius 1 centered at the origin.

15.1.57

a. False. This function has domain $\mathbb{R}^2$.

b. False. The domain of a function of 4 variables is a region in $\mathbb{R}^4$.

c. True. The level curves for the function defined by $z = 2x - 3y$ are lines of the form $2x - 3y = c$ for any constant c.

15.1.58

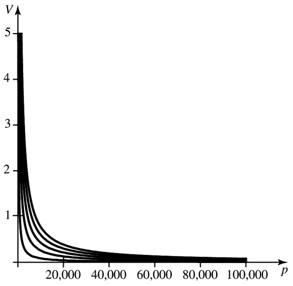
a. His quarterback passer rating is $R = \dfrac{50 + 20 \cdot 71.43 + 80 \cdot 9.18 - 100 \cdot 0 + 100 \cdot 10.35}{24} \approx 135.33.$

b. His quarterback passer rating is $R = \dfrac{50 + 20 \cdot 67.36 + 80 \cdot 6.48 - 100 \cdot 0.46 + 100 \cdot 8.23}{24} \approx 112.2.$

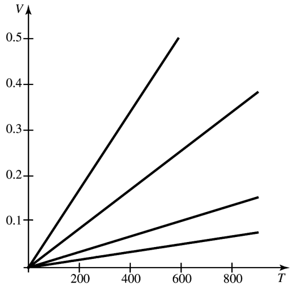
c. If c, t and y are fixed then R is a decreasing linear function of t. This makes sense since a quarterback's rating should decrease if his interception percentage increases.

15.1.59

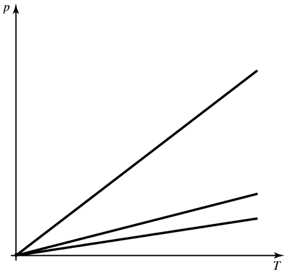
a.



b.

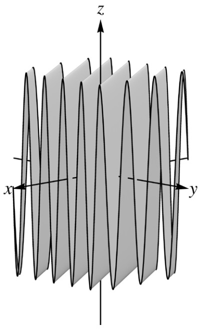


c.



15.1.60

a.



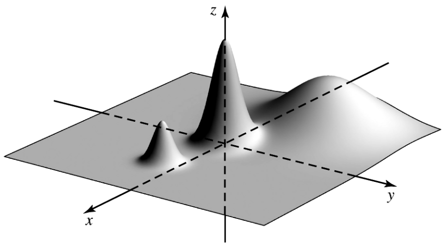
b. The domain of this function is $\mathbb{R}^2$.

c. The maximum and minimum water heights are ±10.

d. The maximum and minimum heights occur along the lines $2x - 3y = \dfrac{\pi}{2} + k\pi$, where k is any integer. A vector orthogonal to these lines is $\mathrm{v} = \langle 2, -3 \rangle$.

15.1.61

a.



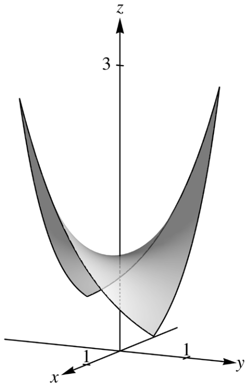
b. The peaks occur near the points $(0, 0)$, $(-5, 3)$ and $(4, -1)$.

c. We have $f(0, 0) \approx 10.17$, $f(-5, 3) \approx 5.00$, $f(4, -1) \approx 4.00$.

15.1.62

a. The domain is $\mathbb{R}^2$ and the range is the interval $(0, \infty)$.

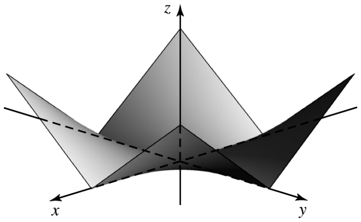
b.



15.1.63

a. The domain is $\mathbb{R}^2$ and the range is the interval $[0, \infty)$.

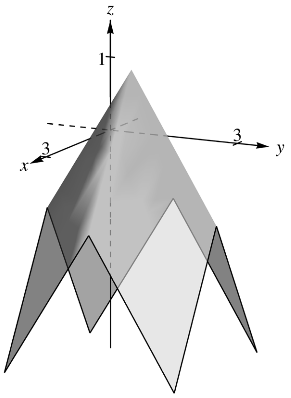
b.



15.1.64

a. The domain is $\mathbb{R}^2$ and the range is $\mathbb{R}$.

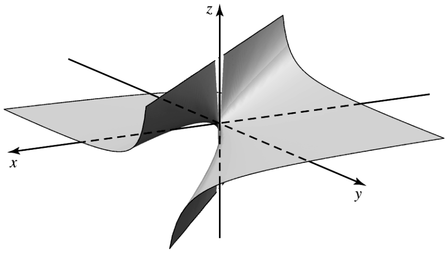
b.



15.1.65

a. The domain is $\{(x,\, y) : x \neq y\}$ and the range is $\mathbb{R}$.

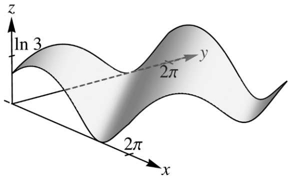
b.



15.1.66

a. The domain is $\mathbb{R}^2$ and the range is the interval $[0,\, \ln 3]$.

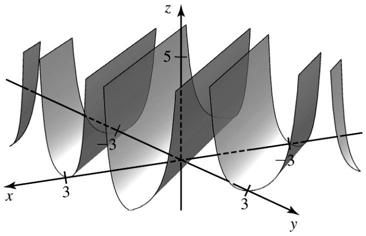
b.



15.1.67

a. The domain is $\{(x,\, y) : y \neq x + \dfrac{\pi}{2} + n\pi$ for any integer $n\}$ and the range is the interval $[0,\, \infty)$.

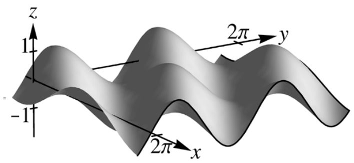
b.



15.1.68

a. The domain is $\mathbb{R}^2$ and the range is the interval $[-1,\, 1]$.

b.



15.1.69 This function has a peak at the origin.

15.1.70 This function has a peak at the point $\left(\frac{1}{2}, -1\right)$.

15.1.71 This function has a depression at the point (1, 0).

15.1.72 This function has a depression at the point (1, 1).

15.1.73 The level curves are the lines given by $ax+by=d-cz_0$, where z_0 is a constant; these lines all have slope $-\frac{a}{b}$ (in the case $b=0$ the lines are all vertical).

15.1.74 If $\frac{1}{x^2+y^2+z^2}=K$, then $x^2+y^2+z^2=\frac{1}{K}=C$ for a constant C. Thus, the level surfaces are spheres centered at the origin.

15.1.75 If $x^2+y^2-z=C$, then $z=x^2+y^2-C$, so the level surfaces are paraboloids with vertices $(0,0,-C)$.

15.1.76 If $x^2-y^2-z=C$, then $z=x^2-y^2-C$, so the level surfaces are hyperbolic paraboloids with a saddle point at $(0,0,-C)$.

15.1.77 If $\sqrt{x^2+2z^2}=K$, then $x^2+2z^2=C$ where $C=K^2$, so the level surfaces are elliptic cylinders parallel to the y-axis.

15.1.78 Factor the equation $x^2-(y+z)\,x+yz=(x-y)\,(x-z)$; hence the domain of g is $\{(x,\, y,\, z) : x \neq y$ and $x \neq z\}$. The domain consists of all points not on the planes given by $x=y$ and $x=z$.

15.1.79 The domain of f is $\{(x,\, y) : x-1 \leq y \leq x+1\}$. This is the region between the two parallel lines given by $y=x-1$ and $y=x+1$.

15.1.80 The domain of f is $\{(x,\, y,\, z) : z > x^2+y^2-2x-3\}$, which is equivalent to $\{(x,\, y,\, z) : z > (x-1)^2+y^2-4\}$. This region consists of all points inside a circular paraboloid with vertex at (1, 0, −4).

15.1.81 Factor the equation $z^2-xz+yz-xy=(z-x)\,(z+y)$; hence the domain of h is $\{(x,\, y,\, z) : (z-x)\,(z+y) \geq 0\}$, which is equivalent to $D=\{(x,\, y,\, z) : (x \leq z$ and $y \geq -z)$ or $(x \geq z$ and $y \leq -z)\}$. The domain consists of all points above or below both the planes given by $z=x$ and $z=-y$ as well as the points on either one of these planes.

15.1.82

a. This "ball" is the solid octahedron with vertices (±1, 0, 0), (0, ±1, 0) and (0, 0, ±1).

b. This "ball" is the solid cube with vertices $(x,\, y,\, z)$ where $x,\, y,\, z = \pm 1$.

15.2 Limits and Continuity

15.2.1 The values of $|f(x,y)-L|$ can be made arbitrarily small if (x,y) is sufficiently close to (a,b).

15.2.2 If $f(x,y)$ has a different limit as (x,y) approaches (a,b) along two different paths, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

15.2.3 If $f(x,y)$ is a polynomial, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)=f(a,b)$; in other words, the limit can be found by plugging in $x=a$, $y=b$ in $f(x,y)$.

15.2.4 We evaluate $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ along all paths that approach (a,b) from within the domain of f.

15.2.5 If the limits along different paths do not agree, then the limit does not exist.

15.2.6 Evaluating $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ along a finite number of paths does not establish that the limit is the same along *all* paths that approach (a,b).

15.2.7 The function f must be defined at (a,b), $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ must exist, and the limit must equal $f\left(a,b\right)$.

15.2.8 Yes, $(0,0)$ is a boundary point for R because every disc centered at $(0,0)$ contains points in R and points not in R (namely $(0,0)$). The set R is neither open nor closed because it contains some of its boundary points (namely points on the unit circle) but not all of them (namely $(0,0)$).

15.2.9 A rational function is continuous at all points where its denominator is nonzero.

15.2.10 Because xy^2z^3 is a polynomial, $\lim\limits_{(x,y,z)\to(1,1,-1)} xy^2z^3 = 1\cdot 1^2\cdot(-1)^3 = -1$.

15.2.11 $\lim\limits_{(x,y)\to(5,-5)} \dfrac{x^2-y^2}{x+y} = \lim\limits_{(x,y)\to(5,-5)} \dfrac{(x-y)(x+y)}{x+y} = \lim\limits_{(x,y)\to(5,-5)} x-y = 5-(-5) = 10$.

15.2.12 We have (along $y=0$) $\lim\limits_{(x,y)\to(1,0)} \dfrac{x^2-2x-y^2+1}{x^2-2x+y^2+1} = \lim\limits_{x\to1} \dfrac{x^2-2x+1}{x^2-2x+1} = \lim\limits_{x\to1} 1 = 1$. But, along $x=1$ we have $\lim\limits_{(x,y)\to(1,0)} \dfrac{x^2-2x-y^2+1}{x^2-2x+y^2+1} = \lim\limits_{y\to0} -\dfrac{y^2}{y^2} = \lim\limits_{y\to0} -1 = -1$. Because we obtain different limits along different paths within the domain, $\lim\limits_{(x,y)\to(1,0)} \dfrac{x^2-2x-y^2+1}{x^2-2x+y^2+1}$ does not exist.

15.2.13 $\lim\limits_{(x,y)\to(2,9)} 101 = 101$.

15.2.14 $\lim\limits_{(x,y)\to(1,-3)} (3x+4y-2) = 3\cdot1+4\left(-3\right)-2 = -11$.

15.2.15 $\lim\limits_{(x,y)\to(-3,3)} \left(4x^2-y^2\right) = 4\cdot\left(-3\right)^2-\left(3\right)^2 = 27$.

15.2.16 $\lim\limits_{(x,y)\to(2,-1)} \left(xy^8-3x^2y^3\right) = 2\left(-1\right)^8-3\cdot2^2\left(-1\right)^3 = 14$.

15.2.17 $\lim\limits_{(x,y)\to(0,\pi)} \dfrac{\cos xy+\sin xy}{2y} = \dfrac{\cos 0+\sin 0}{2\pi} = \dfrac{1}{2\pi}$.

15.2.18 $\lim\limits_{(x,y)\to(e^2,4)} \ln\sqrt{xy} = \ln\sqrt{4e^2} = \ln 2e = 1+\ln 2$.

15.2.19 $\lim\limits_{(x,y)\to(2,0)} \dfrac{x^2-3xy^2}{x+y} = \dfrac{2^2-3\cdot2\cdot0^2}{2+0} = 2$.

15.2.20 $\lim\limits_{(u,v)\to(1,-1)} \dfrac{10uv-2v^2}{u^2+v^2} = \dfrac{10\cdot1\left(-1\right)-2\left(-1\right)^2}{1^2+\left(-1\right)^2} = -6$.

15.2.21 $\lim\limits_{(x,y)\to(6,2)} \dfrac{x^2-3xy}{x-3y} = \lim\limits_{(x,y)\to(6,2)} \dfrac{x\left(x-3y\right)}{x-3y} = \lim\limits_{(x,y)\to(6,2)} x = 6$.

15.2.22 $\lim\limits_{(x,y)\to(1,-2)} \dfrac{y^2+2xy}{y+2x} = \lim\limits_{(x,y)\to(1,-2)} \dfrac{y\left(y+2x\right)}{y+2x} = \lim\limits_{(x,y)\to(1,-2)} y = -2$.

15.2.23 $\lim\limits_{(x,y)\to(3,1)} \dfrac{x^2-7xy+12y^2}{x-3y} = \lim\limits_{(x,y)\to(3,1)} \dfrac{(x-3y)(x-4y)}{x-3y} = \lim\limits_{(x,y)\to(3,1)} (x-4y) = 3-4 = -1$.

15.2.24 $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{2x^2-xy-3y^2}{x+y} = \lim_{(x,y)\to(-1,1)} \frac{(x+y)(2x-3y)}{x+y} = \lim_{(x,y)\to(-1,1)} (2x-3y) = -5.$

15.2.25 $\displaystyle\lim_{(x,y)\to(2,2)} \frac{y^2-4}{xy-2x} = \lim_{(x,y)\to(2,2)} \frac{(y+2)(y-2)}{x(y-2)} = \lim_{(x,y)\to(2,2)} \frac{y+2}{x} = \frac{2+2}{2} = 2.$

15.2.26

$$\begin{aligned}\lim_{(x,y)\to(4,5)} \frac{\sqrt{x+y}-3}{x+y-9} &= \lim_{(x,y)\to(4,5)} \frac{(\sqrt{x+y}-3)(\sqrt{x+y}+3)}{(x+y-9)(\sqrt{x+y}+3)}\\ &= \lim_{(x,y)\to(4,5)} \frac{x+y-9}{(x+y-9)(\sqrt{x+y}+3)}\\ &= \lim_{(x,y)\to(4,5)} \frac{1}{\sqrt{x+y}+3} = \frac{1}{\sqrt{4+5}+3} = \frac{1}{6}.\end{aligned}$$

15.2.27

$$\begin{aligned}\lim_{(x,y)\to(1,2)} \frac{\sqrt{y}-\sqrt{x+1}}{y-x-1} &= \lim_{(x,y)\to(1,2)} \frac{\left(\sqrt{y}-\sqrt{x+1}\right)\left(\sqrt{y}+\sqrt{x+1}\right)}{(y-x-1)\left(\sqrt{y}+\sqrt{x+1}\right)}\\ &= \lim_{(x,y)\to(1,2)} \frac{y-x-1}{(y-x-1)\left(\sqrt{y}+\sqrt{x+1}\right)}\\ &= \lim_{(x,y)\to(1,2)} \frac{1}{\sqrt{y}+\sqrt{x+1}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}.\end{aligned}$$

15.2.28

$$\begin{aligned}\lim_{(u,v)\to(8,8)} \frac{u^{1/3}-v^{1/3}}{u^{2/3}-v^{2/3}} &= \lim_{(u,v)\to(8,8)} \frac{u^{1/3}-v^{1/3}}{\left(u^{1/3}-v^{1/3}\right)\left(u^{1/3}+v^{1/3}\right)}\\ &= \lim_{(u,v)\to(8,8)} \frac{1}{u^{1/3}+v^{1/3}} = \frac{1}{2+2} = \frac{1}{4}.\end{aligned}$$

15.2.29 Observe that along the line $y=0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+2y}{x-2y} = \lim_{x\to0} \frac{x}{x} = 1$, whereas along the line $x=0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+2y}{x-2y} = \lim_{y\to0} \frac{2y}{-2y} = -1.$

15.2.30 Observe that along the line $y=x$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2+y^2} = \lim_{x\to0} \frac{4x^2}{4x^2} = 1$, whereas along the line $y=-x$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2+y^2} = \lim_{x\to0} -\frac{4x^2}{4x^2} = -1.$

15.2.31 Observe that along the line $x=0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^4-2x^2}{y^4+x^2} = \lim_{y\to0} \frac{y^4}{y^4} = 1$, whereas along the line $y=0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^4-2x^2}{y^4+x^2} = \lim_{x\to0} -\frac{2x^2}{x^2} = -2.$

15.2.32 Observe that along the line $y=0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^2}{x^3+y^2} = \lim_{x\to0} \frac{x^3}{x^3} = 1$, whereas along the line $x=0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^2}{x^3+y^2} = \lim_{y\to0} -\frac{y^2}{y^2} = -1.$

15.2.33 Observe that along the line $y=x$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^3+x^3}{xy^2} = \lim_{x\to0} \frac{2x^3}{x^3} = 2$, whereas along the line $y=-x$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^3+x^3}{xy^2} = \lim_{x\to0} \frac{0}{-x^3} = 0.$

15.2.34 Observe that along the line $y = 0$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{y}{\sqrt{x^2-y^2}} = \lim\limits_{x\to 0} \dfrac{0}{|x|} = 0$, whereas along the ray $x = 2y$, $y > 0$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{y}{\sqrt{x^2-y^2}} = \lim\limits_{x\to 0} \dfrac{y}{\sqrt{3}y} = \dfrac{1}{\sqrt{3}}$.

15.2.35 The function f is continuous on $\mathbb{R}^2$.

15.2.36 The function f is continuous on $\mathbb{R}^2$ (the denominator of this rational function is always positive).

15.2.37 The function p is continuous at all points except the origin, where it is undefined.

15.2.38 The function S is continuous at all points except where $x^2 = y^2$, which is along the lines $x = y$ and $x = -y$.

15.2.39 The function f is continuous on $\mathbb{R}^2$ except where $x = 0$.

15.2.40 The function f is continuous on $\mathbb{R}^2$ except where $x = 0$ or $y = \pm 1$.

15.2.41 The function f is continuous on $\mathbb{R}^2$ except the origin. Note that along the line $y = x$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+x^2} = \dfrac{1}{2} \neq 0$.

15.2.42 The function f is continuous on $\mathbb{R}^2$ except the origin. Note that along the line $x = 0$,

$$\lim_{(x,y)\to(0,0)} \frac{y^4-2x^2}{y^4+x^2} = \lim_{(x,y)\to(0,0)} \frac{y^4}{y^4} = 1 \neq 0.$$

15.2.43 The function f is continuous on $\mathbb{R}^2$.

15.2.44 The function f is continuous on $\mathbb{R}^2$.

15.2.45 The function f is continuous on $\mathbb{R}^2$.

15.2.46 The function g is continuous on its domain, which is $D = \{(x, y) : x > y\}$.

15.2.47 The function h is continuous on $\mathbb{R}^2$.

15.2.48 The function p is continuous on $\mathbb{R}^2$.

15.2.49 The function f is continuous on its domain, which is $D = \{(x, y) : (x, y) \neq (0, 0)\}$.

15.2.50 The function f is continuous on its domain, which is $D = \{(x, y) : x^2 + y^2 \leq 4\}$.

15.2.51 The function g is continuous on $\mathbb{R}^2$.

15.2.52 The function h is continuous on its domain, which is $D = \{(x, y) : x \geq y\}$.

15.2.53 The function f is continuous on $\mathbb{R}^2$.

15.2.54 The function f is continuous on $\mathbb{R}^2$.

15.2.55 $\lim\limits_{(x,y,z)\to(1,\ln 2,3)} ze^{xy} = 3e^{1\cdot\ln 2} = 6$.

15.2.56 $\lim\limits_{(x,y,z)\to(0,1,0)} \ln(1+y)\, e^{xz} = (\ln 2)\, e^0 = \ln 2$.

15.2.57 $\lim\limits_{(x,y,z)\to(1,1,1)} \dfrac{yz-xy-xz-x^2}{yz+xy+xz-y^2} = \dfrac{1-1-1-1}{1+1+1-1} = -1$.

15.2.58 $\displaystyle\lim_{(x,y,z)\to(1,1,1)} \frac{x-\sqrt{xz}-\sqrt{xy}+\sqrt{yz}}{x-\sqrt{xz}+\sqrt{xy}-\sqrt{yz}} = \lim_{(x,y,z)\to(1,1,1)} \frac{\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}-\sqrt{z}\right)}{\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{z}\right)} =$
$\displaystyle\lim_{(x,y,z)\to(1,1,1)} \frac{\left(\sqrt{x}-\sqrt{y}\right)}{\left(\sqrt{x}+\sqrt{y}\right)} = \frac{1-1}{1+1} = 0.$

15.2.59 $\displaystyle\lim_{(x,y,z)\to(1,1,1)} \frac{x^2+xy-xz-yz}{x-z} = \lim_{(x,y,z)\to(1,1,1)} \frac{(x-z)(x+y)}{x-z} = \lim_{(x,y,z)\to(1,1,1)} (x+y) = 2.$

15.2.60 $\displaystyle\lim_{(x,y,z)\to(1,-1,1)} \frac{xz+5x+yz+5y}{x+y} = \lim_{(x,y,z)\to(1,-1,1)} \frac{(x+y)(z+5)}{x+y} = \lim_{(x,y,z)\to(1,-1,1)} (z+5) = 6.$

15.2.61

a. False. The limit may be different or not exist along other paths approaching $(0,0)$.

b. False. We may have $f(a,b)$ undefined, or $f(a,b) \neq L$.

c. True. The limit must exist for f to be continuous at (a,b).

d. False. For example, take $P=(0,0)$ and the domain of f to be $\{(x,y):(x,y)\neq(0,0)\}$.

15.2.62 Observe that along the line $y=0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^8+y^2} = \lim_{x\to 0} \frac{0}{x^8} = 0$, whereas along the curve $y=x^4$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^8+y^2} = \lim_{x\to 0} \frac{x^8}{2x^8} = \frac{1}{2}$, therefore this limit does not exist.

15.2.63 $\displaystyle\lim_{(x,y)\to(0,1)} \frac{y\sin x}{x(y+1)} = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{y\to 1} \frac{y}{y+1}\right) = 1\cdot\frac{1}{2} = \frac{1}{2}.$

15.2.64 $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2+xy-2y^2}{2x^2-xy-y^2} = \lim_{(x,y)\to(1,1)} \frac{(x+2y)(x-y)}{(2x+y)(x-y)} = \lim_{(x,y)\to(1,1)} \frac{x+2y}{2x+y} = \frac{1+2}{2+1} = 1.$

15.2.65 $\displaystyle\lim_{(x,y)\to(1,0)} \frac{y\ln y}{x} = \left(\lim_{x\to 1} \frac{1}{x}\right)\left(\lim_{y\to 0} \ln y^y\right) = 1\cdot\ln 1 = 0$

15.2.66 Observe that along the line $y=x$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|xy|}{xy} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$, whereas along the line $y=-x$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|xy|}{xy} = \lim_{x\to 0} \frac{x^2}{-x^2} = -1$, therefore this limit does not exist.

15.2.67 Observe that along the line $y=x$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x-y|}{|x+y|} = \lim_{x\to 0} \frac{0}{2|x|} = 0$, whereas along the line $y=2x$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x-y|}{|x+y|} = \lim_{x\to 0} \frac{|x|}{3|x|} = \frac{1}{3}$, therefore this limit does not exist.

15.2.68 $\displaystyle\lim_{(u,v)\to(-1,0)} \frac{uve^{-v}}{u^2+v^2} = \frac{(-1)\cdot 0\cdot e^o}{(-1)^2+0^2} = 0.$

15.2.69 Observe that $\displaystyle\lim_{(x,y)\to(2,0)} \frac{1-\cos y}{xy^2} = \left(\lim_{x\to 2} \frac{1}{x}\right)\left(\lim_{y\to 0} \frac{1-\cos y}{y^2}\right) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$ where the y-limit is evaluated by two applications of L′Hôpital′s rule.

15.2.70 Let $u=xy$; then $u\to 0$ as $(x,y)\to(4,0)$, so $\displaystyle\lim_{(x,y)\to(4,0)} x^2y\ln xy = \lim_{(x,y)\to(4,0)} x\cdot xy\ln xy =$ $\displaystyle 4\lim_{u\to 0} u\ln u = 4\lim_{u\to 0} \ln u^u = 4\cdot 0 = 0.$

15.2.71 Let $u = xy$; then $u \to 0$ as $(x, y) \to (1, 0)$, so $\displaystyle\lim_{(x,y)\to(1,0)} \frac{\sin xy}{xy} = \lim_{u\to 0} \frac{\sin u}{u} = 1$.

15.2.72 Let $u = xy$; then $u \to 0$ as $(x, y) \to \left(0, \frac{\pi}{2}\right)$, so $\displaystyle\lim_{(x,y)\to(0,\pi/2)} \frac{1-\cos xy}{4x^2y^3} =$ $\displaystyle\lim_{(x,y)\to(0,\pi/2)} \left(\frac{1}{4y} \cdot \frac{1-\cos xy}{x^2y^2}\right) = \frac{1}{2\pi} \lim_{u\to 0} \frac{1-\cos u}{u^2} = \frac{1}{4\pi}$. (Use L'Hôpital's rule twice to see that $\lim_{u\to 0}\left(1-\cos u\right)/u^2 = 1/2$).

15.2.73 Let $u = xy$; then $u \to 0$ as $(x, y) \to (0, 2)$, so $\displaystyle\lim_{(x,y)\to(0,2)} (2xy)^{xy} = \lim_{u\to 0} 2^u u^u = 1$.

15.2.74 Let $u = x + y$. Then $\displaystyle\lim_{(x,y)\to(3,3)} \frac{x^2+2xy-6x+y^2-6y}{x+y-6} = \lim_{(x,y)\to(3,3)} \frac{x^2+2xy+y^2-(6x_6y)}{x+y-6} =$ $\displaystyle\lim_{(x,y)\to(3,3)} \frac{(x+y)^2-6(x+y)}{x+y-6} = \lim_{u\to 6} \frac{u^2-6u}{u-6} = \lim_{u\to 6} u = 6$.

15.2.75 Let $u = x+y$. Then $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2+2xy-x+y^2-y-6}{x+y-3} = \lim_{(x,y)\to(1,2)} \frac{x^2+2xy+y^2-x-y-6}{x+y-3} =$ $\displaystyle\lim_{(x,y)\to(1,2)} \frac{(x+y)^2-(x+y)-6}{x+y-3} = \lim_{u\to 3} \frac{u^2-u-6}{u-3} = \lim_{u\to 3} \frac{(u+2)(u-3)}{u-3} = \lim_{u\to 3}(u+2) = 5$.

15.2.76 Let $u = x^2 + y^2$. Then

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{(2+(x+y)^2+(x-y)^2}{2e^{x^2+y^2}}\right) &= \lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{2+2(x^2+y^2)}{2e^{x^2+y^2}}\right)\\ &= \lim_{u\to 0^+} \tan^{-1}\left(\frac{1+u}{e^u}\right) = \frac{\pi}{4}.\end{aligned}$$

15.2.77 Let $u = x^2 + y^2 - 1$. Then as $x^2 + y^2 \to 1$, $u \to 0$. Because $\displaystyle\lim_{u\to 0} \frac{\sin u}{u} = 1$, we must have $b = 1$ in order for f to be continuous everywhere.

15.2.78 Let $u = xy$. Note that as $xy \to 0$, $u \to 0$. Also, $\displaystyle\lim_{xy\to 0} \frac{1+2xy-\cos(xy)}{xy} = \lim_{u\to 0} \frac{1+2u-\cos u}{u} =$ $\displaystyle\lim_{u\to 0} \frac{2+\sin u}{1} = 2$ by L'Hôpital's rule. Thus, we must have $a = 2$ in order for f to be continuous everywhere.

15.2.79 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta}{r^2} = \lim_{r\to 0} r\cos^2\theta\sin\theta = 0$.

15.2.80 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r\cos\theta - r\sin\theta}{r} = \lim_{r\to 0}(\cos\theta - \sin\theta)$, which does not exist.

15.2.81 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2+x^2y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2+r^4\cos^2\theta\sin^2\theta}{r^2} = \lim_{r\to 0}(1+r^2\cos^2\theta\sin^2\theta) = 1$.

15.2.82 Let $z = x + y$, then

$$\frac{\sin x+\sin y}{x+y} = \frac{\sin x+\sin(z-x)}{z} = \frac{\sin x+\sin z\cos x-\cos z\sin x}{z} = \sin x\left(\frac{1-\cos z}{z}\right) + \cos x\left(\frac{\sin z}{z}\right).$$

As $(x, y) \to (0, 0)$ we have $z \to 0$; $\displaystyle\lim_{z\to 0} \frac{\sin z}{z} = 1$ and $\displaystyle\lim_{z\to 0} \frac{1-\cos z}{z} = 0$, so $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin x+\sin y}{x+y} = 0{\cdot}0+1{\cdot}1 = 1$.

15.2.83 The limit is 0 along the lines $x = 0$ or $y = 0$. However, along the line $x = y$ we have $\lim\limits_{(x,y)\to(0,0)} \dfrac{ax^m y^n}{bx^{m+n} + cy^{m+n}} = \lim\limits_{x\to 0} \dfrac{ax^{m+n}}{bx^{n+m} + cx^{m+n}} = \dfrac{a}{b+c} \neq 0$ because $a \neq 0$. Therefore this limit does not exist.

15.2.84 The limit is 0 along the line $y = 0$. However, along the curve $y = x^2$ we have $\lim\limits_{(x,y)\to(0,0)} \dfrac{ax^{2(p-n)}y^n}{bx^{2p} + cy^p}$ $= \lim\limits_{x\to 0} \dfrac{ay^p}{(b+c)y^p} = \dfrac{a}{b+c} \neq 0$ because $a \neq 0$. Therefore this limit does not exist.

15.2.85 Because $\lim\limits_{(x,y)\to(0,0)} e^{-1/\left(x^2+y^2\right)} = 0$, we should define $f\left(0,0\right) = 0$.

15.2.86 For any $\epsilon > 0$, let $\delta = \epsilon$; then $0 < \sqrt{\left(x-a\right)^2 + \left(y-b\right)^2} < \delta \implies |y-b| < \epsilon$ because $|y-b| \leq \sqrt{\left(x-a\right)^2 + \left(y-b\right)^2}$.

15.2.87 For any $\epsilon > 0$, let $\delta = \frac{\epsilon}{2}$. Then $|x-a|, |y-b| \leq \sqrt{\left(x-a\right)^2 + \left(y-b\right)^2}$, so $0 < \sqrt{\left(x-a\right)^2 + \left(y-b\right)^2} < \delta$ which implies that $|x+y-\left(a+b\right)| \leq |x-a| + |y-b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

15.2.88 Suppose $\lim\limits_{(x,y)\to(a,b)} f\left(x,y\right) = L$, $\lim\limits_{(x,y)\to(a,b)} g\left(x,y\right) = M$. Let $\epsilon > 0$. Then there exist $\delta_1, \delta_2 > 0$ such that $0 < \sqrt{\left(x-a\right)^2 + \left(y-b\right)^2} < \delta_1 \implies |f\left(x,y\right) - L| < \frac{\epsilon}{2}$ and $0 < \sqrt{\left(x-a\right)^2 + \left(y-b\right)^2} < \delta_2 \implies |g\left(x,y\right) - M| < \frac{\epsilon}{2}$. Let $\delta = \min\left(\delta_1, \delta_2\right)$. Then $0 < \sqrt{\left(x-a\right)^2 + \left(y-b\right)^2} < \delta \implies$ $|f\left(x,y\right) + g\left(x,y\right) - \left(L+M\right)| \leq |f\left(x,y\right) - L| + |g\left(x,y\right) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

15.2.89 Observe first that this is trivial when $c = 0$, so assume $c \neq 0$ and let $\epsilon > 0$. Then there exists $\delta > 0$ such that $0 < \sqrt{\left(x-a\right)^2 + \left(y-b\right)^2} < \delta \implies |f\left(x,y\right) - L| < \frac{\epsilon}{|c|}$. Therefore $0 < \sqrt{\left(x-a\right)^2 + \left(y-b\right)^2} < \delta \implies$ $|cf\left(x,y\right) - cL| = |c||f\left(x,y\right) - L| < |c| \cdot \frac{\epsilon}{|c|} = \epsilon$.

15.3 Partial Derivatives

15.3.1 The slope parallel to the x-axis is $f_x\left(a,b\right)$, and the slope parallel to the y-axis is $f_y\left(a,b\right)$.

15.3.2

a. $f_x(x,y) = 6x$ and $f_y(x,y) = 3y^2$.

b. $f_x(1,3) = 6$ and $f_y(1,3) = 27$.

c. $f_{xx}(x,y) = 6$, $f_{yy}(x,y) = 6y$, $f_{xy}(x,y) = f_{yx}(x,y) = 0$.

15.3.3

a. Negative.

b. Negative.

c. Negative.

d. Positive.

15.3.4 $f_x\left(x,y\right) = 12x^5 + 2y$, $f_y\left(x,y\right) = 8y^7 + 2x$.

15.3.5 $f_x\left(x,y\right) = 6xy$, $f_y\left(x,y\right) = 3x^2$.

15.3.6 We have $f_x(x,y) = 2xy^3$, $f_y(x,y) = 3x^2y^2$; therefore $f_{xx}(x,y) = 2y^3$, $f_{yy}(x,y) = 6x^2y$, $f_{xy}(x,y) = f_{yx}(x,y) = 6xy^2$.

15.3.7 Observe that $f_x\left(x, \cos y\right) = 6x^2$, so $f_{xy}\left(x, \cos y\right) = 0$; and $f_y\left(x, \cos y\right) = 6y$, so $f_{yx}\left(x, \cos y\right) = 0$.

15.3.8 Observe that $f_x\left(x, \cos y\right) = e^y$, so $f_{xy}\left(x, \cos y\right) = e^y$; and $f_y\left(x, \cos y\right) = xe^y$, so $f_{yx}\left(x, \cos y\right) = e^y$.

15.3.9 $f_x\left(x, \cos y, \cos z\right) = y + z$; $f_y\left(x, \cos y, \cos z\right) = x + z$; $f_z\left(x, \cos y, \cos z\right) = x + y$.

15.3.10 Note that $\dfrac{\partial V}{\partial r} = 2\pi rh > 0$, so the volume is an increasing function of the radius r if the height h is fixed.

15.3.11 $f_x = \lim\limits_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h} = \lim\limits_{h\to 0} \dfrac{5(x+h)y - 5xy}{h} = \lim\limits_{h\to 0} \dfrac{5hy}{h} = \lim\limits_{h\to 0} 5y = 5y$.
$f_y = \lim\limits_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h} = \lim\limits_{h\to 0} \dfrac{5x(y+h) - 5xy}{h} = \lim\limits_{h\to 0} \dfrac{5xh}{h} = \lim\limits_{h\to 0} 5x = 5x$.

15.3.12 $f_x = \lim\limits_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h} = \lim\limits_{h\to 0} \dfrac{x+h+y^2+4-(x+y^2+4)}{h} = \lim\limits_{h\to 0} \dfrac{h}{h} = \lim\limits_{h\to 0} 1 = 1$.
$f_y = \lim\limits_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h} = \lim\limits_{h\to 0} \dfrac{x+(y+h)^2+4-(x+y^2+4)}{h} = \lim\limits_{h\to 0} \dfrac{y^2+2hy+h^2-y^2}{h} = \lim\limits_{h\to 0} 2y + h = 2y$.

15.3.13 $f_x = \lim\limits_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{x+h}{y} - \frac{x}{y}}{h} = \lim\limits_{h\to 0} \dfrac{h}{hy} = \lim\limits_{h\to 0} \dfrac{1}{y} = \dfrac{1}{y}$.
$f_y = \lim\limits_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{x}{y+h} - \frac{x}{y}}{h} = \lim\limits_{h\to 0} \dfrac{xy - x(y+h)}{y(y+h)h} = \lim\limits_{h\to 0} \dfrac{-x}{y(y+h)} = -\dfrac{x}{y^2}$.

15.3.14

$$\begin{aligned} f_x &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)y} - \sqrt{xy}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{(x+h)y} - \sqrt{xy}}{h} \cdot \frac{\sqrt{(x+h)y} + \sqrt{xy}}{\sqrt{(x+h)y} + \sqrt{xy}} \\ &= \lim_{h\to 0} \frac{xy + hy - xy}{h(\sqrt{(x+h)y} + \sqrt{xy})} = \frac{y}{2\sqrt{xy}}. \end{aligned}$$

$$\begin{aligned} f_y &= \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} = \lim_{h\to 0} \frac{\sqrt{x(y+h)} - \sqrt{xy}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x(y+h)} - \sqrt{xy}}{h} \cdot \frac{\sqrt{x(y+h)} + \sqrt{xy}}{\sqrt{x(y+h)} + \sqrt{xy}} \\ &= \lim_{h\to 0} \frac{xy + xh - xy}{h(\sqrt{x(y+h)} + \sqrt{xy})} = \frac{x}{2\sqrt{xy}}. \end{aligned}$$

15.3.15 $f_x(x,y) = e^y$, $f_y(x,y) = xe^y$.

15.3.16 $f_x(x,y) = 12x^2y^2 + 6xy^3 = 6xy^2(2x+y)$. $f_y(x,y) = 8x^3y + 9x^2y^2 = x^2y(8x+9y)$.

15.3.17 $f_x(x,y) = 2xye^{x^2y}$, $f_y(x,y) = x^2e^{x^2y}$.

15.3.18 $f_x(x,y) = 5(3xy+4y^2+1)^4(3y) = 15y(3xy+4y^2+1)^4$. $f_y(x,y) = 5(3xy+4y^2+1)^4(3x+8y) = 5(3x+8y)(3xy+4y^2+1)^4$.

15.3.19 $f_w(w,z) = \dfrac{\left(w^2+z^2\right)\cdot 1 - w\cdot 2w}{\left(w^2+z^2\right)^2} = \dfrac{z^2-w^2}{\left(w^2+z^2\right)^2}$, $f_z(w,z) = -w\left(w^2+z^2\right)^{-2}\cdot 2z = -\dfrac{2wz}{\left(w^2+z^2\right)^2}$.

15.3.20 $f_s(s,t) = \dfrac{(s+t)(1)-(s-t)(1)}{(s+t)^2} = \dfrac{2t}{(s+t)^2}$. $f_t(s,t) = \dfrac{(s+t)(-1)-(s-t)(1)}{(s+t)^2} = \dfrac{-2s}{(s+t)^2}$.

15.3.21

$$f_x(x,y) = \cos(xy) + x(-\sin(xy))\,y = \cos(xy) - xy\sin(xy),$$
$$f_y(x,y) = x(-\sin(xy))\,x = -x^2\sin(xy).$$

15.3.22 $f_x(x,y) = \dfrac{1}{(x^4/y^4)+1}\cdot\dfrac{2x}{y^2} = \dfrac{2xy^2}{x^4+y^4}$. $f_y(x,y) = \dfrac{1}{(x^4/y^4)+1}\cdot\dfrac{-2x^2}{y^3} = -\dfrac{2x^2y}{x^4+y^4}$.

15.3.23 $s_y(y,z) = z^2\left(\sec^2 yz\right)z = z^3\sec^2 yz$, $s_z(y,z) = 2z\tan yz + z^2\left(\sec^2 yz\right)y = 2z\tan yz + yz^2\sec^2 yz$.

15.3.24 $g_x(x,z) = 1\cdot\ln\left(z^2+x^2\right) + x\cdot\dfrac{2x}{z^2+x^2} = \ln\left(z^2+x^2\right) + \dfrac{2x^2}{z^2+x^2}$, $g_z(x,z) = x\cdot\dfrac{2z}{z^2+x^2} = \dfrac{2xz}{z^2+x^2}$.

15.3.25 $G_s(s,t) = \dfrac{t}{2\sqrt{st}}\cdot\dfrac{1}{s+t} + \sqrt{st}\cdot -\dfrac{1}{(s+t)^2} = \dfrac{\sqrt{st}\,(s+t)-2s\sqrt{st}}{2s(s+t)^2} = \dfrac{\sqrt{st}\,(t-s)}{2s(s+t)^2}$, $G_t(s,t) = \dfrac{s}{2\sqrt{st}}\cdot \dfrac{1}{s+t} + \sqrt{st}\cdot -\dfrac{1}{(s+t)^2} = \dfrac{\sqrt{st}\,(s+t)-2t\sqrt{st}}{2t(s+t)^2} = \dfrac{\sqrt{st}\,(s-t)}{2t(s+t)^2}$.

15.3.26

$$F_p(p,q) = \frac{1}{2}\left(p^2+pq+q^2\right)^{-1/2}(2p+q) = \frac{2p+q}{2\sqrt{p^2+pq+q^2}},$$
$$F_q(p,q) = \frac{1}{2}\left(p^2+pq+q^2\right)^{-1/2}(p+2q) = \frac{p+2q}{2\sqrt{p^2+pq+q^2}}.$$

15.3.27 $f_x(x,y) = 2yx^{2y-1}$, $f_y(x,y) = 2x^{2y}\ln x$.

15.3.28

$$g_x(x,y) = 5\cos^4(x^2y^3)(-\sin(x^2y^3))(2xy^3) = -10xy^3\sin(x^2y^3)\cos^4(x^2y^3)$$

.

$$g_y(x,y) = 5\cos^4(x^2y^3)(-\sin(x^2y^3))(3x^2y^2) = -15x^2y^2\sin(x^2y^3)\cos^4(x^2y^3).$$

15.3.29 $h_x(x,y) = 1 - \dfrac{2x}{2\sqrt{x^2-4y}} = \dfrac{\sqrt{x^2-4y}-x}{\sqrt{x^2-4y}}$. $h_y(x,y) = \dfrac{4}{2\sqrt{x^2-4y}} = \dfrac{2}{\sqrt{x^2-4y}}$.

15.3.30 $h_u(u,v) = \dfrac{1}{2}\left(\dfrac{uv}{u-v}\right)^{-1/2}\left(\dfrac{(u-v)\,v - uv\cdot 1}{(u-v)^2}\right) = -\dfrac{1}{2}u^{-1/2}v^{3/2}(u-v)^{-3/2}$,
$h_{\mathrm{v}}(u,v) = \dfrac{1}{2}\left(\dfrac{uv}{u-v}\right)^{-1/2}\left(\dfrac{(u-v)\,u - uv\cdot(-1)}{(u-v)^2}\right) = \dfrac{1}{2}u^{3/2}v^{-1/2}(u-v)^{-3/2}$

15.3.31 Using the Fundamental Theorem of Calculus, we have $f_x(x,y) = -e^{x^2}$, and $f_y(x,y) = 3y^2e^{y^6}$.

15.3.32 $g_x(x,y) = \dfrac{y}{\sqrt{1-xy}}\cdot\dfrac{1}{2\sqrt{xy}}\cdot y = \dfrac{y^2}{2\sqrt{xy}\sqrt{1-xy}}$. $g_y(x,y) = \sin^{-1}\sqrt{xy} + \dfrac{y}{\sqrt{1-xy}}\cdot\dfrac{1}{2\sqrt{xy}}\cdot x = \dfrac{xy+2\sqrt{xy}\sqrt{1-xy}\sin^{-1}\sqrt{xy}}{2\sqrt{xy}\sqrt{1-xy}}$.

15.3.33 We have $f_x(x,y) = -\dfrac{2x}{1+\left(x^2+y^2\right)^2}$ and $f_y(x,y) = -\dfrac{2y}{1+\left(x^2+y^2\right)^2}$.

15.3.34 We have $f_x(x,y) = -\dfrac{ye^{-xy}}{1+e^{-xy}}$ and $f_y(x,y) = -\dfrac{xe^{-xy}}{1+e^{-xy}}$.

15.3.35 We have $h_x(x,y,z) = (1+2y)^x \ln(1+2y)$, $h_y(x,y,z) = x(1+2y)^{x-1}$, and $h_x(x,y,z) = (1+x+2y)^z \ln(1+x+2y)$.

15.3.36 We have $f_x(x,y) = f_y(x,y) = 2\sin(2(x+y)) - 2\cos(x+y)\sin(x+y)$.

15.3.37 By the Fundamental Theorem of Calculus, $f_x(x,y) = -\dfrac{\partial}{\partial x}\displaystyle\int_x^y h(s)\,ds = -h(x)$ and similarly $f_y(x,y) = h(y)$.

15.3.38 We have $f_x(x,y) = 2x\sin y$, $f_y(x,y) = x^2\cos y$; therefore $f_{xx}(x,y) = 2\sin y$, $f_{yy}(x,y) = -x^2\sin y$, $f_{xy}(x,y) = f_{yx}(y,x) = 2x\cos y$.

15.3.39 We have $h_x(x,y) = 3x^2+y^2$, $h_y(x,y) = 2xy$; therefore $h_{xx}(x,y) = 6x$, $h_{yy}(x,y) = 2x$, $h_{xy}(x,y) = h_{yx}(y,x) = 2y$.

15.3.40 We have $f_x(x,y) = 10x^4y^2 + 2xy$, $f_y(x,y) = 4x^5y + x^2$; therefore $f_{xx}(x,y) = 40x^3y^2 + 2y$, $f_{yy}(x,y) = 4x^5$, $f_{xy}(x,y) = f_{yx}(y,x) = 20x^4y + 2x$.

15.3.41 We have $f_x(x,y) = 4y^3\cos 4x$, $f_y(x,y) = 3y^2\sin 4x$; therefore $f_{xx}(x,y) = -16y^3\sin 4x$, $f_{yy}(x,y) = 6y\sin 4x$, $f_{xy}(x,y) = f_{yx}(y,x) = 12y^2\cos 4x$.

15.3.42

$$f_x(x,y) = 2\sin(x^3y)\cos(x^3y)(3x^2y) = 6x^2y\sin(x^3y)\cos(x^3y) = 3x^2y\sin(2x^3y),$$

while

$$f_y(x,y) = 2\sin(x^3y)\cos(x^3y)(x^3) = 2x^3\sin(x^3y)\cos(x^3y) = x^3\sin(2x^3y).$$

Then,

$$f_{xx}(x,y) = 6xy\sin(2x^3y) + 3x^2y\cos(2x^3y)(6x^2y) = 6xy(\sin(2x^3y) + 3x^3y\cos(2x^3y)),$$

and

$$f_{xy}(x,y) = 3x^2\sin(2x^3y) + 3x^2y\cos(2x^3y)(2x^3) = 3x^2(\sin(2x^3y) + 2x^3y\cos(2x^3y)),$$

and

$$f_{yx}(x,y) = 3x^2\sin(2x^3y) + x^3\cos(2x^3y)(6x^2y) = 3x^2(\sin(2x^3y) + 2x^3y\cos(2x^3y)),$$

and

$$f_{yy}(x,y) = x^3\cos(2x^3y)(2x^3) = 2x^6\cos(2x^3y).$$

15.3.43 We have

$$p_u(u,v) = \frac{2u}{u^2+v^2+4},$$

$$p_v(u,v) = \frac{2v}{u^2+v^2+4}.$$

Therefore

$$p_{uu}(u,v) = \frac{\left(u^2+v^2+4\right)\cdot 2 - 2u\cdot 2u}{\left(u^2+v^2+4\right)^2} = \frac{-2u^2+2v^2+8}{\left(u^2+v^2+4\right)^2},$$

$$p_{vv}(u,v) = \frac{\left(u^2+v^2+4\right)\cdot 2 - 2v\cdot 2v}{\left(u^2+v^2+4\right)^2} = \frac{2u^2-2v^2+8}{\left(u^2+v^2+4\right)^2},$$

and

$$p_{uv}(u,v) = p_{vu}(u,v) = -2u\left(u^2+v^2+4\right)^{-2}\cdot 2v = -\frac{4uv}{\left(u^2+v^2+4\right)^2}.$$

15.3.44

$$Q_r(r,s) = \frac{3r^2se^{r^3s}}{s} = 3r^2e^{r^3s},$$

while

$$Q_s(r,s) = \frac{se^{r^3s}r^3 - e^{r^3s}}{s^2} = \frac{e^{r^3s}(r^3s-1)}{s^2}.$$

So

$$Q_{rr}(r,s) = 6re^{r^3s} + 3r^2e^{r^3s}(3r^2s) = 3re^{r^3s}(2+3r^3s),$$

while

$$Q_{rs}(r,s) = Q_{sr}(r,s) = 3r^2e^{r^3s}(r^3) = 3r^5e^{r^3s},$$

and

$$Q_{ss}(r,s) = \frac{s^2(r^3e^{r^3s}(r^3s-1) + e^{r^3s}(r^3)) - e^{r^3s}(r^3s-1)(2s)}{s^4} = \frac{e^{r^3s}(r^6s^2 - 2r^3s + 2)}{s^3}.$$

15.3.45 We have $F_r(r,s) = e^s$, $F_s(r,s) = re^s$; therefore $F_{rr}(r,s) = 0$, $F_{ss}(r,s) = re^s$, $F_{rs}(r,s) = F_{sr}(r,s) = e^s$.

15.3.46 We have

$$H_x(x,y) = \frac{x}{\sqrt{4+x^2+y^2}},$$
$$H_y(x,y) = \frac{y}{\sqrt{4+x^2+y^2}}.$$

Therefore

$$H_{xx}(x,y) = \frac{\left(\sqrt{4+x^2+y^2}\right)\cdot 1 - x\dfrac{x}{\sqrt{4+x^2+y^2}}}{4+x^2+y^2} = \frac{4+y^2}{(4+x^2+y^2)^{3/2}},$$

and

$$H_{yy}(x,y) = \frac{\left(\sqrt{4+x^2+y^2}\right)\cdot 1 - y\dfrac{y}{\sqrt{4+x^2+y^2}}}{4+x^2+y^2} = \frac{4+x^2}{(4+x^2+y^2)^{3/2}},$$

and

$$H_{xy}(x,y) = H_{yx}(x,y) = -\frac{1}{2}x\left(4+x^2+y^2\right)^{-3/2}\cdot 2y = -\frac{xy}{(4+x^2+y^2)^{3/2}}.$$

15.3.47

$$f_x(x,y) = \frac{1}{1+x^6y^4}\cdot(3x^2y^2) = \frac{3x^2y^2}{1+x^6y^4},$$

and

$$f_y(x,y) = \frac{1}{1+x^6y^4}\cdot(2x^3y) = \frac{2x^3y}{1+x^6y^4}.$$

Then

$$f_{xx}(x,y) = \frac{(1+x^6y^4)(6xy^2) - (3x^2y^2)(6x^5y^4)}{(1+x^6y^4)^2} = \frac{6xy^2(1-2x^6y^4)}{(1+x^6y^4)^2},$$

and

$$f_{xy}(x,y) = f_{yx}(x,y) = \frac{(1+x^6y^4)(6x^2y) - (3x^2y^2)(4x^6y^3)}{(1+x^6y^4)^2} = \frac{6x^2y(1-x^6y^4)}{(1+x^6y^4)^2},$$

and

$$f_{yy}(x,y) = \frac{(1+x^6y^4)(2x^3) - (2x^3y)(4x^6y^3)}{(1+x^6y^4)^2} = \frac{2x^3(1-3x^6y^4)}{(1+x^6y^4)^2}.$$

15.3.48 Observe that $f_x\,(x,y) = 6xy^{-1}+2x^{-2}y^2$, so $f_{xy}\,(x,y) = -6xy^{-2}+4x^{-2}y$; and $f_y\,(x,y) = -3x^2y^{-2}-4x^{-1}y$, so $f_{yx}\,(x,y) = -6xy^{-2}+4x^{-2}y$.

15.3.49 Observe that $f_x\,(x,y) = e^{x+y}$, so $f_{xy}\,(x,y) = e^{x+y}$; and $f_y\,(x,y) = e^{x+y}$, so $f_{yx}\,(x,y) = e^{x+y}$.

15.3.50 Observe that $f_x\,(x,y) = \dfrac{1}{2}x^{-1/2}y^{1/2}$, so $f_{xy}\,(x,y) = \dfrac{1}{4}x^{-1/2}y^{-1/2} = \dfrac{1}{\sqrt{4xy}}$; and $f_y\,(x,y) = \dfrac{1}{2}x^{1/2}y^{-1/2}$, so $f_{yx}\,(x,y) = \dfrac{1}{4}x^{-1/2}y^{-1/2} = \dfrac{1}{\sqrt{4xy}}$.

15.3.51 Observe that $f_x\,(x,y) = -y\sin xy$, so $f_{xy}\,(x,y) = -\sin xy - xy\cos xy$; and $f_y\,(x,y) = -x\sin xy$, so $f_{yx}\,(x,y) = -\sin xy - xy\cos xy$.

15.3.52 Note that $f_x(x,y) = e^{\sin xy}y\cos xy$, so

$$f_{xy} = e^{\sin xy}x\cos xy(y\cos xy) + e^{\sin xy}(\cos xy - yx\sin xy) = e^{\sin xy}(xy\cos^2 xy + \cos xy - yx\sin xy).$$

Also $f_y(x,y) = e^{\sin xy}(x\cos xy)$. So

$$f_{yx}(x,y) = e^{\sin xy}y\cos xy(x\cos xy) + e^{\sin xy}(\cos xy - yx\sin xy) = e^{\sin xy}(xy\cos^2 xy + \cos xy - yx\sin xy).$$

15.3.53 Note that $f_x(x,y) = 4(2x-y^3)^3(2) = 8(2x-y^3)^3$, so $f_{xy} = 24(2x-y^3)^2(-3y^2) = -72y^2(2x-y^3)^2$. Also, $f_y(x,y) = 4(2x-y^3)^3(-3y^2) = -12y^2(2x-y^3)^3$. Then $f_{yx}(x,y) = -12y^2(3(2x-y^3)^2(2)) = -72y^2(2x-y^3)^2$.

15.3.54 $G_r\,(r,s,t) = \dfrac{s+t}{2\sqrt{rs+rt+st}}$; $G_s\,(r,s,t) = \dfrac{r+t}{2\sqrt{rs+rt+st}}$; $G_t\,(r,s,t) = \dfrac{r+s}{2\sqrt{rs+rt+st}}$.

15.3.55 $h_x\,(x,y,z) = h_y\,(x,y,z) = h_z\,(x,y,z) = -\sin(x+y+z)$.

15.3.56 $g_x\,(x,y,z) = 4xy-3z^4$; $f_y\,(x,y,z) = 2x^2+20yz^2$; $f_z\,(x,y,z) = -12xz^3+20y^2z$.

15.3.57 $F_u\,(u,v,w) = \dfrac{1}{v+w}$; $F_v\,(u,v,w) = -\dfrac{u}{(v+w)^2}$; $F_w\,(u,v,w) = -\dfrac{u}{(v+w)^2}$.

15.3.58 $Q_x\,(x,y,z) = yz\sec^2 xyz$; $Q_y\,(x,y,z) = xz\sec^2 xyz$; $Q_z\,(x,y,z) = xy\sec^2 xyz$.

15.3.59 $G_r(r,s,t) = \dfrac{s^3t^5}{2\sqrt{rs^3t^5}}$; $G_s(r,s,t) = \dfrac{3rs^2t^5}{2\sqrt{rs^3t^5}}$; $G_t(r,s,t) = \dfrac{5rs^3t^4}{2\sqrt{rs^3t^5}}$.

15.3.60 $g_w\,(w,x,y,z) = g_x\,(w,x,y,z) = -\sin(w+x)\sin(y-z)$; $g_y\,(w,x,y,z) = \cos(w+x)\cos(y-z)$; $g_z\,(w,x,y,z) = -\cos(w+x)\cos(y-z)$.

15.3.61 $h_w\,(w,x,y,z) = \dfrac{z}{xy}$; $h_x\,(w,x,y,z) = -\dfrac{wz}{x^2y}$; $h_y\,(w,x,y,z) = -\dfrac{wz}{xy^2}$; $h_z\,(w,x,y,z) = \dfrac{w}{xy}$.

15.3.62 $F_w\,(w,x,y,z) = \sqrt{x+2y+3z}$; $F_x\,(w,x,y,z) = \dfrac{w}{2\sqrt{x+2y+3z}}$; $F_y\,(w,x,y,z) = \dfrac{w}{\sqrt{x+2y+3z}}$; $F_z\,(w,x,y,z) = \dfrac{3w}{2\sqrt{x+2y+3z}}$.

15.3.63

a. $R'(t) = \dfrac{(ct+d)a-(at+b)c}{(ct+d)^2} = \dfrac{ad-bc}{(ct+d)^2}$.

b. $g_x(x,y,z) = -\dfrac{8z}{3(2x-y+z)^2}$; $g_y(x,y,z) = \dfrac{4z}{3(2x-y+z)^2}$; $g_z(x,y,z) = \dfrac{4(2x-y)}{3(2x-y+x)^2}$.

15.3.64 $f_x(2,3) \approx \dfrac{f(2.1,3)-f(2,3)}{0.1} = \dfrac{4.347-4.243}{0.1} = 1.04.$

15.3.65 $f_y(2,3) \approx \dfrac{f(2,3.1)-f(2,3)}{0.1} = \dfrac{4.384-4.243}{0.1} = 1.41.$

15.3.66 $f_x(2.2,3.4) \approx \dfrac{f(2.3,3.4)-f(2.2,3.4)}{0.1} = \dfrac{5.156-5.043}{0.1} = 1.13.$ Answers may vary.

15.3.67 $f_y(2.4,3.3) \approx \dfrac{f(2.4,3.4)-f(2.4,3.3)}{0.1} = \dfrac{5.267-5.112}{0.1} = 1.55.$ Answers may vary.

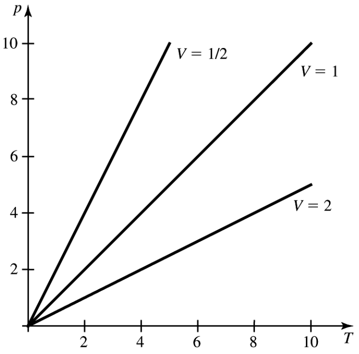
15.3.68 $f_x(0.42,0.5) \approx \dfrac{1.75-1.5}{0.5-0.42} \approx 3.125$; $f_y(0.42,0.5) \approx \dfrac{1.75-1.5}{0.7-0.5} \approx 1.25.$ Answers may vary.

15.3.69

a. We have $V = \dfrac{kT}{P}$, so $\dfrac{\partial V}{\partial P} = -\dfrac{kT}{P^2}$. Because this partial derivative is negative, the volume decreases as the pressure increases at a fixed temperature.

b. We have $\dfrac{\partial V}{\partial T} = \dfrac{k}{P}$. Because this partial derivative is positive, the volume increases as the temperature increases at a fixed pressure.

c.



15.3.70

a. We have $B_w = \dfrac{1}{h^2}$.

b. The partial derivative $B_w > 0$ so B is an increasing function of w.

c. We have $B_h = -\dfrac{2w}{h^3}$.

d. The partial derivative $B_h < 0$ so B is a decreasing function of h.

15.3.71

a. Solving for R gives $R = \left(R_1^{-1} + R_2^{-1}\right)^{-1}$, so $\dfrac{\partial R}{\partial R_1} = -\left(R_1^{-1}+R_2^{-1}\right)^{-2}\left(-R_1^{-2}\right) = \dfrac{R_2^2}{\left(R_1+R_2\right)^2}$ and similarly $\dfrac{\partial R}{\partial R_2} = \dfrac{R_1^2}{\left(R_1+R_2\right)^2}$.

b. We have $-R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2} \Longrightarrow \dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$ and similarly $\dfrac{\partial R}{\partial R_2} = \dfrac{R^2}{R_2^2}$.

c. Because $\dfrac{\partial R}{\partial R_1} > 0$, an increase in R_1 causes an increase in R.

d. Because $\dfrac{\partial R}{\partial R_2} > 0$, a decrease in R_2 causes a decrease in R.

15.3.72

a. We have $V_h = \frac{\pi}{3}\left(3r \cdot 2h - 3h^2\right) = \pi\left(2rh - h^2\right)$ and $V_r = \pi h^2$.

b. Because $V_r = \pi h^2$, V_r is greater when $h = 0.8r$.

c. Solve $V_r = \pi h^2 = 1$ to obtain $h = \pi^{-1/2}$.

d. The maximum value of $2rh - h^2$ as a function of h occurs when $2r - 2h = 0$, which gives $h = r$.

15.3.73 We see that $u_t = -16e^{-4t}\cos 2x = u_{xx}$.

15.3.74 We see that $u_t = -10e^{-t}\sin x = u_{xx}$.

15.3.75 We see that $u_t = -a^2Ae^{-a^2t}\cos ax = u_{xx}$.

15.3.76 We see that $u_t = -e^{-t}\left(2\sin x + 3\cos x\right) = u_{xx}$.

15.3.77

a. Observe that as $f(x,y) = 0$ along either coordinate axis but on the line $y = x$, $f(x,y) = -\frac{x^2}{2x^2} = -\frac{1}{2}$, so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, and hence f is not continuous at $(0,0)$.

b. By Theorem 15.6, f is not differentiable at $(0,0)$.

c. Because f is identically 0 on the coordinate axes, $f_x(0,0) = f_y(0,0) = 0$.

d. We have $f_x(x,y) = -\left(\frac{\left(x^2+y^2\right)y - xy \cdot 2x}{\left(x^2+y^2\right)^2}\right) = \frac{\left(x^2-y^2\right)y}{\left(x^2+y^2\right)^2}$. Along the line $x = 2y$, $f_x(x,y) = \frac{3y^3}{25y^4} = \frac{3}{25}\cdot\frac{1}{y}$, which does not converge to 0 as $y \to 0$. Hence f_x is not continuous at $(0,0)$. A similar argument shows that f_y is also not continuous at $(0,0)$.

e. Theorem 15.5 does not apply because the partials f_x and f_y are not continuous at $(0,0)$, and Theorem 15.6 does not apply because f is not differentiable at $(0,0)$.

15.3.78

a. Observe that as $f(x,y) = 0$ along either coordinate axis but on the curve $x = y^2$, $f(x,y) = \frac{2y^4}{2y^4} = 1$, so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, and hence f is not continuous at $(0,0)$.

b. By Theorem 15.6, f is not differentiable at $(0,0)$.

c. Because f is identically 0 on the coordinate axes, $f_x(0,0) = f_y(0,0) = 0$.

d. We have $f_x(x,y) = \frac{\left(x^2+y^4\right)2y^2 - 2xy^2\left(2x\right)}{\left(x^2+y^4\right)^2} = \frac{2y^2\left(y^4-x^2\right)}{\left(x^2+y^4\right)^2}$. Along the curve $x = 2y^2$, $f_x(x,y) = -\frac{6y^6}{25y^8} = -\frac{6}{25}\cdot\frac{1}{y^2}$, which does not converge to 0 as $y \to 0$. Hence f_x is not continuous at $(0,0)$. We also have $f_y(x,y) = \frac{\left(x^2+y^4\right)4xy - 2xy^2\left(4y^3\right)}{\left(x^2+y^4\right)^2} = \frac{4xy\left(x^2-y^4\right)}{\left(x^2+y^4\right)^2}$. Along the curve $x = 2y^2$, $f_y(x,y) = -\frac{24y^7}{25y^8} = -\frac{24}{25}\cdot\frac{1}{y}$, which does not converge to 0 as $y \to 0$. Hence f_y is not continuous at $(0,0)$.

e. Theorem 15.5 does not apply because the partials f_x and f_y are not continuous at $(0,0)$, and Theorem 15.6 does not apply because f is not differentiable at $(0,0)$.

15.3.79

a. False. $\frac{\partial}{\partial x} y^{10} = 0$ because x and y are independent variables.

b. False. $\frac{\partial^2}{\partial x\, \partial y}(xy)^{1/2} = \frac{1}{2} \cdot x^{-1/2} \cdot \frac{1}{2} y^{-1/2} = \frac{1}{4\sqrt{xy}}$.

c. True. If f has continuous partial derivatives of all orders, then the order of differentiation for mixed partials can be exchanged.

15.3.80

a. There are 3 choices for each of the variables we differentiate with respect to, so there are 9 possible second partial derivatives: w_{xx}, w_{yy}, w_{zz}, w_{xy}, w_{yx}, w_{xz}, w_{zx}, w_{yz}, w_{zy}.

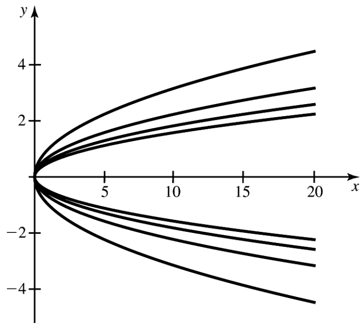
b. This function has continuous partial derivatives of all orders, so $w_{xy} = w_{yx}$, $w_{xz} = w_{zx}$ and $w_{yz} = w_{zy}$.

c. There are 4 choices for each of the variables we differentiate with respect to, so there are 16 possible second partial derivatives.

15.3.81

a. We have $z_x = \frac{1}{y^2}$ and $z_y = -\frac{2x}{y^3}$.

b.



c. We observe that z increases at the same rate as x, which makes sense because $z_x = 1$ along this line.

d. We observe that z increases when $y < 0$, is undefined when $y = 0$ and decreases when $y > 0$, which is consistent with $z_y = -\frac{2}{y^3}$ along this line.

15.3.82

a. $V_x = 2xh$, $V_h = x^2$.

b. $\triangle V \approx 2xh\triangle x = 2 \cdot 0.5 \cdot 1.5 \cdot 0.01 = 0.015\,\text{m}^3$.

c. $\triangle V = x^2\triangle h = (0.5)^2(-0.01) = -0.0025\,\text{m}^3$ (notice that because V is a linear function of h, the linear approximation is exact).

d. Notice that for fixed height h, $\frac{\triangle V}{V} \approx \frac{2xh\triangle x}{x^2 h} = 2\frac{\triangle x}{x}$. Therefore a 10% change in x will produce (approximately) a 20% change in V .

e. Notice that for fixed base x, $\frac{\triangle V}{V} = \frac{\triangle h}{h}$, so a 10% change in h will produce a 10% change in V .

15.3.83

a. We have $\varphi_x(x,y) = -\dfrac{2x}{\left(x^2+(y-1)^2\right)^{3/2}} - \dfrac{x}{\left(x^2+(y+1)^2\right)^{3/2}}$ and $\varphi_y(x,y) = -\dfrac{2(y-1)}{\left(x^2+(y-1)^2\right)^{3/2}} - \dfrac{y+1}{\left(x^2+(y+1)^2\right)^{3/2}}$.

b. Observe that $|\varphi_x(x,y)| \le \dfrac{2|x|}{|x|^{3/2}} + \dfrac{|x|}{|x|^{3/2}} = \dfrac{3}{|x|^{1/2}}$ and similarly $|\varphi_y(x,y)| \le \dfrac{2|y-1|}{|y-1|^{3/2}} + \dfrac{|y+1|}{|y+1|^{3/2}} = \dfrac{2}{|y-1|^{1/2}} + \dfrac{1}{|y+1|^{1/2}}$, which both converge to 0 as $x, y \to \infty$.

c. We see that $\varphi_x(0,y) = 0$ as long as $y \neq \pm 1$. This is consistent with the observation that along horizontal lines $y = y_0$ the potential function takes its maximum at $x = 0$.

d. We see that $\varphi_y(x,0) = \dfrac{1}{(x^2+1)^{3/2}}$. This implies that if we cross the x-axis at any point from below to above, the potential function is increasing.

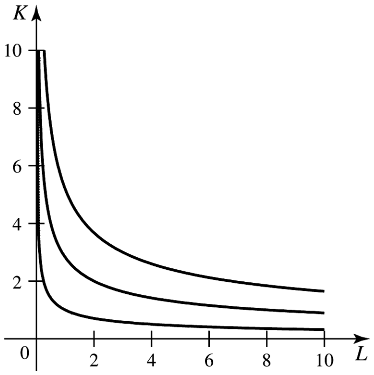
15.3.84

a. We have $Q_L = \dfrac{1}{3}L^{-2/3}K^{2/3}$ and $Q_K = \dfrac{2}{3}L^{1/3}K^{-1/3}$.

b. We have $\triangle Q \approx Q_K(10,20)\triangle K = \dfrac{2}{3}(10)^{1/3}(20)^{-1/3} \cdot 0.5 \approx 0.2646$.

c. We have $\triangle Q \approx Q_L(10,20)\triangle L = \dfrac{1}{3}(10)^{-2/3}(20)^{2/3} \cdot (-0.5) \approx -0.2646$.

d.



e. As we move along the vertical line $L = 2$ in the positive K-direction, Q increases, which is consistent with $Q_K > 0$.

f. As we move along the horizontal line $K = 2$ in the positive L-direction, Q increases, which is consistent with $Q_L > 0$.

15.3.85 Observe that $f_x(x,y) = \dfrac{(cx+dy)a - (ax+by)c}{(cx+dy)^2} = \dfrac{y(ad-bc)}{(cx+dy)^2} = 0$; similarly, $f_y(x,y) = 0$. Suppose $a \neq 0$ (other cases can be handled similarly). Then $ad = bc$ implies $d = \dfrac{bc}{a}$; this also shows that $c \neq 0$, otherwise both c and $d = 0$ and the function is undefined. Hence $ax + by = \dfrac{a}{c}\left(cx + \dfrac{bc}{a}y\right) = \dfrac{a}{c}(cx+dy)$, so $f(x,y) = \dfrac{a}{c}$ is a constant function which implies $f_x = f_y = 0$.

15.3.86

a. The period T is found by solving $\dfrac{\pi t}{2} = 2\pi$, so $t = 4$.

b. We have $u_t = 2\sin(\pi x)\cos\left(\dfrac{\pi t}{2}\right) \cdot \dfrac{\pi}{2} = \pi\,\sin(\pi x)\cos\left(\dfrac{\pi t}{2}\right)$.

c. For fixed t, the quantity u_t is largest when $\sin(\pi x) = \pm 1$, which occurs at $x = \dfrac{1}{2}$ (the middle of the string).

d. For fixed x, the quantity u_t is largest when $\cos\left(\dfrac{\pi t}{2}\right) = \pm 1$, which occurs at $t = 2k$ for any integer k. At these times the string is in its rest position (no displacement).

e. We have $u_x = 2\pi\cos(\pi x)\cos\left(\dfrac{\pi t}{2}\right)$.

f. For fixed t, the slope is greatest when $\cos(\pi x) = \pm 1$, which occurs at the endpoints of the string $(x = 0, 1)$.

15.3.87 Observe that $\dfrac{\partial^2 u}{\partial t^2} = -4c^2\cos(2(x+ct)) = c^2\dfrac{\partial^2 u}{\partial x^2}$.

15.3.88 Observe that $\dfrac{\partial^2 u}{\partial t^2} = -20c^2\cos(2(x+ct)) - 3c^2\sin(x-ct) = c^2\dfrac{\partial^2 u}{\partial x^2}$.

15.3.89 Observe that $\dfrac{\partial^2 u}{\partial t^2} = Ac^2 f''(x+ct) + Bc^2 g''(x-ct) = c^2\dfrac{\partial^2 u}{\partial x^2}$.

15.3.90 Observe that $u_{xx} + u_{yy} = e^{-x}\sin\,y + e^{-x}(-\sin\,y) = 0$.

15.3.91 Observe that $u_{xx} + u_{yy} = 6x - 6x = 0$.

15.3.92 Observe that $u_{xx} + u_{yy} = a^2 e^{ax}\cos\,ay + e^{ax}\left(-a^2\cos\,ay\right) = 0$

15.3.93 Observe that

$$u_{xx} = \frac{2(x-1)y}{\left[(x-1)^2+y^2\right]^2} - \frac{2(x+1)y}{\left[(x+1)^2+y^2\right]^2},$$

and

$$u_{yy} = -\frac{2(x-1)y}{\left[(x-1)^2+y^2\right]^2} + \frac{2(x+1)y}{\left[(x+1)^2+y^2\right]^2};$$

so

$$u_{xx} + u_{yy} = 0.$$

15.3.94 We have $f(0,0) = 0$, $f_x(0,0) = f_y(0,0) = 1$, and $f(\triangle x, \triangle y) = 1 \cdot \triangle x + 1 \cdot \triangle y$, so we can take $\epsilon_1 = \epsilon_2 = 0$.

15.3.95 We have $f(0,0) = 0$, $f_x(0,0) = f_y(0,0) = 0$, and $f(\triangle x, \triangle y) = \triangle x \cdot \triangle y$, so we can take $\epsilon_1 = \triangle y$, $\epsilon_2 = 0$ or $\epsilon_1 = 0$, $\epsilon_2 = \triangle x$.

15.3.96

a. Observe that $\lim\limits_{(x,y)\to(0,0)} (1 - |\,xy\,|) = 1 = f(0,0)$, so f is continuous at $(0,0)$.

b. Let $(a,b) = (0,0)$; then $f(a + \triangle x, b + \triangle y) - f(a,b) = -|\triangle x\,||\triangle y\,| = \epsilon_1 \triangle x$ where $\epsilon_1 = \pm\triangle y$ (depending on the sign of x). Because $\epsilon_1 \to 0$ as $\triangle y \to 0$, we see that f is differentiable at $(0,0)$.

c. Because f is identically equal to 1 on the coordinate axes, $f_x(0,0) = f_y(0,0) = 0$.

d. Because $f(x,y) = 1 - |x|\,|y|$,

$$f_x(x,y) = \begin{cases} |y| & \text{if } x < 0, \\ -|y| & \text{if } x > 0. \end{cases}$$

So $|f_x(x,y)| = |y|$ if $x \neq 0$. If $y \neq 0$, $f_x(0,y)$ is undefined because $\frac{d}{dx}|x|$ is undefined at $x = 0$. By part c), $f_x(0,0)$ is defined and equals 0. So the domain of f_x is all of $\mathbb{R}^2$ except points (x,y) where $x = 0$ and $y \neq 0$.

Let $\epsilon > 0$ be given. Assume (x,y) is in the domain of f_x and $\sqrt{(x-0)^2 + (y-0)^2} < \delta$, where $\delta = \epsilon$. Then $x \neq 0$, $y \neq 0$ and $|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2} < \epsilon$. This implies that

$$|f_x(x,y) - f_x(0,0)| = |f_x(x,y)| = |y| < \epsilon.$$

So $\lim\limits_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0) = 0$. Therefore f_x is continuous at $(0,0)$. A similar argument shows that f_y is continuous at $(0,0)$.

e. Theorem 15.5 does not apply because we have shown that f_x and f_y are undefined at points in any open set containing $(0,0)$. Theorem 15.6 holds by parts c) and d). This exercise shows that the converse of Theorem 15.5 doesn't hold in general.

15.3.97

a. Observe that $\lim\limits_{(x,y)\to(0,0)} \sqrt{|xy|} = 0 = f(0,0)$, so f is continuous at $(0,0)$.

b. Let $(a,b) = (0,0)$, and suppose that $f(a + \triangle x, b + \triangle y) - f(a,b) = \sqrt{|\triangle x \triangle y|} = \epsilon_1 \triangle x + \epsilon_2 \triangle y$. Let $\triangle x = \triangle y$; then we obtain $\sqrt{|\triangle x|^2} = |\triangle x| = (\epsilon_1 + \epsilon_2)\triangle x$ which implies that $\epsilon_1 + \epsilon_2 = \pm 1$, and so we cannot have $\epsilon_1, \epsilon_2 \to 0$, as $\triangle x, \triangle y \to 0$. Therefore f is not differentiable at $(0,0)$.

c. Because f is identically equal to 0 on the coordinate axes, $f_x(0,0) = f_y(0,0) = 0$.

d. Because $f_x(x,y) = 1 - \sqrt{|x|}\sqrt{|y|}$,

$$f_x(x,y) = \begin{cases} -\frac{\sqrt{|y|}}{2\sqrt{|x|}} & \text{if } x < 0 \\ \frac{\sqrt{|y|}.}{2\sqrt{|x|}} & \text{if } x > 0. \end{cases}$$

Observe that along $y = x$, $x > 0$, $\lim\limits_{(x,y)\to(0,0)} f_x(x,y) = \dfrac{1}{2}$, and along $y = x$, $x < 0$, $\lim\limits_{(x,y)\to(0,0)} f_x(x,y) = -\dfrac{1}{2}$. So by the Two Path test, $\lim\limits_{(x,y)\to(0,0)} f_x(x,y)$ doesn't exist. Therefore f_x is not continuous at $(0,0)$.

e. Theorem 15.5 does not apply because the partials f_x and f_y are not defined at $(0,0)$. Theorem 15.6 does not apply because f is not differentiable at $(0,0)$. This exercise shows that the converse of Theorem 15.6 isn't true in general.

15.3.98

a. Observe that $u_x = 2x = v_y$ and $u_y = -2y = -v_x$.

b. Observe that $u_x = 3x^2 - 3y^2 = v_y$ and $u_y = -6xy = -v_x$.

c. We have $u_{xx} = v_{yx} = v_{xy} = -u_{yy}$, so $u_{xx} + u_{yy} = 0$. The proof that $v_{xx} + v_{yy} = 0$ is similar.

15.3.99 Let $H(s)$ be an antiderivative of $h(s)$; then $f(x,y) = H(xy) - H(1)$, so $f_x(x,y) = y\,h(xy)$, $f_y(x,y) = x\,h(xy)$.

15.4 The Chain Rule

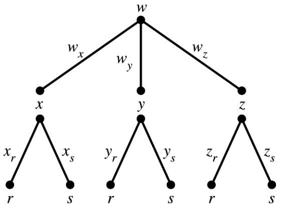
15.4.1 There is one dependent variable (z), two intermediate variables (x and y) and one independent variable (t).

15.4.2 Multiply each of the partial derivatives of z by the t-derivative of the corresponding function, and add all these expressions.

15.4.3 Multiply each of the partial derivatives of w by the t-derivative of the corresponding function, and add all these expressions.

15.4.4 Multiply each of the partial derivatives of z by the t-partial derivative of the corresponding function, and add all these expressions.

15.4.5



15.4.6 Use Theorem 15.9: $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.

15.4.7 We have $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = 2x(2t) + 3y^2(1) = 4t^3 + 3t^2$.

To check, we write $z = x^2 + y^3 = t^4 + t^3$, so $\dfrac{dz}{dt} = 4t^3 + 3t^2$.

15.4.8 We have $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = y^2(2t) + 2xy(1) = 2t^3 + 2t^3 = 4t^3$.

To check, we write $z = t^2t^2 = t^4$, so $\dfrac{dz}{dt} = 4t^3$.

15.4.9 We have $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (\sin y)\,2t + (x\,\cos y)\,12t^2 = 2t\,\sin\left(4t^3\right) + 12t^4\cos\left(4t^3\right)$.

15.4.10 We have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \left(2xy - y^3\right)2t + \left(x^2 - 3xy^2\right)\left(-2t^{-3}\right) = 2t\left(2 - t^{-6}\right) - 2t^{-3}\left(t^4 - 3t^{-2}\right) = 2t + 4t^{-5}.$$

15.4.11 We have $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = (-2\sin 2x\,\sin 3y)\left(\dfrac{1}{2}\right) + (3\,\cos 2x\,\cos 3y)\,4t^3 = -\sin t\,\sin 3t^4 + 12t^3\cos t\,\cos 3t^4$.

15.4.12 We have

$$\frac{dz}{dt} = \frac{\partial z}{\partial r}\frac{dr}{dt} + \frac{\partial z}{\partial s}\frac{ds}{dt} = \frac{r}{\sqrt{r^2+s^2}}(-2\sin 2t) + \frac{s}{\sqrt{r^2+s^2}}(2\cos 2t) = \frac{-2\cos 2t\,\sin 2t + 2\sin 2t\cos 2t}{1} = 0.$$

15.4.13

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= 10(x+2y)^9 2\sin t\cos t + 10(x+2y)^9\cdot 2\cdot 5(3t+4)^4(3)\\ &= 20(\sin^2 t + 2(3t+4)^5)^9(\sin t\cos t + 15(3t+4)^4).\end{aligned}$$

15.4.14

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= \frac{20x^{19}}{y^{10}} \cdot \frac{1}{1+t^2} + \frac{-10x^{20}}{y^{11}} \cdot \frac{2t}{t^2+1} \\ &= \frac{20x^{19}y - 20x^{20}t}{y^{11}(t^2+1)} = \frac{20x^{19}(y-tx)}{y^{11}(t^2+1)} \\ &= \frac{20(\tan^{-1} t)^{19}(\ln(t^2+1) - t\tan^{-1} t)}{(t^2+1)\ln^{11}(t^2+1)}\end{aligned}$$

15.4.15 We have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\begin{aligned}&= (y\sin z)\,2t + (x\sin z)\,12t^2 + (xy\cos z)\cdot 1 \\ &= \left(2ty + 12t^2x\right)\sin z + xy\cos z \\ &= 20t^4\sin(t+1) + 4t^5\cos(t+1)\,.\end{aligned}$$

15.4.16 We have

$$\begin{aligned}\frac{dQ}{dt} &= \frac{\partial Q}{\partial x}\frac{dx}{dt} + \frac{\partial Q}{\partial y}\frac{dy}{dt} + \frac{\partial Q}{\partial z}\frac{dz}{dt} \\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\cos t + \frac{y}{\sqrt{x^2+y^2+z^2}}(-\sin t) + \frac{z}{\sqrt{x^2+y^2+z^2}}(-\sin t) \\ &= -\frac{\cos t\,\sin t}{\sqrt{1+\cos^2 t}}.\end{aligned}$$

15.4.17

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} \\ &= yze^t + 2xz + xy\cos t = e^t(2t+3)\sin t + 2e^t\sin t + e^t(2t+3)\cos t \\ &= e^t((2t+5)\sin t + (2t+3)\cos t).\end{aligned}$$

15.4.18

$$\begin{aligned}\frac{dU}{dt} &= \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} \\ &= \frac{y^2}{z^8}e^t + \frac{2xy}{z^8}3\cos 3t - \frac{8xy^2}{z^9}\cdot 4 \\ &= \frac{xy(yz + 6z\cos 3t - 32y)}{z^9} \\ &= \frac{e^t\sin 3t((4t+1)\sin 3t + 6(4t+1)\cos 3t - 32\sin 3t)}{z^9} \\ &= \frac{e^t\sin 3t(4t\sin 3t - 31\sin 3t + 24t\cos 3t + 6\cos 3t)}{(4t+1)^9}.\end{aligned}$$

15.4.19 $z_s = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = 2x\sin y + x^2\cos y\cdot 0 = 2(s-t)\sin t^2$ and $z_t = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = (2x\sin y)(-1) + x^2\cos y(2t) = 2(t-s)\sin t^2 + 2t(s-t)^2\cos t^2$.

15.4.20 $z_s = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = 2\cos(2x+y)(2s) + \cos(2x+y)(2s) = 6s\cos(3s^2-t^2)$ and $z_t = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = 2\cos(2x+y)(-2t) + \cos(2x+y)(2t) = -2t\cos(2(s^2-t^2) + s^2 + t^2) = -2t\cos(3s^2-t^2)$.

15.4.21 $z_s = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (y-2xy)\cdot 1+\left(x-x^2\right)\cdot 1 = s-t-2\left(s^2-t^2\right)+(s+t)-(s+t)^2 = 2s-3s^2-2st+t^2$ and $z_t = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (y-2xy)\cdot 1+\left(x-x^2\right)\cdot(-1) = s-t-2\left(s^2-t^2\right)-(s+t)+(s+t)^2 = -s^2-2t+2st+3t^2$.

15.4.22

$$z_s = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \cos x\,\cos 2y - 2\sin x\,\sin 2y = \cos(s+t)\cos[2(s-t)] - 2\sin(s+t)\sin[2(s-t)]$$

and

$$z_t = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \cos x\,\cos 2y - 2\sin x\,\sin 2y\cdot(-1) = \cos(s+t)\cos[2(s-t)] + 2\sin(s+t)\sin[2(s-t)]$$

15.4.23 $z_s = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = e^{x+y}\cdot t+e^{x+y}\cdot 1 = (t+1)\,e^{st+s+t}$ and $z_t = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = e^{x+y}\cdot s+e^{x+y}\cdot 1 = (s+1)\,e^{st+s+t}$.

15.4.24 $z_s = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (y\cos xy)(2st)+(x\cos xy)(10(s+t)^9) = (2yst+10s^2t(s+t)^9)(\cos xy) = 2st((s+t)^{10}+5s(s+t)^9)(\cos xy) = 2st(s+t)^9(6s+t)\cos(s^2t(s+t)^{10})$ and
$z_t = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (y\cos xy)s^2+(x\cos xy)10(s+t)^9 = ((s+t)^{10}s^2+10s^2t(s+t)^9)\cos xy = s^2(s+t)^9(s+11t)\cos(s^2t(s+t)^{10})$.

15.4.25 $w_s = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{1}{y+z}\cdot 1+\frac{z-x}{(y+z)^2}\cdot t-\frac{x+y}{(y+z)^2}\cdot 1 = \frac{(1+t)(z-x)}{(y+z)^2} = -\frac{2t(1+t)}{(st+s-t)^2}$ and
$w_t = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = \frac{1}{y+z}\cdot 1+\frac{z-x}{(y+z)^2}\cdot s-\frac{x+y}{(y+z)^2}\cdot(-1) = \frac{(1-s)x+2y+(1+s)z}{(y+z)^2} = \frac{2s}{(st+s-t)^2}$.

15.4.26 $w_r = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = \frac{x}{\sqrt{x^2+y^2+z^2}}\cdot 0+\frac{y}{\sqrt{x^2+y^2+z^2}}\cdot s+\frac{z}{\sqrt{x^2+y^2+z^2}}\cdot t = \frac{r\left(s^2+t^2\right)}{\sqrt{r^2s^2+r^2t^2+s^2t^2}}$.
Similarly, $w_s = \frac{s\left(r^2+t^2\right)}{\sqrt{r^2s^2+r^2t^2+s^2t^2}}$ and $w_{\mathrm{t}} = \frac{t\left(r^2+s^2\right)}{\sqrt{r^2s^2+r^2t^2+s^2t^2}}$.

15.4.27

a. By the chain rule, $V'(t) = 2\pi\,r(t)\,h(t)\,r'(t)+\pi[r(t)]^2h'(t)$.

b. Substituting $r(t)=e^t$ and $h(t)=e^{-2t}$ gives $V'(t) = 2\pi\,e^te^{-2t}e^t+\pi\,e^{2t}\left(-2e^{-2t}\right)=0$.

c. Because $V'(t)=0$, the volume remains constant.

15.4.28

a. By the chain rule, $V'(t) = \frac{2}{3}x(t)\;h(t)\;x'(t)+\frac{1}{3}[x(t)]^2\,h'(t)$.

b. Substituting $x(t)=\frac{t}{t+1}$ and $h(t)=\frac{1}{t+1}$ gives

$$V'(t) = \frac{2}{3}\frac{t}{t+1}\frac{1}{t+1}\frac{1}{(t+1)^2}+\frac{1}{3}\left(\frac{t}{t+1}\right)^2\left(-\frac{1}{(t+1)^2}\right) = \frac{1}{3}\frac{t(2-t)}{(t+1)^4}.$$

c. The volume increases for $0 \le t \le 2$ and decreases for $t \ge 2$.

15.4.29

a. $z(t) = \dfrac{1}{t^2+2t} + \dfrac{1}{t^3-2}$, so $z'(t) = -\dfrac{1}{(t^2+2t)^2}\cdot(2t+2) - \dfrac{1}{(t^3-2)^2}\cdot 3t^2$.

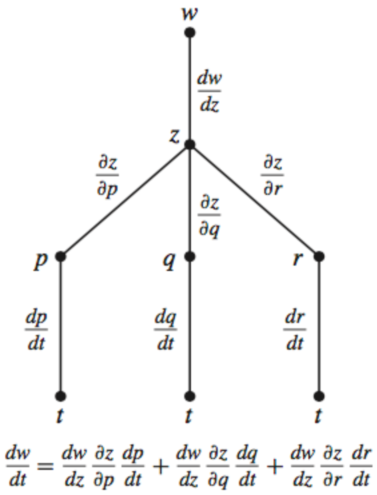
b. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = -\dfrac{1}{x^2}\cdot(2t+2) - \dfrac{1}{y^2}\cdot 3t^2 = -\dfrac{1}{(t^2+2t)^2}\cdot(2t+2) - \dfrac{1}{(t^3-2)^2}\cdot 3t^2$.

15.4.30

a. $z(t) = \ln(te^t + e^t)$, so $z'(t) = \dfrac{1}{te^t+e^t}\left(e^t + te^t + e^t\right) = \dfrac{te^t + 2e^t}{te^t + e^t}$.

b. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = \dfrac{1}{x+y}\cdot(e^t + te^t) + \dfrac{1}{x+y}\cdot e^t = \dfrac{te^t+2e^t}{te^t+e^t}$.

15.4.31 $\dfrac{dw}{dt} = \dfrac{dw}{dz}\dfrac{\partial z}{\partial p}\dfrac{dp}{dt} + \dfrac{dw}{dz}\dfrac{\partial z}{\partial q}\dfrac{dq}{dt} + \dfrac{dw}{dz}\dfrac{\partial z}{\partial r}\dfrac{dr}{dt}$.


w
dw/dz
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∂z/∂p
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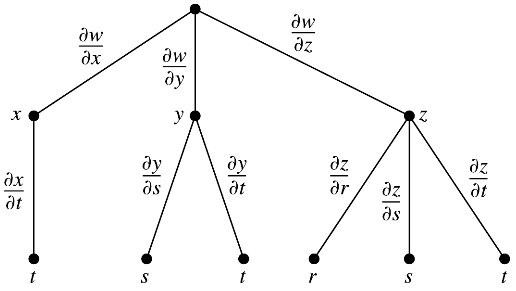
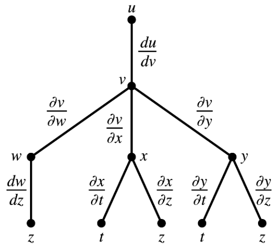
t
t
dw/dt = dw/dz ∂z/∂p dp/dt + dw/dz ∂z/∂q dq/dt + dw/dz ∂z/∂r dr/dt

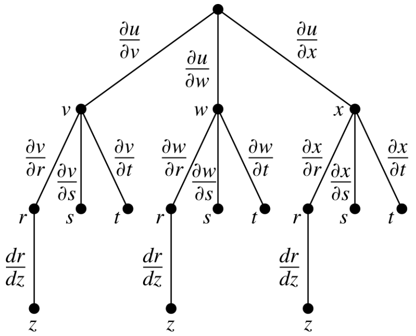
15.4.32 $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial t}$



15.4.33 $\dfrac{\partial u}{\partial z} = \dfrac{du}{dv}\left(\dfrac{\partial v}{\partial w}\dfrac{dw}{dz} + \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial z} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial z}\right)$



15.4.34 $\frac{du}{dz} = \left(\frac{\partial u}{\partial v}\frac{dv}{dr} + \frac{\partial u}{\partial w}\frac{dw}{dr} + \frac{\partial u}{\partial x}\frac{dx}{dr}\right)\frac{dr}{dz}$



15.4.35 Let $F(x, y) = x^2 - 2y^2 - 1$; then by Theorem 15.9, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{-4y} = \frac{x}{2y}$.

15.4.36 Let $F(x, y) = x^3 + 3xy^2 - y^5$; then by Theorem 15.9, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2+3y^2}{6xy-5y^4} = \frac{3x^2+3y^2}{5y^4-6xy}$.

15.4.37 Let $F(x, y) = 2\sin(xy) - 1$; then by Theorem 15.9, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2y\cos(xy)}{2x\cos(xy)} = -\frac{y}{x}$.

15.4.38 Let $F(x, y) = ye^{xy} - 2$; then by Theorem 15.9, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y^2e^{xy}}{e^{xy}+xy\,e^{xy}} = -\frac{y^2}{1+xy}$.

15.4.39 Note that we can simplify this equation to $x^2 + 2xy + y^4 = 9$, so let $F(x, y) = x^2 + 2xy + y^4 - 9$; then by Theorem 15.9, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+2y}{2x+4y^3} = -\frac{x+y}{x+2y^3}$.

15.4.40 Let $F(x,y) = y\ln\left(x^2+y^2+4\right) - 3$; then by Theorem 15.9, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} =$

$$\frac{-\left(\frac{2xy}{x^2+y^2+4}\right)}{\left(\ln(x^2+y^2+4) + \frac{2y^2}{x^2+y^2+4}\right)} = -\frac{2xy}{2y^2+(x^2+y^2+4)\ln(x^2+y^2+4)}.$$

15.4.41 The chain rule gives

$$\frac{\partial s}{\partial x} = \frac{\partial s}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial s}{\partial v}\frac{\partial v}{\partial x} = \frac{u}{\sqrt{u^2+v^2}}\cdot 0 + \frac{v}{\sqrt{u^2+v^2}}\cdot(-2) = \frac{4x}{\sqrt{4(x^2+y^2)}} = \frac{2x}{\sqrt{x^2+y^2}}$$

and

$$\frac{\partial s}{\partial y} = \frac{\partial s}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial s}{\partial v}\frac{\partial v}{\partial y} = \frac{u}{\sqrt{u^2+v^2}}\cdot 2 + \frac{v}{\sqrt{u^2+v^2}}\cdot 0 = \frac{4y}{\sqrt{4(x^2+y^2)}} = \frac{2y}{\sqrt{x^2+y^2}}.$$

15.4.42 The chain rule gives

$$\frac{\partial s}{\partial x}=\frac{\partial s}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial s}{\partial v}\frac{\partial v}{\partial x}=\frac{u}{\sqrt{u^2+v^2}}\cdot(1-2x)\,(1-2y)+\frac{v}{\sqrt{u^2+v^2}}\cdot y\,(y-1)\,(-2)$$
$$=\frac{x\,(1-x)\,(1-2x)\,(1-2y)^2-2y^2\,(1-2x)\,(y-1)^2}{\sqrt{x^2\,(1-x)^2\,(1-2y)^2+y^2\,(y-1)^2\,(1-2x)^2}}$$

and

$$\frac{\partial s}{\partial y}=\frac{\partial s}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial s}{\partial v}\frac{\partial v}{\partial y}$$
$$=\frac{u}{\sqrt{u^2+v^2}}\cdot x\,(1-x)\,(-2)+\frac{v}{\sqrt{u^2+v^2}}\cdot(2y-1)\,(1-2x)$$
$$=\frac{y\,(1-y)\,(1-2y)\,(1-2x)^2-2x^2\,(1-2y)\,(1-x)^2}{\sqrt{x^2\,(1-x)^2\,(1-2y)^2+y^2\,(1-y)^2\,(1-2x)^2}}$$

15.4.43 Using the derivation from Example 7, we have

$$f_{ss}=f_{xx}x_s^2+2f_{xy}x_sy_s+f_{yy}y_s^2+f_xx_{ss}+f_yy_{ss},$$

$$f_{st}=f_{xx}x_sy_t+f_{xy}x_sy_t+f_{xy}x_ty_s+f_{yy}y_sy_t+f_xx_{st}+f_yy_{st},$$

$$f_{tt}=f_{xx}x_t^2+2f_{xy}x_ty_t+f_{yy}y_t^2+f_xx_{tt}+f_yy_{tt}.$$

For $f=x^2y$ and $x=s+t$ and $y=s-t$, we have $f_x=2xy$, $f_{xx}=2y$, $f_y=x^2$, $f_{yx}=2x$, and $f_{yy}=0$. Also, $x_s=1$, $x_t=1$, $y_s=1$, and $y_t=-1$, so all the second partials of x and y are zero. Therefore we have

$$f_{ss}=2y+4x+0+0+0=2s-2t+4s+4t=6s+2t=2(3s+t).$$

$$f_{st}=2y-2x+2x+0+0+0=2y=2(s-t).$$

$$f_{tt}=2y-4x+0+0+0=-2(2x-y)=-2(s+3t).$$

15.4.44 Using the derivation from Example 7, we have

$$f_{ss}=f_{xx}x_s^2+2f_{xy}x_sy_s+f_{yy}y_s^2+f_xx_{ss}+f_yy_{ss},$$

$$f_{st}=f_{xx}x_sy_t+f_{xy}x_sy_t+f_{xy}x_ty_s+f_{yy}y_sy_t+f_xx_{st}+f_yy_{st},$$

$$f_{tt}=f_{xx}x_t^2+2f_{xy}x_ty_t+f_{yy}y_t^2+f_xx_{tt}+f_yy_{tt}.$$

For $f=x^2y-xy^2$ and $x=st$ and $y=\frac{s}{t}$, we have $f_x=2xy-y^2$, $f_{xx}=2y$, $f_y=x^2-2xy$, $f_{yx}=2x-2y$, and $f_{yy}=-2x$. Also, $x_s=t$, $x_t=s$, $y_s=\frac{1}{t}$, and $y_t=-\frac{s}{t^2}$. Also $x_{ss}=0$, $y_{ss}=0$, and $x_{st}=1$, while $y_{ss}=0$, $y_{tt}=\frac{2s}{t^3}$, and $y_{st}=-\frac{1}{t^2}$. Therefore we have

$$f_{ss}=2yt^2+2(2x-2y)-2x\cdot\frac{1}{t^2}=2st+4st-\frac{4s}{t^2}-\frac{2s}{t}=6st-\frac{6s}{t}.$$

$$f_{st}=2y(st)+(2x-2y)\left(-\frac{s}{t}\right)+(2x-2y)\left(\frac{s}{t}\right)-2x\left(-\frac{s}{t^3}\right)+(2xy-y^2)+(x^2-2xy)\left(-\frac{1}{t^2}\right)=3s^2+\frac{3s^2}{t^2}.$$

$$f_{tt}=2ys^2+4(x-y)\left(-\frac{s^2}{t^2}\right)-2x\frac{s^2}{t^4}+0+(x^2-2xy)\frac{2s}{t^3}=-\frac{2s^3}{t^3}.$$

15.4.45 Using the derivation from Example 7, we have

$$f_{ss} = f_{xx}x_s^2 + 2f_{xy}x_sy_s + f_{yy}y_s^2 + f_xx_{ss} + f_yy_{ss},$$

$$f_{st} = f_{xx}x_sy_t + f_{xy}x_sy_t + f_{xy}x_ty_s + f_{yy}y_sy_t + f_xx_{st} + f_yy_{st},$$

$$f_{tt} = f_{xx}x_t^2 + 2f_{xy}x_ty_t + f_{yy}y_t^2 + f_xx_{tt} + f_yy_{tt}.$$

For $f = \frac{y}{x}$ we have $f_x = -\frac{y}{x^2}$, $f_{xy} = -\frac{1}{x^2}$, $f_{xx} = \frac{2y}{x^3}$, $f_y = \frac{1}{x}$, $f_{yy} = 0$, $x_s = 2s$, $x_{ss} = 2$, $x_t = 2t$, $x_{tt} = 2$, $x_{ts} = 0$, $y_s = 2s$, $y_{ss} = 2$, $y_{st} = 0$, $y_t = -2t$, and $y_{tt} = -2$. Then we have

$$f_{ss} = \frac{2y}{x^3} \cdot 4s^2 - \frac{2}{x^2} \cdot 4s^2 + 0 + 2\left(-\frac{y}{x^2}\right) + \frac{2}{x} = \frac{4t^2(t^2 - 3s^2)}{(s^2 + t^2)^3}.$$

$$f_{st} = \frac{2y}{x^3}(2s)(2t) + \frac{-1}{x^2}(-4ts) + \frac{-1}{x^2}(4ts) + 0 - \frac{y}{x^2}(0) + 0 = \frac{2(s^2 - t^2)(4st)}{(s^2 + t^2)^3} = \frac{8st(s^2 - t^2)}{(s^2 + t^2)^3}.$$

$$f_{ttt} = \frac{2y}{x^3}(4t^2) + 2\left(\frac{-1}{x^2}\right)(-4t^2) + 0 + 2\left(\frac{-1}{x^2}\right) + -2\left(\frac{1}{x}\right) = \frac{-4(s^4 - 3s^2t^2)}{(s^2 + t^2)^3}.$$

15.4.46 Using the derivation from Example 7, we have

$$f_{ss} = f_{xx}x_s^2 + 2f_{xy}x_sy_s + f_{yy}y_s^2 + f_xx_{ss} + f_yy_{ss},$$

$$f_{st} = f_{xx}x_sy_t + f_{xy}x_sy_t + f_{xy}x_ty_s + f_{yy}y_sy_t + f_xx_{st} + f_yy_{st},$$

$$f_{tt} = f_{xx}x_t^2 + 2f_{xy}x_ty_t + f_{yy}y_t^2 + f_xx_{tt} + f_yy_{tt}.$$

For $f = e^{x-y}$ and $x = s^2$ and $y = 3t^2$ we have $f_x = e^{x-y}$, $f_y = -e^{x-y}$, $f_{xx} = e^{x-y}$, $f_{xy} = -e^{x-y}$, and $f_{yy} = e^{x-y}$. Also, $x_s = 2s$, $x_t = 0$, $x_{ts} = 0$, $y_t = 6t$, $y_s = 0$, $x_{ss} = 2$, $y_{tt} = 6$ and the others are 0. Then

$$f_{ss} = e^{x-y}(4s^2) + 0 + 0 + 2e^{x-y} + 0 = 4s^2e^{s^2-3t^2} + 2e^{s^2-3t^2}.$$

$$f_{st} = e^{x-y}(12st) + 0 + 0 + 0 + 0 + 0 + 0 = 12ste^{s^2-3t^2}.$$

$$f_{tt} = 0 + 0 + 36t^2e^{x-y} - 6e^{x-y} = 36t^2e^{s^2-3t^2} - 6e^{s^2-3t^2}.$$

15.4.47 Note that $f_x = y + z$, $f_y = x - z$, $f_z = x - y$, $f_{xx} = 0$, $f_{xy} = 1$, $f_{xz} = -1$, $f_{yy} = 0$, $f_{yz} = -1$, and $f_{zz} = 0$. Also, $x_s = 2s - 2$, $y_s = \frac{-4}{s^3}$, $z_s = 6s$, etc.

Omitting the terms that are equal have factors of 0, we have

$$f_{ss} = 2f_{xy}x_sy_s + 2f_{xz}x_sz_s + 2f_{yz}y_sz_s + f_xx_{ss} + f_yy_{ss} + f_zz_{ss}.$$

We have

$$f_{ss} = 2(2s - 2)\left(\frac{-4}{s^3}\right) + 2(2s - 2)(6s) + 2\left(\frac{4}{s^3}\right)(6s) + 2(y + z) + \frac{12}{s^4}(x - z) + (x - y)6.$$

This can be written as

$$4\left(\frac{6}{s^4} - \frac{2}{s^3} - 1 - 9s + 9s^2\right).$$

15.4.48 Note that all of f_{xx}, f_{yy}, and all the mixed second partials for x and y are 0. So we have $f_{ss} = 2f_{xy}x_sy_s = 2$, $f_{st} = f_{xy}x_sy_t + f_{xy}x_ty_s = 2 + 2 = 4$, $f_{su} = f_{xy}x_sy_u + f_{xy}x_uy_s = 1 - 1 = 0$, $f_{tt} = 2f_{xy}x_ty_t = 8$, $f_{tu} = f_{xy}x_ty_u + f_{xy}x_uy_t = 0$, and $f_{uu} = 2f_{xy}x_uy_u = -2$.

15.4.49

a. False. The correct equation is $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{dx}{ds} + \dfrac{\partial z}{\partial y}\dfrac{dy}{ds}$.

b. False. w is a function of both s and t, so the rate of change of w with respect to t is the partial derivative $\frac{\partial w}{\partial t}$.

15.4.50 The chain rule gives $\frac{\partial z}{\partial p} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial p} = \frac{1}{y} \cdot 1 - \frac{x}{y^2} \cdot 1 = \frac{y-x}{y^2} = -\frac{2q}{(p-q)^2}$.

15.4.51 The chain rule gives $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = yz \cdot 8t^3 + xz\left(-3t^{-2}\right) + xy\left(-12t^{-4}\right) = \frac{12}{t^4} \cdot 8t^3 + 8t\left(-\frac{3}{t^2}\right) + 6t^3\left(-\frac{12}{t^4}\right) = 0$.This can also be seen by expressing w in terms of t: $w = 2t^4 3t^{-1} 4t^{-3} = 24$, so $\frac{dw}{dt} = 0$.

15.4.52 Observe that $w = \cos(x+y) - (\cos x \, \cos y - \sin x \, \sin y) = 0$, $\frac{\partial w}{\partial x} = 0$.

15.4.53 The chain rule gives $-\frac{1}{x^2} - \frac{1}{z^2}\frac{\partial z}{\partial x} = 0$, so $\frac{\partial z}{\partial x} = -\frac{z^2}{x^2}$.

15.4.54 Observe that $z = xy - 1$, so $\frac{\partial z}{\partial x} = y$.

15.4.55

a. The chain rule gives $w'(t) = aw_x + bw_y + cw_z$.

b. Using part (a), $w'(t) = ayz + bxz + cxy = 3abct^2$.

c. Using part (a), $w'(t) = \frac{ax}{\sqrt{x^2+y^2+z^2}} + \frac{by}{\sqrt{x^2+y^2+z^2}} + \frac{cz}{\sqrt{x^2+y^2+z^2}} = \frac{ax+by+cz}{\sqrt{x^2+y^2+z^2}} = \sqrt{a^2+b^2+c^2}\,\frac{t}{|t|}$

d. Differentiate the result from part (a) one more time:
$w''(t) = a\left(aw_{xx} + bw_{xy} + cw_{xz}\right) + b\left(aw_{yx} + bw_{yy} + cw_{yz}\right) + c\left(aw_{zx} + bw_{zy} + cw_{zz}\right)$ which simplifies to
$w''(t) = a^2w_{xx} + b^2w_{yy} + c^2w_{zz} + 2abw_{xy} + 2acw_{xz} + 2bcw_{yz}$.

15.4.56 Differentiate $F(x,\, y,\, z(x,\, y)) = 0$ using the chain rule: $F_x \cdot 1 + F_y \cdot 0 + F_z \cdot \frac{\partial z}{\partial x} = 0$, so $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$. The proof that $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$ is similar.

15.4.57 Let $F(x,\, y,\, z) = xy + xz + yz - 3$; then the result from Exercise 56 gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y+z}{x+y}, \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x+z}{x+y}$.

15.4.58 Let $F(x,\, y,\, z) = x^2 + 2y^2 - 3z^2 - 1$; then the result from Exercise 56 gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{-6z} = \frac{x}{3z}, \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y}{-6z} = \frac{2y}{3z}$.

15.4.59 Let $F(x,\, y,\, z) = xyz + x + y - z$; then the result from Exercise 56 gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz+1}{xy-1}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz+1}{xy-1}$.

15.4.60

a. Let $F(x,\, y,\, z) = e^{xyz} - 2$; then the result from Exercise 56 gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yze^{xyz}}{xye^{xyz}} = -\frac{z}{x}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xze^{xyz}}{xye^{xyz}} = -\frac{z}{y}$.

b. Take logarithms of both sides to obtain $xyz - \ln 2 = 0$; then the result from Exercise 56 applied to $F(x,\, y,\, z) = xyz - \ln 2$ gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz}{xy} = -\frac{z}{x}, \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz}{xy} = -\frac{z}{y}$.

c. Solve for z to obtain $z = \frac{\ln 2}{xy}$; therefore $\frac{\partial z}{\partial x} = -\frac{\ln 2}{x^2 y} = -\frac{z}{x}, \frac{\partial z}{\partial y} = -\frac{\ln 2}{xy^2} = -\frac{z}{y}$.

15.4.61

a. The chain rule gives $z'(t) = 2x(-\sin t) + 8y \cos t = 6 \sin t \cos t = 3 \sin 2t$.

b. Observe that for $0 \le t \le 2\pi$, $z'(t) = 3 \sin 2t > 0$ when $0 < t < \frac{\pi}{2}$or $\pi < t < \frac{3\pi}{2}$.

15.4.62

a. The chain rule gives $z'(t) = 8x(-\sin t) - 2y \cos t = -10 \sin t \cos t = -5 \sin 2t$.

b. Observe that for $0 \le t \le 2\pi$, $z'(t) = -5 \sin 2t > 0$ when $\frac{\pi}{2} < t < \pi$ or $\frac{3\pi}{2} < t < 2\pi$.

15.4.63

a. The chain rule gives $z'(t) = -\frac{x}{\sqrt{1 - x^2 - y^2}}\left(-e^{-t}\right) + -\frac{y}{\sqrt{1 - x^2 - y^2}}\left(-e^{-t}\right) = \frac{2e^{-2t}}{\sqrt{1 - e^{-2t}}}$.

b. Observe that $z'(t) > 0$ for all t where defined, so the function $z(t)$ is increasingfor all $t \ge \frac{1}{2} \ln 2$.

15.4.64

a. The chain rule gives $z'(t) = 4x(-\sin t) + 2y \cos t = -4(1 + \cos t) \sin t + 2 \sin t \cos t = -2 \sin t (2 + \cos t)$.

b. Observe that $2 + \cos t > 0$ for all t, so for $0 \le t \le 2\pi$, so $z'(t) > 0$ if and only if $\sin t < 0$, which occurs when $\pi < t < 2\pi$.

15.4.65 The chain rule gives $E'(t) = m(uu' + vv') + mgy' = m(x'x'' + y'y'' + gy') =$
$m(u_0 \cdot 0 + y'(y'' + g)) = 0$. Therefore, the energy of the projectile remains constant during the motion.

15.4.66

a. The marginal utilities are $\frac{\partial U}{\partial x} = ax^{a-1}y^{1-a}$, $\frac{\partial U}{\partial y} = (1 - a)\, x^a y^{-a}$.

b. Using Theorem 15.9, we find that the slope of the indifference curve $U(x,\, y) - c = 0$ is MRS $= -\frac{U_x}{U_y} = -\frac{a}{1-a}\frac{y}{x}$.

c. The result from part (b) above gives MRS $= -\frac{0.4}{0.6}\frac{12}{8} = -1$.

15.4.67

a. If r and R increase at the same rate then $R - r$ is a constant C, so $V = \frac{C^2\pi^2}{4}(R + r)$ is increasing.

b. Similarly, if r and R decrease at the same rate then V is decreasing.

15.4.68

a. The chain rule gives $S'(t) = \frac{1}{60}\left(\frac{w}{2\sqrt{hw}}h'(t) + \frac{h}{2\sqrt{hw}}w'(t)\right) = \frac{w\,h'(t) + h\,w'(t)}{120\sqrt{hw}}$.

b. From part (a), the condition that $S(t)$ is constant is $w\,h'(t) + h\,w'(t) = 0$.

c. If S is constant then we must have hw constant, so h and w are inversely proportional.

15.4.69

a. Consider P as a function of T and V and differentiate with respect to V : $\frac{\partial P}{\partial V}V + P \cdot 1 = 0$, so $\frac{\partial P}{\partial V} = -\frac{P}{V}$. Next, consider T as a function of P and V and differentiate with respect to P: $1 \cdot V = k\frac{\partial T}{\partial P}$, so $\frac{\partial T}{\partial P} = \frac{V}{k}$. Lastly, consider V as a function of T and P and differentiate with respect to T: $P\frac{\partial V}{\partial T} = k$, so $\frac{\partial V}{\partial T} = \frac{k}{P}$.

b. Observe that $\frac{\partial P}{\partial V}\frac{\partial T}{\partial P}\frac{\partial V}{\partial T} = -\frac{P}{V}\frac{V}{k}\frac{k}{P} = -1$.

15.4.70

a. The chain rule gives $\frac{d\rho}{dt} = y(-2\sin t) + x \cdot 2\cos t = 4\left(\cos^2 t - \sin^2 t\right) = 4\cos 2t$.

b. The density as a function of t is given by $\rho(t) = 4 + 4\cos t \sin t = 4 + 2\sin 2t$, which attains its maximum at $t = \frac{\pi}{4}, \frac{5\pi}{4}$. (This also follows from part (a) and the second derivative test.) The corresponding points on the plate are $\pm\left(\sqrt{2}, \sqrt{2}\right)$.

15.4.71

a. The chain rule gives $w'(t) = \frac{yz}{z^2+1}(-\sin t) + \frac{xz}{z^2+1}(\cos t) + \frac{xy\left(1-z^2\right)}{\left(z^2+1\right)^2} \cdot 1 = -\frac{(\sin t)\,t}{t^2+1}\sin t + \frac{(\cos t)\,t}{t^2+1}\cos t + \frac{(\cos t)(\sin t)\left(1-t^2\right)}{\left(t^2+1\right)^2}$.

b. The function $w(t) = \frac{t\,\cos t\,\sin t}{1+t^2}$ takes its maximum value on $[0, \infty)$ approximately at $t = 0.838$, which gives the point $(0.669, 0.743, 0.838)$ on the spiral.

15.4.72

a. We have $x_r = \cos\theta$, $y_r = \sin\theta$, $x_\theta = -r\sin\theta$, $y_\theta = r\cos\theta$.

b. We have $r = \left(x^2+y^2\right)^{1/2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, so $r_x = \frac{x}{\sqrt{x^2+y^2}}$, $r_y = \frac{y}{\sqrt{x^2+y^2}}$, and $\theta_x = -\frac{y}{x^2+y^2}$, $\theta_y = \frac{x}{x^2+y^2}$.

c. The chain rule gives $z_r = f_x\cos\theta + f_y\sin\theta$, $z_\theta = -rf_x\sin\theta + rf_y\cos\theta$.

d. The chain rule gives $z_x = g_r\frac{x}{\sqrt{x^2+y^2}} - g_\theta\frac{y}{x^2+y^2}$, $z_y = g_r\frac{y}{\sqrt{x^2+y^2}} + g_\theta\frac{x}{x^2+y^2}$.

e. Observe that $z_x^2 + z_y^2 = g_r^2\left(\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right) + g_\theta^2\left(\frac{x^2}{\left(x^2+y^2\right)^2} + \frac{y^2}{\left(x^2+y^2\right)^2}\right) = g_r^2 + \frac{1}{r^2}g_\theta^2 = z_r^2 + \frac{1}{r^2}z_\theta^2$.

15.4.73

a. From problem 72 part (d) we have $z_x = \frac{x}{r} z_r - \frac{y}{r^2} z_\theta$, $z_y = \frac{y}{r} z_r + \frac{x}{r^2} z_\theta$

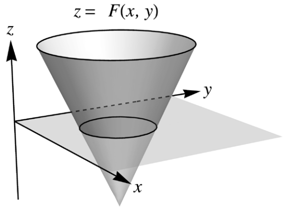
b. Differentiating the equation for z_x in part (a) with respect to x gives $z_{xx} = \frac{1}{r} z_r + x\left(-\frac{1}{r^2}\right) r_x z_r + \frac{x}{r}\left(z_r\right)_x + \frac{2y}{r^3} r_x z_\theta - \frac{y}{r^2}\left(z_\theta\right)_x = \frac{x}{r}\left(z_r\right)_x - \frac{y}{r^2}\left(z_\theta\right)_x + \left(\frac{r^2}{r^3} - \frac{x^2}{r^3}\right) z_r + \frac{2xy}{r^4} z_\theta = \frac{x}{r}\left(\frac{x}{r} z_{rr} - \frac{y}{r^2} z_{r\theta}\right) - \frac{y}{r^2}\left(\frac{x}{r} z_{\theta r} - \frac{y}{r^2} z_{\theta\theta}\right) + \frac{y^2}{r^3} z_r + \frac{2xy}{r^4} z_\theta = \frac{x^2}{r^2} z_{rr} + \frac{y^2}{r^4} z_{\theta\theta} - \frac{2xy}{r^3} z_{r\theta} + \frac{y^2}{r^3} z_r + \frac{2xy}{r^4} z_\theta$.

c. Differentiating the equation for z_y in part (a) with respect to y gives $z_{yy} = \frac{1}{r} z_r + y\left(-\frac{1}{r^2}\right) r_y z_r + \frac{y}{r}\left(z_r\right)_y - \frac{2x}{r^3} r_y z_\theta + \frac{x}{r^2}\left(z_\theta\right)_y = \frac{y}{r}\left(z_r\right)_y + \frac{x}{r^2}\left(z_\theta\right)_y + \left(\frac{r^2}{r^3} - \frac{y^2}{r^3}\right) z_r - \frac{2xy}{r^4} z_\theta = \frac{y}{r}\left(\frac{y}{r} z_{rr} + \frac{x}{r^2} z_{r\theta}\right) + \frac{x}{r^2}\left(\frac{y}{r} z_{\theta r} + \frac{x}{r^2} z_{\theta\theta}\right) + \frac{x^2}{r^3} z_r - \frac{2xy}{r^4} z_\theta = \frac{y^2}{r^2} z_{rr} + \frac{x^2}{r^4} z_{\theta\theta} + \frac{2xy}{r^3} z_{r\theta} + \frac{x^2}{r^3} z_r - \frac{2xy}{r^4} z_\theta$.

d. Adding the results from (b) and (c) gives $z_{xx} + z_{yy} = z_{rr} + \frac{1}{r} z_r + \frac{1}{r^2} z_{\theta\theta}$.

15.4.74

a. The tangent plane to the surface $z = F(x, y)$ has normal vector $\mathrm{n} = \langle F_x, F_y, -1\rangle$, so we see that the projection of n into the xy-plane is orthogonal to the curve $F(x, y) = 0$.



b. The curve $F(x, y) = 0$ has a vertical tangent at a point where $F_y(x, y) = 0$.

15.4.75

a. Assuming y is fixed, the chain rule gives $F_x \cdot 1 + F_y \cdot 0 + F_z \cdot \left(\frac{\partial z}{\partial x}\right)_y = 0$, so $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$.

b. Similarly we find that $\left(\frac{\partial y}{\partial z}\right)_x = -\frac{F_z}{F_y}$ and $\left(\frac{\partial x}{\partial y}\right)_z = -\frac{F_y}{F_x}$

c. From (a) and (b) we see that $\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial x}{\partial y}\right)_z = -1$.

d. Let $\left(\frac{\partial w}{\partial x}\right)_{y,z}$ denote the partial derivative of w with respect to x holding y and z constant, with similar notation for the other possible pairs of variables. A similar derivation as in part (a) and (b) above for $F(w, x, y, z) = 0$ shows that $\left(\frac{\partial w}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial y}\right)_{w,z}\left(\frac{\partial y}{\partial z}\right)_{w,x}\left(\frac{\partial z}{\partial w}\right)_{x,y} = \left(-\frac{F_x}{F_w}\right)\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right)\left(-\frac{F_w}{F_z}\right) = 1$.

15.4.76

a. We know that $y'(x) = -\frac{f_x}{f_y}$; differentiating again with respect to x gives

$$y''(x) = -\frac{f_y\left(f_x\right)' - f_x\left(f_y\right)'}{f_y^2} = -\frac{f_y\left(f_{xx} + f_{xy}\, y'\right) - f_x\left(f_{yx} + f_{yy}\, y'\right)}{f_y^2} = -\frac{f_{xx} f_y^2 - 2 f_x f_y f_{xy} + f_{yy} f_x^2}{f_y^3}.$$

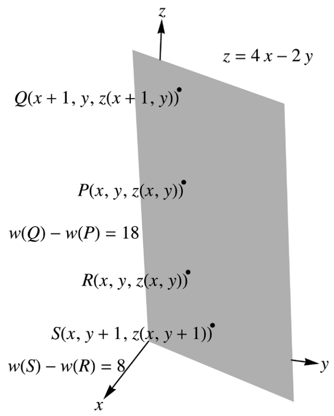
b. For $f(x, y) = xy - 1$ we have $f_x = y$, $f_y = x$, $f_{xy} = 1$, $f_{xx} = f_{yy} = 0$, so we obtain $y''(x) = - -\frac{2xy}{x^3} = \frac{2}{x^3}$ because $xy = 1$. In this case we can solve to find $y(x) = \frac{1}{x}$, so $y''(x) = \frac{2}{x^3}$ is correct.

15.4.77

a. We have $\left(\frac{\partial w}{\partial x}\right)_y = f_x + f_z \frac{dz}{dx} = 2 + 4 \cdot 4 = 18$.

b. Rewrite $z = 4x - 2y$ as $y = 2x - \frac{z}{2}$; therefore $\left(\frac{\partial w}{\partial x}\right)_z = f_x + f_y \frac{dy}{dx} = 2 + 3 \cdot 2 = 8$.

c.



d. Hold x constant; then

$$\left(\frac{\partial w}{\partial y}\right)_x = f_y + f_z \frac{dz}{dy} = 3 + 4(-2) = -5.$$

Hold z constant; then

$$\left(\frac{\partial w}{\partial y}\right)_z = f_x \frac{dx}{dy} + f_y = 2 \cdot \frac{1}{2} + 3 = 4.$$

Hold x constant; then

$$\left(\frac{\partial w}{\partial z}\right)_x = f_y \frac{dy}{dz} + f_z = 3\left(-\frac{1}{2}\right) + 4 = \frac{5}{2}.$$

Hold y constant; then

$$\left(\frac{\partial w}{\partial z}\right)_y = f_x \frac{dx}{dz} + f_z = 2 \cdot \frac{1}{4} + 4 = \frac{9}{2}.$$

15.5 Directional Derivatives and the Gradient

15.5.1 Take the dot product of the unit direction vector u and the gradient of the function.

15.5.2 The gradients $\nabla f = \langle f_x, f_y \rangle$ and $\nabla f = \langle f_x, f_y, f_z \rangle$.

15.5.3 The direction of the gradient vector is the direction in which the function is increasing the most (steepest ascent).

15.5.4 The magnitude of the gradient vector is the largest possible directional derivative of the function at a point.

15.5.5 The gradient is perpendicular to the level curves.

15.5.6 The direction must be perpendicular to the level curves and in a direction in which the function is increasing; therefore the gradients have directions $\pm\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ at the points $\pm(1,1)$ respectively.

15.5.7 $\left\langle -\sqrt{3}, 1\right\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle = -\frac{3}{2} - \frac{1}{2} = -2.$

15.5.8 It is $\langle 3,1\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right\rangle = \frac{2}{\sqrt{2}} = \sqrt{2}.$

15.5.9 Because $\mathbf{v}$ is in the opposite direction of $\mathbf{u}$, $D_{\mathbf{v}}f(3,4) = -7$. $D_{\mathbf{w}}f(3,4) = 0$, because $\mathbf{w}\cdot\nabla f(3,4) = 0$.

15.5.10 We are looking for the slope of a line which is perpendicular to a line of slope $\frac{4}{3}$, so the slope we seek is $-\frac{3}{4}$.

15.5.11

a. Note that $f_x = -x$ and $f_y = -2y$. So $\nabla f(2,0) = \langle -2, 0\rangle$, $\nabla f(0,2) = \langle 0, -4\rangle$, and $\nabla f(1,1) = \langle -1, -2\rangle$.

	$(a,b) = (2,0)$	$(a,b) = (0,2)$	$(a,b) = (1,1)$
$\mathbf{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$	$-\sqrt{2}$	$-2\sqrt{2}$	$-3\sqrt{2}/2$
$\mathbf{v} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$	$\sqrt{2}$	$-2\sqrt{2}$	$-\sqrt{2}/2$
$\mathbf{w} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$	$\sqrt{2}$	$2\sqrt{2}$	$3\sqrt{2}/2$

b. The function is decreasing at $(2,0)$ in the direction of $\mathbf{u}$ and increasing at $(2,0)$ in the direction of $\mathbf{v}$ and $\mathbf{w}$.

15.5.12

a. Note that $f_x = 4x$ and $f_y = 2y$. So $\nabla f(1,0) = \langle 4, 0\rangle$, $\nabla f(1,1) = \langle 4, 2\rangle$, and $\nabla f(1,2) = \langle 4, 4\rangle$.

	$(a,b) = (1,0)$	$(a,b) = (1,1)$	$(a,b) = (1,2)$
$\mathbf{u} = \langle 1, 0\rangle$	4	4	4
$\mathbf{v} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$	$2\sqrt{2}$	$3\sqrt{2}$	$4\sqrt{2}$
$\mathbf{w} = \langle 0, 1\rangle$	0	2	4

b. The function is increasing at $(1,0)$ in the direction of $\mathbf{u}$ and $\mathbf{v}$ and is constant in the direction of $\mathbf{w}$.

15.5.13 $\nabla f(x,y) = \langle 6x, -10y\rangle$ so $\nabla f(2,-1) = \langle 12, 10\rangle$.

15.5.14 $\nabla f(x,y) = \langle 8x - 2y, -2x + 2y\rangle$, so $\nabla f(-1,-5) = \langle 2, -8\rangle$.

15.5.15 $\nabla g(x,y) = \left\langle 2x - 8xy - 8y^2, -4x^2 - 16xy\right\rangle$, so $\nabla g(-1,2) = \langle -18, 28\rangle$.

15.5.16 $\nabla p(x,y) = \left\langle -\frac{4x}{\sqrt{12 - 4x^2 - y^2}}, -\frac{y}{\sqrt{12 - 4x^2 - y^2}}\right\rangle$, so $\nabla p(-1,-1) = \left\langle \frac{4}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right\rangle$.

15.5.17 $\nabla f(x,y) = \left\langle 2xye^{2xy} + e^{2xy}, 2x^2e^{2xy}\right\rangle$, so $\nabla f(1,0) = \langle 1,2\rangle$.

15.5.18 $\nabla f(x,y) = \langle 3\cos(3x+2y), 2\cos(3x+2y)\rangle$, so $\nabla f(\pi, 3\pi/2) = \langle 3,2\rangle$.

15.5.19 $\nabla F(x,y) = \left\langle -2xe^{-x^2-2y^2}, -4ye^{-x^2-2y^2}\right\rangle$, so $\nabla F(-1,2) = 2e^{-9}\langle 1,-4\rangle$.

15.5.20 $\nabla h(x,y) = \left\langle \frac{2x}{1+x^2+2y^2}, \frac{4y}{1+x^2+2y^2}\right\rangle$, so $\nabla h(2,-3) = \frac{2}{23}\langle 2,-6\rangle$.

15.5.21 $\nabla f(x,y) = \langle 2x,-2y\rangle$, so $\nabla f(-1,-3) = \langle -2,6\rangle$. We have $D_u f(-1,-3) = \langle -2,6\rangle \cdot \langle 3/5,-4/5\rangle = -6/5 - 24/5 = -30/5 = -6$.

15.5.22 $\nabla f(x,y) = \left\langle 6x, 3y^2\right\rangle$, so $\nabla f(3,2) = \langle 18,12\rangle$. We have $D_u f(3,2) = \langle 18,12\rangle \cdot \langle 5/13, 12/13\rangle = 90/13 + 144/13 = 234/13 = 18$.

15.5.23 $\nabla f(x,y) = \left\langle -6x, y^3\right\rangle$, so $\nabla f(2,-3) = \langle -12,-27\rangle$. We have

$$D_u f(2,-3) = \langle -12,-27\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle = \frac{27}{2} - 6\sqrt{3}.$$

15.5.24 $\nabla g(x,y) = \langle 2\pi\cos\pi(2x-y), -\pi\cos\pi(2x-y)\rangle$, so $\nabla g(-1,-1) = \langle -2\pi, \pi\rangle$. We have $D_u g(-1,-1) = \langle -2\pi,\pi\rangle \cdot \left\langle \frac{5}{13}, -\frac{12}{13}\right\rangle = -\frac{22}{13}\pi$.

15.5.25 $\nabla f(x,y) = \left\langle -\frac{x}{\sqrt{4-x^2-2y}}, -\frac{1}{\sqrt{4-x^2-2y}}\right\rangle$, so $\nabla f(2,-2) = \left\langle -1, -\frac{1}{2}\right\rangle$. Thus,

$$D_u f(2,-2) = \left\langle -1, -\frac{1}{2}\right\rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle = -\frac{2}{\sqrt{5}}.$$

15.5.26 $\nabla f(x,y) = \langle 13ye^{xy}, 13xe^{xy}\rangle$, so $\nabla f(1,0) = \langle 0,13\rangle$. A unit vector in the direction given is $\langle 5/13, 12/13\rangle$, so $D_u f = \langle 0,13\rangle \cdot \langle 5/13,12/13\rangle = 12$.

15.5.27 $\nabla f(x,y) = \langle 6x,2\rangle$, so $\nabla f(1,2) = \langle 6,2\rangle$. A unit vector in the direction given is $\langle -3/5, 4/5\rangle$, so $D_u f = \langle 6,2\rangle \cdot \langle -3/5,4/5\rangle = -2$.

15.5.28 $\nabla h(x,y) = \langle -e^{-x-y}, -e^{-x-y}\rangle$, so $\nabla h(\ln 2, \ln 3) = -\left\langle \frac{1}{6}, \frac{1}{6}\right\rangle$ A unit vector in the direction given is $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$. We have $D_u h(\ln 2, \ln 3) = -\left\langle \frac{1}{6},\frac{1}{6}\right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle = -\frac{2}{6\sqrt{2}} = -\frac{1}{3\sqrt{2}}$.

15.5.29 $\nabla g(x,y) = \left\langle \frac{2x}{4+x^2+y^2}, \frac{2y}{4+x^2+y^2}\right\rangle$, so $\nabla g(-1,2) = \left\langle -\frac{2}{9}, \frac{4}{9}\right\rangle$. A unit vector in the direction given is $\left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\rangle$. We have $D_u g(-1,2) = \left\langle -\frac{2}{9}, \frac{4}{9}\right\rangle \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\rangle = 0$.

15.5.30 $\nabla f(x,y) = \left\langle -\frac{y}{(x-y)^2}, \frac{x}{(x-y)^2}\right\rangle$, so $\nabla f(4,1) = \left\langle -\frac{1}{9}, \frac{4}{9}\right\rangle$. A unit vector in the direction given is $\left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$. We have $D_u f(4,1) = \left\langle -\frac{1}{9}, \frac{4}{9}\right\rangle \cdot \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle = \frac{1}{\sqrt{5}}$.

15.5.31

a. At the point $(1,-2)$ the value of the gradient is $\nabla f(1,-2) = \langle 2x,-8y\rangle\Big|_{(1,-2)} = \langle 2,16\rangle$. Therefore, the direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{65}}\langle 1,8\rangle$ and the direction of steepest descent is $-\mathbf{u}$.

b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{65}}\langle -8, 1\rangle$.

15.5.32

a. At the point $(2, 1)$ the value of the gradient is $\nabla f(2,1) = \langle 2x + 4y, 4x - 2y\rangle \Big|_{(2,1)} = \langle 8, 6\rangle$. Therefore, the direction of steepest ascent is $\mathbf{u} = \langle 4/5, 3/5\rangle$ and the direction of steepest descent is $-\mathbf{u}$.

b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \langle -3/5, 4/5\rangle$.

15.5.33

a. At the point $(-1, 1)$ the value of the gradient is $\nabla f(-1,1) = \left\langle 4x^3 - 2xy, -x^2 + 2y\right\rangle \Big|_{(-1,1)} = \langle -2, 1\rangle$. Therefore, the direction of steepest ascent is $\mathbf{u} = \dfrac{1}{\sqrt{5}}\langle -2, 1\rangle$ and the direction of steepest descent is $-\mathbf{u}$.

b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \dfrac{1}{\sqrt{5}}\langle 1, 2\rangle$.

15.5.34

a. At the point $(1, 2)$ the value of the gradient is

$$\nabla P(1,2) = \left\langle \frac{x+y}{\sqrt{20 + x^2 + 2xy - y^2}}, \frac{x-y}{\sqrt{20 + x^2 + 2xy - y^2}} \right\rangle \Big|_{(1,2)} = \left\langle \frac{3}{\sqrt{21}}, -\frac{1}{\sqrt{21}} \right\rangle.$$

Therefore, the direction of steepest ascent is $\mathbf{u} = \dfrac{1}{\sqrt{10}}\langle 3, -1\rangle$ and the direction of steepest descent is $-\mathbf{u}$.

b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{10}}\langle 1, 3\rangle$.

15.5.35

a. At the point $(-1, 1)$ the value of the gradient is

$$\nabla f(-1,1) = \left\langle -xe^{-x^2/2-y^2/2}, -ye^{-x^2/2-y^2/2} \right\rangle \Big|_{(-1,1)} = e^{-1}\langle 1, -1\rangle.$$

Therefore, the direction of steepest ascent is $\mathbf{u} = \dfrac{1}{\sqrt{2}}\langle 1, -1\rangle$ and the direction of steepest descent is $-\mathbf{u}$.

b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$.

15.5.36

a. At the point $(0, \pi)$ the value of the gradient is

$$\nabla f(0,\pi) = \langle 4\cos(2x - 3y), -6\cos(2x - 3y)\rangle \Big|_{(0,\pi)} = \langle -4, 6\rangle.$$

Therefore, the direction of steepest ascent is $\mathbf{u} = \dfrac{1}{\sqrt{13}}\langle -2, 3\rangle$ and the direction of steepest descent is $-\mathbf{u}$.

b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{13}}\langle 3, 2\rangle$.

15.5.37

a. The gradient of $f$at P is $\nabla f(3,2) = \langle -4x, -6y\rangle \Big|_{(3,2)} = \langle -12, -12\rangle$.

b. The direction of steepest ascent is$\mathbf{u} = \frac{1}{\sqrt{2}}\langle -1,-1\rangle$ which makes angle $\theta = \frac{5\pi}{4}$ with the x-axis; therefore, the angle of maximum decrease is $\theta = \frac{\pi}{4}$ and the angles of zero change are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

c. We have $g(\theta) = \langle -12,-12\rangle \cdot \langle \cos\theta, \sin\theta\rangle = -12\cos\theta - 12\sin\theta$.

d. The critical points for $g(\theta)$ satisfy $g'(\theta) = 12(\sin\theta - \cos\theta) = 0$, which gives $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$. By inspection we see that the maximum occurs at $\frac{5\pi}{4}$, and we have $g\left(\frac{5\pi}{4}\right) = 12\sqrt{2}$.

e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that $|\nabla f(3,2)| = 12\sqrt{2} = g\left(\frac{5\pi}{4}\right)$.

15.5.38

a. The gradient of f at p is $\nabla f(-3,-1) = \langle 2x, 6y\rangle \Big|_{(-3,-1)} = \langle -6,-6\rangle$.

b. The direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{2}}\langle -1,-1\rangle$ which makes angle $\theta = \frac{5\pi}{4}$ with the x-axis; therefore, the angle of maximum decrease is $\theta = \frac{\pi}{4}$ and the angles of zero change are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

c. We have $g(\theta) = \langle -6,-6\rangle \cdot \langle \cos\theta, \sin\theta\rangle = -6\cos\theta - 6\sin\theta$.

d. The critical points for $g(\theta)$ satisfy $g'(\theta) = 6(\sin\theta - \cos\theta) = 0$, which gives $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$. By inspection we see that the maximum occurs at $\frac{5\pi}{4}$, and we have $g\left(\frac{5\pi}{4}\right) = 6\sqrt{2}$.

e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that$|\nabla f(-3,-1)| = 6\sqrt{2} = g\left(\frac{5\pi}{4}\right)$.

15.5.39

a. The gradient of f at P is $\nabla f\left(\sqrt{3},1\right) = \left\langle \frac{x}{\sqrt{2+x^2+y^2}}, \frac{y}{\sqrt{2+x^2+y^2}} \right\rangle \Big|_{(\sqrt{3},1)} = \left\langle \frac{\sqrt{3}}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle = \frac{\sqrt{6}}{6}\left\langle \sqrt{3},1\right\rangle$.

b. The direction of steepest ascent is $\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$ which makes angle $\theta = \frac{\pi}{6}$ with the x-axis; therefore, the angle of maximum decrease is $\theta = \frac{7\pi}{6}$ and the angles of zero change are $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

c. We have $g(\theta) = \frac{\sqrt{6}}{6}\left\langle \sqrt{3},1\right\rangle \cdot \langle \cos\theta, \sin\theta\rangle = \frac{\sqrt{18}}{6}\cos\theta + \frac{\sqrt{6}}{6}\sin\theta$.

d. The critical points for $g(\theta)$ satisfy $g'(\theta) = \frac{\sqrt{6}}{6}\left(-\sqrt{3}\sin\theta + \cos\theta\right) = 0$, which gives $\tan\theta = \frac{1}{\sqrt{3}}$, so $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$. By inspection we see that the maximum occurs at $\frac{\pi}{6}$, and we have $g\left(\frac{\pi}{6}\right) = \frac{\sqrt{6}}{3}$.

e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that $|\nabla f(3,1)| = \frac{\sqrt{6}}{3} = g\left(\frac{\pi}{6}\right)$.

15.5.40

a. The gradient of f at P is $\nabla f\left(-1,-\dfrac{1}{\sqrt{3}}\right) = \left\langle -\dfrac{x}{\sqrt{12-x^2-y^2}}, -\dfrac{y}{\sqrt{12-x^2-y^2}}\right\rangle \Big|_{(-1,-1/\sqrt{3})} = \dfrac{1}{4\sqrt{2}}\left\langle \sqrt{3},1\right\rangle$.

b. The direction of steepest ascent is u $= \left\langle \dfrac{\sqrt{3}}{2},\dfrac{1}{2}\right\rangle$ which makes angle $\theta = \dfrac{\pi}{6}$ with the x-axis; therefore, the angle of maximum decrease is $\theta = \dfrac{7\pi}{6}$ and the angles of zero change are $\dfrac{2\pi}{3}$ and $\dfrac{5\pi}{3}$.

c. We have $g(\theta) = \dfrac{1}{4\sqrt{2}}\left\langle \sqrt{3},1\right\rangle \cdot \langle \cos\theta, \sin\theta\rangle = \dfrac{1}{4\sqrt{2}}\left(\sqrt{3}\cos\theta + \sin\theta\right)$.

d. The critical points for $g(\theta)$ satisfy $g'(\theta) = \dfrac{1}{4\sqrt{2}}\left(-\sqrt{3}\sin\theta + \cos\theta\right) = 0$,which gives $\tan\theta = \dfrac{1}{\sqrt{3}}$, so $\theta = \dfrac{\pi}{6}, \dfrac{7\pi}{6}$. By inspection we see that the maximum occurs at $\dfrac{\pi}{6}$, and we have $g\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{2}}{4}$.

e. Observe that the maximum value of $g(\theta)$ occurs at theangle found in part (d), and that $\left|\nabla f\left(-1,-\dfrac{1}{\sqrt{3}}\right)\right| = \dfrac{\sqrt{2}}{4} = g\left(\dfrac{\pi}{6}\right)$.

15.5.41

a. The gradient of f at P is $\nabla f(-1,0) = \left\langle -2xe^{-x^2-2y^2}, -4ye^{-x^2-2y^2}\right\rangle \big|_{(-1,0)} = \left\langle 2e^{-1},0\right\rangle$.

b. The direction of steepest ascent is $\mathbf{u} = \langle 1,0\rangle$ which makes angle $\theta = 0$ with the x-axis; therefore the angle of maximum decrease is $\theta = \pi$ and the angles of zero change are $\pm\dfrac{\pi}{2}$.

c. We have $g(\theta) = \dfrac{2}{\mathrm{e}}\langle 1,0\rangle \cdot \langle \cos\theta, \sin\theta\rangle = \dfrac{2}{\mathrm{e}}\cos\theta$.

d. The maximum value of $g(\theta)$ occurs at $\theta = 0$, and we have $g(0) = \dfrac{2}{\mathrm{e}}$.

e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that $|\nabla f(-1,0)| = \dfrac{2}{\mathrm{e}} = g(0)$.

15.5.42

a. The gradient of f at P is

$$\nabla f\left(\frac{3}{4},-\sqrt{3}\right) = \left\langle \frac{4x}{1+2x^2+3y^2}, \frac{6y}{1+2x^2+3y^2}\right\rangle \Big|_{(3/4,-\sqrt{3})} = \frac{24}{89}\left\langle 1,-2\sqrt{3}\right\rangle.$$

b. The direction of steepest ascent is u $= \left\langle \dfrac{1}{\sqrt{13}}, -\dfrac{2\sqrt{3}}{\sqrt{13}}\right\rangle$ which makes angle $\theta = -\tan^{-1}\left(2\sqrt{3}\right)$ with the x-axis; therefore, the angle of maximum decrease is $\theta = \pi - \tan^{-1}\left(2\sqrt{3}\right)$ and the angles of zero change are $\pm\dfrac{\pi}{2} - \tan^{-1}\left(2\sqrt{3}\right)$.

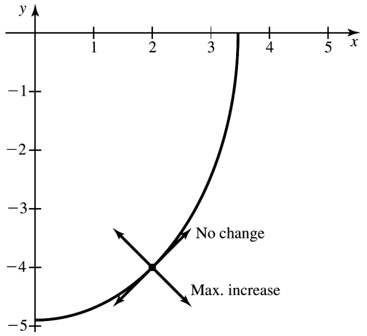
c. We have $g(\theta) = \dfrac{24}{89}\left\langle 1,-2\sqrt{3}\right\rangle \cdot \langle \cos\theta, \sin\theta\rangle = \dfrac{24}{89}\left(\cos\theta - 2\sqrt{3}\sin\theta\right)$.

d. The critical points for $g(\theta)$ satisfy $g'(\theta) = \dfrac{24}{89}\left(-\sin\theta - 2\sqrt{3}\cos\theta\right) = 0$, which gives $\tan\theta = -2\sqrt{3}$, so $\theta = -\tan^{-1}\left(2\sqrt{3}\right), \pi - \tan^{-1}\left(2\sqrt{3}\right)$. By inspection we see that the maximum occurs at $-\tan^{-1}\left(2\sqrt{3}\right)$, and we have $g\left(-\tan^{-1}\left(2\sqrt{3}\right)\right) = \dfrac{24\sqrt{13}}{89}$.

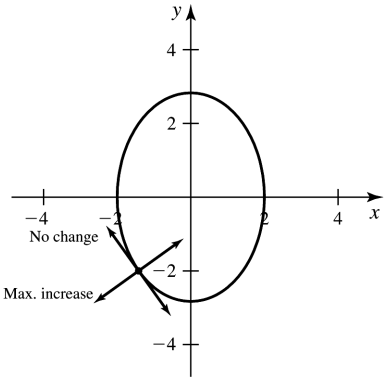
e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that

$$\left|\nabla f\left(\frac{3}{4}, -\sqrt{3}\right)\right| = \frac{24\sqrt{13}}{89} = g\left(-\tan^{-1}\left(2\sqrt{3}\right)\right).$$

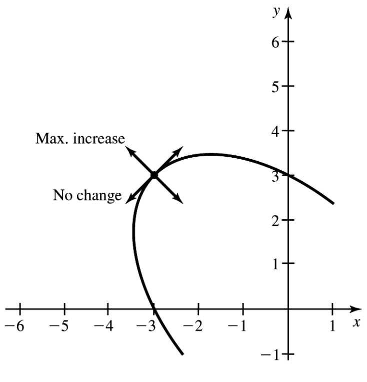
15.5.43 The gradient of f at P is $\nabla f(2,-4) = \langle 8x, 4y\rangle\Big|_{(2,-4)} = \langle 16, -16\rangle$, which gives the direction of maximum increase.


y
x
1
2
3
4
5
−1
−2
−3
−4
−5
No change
Max. increase

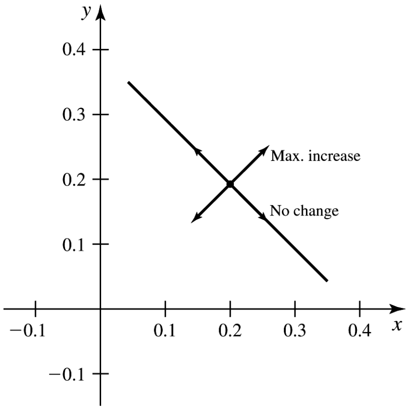
15.5.44 The gradient of f at P is $\nabla f(-1,-2) = \langle 12x, 6y\rangle\Big|_{(-1,-2)} = \langle -12, -12\rangle$, which gives the direction of maximum increase.


y
x
4
2
−2
−4
−4
−2
2
4
No change
Max. increase

15.5.45 The gradient of f at P is $\nabla f(-3,3) = \langle 2x+y, x+2y\rangle\Big|_{(-3,3)} = \langle -3, 3\rangle$, which gives the direction of maximum increase.


y
x
6
5
4
3
2
1
−1
−6
−5
−4
−3
−2
−1
1
Max. increase
No change

15.5.46 The gradient of f at P is $\nabla f\left(\frac{\pi}{16},\frac{\pi}{16}\right)=\left\langle 2\sec^2(2x+2y),2\sec^2(2x+2y)\right\rangle\Big|_{(\pi/16,\pi/16)}=\langle 4,4\rangle$, which gives the direction of maximum increase.



15.5.47 The slope of the level curves for $f(x,y)$ is given by $y'(x)=-\frac{f_x}{f_y}=-\frac{4x}{y}$, so the tangent line has slope 0 at $(0,16)$. The gradient of f at this point is $\nabla f(0,16)=\left\langle -\frac{x}{2},-\frac{y}{8}\right\rangle\Big|_{(0,16)}=\langle 0,-2\rangle$, which is perpendicular to the tangent line.

15.5.48 The slope of the level curves for $f(x,y)$ is given by $y'(x)=-\frac{f_x}{f_y}=-\frac{4x}{y}$, which is undefined at $(8,0)$, so the tangent line is vertical at $(8,0)$. The gradient of f at this point is $\nabla f(8,0)=\left\langle -\frac{x}{2},-\frac{y}{8}\right\rangle\Big|_{(8,0)}=\langle -4,0\rangle$, which is perpendicular to the tangent line.

15.5.49 The slope of the level curves for $f(x,y)$ is given by $y'(x)=-\frac{f_x}{f_y}=-\frac{4x}{y}$, which is undefined at $(4,0)$, so the tangent line is vertical at $(4,0)$. The gradient of f at this point is $\nabla f(4,0)=\left\langle -\frac{x}{2},-\frac{y}{8}\right\rangle\Big|_{(4,0)}=\langle -2,0\rangle$, which is perpendicular to the tangent line.

15.5.50 The slope of the level curves for $f(x,y)$ is given by $y'(x)=-\frac{f_x}{f_y}=-\frac{4x}{y}$, so the tangent line has slope $-2\sqrt{3}$ at $\left(2\sqrt{3},4\right)$, and its direction is parallel to the vector $\langle 1,-2\sqrt{3}\rangle$. The gradient of f at this point is $\nabla f\left(2\sqrt{3},4\right)=\left\langle -\frac{x}{2},-\frac{y}{8}\right\rangle\Big|_{(2\sqrt{3},4)}=\left\langle -\sqrt{3},-\frac{1}{2}\right\rangle$, which is perpendicular to the tangent direction.

15.5.51 Let $z=f(x,y)$. Then $z^2=1-\frac{x^2}{4}-\frac{y^2}{16}$, so $2zz_x=-\frac{x}{2}$, which implies that $z_x=f_x=-\frac{x}{4z}$. Also $2zz_y=-\frac{y}{8}$, which implies that $z_y=f_y=-\frac{y}{16z}$. The slope of the level curves for $f(x,y)$ is given by $y'(x)=-\frac{f_x}{f_y}=-\frac{4x}{y}$, so the tangent line has slope $-\frac{2}{\sqrt{3}}$ at $\left(\frac{1}{2},\sqrt{3}\right)$, and its direction is parallel to the vector $\left\langle 1,-\frac{2}{\sqrt{3}}\right\rangle$. The gradient of f at this point is $\nabla f\left(\frac{1}{2},\sqrt{3}\right)=\left\langle -\frac{x}{4z},-\frac{y}{16z}\right\rangle\Big|_{(1/2,\sqrt{3})}=-\frac{1}{8}\left\langle \frac{2}{\sqrt{3}},1\right\rangle$, which is perpendicular to the tangent direction.

15.5.52 Let $z=f(x,y)$. Then $z^2=1-\frac{x^2}{4}-\frac{y^2}{16}$, so $2zz_x=-\frac{x}{2}$, which implies that $z_x=f_x=-\frac{x}{4z}$. Also $2zz_y=-\frac{y}{8}$, which implies that $z_y=f_y=-\frac{y}{16z}$. The slope of the level curves for $f(x,y)$ is given by $y'(x)=-\frac{f_x}{f_y}=-\frac{4x}{y}$, so the tangent line has slope 0 at $\left(0,\sqrt{8}\right)$, and its direction is parallel to the vector $\langle 1,0\rangle$. The gradient of f at this point is $\nabla f\left(0,\sqrt{8}\right) = \left\langle -\frac{x}{4z}, -\frac{y}{16z}\right\rangle\Big|_{\left(0,\sqrt{8}\right)} = \left\langle 0, -\frac{1}{4}\right\rangle$,which is perpendicular to the tangent direction.

15.5.53 Let $z = f\left(x,y\right)$. Then $z^2 = 1 - \frac{x^2}{4} - \frac{y^2}{16}$, so $2zz_x = -\frac{x}{2}$, which implies that $z_x = f_x = -\frac{x}{4z}$. Also, $2zz_y = -\frac{y}{8}$, which implies that $z_y = f_y = -\frac{y}{16z}$.The slope of the level curves for $f\left(x,y\right)$ is given by $y'\left(x\right) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line is vertical at $\left(\sqrt{2},0\right)$, and its direction is parallel to the vector $\langle 0,1\rangle$. The gradient of f at this point is $\nabla f\left(\sqrt{2},0\right) = \left\langle -\frac{x}{4z}, -\frac{y}{16z}\right\rangle\Big|_{\left(\sqrt{2},0\right)} = \left\langle -\frac{1}{2}, 0\right\rangle$, which is perpendicular to the tangent direction.

15.5.54 Let $z = f\left(x,y\right)$. Then $z^2 = 1 - \frac{x^2}{4} - \frac{y^2}{16}$, so $2zz_x = -\frac{x}{2}$, which implies that $z_x = f_x = -\frac{x}{4z}$. Also, $2zz_y = -\frac{y}{8}$, which implies that $z_y = f_y = -\frac{y}{16z}$.The slope of the level curves for $f\left(x,y\right)$ is given by $y'\left(x\right) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line has slope -2 at $(1,2)$, and its direction is parallel to the vector $\langle 1,-2\rangle$. The gradient of f at this point is $\nabla f\left(1,2\right) = \left\langle -\frac{x}{4z}, -\frac{y}{16z}\right\rangle\Big|_{(1,2)} = -\frac{\sqrt{2}}{8}\left\langle 2,1\right\rangle$, which is perpendicular to the tangent direction.

15.5.55

a. We have $\nabla f = \langle f_x, f_y\rangle = \langle 1,0\rangle$.

b. Let $\left(x\left(t\right), y\left(t\right)\right)$ be the projection into the xy-plane of the path of steepest descent starting at $\left(x\left(0\right), y\left(0\right)\right) = (4,4)$. We solve $\left(x'\left(t\right), y'\left(t\right)\right) = -\nabla f = \langle -1,0\rangle$ which together with the initial conditions gives $x = 4 - t$, $y = 4$ for $t \geq 0$.

c. We have $z = 4 + x$, so $\mathbf{r}(t) = \langle 4-t, 4, 8-t\rangle$.

15.5.56

a. We have $\nabla f = \langle f_x, f_y\rangle = \langle 1,1\rangle$.

b. Let $\left(x\left(t\right), y\left(t\right)\right)$ be the projection into the xy-plane of the path of steepest descent starting at $\left(x\left(0\right), y\left(0\right)\right) = (2,2)$. We solve $\left(x'\left(t\right), y'\left(t\right)\right) = -\nabla f = \langle -1,-1\rangle$ which together with the initial conditions gives $x = 2 - t$, $y = 2 - t$ for $t \geq 0$.

c. We have $z = x + y$, so $\mathbf{r}(t) = \langle 2-t, 2t, 2-t\rangle$.

15.5.57

a. We have $\nabla f = \langle f_x, f_y\rangle = \langle -2x, -4y\rangle$.

b. Let $\left(x\left(t\right), y\left(t\right)\right)$ be the projection into the xy-plane of the path of steepest descent starting at $\left(x\left(0\right), y\left(0\right)\right) = \left(\frac{\pi}{2}, 1\right)$. We solve $\left(x'\left(t\right), y'\left(t\right)\right) = -\nabla f = \langle 2x, 4y\rangle$ with the initial conditions $x\left(0\right) = 1$ and $y\left(0\right) = 1$. The differential equation $\frac{dx}{dt} = -2x$ is separable: we have $\int \frac{1}{x}\,dx = \int 2\,dt$ which implies that $\ln|x| = 2t + C$. The initial condition $x\left(0\right) = 1$ gives $C = 0$ so $x\left(t\right) = e^{2t}$. Similarly, $y\left(t\right) = e^{4t}$. Thus, $y = x^2$ for $x \geq 1$.

c. So if we let $x = t$, we have $y = t^2$ and $z = 4 - x^2 - 2y^2$, so $\mathbf{r}(t) = \left\langle t, t^2, 4 - t^2 - 2t^4\right\rangle$.

15.5.58

a. We have $\nabla f = \langle f_x, f_y\rangle = \left\langle -x^{-2}, 1\right\rangle$.

b. Let $(x(t), y(t))$ be the projection into the xy-plane of the path of steepest descent starting at $(x(0), y(0)) = (1, 2)$. We solve $(x'(t), y'(t)) = -\nabla f = \langle x^{-2}, -1 \rangle$ with the initial conditions $x(0) = 1$ and $y(0) = 2$. The differential equation $\dfrac{dx}{dt} = \dfrac{1}{x^2}$ is separable: we have $\int x^2\,dx = \int dt$, which implies that $\dfrac{x^3}{3} = t + C$; the initlal condition gives $C = \dfrac{1}{3}$ so $x(t) = (3t+1)^{1/3}$ and $y(t) = 2 - t$ for $t \geq 0$.

c. $z = y + \dfrac{1}{x}$ so $\mathbf{r}(t) = \left\langle (3t+1)^{1/3}, 2 - t, 2 - t + \dfrac{1}{(3t+1)^{1/3}} \right\rangle$.

15.5.59

a. We have $\nabla f = 2x\mathbf{i} + 4y\mathbf{j} + 8z\mathbf{k}$; $\nabla f(1,0,4) = 2\mathbf{i} + 32\mathbf{k}$.

b. The vector $2\mathbf{i} + 32\mathbf{k}$ has length $2\sqrt{257}$, so the unit vector in this direction is $\mathbf{u} = \dfrac{1}{\sqrt{257}}(\mathbf{i} + 16\mathbf{k})$.

c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(1,0,4)| = 2\sqrt{257}$.

d. The vector $\mathbf{u} = \left\langle \dfrac{1}{\sqrt{2}}, 0, \dfrac{1}{\sqrt{2}} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f(1,0,4) = \langle 2, 0, 32 \rangle \cdot \left\langle \dfrac{1}{\sqrt{2}}, 0, \dfrac{1}{\sqrt{2}} \right\rangle = \dfrac{34}{\sqrt{2}} = 17\sqrt{2}$.

15.5.60

a. We have $\nabla f = -2x\mathbf{i} + 6y\mathbf{j} + z\mathbf{k}$; $\nabla f(0,2,-1) = 12\mathbf{j} - \mathbf{k}$.

b. The vector $12\mathbf{j} - \mathbf{k}$ has length $\sqrt{145}$, so the unit vector in this direction is $\mathbf{u} = \dfrac{1}{\sqrt{145}}(12\mathbf{j} - \mathbf{k})$.

c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(0,2,-1)| = \sqrt{145}$.

d. The vector $\mathbf{u} = \left\langle 0, \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f(0,2,-1) = \langle 0, 12, -1 \rangle \cdot \left\langle 0, \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}} \right\rangle = \dfrac{13}{\sqrt{2}}$.

15.5.61

a. We have $\nabla f = 4yz\mathbf{i} + 4xz\mathbf{j} + 4xy\mathbf{k}$; $\nabla f(1,-1,-1) = 4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$.

b. The vector $4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ has length $4\sqrt{3}$, so the unit vector in this direction is $\mathbf{u} = \dfrac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} - \mathbf{k})$.

c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(1,-1,-1)| = 4\sqrt{3}$.

d. The vector $\mathbf{u} = \left\langle \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f(1,-1,-1) = \langle 4, -4, -4 \rangle \cdot \left\langle \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}} \right\rangle = \dfrac{4}{\sqrt{3}}$.

15.5.62

a. We have $\nabla f = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$; $\nabla f(2,-2,1) = -\mathbf{i} + 3\mathbf{j}$.

b. The vector $-\mathbf{i} + 3\mathbf{j}$ has length 10, so the unit vector in this direction is $\mathbf{u} = \dfrac{1}{\sqrt{10}}(-\mathbf{i} + 3\mathbf{j})$.

c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(2,-2,1)| = \sqrt{10}$.

d. The vector $\mathbf{u} = \left\langle 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f(2,-2,1) = \langle -1, 3, 0\rangle \cdot \left\langle 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = -\frac{3}{\sqrt{2}}$.

15.5.63

a. We have $\nabla f = \cos(x+2y-z)(\mathbf{i}+2\mathbf{j}-\mathbf{k})$; $\nabla f\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right) = \cos\left(\frac{2\pi}{3}\right)(\mathbf{i}+2\mathbf{j}-\mathbf{k}) = -\frac{1}{2}\mathbf{i}-\mathbf{j}+\frac{1}{2}\mathbf{k}$.

b. The vector $-\frac{1}{2}\mathbf{i}-\mathbf{j}+\frac{1}{2}\mathbf{k}$ has length $\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{6}}(-\mathbf{i}-2\mathbf{j}+\mathbf{k})$.

c. The rate of change of f in the direction of maximum increase at P is $\left|\nabla f\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right)\right| = \frac{\sqrt{6}}{2}$.

d. The vector $\mathbf{u} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle \cdot \left\langle -\frac{1}{2}, -1, \frac{1}{2} \right\rangle = -\frac{1}{2}$.

15.5.64

a. We have $\nabla f = e^{xyz-1}(yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k})$; $\nabla f(0,1,-1) = -e^{-1}\mathbf{i}$.

b. The unit vector in the direction of $-e^{-1}\mathbf{i}$ is $\mathbf{u} = -\mathbf{i}$.

c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(0,1,-1)| = e^{-1}$.

d. The vector $\mathbf{u} = \left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f(0,1,-1) = \left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle \cdot \langle -e^{-1}, 0, 0\rangle = \frac{2}{3e}$.

15.5.65

a. We have $\nabla f = \frac{2}{1+x^2+y^2+z^2}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$; $\nabla f(1,1,-1) = \frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}-\frac{1}{2}\mathbf{k}$.

b. The vector $\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}-\frac{1}{2}\mathbf{k}$ has length $\frac{\sqrt{3}}{2}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}-\mathbf{k})$.

c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(1,1,-1)| = \frac{\sqrt{3}}{2}$.

d. The vector $\mathbf{u} = \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f(1,1,-1) = \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle \cdot \left\langle \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle = \frac{5}{6}$.

15.5.66

a. We have $\nabla f = \frac{1}{(y-z)^2}((y-z)\mathbf{i}+(z-x)\mathbf{j}+(x-y)\mathbf{k})$; $\nabla f(3,2,-1) = \frac{1}{3}\mathbf{i}-\frac{4}{9}\mathbf{j}+\frac{1}{9}\mathbf{k}$.

b. The vector $\frac{1}{3}\mathbf{i}-\frac{4}{9}\mathbf{j}+\frac{1}{9}\mathbf{k}$ has length $\frac{\sqrt{26}}{9}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{26}}(3\mathbf{i}-4\mathbf{j}+\mathbf{k})$.

c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(3,2,-1)| = \frac{\sqrt{26}}{9}$.

d. The vector $\mathbf{u} = \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f(3,2,-1) = \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle \cdot \left\langle \frac{1}{3}, -\frac{4}{9}, \frac{1}{9} \right\rangle = -\frac{7}{27}$.

15.5.67

a. False. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$.

b. False. The gradient is a vector, so it does not make sense to say that it is positive.

c. False. f is a function of three variables, so ∇f has three components.

d. True. This is because $f_x = f_y = f_z = 0$.

15.5.68

a. We can express $F(x,y,z) = f(g(x,y,z))$ where $g(x,y,z) = xyz$ and $f(w) = e^w$.

b. Observe that $\nabla F(x,y,z) = e^{xyz}(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) = e^{xyz}\nabla g = f'(g(x,y,z))\nabla g(x,y,z)$.

15.5.69 Observe that $\nabla f(x,y) = -8x\mathbf{i} - 2y\mathbf{j}$, so $\nabla f(1,2) = -8\mathbf{i} - 4\mathbf{j}$. A vector perpendicular to $\nabla f(1,2)$ is $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$, so the unit vectors perpendicular to $\nabla f(1,2)$ are $\mathbf{u} = \pm\frac{1}{\sqrt{5}}(\mathbf{i} - 2\mathbf{j})$.

15.5.70 Observe that $\nabla f(x,y) = 2x\mathbf{i} - 8y\mathbf{j}$, so $\nabla f(4,1) = 8\mathbf{i} - 8\mathbf{j}$. A vector perpendicular to $\nabla f(4,1)$ is $\mathbf{v} = \mathbf{i} + \mathbf{j}$, so the unit vectors perpendicular to $\nabla f(4,1)$ are $\mathbf{u} = \pm\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$.

15.5.71 Observe that $\nabla f(x,y) = \frac{1}{\sqrt{3 + 2x^2 + y^2}}(2x\mathbf{i} + y\mathbf{j})$, so $\nabla f(1,-2) = \frac{2}{3}(\mathbf{i} - \mathbf{j})$. A vector perpendicular to $\nabla f(1,-2)$ is $\mathbf{v} = \mathbf{i} + \mathbf{j}$, so the unit vectors perpendicular to $\nabla f(1,-2)$ are $\mathbf{u} = \pm\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$.

15.5.72 Observe that $\nabla f(x,y) = -e^{1-xy}(y\mathbf{i} + x\mathbf{j})$, so $\nabla f(1,0) = -e\mathbf{j}$, so the unit vectors perpendicular to $\nabla f(1,0)$ are $\mathbf{u} = \pm\mathbf{i}$.

15.5.73 The function $f(x,y) = ax + by + c$ has $\nabla f(x,y) = a\mathbf{i} + b\mathbf{j}$, so the path in the xy-plane corresponding to the path of steepest ascent on the plane is given by $x = x_0 + at$, $y = y_0 + bt$.

15.5.74

a. The function $z = f(x,y)$ satisfies the relation $z^2 = (x-a)^2 + (y-b)^2$, which is the equation of a cone with vertex at $(a,b,0)$.

b. We have $\nabla f(x,y) = \frac{1}{\sqrt{(x-a)^2 + (y-b)^2}}((x-a)\mathbf{i} + (y-b)\mathbf{j})$, which implies that $|\nabla f(x,y)| = 1$ for all points $(x,y) \neq (a,b)$.

c. At any point $(x,y) \neq (a,b)$, the greatest rate of change of f is 1, which occurs in the direction of the vector from (a,b) to (x,y).

15.5.75

a. We have $\nabla f(x,y,z) = \langle 2x, 2y, 2z\rangle$, so $\nabla f(1,1,1) = \langle 2,2,2\rangle$.

b. Points (x,y,z) on the plane P satisfy $\langle x-1, y-1, z-1\rangle \cdot \langle 2,2,2\rangle = 0$, so $x + y + z = 3$ is an equation for P.

15.5.76

a. We have $\nabla f(x,y,z) = \langle -yz, -xz, -xy\rangle$, so $\nabla f(2,2,2) = \langle -4,-4,-4\rangle$.

b. Points (x,y,z) on the plane P satisfy $\langle x-2, y-2, z-2\rangle \cdot \langle -4,-4,-4\rangle = 0$, so $x+y+z=6$ is an equation for P.

15.5.77

a. We have $\nabla f(x,y,z) = e^{x+y-z}\langle 1,1,-1\rangle$, so $\nabla f(1,1,2) = \langle 1,1,-1\rangle$.

b. Points (x,y,z) on the plane P satisfy $\langle x-1, y-1, z-2\rangle \cdot \langle 1,1,-1\rangle = 0$, so $x+y-z=0$ is an equation for P.

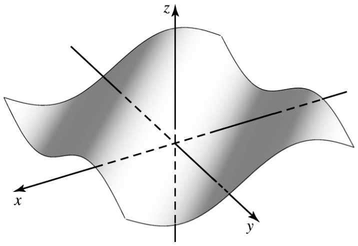
15.5.78

a. We have $\nabla f(x,y,z) = \langle y+z, x-z, x-y\rangle$, so $\nabla f(1,1,1) = \langle 2,0,0\rangle$.

b. Points (x,y,z) on the plane P satisfy $\langle x-1, y-1, z-1\rangle \cdot \langle 2,0,0\rangle = 0$, so $x=1$ is an equation for P.

15.5.79

a.



b. The height function has gradient $\nabla z = \cos(x-y)\langle 1,-1\rangle$, so the directions in which the height function has zero change are $\mathbf{v} = \pm\left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle$.

c. The direction $\mathbf{v}$ would be the opposite of the direction of ∇z, so $\mathbf{v} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$.

d. These answers are consistent with the graph in part (a).

15.5.80

a. The height function has gradient $\nabla z = \cos(ax-b)\langle a,-b\rangle$, so the directions in which the height function has zero change are $\mathbf{v} = \pm\left\langle \frac{b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\right\rangle$.

b. The direction $\mathbf{v}$ would be the opposite of the direction of ∇z,so $\mathbf{v} = \left\langle -\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right\rangle$.

15.5.81

a. Observe that $\varphi_x = -\frac{kQ}{r^2} r_x = -\frac{kQ}{r^2}\frac{x}{\sqrt{x^2+y^2+z^2}} = -\frac{kQx}{r^3}$; similarly $\varphi_y = -\frac{kQy}{r^3}$ and $\varphi_z = -\frac{kQz}{r^3}$. Therefore $\mathrm{E}(x,y,z) = -\nabla\varphi(x,y,z) = kQ\left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right\rangle$.

b. We have $|\mathrm{E}| = \frac{kQ}{r^3}|\langle x,y,z\rangle| = \frac{kQ}{r^2}$. Therefore the magnitude of the electric field is inversely proportional to the square of the distance to the point charge.

15.5.82 Observe that

$$\varphi_x = \frac{GMm}{r^2} r_x = \frac{GMm}{r^2} \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{GMmx}{r^3}.$$

Similarly, $\varphi_y = \dfrac{GMmy}{r^3}$ and $\varphi_z = \dfrac{GMmz}{r^3}$. Therefore

$$\mathrm{F}(x,y,z) = -\nabla\varphi(x,y,z) = -\frac{GMm}{r^3}\langle x,y,z\rangle,$$

and we have $|\mathrm{F}| = \dfrac{GMm}{r^3}|\langle x,y,z\rangle| = \dfrac{GMm}{r^2}$. Therefore the magnitude of the gravitational field is inversely proportional to the square of the distance between the two objects.

15.5.83 We have $\langle u,v\rangle = \nabla\varphi = \langle \pi\cos\pi x\sin 2\pi y, 2\pi\sin\pi x\cos 2\pi y\rangle$.

15.5.84 Observe that $\nabla f(x,y) = A\mathbf{i}+B\mathbf{j}$, which is a constant vector. Therefore the direction and magnitude of the greatest directional derivative of $f(x,y,z)$ is the same at all points (x,y).

15.5.85 We give the proofs for functions on $\mathbb{R}^2$. The proofs for functions on $\mathbb{R}^3$ are similar.

a. $\nabla(cf) = \left\langle (cf)_x, (cf)_y\right\rangle = \langle cf_x, cf_y\rangle = c\nabla f$.

b. $\nabla(f+g) = \left\langle (f+g)_x, (f+g)_y\right\rangle = \langle f_x+g_x, f_y+g_y\rangle = \nabla f + \nabla g$.

c. $\nabla(fg) = \left\langle (fg)_x, (fg)_y\right\rangle = \langle f_xg+fg_x, f_yg+fg_y\rangle = (\nabla f)g + f\nabla g$.

d. $\nabla\left(\dfrac{f}{g}\right) = \left\langle \left(\dfrac{f}{g}\right)_x, \left(\dfrac{f}{g}\right)_y\right\rangle = \left\langle \dfrac{gf_x - fg_x}{g^2}, \dfrac{gf_y-fg_y}{g^2}\right\rangle = \dfrac{g(\nabla f) - f\nabla g}{g^2}$.

e. $\nabla(f\circ g) = \langle f'(g)g_x, f'(g)g_y\rangle = f'(g)\nabla g$.

15.5.86 Using the product and chain rules from Exercise 85 gives $\nabla(xy\cos(xy)) = \cos(xy)\nabla(xy) + xy(-\sin(xy))\nabla(xy) = (\cos(xy) - xy\sin(xy))\langle y,x\rangle$.

15.5.87 Using the quotient rule from Exercise 85 gives

$$\begin{aligned}\nabla\left(\frac{x+y}{x^2+y^2}\right) &= \frac{\left(x^2+y^2\right)\nabla(x+y) - (x+y)\nabla\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}\\ &= \frac{1}{\left(x^2+y^2\right)^2}\left(\left(x^2+y^2\right)\langle 1,1\rangle - (x+y)\langle 2x,2y\rangle\right)\\ &= \frac{1}{\left(x^2+y^2\right)^2}\left\langle y^2-x^2-2xy, x^2-y^2-2xy\right\rangle.\end{aligned}$$

15.5.88 Using the chain rule from Exercise 85 gives $\nabla\left(\ln\left(1+x^2+y^2\right)\right) = \dfrac{1}{1+x^2+y^2}\nabla\left(1+x^2+y^2\right) = \dfrac{1}{1+x^2+y^2}\langle 2x,2y\rangle$.

15.5.89 Using the chain rule from Exercise 85 gives

$$\begin{aligned}\nabla\left(\sqrt{25-x^2-y^2-z^2}\right) &= \frac{1}{2\sqrt{25-x^2-y^2-z^2}}\nabla\left(25-x^2-y^2-z^2\right)\\ &= \frac{1}{2\sqrt{25-x^2-y^2-z^2}}\langle -2x,-2y,-2z\rangle\\ &= -\frac{1}{\sqrt{25-x^2-y^2-z^2}}\langle x,y,z\rangle.\end{aligned}$$

15.5.90 Using the product and chain rules from Exercise 85 gives

$$\begin{aligned}\nabla\left((x+y+z)\,e^{xyz}\right) &= e^{xyz}\nabla\left(x+y+z\right)+\left(x+y+z\right)\nabla e^{xyz}\\ &= e^{xyz}\left(\langle 1,1,1\rangle+\left(x+y+z\right)\langle yz,xz,xy\rangle\right)\\ &= e^{xyz}\left\langle 1+\left(x+y+z\right)yz,1+\left(x+y+z\right)xz,1+\left(x+y+z\right)xy\right\rangle.\end{aligned}$$

15.5.91 Using the quotient rule from Exercise 85 gives

$$\begin{aligned}\nabla\left(\frac{x+yz}{y+xz}\right) &= \frac{\left(y+xz\right)\nabla\left(x+yz\right)-\left(x+yz\right)\nabla\left(y+xz\right)}{\left(y+xz\right)^2}\\ &= \frac{1}{\left(y+xz\right)^2}\left(\left(y+xz\right)\langle 1,z,y\rangle-\left(x+yz\right)\langle z,1,x\rangle\right)\\ &= \frac{1}{\left(y+xz\right)^2}\left\langle y\left(1-z^2\right),x\left(z^2-1\right),y^2-x^2\right\rangle.\end{aligned}$$

15.6 Tangent Planes and Linear Approximation

15.6.1 The gradient of F is a multiple of n.

15.6.2 Let $F\left(x,y,z\right)=z-f\left(x,y\right)=z-xy^2-x^2y+10$.

15.6.3 The tangent plane has equation $F_x\left(a,b,c\right)\left(x-a\right)+F_y\left(a,b,c\right)\left(y-b\right)+F_z\left(a,b,c\right)\left(z-c\right)=0$.

15.6.4 The tangent plane has equation $z=f_x\left(a,b\right)\left(x-a\right)+f_y\left(a,b\right)\left(y-b\right)+f\left(a,b\right)$.

15.6.5 Multiply the change in x by f_x and the change in y by f_y and add both terms to $f\left(a,b\right)$.

15.6.6 The change in f is approximately $f_x\left(a,b\right)\triangle x+f_y\left(a,b\right)\triangle y$.

15.6.7 In terms of differentials, $dz=f_x\left(a,b\right)dx+f_y\left(a,b\right)dy$.

15.6.8 The differential for the function $w=f\left(x,y,z\right)$ at the point $\left(a,b,c\right)$ is

$$dw=f\left(a,b,c\right)dx+f_y\left(a,b,c\right)dy+f_z\left(a,b,c\right)dz.$$

15.6.9 The tangent plane is $z=5(x-1)-3(y-2)+4$ or $z=5x-3y+5$.

15.6.10 The linear approximation is $z=5x-3y+5$. Evaluating this at $(1.01,1.99)$ gives 4.08.

15.6.11 $3(x-0)-(y-2)+6(z-1)=0$, or $3x-y+6z=4$.

15.6.12 Let $f(x,y,z)=3x-(y-2)+6(z-1)$. Then $f(0.1,2,0.99)=0.24$.

15.6.13 $\nabla F=\langle 2x,1,1\rangle$, so $\nabla F(1,1,1)=\langle 2,1,1\rangle$. The tangent plane at $(1,1,1)$ has equation $2(x-1)+(y-1)+(z-1)=0$, or $2x+y+z=4$. Also, $\nabla F(2,0,-1)=\langle 4,1,1\rangle$, so the equation of the tangent plane there is $4(x-2)+y+(z+1)=0$, or $4x+y+z=7$.

15.6.14 $\nabla F=\left\langle 2x,3y^2,4z^3\right\rangle$, so $\nabla F(1,0,1)=\langle 2,0,4\rangle$. The tangent plane at $(1,0,1)$ is given by $2(x-1)+4(z-1)=0$, or $x+2z=3$. Also, $\nabla F(-1,0,1)=\langle -2,0,4\rangle$, so the tangent plane there is given by $-2(x+1)+4(z-1)=0$, or $x-2z=-3$.

15.6.15 $\nabla F=\langle y+z,x+z,x+y\rangle$, so $\nabla F(2,2,2)=\langle 4,4,4\rangle$. The tangent plane at $(2,2,2)$ is given by $4(x-2)+4(y-2)+4(z-2)=0$, or $x+y+z=6$. At the point $(2,0,6)$ we have $\nabla F(2,0,6)=\langle 6,8,2\rangle$, so the equation of the tangent plane is $6(x-2)+8y+2(z-6)=0$, or $3x+4y+z=12$.

15.6.16 $\nabla F=\langle 2x,2y,-2z\rangle$, so $\nabla F\left(3,4,5\right)=\langle 6,8,-10\rangle$ and the tangent plane at $(3,4,5)$ has equation $8\left(x-3\right)+8\left(y-4\right)-10\left(z-5\right)=0$, or $3x+4y-5z=0$. $\nabla F\left(-4,-3,5\right)=\langle -8,-6,-10\rangle$ so the tangent plane at $(-4,-3,5)$ has equation $-8\left(x+4\right)-6\left(y+3\right)-10\left(z+5\right)=0$ or $4x+3y+5z=0$.

15.6.17 We have $f_x = -4x$, $f_y = -2y$, so the tangent plane at $(2, 2, -8)$ has equation $z = f(2, 2) + f_x(2, 2)(x - 2) + f_y(2, 2)(y - 2) = -8 - 8(x - 2) - 4(y - 2)$, or $z = -8x - 4y + 16$, and the equation of the tangent plane at $(-1, -1, 1)$ is $z = f(-1, -1) + f_x(-1, -1)(x + 1) + f_y(-1, -1)(y + 1) = 1 + 4(x + 1) + 2(y + 1)$, or $z = 4x + 2y + 7$.

15.6.18 We have $f_x = 4x$, $f_y = y$, so the tangent plane at $\left(-\frac{1}{2}, 1, 3\right)$ has equation $z = f\left(-\frac{1}{2}, 1\right) + f_x\left(-\frac{1}{2}, 1\right)\left(x + \frac{1}{2}\right) + f_y\left(-\frac{1}{2}, 1\right)(y - 1) = 3 - 2\left(x + \frac{1}{2}\right) + (y - 1)$, or $z = -2x + y + 1$, and the equation of the tangent plane at $(3, -2, 22)$ is $z = f(3, -2) + f_x(3, -2)(x - 3) + f_y(3, -2)(y + 2) = 22 + 12(x - 3) - 2(y + 2)$, or $z = 12x - 2y - 18$.

15.6.19 We have $f_x = ye^{xy}$, $f_y = xe^{xy}$, so the tangent plane at $(1, 0, 1)$ has equation $z = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y) = 1 + 0 + 1(y)$, or $z = 1 + y$. The equation of the tangent plane at $(0, 1, 1)$ is $z = 1 + 1(x) + 0(y - 1)$, or $z = 1 + x$.

15.6.20 We have $f_x = y\cos(xy)$ and $f_y = x\cos(xy)$, so the tangent plane at $(1, 0, 2)$ has equation $z = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y) = 2 + 0(x - 1) + 1(y)$, or $z = 2 + y$. The equation of the tangent plane at $(0, 5, 2)$ is $z = 2 + 5(x) + 0(y - 5)$, or $z = 2 + 5x$.

15.6.21 $\nabla F = \langle y\sin z, x\sin z, yz\cos z\rangle$, so $\nabla F\left(1, 2, \frac{\pi}{6}\right) = \langle 1, \frac{1}{2}, \sqrt{3}\rangle$ and the tangent plane at $\left(1, 2, \frac{\pi}{6}\right)$ has equation $(x - 1) + \frac{1}{2}(y - 2) + \sqrt{3}\left(z - \frac{\pi}{6}\right) = 0$, or $x + \frac{1}{2}y + \sqrt{3}z = 2 + \frac{\sqrt{3}\pi}{6}$. Also, $\nabla F\left(-2, -1, \frac{5\pi}{6}\right) = \langle -\frac{1}{2}, -1, -\sqrt{3}\rangle$, so the tangent plane at $\left(-2, -1, \frac{5\pi}{6}\right)$ has equation $-\frac{1}{2}(x + 2) - (y + 1) - \sqrt{3}\left(z - \frac{5\pi}{6}\right) = 0$ or $\frac{1}{2}x + y + \sqrt{3}z = \frac{5\sqrt{3}\pi}{6} - 2$.

15.6.22 $\nabla F = e^{xz}\langle yz^2, z, y(1 + x)z\rangle$, so $\nabla F(0, 2, 4) = \langle 32, 4, 8\rangle$ and the tangent plane at $(0, 2, 4)$ has equation $32x + 4(y - 2) + 8(z - 4) = 0$, or $8x + y + 2z = 10$. Also, $\nabla F(0, -8, -1) = \langle -8, -1, 8\rangle$, so the tangent plane at $(0, -8, -1)$ has equation $-8x - (y + 8) + 8(z + 1) = 0$ or $8x + y - 8z = 0$.

15.6.23 $\nabla F = \langle -\frac{x}{8}, -\frac{2y}{9}, 2z\rangle$, so $\nabla F\left(4, 3, -\sqrt{3}\right) = \langle -\frac{1}{2}, -\frac{2}{3}, -2\sqrt{3}\rangle$ and the tangent plane at $\left(4, 3, -\sqrt{3}\right)$ has equation $-\frac{1}{2}(x - 4) - \frac{2}{3}(y - 3) - 2\sqrt{3}\left(z + \sqrt{3}\right) = 0$, or $\frac{1}{2}x + \frac{2}{3}y + 2\sqrt{3}z = -2$. Also, $\nabla F\left(-8, 9, \sqrt{14}\right) = \left\langle 1, -2, 2\sqrt{14}\right\rangle$, so the tangent plane at $\left(-8, 9, \sqrt{14}\right)$ has equation $(x + 8) - 2(y - 9) + 2\sqrt{14}\left(z - \sqrt{14}\right) = 0$ or $x - 2y + 2\sqrt{14}z = 2$.

15.6.24 $\nabla F = \langle 2, 2y, -2z\rangle$, so $\nabla F(0, 1, 1) = \langle 2, 2, -2\rangle$ and the tangent plane at $(0, 1, 1)$ has equation $2x + 2(y - 1) - 2(z - 1) = 0$, or $x + y - z = 0$. Also, $\nabla F(4, 1, -3) = \langle 2, 2, 6\rangle$, so the tangent plane at $(4, 1, -3)$ has equation $2(x - 4) + 2(y - 1) + 6(z + 3) = 0$ or $x + y + 3z = -4$.

15.6.25 We have $f_x = \left(2x + x^2\right)e^{x-y}$, $f_y = -x^2e^{x-y}$, so the tangent plane at $(2, 2, 4)$ has equation $z = f(2, 2) + f_x(2, 2)(x - 2) + f_y(2, 2)(y - 2) = 4 + 8(x - 2) - 4(y - 2)$, or $z = 8x - 4y - 4$, and the equation of the tangent plane at $(-1, -1, 1)$ is $z = f(-1, -1) + f_x(-1, -1)(x + 1) + f_y(-1, -1)(y + 1) = 1 - (x + 1) - (y + 1)$, or $z = -x - y - 1$.

15.6.26 We have $f_x = \frac{y}{1 + xy}$, $f_y = \frac{x}{1 + xy}$, so the tangent plane at $(1, 2, \ln 3)$ has equation $z = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) = \ln 3 + \frac{2}{3}(x - 1) + \frac{1}{3}(y - 2)$, or $z = \frac{2}{3}x + \frac{1}{3}y + \ln 3 - \frac{4}{3}$, and the equation of the tangent plane at $(-2, -1, \ln 3)$ is $z = f(-2, -1) + f_x(-2, -1)(x + 2) + f_y(-2, -1)(y + 1) = \ln 3 - \frac{1}{3}(x + 2) - \frac{2}{3}(y + 1)$, or $z = -\frac{1}{3}x - \frac{2}{3}y + \ln 3 - \frac{4}{3}$.

15.6.27 We have $f_x = \dfrac{y^2+2xy-x^2}{\left(x^2+y^2\right)^2}$, $f_y = \dfrac{y^2-2xy-x^2}{\left(x^2+y^2\right)^2}$, so the tangent plane at $\left(1,2,-\dfrac{1}{5}\right)$ has equation $z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = -\dfrac{1}{5} + \dfrac{7}{25}(x-1) - \dfrac{1}{25}(y-2)$, or $z = \dfrac{7}{25}x - \dfrac{1}{25}y - \dfrac{2}{5}$, and the equation of the tangent plane at $\left(2,-1,\dfrac{3}{5}\right)$ is $z = f(2,-1) + f_x(2,-1)(x-2) + f_y(2,-1)(y+1) = \dfrac{3}{5} - \dfrac{7}{25}(x-2) + \dfrac{1}{25}(y+1)$, or $z = -\dfrac{7}{25}x + \dfrac{1}{25}y + \dfrac{6}{5}$.

15.6.28 We have $f_x = -2\sin(x-y)$, $f_y = 2\sin(x-y)$, so the tangent plane at $\left(\dfrac{\pi}{6}, -\dfrac{\pi}{6}, 3\right)$ has equation $z = f\left(\dfrac{\pi}{6},-\dfrac{\pi}{6}\right) + f_x\left(\dfrac{\pi}{6},-\dfrac{\pi}{6}\right)\left(x-\dfrac{\pi}{6}\right) + f_y\left(\dfrac{\pi}{6},-\dfrac{\pi}{6}\right)\left(y+\dfrac{\pi}{6}\right) = 3 - \sqrt{3}\left(x-\dfrac{\pi}{6}\right) + \sqrt{3}\left(y+\dfrac{\pi}{6}\right)$, or $z = \sqrt{3}(y-x) + 3 + \dfrac{\sqrt{3}\pi}{3}$, and the equation of the tangent plane at $\left(\dfrac{\pi}{3},\dfrac{\pi}{3},4\right)$ is $z = f\left(\dfrac{\pi}{3},\dfrac{\pi}{3}\right) + f_x\left(\dfrac{\pi}{3},\dfrac{\pi}{3}\right)\left(x-\dfrac{\pi}{3}\right) + f_y\left(\dfrac{\pi}{3},\dfrac{\pi}{3}\right)\left(y-\dfrac{\pi}{3}\right) = 4$, or $z = 4$.

15.6.29 Let $f(x,y) = \tan^{-1}(xy)$; then $f_x(x,y) = \dfrac{y}{1+(xy)^2}$, $f_y(x,y) = \dfrac{x}{1+(xy)^2}$ and the tangent plane at $\left(1,1,\dfrac{\pi}{4}\right)$ has equation $z = f_x(1,1)(x-1) + f_y(1,1)(y-1) + f(1,1) = \dfrac{1}{2}x + \dfrac{1}{2}y + \dfrac{\pi}{4} - 1$.

15.6.30 Let $f(x,y) = \tan^{-1}(x+y)$; then $f_x(x,y) = f_y(x,y) = \dfrac{1}{1+(x+y)^2}$ and the tangent plane at $(0,0,0)$ has equation $z = f_x(0,0)x + f_y(0,0)y + f(0,0) = x+y$.

15.6.31 The branch of the surface $\sin(xyz) = \dfrac{1}{2}$ at the point $\left(\pi,1,\dfrac{1}{6}\right)$ can be described more simply by the equation $F(x,y,z) = xyz = \dfrac{\pi}{6}$. We have $F_x(x,y,z) = yz$, $F_y(x,y,z) = xz$, $Fz(x,y,z) = xy$ so the tangent plane at $\left(\pi,1,\dfrac{1}{6}\right)$ has equation $F_x\left(\pi,1,\dfrac{1}{6}\right)(x-\pi) + F_y\left(\pi,1,\dfrac{1}{6}\right)(y-1) + Fz\left(\pi,1,\dfrac{1}{6}\right)\left(z-\dfrac{1}{6}\right) = 0$, or $\dfrac{1}{6}(x-\pi) + \dfrac{\pi}{6}(y-1) + \pi\left(z-\dfrac{1}{6}\right) = 0$.

15.6.32 Rewrite this equation as $x+z = 2(y-z)$ or equivalently, $x-2y+3z=0$; therefore the surface is a plane and hence is identical to its tangent plane at any point.

15.6.33

a. We have $f_x = y+1$, $f_y = x-1$, so the linear approximation for f at $(2,3)$ is $L(x,y) = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3) = 5 + 4(x-2) + (y-3)$,or $L(x,y) = 4x+y-6$.

b. We have $L(2.1, 2.99) = 5.39$.

15.6.34

a. We have $f_x = -8x$, $f_y = -16y$, so the linear approximation for f at $(-1,4)$ is $L(x,y) = f(-1,4) + f_x(-1,4)(x+1) + f_y(-1,4)(y-4) = -120 + 8(x+1) - 64(y-4)$ or $L(x,y) = 8x - 64y + 144$.

b. We have $L(-1.05, 3.95) = -117.20$.

15.6.35

a. We have $f_x = -2x$, $f_y = 4y$, so the linear approximation for f at $(3,-1)$ is $L(x,y) = f(3,-1) + f_x(3,-1)(x-3) + f_y(3,-1)(y+1) = -7 - 6(x-3) - 4(y+1)$ or $L(x,y) = -6x - 4y + 7$.

b. We have $L(3.1, -1.04) = -7.44$.

15.6.36

a. We have $f_x = \dfrac{x}{\sqrt{x^2+y^2}}$, $f_y = \dfrac{y}{\sqrt{x^2+y^2}}$, so the linear approximation for f at $(3,-4)$ is $L(x,y) = f(3,-4) + f_x(3,-4)(x-3) + f_y(3,-4)(y+4) = 5 + \dfrac{3}{5}(x-3) + \dfrac{4}{5}(y+4)$,or $L(x,y) = \dfrac{3}{5}x + \dfrac{4}{5}y$.

b. We have $L(3.06, -3.92) = 4.972$.

15.6.37

a. $f_x = \dfrac{1}{1+x+y+2z}$, $f_y = \dfrac{1}{1+x+y+2z}$, $f_z = \frac{2}{1+x+y+2z}$, so at $(0,0,0)$ we have $f_x = 1$, $f_y = 1$, and $f_z = 2$. So $L(x,y,z) = x + y + 2z$.

b. $L(0.1, -0.2, 0.2) = 0.1 - 0.2 + 0.4 = 0.3$.

15.6.38

a. $f_x = \dfrac{x-z-(x+y)}{(x-z)^2} = \dfrac{-z-y}{(x-z)^2}$, $f_y = \dfrac{1}{x-z}$, and $f_z = \dfrac{x+y}{(x-z)^2}$, so at $(3,2,4)$ we have $f_x = -6$, $f_y = -1$, and $f_z = 5$. So $L(x,y,z) = -6(x-3) - 1(y-2) + 5(z-4) - 5 = -6x - y + 5z - 5$.

b. $L(2.95, 2.05, 4.02) = -6(2.95) - (2.5) + 5(4.02) - 5 = -5.1$.

15.6.39 We have $f_x = 2 - 2y$, $f_y = -3 - 2x$, and $dx = 0.1$, $dy = -0.1$, so $dz = f_x(1,4)\,dx + f_y(1,4)\,dy = -6dx - 5dy = -0.1$.

15.6.40 We have $f_x = -2x$, $f_y = 6y$, and $dx = -0.05$, $dy = -0.1$, so $dz = f_x(-1,2)\,dx + f_y(-1,2)\,dy = 2dx + 12dy = -1.30$.

15.6.41 We have $f_x = e^{x+y}$, $f_y = e^{x+y}$, and $dx = 0.1$, $dy = -0.05$, so $dz = f_x(0,0)\,dx + f_y(0,0)\,dy = dx + dy = 0.05$.

15.6.42 We have $f_x = \dfrac{1}{1+x+y}$, $f_y = \dfrac{1}{1+x+y}$, and $dx = -0.1$, $dy = 0.03$, so $dz = f_x(0,0)\,dx + f_y(0,0)\,dy = dx + dy = -0.07$.

15.6.43

a. If r increases and R decreases then $R^2 - r^2$ decreases, so S decreases.

b. If both R and r increase, then it is impossible to say whether $R^2 - r^2$ increases or decreases.

c. We have $dS = 8\pi^2(RdR - rdr) = 8\pi^2(5.50 \cdot 0.15 - 3 \cdot 0.05) = 5.4\pi^2 \approx 53.296$.

d. We have $dS = 8\pi^2(RdR - rdr) = 8\pi^2(7 \cdot 0.04 - 3 \cdot (-0.05)) = 3.44\pi^2 \approx 33.951$.

e. The surface area is approximately unchanged when $RdR = rdr$.

15.6.44

a. We have $dV = \dfrac{\pi}{3}\left(2rhdr + r^2dh\right) = \dfrac{\pi}{3}\left(2 \cdot 6.5 \cdot 4.2 \cdot 0.1 + 6.5^2(-0.05)\right) \approx 3.505$.

b. We have $dV = \dfrac{\pi}{3}\left(2rhdr + r^2dh\right) = \dfrac{\pi}{3}\left(2 \cdot 5.4 \cdot 12(-0.03) + 6.5^2(-0.04)\right) \approx -5.293$.

15.6.45 Observe that $dA = \pi(bda + adb)$ so $\dfrac{dA}{A} = \dfrac{da}{a} + \dfrac{db}{b}$, and hence the percentage increase in the area is approximately 2% + 1.5%= 3.5%.

15.6.46 Observe that $dV = (\pi/2)\left(2rhdr + r^2dh\right)$ so $\dfrac{dV}{V} = 2\dfrac{dr}{r} + \dfrac{dh}{h}$, and hence the percentage decrease in the area is approximately $2(-1.5\%) + 2.2\% = -0.8\%$.

15.6.47 We have $dw = \left(y^2 + 2xz\right) dx + \left(2xy + z^2\right) dy + \left(x^2 + 2yz\right) dz$.

15.6.48 We have $dw = \cos\left(x + y - z\right)\left(dx + dy - dz\right)$.

15.6.49 We have $dw = \dfrac{dx}{y+z} - \dfrac{u+x}{\left(y+z\right)^2}dy - \dfrac{u+x}{\left(y+z\right)^2}dz + \dfrac{du}{y+z}$.

15.6.50 We have $dw = \dfrac{q}{rs}dp + \dfrac{p}{rs}dq - \dfrac{pq}{r^2 s}dr - \dfrac{pq}{rs^2}ds$.

15.6.51

a. Observe that
$$2cdc = \left(2a - 2b\cos\theta\right) da + \left(2b - 2a\cos\theta\right) db + 2ab\sin\theta\, d\theta$$
so $dc = \dfrac{a - b\cos\theta}{c}da + \dfrac{b - a\cos\theta}{c}db + \dfrac{ab\sin\theta}{c}d\theta$. We have $a = 2$, $b = 4$, $\theta = \dfrac{\pi}{3}$ which gives $c = \sqrt{12}$, and $da = 0.03$, $db = -0.04$, $d\theta = \dfrac{\pi}{90}$; substituting in the equation above gives $dc \approx 0.035$.

b. We have $a = 2$, $b = 4$, $da = 0.03$, $db = -0.04$, $d\theta = 0$, so $dc = -\dfrac{0.01 + 0.04\cos\theta}{c}$; comparing the cases $\theta = \dfrac{\pi}{20}$ and $\theta = \dfrac{9\pi}{20}$ we see that c is smaller and $\cos\theta$ is larger in the first case; therefore the change in c is greater when $\theta = \dfrac{\pi}{20}$.

15.6.52

a. The cost function is obtained by multiplying the distance L by the cost per mile, which is $\dfrac{p}{m}$.

b. We have $C_L = \dfrac{p}{m}$, $C_m = -\dfrac{Lp}{m^2}$, $C_p = \dfrac{L}{m}$.This shows that C is an increasing function of L and p and a decreasing function of m, which makes sense.

c. We have $dC = \dfrac{p}{m}dL - \dfrac{Lp}{m^2}dm + \dfrac{L}{m}dp \approx \$10.29$.

d. Observe that $\dfrac{dC}{C} = \dfrac{dL}{L} - \dfrac{dm}{m} + \dfrac{dp}{p}$; therefore the cost is equally sensitive to a 1% change in any of the variables.

15.6.53

a. True. This is because the function $F\left(x, y, z\right) = x^2 + z^2$ has $F_y = 0$.

b. True. As $z > 0$ decreases, $\dfrac{1}{z}$ increases.

c. False. The gradient $\nabla F\left(a, b, c\right)$ is perpendicular to the tangent plane for the surface $F\left(x, y, z\right) = 0$ at $\left(a, b, c\right)$.

15.6.54 Let $F\left(x, y, z\right) = x^2 + 2y^2 + z^2 - 2x - 2z - 2$; then $F_x\left(x, y, z\right) = 2x - 2$, $F_y\left(x, y, z\right) = 4y$. The tangent plane to the surface $F\left(x, y, z\right) = 0$ at $\left(a, b, c\right)$ is horizontal if and only if $F_x\left(a, b, c\right) = F_y\left(a, b, c\right) = 0$, which gives $a = 1$, $b = 0$. This implies $c^2 - 2c - 3 = 0$, which gives $c = 3, -1$ so the points are $\left(1, 0, 3\right)$ and $\left(1, 0, -1\right)$.

15.6.55 Let $F\left(x, y, z\right) = x^2 + y^2 - z^2 - 2x + 2y + 3$; then $F_x\left(x, y, z\right) = 2x - 2$, $F_y\left(x, y, z\right) = 2y + 2$. The tangent plane to the surface $F\left(x, y, z\right) = 0$ at $\left(a, b, c\right)$ is horizontal if and only if $F_x\left(a, b, c\right) = F_y\left(a, b, c\right) = 0$, which gives $a = 1$, $b = -1$. This implies $c^2 = 1$, so the points are $\left(1, -1, 1\right)$ and $\left(1, -1, -1\right)$.

15.6.56 Let $f\left(x, y\right) = \sin\left(x - y\right)$; then $f_x\left(x, y\right) = \cos\left(x - y\right)$, $f_y\left(x, y\right) = -\cos\left(x - y\right)$ so the tangent plane at $\left(a, b, \sin\left(a - b\right)\right)$ is horizontal at all points where $b - a = \dfrac{\pi}{2} + k\pi$ for some integer k.

15.6.57 Let $f(x,y) = \cos 2x \sin y$; then $f_x(x,y) = -2\sin 2x \sin y$, $f_y(x,y) = \cos 2x \cos y$; so the tangent plane at $(a, b, \cos 2a \sin b)$ is horizontal at all points where $\sin 2a = \cos b = 0$ or $\sin b = \cos 2a = 0$. In the region $-\pi \le x \le \pi$, $-\pi \le y \le \pi$ the points are $a = 0, \pm\frac{\pi}{2}, \pm\pi$ and $b = \pm\frac{\pi}{2}$, or $a = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ and $b = 0, \pm\pi$.

15.6.58

a. Apply logarithmic differentiation to $A^2 = s(s-a)(s-b)(s-c)$: $\frac{2A_a}{A} = \frac{\frac{1}{2}}{s} + \frac{-\frac{1}{2}}{s-a} + \frac{\frac{1}{2}}{s-b} + \frac{\frac{1}{2}}{s-c}$, so $A_a = \frac{A}{4}\left(\frac{1}{s} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s-a}\right)$, and similarly $A_b = \frac{A}{4}\left(\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-c} - \frac{1}{s-b}\right)$, and $A_c = \frac{A}{4}\left(\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right)$.

b. The change is approximately $dA = A_a da + A_b db + A_c dc \approx -0.5573$.

c. In this case $s = \frac{3}{2}a$, $s-a = s-b = s-c = \frac{1}{2}a$,and $A_a = A_b = A_c = \frac{2A}{3a}$ so if all sides change by the same amount da, then $\frac{dA}{A} = 3 \cdot \frac{2}{3a} da = 2\frac{da}{a}$. Therefore if all sides increase by $p\%$, the area will increase by approximately $2p\%$. (This result can also be obtained more directly using the formula $A = \frac{\sqrt{3}a^2}{4}$ for the area of an equilateral triangle with side a.)

15.6.59

a. We have $S_r = \pi\sqrt{r^2+h^2} + \frac{\pi r^2}{\sqrt{r^2+h^2}}$, $S_h = \frac{\pi r h}{\sqrt{r^2+h^2}}$; using the values $r = 2.5$, $h = 0.6$, $dr = 0.05$, $dh = -0.02$ gives $dS = S_r dr + S_h dh = \frac{\pi}{\sqrt{r^2+h^2}}\left(\left(2r^2+h^2\right)dr + rhdh\right) \approx 0.749$.

b. If $r = 100$, $h = 200$ then $dS = 40\sqrt{5}\pi\,(3dr + dh)$, so the surface area is more sensitive to small changes in r.

15.6.60

a. The vector $\mathbf{n}_1 = \langle 1, 1, -1\rangle$ is normal to this plane.

b. Rewrite the equation of the paraboloid as $F(x,y,z) = z - x^2 - 3y^2 = 0$; then $\nabla F(2,1,7) = \langle 4, 6, -1\rangle$, so $\mathbf{n}_2 = \langle 4, 6, -1\rangle$ is normal to the tangent plane.

c. The line tangent to C lies in both of these planes and hence must be orthogonal to both vectors n_1 and n_2. Therefore a direction vector for this line is given by $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \left|\mathbf{ijk}11-146-1\right| = 5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

d. The tangent line has parametric equation $\mathbf{r}(t) = \langle 2, 1, 7\rangle + t\langle 5, -3, 2\rangle = \langle 2+5t, 1-3t, 7+2t\rangle$, or $x = 2+5t, y = 1-3t$, $z = 7+2t$.

15.6.61

a. The differential of A is given by $dA = \frac{dx}{y} - \frac{xdy}{y^2}$; substituting $x = 60$, $y = 175$, $dx = 2$ and $dy = 5$ gives $dA = \frac{2}{1225} \approx 0.00163$.

b. If the batter fails to get a hit, the average decreases by $\frac{x}{y} - \frac{x}{y+1} = \frac{x}{y(y+1)} = \frac{A}{y+1}$, whereas if the batter gets a hit, the average increases by $\frac{x+1}{y+1} - \frac{x}{y} = \frac{y-x}{y(y+1)} = \frac{1-A}{y+1}$. If $A = 0.350$ the second of these quantities is larger so the answer is no; the batting average changes more if the batter gets a hit than if he fails to get a hit.

c. The answer depends on whether A is less than or greater than 0.500.

15.6.62 The volume of the tank is $V = \dfrac{\pi r^2 h}{3}$, so $dV = \dfrac{2\pi}{3} rhdr + \dfrac{\pi}{3} r^2 dh$ or equivalently $\dfrac{dV}{V} = 2\dfrac{dr}{r} + \dfrac{dh}{h}$. If the water level drops from 2.00 to 1.95, then the radius changes proportionately so $\dfrac{dV}{V} = 3\dfrac{dh}{h} = -0.075$, which gives $dV \approx -0.039\text{m}^2$.

15.6.63

a. The centerline velocity is given by $V = \dfrac{R^2}{L}$ so $dV = \dfrac{2R}{L}dR - \dfrac{R^2}{L^2}dL$; evaluate this with $R = 3$, $dR = 0.05$, $L = 50$, $dL = 0.5$ to obtain $dV = \dfrac{21}{5000} = 0.0042\text{cm}^3$.

b. Rewrite the formula for dV as $\dfrac{dV}{V} = \dfrac{2dR}{R} - \dfrac{dL}{L}$; hence if R decreases 1% and L increases 2%, then V will decrease by approximately 4%.

c. If the radius of the cylinder increases by $p\%$, then the length of the cylinder must decrease by approximately $2p\%$ in order for the velocity to remain constant.

15.6.64

a. Let $z = xy$; then $dz = ydx + xdy$, so the absolute error is at most $(|x|+|y|) \cdot 10^{-16}$ and the percentage error at most $\dfrac{100 \cdot (|x|+|y|) \cdot 10^{-16}}{|xy|}\% = \left(\dfrac{1}{|x|} + \dfrac{1}{|y|}\right) \cdot 10^{-14}\%$.

b. Let $z = \dfrac{x}{y}$; then $dz = \dfrac{ydx - xdy}{y^2}$, so the absolute error is at most $\left(\dfrac{|x|+|y|}{y^2}\right) \cdot 10^{-16}$ and the percentage error at most $100 \cdot \left(\dfrac{|x|+|y|}{y^2}\right) \cdot \dfrac{|y|}{|x|} \cdot 10^{-16}\% = \left(\dfrac{1}{|x|} + \dfrac{1}{|y|}\right) \cdot 10^{-14}\%$.

c. Let $w = xyz$; then $dw = yzdx + xzdy + xydz$, so the absolute error is at most $(|xy|+|yz|+|xz|) \cdot 10^{-16}$ and the percentage error at most

$$\frac{100 \cdot (|xy|+|yz|+|xz|) \cdot 10^{-16}}{|xyz|}\% = \left(\frac{1}{|x|} + \frac{1}{|y|} + \frac{1}{|z|}\right) \cdot 10^{-14}\%.$$

d. Let $z = \dfrac{x/y}{z} = \dfrac{x}{yz}$; then $dz = \dfrac{yzdx - xzdy - xydz}{y^2 z^2}$, so the absolute error is at most

$$\left(\frac{|xy|+|yz|+|xz|}{y^2 z^2}\right) \cdot 10^{-16}$$

and the percentage error at most

$$100 \cdot \left(\frac{|xy|+|yz|+|xz|}{y^2 z^2}\right) \cdot \frac{|yz|}{|x|} \cdot 10^{-16}\% = \left(\frac{1}{|x|} + \frac{1}{|y|} + \frac{1}{|z|}\right) \cdot 10^{-14}\%.$$

15.6.65

a. We have $f_r = n\,(1-r)^{n-1}$ and $f_n = -\,(1-r)^n \ln(1-r)$.

b. We have $\triangle P \approx f\,(20, 0.1) \cdot 0.01 = 20 \cdot (0.9)^{19}(0.01) \approx 0.027$.

c. We have $\triangle P \approx f\,(20, 0.9) \cdot 0.01 = 20 \cdot (0.1)^{19}(0.01) \approx 2 \times 10^{-20}$.

d. Small changes in the flu rate have a greater effect on the probability of catching the flu when the flu rate is small compared to when the flu rate is large.

15.6.66

a. We have $R = \dfrac{R_1R_2}{R_1+R_2}$ so $dR = \dfrac{R_2^2dR_1 + R_1^2dR_2}{(R_1+R_2)^2} = R^2\left(\dfrac{dR_1}{R_1^2} + \dfrac{dR_2}{R_2^2}\right)$. Substituting $R_1 = 2$, $dR_1 = 0.05$,$R_2 = 3$, $dR_2 = -0.05$ gives $dR = 0.01$ ohms.

b. True; if $R_1 = R_2$ and $dR_1 = -dR_2$ then $dR = 0$.

c. Yes, because in this case $dR > 0$.

d. The formula in (a) shows that if $R_1 > R_2$ then R is more sensitive to changes in R_2, so if R_1 increases by the same small amount that R_2 decreases, then R will decrease.

15.6.67 From the equation $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$ we obtain $-\dfrac{1}{R^2}\dfrac{\partial R}{\partial R_1} = -\dfrac{1}{R_1^2}$, which implies that $\dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$, with similar formulas for the other partials. Hence $dR = R^2\left(\dfrac{dR_1}{R_1^2} + \dfrac{dR_2}{R_2^2} + \dfrac{dR_3}{R_3^2}\right)$. Substituting the values $R_1 = 2$, $dR_1 = 0.05$, $R_2 = 3$, $dR_2 = -0.05$, $R_3 = 1.5$, $dR_3 = 0.05$ gives $R = \dfrac{2}{3}$ and $dR = \dfrac{7}{540} \approx 0.0130$ ohms.

15.6.68 Observe that $f_x = ax^{a-1}y^b$ and $f_y = bx^ay^{b-1}$, so $\dfrac{dz}{z} = \dfrac{ax^{a-1}y^bdx + bx^ay^{b-1}dy}{x^ay^b} = a\dfrac{dx}{x} + b\dfrac{dy}{y}$.

15.6.69

a. Suppose f is a function of x and y; then $d(\ln f) = (\ln f)_x\, dx + (\ln f)_y\, dy = \dfrac{f_x}{f}dx + \dfrac{f_y}{f}dy = \dfrac{df}{f}$.

b. The absolute change in ln f is approximately $d(\ln f)$ and the relative change in f is approximately $\dfrac{df}{f}$; from part (a) these agree.

c. Observe that $df = ydx + xdy$, so $\dfrac{df}{f} = \dfrac{dx}{x} + \dfrac{dy}{y}$.

d. Observe that $df = \dfrac{ydx - xdy}{y^2}$, so $\dfrac{df}{f} = \dfrac{dx}{x} - \dfrac{dy}{y}$.

e. If $f = x_1x_2\cdots x_n$ then $\ln f = \ln x_1 + \ln x_2 + \cdots + \ln x_n$ and therefore $\dfrac{df}{f} = \dfrac{dx_1}{x_1} + \dfrac{dx_2}{x_2} + \cdots + \dfrac{dx_n}{x_n}$.

15.6.70

a. Let $F(x,y,z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1$; then $\nabla F = \langle \dfrac{2x}{a^2}, \dfrac{2y}{b^2}, \dfrac{2z}{c^2}\rangle$ so the tangent plane at (p,q,r) has equation $\dfrac{p}{a^2}(x-p) + \dfrac{q}{b^2}(y-q) + \dfrac{r}{c^2}(z-r) = 0$, which simplifies to $\dfrac{px}{a^2} + \dfrac{qy}{b^2} + \dfrac{rz}{c^2} = 1$.

b. The vectors $\langle A,B,C\rangle$ and $\langle \dfrac{p}{a^2}, \dfrac{q}{b^2}, \dfrac{r}{c^2}\rangle$ must be proportional; therefore we can express $\langle p,q,r\rangle = \lambda\langle Aa^2, Bb^2, Cc^2\rangle$ for some scalar λ. Substituting in the equation of the ellipsoid gives $\dfrac{p^2}{a^2} + \dfrac{q^2}{b^2} + \dfrac{r^2}{c^2} = \lambda^2\left(A^2a^2 + B^2b^2 + C^2c^2\right) = 1$, so $\lambda = \pm\dfrac{1}{m}$ and $(p,q,r) = \pm\left(Aa^2, Bb^2, Cc^2\right)$ where $m = \sqrt{A^2a^2 + B^2b^2 + C^2c^2}$.

c. The distance of the plane $Ax + By + Cz = 1$ to the origin is $d = \dfrac{1}{\sqrt{A^2+B^2+C^2}} = h$ (this formula is derived in a previous exercise).

d. The tangent plane at $(p,q,r) = \pm\left(Aa^2, Bb^2, Cc^2\right)$ has equation $Ax + By + Cz = \pm m$, which has distance to the origin $d = \dfrac{m}{\sqrt{A^2+B^2+C^2}} = mh$ using the formula from part c.

e. The plane P does not intersect the ellipsoid if and only if the two tangent planes parallel to P are closer to the origin than P; this is equivalent to the condition $m < 1$.

15.7 Maximum/Minimum Problems

15.7.1 It is locally the highest point on the surface; you cannot get toa higher point in any direction.

15.7.2 The surface looks similar to the hyperbolic paraboloid $z = x^2 - y^2$ at the origin.

15.7.3 The partial derivatives are both zero or one or both do not exist.

15.7.4 No; for example f may have a saddle point at (a, b).

15.7.5 The discriminant of f at (a, b) is the determinant given by $D\left(a,b\right) = f_{xx}\left(a,b\right) f_{yy}\left(a,b\right) - f_{xy}\left(a,b\right)^2$.

15.7.6 The second derivative test may be used to determine whether a critical point is a local maximum, minimum or saddle point (see Theorem 15.14).

15.7.7 The function f has an absolute minimum value at $(a, b) \in R$ if $f\left(x,y\right) \geq f\left(a,b\right)$ for all $(x, y) \in R$.

15.7.8 Determine the values of f at all critical points in the closed bounded domain R, and find the maximum/minimum values of f on the boundary of R; the greatest/smallest of these values is the absolute maximum/minimum of f on R.

15.7.9 The discriminant is $5(3) - (-4)^2 = 15 - 16 = -1 < 0$, so this is a saddle point.

15.7.10 The discriminant is $(-6)(-3) - 4^2 = 18 - 16 = 2 > 0$, and $f_{xx}(0, 0) < 0$, so f has a local maximum at $(0, 0)$.

15.7.11 The discriminant is $8 \cdot 5 - 36 = 4 > 0$, and $f_{xx}(0, 0) > 0$, so f has a local minimum at $(0, 0)$.

15.7.12 The discriminant is $(-9)(-4) - (-6)^2 = 0$, so the test is inconclusive.

15.7.13 We have $f_x = 6x$, $f_y = -8y$ so $(0, 0)$ is the only critical point of f.

15.7.14 We have $f_x = 2x - 6$, $f_y = 2y + 8$ so $(3, -4)$ is the only critical point of f.

15.7.15 $f_x = 6x$ and $f_y = 3 - 3y^2$. So the critical points are $(0, 1)$ and $(0, -1)$.

15.7.16 $f_x = 3x^2 - 12 = 3(x^2 - 4)$ and $f_y = 12y$, so the critical points are $(2, 0)$ and $(-2, 0)$.

15.7.17 We have $f_x = 4x^3 - 16y$, $f_y = 4y^3 - 16x$; solving $f_x = f_y = 0$ gives $y = \dfrac{x^3}{4}$ and $x = \dfrac{y^3}{4}$; therefore $x = \dfrac{x^9}{4^4}$ which gives $x = 0, \pm 2$, and the critical points are $(0, 0)$, $(2, 2)$ and $(-2, -2)$.

15.7.18 We have $f_x = x^2 + 3y$, $f_y = -y^2 + 3x$; solving $f_x = f_y = 0$ gives $y = -\dfrac{x^2}{3}$ and $x = \dfrac{y^2}{3}$; therefore $x = \dfrac{x^4}{3^3}$ which gives $x = 0, 3$, and the critical points are $(0, 0)$ and$(3, -3)$.

15.7.19 We have $f_x = 4x^3 - 4x$ and $f_y = 2y - 4$; solving $f_x = f_y = 0$ gives $x = 0$, ± 1 and $y = 2$; and the critical points of are $(0, 2)$ and $(\pm 1, 2)$.

15.7.20 $f_x = 3x^2 + 6y - 6$ and $f_y = 6x + 2y - 2$. Setting these both equal to 0 and subtracting $f_x - 3f_y$ gives $3x^2 - 18x = 0$. So $x = 0$ or $x = 6$. When $x = 0$ we have $6y - 6 = 0$ so $y = 1$, and when $x = 6$ we have $36 + 2y - 2 = 0$, so $y = -17$. The critical points are $(0, 1)$ and $(6, -17)$.

15.7.21 $f_x = 6y + 2x - 6$ and $f_y = 3y^2 + 6x - 18$. Setting these both equal to 0 and subtracting $f_y - 3f_x$ gives $3y^2 - 18y = 0$. So $y = 0$ or $y = 6$. When $y = 0$ we have $2x - 6 = 0$ so $x = 3$, and when $y = 6$ we have $36 + 2x - 6 = 0$, so $x = -15$. The critical points are $(3, 0)$ and $(-15, 6)$.

15.7.22 $f_x = e^{8x^2y^2-24x^2-8xy^4}(16xy^2 - 48x - 8y^4)$, and $f_y = e^{8x^2y^2-24x^2-8xy^4}(16x^2y - 32xy^3)$. Other than at the origin, $f_y = 0$ for $x - 2y^2 = 0$. But if $x = 2y^2$, then $f_x = 0$ for $16xy^2 - 48x - 8y^4 = 0$, or $4y^4 - 12y^2 - y^4 = 0$, or $3y^2(y^2 - 4) = 0$. So $y = \pm 2$, and the critical points are $(0, 0)$, $(8, 2)$, and $(8, -2)$.

15.7.23 We have $f_x = -8x$, $f_y = 16y$; therefore, $(0, 0)$ is the only critical point. We also have $f_{xx} = -8$, $f_{yy} = 16$ and $f_{xy} = 0$; hence $D\,(0, 0) = -8 \cdot 16 < 0$, which by the Second Derivative Test implies that f has a saddle point at $(0, 0)$.

15.7.24 We have $f_x = 4x^3 - 4$, $f_y = 4y^3 - 32$; therefore, $(1, 2)$ is the only critical point. We also have $f_{xx} = 12x^2$, $f_{yy} = 12y^2$ and $f_{xy} = 0$; hence $D\,(1, 2) = 12 \cdot 48 > 0$ and $f_{xx}\,(1, 2) > 0$, which by the Second Derivative Test implies that f has a local minimum at $(1, 2)$.

15.7.25 We have $f_x = 4x$, $f_y = 6y$; therefore $(0, 0)$ is the only critical point. We also have $f_{xx} = 4$, $f_{yy} = 6$ and $f_{xy} = 0$; hence $D\,(0, 0) = 4 \cdot 6 - 0^2 = 24 > 0$ and $f_{xx}\,(0, 0) > 0$, which by the Second Derivative Test implies that f has a local minimum at $(0, 0)$.

15.7.26 We have $f_x = (1 - x)\,ye^{-x-y}$, $f_y = (1 - y)\,xe^{-x-y}$; therefore, the critical points are $(0, 0)$ and$(1, 1)$. We also have $f_{xx} = (x - 2)\,ye^{-x-y}$, $f_{yy} = (y - 2)\,xe^{-x-y}$ and $f_{xy} = (1 - x)\,(1 - y)\,e^{-x-y}$; hence $D\,(0, 0) = -1 < 0$ so f has a saddle point at $(0, 0)$; and $D\,(1, 1) = e^{-2} > 0$, $f_{xx}\,(1, 1) = -e^{-2} < 0$ so f has a local maximum at $(1, 1)$.

15.7.27 We have $f_x = 4x^3 - 4y$, $f_y = 4y - 4x$; therefore, the critical points satisfy $y = x$ and $y = x^3$, which gives $x^3 = x$ and therefore $x = 0, \pm 1$, so the critical points are $(0, 0)$, $(1, 1)$ and $(-1, -1)$. We also have $f_{xx} = 12x^2$, $f_{yy} = 4$ and $f_{xy} = -4$; hence $D\,(x, y) = 48x^2 - 16$. Thus $D\,(0, 0) = -16 < 0$, so f has a saddle point at $(0, 0)$; and $D\,(1, 1) = D\,(-1, -1) = 32 > 0$, $f_{xx}\,(1, 1) = f_{xx}\,(-1, -1) = 12 > 0$ so f has a local minimum at $(1, 1)$ and $(-1, -1)$.

15.7.28 We have $f_x = 8\,(4x - 1)$, $f_y = 4\,(2y + 4)$; therefore, $\left(\frac{1}{4}, -2\right)$ is the only critical point. We also have $f_{xx} = 32$, $f_{yy} = 8$ and $f_{xy} = 0$; hence $D\left(\frac{1}{4}, -2\right) = 32 \cdot 8 > 0$ and $f_{xx}\left(\frac{1}{4}, -2\right) > 0$, which by the Second Derivative Test implies that f has a local minimum at $\left(\frac{1}{4}, -2\right)$.

15.7.29 Observe that $f_x = 4x^3$, $f_y = 12y^3$ and $f_{xx} = 12x^2$, $f_{yy} = 36y^2$ and $f_{xy} = 0$; therefore $D\,(0, 0) = 0$ and the Second Derivative Test is inconclusive. Observe that both x^4 and $3y^4$ have absolute minima at 0; therefore, f has an absolute minimum of 4 at $(0, 0)$.

15.7.30 Observe that $f_x = 4x^3y^2$, $f_y = 2x^4y$ and $f_{xx} = 12x^2y^2$, $f_{yy} = 2x^4$ and $f_{xy} = 8x^3y$. Note that every point on the x-axis and on the y-axis is a critical point, and that $D = 24x^6y^2 - 64x^6y^2 = -40x^6y^2$, which has value 0 at every point on the axes, so the Second Derivative Test is inconclusive. Note that the function is always nonnegative, so the value of 0 along the coordinate axes represents the minimum value.

15.7.31 Note that $f\,(x, y)$ has the same critical points as the simpler function

$$g\,(x, y) = x^2 + y^2 - 4x + 5 = (x - 2)^2 + y^2 + 1.$$

We have $g_x = 2\,(x - 2)$, $g_y = 2y$; therefore, $(2, 0)$ is the only critical point. We also have $g_{xx} = 2$, $g_{yy} = 2$ and $g_{xy} = 0$; hence $D\,(2, 0) = 4 > 0$ and $g_{xx}\,(2, 0) > 0$, which by the Second Derivative Test implies that g (and hence f) has a local minimum at $(2, 0)$.

15.7.32 We have $f_x = \frac{y}{1 + x^2y^2}$, $f_y = \frac{x}{1 + x^2y^2}$; therefore, $(0, 0)$ is the only critical point. We also have $f_{xx} = -\frac{2xy^3}{(1 + x^2y^2)^2}$, $f_{yy} = -\frac{2x^3y}{(1 + x^2y^2)^2}$ and $f_{xy} = \frac{1 - x^2y^2}{(1 + x^2y^2)^2}$;hence $D\,(0, 0) = -1 < 0$, which by the Second Derivative Test implies that f has a saddle point at $(0, 0)$.

15.7.33 We have $f_x = 2\left(1-2x^2\right)ye^{-x^2-y^2}$, $f_y = 2\left(1-2y^2\right)xe^{-x^2-y^2}$. Therefore, the critical points are $(0,0)$, $\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Also, $f_{xx} = 4\left(2x^2-3\right)xye^{-x^2-y^2}$, $f_{yy} = 4\left(2y^2-3\right)xye^{-x^2-y^2}$ and $f_{xy} = 2\left(1-2x^2\right)\left(1-2y^2\right)e^{-x^2-y^2}$. Hence $D\,(0,0) = -4 < 0$, so f has a saddle point at $(0,0)$ by the Second Derivative Test. We also see that $D\,(x,y) > 0$ at the four other critical points; also $f_{xx}\left(\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) < 0$ so f has a local maximum at $\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and $f_{xx}\left(\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) > 0$ so f has a local minimum at $\pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

15.7.34 We have $f_x = 2x+y^2-2$, $f_y = 2xy$. Solving $f_x = f_y = 0$ gives critical points at $(1,0)$ and $(0,\pm\sqrt{2})$. We have $f_{xx} = 2$, $f_{yy} = 2x$ and $f_{xy} = 2y$. Hence, $D(1,0) = 4$, so f has a local minimum at $(1,0)$. Also, $D(0,\pm\sqrt{2}) = 0 - (2\sqrt{2}) < 0$, so there are saddle points at $(0,\pm\sqrt{2})$.

15.7.35 We have $f_x = \frac{1+y^2-x^2}{(x^2+y^2+1)^2}$ and $f_y = \frac{-2xy}{(x^2+y^2+1)^2}$, so the critical points are $(\pm 1, 0)$. We have $f_{xx} = \dfrac{2x\left(x^2-3\left(y^2+1\right)\right)}{\left(x^2+y^2+1\right)^3}$, so $f_{xx}(\pm 1,0) = \mp\frac{1}{2}$. We have $f_{yy} = -\dfrac{2x\left(x^2-3y^2+1\right)}{\left(x^2+y^2+1\right)^3}$, so $f_{yy}(\pm 1,0) = \mp\frac{1}{2}$. Also, $f_{xy} = -\dfrac{2y\left(-3x^2+y^2+1\right)}{\left(x^2+y^2+1\right)^3}$, so $f_{xy}(\pm 1,0) = 0$. Thus, $D(\pm 1,0) = \frac{1}{4}$, so there is a local maximum at $(1,0)$ and a local minimum at $(-1,0)$.

15.7.36 We have $f_x = \dfrac{-x^2+2x+y^2}{\left(x^2+y^2\right)^2}$ and $f_y = -\dfrac{2(x-1)y}{\left(x^2+y^2\right)^2}$. The only critical point in the domain of f is $(2,0)$. Note that $f_{xx} = \dfrac{2\left(x^3-3x^2-3xy^2+y^2\right)}{\left(x^2+y^2\right)^3}$, so $f_{xx}(2,0) = -1/8$. Also, $f_{yy} = -\dfrac{2(x-1)\left(x^2-3y^2\right)}{\left(x^2+y^2\right)^3}$ and $f_{yy}(2,0) = -1/8$. Also, $f_{xy} = -\dfrac{2y\left(-3x^2+4x+y^2\right)}{\left(x^2+y^2\right)^3}$ and $f_{xy}(2,0) = 0$. Thus $D(2,0) > 0$, so there is a local maximum at $(2,0)$.

15.7.37 We have $f_x = 4x^3+8x(y-2)$ and $f_y = 4x^2+16(y-1)$. Solving $f_x = f_y = 0$ yields critical points $(0,1)$ and $(\pm 2,0)$. We have $f_{xx} = 12x^2+8(y-2)$, $f_{yy} = 16$, and $f_{xy} = 8x$. Thus $D(0,1) < 0$, $D(\pm 2,0) > 0$. There is a saddle point at $(0,1)$ and local minimums at $(\pm 2,0)$.

15.7.38 We have $f_x = e^{(-x-y)}(\sin y) - xe^{(-x-y)}(\sin y) = e^{(-x-y)}(\sin y)(1-x)$ and $f_y = xe^{(-x-y)}\cos y - xe^{(-x-y)}(\sin y) = xe^{(-x-y)}(\cos y - \sin y)$. We have critical points at $(1,\pi/4)$ and $(0,0)$. $f_{xx} = (x-2)e^{(-x-y)}\sin y$, $f_{yy} = -2xe^{(-x-y)}\cos y$ and $f_{xy} = (x-1)\left(-e^{(-x-y)}\right)(\cos y - \sin y)$. We have $D(0,0) = -1 < 0$ so there is a saddle point at $(0,0)$. $D(1,\pi/4) > 0$ and $f_{xx}(1,\pi/4) < 0$, so there is a local maximum at $(1,\pi/4)$.

15.7.39 We have $f_x = ye^x$, $f_y = e^x - e^y$; therefore, the critical points must satisfy $y = 0$ and $y = x$, so $(0,0)$ is the only critical point. We also have $f_{xx} = ye^x$, $f_{yy} = -e^y$ and $f_{xy} = e^x$; hence $D\,(0,0) = -1 < 0$ so f has a saddle point at $(0,0)$.

15.7.40 We have $f_x = 2\pi\cos(2\pi x)\ \cos(\pi y)$, $f_y = -\pi\sin(2\pi x)\ \sin(\pi y)$, so the critical points must satisfy $\cos(2\pi x) = \sin(\pi y) = 0$, or $\sin(2\pi x) = \cos(\pi y) = 0$; hence, the only critical points in the interior of the domain of f are $\pm\left(\frac{1}{4}, 0\right)$. We also have $f_{xx} = -4\pi^2\sin(2\pi x)\ \cos(\pi y)$, $f_{yy} = -\pi^2\sin(2\pi x)\ \cos(\pi y)$, $f_{xy} = -2\pi^2\cos(2\pi x)\ \sin(\pi y)$; hence $D\left(\pm\left(\frac{1}{4},0\right)\right) > 0$. We also have $f_{xx}\left(\frac{1}{4},0\right) < 0$ and$f_{xx}\left(-\frac{1}{4},0\right) > 0$, so f has a local maximum at $\left(\frac{1}{4},0\right)$ and a local minimum at $\left(-\frac{1}{4},0\right)$.

15.7.41 Observe that $f_x = 2xy$, $f_y = x^2$ and $f_{xx} = 2y$, $f_{yy} = 0$ and $f_{xy} = 2x$; therefore $D\,(0,0) = 0$ and the Second Derivative Test is inconclusive. Observe that x^2y takes on both negative and positive values in any disc centered at $(0,0)$; therefore, f has a saddle at $(0,0)$.

15.7.42 Observe that $f_x = 2xy^2\cos\left(x^2y^2\right)$, $f_y = 2x^2y\cos\left(x^2y^2\right)$, $f_{xx} = 2y^2\cos\left(x^2y^2\right) - 4x^2y^4\sin\left(x^2y^2\right)$, $f_{yy} = 2x^2\cos\left(x^2y^2\right) - 4x^4y^2\sin\left(x^2y^2\right)$ and $f_{xy} = 4xy\cos\left(x^2y^2\right) - 4x^3y^3\sin\left(x^2y^2\right)$. Therefore, $D(0,0) \ = 0$ and the Second Derivative Test is inconclusive. Observe that both x^2,$y^2 \geq 0$ so $\sin\left(x^2y^2\right) \geq 0$ in sufficiently small discs centered at $(0,0)$; therefore, f has an absolute minimum at $(0,0)$ (and in fact along both coordinate axes).

15.7.43 Let x, $y \geq 0$ be the dimensions of the base of the box; then for any given x, y the maximum allowable height is given by $h = 96 - 2x - 2y$, and we must have $h \geq 0$ which implies $x + y \leq 48$. The volume of the box is given by $V = 2xy\,(48 - x - y)$, which we must maximize over the domain R given by x, $y \geq 0$ and $x + y \leq 48$. The critical points of V satisfy $V_x = 2\,(48 - 2x - y)\,y = 0$, $V_y = 2\,(48 - x - 2y)\,x = 0$; hence the critical points in the interior of R satisfy $2x + y = 2y + x = 48$, which gives $x = y = 16$. Furthermore $V(x,y) = 0$ on the boundary of R, so the maximum volume must occur at the point $(16,16)$. Therefore the box with largest volume has height 32 in and base 16 in×16 in with a volume of 8192 in^3.

15.7.44 Let x, $y \geq 0$ be the dimensions of the base and $h \geq 0$ be the height of the box; then the four sides of the box plus the base have total area $2xh + 2yh + xy = 2$, so $h = \dfrac{2 - xy}{2x + 2y}$. The volume of the box is given by $V = xyh = \dfrac{1}{2}\dfrac{(2 - xy)\,xy}{x + y}$, which we must maximize over the domain R given by x, $y \geq 0$ and $xy \leq 2$. The critical points of V satisfy $V_x = \dfrac{1}{2}\dfrac{\left(2 - x^2 - 2xy\right)y^2}{(x+y)^2} = 0$, $V_y = \dfrac{1}{2}\dfrac{\left(2 - y^2 - 2xy\right)x^2}{(x+y)^2} = 0$; hence the critical points in R satisfy $x = y$ and $3x^2 = 2$, which gives $x = y = \dfrac{\sqrt{6}}{3}$, and the corresponding height is $\dfrac{\sqrt{6}}{6}$. Furthermore $V(x,y) \to 0$ as (x,y) approaches the boundary of R, so the maximum volume must occur at the point $\left(\dfrac{\sqrt{6}}{3}, \dfrac{\sqrt{6}}{3}\right)$. Therefore, the box with largest volume has height $\dfrac{\sqrt{6}}{6}$ m and base $\dfrac{\sqrt{6}}{3} \times \dfrac{\sqrt{6}}{3}$ m.

15.7.45 Let x, $y \geq 0$ be the dimensions of the base and $h \geq 0$ be the height of the box; then the volume of the box is $xyh = 4$, so $h = \dfrac{4}{xy}$. The four sides of the box plus the base have total area $A = 2xh + 2yh + xy = \dfrac{8}{x} + \dfrac{8}{y} + xy$, which we must maximize over the domain R given by x, $y \geq 0$. The critical points of A satisfy $A_x = y - \dfrac{8}{x^2} = 0, A_y = x - \dfrac{8}{y^2} = 0$; hence the critical points in R satisfy $x^2y = xy^2 = 8$, which gives $x = y$ and hence $x, y = 2$; the corresponding height is 1. Furthermore $V(x,y) \to \infty$ as (x,y) approaches the boundary of R or as either $x, y \to \infty$, so the minimum area must occur at the point $(2,2)$. Therefore the box with smallest area has dimensions 2m × 2m × 1m.

15.7.46 Let x, $y \geq 0$ be the dimensions of the base and $z \geq 0$ be the height of the box; then $z = 2 - \dfrac{x}{3} - \dfrac{2y}{3}$. The box has volume $V = xyz = xy\left(2 - \dfrac{x}{3} - \dfrac{2y}{3}\right)$, which we must maximize over the domain R given by x, $y \geq 0$ and $x + 2y \leq 6$. The critical points of V satisfy $V_x = 2y - \dfrac{2xy}{3} - \dfrac{2y^2}{3} = 0$, $V_y = 2x - \dfrac{x^2}{3} - \dfrac{4xy}{3} = 0$; hence the critical points in the interior of R satisfy $x + y = 3$ and $x + 4y = 6$, which gives $x = 2$ and $y = 1$; the corresponding height is $z = \dfrac{2}{3}$. Furthermore $V(x,y) = 0$ on the boundary of R, so the maximum area must occur at the point $(2,1)$. Therefore the box with largest volume has dimensions $x = 2$, $y = 1$, $z = \dfrac{2}{3}$.

15.7.47 First find the values of f at all critical points in the interior of $R = \{x^2 + y^2 \leq 4\}$; we have $f_x = 2x$, $f_y = 2y - 2$, so $(0,1)$ is the only critical point in R, and $f(0,1) = 0$. Next, find the minimum and maximum values of f on the boundary of R, which we can parameterize by $x = 2\cos\theta$, $y = 2\sin\theta$ for $0 \leq \theta \leq 2\pi$. Then $f(2\cos\theta, 2\sin\theta) = 5 - 4\sin\theta$, which has maximum value 9 at $\theta = \dfrac{3\pi}{2}$ and minimum value 1 at $\theta = \dfrac{\pi}{2}$. Therefore, the maximum value of f on R is $f(0,-2) = 9$ and the minimum value is $f(0,1) = 0$.

15.7.48 First, find the values of f at all critical points in the interior of R. We have $f_x = 4x$ and $f_y = 2y$, so the only critical point in the interior of R is $(0,0)$ and the value of f there is 0 (which is clearly the absolute minimum for f). We parameterize the boundary by letting $x = 4\cos t$ and $y = 4\sin t$ for $0 \le t \le 2\pi$. On the boundary we have $f(x,y) = f(t) = 32\cos^2 t + 16\sin^2 t = 16\cos^2 t + 16$. Note that $f(t)$ has a maximum value of 32 at $t = 0$ and $t = \pi$, which corresponds to the points $(\pm 4, 0)$ in R. Thus, the absolute minimum of f is 0 at $(0,0)$ and the absolute maximum is 32 at $(\pm 4, 0)$.

15.7.49 First find the values of f at all critical points in the interior of R; we have $f_x = 4x$, $f_y = 2y$, so $(0,0)$ is the only critical point in R, and $f(0,0) = 4$. Next, find the minimum and maximum values of f on the boundary of R, which is a square. On the sides $y = \pm 1$, $-1 \le x \le 1$ we have $f(x,\pm 1) = 4 + 2x^2 + 1 = 2x^2 + 5$, which has extreme values 5 and 7 on $[-1,1]$. On the sides $x = \pm 1$, $-1 \le y \le 1$ we have$f(\pm 1, y) = 4 + 2 + y^2 = y^2 + 6$,which has extreme values 6 and 7 on $[-1,1]$. Therefore, the maximum value of f on R is 7 and the minimum value is 4.

15.7.50 First find the values of f at all critical points in the interior of R; we have $f_x = -2x$, $f_y = -8y$, so $(0,0)$ is the only critical point in R, and $f(0,0) = 6$. Next, find the minimum and maximum values of f on the boundary of R, which is a rectangle. On the sides $y = \pm 1$, $-2 \le x \le 2$ we have $f(x,\pm 1) = 6 - x^2 - 4 = 2 - x^2$, which has extreme values ± 2 on $[-2,2]$. On the sides $x = \pm 2$, $-1 \le y \le 1$ we have $f(\pm 2, y) = 6 - 4 - 4y^2 = 2 - 4y^2$, which has extreme values ± 2 on $[-1,1]$. Therefore, the maximum value of f on R is 6 and the minimum value is -2.

15.7.51 First find the values of f at all critical points in the interior of R; we have $f_x = 4x - 4$ and $f_y = 6y$, so the only critical point is $(1,0)$. Note that $f(1,0) = 0$. Parameterize the boundary of R by letting $x = \cos t + 1$ and $y = \sin t$ for $0 \le t \le 2\pi$. Note that $f(x,y) = (2x^2 - 4x + 2) + 3y^2 = 2(x-1)^2 + 3y^2$, so $f(t) = 2\cos^2 t + 3\sin^2 t = 2 + \sin^2 t$, and f has a maximum of 3 on the boundary of R at $(1,\pm 1)$ and an absolute minimum of 2 on the boundary of R. Thus, the absolute maximum of f on all of R is 3 and the absolute minimum is 0.

15.7.52 First find the values of f at all critical points in the interior of R; we have $f_x = 2x - 2$, $f_y = 2y - 2$, so $(1,1)$ is the only critical point for f, and hence there are no critical points in the interior of R (the point $(1,1)$ is on the boundary of R). Next, find the minimum and maximum values of f on the boundary of R, which is a triangle. On the side $y = 0$, $0 \le x \le 2$ we have $f(x,0) = x^2 - 2x = (x-1)^2 - 1$ which has extreme values -1, 0 on $[0,2]$. Similarly on the side $x = 0$, $0 \le y \le 2$ we have $f(0,y) = y^2 - 2y = (y-1)^2 - 1$, which is the same function as above. The third side is given by $y = 2 - x$ where $0 \le x \le 2$; we have $f(x, 2-x) = x^2 + (2-x)^2 - 2x - 2(2-x) = 2(x-1)^2 - 2$, which has extreme values -2, 0 on $[0,2]$. Therefore, the maximum value of f on R is 0 and the minimum value is -2.

15.7.53 First find the values of f at all critical points in the interior of R; we have $f_x = -4x + 4$ and $f_y = -6y - 6$, so the only critical point in the interior of R is $(1,-1)$. Note that $f(1,-1) = 4$. We parameterize the boundary of R by letting $x = 1 + \cos t$ and $y = -1 + \sin t$ for $0 \le t \le 2\pi$. Note that $f(x,y) = -2(x^2 - 2x + 1) + -3(y^2 + 2y + 1) - 1 + 2 + 3 = -2(x-1)^2 + -3(y+1)^2 + 4$. Thus $f(t) = -2\cos^2 t - 3\sin^2 t + 3\cos^2 t + 3\sin^2 t + 1 = \cos^2 t + 1$. This has a maximum of 2 at $t = 0$ and $t = \pi$ and a minimum of 1 at $t = \pi/2$ and $t = 3\pi/2$. The original function therefore has an absolute maximum of 4 at $(1,-1)$ and an absolute minimum of 1 at $(1,0)$ and at $(1,-2)$.

15.7.54 Observe that it is easier to find the extreme values of the function $g(x,y) = x^2 + y^2 - 2x + 2$; then the extreme values of f on R will be the square roots of the extreme values of g on R. We first find the values of g at all critical points in the interior of R; we have $g_x = 2x - 2$, $g_y = 2y$, so $(1,0)$ is the only critical point for g, and hence there are no critical points in the interior of R (the point $(1,0)$ is on the boundary of R). Next, find the minimum and maximum values of g on the boundary of R. On the side $y = 0$, $-2 \le x \le 2$ we have $g(x,0) = x^2 - 2x + 2 = (x-1)^2 + 1$, which has extreme values 1, 10 on $[-2,2]$. We can parameterize the semicircular part of the boundary by $x = 2\cos\theta$, $y = 2\sin\theta$ for $0 \le \theta \le \pi$. Then $g(2\cos\theta, 2\sin\theta) = 4 - 4\cos\theta + 2 = 6 - 4\cos\theta$, which has extreme values 2, 10 on $[0,\pi]$. Therefore, the maximum value of g on R is 10 and the minimum value is 1 hence, the maximum and minimum values of f are $\sqrt{10}$ and 1.

15.7.55 $f_x = -\dfrac{x\left(2y^4+1\right)}{\left(x^2y^2+1\right)^2}$, which is 0 for $x=0$, and $f_y = \dfrac{\left(x^4+2\right)y}{\left(x^2y^2+1\right)^2}$, which is 0 for $y=0$. On the boundary $y=2$ we have $f(x,y) = \dfrac{8-x^2}{2+8x^2}$ which is maximized on $[1,2]$ at $x=1$ (with value $f(1,2)=\dfrac{7}{9}$) and minimized at $x=2$ (with value $f(2,2)=\dfrac{1}{3}$). On the boundary $y=x$ we have $f(x,y)=\dfrac{x^2}{2+2x^2}$ which has minimum value 0 at $(0,0)$ and maximum value $\dfrac{1}{4}$ at $(1,1)$. On the boundary $y=2x$ we have $f(x,y)=\dfrac{7x^2}{2+8x^4}$ which has derivative $\dfrac{7x-28x^5}{\left(4x^4+1\right)^2}$. This is zero for $x=0$ and for $x=\dfrac{1}{\sqrt{2}}$. These points lead to a minimum of 0 at $(0,0)$ and a maximum of $\dfrac{7}{8}$ at $\left(\dfrac{1}{\sqrt{2}},\sqrt{2}\right)$. Therefore the global minimum of f on the given set is $0=f(0,0)$ and the global maximum is $\dfrac{7}{8}=f\left(\dfrac{1}{\sqrt{2}},\sqrt{2}\right)$.

15.7.56 $f_x = \dfrac{x}{\sqrt{x^2+y^2}}$ and $f_y = \dfrac{y}{\sqrt{x^2+y^2}}$, so the critical point in the interior of the ellipse is $(0,0)$ which yields the value $f(0,0)=0$ which is clearly the minimum for the function. We can parameterize the boundary with $x=2\cos\theta$, $y=\sin\theta$ for $0\le\theta\le 2\pi$. Then $f(x,y)=\sqrt{x^2+y^2}=\sqrt{4\cos^2\theta+\sin^2\theta}=\sqrt{3\cos^2\theta+1}$. This has derivative $\dfrac{-3\cos\theta\sin\theta}{\sqrt{3\cos^2\theta+1}}$ which is 0 for $\theta=0,\dfrac{\pi}{2},\pi$, and $\dfrac{3\pi}{2}$. Checking these points, the maximum for f is 2 which occurs for $\theta=0$ and $\theta=\pi$, which correspond with the points $(-2,0)$ and $(2,0)$ on the ellipse.

15.7.57

a. $F_H = 58.729 - 22.438H - 0.213T$ and $F_T = 7.327 - 0.213H - 0.084T$. Setting these equal to 0 and solving the resulting system of linear equations gives $H\approx 1.83$ and $T\approx 82.6$. So the only critical point on the interior is $(1.83, 82.6)$.

b. $F(1.83, 82.6)\approx 13.67\approx 14\%$.

15.7.58 Because $-1<x<1$ and $-2<y<2$ on R, the function $f(x,y)=x+3y$ takes on all values between $-1+3(-2)=-7$ and $1+3\cdot 2=7$; in other words, the range of f on R is the interval $(-7,7)$, and we see that f has neither an absolute minimum or maximum on R.

15.7.59 Observe that $f(0,0)=-4$ and $f(x,y)\ge -4$ for all points $(x,y)\in R$; hence the absolute minimum value of f on R is -4. The function f on R takes on all values in the interval $[-4,0)$; therefore f has no absolute maximum on R.

15.7.60 Because $-1<x$, $y<1$ on R, the function $f(x,y)=x^2-y^2$ takes on all values between -1 and 1; in other words, the range of f on R is the interval $(-1,1)$, and we see that f has neither an absolute minimum or maximum on R.

15.7.61 Observe that $f(0,0)=2$ and $f(x,y)\le 2$ for all points $(x,y)\in R$; hence, the absolute maximum value of f on R is 2. The function f on R takes on all values in the interval $(0,2]$; therefore, f has no absolute minimum on R.

15.7.62 The equation of the plane can be written as $z=4-x-y$; so it suffices to minimize the square of the distance from $(x,y,4-x-y)$ to $(5,4,4)$, which is given by $w=(x-5)^2+(y-4)^2+(x+y)^2$. We have $w_x=2x-10+2x+2y$, $w_y=2y-8+2x+2y$, so the critical points of w satisfy $4x+2y=10$, $4y+2x=8$ which gives $(x,y,z)=(2,1,1)$. Because we know that there is some point on the plane closest to $(2,1,1)$, this critical point must be that point.

15.7.63 The plane has equation $z=2-x+y$; the distance from a point $(x,y,2-x+y)$ on the plane to the point $(1,1,1)$ is given by $d^2=(x-1)^2+(y-1)^2+(x-y-1)^2$. It suffices to minimize the function

$f(x,y) = (x-1)^2 + (y-1)^2 + (x-y-1)^2$ on R^2. We have $f_x = 2(x-1+x-y-1) = 2(2x-y-2)$, $f_y = 2(y-1+y-x+1) = 2(-x+2y)$, so the critical point of f satisfies $2x-y=2$, $x-2y=0$ which gives $x = \frac{4}{3}$, $y = \frac{2}{3}$. The corresponding point on the plane is $\left(\frac{4}{3}, \frac{2}{3}, \frac{4}{3}\right)$. Because there is a point on the plane closest to the point $(1,1,1)$, this must be the point we found.

15.7.64 We need to minimize the square of the distance from (x, y, x^2+y^2) to $(3,3,1)$, which is

$$w = (x-3)^2 + (y-3)^2 + (x^2+y^2-1)^2.$$

We have $w_x = 2x - 6 + (2x^2+2y^2-2)(2x)$ and $w_y = 2y - 6 + (2x^2+2y^2-2)(2y)$. Setting these equal to 0 and solving gives $x = 1$ and $y = 1$. The closest point is $(1,1,2)$.

15.7.65 It suffices to minimize the square of the distance from points (x,y,z) on the cone to $(6,8,0)$, which is given by $w = (x-6)^2 + (y-8)^2 + z^2 = (x-6)^2 + (y-8)^2 + x^2 + y^2$. We have $w_x = 2x - 12 + 2x$, $w_y = 2y - 16 + 2y$, so $(3,4)$ is the only critical point of w; the corresponding points on the cone are $(3,4,5)$ and $(3,4,-5)$. These two points are tied for being the closest point on the cone to (6,8,0).

15.7.66 Let $x, y > 0$ be the dimensions of the base and $z > 0$ be the height of the box; then $xyz = 10$ so $z = \frac{10}{xy}$. The cost to produce the box is $C = 2 \cdot 10 \cdot xy + 2 \cdot 1 \cdot xz + 2 \cdot 1 \cdot yz = 20xy + \frac{20}{x} + \frac{20}{y}$, which we must minimize over the domain R given by $x, y > 0$. The critical points of C satisfy $C_x = 20y - \frac{20}{x^2} = 0$, $C_y = 20x - \frac{20}{y^2} = 0$; hence the critical points satisfy $x^2y = xy^2 = 1$, which gives $x = y = 1$; the corresponding height is $z = 10$. Furthermore $C(x,y) \to 0$ as (x,y) approaches the boundary of R or as either x or y approach ∞, so the minimum cost must occur at the critical point. Therefore, the box with least cost has dimensions 1m × 1m × 10m.

15.7.67

a. True. This is because our definition of saddle point is a critical point which is neither a local maximum or local minimum.

b. False. A necessary condition for a local maximum at (a,b) is that both $f_x = f_y = 0$ at (a,b), assuming both partials exist.

c. True. This is because f may take on its absolute maximum or minimum at a point on the boundary of its domain.

d. True. The equation of the tangent plane at a critical point (a,b) is $z = f(a,b)$.

15.7.68 This function has a local maximum at $\left(\frac{1}{2}, -1\right)$ and saddle points at $(0,0)$, $(1,0)$, $(-2,0)$ and $(1,-2)$.

15.7.69 This function has a local minimum near $(0.3, -0.3)$ and a saddle point at $(0,0)$.

15.7.70 Let (x,y,z) be the point on the ellipsoid; then the box has volume $V = xyz$. We have the relation $36x^2 + 4y^2 + 9z^2 = 36$ which implies that $z^2 = 4\left(1 - x^2 - \frac{y^2}{9}\right)$, and it suffices to maximize the square of the volume, which is given by $w = 4x^2y^2\left(1 - x^2 - \frac{y^2}{9}\right)$ over the region R given by $x, y \geq 0$ and $x^2 - \frac{y^2}{9} \leq 1$. The critical points in the interior of R satisfy $w_x = 8xy^2\left(1 - 2x^2 - \frac{y^2}{9}\right) = 0$, $w_y = 8x^2y\left(1 - x^2 - \frac{2y^2}{9}\right) = 0$, which gives $x^2 = \frac{1}{3}$, $y^2 = 3$ so $x = \frac{\sqrt{3}}{3}$, $y = \sqrt{3}$; The corresponding value of $z = \frac{2\sqrt{3}}{3}$. Notice that $w = 0$ on the boundary of the closed and bounded region R; therefore the critical point we found must give the absolute maximum value of w, so the box with maximum volume has dimensions $\frac{\sqrt{3}}{3} \times \sqrt{3} \times \frac{2\sqrt{3}}{3}$.

15.7.71

a. Using the relation $z = 200 - x - y$, we see that it suffices to minimize the function $f(x,y) = x^2 + y^2 + (200 - x - y)^2$ over the closed bounded region R given by $x, y \geq 0$ and $x + y \leq 200$. We have $f_x = 2(x + x + y - 200) = 2(2x + y - 200)$, $f_y = 2(y + x + y - 200) = 2(x + 2y - 200)$; therefore, the critical point of f satisfies $2x + y = 200$, $x + 2y = 200$ which gives $x = y = \dfrac{200}{3}$ (the corresponding value of $z = \dfrac{200}{3}$ as well), and we have $f\left(\dfrac{200}{3}, \dfrac{200}{3}\right) = \dfrac{40,000}{3}$. We must also find the extreme values of f on the boundary of R. Along the segment $y = 0$, $0 \leq x \leq 200$ we have $f(x,0) = x^2 + (200 - x)^2$ which has range $[20,000, 40,000]$, and we get the same result along the other two segments that make up the boundary of R. Therefore, the minimum value of $x^2 + y^2 + z^2$ is given by $x = y = z = \dfrac{200}{3}$.

b. The function $\sqrt{x^2 + y^2 + z^2}$ takes its minimum at the same point that minimizes $x^2 + y^2 + z^2$, which we saw in part (a) is $x = y = z = \dfrac{200}{3}$.

c. Using the relation $z = 200 - x - y$, we see that it suffices to minimize the function $f(x,y) = xy(200 - x - y) = 200xy - x^2y - xy^2$ over the closed bounded region R given by $x, y \geq 0$ and $x + y \leq 200$. We have $f_x = y(200 - 2x - y)$, $f_y = x(200 - x - 2y)$; therefore, the critical points of f in the interior of R satisfy $2x + y = 200$, $x + 2y = 200$ which gives $x = y = \dfrac{200}{3}$ (the corresponding value of $z = \dfrac{200}{3}$ as well), and we have $f\left(\dfrac{200}{3}, \dfrac{200}{3}\right) = \left(\dfrac{200}{3}\right)^2$. We also observe that $f(x,y) = 0$ on the boundary of R; therefore, the maximum value of xyz is given by $x = y = z = \dfrac{200}{3}$.

d. The function $x^2y^2z^2$ takes its maximum at the same point that maximizes xyz, which we saw in part (c) is $x = y = z = \dfrac{200}{3}$.

15.7.72 The level curves of a linear function $f(x,y)$ are a family of parallel lines; therefore at any interior point of a region R, one can move to a nearby level curve in such a way as to either increase or decrease the value of f. This shows that the absolute maximum or minimum of f on R cannot occur at an interior point, and hence must occur along the boundary of R.

15.7.73

a. The function to be minimized is $f(x,y) = x^2 + y^2 + (x - 2)^2 + y^2 + (x - 1)^2 + (y - 1)^2 = 3x^2 - 6x + 3y^2 - 2y$; we have $f_x = 6(x - 1)$, $f_y = 2(3y - 1)$ so the optimal location is the unique critical point $\left(1, \dfrac{1}{3}\right)$.

b. The function to be minimized is now $f(x,y) = (x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 + (x - x_3)^2 + (y - y_3)^2 = 3\left(x^2 - 2\overline{x}x + y^2 - 2\overline{y}y\right) + \text{const}$; where $\overline{x} = \dfrac{x_1 + x_2 + x_3}{3}$, $\overline{y} = \dfrac{y_1 + y_2 + y_3}{3}$. we have $f_x = 6(x - \overline{x})$, $f_y = 6(y - \overline{y})$ so the optimal location is the unique critical point $(\overline{x}, \overline{y})$.

c. The function to be minimized is now

$$f(x,y) = \sum_{i=1}^{n} (x - x_i)^2 + \sum_{i=1}^{n} (y - y_i)^2 = n\left(x^2 - 2\overline{x}x + y^2 - 2\overline{y}y\right) + \text{const}$$

where $\overline{x} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i$, $\overline{y} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} y_i$. We have $f_x = n(x - \overline{x})$, $f_y = n(y - \overline{y})$ so the optimal location is the unique critical point $(\overline{x}, \overline{y})$.

d. The actual sum of the distances is given by the function $f(x,y) = \sqrt{x^2+y^2} + \sqrt{(x-2)^2+y^2} + \sqrt{(x-1)^2+(y-1)^2}$ We can minimize this function as follows. First, fix $y = y_0$ and consider the function $g(x) = f(x, y_0)$. Each of the three terms in this function has positive second derivative, and approaches ∞ as $x \to \pm\infty$, so the same is true for their sum; therefore $g(x)$ must have a unique absolute minimum. Observe in addition that g is symmetric in the line $x = 1$, which implies that the absolute minimum must occur at $x = 1$. So to minimize $f(x,y)$, we can set $x = 1$ and reduce to minimizing the function $h(y) = f(1,y) = 2\sqrt{1+y^2} + |y-1|$, which has absolute minimum at $y = \dfrac{1}{\sqrt{3}}$. Therefore, the optimal location is $\left(1, \dfrac{1}{\sqrt{3}}\right)$, which is different from the point found in part (a).

15.7.74 The sum of the squares of the vertical distances is $E(m,b) = [(m+b)-2]^2 + [(3m+b)-5]^2 + [(4m+b)-6]^2 = 26m^2 + 16mb + 3b^2 - 82m - 26b + 65$. We have $E_m = 52m + 16b - 82$, $E_b = 16m + 6b - 26$, and solving the simultaneous equations $52m + 16b = 82$, $16m + 6b = 26$ gives $m = \dfrac{19}{14}$, $b = \dfrac{10}{14}$.

15.7.75 Given the n data points $(x_1, y_1), \ldots, (x_n, y_n)$, we seek to minimize the function $E(m,b) = \sum_{k=1}^{n} (mx_k + b - y_k)^2 = \left(\sum x_k^2\right) m^2 + 2\left(\sum x_k\right) mb + nb^2 - 2\left(\sum x_k y_k\right) m - 2\left(\sum y_k\right) b + \sum y_k^2$. We have $E_m = 2\left(\sum x_k^2\right) m + 2\left(\sum x_k\right) b - 2\left(\sum x_k y_k\right)$, $E_b = 2\left(\sum x_k\right) m + 2nb - 2\left(\sum y_k\right)$ and solving $E_m = E_b = 0$ gives $m = \dfrac{\left(\sum x_k\right)\left(\sum y_k\right) - n\sum x_k y_k}{\left(\sum x_k\right)^2 - n\sum x_k^2}$, $b = \dfrac{1}{n}\left(\sum y_k - m\sum x_k\right)$.

15.7.76 Using the result in Exercise 75, we obtain $m = \dfrac{6\cdot 8 - 3\cdot 26}{6^2 - 3\cdot 20} = \dfrac{5}{4}$, $b = \dfrac{1}{3}\left(8 - \dfrac{5}{4}\cdot 6\right) = \dfrac{1}{6}$ so the line has equation $y = \dfrac{5}{4}x + \dfrac{1}{6}$.

15.7.77 Using the result in Exercise 75, we obtain $m = \dfrac{2\cdot 14 - 3\cdot 24}{2^2 - 3\cdot 10} = \dfrac{22}{13}$, $b = \dfrac{1}{3}\left(14 - \dfrac{22}{13}\cdot 2\right) = \dfrac{46}{13}$ so the line has equation $y = \dfrac{22}{13}x + \dfrac{46}{13}$.

15.7.78 Observe that $D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2 < 0$ because $f_{xx}(a,b) f_{yy}(a,b) < 0$ in both cases. Therefore f has a saddle at (a,b).

15.7.79 Let $s = \dfrac{a+b+c}{2}$ be the semi-perimeter, which we assume is constant. Then we may express $c = 2s - a - b$ and therefore $A^2 = s(s-a)(s-b)(s-(2s-a-b)) = s(s-a)(s-b)(a+b-s)$, so it suffices to maximize the simpler function $f(a,b) = s(s-a)(s-b)(a+b-s)$ over the closed bounded region given by $0 \le a, b \le s$ and $a + b \ge s$. We have $f_a = s(s-b)(-(a+b-s) + s - a) = s(s-b)(2s - 2a - b)$ and similarly $f_b = s(s-a)(2s - a - 2b)$, so the critical points of f in the interior of R satisfy $2a + b = 2s$, $a + 2b = 2s$ which gives $a = b = \dfrac{2}{3}s$, and therefore $c = \dfrac{2}{3}s$ as well, so we get an equilateral triangle. We also observe that $f(a,b) = 0$ on the boundary of R, so this critical point must give the absolute maximum of f. We conclude that of all triangles with a given perimeter, the maximum area is obtained when all three sides are equal (in the special case of perimeter 9 units, each side length is 3 units).

15.7.80 Write the equation of the plane in the form $ax + by + cz = 1$, with $a, b, c > 0$; then a, b, c must satisfy the condition $3a + 2b + c = 1$. This plane meets the coordinate axes in the points $\left(\dfrac{1}{a}, 0, 0\right)$, $\left(0, \dfrac{1}{b}, 0\right)$, $\left(0, 0, \dfrac{1}{c}\right)$. The region between then plane and the coordinate planes is a tetrahedron T which has base area $A = \dfrac{1}{2ab}$ and height $h = \dfrac{1}{c}$, so its volume is $V = \dfrac{1}{6abc}$. Minimizing this function is equivalent to maximizing its reciprocal, or equivalently the quantity $Q = abc$. Now if we let $u = 3a$, $v = 2b$, $w = c$, then an equivalent (but simpler) problem is to maximize $Q = uvw$ where u, v,$w \geq 0$ and $u + v + w = 1$. The maximum occurs when $u = v = w = \dfrac{1}{3}$ (see problem 65c), so the tetrahedron of minimal volume is given by $3a = 2b = c = \dfrac{1}{3}$, which gives the plane $\dfrac{x}{9} + \dfrac{y}{6} + \dfrac{z}{3} = 1$.

15.7.81

a. We have $d_1(x, y) = \sqrt{(x - x_1)^2 + (y - y_1)^2}$, so $\nabla d_1(x, y) = \dfrac{x - x_1}{d_1(x, y)}\mathrm{i} + \dfrac{y - y_1}{d_1(x, y)}\mathrm{j}$; observe that this is a unit vector in the direction of the vector joining (x_1, y_1) to (x, y).

b. Similarly, we have $\nabla d_2(x, y) = \dfrac{x - x_2}{d_2(x, y)}\mathrm{i} + \dfrac{y - y_2}{d_2(x, y)}\mathrm{j}$ and $\nabla d_3(x, y) = \dfrac{x - x_3}{d_3(x, y)}\mathrm{i} + \dfrac{y - y_3}{d_3(x, y)}\mathrm{j}$.

c. Because $\nabla f = \nabla d_1 + \nabla d_2 + \nabla d_3$, the condition $f_x = f_y = 0$ is equivalent to $\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$.

d. If three unit vectors add to 0, they must make angles of $\pm\dfrac{2\pi}{3}$ with each other.

e. In this case the optimal point is the vertex at the large angle.

f. Solving the equations $f_x = f_y = 0$ numerically gives $(0.255457, 0.304504)$.

15.7.82

a. We have $f_x = 3e^y - 3x^2$, $f_y = 3xe^y - 3e^{3y}$, so the critical points must satisfy $x = e^{2y}$, and then the equation $f_x = 0$ reduces to $e^{3y} = 1$, so the only critical point is $(1, 0)$. Next we compute $f_{xx} = -6x$, $f_{yy} = 3xe^y - 9e^{3y}$, $f_{xy} = 3e^y$, so $f_{xx}f_{yy} - f_{xy}^2 = (-6)(-6) - 3^2 = 27 > 0$ at the critical point $(1, 0)$; we also have $f_{xx} < 0$ at $(1, 0)$, so the critical point is a local maxima, and we conclude that f has a unique local maximum. However observe that if we fix $y = 0$, then $f(x, 0) = -x^3 + 3x - 1$ takes on all real values, so f does not have an absolute maximum or minimum.

b. We have $f_x = (2y^2 - y^4)e^x - (2y^2 - y^4 - 1)\dfrac{2x}{(1 + x^2)^2}$, $f_y = (4y - 4y^3)\left(e^x + \dfrac{1}{1 + x^2}\right)$, so the critical points must satisfy $y = 0, \pm 1$ from the equation $f_y = 0$. Substituting $y = 0$ into the equation $f_x = 0$ gives $x = 0$, whereas substituting $y = \pm 1$ into $f_x = 0$ gives no solutions; therefore, $(0, 0)$ is the only critical point. We compute $f_{xx}(0, 0) = 2$, $f_{yy}(0, 0) = 8$, $f_{xy} = 0$, so $D(0, 0) > 0$, and $f_{xx}(0, 0) > 0$ implies that f has a local minimum at $(0, 0)$. However, observe that if we fix $y = 1$, then $f(x, 1) = e^x$ takes on all positive values, and if we fix $y = 2$, then $f(x, 2) = -8e^x - \dfrac{9}{1 + x^2}$ takes on all negative values. Hence f must take on all real values on R^2, and therefore has no absolute maximum or minimum.

15.7.83

a. We have $f(x, y) = -2x^4 + 2x^2(e^y + 1) - e^{2y} - 1$; hence $f_x = -8x^3 + 4x(e^y + 1)$, $f_y = 2x^2e^y - 2e^{2y}$, so the critical points must satisfy $x^2 = e^y$, and then the equation $f_x = 0$ reduces to $e^y = 1$, so the critical points are $(\pm 1, 0)$. Next we compute$f_{xx} = -24x^2 + 4(1 + e^y)$, $f_{yy} = 2x^2e^y - 4e^{2y}$, $f_{xy} = 4e^y$ so $f_{xx}f_{yy} - f_{xy}^2 = (-16)(-2) - (4)^2 = 16$ at both critical points; we also have $f_{xx} < 0$ at both critical points, so the critical points both yield local maxima.

b. We have $f_x = 8xe^y - 8x^3$, $f_y = 4x^2e^y - 4e^{4y}$, so the critical points must satisfy $x^2 = e^{3y}$, and then the equation $f_x = 0$ reduces to $e^y = 1$, so the critical points are $(\pm 1, 0)$. Next we compute $f_{xx} = 8e^y - 24x^2$, $f_{yy} = 4x^2e^y - 16e^{4y}$, $f_{xy} = 8xe^y$, so $f_{xx}f_{yy} - f_{xy}^2 = (-16)(-12) - (\pm 8)^2 = 128$ at both critical points; we also have $f_{xx} < 0$ at both critical points, so the critical points both yield local maxima.

15.7.84

a. Using the relation $z = 1 - x - y$, we see that it suffices to minimize the function $f(x, y) = \left(1 + x^2\right)\left(1 + y^2\right)\left(1 + z^2\right) = \left(1 + x^2\right)\left(1 + y^2\right)\left(1 + (1 - x - y)^2\right)$ over the closed bounded region R given by $x, y \geq 0$ and $x+y \leq 1$. Using the chain rule, we have $f_x = 2\left(1 + y^2\right)\left(x\left(1 + z^2\right) - z\left(1 + x^2\right)\right)$, $f_y = 2\left(1 + x^2\right)\left(y\left(1 + z^2\right) - z\left(1 + y^2\right)\right)$. The critical points of f in the interior of R satisfy $f_x = f_y = 0$; the first equation gives $x\left(1 + z^2\right) - z\left(1 + x^2\right) = (x - z)(1 - xz) = 0$, and we observe that $xz = 1$ is impossible, so we conclude that $x = z$. Similarly $y = z$ so the critical point is $\left(\frac{1}{3}, \frac{1}{3}\right)$, and we have $f\left(\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{10}{9}\right)^3$. On the boundary segment given by $y = 0$, $0 \leq x \leq 1$ we have $f(x, 0) = \left(1 + x^2\right)\left(1 + (1 - x)^2\right)$ and this function has a maximum value $f(0) = f(1) = 2$, and minimum value $f\left(\frac{1}{2}\right) = \left(\frac{5}{4}\right)^2$. The range of f on the other two boundary components is the same. Therefore the maximum value of f is 2, and the minimum is $\left(\frac{10}{9}\right)^3$.

b. Using the relation $z = 1 - x - y$, we see that it suffices to minimize the function $f(x, y) = (1 + \sqrt{x})(1 + \sqrt{y})(1 + \sqrt{z}) = (1 + \sqrt{x})(1 + \sqrt{y})\left(1 + \sqrt{1 - x - y}\right)$ over the closed bounded region R given by $x, y \geq 0$ and $x + y \leq 1$. Using the chain rule, we have

$$f_x = (1 + \sqrt{y})\left(\left(1 + \frac{1}{2\sqrt{x}}\right)(1 + \sqrt{z}) - (1 + \sqrt{x})\left(1 + \frac{1}{2\sqrt{z}}\right)\right)$$

$$= \left(\frac{1 + \sqrt{y}}{4\sqrt{x}\sqrt{z}}\right)\left((2\sqrt{x} + 1)(\sqrt{z} + 1) - (2\sqrt{z} + 1)(\sqrt{x} + 1)\right) = \left(\frac{1 + \sqrt{y}}{4\sqrt{x}\sqrt{z}}\right)(\sqrt{x} - \sqrt{z}),$$

and similarly $f_y = \left(\frac{1 + \sqrt{x}}{4\sqrt{y}\sqrt{z}}\right)(\sqrt{y} - \sqrt{z})$. The critical points of f in the interior of R satisfy $f_x = f_y = 0$ which gives $x = y = \frac{1}{3}$, so the critical point is $\left(\frac{1}{3}, \frac{1}{3}\right)$, and we have $f\left(\frac{1}{3}, \frac{1}{3}\right) \approx 3.925$. On the boundary segment given by $y = 0$, $0 \leq x \leq 1$ we have $f(x, 0) = (1 + \sqrt{x})\left(1 + \sqrt{1 - x}\right)$, and this function has a maximum value $f(0) = f(1) = 2$, and minimum value $f\left(\frac{1}{2}\right) \approx 2.914$. The range of f on the other two boundary components is the same. Therefore the minimum value of f is 2, and the maximum is 3.925.

15.7.85 Let (x_0, y_0, z_0) be a point on the ellipsoid; then the tangent plane P at this point has equation $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1$, and this plane meets the coordinate axes in the points $\left(\frac{a^2}{x_0}, 0, 0\right)$, $\left(0, \frac{b^2}{y_0}, 0\right)$, $\left(0, 0, \frac{c^2}{z_0}\right)$. Therefore, the tetrahedron T has base area $A = \frac{a^2 b^2}{2x_0 y_0}$ and height $h = \frac{c^2}{z_0}$, so its volume is $V = \frac{a^2 b^2 c^2}{6x_0 y_0 z_0}$. Minimizing this function is equivalent to maximizing its reciprocal, or equivalently the quantity $Q = \frac{x_0^2}{a^2} \cdot \frac{y_0^2}{b^2} \cdot \frac{z_0^2}{c^2}$. Now if we let $u = \frac{x_0^2}{a^2}$, $v = \frac{y_0^2}{b^2}$, $w = \frac{z_0^2}{c^2}$, then an equivalent (but simpler) problem is to maximize $Q = uvw$ where u, v,$w \geq 0$ and $u + v + w = 1$. The maximum occurs when $u = v = w = \frac{1}{3}$ (see problem 65 c), so the tetrahedron of minimal volume is given by $\frac{x_0^2}{a^2} = \frac{y_0^2}{b^2} = \frac{z_0^2}{c^2} = \frac{1}{3}$ which gives $V = \frac{abc}{6}\left(\frac{a}{x_0}\right)^3 = \frac{abc\sqrt{3}}{2}$.

15.8 Lagrange Multipliers

15.8.1 The level curves of f must be tangential to the level curves of g at the optimal point; thus, the gradients are parallel.

15.8.2 Let $g(x, y) = x^2 + y^2 - 1$. Solve the system $\nabla f = \lambda \nabla g$ and $g(x, y) = 0$. Among the solutions, find the corresponding maximum and minimum values of f.

15.8.3 $1 = 2\lambda x$ and $4 = 2\lambda y$, and $x^2 + y^2 - 1 = 0$.

15.8.4 $yz = 2\lambda x$, $xz = 4\lambda y$, $xy = 6\lambda z$, and $x^2 + 2y^2 + 3z^2 - 1 = 0$.

15.8.5 The maximum and minimum values of f along the curve $g\,(x, y) = 0$ occur at points where the level curves of f are tangent to the curve $g\,(x, y) = 0$; using this, we see that the minimum and maximum values are 1 and 8.

15.8.6 The maximum and minimum values of f along the curve $g\,(x, y) = 0$ occur at points where the level curves of f are tangent to the curve $g\,(x, y) = 0$; using this, we see that the minimum and maximum values are 2 and 6.

15.8.7 We have $\nabla f = \langle 1, 2\rangle$, $\nabla g = \langle 2x, 2y\rangle$ so the Lagrange multiplier conditions are $1 = 2\lambda x$, $2 = 2\lambda y$, $x^2 + y^2 - 4 = 0$. Hence $\dfrac{1}{x} = 2\lambda = \dfrac{2}{y} \implies y = 2x$; substituting this in the constraint gives $5x^2 = 4$, so $x = \pm\dfrac{2}{\sqrt{5}}$ and the extreme values of f on the circle $x^2 + y^2 = 4$ must occur at the points $\pm\left(\dfrac{2}{\sqrt{5}}, \dfrac{4}{\sqrt{5}}\right)$. We see that $f\left(\dfrac{2}{\sqrt{5}}, \dfrac{4}{\sqrt{5}}\right) = 2\sqrt{5}$, $f\left(-\dfrac{2}{\sqrt{5}}, -\dfrac{4}{\sqrt{5}}\right) = -2\sqrt{5}$, so these are the maximum and minimum values.

15.8.8 We have $\nabla f = \langle y^2, 2xy\rangle$, $\nabla g = \langle 2x, 2y\rangle$ so the Lagrange multiplier conditions are $y^2 = 2\lambda x$, $2xy = 2\lambda y$, $x^2 + y^2 - 1 = 0$. One possible solution is $y = \lambda = 0$, which gives the points $(\pm 1, 0)$. All other solutions will have $x,\ y,\ \ \lambda \neq 0$, so we can eliminate λ and obtain $2\lambda = \dfrac{y^2}{x} = 2x \implies y^2 = 2x^2$; substituting this in the constraint gives $3x^2 = 1$, so $x = \pm\dfrac{1}{\sqrt{3}}$, hence $y = \pm\dfrac{\sqrt{2}}{\sqrt{3}}$ and the extreme values of f on the circle $x^2 + y^2 = 1$ must occur at the points $(\pm 1, 0)$, $\left(\dfrac{1}{\sqrt{3}}, \pm\dfrac{\sqrt{2}}{\sqrt{3}}\right)$ and $\left(-\dfrac{1}{\sqrt{3}}, \pm\dfrac{\sqrt{2}}{\sqrt{3}}\right)$. We see that $f\,(\pm 1, 0) = 0$, $f\left(\dfrac{1}{\sqrt{3}}, \pm\dfrac{\sqrt{2}}{\sqrt{3}}\right) = \dfrac{2\sqrt{3}}{9}$, $f\left(-\dfrac{1}{\sqrt{3}}, \pm\dfrac{\sqrt{2}}{\sqrt{3}}\right) = -\dfrac{2\sqrt{3}}{9}$, so the maximum and minimum values are $\pm\dfrac{2\sqrt{3}}{9}$.

15.8.9 We have $\nabla f = \langle 1, 1\rangle$, $\nabla g = \langle 2x - y, 2y - x\rangle$, so the Lagrange multiplier conditions are $2x - y = 1/\lambda$, $2y - x = 1/\lambda$, $x^2 - xy + y^2 = 1$. Subtracting the first two equations gives $3x - 3y = 0$, so $x = y = 1/\lambda$, which yields the points $(\pm 1, \pm 1)$. The maximum value is 2 at $(1, 1)$ and the minimum is -2 at $(-1, -1)$.

15.8.10 We have $\nabla f = \langle 2x, 2y\rangle$, $\nabla g = \langle 4x + 3y, 4y + 3x\rangle$, so the Lagrange multiplier conditions are $2x = \lambda(4x + 3y)$, $2y = \lambda(4y + 3x)$, $2x^2 + 3xy + 2y^2 = 7$. Multiplying the first equation by y and the second by x and subtracting gives $0 = \lambda(3y^2 - 3x^2)$. Note that if $\lambda = 0$, then $x = 0$ and $y = 0$ which does not satisfy the constraint. If $\lambda \neq 0$, we have $x = \pm y$. If $x = y$ we obtain the points $x = y = \pm 1$, if $x = -y$, we obtain the point $(\pm\sqrt{7}, \mp\sqrt{7})$. The minimum value of f is 2 at $(\pm 1, \pm 1)$ and the maximum is 14 at $(\pm\sqrt{7}, \mp\sqrt{7})$.

15.8.11 We have $\nabla f = \langle y, x\rangle$ and $\nabla g = \langle 2x - y, 2y - x\rangle$, so the Lagrange multiplier conditions are $y = \lambda(2x - y)$, $x = \lambda(2y - x)$, $x^2 + y^2 - xy = 9$. Multiplying the first equation by x and the second by y and subtracting leads to $\lambda(2x^2 - 2y^2) = 0$, so either $x = \pm y$ or $\lambda = 0$. If $\lambda = 0$, we have $x = y = 0$ which doesn't meet the constraint. If $x = y$, we have $x = \pm 3 = y$, if $x = -y$, we have $x = \pm\sqrt{3}$, $y = \mp\sqrt{3}$. There is a minimum value of -3 at $(\pm\sqrt{3}, \mp\sqrt{3})$ and a maximum value of 9 at $(\pm 3, \pm 3)$.

15.8.12 We have $\nabla f = \langle 1, -1\rangle$ and $\nabla g = \langle 2x-3y, 2y-3x\rangle$, so the Lagrange multiplier conditions are $2x-3y = 1/\lambda$, $2y-3x = -1/\lambda$, $x^2+y^2-3xy = 20$. Adding the first two equations gives $-y-x = 0$, so $x = -y$ and we have the points $(\pm 2, \mp 2)$. There is a maximum value of 4 at $(2, -2)$ and a minimum value of -4 at $(-2, 2)$.

15.8.13 We have $\nabla f = e^{xy}\langle y, x\rangle$ and $\nabla g = \langle 2x+y, 2y+x\rangle$. Then the conditions are $ye^{xy} = \lambda(2x+y)$, $xe^{xy} = \lambda(2y+x)$, and $x^2+xy+y^2 = 9$. Subtracting the second from the first gives $(y-x)e^{xy} = \lambda(x-y)$, so either $y = x$ or $\lambda = -e^{xy}$. The former leads to $x = \pm\sqrt{3}$, $y = \pm\sqrt{3}$ while the second leads to $x = -y$, which in turn leads to $x = \pm 3$, $y = \mp 3$. We have an absolute maximum of e^3 at $(\sqrt{3}, \sqrt{3})$ and $(-\sqrt{3}, -\sqrt{3})$ and an absolute minimum of e^{-9} at $(-3, 3)$ and at $(3, -3)$.

15.8.14 We have $\nabla f = \left\langle 2xy, x^2\right\rangle$ and $\nabla g = \langle 2x, 2y\rangle$ so our conditions are $2xy = 2\lambda x$ and $x^2 = 2\lambda y$ and $x^2+y^2 = 9$. The first equation tells us that either $x = 0$ or $y = \lambda$, from which we can deduce critical points of $(0, \pm 3)$ or in the second case, we have $x^2 = 2\lambda^2$, so $y = \lambda$, and we can deduce that $\lambda = \pm\sqrt{3}$ and then $x = \pm\sqrt{6}$ and in each case, $y = \pm\sqrt{3}$. So the complete set of points to check includes $(0, \pm 3)$, $(\pm\sqrt{6}, \sqrt{3})$, $(\pm\sqrt{6}, -\sqrt{3})$. We obtain a minimum of $-6\sqrt{3}$ at $(\pm\sqrt{6}, -\sqrt{3})$ and a maximum of $6\sqrt{3}$ at $(\pm\sqrt{6}, \sqrt{3})$.

15.8.15 We have $\nabla f = \langle 4x, 2y\rangle$ and $\nabla g = \langle 2x, 2+2y\rangle$ so our conditions are $4x = 2\lambda x$ and $2y = \lambda(2y+2)$ and $x^2+2y+y^2 = 15$. The first equation tells us that either $x = 0$ or $\lambda = 2$. If $x = 0$ we can deduce that $y = -5$ or $y = 3$. From $\lambda = 2$ we can deduce that $2y = 4y+4$, so $y = -2$ and $x = \pm\sqrt{15}$. We obtain a minimum of 9 at $(0, 3)$ and a maximum of 34 at $(\pm\sqrt{15}, -2)$.

15.8.16 We have $\nabla f = \langle 2x, 0\rangle$ and $\nabla g = \langle 2x+y, x+2y\rangle$, so our conditions become $2x = \lambda(2x+y)$ and $0 = \lambda(2y+x)$. and $x^2+xy+y^2 = 3$. From the second equation, we can deduce that either $\lambda = 0$ (which leads to $x = 0$ and $y = \pm\sqrt{3}$), or $x = -2y$ (from which we can deduce that $-4y = \lambda(-3y)$, so either $y = 0$ which leads to a contradiction or $\lambda = 4/3$ and $y = \pm 1$). The latter case results in the critical points $(2, -1)$ and $(-2, 1)$. The absolute minimum is 0 at $(0, \sqrt{3})$ and $(0, -\sqrt{3})$, and the absolute maximum is 4 at $(2, -1)$ and $(-2, 1)$.

15.8.17 We have $\nabla f = \langle 1, 3, -1\rangle$, $\nabla g = \langle 2x, 2y, 2z\rangle$ so the Lagrange multiplier conditions are $1 = 2\lambda x$, $3 = 2\lambda y$, $-1 = 2\lambda z$, $x^2+y^2+z^2-4 = 0$. These equations imply x, y, $z \neq 0$, so we can eliminate λ and obtain $y = 3x$, $z = -x$. Then the constraint gives $11x^2 = 4$, so $x = \pm\dfrac{2}{\sqrt{11}}$ and the solutions are the points $\pm\left(\dfrac{2}{\sqrt{11}}, \dfrac{6}{\sqrt{11}}, -\dfrac{2}{\sqrt{11}}\right)$. We compute $f\left(\dfrac{2}{\sqrt{11}}, \dfrac{6}{\sqrt{11}}, -\dfrac{2}{\sqrt{11}}\right) = 2\sqrt{11}$, $f\left(-\dfrac{2}{\sqrt{11}}, -\dfrac{6}{\sqrt{11}}, \dfrac{2}{\sqrt{11}}\right) = -2\sqrt{11}$, and hence the minimum and maximum values of f on the closed bounded set given by $x^2+y^2+z^2 = 4$ are $-2\sqrt{11}$ and $2\sqrt{11}$.

15.8.18 We have $\nabla f = \langle yz, xz, xy\rangle$, $\nabla g = \langle 2x, 4y, 8z\rangle$ so the Lagrange multiplier conditions are $yz = 2\lambda x$, $xz = 4\lambda y$, $xy = 8\lambda z$, $x^2+2y^2+4z^2-9 = 0$. Assume first that x, y, $z \neq 0$, so we can eliminate λ and obtain $\dfrac{xyz}{2\lambda} = x^2 = 2y^2 = 4z^2$. Then using the constraint we obtain $3x^2 = 9$ so $x = \pm\sqrt{3}$, $y = \pm\dfrac{\sqrt{3}}{\sqrt{2}}$ and $z = \pm\dfrac{\sqrt{3}}{2}$. The value of xyz at any of these points is $\pm\sqrt{3}\cdot\dfrac{\sqrt{3}}{\sqrt{2}}\cdot\dfrac{\sqrt{3}}{2} = \pm\dfrac{3\sqrt{6}}{4}$. We also observe that if any of x, y, $z = 0$ then $xyz = 0$. Hence, the minimum and maximum values of f on the closed bounded set given by $x^2+2y^2+4z^2 = 9$ are $\pm\dfrac{3\sqrt{6}}{4}$.

15.8.19 We have $\nabla f = \langle 1, 0, 0\rangle$ and $\nabla g = \langle 2x, 2y, 2z-1\rangle$. The Lagrange multiplier conditions are $1 = 2x\lambda$, $0 = 2y\lambda$, $0 = (2z-1)\lambda$, and $x^2+y^2+z^2-z = 1$. Clearly $\lambda \neq 0$, so we must have $y = 0$ and $z = 1/2$. Solving the constraint for x gives $x = \pm\sqrt{5}/2$. There is a maximum of $\sqrt{5}/2$ at $(\sqrt{5}/2, 0, 1/2)$ and a minimum of $-\sqrt{5}/2$ at $(-\sqrt{5}/2, 0, 1/2)$.

15.8.20 We have $\nabla f = \langle 1, 0, -1\rangle$ and $\nabla g = \langle 2x, 2y-1, 2z\rangle$. The Lagrange multiplier conditions are $1 = 2x\lambda$, $0 = (2y-1)\lambda$, and $-1 = 2z\lambda$. Clearly $\lambda \neq 0$, so we must have $y = 1/2$. Also, $\frac{1}{x} = \frac{-1}{z} = 2\lambda$, so $x = -z$. The constraint equation then gives $x = \pm 3/(2\sqrt{2})$ and $z = \mp 3/(2\sqrt{2})$. There is a minimum of $-3/\sqrt{2}$ at $(-3/(2\sqrt{2}), 1/2, 3/(2\sqrt{2}))$ and a maximum of $3/\sqrt{2}$ at $(3/(2\sqrt{2}), 1/2, -3/(2\sqrt{2}))$.

15.8.21 We have $\nabla f = \langle 1,1,1\rangle$ and $\nabla g = \langle 2x-y, 2y-x, 2z\rangle$. The Lagrange multiplier conditions are $1 = \lambda(2x-y)$, $1 = \lambda(2y-x)$, and $1 = 2\lambda z$ and $x^2+y^2+z^2-xy = 5$. Combining the first two equations gives us that $x = y$, from which we can deduce that $x = y = 1/\lambda$ and $z = 1/(2\lambda)$. Then the last equation gives us $5/(4\lambda^2) = 5$, so $\lambda = \pm 1/2$. Then we can deduce that $x = y = 2$ and $z = 1$, or $x = y = -2$, and $z = -1$. There is an absolute minimum of -5 at $(-2,-2,-1)$ and an absolute maximum of 5 at $(2,2,1)$.

15.8.22 We have $\nabla f = \langle 1,1,1\rangle$ and $\nabla g = \langle 2x-2, 2y-2, 2z\rangle$. The Lagrange multiplier conditions are $1 = \lambda(2x-2)$, $1 = \lambda(2y-2)$, $1 = 2\lambda z$, and $x^2+y^2+z^2-2x-2y = 1$. The first two equations imply that $x = y$ and the third together with the first two implies that $x = y = z+1$. The constraint then implies that $2x^2+(x-1)^2-4x = 1$, or $3x^2-6x = 0$. So $x = 0$ or $x = 2$. There is a maximum of 5 at $(2,2,1)$ and a minimum of -1 at $(0,0,-1)$.

15.8.23 $\nabla f = \langle 2,0,2z\rangle$ and $\nabla g = \langle 2x,2y,4z\rangle$. The Lagrange multiplier conditions are $2 = 2\lambda x$, $0 = 2\lambda y$, $2z = 4\lambda z$, and $x^2+y^2+2z^2 = 25$. Note that $\lambda \neq 0$ so $y = 0$ and if $z \neq 0$ then $\lambda = 1/2$ so $x = 2$ and we have the point $(2,0,\sqrt{21/2})$. If $z = 0$, then we have $x = \pm 5$. The minimum is -10 at $(-5,0,0)$ and the maximum is 14.5 at $(2,0,\sqrt{21/2})$.

15.8.24 We have $\nabla f = \langle y,x,-1\rangle$ and $\nabla g = \langle 2x-y, 2y-x, 2z\rangle$. The Lagrange multiplier conditions are $y = \lambda(2x-y)$, $x = \lambda(2y-x)$, $-1 = 2\lambda z$, and $x^2+y^2+z^2-xy = 1$. Solving the first two equations for λ gives

$$\lambda = \frac{y}{2x-y} = \frac{x}{2y-x},$$

so $2y^2-xy = 2x^2-xy$, and $x = \pm y$. If $x = y$ we obtain the points $(0,0,\pm1)$, or (if $\lambda = 1$) we obtain $z = -1/2$ and then $x = y = \pm\sqrt{3}/2$. In the case where $x = -y$, we have $-y = \lambda(2y+y)$, so if $y \neq 0$, we have $\lambda = -1/3$, which leads to $z = 3/2$ and no solutions for the last constraint equation. Therefore our only critical points are $(0,0,\pm1)$ and $(\pm\sqrt{3}/2, \pm\sqrt{3}/2, -1/2)$. There is an absolute minimum of -1 at $(0,0,1)$ and an absolute maximum of $5/4$ at both $(\sqrt{3}/2,\sqrt{3}/2,-1/2)$ and $(-\sqrt{3}/2,-\sqrt{3}/2,-1/2)$.

15.8.25 We have $\nabla f = \langle 2x,1,1\rangle$ and $\nabla g = \langle 4x,4y,2z\rangle$, so the Lagrange multiplier conditions are $2x = 4\lambda x$, $1 = 4\lambda y$, $1 = 2\lambda z$, and $2x^2+2y^2+z^2 = 2$. The second and third equations tell us that $z = 2y$. The first equation tells us that either $x = 0$ or $\lambda = \frac{1}{2}$. If $\lambda = \frac{1}{2}$, we can deduce that $y = \frac{1}{2}$ and $z = 1$. If $x = 0$, we can deduce from the fact that $z = 2y$ that $y = \pm 1/\sqrt{3}$ and $z = \pm 2/\sqrt{3}$. The critical points are $(0, 1/\sqrt{3}, 2/\sqrt{3})$, $(0,-1/\sqrt{3},2/\sqrt{3})$ and $(1/2,1/2,1)$ and $(-1/2,1/2,1)$. The absolute minimum is $-\sqrt{3}$ at $(0,-1/\sqrt{3},-2/\sqrt{3})$ and the absolute maximum is $\frac{7}{4}$ at both $(1/2,1/2,1)$ and $(-1/2,1/2,1)$.

15.8.26 Observe first that it is equivalent to find the extreme values of the simpler function $h(x,y,z) = f(x,y,z)^2 = xyz$ subject to the constraints $g(x,y,z) = x+y+z-1 = 0$ and x, y, $z \geq 0$. We have $\nabla h = \langle yz,xz,xy\rangle$, $\nabla g = \langle 1,1,1\rangle$, so the Lagrange multiplier conditions are $yz = \lambda$, $xz = \lambda$, $xy = \lambda$, $x+y+z-1 = 0$. Assume first that x, y, $z \neq 0$; then first three equations give $x = y = z$, and the constraint implies $3x = 1$, so x, y, $z = \dfrac{1}{3}$. The domain of f (or g) is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, which is closed and bounded. We also note that $f = h = 0$ along any of the edges of the triangle. Therefore, the maximum value of f is $f\left(\dfrac{1}{3},\dfrac{1}{3},\dfrac{1}{3}\right) = \dfrac{1}{3\sqrt{3}}$ and the minimum value is 0.

15.8.27 Let x, y, $z \geq 0$ denote the lengths of the sides of the box, with z the longest side. Then the length plus girth of the box is $2x+2y+z$ so we must maximize the volume $f(x,y,z) = xyz$ subject to the constraint $g(x,y,z) = 2x+2y+z-108 = 0$. We have $\nabla f = \langle yz,xz,xy\rangle$, $\nabla g = \langle 2,2,1\rangle$ so the Lagrange multiplier conditions are $yz = 2\lambda$, $xz = 2\lambda$, $xy = \lambda$, $2x+2y+z-108 = 0$. Assume that x, y, $z > 0$ (otherwise the volume is 0); then the first three equations give $\dfrac{xyz}{2\lambda} = x = y = \dfrac{z}{2}$, and the constraint gives $6x = 108$, so $x = 18$ and the box has dimensions 18 in×18 in×36 in. The domain of f is the triangle with vertices $(54,0,0)$, $(0,54,0)$, $(0,0,108)$, which is closed and bounded. We also note that $f = 0$ along any of the edges of the triangle. Therefore, the maximum value of f occurs at the point we found, and the minimum value is 0.

15.8.28 Let x, y, $z > 0$ denote the lengths of the sides of the box; then its surface area is given by $f(x,y,z) = 2xy + 2yz + 2xz$, which we must minimize subject to the constraint $g(x,y,z) = xyz - 16 = 0$. We have $\nabla f = \langle 2(y+z), 2(x+z), 2(x+y)\rangle$, $\nabla g = \langle yz, xz, xy\rangle$ so the Lagrange multiplier conditions are $2(y+z) = \lambda yz$, $2(x+z) = \lambda xz$, $2(x+y) = \lambda xy$, $xyz - 16 = 0$. The first three equations give $\frac{\lambda}{2} = \frac{1}{y} + \frac{1}{z} = \frac{1}{x} + \frac{1}{z} = \frac{1}{x} + \frac{1}{y}$ which implies $x = y = z$; the constraint gives $x^3 = 16$, so $x = y = z = 2\sqrt[3]{2}$,and the box is a cube with side length $2\sqrt[3]{2}$ ft. The domain of f is the surface in the first quadrant given by $xyz = 16$, which is unbounded. We can rewrite $f(x,y,z) = 2xy + 2yz + 2xz = 2xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 32\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$ on this surface. Now if, say, $x \to \infty$ on this surface either $y \to 0$ or $z \to 0$ (or both) because of the relation $xyz = 16$; therefore, $f(x,y,z) \to \infty$. This shows that the point we found above must give the absolute minimum value of f on the surface.

15.8.29 It suffices to find the extreme values of the function $f(x,y) = x^2 + y^2$ subject to the constraint $g(x,y) = x^2 + xy + 2y^2 - 1 = 0$. We have $\nabla f = \langle 2x, 2y\rangle$, $\nabla g = \langle 2x + y, 4y + x\rangle$, so the Lagrange multiplier conditions are $2x = \lambda(2x + y)$, $2y = \lambda(4y + x)$, $x^2 + xy + 2y^2 - 1 = 0$. The first two equations give $2x(4y + x) = \lambda(2x + y)(4y + x) = 2y(2x + y) \Longrightarrow x^2 + 4xy = y^2 + 2xy$, or $x^2 + 2xy - y^2$. This implies that both x, $y \neq 0$, for if, say, $x = 0$ then this condition gives $y = 0$ as well, which violates the constraint (same argument for $y = 0$). Let $r = \frac{y}{x}$, and rewrite this equation as $\frac{y^2}{x^2} - \frac{2y}{x} - 1 = r^2 - 2r - 1 = 0$, which we can solve to obtain $r = 1 \pm \sqrt{2}$. We now use the constraint and the relation $y = rx$ to obtain $x^2\left(1 + r + 2r^2\right) = 1$or $x^2 = \frac{1}{2r^2 + r + 1}$. Then the values of the function f are given by $f(x,y) = f(x,rx) = x^2\left(1 + r^2\right) = \frac{1 + r^2}{2r^2 + r + 1} = \frac{6 \pm 2\sqrt{2}}{7} \approx 0.4531, 1.2612$, and the corresponding minimum and maximum distances are $\sqrt{\frac{6 \pm 2\sqrt{2}}{7}} \approx 0.6731, 1.1230$.

15.8.30 Let (x,y) be the vertex of the rectangle in the first quadrant; then the area of the rectangle is $4xy$, so it suffices to maximize the function $f(x,y) = xy$ subject to the constraint $g(x,y) = x^2 + 4y^2 - 4 = 0$ and x, $y \geq 0$. We have $\nabla f = \langle y, x\rangle$, $\nabla g = \langle 2x, 8y\rangle$ so the Lagrange multiplier conditions are $y = 2\lambda x$, $x = 8\lambda y$, $x^2 + 4y^2 - 4 = 0$. The first two equations give $4y^2 = 8\lambda xy = x^2$, and substituting in the constraint gives $2x^2 = 4$ so $x = \sqrt{2}$ and $y = \frac{\sqrt{2}}{2}$. The domain given by the constraint and x, $y \geq 0$ is a closed and bounded arc, and we observe that $f = 0$ at the boundary points $(2,0)$ and $(1,0)$. Therefore, f takes its maximum at $\left(\sqrt{2}, \frac{\sqrt{2}}{2}\right)$, and the corresponding area of the rectangle is $A = 4 \cdot \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 4$.

15.8.31 Let (x,y) be the vertex of the rectangle in the first quadrant; then the perimeter of the rectangle is $4(x+y)$, so it suffices to maximize the function $f(x,y) = x + y$ subject to the constraint $g(x,y) = 2x^2 + 4y^2 - 3 = 0$ and x, $y \geq 0$. We have $\nabla f = \langle 1, 1\rangle$, $\nabla g = \langle 4x, 8y\rangle$, so the Lagrange multiplier conditions are $1 = 4\lambda x$, $1 = 8\lambda y$, $2x^2 + 4y^2 - 3 = 0$. The first two equations give $4x = \frac{1}{\lambda} = 8y$; so $x = 2y$. Substituting in the constraint gives $8y^2 + 4y^2 = 3$, so $y^2 = 1/4$ and $y = 1/2$ and thus $x = 1$. Note that $f(1, 1/2) = 1.5$. The domain given by the constraint and x, $y \geq 0$ is a closed and bounded arc, and we observe that at the boundary points we have $f(0, \sqrt{3}/2) = \sqrt{3}/2 \approx .87$ and $f(\sqrt{3/2}, 0) = \sqrt{3/2} \approx 1.25$. Thus, the dimensions of the rectangle of maximum perimeter is 2 × 1.

15.8.32 It suffices to maximize the function $f(x,y,z) = (x+2)^2 + (y-5)^2 + (z-1)^2$ subject to the constraint $g(x,y,z) = 2x + 3y + 6z - 10 = 0$. We have $\nabla f = \langle 2x + 4, 2y - 10, 2z - 2\rangle$, $\nabla g = \langle 2, 3, 6\rangle$, so the Lagrange multiplier conditions are $2x + 4 = 2\lambda$, $2y - 10 = 3\lambda$, $2z - 2 = 6\lambda$, $2x + 3y + 6z - 10 = 0$. The first three equations give $x + 2 = \lambda = \frac{2y - 10}{3} = \frac{z - 1}{3}$, which gives $y = \frac{3}{2}x + 8$, $z = 3x + 7$. Substituting in the constraint and solving for x gives $x = -\frac{16}{7}$, so the closest point is $\left(-\frac{16}{7}, \frac{32}{7}, \frac{1}{7}\right)$; $f\left(-\frac{16}{7}, \frac{32}{7}, \frac{1}{7}\right) = 1$ so the distance is 1.

15.8.33 It suffices to minimize the function $f(x, y, z) = (x-1)^2 + (y-2)^2 + (z+3)^2$ subject to the constraint $4x + y - 1 = 0$. We have $\nabla f = \langle 2(x-1), 2(y-2), 2(z+3)\rangle$ and $\nabla g = \langle 4, 1, 0\rangle$. Then the Lagrange multiplier conditions are $2x - 2 = 4\lambda$, $2y - 4 = \lambda$, $2z + 6 = 0$, $4x + y - 1 = 0$. Solving this system gives $x = -3/17$, $y = 29/17$, $z = -3$, and $\lambda = -10/17$, so the closest point on the surface to the given point is $(-3/17, 29/17, -3)$.

15.8.34 It suffices to minimize the function $f(x, y, z) = (x-1)^2 + (y-2)^2 + z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 - z^2 = 0$. We have $\nabla f = \langle 2x-2, 2y-4, 2z\rangle$, $\nabla g = \langle 2x, 2y, -2z\rangle$ so the Lagrange multiplier conditions are $2x - 2 = 2\lambda x$, $2y - 4 = 2\lambda y, 2z = -2\lambda z$, $x^2 + y^2 - z^2 = 0$. Observe that $z \neq 0$; otherwise $x = y = z = 0$, which is not a solution to the Lagrange multiplier equations. Therefore, the third equation gives $\lambda = -1$, the first two equations give $x = \frac{1}{2}$, $y = 1$ and the constraint gives $z = \pm\frac{\sqrt{5}}{2}$. Therefore, the points on the cone closest to $(1, 2, 0)$ is $\left(\frac{1}{2}, 1, \pm\frac{\sqrt{5}}{2}\right)$.

15.8.35 It suffices to find the extreme values the function $f(x, y, z) = (x-2)^2 + (y-3)^2 + (z-4)^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$. We have $\nabla f = \langle 2x-4, 2y-6, 2z-8\rangle$, $\nabla g = \langle 2x, 2y, 2z\rangle$ so the Lagrange multiplier conditions are $2x-4 = 2\lambda x$, $2y-6 = 2\lambda y$, $2z-8 = 2\lambda z$, $x^2+y^2+z^2-9 = 0$. We can write the first three equations in the form $(1-\lambda)\langle x, y, z\rangle = \langle 2, 3, 4\rangle$ so $\langle x, y, z\rangle = c\langle 2, 3, 4\rangle$ for some scalar c; using the constraint, we find that $c = \pm\frac{3}{\sqrt{29}}$, and hence $\langle x, y, z\rangle = \pm\frac{3}{\sqrt{29}}\langle 2, 3, 4\rangle$; the corresponding values of f are $f\left(\pm\left(\frac{6}{\sqrt{29}}, \frac{9}{\sqrt{29}}, \frac{12}{\sqrt{29}}\right)\right) = 38 \mp 6\sqrt{29} = \left(\sqrt{29} \mp 3\right)^2$, so the minimum distance is $\sqrt{29} - 3$ and the maximum distance is $\sqrt{29} + 3$.

15.8.36 Assume the cylinder is the region $x^2 + y^2 \leq r^2$, $-\frac{h}{2} \leq z \leq \frac{h}{2}$; then r and h are the radius and height respectively of the cylinder, and the points on the cylinder furthest from the origin have distance $\left(r^2 + \left(\frac{h}{2}\right)^2\right)^{1/2}$; therefore, we must maximize the function $V = f(r, h) = \pi r^2 h$ subject to the constrain $g(r, h) = r^2 + \left(\frac{h}{2}\right)^2 - 16^2 = 0$. We have $\nabla f = \pi\langle 2rh, r^2\rangle$, $\nabla g = \left\langle 2r, \frac{h}{2}\right\rangle$, so the Lagrange multiplier conditions are equivalent to $2rh = 2\lambda r$, $r^2 = \lambda\frac{h}{2}$, $r^2 + \left(\frac{h}{2}\right)^2 - 16^2 = 0$. Because we are interested in maximizing the volume we may assume that r, $h > 0$; then the first equation gives $\lambda = h$ and the second equation gives $h^2 = 2r^2$; substituting in the constraint then gives dimensions $r = \frac{16\sqrt{6}}{3}$, $h = \frac{32\sqrt{3}}{3}$.

15.8.37 Notice that the constraint is equivalent to $\ell + 2g = 6$. We have $\nabla U = 5\langle \ell^{-1/2}g^{1/2}, \ell^{1/2}g^{-1/2}\rangle$ so the Lagrange multiplier conditions are equivalent to $\ell^{-1/2}g^{1/2} = \lambda$, $\ell^{1/2}g^{-1/2} = 2\lambda$, $\ell + 2g = 6$. Eliminating λ from the first two equations gives $\ell^{1/2}g^{-1/2} = 2\ell^{-1/2}g^{1/2}$, which simplifies to $g = \frac{\ell}{2}$; substituting in the constraint then gives $\ell = 3$ and $g = \frac{3}{2}$. The value of the utility function at this point is $U = 15\sqrt{2}$.

15.8.38 Notice that the constraint is equivalent to $2\ell + g = 6$. We have $\nabla U = \frac{32}{3}\langle 2\ell^{-1/3}g^{1/3}, \ell^{2/3}g^{-2/3}\rangle$, so the Lagrange multiplier conditions are equivalent to $2\ell^{-1/3}g^{1/3} = 2\lambda$, $\ell^{2/3}g^{-2/3} = \lambda$, $2\ell + g = 6$. Eliminating λ from the first two equations gives $\ell^{-1/3}g^{1/3} = \ell^{2/3}g^{-2/3}$, which simplifies to $g = \ell$; substituting in the constraint then gives $\ell = g = 2$. The value of the utility function at this point is $U = 64$.

15.8.39 Notice that the constraint is equivalent to $5\ell + 4g = 20$. We have $\nabla U = \frac{8}{5}\langle 4\ell^{-1/5}g^{1/5}, \ell^{4/5}g^{-4/5}\rangle$, so the Lagrange multiplier conditions are equivalent to $4\ell^{-1/5}g^{1/5} = 5\lambda$, $\ell^{4/5}g^{-4/5} = 4\lambda$, $5\ell + 4g = 20$. Eliminating λ from the first two equations gives $16\ell^{-1/5}g^{1/5} = 5\ell^{4/5}g^{-4/5}$, which simplifies to $g = \frac{5\ell}{16}$;

substituting in the constraint then gives $\ell = \dfrac{16}{5}$, $g = 1$. The value of the utility function at this point is $U = 8 \cdot \left(\dfrac{16}{5}\right)^{4/5} \approx 20.287$.

15.8.40 We have $\nabla U = \dfrac{1}{6}\langle \ell^{-5/6}g^{5/6}, 5\ell^{1/6}g^{-1/6}\rangle$, so the Lagrange multiplier conditions are equivalent to $\ell^{-5/6}g^{5/6} = 4\lambda$, $5\ell^{1/6}g^{-1/6} = 5\lambda$, $4\ell + 5g = 20$. Eliminating from the first two equations gives $\ell^{-5/6}g^{5/6} = 4\ell^{1/6}g^{-1/6}$, which simplifies to $g = 4\ell$; substituting in the constraint then gives $\ell = \dfrac{5}{6}$, $g = \dfrac{10}{3}$. The value of the utility function at this point is $U = \left(\dfrac{5}{6}\right)^{1/6}\left(\dfrac{10}{3}\right)^{5/6} \approx 2.646$.

15.8.41

a. True. This is because the tangent plane to a sphere at any point has normal vector in the direction of the line joining the point to the center of the sphere.

b. False in general. In fact, the two vectors ∇f and ∇g are in the same direction, so $\nabla f \cdot \nabla g = 0$ only if one of these vectors is zero.

15.8.42 Let x, $y \geq 0$ be the dimensions of the base of the box and h be the height; then x, y, h satisfy the constraint $2x + 2y + h = 96$. The volume of the box is $V = xyh$, so the Lagrange multiplier conditions are $yh = 2\lambda$, $xh = 2\lambda$, $xy = \lambda$, $2x + 2y + h = 96$. Eliminating λ from the first three equations gives $y = x$ and $h = 2x$; substituting in the constraint then gives $x = 16$, $y = 16$, $h = 32$ in.

15.8.43 Let x, $y > 0$ be the dimensions of the base and $h > 0$ be the height of the box; then the four sides of the box plus the base have total area $2xh + 2yh + xy = 2$, which is our constraint. The volume of the box is $V = xyh$, so the Lagrange multiplier conditions are $yh = \lambda(2h + y)$, $xh = \lambda(2h + x)$, $xy = 2\lambda(x + y)$, $2xh + 2yh + xy = 2$. The first two equations give $\dfrac{2h + y}{yh} = \dfrac{2h + x}{xh} \Longrightarrow \dfrac{2}{y} + \dfrac{1}{h} = \dfrac{2}{x} + \dfrac{1}{h}$, so $y = x$. The second and third equations give $\dfrac{2h + x}{xh} = \dfrac{2x + 2y}{xy} \Longrightarrow \dfrac{2}{x} + \dfrac{1}{h} = \dfrac{2}{x} + \dfrac{2}{y} = \dfrac{4}{x}$, so $h = \dfrac{x}{2}$. Then substituting in the constraint gives $3x^2 = 2$, so $x = \dfrac{\sqrt{6}}{3}$. Therefore, the box with largest volume has height $\dfrac{\sqrt{6}}{6}$ m and base $\dfrac{\sqrt{6}}{3} \times \dfrac{\sqrt{6}}{3}$ m.

15.8.44 Let x, $y > 0$ be the dimensions of the base and $h > 0$ be the height of the box; then the volume of the box is $xyh = 4$, which is our constraint. The four sides of the box plus the base have total area $A = 2xh + 2yh + xy$, so the Lagrange multiplier conditions are $2h + y = \lambda yh$, $2h + x = \lambda xh$, $2x + 2y = \lambda xy$, $xyh = 4$. The first two equations give $\dfrac{2h + y}{yh} = \dfrac{2h + x}{xh} \Longrightarrow \dfrac{2}{y} + \dfrac{1}{h} = \dfrac{2}{x} + \dfrac{1}{h}$, so $y = x$. The second and third equations give $\dfrac{2h + x}{xh} = \dfrac{2x + 2y}{xy} \Longrightarrow \dfrac{2}{x} + \dfrac{1}{h} = \dfrac{2}{x} + \dfrac{2}{y} = \dfrac{4}{x}$, so $h = \dfrac{x}{2}$. Then substituting in the constraint gives $x^3 = 8$, so $x = 2$. Therefore, the box with smallest area has dimensions 2 m×2m×1m.

15.8.45 Let x, $y \geq 0$ be the dimensions of the base and $z \geq 0$ be the height of the box; then $x + 2y + 3z = 6$ is our constraint. The box has volume $V = xyz$, so the Lagrange multiplier conditions are $yz = \lambda$, $xz = 2\lambda$, $xy = 3\lambda$, $x + 2y + 3z = 6$. The first two equations give $y = \dfrac{x}{2}$, the first and third equations give $z = \dfrac{x}{3}$, and substituting in the constraint gives $x = 2$. Therefore, the box with largest volume has dimensions $x = 2$, $y = 1$, $z = \dfrac{2}{3}$.

15.8.46 Let (x, y, z) be the point on the ellipsoid; then x, y, $z > 0$ and the box has volume $V = xyz$; the relation $36x^2 + 4y^2 + 9z^2 = 36$ is our constraint. The Lagrange multiplier conditions are $yz = 72\lambda x$, $xz = 8\lambda y$, $xy = 18\lambda z$, $36x^2 + 4y^2 + 9z^2 = 36$. The first two equations give $\lambda = \dfrac{yz}{72x} = \dfrac{xz}{8y}$, which implies

$y^2 = 9x^2$ or $y = 3x$. Similarly the first and third equations give $z = 2x$ and then the constraint gives $3x^2 = 1$, so $x = \dfrac{\sqrt{3}}{3}$, and the box with maximum volume has dimensions $\dfrac{\sqrt{3}}{3} \times \sqrt{3} \times \dfrac{2\sqrt{3}}{3}$.

15.8.47 The constraint is the equation of the plane $x - y + z = 2$; the distance from a point (x, y, z) to the point $(1, 1, 1)$ is given by $d^2 = (x-1)^2 + (y-1)^2 + (x-y-1)^2$, so it suffices to minimize the function $f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ subject to the constraint $x - y + z = 2$. The Lagrange multiplier conditions are $2x - 2 = \lambda$, $2y - 2 = -\lambda$, $2z - 2 = \lambda$, $x - y + z = 2$. The first two equations give $2(x+y) = 4$, so $y = 2 - x$, the first and third equations give $z = x$, and then substituting in the constraint gives $x = \dfrac{4}{3}$, so the closest point on the plane to $(1, 1, 1)$ is $\left(\dfrac{4}{3}, \dfrac{2}{3}, \dfrac{4}{3}\right)$.

15.8.48 The function $f(x, y) = x^2 + 4y^2 + 1$ has a unique critical point at $(0, 0)$, and $f(0, 0) = 1$. On the boundary $\{x^2 + 4y^2 = 1\}$ of R, $f(x, y) = 2$ at all points. Hence, the absolute minimum and maximum values of f on R are 1 and 2.

15.8.49 $\nabla f = \langle 2x, 2y-2\rangle$ and $\nabla g = \langle 2x, 2y\rangle$. Note that $\nabla f = \langle 0, 0\rangle$ at $(0, 1)$, and $f(0, 1) = 0$. Consider $\nabla f = \lambda \nabla g$. If $x = 0$ then $y = \pm 2$. If $x \neq 0$, then because $2x = 2\lambda x$, we have $\lambda = 1$ which leads to a contradiction with the equation $2y - 2 = 2\lambda y$. So we need to check $(0, \pm 2)$. The absolute minimum is 0 at $(0, 1)$. At $(0, -2)$ we have the absolute maximum of 9.

15.8.50 $\nabla f = \langle 4x, 2y\rangle$ and $\nabla g = \langle 2x, 2y\rangle$. If $x = 0$ then $y = \pm 4$. If $x \neq 0$, then the equation $4x = 2\lambda x$ tells us that $\lambda = 2$, in which case $2y = 4y$ so $y = 0$ and $x = \pm 4$. So we must check $(0, \pm 4)$ and $(\pm 4, 0)$ on the boundary. Also, $\nabla f = \langle 0, 0\rangle$ at $(0, 0)$. There is an absolute minimum of 0 at $(0, 0)$ and an absolute maximum of 32 at $(\pm 4, 0)$.

15.8.51 $\nabla f = \langle 4x - 4, 6y\rangle$, which is equal to $\langle 0, 0\rangle$ at $(1, 0)$. $\nabla g = \langle 2x - 2, 2y\rangle$. We have $6y = 2\lambda y$, so either $y = 0$ (which means either $x = 0$ or $x = 2$), or $\lambda = 3$. If $\lambda = 3$ then $4x - 4 = 6(x-1)$, so $2 = 2x$ and $x = 1$ (and therefore $y = \pm 1$.) We must check $(1, 0)$, $(2, 0)$, $(0, 0)$, $(1, 1)$ and $(1, -1)$. There is an absolute minimum of 0 at $(1, 0)$ and an absolute maximum of 3 at $(1, \pm 1)$.

15.8.52

a. It suffices to maximize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = x^4 + y^4 + z^4 = 1$. The Lagrange multiplier conditions give $2x = 4\lambda x^3$, $2y = 4\lambda y^3$, $2z = 4\lambda z^3$, $x^4 + y^4 + z^4 = 1$. Assume first that x, y, $z \neq 0$; then we can eliminate λ and conclude that $x^2 = y^2 = z^2$; the constraint gives $3x^4 = 1$ so $x^2 = \dfrac{1}{\sqrt{3}}$ and $f(x, y, z) = \sqrt{3}$. There are also solutions to the Lagrange conditions with x, y or $z = 0$, but one can check that these do not give larger values of f. Therefore, the extreme points have coordinates x, y, $z = \pm 3^{-1/4}$.

b. It suffices to maximize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = x^{2n} + y^{2n} + z^{2n} = 1$. The Lagrange multiplier conditions give $2x = 2\lambda n x^{2n-1}$, $2y = 2\lambda n y^{2n-1}$, $2z = 2\lambda n z^{2n-1}$, $x^{2n} + y^{2n} + z^{2n} = 1$. As above, assume that x, y, $z \neq 0$; then we can eliminate λ and conclude that $x^{2n-2} = y^{2n-2} = z^{2n-2}$, which implies $x^{2n} = y^{2n} = z^{2n}$; the constraint gives $3x^{2n} = 1$ so $x^2 = 3^{-1/n} \implies f(x, y, z) = 3^{\frac{n-1}{n}}$. There are also solutions to the Lagrange conditions with x, y or $z = 0$, but one can check that these do not give larger values of f. Therefore, the extreme points have coordinates x, y, $z = \pm 3^{-\frac{1}{2n}}$.

c. As $n \to \infty$, the coordinates of the extreme points x, y, $z \to \pm 1$, and the limiting distance of these points to the origin is $\sqrt{3}$.

15.8.53 Notice that the constraint is equivalent to $2K + 3L = 30$. We have

$$\nabla f = \frac{1}{2}\langle K^{-1/2}L^{1/2}, K^{1/2}L^{-1/2}\rangle,$$

so the Lagrange multiplier conditions are equivalent to $K^{-1/2}L^{1/2} = 2\lambda$, $K^{1/2}L^{-1/2} = 3\lambda$, $2K + 3L = 30$. Eliminating λ from the first two equations gives $3K^{-1/2}L^{1/2} = 2K^{1/2}L^{-1/2}$, which simplifies to $3L = 2K$; substituting in the constraint then gives $K = 7.5$ and $L = 5$. The domain over which f is to be maximized is a closed line segment; K or L is 0 at the endpoints, and hence so is f. Therefore, the values of K and L found above must maximize f.

15.8.54 Notice that the constraint is equivalent to $K + 2L = 12$. We have

$$\nabla f = \frac{10}{3}\langle K^{-2/3}L^{2/3}, 2K^{1/3}L^{-1/3}\rangle,$$

so the Lagrange multiplier conditions are equivalent to $K^{-2/3}L^{2/3} = \lambda$, $2K^{1/3}L^{-1/3} = 2\lambda$, $K + 2L = 12$. Eliminating λ from the first two equations gives $K^{-2/3}L^{2/3} = K^{1/3}L^{-1/3}$, which simplifies to $K = L$; substituting in the constraint then gives $K = L = 4$. The domain over which f is to be maximized is a closed line segment; K or L is 0 at the endpoints, and hence so is f. Therefore, the values of K and L found above must maximize f.

15.8.55 We have $\nabla f = \langle aK^{a-1}L^{1-a}, (1-a)\,K^aL^{-a}\rangle$, so the Lagrange multiplier conditions are equivalent to $aK^{a-1}L^{1-a} = \lambda p$, $(1-a)\,K^aL^{-a} = \lambda q$, $pK + qL = B$. Eliminating λ from the first two equations gives $aqK^{a-1}L^{1-a} = (1-a)\,pK^aL^{-a}$, which simplifies to $aqL = (1-a)\,pK$; substituting in the constraint then gives $K = \dfrac{aB}{p}$ and $L = \dfrac{(1-a)\,B}{q}$. The domain over which f is to be maximized is a closed line segment; K or L is 0 at the endpoints, and hence so is f. Therefore, the values of K and L found above must maximize f.

15.8.56 We have $\nabla T = \langle 50x, 50y\rangle$ and $\nabla g = \langle 2x + y, 2y + x\rangle$ where $g(x) = x^2 + y^2 + xy = 1$. The Lagrange multiplier conditions are $50x = \lambda(2x + y)$, $50y = 2y + x$, and $x^2 + y^2 + xy = 1$. Multiplying the first equation by y and the second by x and subtracting gives $0 = y^2 - x^2$, so $x = \pm y$. If $x = y$, then the constraint gives $3x^2 = 1$, so $x = \pm\frac{1}{\sqrt{3}}$. Note that $T(\pm(1/\sqrt{3}), \pm(1/\sqrt{3})) = 50/3$. If $x = -y$, the constraint gives $x = \pm 1$, and note that $T(1, -1) = T(-1, 1) = 50$. The hottest temperature on the edge of the plate is 50 and the coldest is $\frac{50}{3}$.

15.8.57 The function to be maximized is $f(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + x_4$, subject to the constraint $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$. The Lagrange multiplier conditions are $1 = 2\lambda x_1$, $1 = 2\lambda x_2$, $1 = 2\lambda x_3$, $1 = 2\lambda x_4$, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$. The first four equations give $x_1 = x_2 = x_3 = x_4$, and then the constraint gives $4x_1^2 = 16$, so $x_1 = \pm 2$. Therefore, the maximum of f on the closed, bounded set given by $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$ is $f(2, 2, 2, 2) = 8$ (and the minimum is $f(-2, -2, -2, -2) = -8$).

15.8.58 The function to be maximized is $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$, subject to the constraint $x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$. The Lagrange multiplier conditions are $1 = 2\lambda x_1$, $1 = 2\lambda x_2$, …, $1 = 2\lambda x_n$, $x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$. The first n equations give $x_1 = x_2 = \cdots = x_n$, and then the constraint gives $nx_1^2 = c^2$, so $x_1 = \pm\dfrac{c}{\sqrt{n}}$. Therefore, the maximum of f on the closed, bounded set given by $x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$ is $f\left(\dfrac{c}{\sqrt{n}}, \ldots, \dfrac{c}{\sqrt{n}}\right) = c\sqrt{n}$ (and the minimum is $-c\sqrt{n}$).

15.8.59 The function to be maximized is $f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$, subject to the constraint $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. The Lagrange multiplier conditions are $a_1 = 2\lambda x_1$, $a_2 = 2\lambda x_2$, …, $a_n = 2\lambda x_n$, $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. The first n equations are equivalent to $c\,(x_1, x_2, \ldots, x_n) = (a_1, a_2, \ldots, a_n)$ for some c, and then the constraint gives $c^2 = a_1^2 + a_2^2 + \cdots + a_n^2$. Therefore, the maximum of f on the closed, bounded set given by $f\left(\dfrac{a_1}{c}, \dfrac{a_2}{c}, \ldots, \dfrac{a_n}{c}\right) = \dfrac{1}{c}\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$ (and the minimum is $-\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$).

15.8.60

a. The function to be maximized is $f(x, y, z) = xyz$, subject to the constraints $x + y + z = k$ and x, y, $z > 0$. The Lagrange multiplier conditions are $yz = \lambda$, $xz = \lambda$, $xy = \lambda$, $x + y + z = k$; the first three equations imply $\frac{xyz}{\lambda} = x = y = z$, and the constraint gives $x = y = z = \frac{k}{3}$. The set determined by the constraints $x + y + z = k$ and x, y, $z \geq 0$ is a triangle, and $f(x, y, z) = 0$ along its edges. Therefore, the maximum of f must occur at the point we found. This means that if x, y, $z > 0$ and $x + y + z = k$, then $xyz \leq \left(\frac{k}{3}\right)^3$ if and only if $(xyz)^{1/3} \leq \frac{x + y + z}{3}$.

b. Generalizing the case above, the function to be maximized is $f(x_1, \ldots, x_n) = x_1 \cdots x_n$, subject to the constraints $x_1 + \cdots + x_n = k$ and $x_1, \ldots, x_n > 0$. The Lagrange multiplier conditions are $x_2 \cdots x_n = \lambda$, $x_1 x_3 \cdots x_n = \lambda, \ldots,$ $x_2 \cdots x_{n-1} = \lambda$, $x_1 + \cdots + x_n = k$; the first n equations imply $\frac{x_1 \cdots x_n}{\lambda} = x_1 = x_2 = \cdots = x_n$, and the constraint gives $x_1 = x_2 = \cdots = x_n = \frac{k}{n}$. The set determined by the constraints $x_1 + \cdots + x_n = k$ and $x_1, \ldots, x_n \geq 0$ is closed and bounded, and $f(x_1, \ldots, x_n) = 0$ along its boundary. Therefore, the maximum of f must occur at the point we found. This means that if $x_1, \ldots, x_n > 0$ and $x_1 + \cdots + x_n = k$, then $x_1 \cdots x_n \leq \left(\frac{k}{n}\right)^n \iff (x_1 \cdots x_n)^{1/n} \leq \frac{x_1 + \cdots + x_n}{k}$.

15.8.61

a. Gradients are perpendicular to level surfaces.

b. If ∇f was not in the plane spanned by ∇g and ∇h, then f could be increased or decreased by moving the point P slightly along the curve C.

c. Because ∇f is in the plane spanned by ∇g and ∇h, we can express ∇f as a linear combination of ∇g and ∇h.

d. The gradient condition from part (c), as well as the constraints, must be satisfied.

15.8.62 It suffices to minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $g(x, y, z) = x + 2z - 12 = 0$ and $h(x, y, z) = x + y - 6 = 0$. Using the method described in problem 61 above, we solve the equation $\nabla f = \lambda \nabla g + \mu \nabla h$, together with the constraints. This gives the conditions $2x = \lambda + \mu$, $2y = \mu$, $2z = 2\lambda$, $x + 2z = 12$, $x + y = 6$. The first three equations imply $2x = 2y + z$, and the constraints give $y = 6 - x$, $z = 6 - \frac{1}{2}x$; therefore $2x = 2(6 - x) + 6 - \frac{1}{2}x$, so $x = 4$, which gives $y = 2$, $z = 4$. There is a point on the line closest to the origin, so this point must be $(4, 2, 4)$.

15.8.63 We wish to find the extreme values of the function $f(x, y, z) = xyz$ subject to the constraints $g(x, y, z) = x^2 + y^2 - 4 = 0$ and $h(x, y, z) = x + y + z - 1 = 0$. Using the method described in problem 61 above, we solve the equation $\nabla f = \lambda \nabla g + \mu \nabla h$, together with the constraints. This gives the conditions $yz = 2\lambda x + \mu$, $xz = 2\lambda y + \mu$, $xy = \mu$, $x^2 + y^2 = 4$, $x + y + z = 1$. The first and second equations together give $z(y - x) = 2\lambda(x - y)$, which implies either $x = y$ or $z = -2\lambda$. In the former case the first constraint gives $2x^2 = 4$, so $x = y = \pm\sqrt{2}$, solving for z gives $z = 1 \mp 2\sqrt{2}$ and $f(x, y, z) = 2 \pm 4\sqrt{2}$. In the latter case we obtain $(x + y)z = \mu = xy$ from the first and third equations, and then solving for $z = 1 - x - y$ gives $(x + y)(1 - x - y) - xy = 0 \iff x + y - 3xy - 4 = 0$, using the relation $x^2 + y^2 = 4$. Therefore, $x - 4 = (3x - 1)\, y\, (x - 4)^2 = (3x - 1)^2 \left(4 - x^2\right) x^2 - 8x + 16 = \left(9x^2 - 6x + 1\right)\left(4 - x^2\right)$ which simplifies to $9x^4 - 6x^3 - 34x^2 + 16x + 12 = 0$. This equation has roots $x \approx -0.42, -1.78, 1.96, 0.91$ in the interval $(-2, 2)$; one can check that the corresponding solutions (x, y, z) to the Lagrange conditions do not give values larger or smaller resp. than $2 + 4\sqrt{2}$, $2 - 4\sqrt{2}$, so these are in fact the maximum and minimum values of f subject to the constraints.

15.8.64 We wish to find the extreme values of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $g(x, y, z) = 4x^2 + 4y^2 - z^2 = 0$ and $h(x, y, z) = 2x + 4z - 5 = 0$. Using the method described in problem 61 above, we solve the equation $\nabla f = \lambda \nabla g + \mu \nabla h$, together with the constraints. This gives the conditions

$2x = 8\lambda x + 2\mu$, $2y = 8\lambda y$, $2z = -2\lambda z + 4\mu$, $z^2 = 4x^2 + 4y^2$, $2x + 4z = 5$. The second equation gives $y(1 - 4\lambda) = 0$, so either $y = 0$ or $\lambda = \frac{1}{4}$. Consider first the case $y = 0$; then the first constraint equation gives $z = \pm 2x$. If $z = 2x$ then the second constraint equation gives $x = \frac{1}{2}$, $z = 1$ and we obtain the point $\left(\frac{1}{2}, 0, 1\right)$; similarly if $z = -2x$ then we obtain the point $\left(-\frac{5}{6}, 0, \frac{5}{3}\right)$. In the case $\lambda = \frac{1}{4}$ the first equation gives $\mu = 0$ and then the third equation gives $z = 0$; but then the first of the constraints implies that $x = y = z = 0$, which violates the second constraint. Hence there are no solutions to the Lagrange conditions in this case, and the minimum and maximum values of the function f along this curve are $f\left(\frac{1}{2}, 0, 1\right) = \frac{5}{4}$ and $f\left(-\frac{5}{6}, 0, \frac{5}{3}\right) = \frac{125}{36}$.

15.8.65

a. $\nabla f = \langle y + 1, x + 1\rangle$ and $\nabla g = \langle y, x\rangle$, so the system of equations is $y + 1 = \lambda y$, $x + 1 = \lambda x$, and $xy = 4$.

b. Subtracting the first equation from the second equation, we obtain $x - y = \lambda(x - y)$, which implies that $x - y - \lambda(x - y) = 0$, or $(x - y)(1 - \lambda) = 0$. So either $x = y$ or $\lambda = 1$. Substituting $\lambda = 1$ into either of the first two equations in part (a) results in the equality $1 = 0$, which means that there is no solution to the system of equations in this case. So it must by the case that $x = y$. Replacing y with x in the third equation in the solution to part (a), we find that $x^2 = 4$, or $x = \pm 2$. Therefore the method of Lagrange multipliers gives us the points $(x, y) = (\pm 2, \pm 2)$.

c. We have $f(2, 2) = 108$. Replacing y with $4/x$ in the function f, we find that $h_1(x) = x + \frac{4}{x} + 104$. The first and second derivatives of h_1 are $h_1'(x) = 1 - \frac{4}{x^2}$ and $h_1''(x) = \frac{8}{x^3}$. Because $h_1'(2) = 0$ and $h_1''(x) > 0$ for $x > 0$, it follows that f has an absolute minimum of $h_1(2) = f(2, 2) = 108$ along the curve C_1 and it does not have an absolute maximum value along C_1.

d. We have $f(-2, -2) = 100$. By part (c), $h_2(x) = x + 4/x + 104$, $h_2'(x) = 1 - 4/x^2$ and $h_2''(x) = 8/x^3$, for $x < 0$. Because $h_1'(-2) = 0$ and $h_2''(x) < 0$ for $x < 0$, it follows that f has an absolute maximum of $h_2(-2) = f(-2, -2) = 100$ along the curve C_2 and it does not have an absolute minimum value along C_2.

e. By parts (c) and (d), it follows that f has a local minimum of 108 at (2,2) and a local maximum of 100 at (-2,-2) over the constraint curve $xy = 4$. Because f has just two local extrema, with the local minimum being larger than the local maximum, f does not have an absolute maximum or an absolute minimum over C. Because the constraint curve is unbounded, the values of f get arbitrarily large as we move away from (2,2) along the curve C_1 and values of f get arbitrarily small as we move away from $(-2, -2)$ along the curve C_2.

Chapter Fifteen Review

1

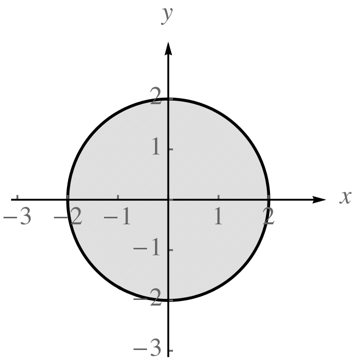
a. True. If $e^{x+y} = k$, then $x + y = \ln k$, which is the equation of a line.

b. False. If $2x^2 - 6y^2 > 0$ then $z = \sqrt{2x^2 - 6y^2}$ or $z = -\sqrt{2x^2 - 6y^2}$.

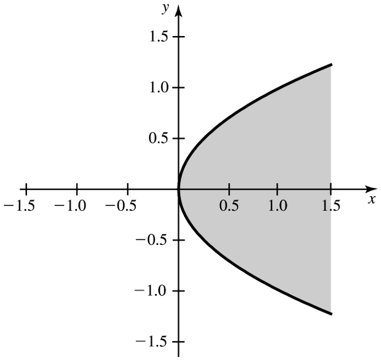
c. False. For example $f(x, y) = x^2 y$ has $f_{xxy} = 2$, $f_{xyy} = 0$.

d. False. ∇f lies in the xy-plane.

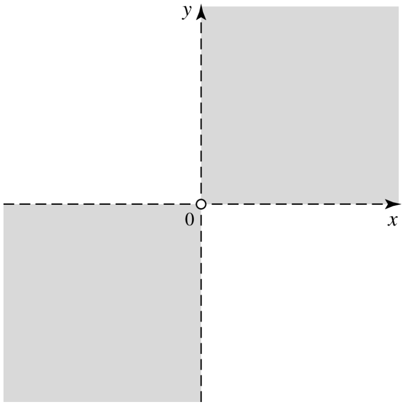
2 The domain is $D = \{(x, y) : x^2 + y^2 \le 4$.



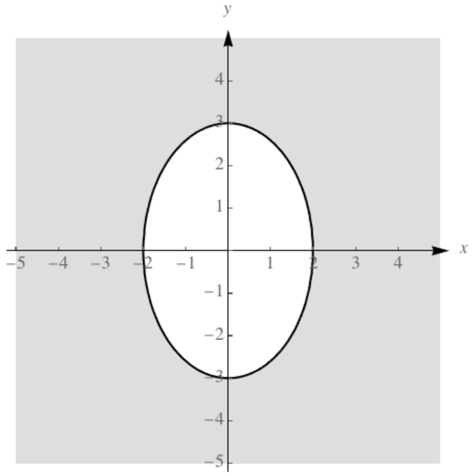
3 The domain is $D = \{(x, y) : x \geq y^2\}$.



4 The domain is $D = \{(x, y) : xy > 0\}$.



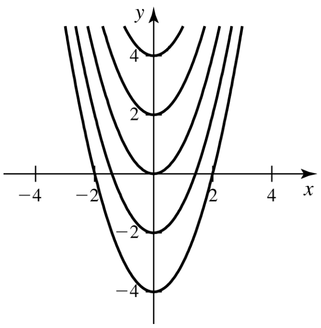
5 The domain is $D = \{(x, y) : \dfrac{x^2}{4} + \dfrac{y^2}{9} \geq 1$.



6 The domain is $\mathbb{R}$ and the range is $(-\infty, 0]$. This is the lower half of the cone $z^2 = x^2 + y^2$.

7 The domain is $D = \{(x, y) : x^2 + y^2 \geq 1\}$. The range is $(-\infty, 0]$. This is the lower half of the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$.

8



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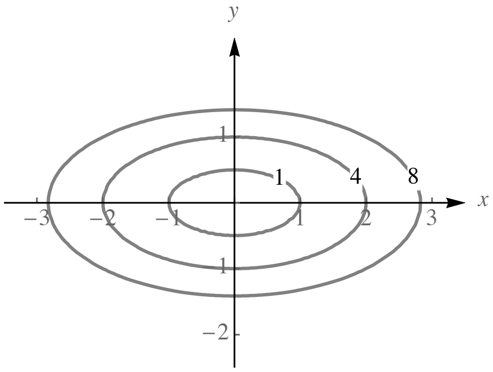


10

a. This matches A.

b. This matches C.

c. This matches D.

d. This matches B.

11 This limit may be evaluated directly by substitution: $\lim\limits_{(x,y)\to(4,-2)} (10x - 5y + 6xy) = 10 \cdot 4 - 5\,(-2) + 6 \cdot 4\,(-2) = 2$.

12 This limit may be evaluated directly by substitution: $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy}{x+y} = \dfrac{1 \cdot 1}{1+1} = \dfrac{1}{2}$.

13 Along the path $y = -x$ we have $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{0}{-x^2} = 0$, but along the path $y = x$ we have $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{2x}{x^2} = \lim\limits_{(x,y)\to(0,0)} \dfrac{2}{x} \neq 0$, so the limit does not exist.

14 Along the path $y = x$ we have $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2)}{2x^2} = \dfrac{1}{2}$, while along the line $y = -x$ we have $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(-x^2)}{2x^2} = \dfrac{-1}{2} \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2)}{x^2} = \dfrac{-1}{2}$, so the limit does not exist.

15 This limit may be evaluated by factoring the numerator and denominator and canceling the common factor $x+y$: $\lim\limits_{(x,y)\to(-1,1)} \dfrac{x^2-y^2}{x^2-xy-2y^2} = \lim\limits_{(x,y)\to(-1,1)} \dfrac{(x-y)\,(x+y)}{(x-2y)\,(x+y)} = \lim\limits_{(x,y)\to(-1,1)} \dfrac{x-y}{x-2y} = \dfrac{2}{3}$.

16 $\lim\limits_{(x,y)\to(0,0)} 25 \cdot \dfrac{x^3y^2}{\sin x^3y^2} \cdot \dfrac{x^3y^2}{\sin x^3y^2} = 25 \cdot \lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y^2}{\sin x^3y^2} \cdot \lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y^2}{\sin x^3y^2} = 25 \cdot 1 \cdot 1 = 25.$

17 $\lim\limits_{(x,y,z)\to(2,2,3)} \dfrac{x^2(z-3)-y^2(z-3)}{x(z-3)-y(z-3)} = \lim\limits_{(x,y,z)\to(2,2,3)} \dfrac{x^2-y^2}{x-y} = \lim\limits_{(x,y,z)\to(2,2,3)} x+y = 2+2 = 4.$

18 $\lim\limits_{(x,y,z)\to(3,4,7)} \dfrac{(\sqrt{x+y}-\sqrt{z})}{(x+y-z)} \cdot \dfrac{(\sqrt{x+y}+\sqrt{z})}{(\sqrt{x+y}+\sqrt{z})} = \lim\limits_{(x,y,z)\to(3,4,7)} \dfrac{x+y-z}{(x+y-z)(\sqrt{x+y}+\sqrt{z})} =$
$\lim\limits_{(x,y,z)\to(3,4,7)} \dfrac{1}{\sqrt{x+y}+\sqrt{z}} = \dfrac{1}{2\sqrt{7}}$.

19 This is continuous where $y - x^2 - 1 > 0$, so for $\{(x,y) : y > x^2+1\}$.

20 This is defined and continuous for $\{(x,y) : (x,y) \neq (0,0)\}$.

21 $f_x = 6xy^5$, $f_y = 15x^2y^4$.

22 $g_x = 4yz^2 - 3/y$, $g_y = 4xz^2 + 3x/y^2$, $g_z = 8xyz$.

23 $f_x = \frac{(x^2+y^2)(2x)-x^2(2x)}{(x^2+y^2)^2} = \frac{2xy^2}{(x^2+y^2)^2}$. $f_y = \frac{0-x^2(2y)}{(x^2+y^2)^2} = \frac{-2x^2y}{(x^2+y^2)^2}$.

24 $g_x = \frac{(x+y)yz-(xyz)}{(x+y)^2} = \frac{y^2z}{(x+y)^2}$. $g_y = \frac{(x+y)xz-(xyz)}{(x+y)^2} = \frac{x^2z}{(x+y)^2}$, $g_z = \frac{xy}{x+y}$.

25 $\dfrac{\partial}{\partial x}\left[xye^{xy}\right] = ye^{xy} + xy^2e^{xy} = y\,(1+xy)\,e^{xy}$, $\dfrac{\partial}{\partial y}\left[xye^{xy}\right] = xe^{xy} + x^2ye^{xy} = x\,(1+xy)\,e^{xy}$.

26 $\dfrac{\partial}{\partial u}[u\cos v - v\sin u] = \cos v - v\cos u$, $\dfrac{\partial}{\partial v}[u\cos v - v\sin u] = -u\sin v - \sin u$.

27 $f_x = 2ye^{2xy}$ and $f_y = 2xe^{2xy}$. Then $f_{xx} = 4y^2e^{2xy}$, $f_{yy} = 4x^2e^{2xy}$, and $f_{xy} = f_{yx} = 2e^{2xy} + 2y(2xe^{2xy}) = 2e^{2xy}(1+2xy)$.

28 Note that $H = (p^5 + 2p^4r)^{1/2}$. Then $H_p = \dfrac{5p^4+8p^3r}{2p^2\sqrt{p+2r}} = \dfrac{5p^2+8pr}{2\sqrt{p+2r}}$, and $H_r = \dfrac{p^2}{\sqrt{p+2r}}$. Then $H_{pp} =$
$\dfrac{2\sqrt{p+2r}(10p+84)-(5p^2+8pr)(p+2r)^{-1/2}}{4(p+2r)} = \dfrac{2(p+2r)(10p+8r)-(5p^2+8pr)}{4(p+2r)^{3/2}} = \dfrac{15p^2+48pr+32r^2}{4(p+2r)^{3/2}}$,
$H_{pr} = H_{rp} = \dfrac{\sqrt{p+2r}(2p) - p^2 \cdot \dfrac{1}{2} \cdot (p+2r)^{-1/2}}{p+2r} = \dfrac{(p+2r)(4p)-p^2}{2(p+2r)^{3/2}} = \dfrac{2p^2+8pr}{2(p+2r)^{3/2}}$, and
$H_{rr} = \dfrac{0-p^2(p+2r)^{-1/2}}{p+2r} = \dfrac{-p^2}{(p+2r)^{3/2}}$.

29 $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 6y - 6y = 0.$

30 $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{2\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2} + \dfrac{2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} = 0.$

31

a. If r is held fixed and R increases then V increases, so $V_R > 0$, whereas, if R is held fixed and r increases then V decreases, so $V_r < 0$.

b. We have $V_R = 4\pi R^2 > 0$ and $V_r = -4\pi r^2 < 0$, consistent with the predictions in part (a).

c. If $R = 3$, $r = 1$ and R is increased by $\triangle R = 0.1$, then $\triangle V \approx 4\pi \cdot 3^2 \cdot 0.1 = 3.6\pi$; if r is decreased by 0.1 then $\triangle V \approx -4\pi \cdot 1^2 \cdot (-0.1) = 0.4\pi$. Therefore, the volume changes more if R is increased.

32 The chain rule gives $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = 2t\cos yz - xz\sin yz - xy(3t^2)\sin yz = 2t\cos t^4 - t^3(t^2+1)\sin t^4 - 3t^3(t^2+1)\sin t^4 = 2t\cos t^4 - 4t^3(t^2+1)\sin t^4$.

33 The chain rule gives $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = \dfrac{2xz}{x^2+y^2}(3e^t) + \dfrac{2yz}{x^2+y^2}(4e^t) + \ln(x^2+y^2) = \dfrac{18te^{2t}}{25e^{2t}} + \dfrac{32te^{2t}}{25e^{2t}} + \ln 25e^{2t} = \dfrac{18t}{25} + \dfrac{32t}{25} + \ln 25 + \ln e^{2t} = 2t + \ln 25 + 2t = 4t + 5\ln 2$.

34 The chain rule gives $w_s = w_x x_s + w_y y_s + w_z z_s = yz \cdot 2t + xz \cdot t^2 + xy \cdot 2st = st^2 \cdot s^2t \cdot 2t + 2st \cdot s^2t \cdot t^2 + 2st \cdot st^2 \cdot 2st = 8s^3t^4$ and similarly $w_t = w_x x_t + w_y y_t + w_z z_t = yz \cdot 2s + xz \cdot 2st + xy \cdot s^2 = st^2 \cdot s^2t \cdot 2s + 2st \cdot s^2t \cdot 2st + 2st \cdot st^2 \cdot s^2 = 8s^4t^3$

35 The chain rule gives $w_r = w_x x_r + w_y y_r = \dfrac{1}{x}(st) + \dfrac{2}{y}(1) = \dfrac{1}{r} + \dfrac{2}{r+s} = \dfrac{3r+s}{r(r+s)}$. $w_s = w_x x_s + w_y y_s = \dfrac{1}{x}(rt) + \dfrac{2}{y}(1) = \dfrac{1}{s} + \dfrac{2}{r+s} = \dfrac{r+3s}{s(r+s)}$. $w_t = w_x x_t + w_y y_t = \dfrac{1}{x}(rs) + \dfrac{2}{y} \cdot 0 = \dfrac{1}{t}$.

36 Let $F(x,y) = 2x^2 + 3xy - 3y^4 - 2$; then if y is determined by $F(x,y) = 0$, we have $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{4x+3y}{3x-12y^3} = \dfrac{4x+3y}{12y^3-3x}$.

37 Let $F(x,y) = y\ln\left(x^2+y^2\right) - 4$; then if y is determined by $F(x,y) = 0$, we have $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} =$

$$-\frac{\dfrac{2xy}{x^2+y^2}}{\ln\left(x^2+y^2\right) + \dfrac{2y^2}{x^2+y^2}} = -\frac{2xy}{2y^2 + \left(x^2+y^2\right)\ln\left(x^2+y^2\right)} .$$

38

a. The chain rule gives $z'(t) = 8xx'(t) + 2yy'(t) = -6\cos t\sin t = -3\sin 2t$.

b. Walking uphill corresponds to $z'(t) > 0$, which occurs when $\dfrac{\pi}{2} < t < \pi$ and $\dfrac{3\pi}{2} < t < 2\pi$.

39

a. The chain rule gives $z'(t) = 2xx'(t) - 4yy'(t) = -24\cos t\sin t = -12\sin 2t$

b. Walking uphill corresponds to $z'(t) > 0$, which occurs when $\dfrac{\pi}{2} < t < \pi$ and $\dfrac{3\pi}{2} < t < 2\pi$.

40 Observe that $r'(t) = \dfrac{ar}{t}$ and $h'(t) = -\dfrac{bh}{t}$; hence, by the chain rule, $V'(t) = \dfrac{\pi}{3}\left(2rhr'(t) + r^2h'(t)\right)$ $= \dfrac{\pi r^2 h}{3t}(2a-b)$. Therefore, the volume of the cone remains constant if and only if $2a = b$. (This can also be seen by substituting the formulas for $r(t)$ and $h(t)$ in the formula for the volume of a cone to obtain $V(t) = \dfrac{\pi}{3}t^{2a-b}$.)

41

a.

	$(a,b)=(0,0)$	$(a,b)=(2,0)$	$(a,b)=(1,1)$
$\mathbf{u}=\left\langle\sqrt{2}/2,\sqrt{2}/2\right\rangle$	0	$4\sqrt{2}$	$-2\sqrt{2}$
$\mathbf{v}=\left\langle-\sqrt{2}/2,\sqrt{2}/2\right\rangle$	0	$-4\sqrt{2}$	$-6\sqrt{2}$
$\mathbf{w}=\left\langle-\sqrt{2}/2,-\sqrt{2}/2\right\rangle$	0	$-4\sqrt{2}$	$2\sqrt{2}$

b. The function is increasing at $(2,0)$ in the direction of $\mathbf{u}$ and decreasing at $(2,0)$ in the directions of $\mathbf{v}$ and $\mathbf{w}$.

42 $\nabla f=\langle 2x,0\rangle$, so $\nabla f(1,2)=\langle 2,0\rangle$. $D_{\mathbf{u}}f(1,2)=\langle 2,0\rangle\cdot\mathbf{u}=\dfrac{2}{\sqrt{2}}=\sqrt{2}$.

43 $\nabla g=\left\langle 2xy^3,3x^2y^2\right\rangle$, so $\nabla g(-1,1)=\langle -2,3\rangle$. $D_u g(-1,1)=\langle -2,3\rangle\cdot\mathbf{u}=\dfrac{-10}{13}+\dfrac{36}{13}=2$.

44 $\nabla f=\left\langle 1/y^2,-2x/y^3\right\rangle$, so $\nabla f(0,3)=\langle 1/9,0\rangle$. $D_{\mathbf{u}}f(0,3)=\langle 0,3\rangle\cdot\mathbf{u}=\sqrt{3}/18$.

45 The gradient of h is given by $\nabla h\left(x,y\right)=h_x\mathbf{i}+h_y\mathbf{j}=\dfrac{1}{\sqrt{2+x^2+2y^2}}\left(x\mathbf{i}+2y\mathbf{j}\right)$; therefore, $\nabla h\left(2,1\right)=\dfrac{\sqrt{2}}{2}\left(\mathbf{i}+\mathbf{j}\right)$ and the directional derivative in the direction of $\mathbf{u}$ is given by $\nabla h\left(2,1\right)\cdot\mathbf{u}=\dfrac{\sqrt{2}}{10}\left(\mathbf{i}+\mathbf{j}\right)\cdot\left(3\mathbf{i}+4\mathbf{j}\right)=\dfrac{7\sqrt{2}}{10}\approx 0.9899$.

46 The gradient of f is given by $\nabla f\left(x,y,z\right)=f_x\mathbf{i}+f_y\mathbf{j}+f_z\mathbf{k}=\left\langle e^{1+y^2+z^2},2xye^{1+y^2+z^2},2xze^{1+y^2+z^2}\right\rangle$; therefore, $\nabla f\left(0,1,-2\right)=e^6\mathbf{i}$ and the directional derivative in the direction of $\mathbf{u}$ is given by $\frac{4e^6}{9}+0+0=\frac{4e^6}{9}$.

47 The gradient of f is given by $\nabla f\left(x,y,z\right)=f_x\mathbf{i}+f_y\mathbf{j}+f_z\mathbf{k}=\langle y\cos xy,x\cos xy,-\sin z\rangle$, so $\nabla f(1,\pi,0)=\langle -\pi,-1,0\rangle$. The directional derivative is therefore

$$\langle -\pi,-1,0\rangle\cdot\langle 2/7,3/7,6/7\rangle=-2\pi/7-3/7=-\frac{1}{7}\left(2\pi+3\right).$$

48

a. The gradient of f is given by $\nabla f\left(x,y\right)=f_x\mathbf{i}+f_y\mathbf{j}=\dfrac{1}{1+xy}\left(y\mathbf{i}+x\mathbf{j}\right)$, so $\nabla f\left(2,3\right)=\dfrac{1}{7}\left(3\mathbf{i}+2\mathbf{j}\right)$. The direction of steepest ascent is the unit vector in this direction, $\mathbf{u}=\dfrac{3}{\sqrt{13}}\mathbf{i}+\dfrac{2}{\sqrt{13}}\mathbf{j}$, and the direction of steepest descent is $-\mathbf{u}$.

b. The unit vectors that point in the direction of no change are v= $\pm\left(\dfrac{2}{\sqrt{13}}\mathbf{i}-\dfrac{3}{\sqrt{13}}\mathbf{j}\right)$, because $\mathbf{u}\cdot\mathbf{v}=0$.

49

a. The gradient of f is given by $\nabla f\left(x,y\right)=f_x\mathbf{i}+f_y\mathbf{j}=-\dfrac{1}{\sqrt{4-x^2-y^2}}\left(x\mathbf{i}+y\mathbf{j}\right)$, so $\nabla f\left(-1,1\right)=\dfrac{1}{\sqrt{2}}\left(\mathbf{i}-\mathbf{j}\right)$. The direction of steepest ascent is the unit vector in this direction, $\mathbf{u}=\dfrac{\sqrt{2}}{2}\mathrm{i}-\dfrac{\sqrt{2}}{2}\mathrm{j}$, and the direction of steepest descent is $-\mathbf{u}$.

b. The unit vectors that point in the direction of no change are v= $\pm\left(\dfrac{\sqrt{2}}{2}\mathbf{i}+\dfrac{\sqrt{2}}{2}\mathbf{j}\right)$, because $\mathbf{u}\cdot\mathbf{v}=0$.

50 If y is determined by $f(x,y) = C$ we have $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{-4x}{-2y} = -\dfrac{2x}{y}$. Therefore, the level curve $f(x,y) = 5$ has slope $m = -2$ at the point $(1,1)$, so the tangent line has direction $\mathbf{i} - 2\mathbf{j}$. The gradient of f at this point is $\nabla f(1,1) = (-4x\mathbf{i} - 2y\mathbf{j})\Big|_{(1,1)} = -4\mathbf{i} - 2\mathbf{j}$, which is perpendicular to the tangent direction.

51 If x is determined by $f(x,y) = C$ we have $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{-4x}{-2y} = -\dfrac{2x}{y}$. Therefore, the level curve $f(x,y) = 0$ has a vertical tangent at the point $(2,0)$, so the tangent line has direction $\mathbf{j}$. The gradient of f at this point is $\nabla f(2,0) = (-4x\mathbf{i} - 2y\mathbf{j})\Big|_{(2,0)} = -8\mathbf{i}$, which is perpendicular to the tangent direction.

52 The gradient of f at $(1,1)$ is given by $\nabla f(1,1) = (8x\mathbf{i} - 2y\mathbf{j})\Big|_{(1,1)} = 8\mathbf{i} - 2\mathbf{j} = 2(4\mathbf{i} - \mathbf{j})$, so the unit vectors in the direction of no change are $\mathbf{u} = \pm\dfrac{1}{\sqrt{17}}(\mathbf{i} + 4\mathbf{j})$.

53 Observe that $V = -\dfrac{k}{2}\left(\ln\left(x^2+y^2\right) - \ln R^2\right)$; therefore $\mathbf{E} = -\nabla V = \dfrac{k}{2}\left(\dfrac{2x}{x^2+y^2}\mathbf{i} + \dfrac{2y}{x^2+y^2}\mathbf{j}\right) = \dfrac{kx}{x^2+y^2}\mathbf{i} + \dfrac{ky}{x^2+y^2}\mathbf{j}$.

54 Let $f(x,y,z) = 2x^2 + y^2 - z$. $\nabla f = \langle 4x, 2y, -1\rangle$ and $\nabla f(1,1,3) = \langle 4, 2, -1\rangle$. The equation of the tangent plane at $(1,1,3)$ is $4(x-1) + 2(y-1) - (z-3) = 0$, or $4x + 2y - z = 3$. At the other given point we have $\nabla f(0,2,4) = \langle 0, 4, -1\rangle$, so the equation of the tangent plane is $4(y-2) - (z-4) = 0$, or $4y - z = 4$.

55 Let $f(x,y,z) = x^2 + y^2/4 - z^2/9$. $\nabla f = \langle 2x, y/2, -2z/9\rangle$, so $\nabla f(0,2,0) = \langle 0,1,0\rangle$. The equation of the tangent plane at $(0,2,0)$ is given by $1(y-2) = 0$, or $y = 2$. At the point $(1,1,3/2)$ we have $\nabla f = \langle 2, 1/2, -1/3\rangle$, so the equation of the tangent plane is $2(x-1) + (1/2)(y-1) + (-1/3)(z-3/2) = 0$, or $12x + 3y - 2z = 12$.

56 Let $F(x,y,z) = x^2 - 2x_y^2 + 4y + 3z^2$; then $\nabla F = \langle 2x-2, 2y+4, 6z\rangle$ and $\nabla F(3,-1,1) = \langle 4, 0, 6\rangle$. The tangent plane is given by $4(x-3) + 6(z-1) = 0$, or $2x + 3z = 9$. At the point $(-1,-2,1)$ we have $\nabla F = \langle -4, 0, 6\rangle$, so the tangent plane there is given by $-4(x+1) + 6(z-1) = 0$, or $-2x + 3z = 5$.

57 Let $F(x,y,z) = e^{xy^2z^3-1}$. Then

$$\nabla F(x,y,z) = \left\langle y^2z^3e^{xy^2z^3-1}, 2xyz^3e^{xy^2z^3-1}, 3xy^2z^3e^{xy^2z^3-1}\right\rangle.$$

$\nabla F(1,1,1) = \langle 1,2,3\rangle$, so the plane is $x - 1 + 2(y-1) + 3(z-1) = 0$, or $x + 2y + 3z = 6$. Also, $\nabla F(1,-1,1) = \langle 1,-2,3\rangle$, so the plane is $x - 1 - 2(y+1) + 3(z-1) = 0$, or $x - 2y + 3z = 6$.

58 Let $F(x,y,z) = z - \tan^{-1} xy$. Then $\nabla F = \left\langle \frac{-y}{1+(xy)^2}, \frac{-x}{1+(xy)^2}, 1\right\rangle$. $\nabla F(1,1,\pi/4) = \langle -1/2, -1/2, 1\rangle$, so the plane is $-(x-1) - (y-1) + 2(z-\pi/4) = 0$, or $x + y - 2z = 2 - \pi/2$. Also, $\nabla F(1,\sqrt{3},\pi/3) = \langle -\sqrt{3}/4, -1/4, 1\rangle$, so the plane is $-(\sqrt{3}/4)(x-1) - (1/4)(y-\sqrt{3}) + (z-\pi/3) = 0$, or $-\sqrt{3}x - y + 4z = 4\pi/3 - 2\sqrt{3}$.

59 Let $F(x,y) = \sqrt{\frac{x+y}{z}} - 1$. Then $\nabla F = \left\langle \frac{\sqrt{z}}{2z\sqrt{x+y}}, \frac{\sqrt{z}}{2z\sqrt{x+y}}, -\frac{\sqrt{x+y}}{2z^{3/2}}\right\rangle$. At the point $(2,2,4)$ we have $\nabla F = \langle 1/8, 1/8, -1/8\rangle = (1/8)\langle 1,1,-1\rangle$. So the equation of the tangent plane is $x - 2 + y - 2 - (z-4) = 0$, or $x + y - z = 0$. At the point $(10,-1,9)$ we have $\nabla F = \langle 1/18, 1/18, -1/18\rangle = (1/18)\langle 1,1,-1\rangle$, so the equation of the tangent plane is $x - 10 + y + 1 - (z-9) = 0$, or $x + y - z = 0$.

60

a. The linear approximation is given by $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = 2\sqrt{2} - 8\sin\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + 4\sin\left(\frac{\pi}{4}\right)\left(y - \frac{\pi}{4}\right) = -4\sqrt{2}x + 2\sqrt{2}y + 2\sqrt{2} + \dfrac{\sqrt{2}\pi}{2}$.

b. This gives the estimate $f(0.8, 0.8) \approx -3.2+2\sqrt{2}+\pi \approx 2.787$ (the actual answer is 2.787 to three decimal places).

61

a. The linear approximation is given by $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = 2 + (x-2) + 5y = x + 5y$.

b. This gives the estimate $f(1.95, 0.05) \approx 2.2$ (the actual answer is 2.205 to three decimal places).

62 We have $\triangle f \approx f_x(1,-2)\triangle x + f_y(1,-2)\triangle y = 4\cdot 0.05 + 9\cdot 0.1 = 1.1$ (the actual change is 1.093 to three decimal places).

63 We have $dV = \pi\left(2rhdr + r^2dh\right)$, so $\dfrac{dV}{V} = 2\dfrac{dr}{r} + \dfrac{dh}{h}$. Therefore, if the radius decreases by 3% and the height increases by 2%, the approximate change in volume is -4%.

64 We have $dV = \pi\left(bcda + acdb + abdc\right)$, so $\dfrac{dV}{V} = \dfrac{da}{a} + \dfrac{db}{b} + \dfrac{dc}{c}$. Therefore, if a, b, c change by 2%, 1.5%, -2.5% respectively, the approximate change in the volume is 1%.

65

a. We have $dV = \pi\left(2rh - h^2\right)dh = 2\pi(-0.05) = -0.1\pi\text{m}^3$ (notice that $r = 1.50$ m is constant, so there is no contribution from the dr term in dV).

b. The surface of the water is a disc with radius $s = \sqrt{2rh - h^2}$, so the surface area is $S = \pi\left(2rh - h^2\right)$. Therefore $dS = 2\pi(r-h)\,dh = -0.05\pi\text{m}^2$.

66 We have $f_x = 4x^3 - 16y$, $f_y = 4y^3 - 16x$; therefore, the critical points satisfy the equations $y = \dfrac{x^3}{4}$ and $x = \dfrac{y^3}{4}$. Eliminating y gives $x^9 = 2^8x$, so $x = 0, \pm 2$, and the critical points are $(0,0)$,$\pm(2,2)$. We also have $f_{xx} = 12x^2$, $f_{yy} = 12y^2$ and $f_{xy} = -16$; hence $D(x,y) = 16\left(9x^2y^2 - 1\right)$. We see that $D(0,0) < 0$ so $(0,0)$ is a saddle. We have $D(2,2) = D(-2,-2) > 0$; $f_{xx}(2,2) = f_{xx}(-2,-2) > 0$, which by the Second Derivative Test implies that f has local minima at $\pm(2,2)$.

67 We have $f_x = x^2 + 2y$, $f_y = -y^2 + 2x$; therefore, the critical points satisfy the equations $y = -\dfrac{x^2}{2}$and $x = \dfrac{y^2}{2}$. Eliminating y gives $x^4 = 8x$ so $x = 0, 2$, and the critical points are $(0,0)$, $(2,-2)$. We also have $f_{xx} = 2x$, $f_{yy} = -2y$ and $f_{xy} = 2$; hence $D(x,y) = -4(1 + xy)$. We see that $D(0,0) < 0$ so $(0,0)$ is a saddle. We have $D(2,-2) > 0$, $f_{xx}(2,-2) > 0$, which by the Second Derivative Test implies that f has a local minimum at $(2,-2)$.

68 We have $f_x = 2xy^2 - 6xy + 2y^2 - 6y$, $f_y = 2x^2y - 3x^2 + 4xy - 6x$; therefore, the critical points satisfy the equations $y(2xy - 6x + 2y - 6) = 0$, $x(2xy - 3x + 4y - 6) = 0$. If $y = 0$ then the second equation gives $x = 0, -2$, and if $x = 0$ then the first equation gives $y = 0, 3$. If both $x, y \neq 0$ then we can divide the equations by x, y respectively and subtract to obtain $3x + 2y = 0$, so $y = -\dfrac{3x}{2}$; substituting this in the first equation gives $x^2 + 3x + 2 = 0$, which has roots $x = -1, -2$. Therefore, the critical point are $(0,0)$, $(0,3)$, $(-2,0)$, $\left(-1, \dfrac{3}{2}\right)$ and $(-2,3)$. We also have $f_{xx} = 2y^2 - 6y$, $f_{yy} = 2x^2 + 4x$ and $f_{xy} = 4xy - 6x + 4y - 6$. We see that $D(0,0)$, $D(0,3)$, $D(-2,0)$, and $D(-2,3) < 0$ so $(0,0)$, $(0,3)$, $(-2,0)$and $(-2,3)$ are saddles. We have $D\left(-1, \dfrac{3}{2}\right) > 0$, $f_{xx}\left(-1, \dfrac{3}{2}\right) < 0$, which by the Second Derivative Test implies that f has a local maximum at $\left(-1, \dfrac{3}{2}\right)$.

69 We have $f_x = -3x^2 - 6x$, $f_y = -3y^2 + 6y$; therefore the critical points must have $x = 0, -2$ and $y = 0, 2$. We also have $f_{xx} = -6x - 6$, $f_{yy} = -6y + 6$ and $f_{xy} = 0$; hence $D\,(x, y) = 36\,(x + 1)\,(y - 1)$. We see that $D\,(0, 0)$, $D\,(-2, 2) < 0$ so $(0, 0)$ and $(-2, 2)$ are saddles. We have $D\,(0, 2) > 0$, $f_{xx}\,(0, 2) < 0$, which by the Second Derivative Test implies that f has a local maximum at $(0, 2)$; $D\,(-2, 0) > 0$, $f_{xx}\,(-2, 0) > 0$, which by the Second Derivative Test implies that f has a local minimum at $(-2, 0)$.

70 First we find the critical points of f inside the rectangle R: we have $f_x = x^2 + 2y$, $f_y = -y^2 + 2x$; the equation $f_x = 0$ gives $y = -\dfrac{x^2}{2}$, and then substituting this in the equation $f_y = 0$ gives $x^4 = 8x$, or $x = 0, 2$. Hence the critical points are $(0, 0)$ and $(2, -2)$, neither of which is in the interior of R. Therefore, we must find the maximum and minimum values of f on the boundary of R, which consists of four segments. On the segment $0 \le x \le 3$, $y = -1$, let $g\,(x) = f\,(x, -1) = \dfrac{x^3}{3} - 2x + \dfrac{1}{3}$: then g has a critical point at $x = \sqrt{2}$, and find that g has extreme values $\dfrac{1 - 4\sqrt{2}}{3}$ and $\dfrac{10}{3}$ on $[0, 3]$. Similarly, we find that on the segment $0 \le x \le 3$, $y = 1$, $g\,(x) = f\,(x, 1) = \dfrac{x^3}{3} + 2x - \dfrac{1}{3}$ has extreme values $-\dfrac{1}{3}$ and $\dfrac{44}{3}$. On the segment $-1 \le y \le 1$, $x = 0$, let $h\,(y) = f\,(0, y) = -\dfrac{y^3}{3}$: then h has extreme values $\pm\dfrac{1}{3}$ on this segment. Similarly, we find that on the segment $-1 \le y \le 1$, $x = 3$, $h\,(y) = f\,(3, y) = -\dfrac{y^3}{3} + 6y + 9$ has a critical point at $y = \sqrt{6}$, and h has extreme values $\dfrac{10}{3}$ and $4\sqrt{6} + 9$. Therefore the absolute minimum and maximum values of f on R are $f\left(\sqrt{2}, 1\right) = \dfrac{1 - 4\sqrt{2}}{3} \approx -0.886$ and $f\left(3, \sqrt{6}\right) = 4\sqrt{6} + 9 \approx 18.798$.

71 First we find the critical points of f inside the square R: we have $f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$; the equation $f_x = 0$ gives $y = x^3$, and then substituting this in the equation $f_y = 0$ gives $x^9 = x$, or $x = 0$,± 1. Hence, the critical points are $(0, 0)$ and $\pm\,(1, 1)$, which are all in the interior of R. We observe that the values of f at these points are $f\,(0, 0) = 1$, $f\,(1, 1) = f\,(-1, -1) = -1$. Next we must find the maximum and minimum values of f on the boundary of R, which consists of four segments. On the segment $-2 \le x \le 2$, $y = 2$, let $g\,(x) = f\,(x, 2) = x^4 - 8x + 17$: then g has a critical point at $x = \sqrt[3]{2}$, and find that g has extreme values $17 - 6\sqrt[3]{2}$ and 49 on $[0, 3]$. We also note that $f\,(y, x) = f\,(-x, -y) = f\,(x, y)$; therefore f takes the same values on all four segments of the square. Hence, the absolute minimum and maximum values of f on R are $f\,(1, 1) = f\,(-1, -1) = -1$ and $f\,(2, -2) = f\,(-2, 2) = 49$.

72 First we find the critical points of f inside the triangle. We have $f_x = 2xy$ and $f_y = x^2 - 3y^2$, so there are no critical points inside the triangle. Now consider the boundary triangle. On the vertical side $x = 0$, we have $f(x, y) = -y^3$ which has a maximum for $0 \le y \le 2$ of 0 and a minimum of -8. On the horizontal side $y = 0$, we have $f(x, y) = 0$, so the minimum and maximum are both 0. On the side $y = 2 - x$, we have $f(x, y) = g(x) = 2x^2 - x^3 - (2 - x)^3$. Note that $g'(x) = 4x - 3x^2 + 3(2 - x)^2 = 12 - 8x$, which is zero for $x = 3/2$. The value of g at $x = 3/2$ is $g(3/2) = 1$. Thus, the absolute maximum of f is 1 at $(3/2, 1/2)$ and the absolute minimum is -8 at $(0, 2)$.

73 First we find the critical points of f inside the semicircle. We have $f_x = y$ and $f_y = x$, so the only critical point is $(0, 0)$, where the value of f is 0. On the flat part of the semicircle where $y = 0$, the value of f is also 0. Now we parametrize the circular part of the boundary by letting $x = \cos t$, $y = \sin t$ for $0 \le t \le \pi$. Then $f(x, y) = g(t) = \cos t \sin t$, and $g'(t) = -\sin^2 t + \cos^2 t$. This is zero for $t = \pi/4$ and $t = 3\pi/4$, where the value of f is $1/2$ and $-1/2$, respectively. Thus, the absolute maximum of f is $1/2$ at $(\sqrt{2}/2, \sqrt{2}/2)$ and the absolute minimum is $-1/2$ at $(-\sqrt{2}/2, \sqrt{2}/2)$.

74 First, observe that there is a unique point on any plane that is closest to the origin. To find this point, it suffices to minimize the function $f\,(y, z) = x^2 + y^2 + z^2 = (8 - y - 4z)^2 + y^2 + z^2$ over all points (y, z). We have $f_y = 2\,(y + 4z - 8) + 2y$, $f_z = 2 \cdot 4\,(4z + y - 8) + 2z$ and setting $f_y = f_z = 0$ gives the equations $y + 2z = 4$, $4y + 17z = 32$; solving these simultaneously gives $y = \dfrac{4}{9}$, $z = \dfrac{16}{9}$ and hence $x = \dfrac{4}{9}$; therefore, the closest point is $\left(\dfrac{4}{9}, \dfrac{4}{9}, \dfrac{16}{9}\right)$.

75 We have $\nabla f = \langle 2, 1\rangle$ and $\nabla g = \langle 4(x-1), 8(y-1)\rangle$, where $g(x, y) = 2(x-1)^2 + 4(y-1)^2$. The Lagrange equations are thus $2 = 4\lambda(x-1)$, $1 = 8\lambda(y-1)$, and $2(x-1)^2+4(y-1)^2 = 1$. Solving gives $\lambda = 3/4$, $x = 5/3$, and $y = 7/6$, or $\lambda = -3/4$, $x = 1/3$, and $y = 5/6$. The absolute maximum for f is $f(5/3, 7/6) = 29/2$ and the absolute minimum for f is $f(1/3, 5/6) = 23/2$.

76 We have $\nabla f = \langle y, x\rangle$ and $\nabla g = \langle 6x - 2y, -2x + 6y\rangle$ where $g(x, y) = 3x^2 - 2xy + 3y^2$. Lagrange equations are therefore $y = \lambda(6x - 2y)$ and $x = \lambda(2x + 6y)$, and $3x^2 - 2xy + 3y^2 = 4$. Solving for λ, we have

$$\lambda = \frac{y}{6x-2y} = \frac{x}{6y-2x},$$

and from this we can deduce that $x = \pm y$. Then using the constraint equation, we have that $x = \pm 1$ or $x = \pm\frac{1}{\sqrt{2}}$. Checking these, we find an absolute maximum of 1 at $\pm(1, 1)$ and an absolute minimum of -1 at $\pm(1, -1)$.

77 The Lagrange multiplier conditions are $1 = 2\lambda x$, $2 = 2\lambda y$, $-1 = 2\lambda z$, $x^2 + y^2 + z^2 = 1$. Hence $2\lambda = \dfrac{1}{x} = \dfrac{2}{y} = -\dfrac{1}{z} \Longrightarrow y = 2x$, $z = -x$; substituting these in the constraint gives $6x^2 = 1$ so $x = \pm\dfrac{\sqrt{6}}{6}$ and we obtain solutions $\pm\left(\dfrac{\sqrt{6}}{6}, \dfrac{\sqrt{6}}{3}, -\dfrac{\sqrt{6}}{6}\right)$. Therefore the extreme values of f on the closed bounded set given by $x^2 + y^2 + z^2 = 1$ are an absolute maximum of $f\left(\dfrac{\sqrt{6}}{6}, \dfrac{\sqrt{6}}{3}, -\dfrac{\sqrt{6}}{6}\right) = \sqrt{6}$, and an absolute minimum of $f\left(-\dfrac{\sqrt{6}}{6}, -\dfrac{\sqrt{6}}{3}, \dfrac{\sqrt{6}}{6}\right) = -\sqrt{6}$.

78 The Lagrange multiplier conditions are $2xy^2z = 4\lambda x$, $2x^2yz = 2\lambda y$, $x^2y^2 = 2\lambda z$, $2x^2 + y^2 + z^2 = 25$. If x, y or $z = 0$ then $f(x, y, z) = 0$, which is neither the maximum or minimum of f on the ellipsoid given by the constraint (f takes both positive and negative values on this set). Hence, we can assume x, y, $z \neq 0$ in the Lagrange conditions, and eliminating λ in the first three equations gives $\lambda = \dfrac{y^2z}{2} = x^2z = \dfrac{x^2y^2}{2z} \Longrightarrow y^2 = 2z^2 = 2x^2$; substituting this in the constraint gives $5x^2 = 25$, so $x = \pm\sqrt{5}$, $y = \pm\sqrt{10}$ and $z = \pm\sqrt{5}$. Therefore the maximum value of f on the closed bounded set given by $2x^2 + y^2 + z^2 = 25$ is $f\left(\pm\sqrt{5}, \pm\sqrt{10}, \sqrt{5}\right) = 50\sqrt{5}$, and the minimum value is $f\left(\pm\sqrt{5}, \pm\sqrt{10}, -\sqrt{5}\right) = -50\sqrt{5}$.

79 Let (x, y) be the corner of the rectangle in the first quadrant; then the perimeter of the rectangle is $4(x + y)$, so it suffices to find the maximum value of $x + y$ subject to the constraint $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The Lagrange multiplier conditions are $1 = \dfrac{2\lambda x}{a^2}$, $1 = \dfrac{2\lambda y}{b^2}$, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Hence, $2\lambda = \dfrac{a^2}{x} = \dfrac{b^2}{y} \Longrightarrow y = \dfrac{b^2x}{a^2}$; substituting in the constraint gives $x^2\left(a^2 + b^2\right) = a^4$ so $x = \dfrac{a^2}{\sqrt{a^2+b^2}}$, $y = \dfrac{b^2}{\sqrt{a^2+b^2}}$, and the dimensions of the rectangle with greatest perimeter are $\dfrac{2a^2}{\sqrt{a^2+b^2}}$ by $\dfrac{2b^2}{\sqrt{a^2+b^2}}$.

80 Let r be the radius and h the height of the cylinder; then the surface area is $A = 2\pi r^2 + 2\pi rh$ and the volume is πr^2h, so it suffices to minimize $r^2 + rh$ subject to the constraint $\pi r^2h = 32$. The Lagrange multiplier conditions are $2r + h = 2\lambda rh$, $r = \lambda r^2$, $\pi r^2h = 32$. We must have r, $h > 0$, so the second equation gives $\lambda = \dfrac{1}{r}$, and then the first equation reduces to $2r = h$. Substituting this in the constraint gives $r^3 = \dfrac{16}{\pi}$, so $r = 2\sqrt[3]{\dfrac{2}{\pi}}$ in. and $h = 4\sqrt[3]{\dfrac{2}{\pi}}$ in.

81 It suffices to minimize the function $f(x, y, z) = (x-1)^2 + (y-3)^2 + (z-1)^2$ subject to the constraint $x^2 + y^2 - z^2 = 0$. The Lagrange multiplier conditions are equivalent to $x - 1 = \lambda x$, $y - 3 = \lambda y$, $z - 1 = -\lambda z$,

$x^2+y^2-z^2=0$. The first two equations give $\lambda xy=(x-1)\,y=(y-3)\,x \Longrightarrow y=3x$ and similarly, the first and third equations give $\lambda xz=(x-1)\,z=-x\,(z-1) \Longrightarrow (2x-1)\,z=x \Rightarrow z=\sqrt{10}x$. Substituting these equations in the constraint gives $(2x-1)^2\cdot 10x^2=x^2$, so either $x=0$ (and hence $y=z=0$ as well) or $10\,(2x-1)^2=1$, which has solutions $x=\dfrac{1}{2}\pm\dfrac{\sqrt{10}}{20}$. Therefore, there are three solutions to the Lagrange conditions: $(0,0,0)$, $\left(\dfrac{1}{2}\pm\dfrac{\sqrt{10}}{20},\dfrac{3}{2}\pm\dfrac{3\sqrt{10}}{20},\dfrac{1}{2}\pm\dfrac{\sqrt{10}}{2}\right)$. We see that $f\,(0,0,0)=11$, $f\left(\dfrac{1}{2}\pm\dfrac{\sqrt{10}}{20},\dfrac{3}{2}\pm\dfrac{3\sqrt{10}}{20},\dfrac{1}{2}\pm\dfrac{\sqrt{10}}{2}\right)=\dfrac{11}{2}\mp\sqrt{10}$, so the closest point is $\left(\dfrac{1}{2}+\dfrac{\sqrt{10}}{20},\dfrac{3}{2}+\dfrac{3\sqrt{10}}{20},\dfrac{1}{2}+\dfrac{\sqrt{10}}{2}\right)$. (The function $f\,(x,y,z)\to\infty$ as either x, y or $z\to\infty$ on the cone; therefore, f must have minimum somewhere on the cone, which corresponds to the point we found.)

82

a. We have $d\,(x,y,z)=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$, so

$$\nabla d\,(x,y,z)=\langle\frac{x-a}{d\,(x,y,z)},\frac{y-b}{d\,(x,y,z)},\frac{z-c}{d\,(x,y,z)}\rangle.$$

b. Observe that $\nabla d\,(x,y,z)=\dfrac{1}{d\,(x,y,z)}\overrightarrow{PP_0}$ and $\left|\overrightarrow{PP_0}\right|=d\,(x,y,z)$; therefore, $\nabla d\,(x,y,z)$ is a unit vector.

c. The level surfaces of d are spheres centered at a,b,c, and $\nabla d\,(x,y,z)$ is perpendicular to these spheres (pointing outwards).

d. If $P\to P_0$ in the direction of a unit vector $\mathbf{u}$, we have $\nabla d\,(x,y,z)=\pm\mathbf{u}$; but because the limit must be the same in all directions, we conclude that this limit does not exist.

83 It suffices to minimize the function $f\,(x,y,z)=(x-5)^2+(y-10)^2+(z-3)^2$ subject to the constraint $x^2+y^2-z=0$. The Lagrange multiplier conditions are equivalent to $x-5=\lambda x$, $y-10=\lambda y$, $2(z-3)=-\lambda$, and $z=x^2+y^2$. Combining the first two equations gives that $y=2x$, and this combined with the fourth equation gives that $z=5x^2$. Then the first and third equations give that $\lambda=\frac{x-5}{x}=2(3-5x^2)$, and solving this equation gives $x=1$. Therefore we have $x=1$, $y=2$, and $z=5$. (The function $f\,(x,y,z)\to\infty$ as either x, y or $z\to\infty$ on the paraboloid; therefore, f must have minimum somewhere on the paraboloid, which corresponds to the point we found.)

Chapter 16

Multiple Integration

16.1 Double Integrals over Rectangular Regions

16.1.1 $\int_0^2 \int_1^3 xy\,dy\,dx$ or $\int_1^3 \int_0^2 xy\,dx\,dy$.

16.1.2 Two solutions: $\int_0^5 \int_{-2}^4 10\ dy\ dx$ or $\int_{-2}^4 \int_0^5 10\,dx\,dy$.

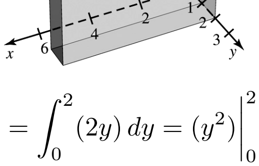
16.1.3 With respect to x first: $\int_1^5 \int_{-2}^4 f\,(x,\,y)\ dx\,dy$; with respect to y first: $\int_{-2}^4 \int_1^5 f\,(x,\,y)\ dy\,dx$.

16.1.4 y is the first (inner) varaiable and has limits $-1 \le y \le 1$, x is the second (outer) variable and has limits $1 \le x \le 3$.

16.1.5 Note that $\Delta A_k = 4$. We have a value for the Riemann sum of

$$4(f(1,1)+f(1,3)+f(1,5)+f(3,5)+f(3,3)+f(3,1)) = 4(0.5+1.5+2.5+3.5+2.5+1.5) = 48.$$

16.1.6 $V = \int_0^6 \int_1^2 10\,dy\,dx = \int_0^6 (10y)\Big|_1^2 dx = \int_0^6 10\,dx = (10x)\Big|_0^6 = 60.$



16.1.7 $\int_0^2 \int_0^1 (4xy)\,dx\,dy = \int_0^2 (2x^2y)\Big|_0^1 dy = \int_0^2 (2y)\,dy = (y^2)\Big|_0^2 = 4.$

16.1.8 $\int_1^2 \int_0^1 (3x^2+4y^3)\,dy\,dx = \int_1^2 (3x^2y+y^4)\Big|_0^1 dx = \int_1^2 (3x^2+1)\,dx = (x^3+x)\Big|_1^2 = 10-2 = 8.$

16.1.9 $\int_1^3 \int_0^2 x^2\,y\,dx\,dy = \int_1^3 \left(\frac{x^3y}{3}\right)\Big|_0^2 dy = \frac{8}{3}\int_1^3 y\,dy = \frac{8}{3}\left(\frac{y^2}{2}\right)\Big|_1^3 = \frac{32}{3}.$

16.1.10 $\displaystyle\int_0^3\int_{-2}^1 (2x+3\,y)\;dx\,dy = \int_0^3 \left(x^2+3\,x\,y\right)\Big|_{-2}^1 dy = \int_0^3 (9y-3)\;dy = \left(\frac{9}{2}y^2-3y\right)\Big|_0^3 = \frac{63}{2}.$

16.1.11 $\displaystyle\int_1^3\int_0^{\pi/2} x\,\sin y\,dy\,dx = \int_1^3 (-x\,\cos y)\Big|_0^{\pi/2} dx = \int_1^3 x\,dx = \left(\frac{x^2}{2}\right)\Big|_1^3 = 4.$

16.1.12 $\displaystyle\int_1^3\int_1^2 \left(y^2+y\,\right)\;dx\,dy = \int_1^3 \left(x\,y^2+x\,y\,\right)\Big|_1^2 dy = \int_1^3 \left(y^2+y\,\right)\;dy = \left(\frac{y^3}{3}+\frac{y^2}{2}\right)\Big|_1^3 = \frac{38}{3}.$

16.1.13 $\displaystyle\int_1^4\int_0^4 \sqrt{u\,v}\,du\,dv = \int_1^4 \left(\frac{2}{3}u^{3/2}v^{1/2}\right)\Big|_{u=0}^{u=4} dv = \int_1^4 \left(\frac{16}{3}v^{1/2}\right)\;dv = \left(\frac{32}{9}v^{3/2}\right)\Big|_{v=1}^{v=4} = \frac{224}{9}.$

16.1.14 $\displaystyle\int_0^{\pi/4}\int_0^3 (\sec\theta) r\,dr\,d\theta = \int_0^{\pi/4} \left((\sec\,\theta)\frac{r^2}{2}\right)\Big|_{r=0}^{r=3} d\theta = \int_0^{\pi/4} \left(\frac{9}{2}\sec\theta\right)\,d\theta =$ $\displaystyle\left(\frac{9}{2}\cdot\ln\Big|\sec\,\theta+\tan\theta\Big|\right)\Big|_{\theta=0}^{\theta=\pi/4} = \frac{9}{2}\left(\ln\Big|\sqrt{2}+1\Big|-\ln\Big|1+0\Big|\right) = \frac{9}{2}\ln\left(\sqrt{2}+1\right).$

16.1.15 $\displaystyle\int_1^{\ln 5}\int_0^{\ln 3} e^{x+y}\,dx\,dy = \int_1^{\ln 5}\int_0^{\ln 3} (e^x)\;(e^y)\;dx\,dy = \int_1^{\ln 5} (e^x\cdot e^y\,)\Big|_0^{\ln 3} dy = \int_1^{\ln 5} (2e^y\,)\;dy =$ $\displaystyle(2e^y)\Big|_1^{\ln 5} = 10-2e.$

16.1.16 Let $w=u^2v$. Then $dw=2uv\,du$. Then we have $\displaystyle\frac{1}{2}\int_0^{\pi/2}\int_0^v \cos w\,dw\,dv = \frac{1}{2}\int_0^{\pi/2}\sin v\,dv =$ $\displaystyle-\frac{1}{2}\left(\cos\pi/2-\cos 0\right) = \frac{1}{2}.$

16.1.17 We have $\displaystyle\int_0^1 te^{st}\Big|_{s=0}^{s=1} dt = \int_0^1 \left(te^t-t\right)\,dt = \left(e^t(t-1)-\frac{t^2}{2}\right)\Big|_0^1 = -\frac{1}{2}-(-1) = \frac{1}{2}$, where we use the fact that an antiderivative of te^t is $e^t(t-1)$ which follows via integration by parts.

16.1.18 This is equal to $\displaystyle\int_0^2 4y\tan^{-1}x^2\Big|_{x=0}^{x=1} dy = \int_0^2 \pi y\,dy = \frac{\pi y^2}{2}\Big|_0^2 = 2\pi.$

16.1.19 Note that $\displaystyle\int 4(p+q)\ln q\,dp = 4\left(\frac{p^2}{2}+pq\right)\ln q + C$, so we have $\displaystyle\int_1^e\int_0^1 4(p+q)\ln q\,dp\,dq =$ $\displaystyle\int_1^e (2+4q)\ln q\,dq$. Using integration by parts, we see that an antiderivative of $q\ln q$ is $\displaystyle\frac{q^2\ln q}{2}-\frac{q^2}{4}$, so our integral is equal to $\displaystyle\left(2q\ln q-2q+2q^2\ln q-q^2\right)\Big|_1^e = e^2+3.$

16.1.20 This is equal to $\displaystyle\int_0^1 y\sin xy\Big|_{x=0}^{x=\pi} dy = \int_0^1 y\sin\pi y\,dy = \left(\frac{\sin\pi y}{\pi^2}-\frac{y\cos\pi y}{\pi}\right)\Big|_0^1 = \frac{1}{\pi}.$

16.1.21 $\displaystyle\int_1^2\int_1^2 \frac{x}{x+y}\,dy\,dx = \int_1^2 \left(x\,\ln\left(x+y\right)\right)\Big|_1^2 dx = \int_1^2 (x\ln(x+2)-x\ln(x+1))\,dx = (-\frac{1}{2}x^2\ln(x+1)+$ $\displaystyle\frac{1}{2}x^2\ln(x+2)+\frac{x}{2}+\frac{1}{2}\ln(x+1)-2\ln(x+2))\Big|_1^2 = -2\ln 3+2\ln 4+1+\frac{\ln 3}{2}-2\ln 4-(\frac{-\ln 2}{2}+\frac{\ln 3}{2}+\frac{1}{2}+\frac{\ln 2}{2}-2\ln 3) =$ $\displaystyle 1-\frac{1}{2} = \frac{1}{2}.$

16.1.22 $\displaystyle\int_0^2\int_0^1 x^5\,y^2 e^{x^3y^3}\,dy\,dx = \int_0^2\left(\frac{x^5}{3x^3}\,e^{x^5y^3}\right)\Bigg|_0^1 dx = \frac{1}{3}\int_0^2\left(x^2\,e^{x^3}-x^2\right)dx = \frac{1}{3}\left(\frac{1}{3}e^{x^3}-\frac{1}{3}x^3\right)\Bigg|_0^2 =$ $\displaystyle\frac{1}{9}\left(e^8-9\right).$

16.1.23 $\displaystyle\int_0^1\int_1^4 \frac{3y}{\sqrt{x+y^2}}\,dx\,dy = \int_0^1\left(6y\sqrt{x+y^2}\right)\Bigg|_1^4 dy = 6\int_0^1\left(y\sqrt{4+y^2}-y\sqrt{1+y^2}\right)dy =$ $\displaystyle 6\left(\frac{1}{3}\left(4+y^2\right)^{3/2}-\frac{1}{3}\left(1+y^2\right)^{3/2}\right)\Bigg|_0^1 = 2\left(5\sqrt{5}-2\sqrt{2}-7\right).$

16.1.24 $\displaystyle\int_0^1 \frac{1}{3}e^{x^3y}y\Bigg|_{x=0}^{x=1} dy = \frac{1}{3}\int_0^1 y(e^y-1)\,dy = \frac{1}{3}\left(e^y(y-1)-\frac{y^2}{2}\right)\Bigg|_0^1 = \frac{1}{3}\left(-\frac{1}{2}+1\right) = \frac{1}{6}.$

16.1.25 $\displaystyle\iint_R (x+2y)\ dA = \int_1^4\int_0^3 (x+2y)\ dx\,dy = \int_1^4\left(\frac{x^2}{2}+2\,x\,y\right)\Bigg|_0^3 dy = \int_1^4\left(\frac{9}{2}+6y\right)\,dy =$ $\displaystyle\left(\frac{9}{2}y+3y^2\right)\Bigg|_1^4 = \frac{117}{2}.$

16.1.26 $\displaystyle\iint_R \left(x^2+x\,y\right)\ dA = \int_{-1}^1\int_1^2 \left(x^2+x\,y\right)\ dx\,dy = \int_{-1}^1\left(\frac{x^3}{3}+\frac{x^2y}{2}\right)\Bigg|_1^2 dy = \int_{-1}^1\left(\frac{7}{3}+\frac{3y}{2}\right)\,dy =$ $\displaystyle\left(\frac{7}{3}y+\frac{3}{4}y^2\right)\Bigg|_{-1}^1 = \frac{14}{3}.$

16.1.27 This is equal to $\displaystyle\int_0^\pi\int_0^1 s^2t\sin(st^2)\,dt\,ds = \int_0^\pi -\frac{1}{2}s\cos(st^2)\Bigg|_{t=0}^{t=1} ds = -\frac{1}{2}\int_0^\pi (s\cos s - s)\,ds =$ $\displaystyle -\frac{1}{2}\left(\cos s + s\sin s - \frac{s^2}{2}\right)\Bigg|_0^\pi = -\frac{1}{2}\left(-1-\frac{\pi^2}{2}-1\right) = 1+\frac{\pi^2}{4}.$

16.1.28 This is equal to $\displaystyle\int_0^1 \ln(1+xy)\Bigg|_0^1 dx = \int_0^1 \ln(1+x)\,dx = ((1+x)\ln(1+x)-(1+x))\Bigg|_0^1 = 2\ln 2 -$ $2-(-1) = 2\ln 2 - 1.$

16.1.29 $\displaystyle\iint_R \sqrt{\frac{x}{y}}\ dA = \int_1^4\int_0^1 \left(\frac{x}{y}\right)^{1/2}\ dx\,dy = \int_1^4\left(\frac{2}{3}\cdot\frac{x^{3/2}}{y^{1/2}}\right)\Bigg|_0^1 dy = \int_1^4\left(\frac{2}{3\ y^{1/2}}\right)\,dy =$ $\displaystyle\left(\frac{4}{3}y^{1/2}\right)\Bigg|_1^4 = \frac{4}{3}.$

16.1.30 $\displaystyle\iint_R xy\sin x^2\ dA = \int_0^{\sqrt{\pi/2}}\int_0^1 x\,y\,\sin x^2\,dy\,dx = \int_0^{\sqrt{\pi/2}}\left(\frac{x\,y^2}{2}\sin x^2\right)\Bigg|_0^1 dx =$ $\displaystyle\frac{1}{2}\int_0^{\sqrt{\pi/2}}\left(x\,\sin x^2\right)\,dx = -\frac{1}{4}\left(\cos x^2\right)\Bigg|_0^{\sqrt{\pi/2}} = -\frac{1}{4}\left(\cos\left(\frac{\pi}{2}\right)-1\right) = -\frac{1}{4}\,(-1) = \frac{1}{4}.$

16.1.31 $\displaystyle\iint_R e^{x+2y}\ dA = \int_1^{\ln 3}\int_0^{\ln 2}\left(e^x\cdot e^{2y}\right)\ dx\,dy = \int_1^{\ln 3}\left(e^x\cdot e^{2y}\right)\Bigg|_0^{\ln 2} dy = \int_1^{\ln 3} e^{2y}\,dy =$ $\displaystyle\left(\frac{1}{2}e^{2y}\right)\Bigg|_1^{\ln 3} = \frac{1}{2}\left(9-e^2\right).$

16.1.32 $\displaystyle\iint\limits_R (x^2-y^2)^2\,dA = \int_{-1}^{2}\int_0^1 (x^4-2x^2y^2+y^4)\,dy\,dx = \int_{-1}^{2}(x^4y-2x^2y^3/3+y^5/5)\Big|_0^1\,dx =$ $\displaystyle\int_{-1}^{2}(x^4-2x^2/3+1/5)\,dx = (x^5/5-2x^3/9+x/5)\Big|_{-1}^{2} = 32/5-16/9+2/5-(-1/5+2/9-1/5) = 26/5.$

16.1.33 $\displaystyle\iint\limits_R \left(x^5-y^5\right)^2\,dA = \int_{-1}^{1}\int_0^1\left(x^{10}-2x^5y^5+y^{10}\right)\,dx\,dy = \int_{-1}^{1}\left(\frac{x^{11}}{11}-\frac{1}{3}x^6y^5+x\,y^{10}\right)\Big|_0^1\,dy =$ $\displaystyle\int_{-1}^{1}\left(\frac{1}{11}-\frac{1}{3}y^5+y^{10}\right)dy = \left(\frac{y}{11}-\frac{y^6}{18}+\frac{y^{11}}{11}\right)\Big|_{-1}^{1} = \frac{4}{11}.$

16.1.34 $\displaystyle\int_0^{\pi^2/4}\int_0^1\cos\left(x\sqrt{y}\,\right)\,dx\,dy = \int_0^{\pi^2/4}\left(\frac{\sin\left(x\sqrt{y}\right)}{\sqrt{y}}\right)\Big|_0^1\,dy = \int_0^{\pi^2/4}\frac{\sin\left(\sqrt{y}\,\right)}{\sqrt{y}}\,dy =$ $\displaystyle(-2\cos\sqrt{y})\Big|_0^{\pi^2/4} = 2.$

16.1.35 $\displaystyle\int_0^{\sqrt{\pi/2}}\int_0^1 x^3y\cos\left(x^2\,y^2\right)\,dy\,dx = \int_0^{\sqrt{\pi/2}}\left(\frac{x^3}{2x^2}\,\sin\left(x^2y^2\right)\right)\Big|_0^1\,dx = \int_0^{\sqrt{\pi/2}}\frac{1}{2}x\,\sin\left(x^2\right)\,dx =$ $\displaystyle\left(-\frac{1}{4}\cos\left(x^2\right)\right)\Big|_0^{\sqrt{\pi/2}} = \frac{1}{4}\,(1-0) = \frac{1}{4}.$

16.1.36 $\displaystyle\int_{-2}^{2}\int_0^{\ln 4}e^{-x}\,dx\,dy = \int_{-2}^{2}\left(-e^{-x}\right)\Big|_0^{\ln 4}\,dy = \int_{-2}^{2}\frac{3}{4}\,dy = \left(\frac{3}{4}y\right)\Big|_{-2}^{2} = 3.$

16.1.37 $\displaystyle\int_{-1}^{3}\int_0^2(24-3x-4y)\,dy\,dx = \int_{-1}^{3}\left(24y-3x\,y-2y^2\right)\Big|_0^2\,dx = \int_{-1}^{3}(40-6x)\,dx =$ $\displaystyle\left(40x-3x^2\right)\Big|_{-1}^{3} = 136.$

16.1.38 This is given by $\displaystyle\int_0^2\int_0^1 9xy\sqrt{1-x^2}\sqrt{4-y^2}\,dx\,dy = \int_0^2(-3)(1-x^2)^{3/2}y\sqrt{4-y^2}\Big|_{x=0}^{x=1}\,dy =$ $\displaystyle\int_0^2 3y\sqrt{4-y^2}\,dy = -(4-y^2)^{3/2}\Big|_0^2 = -(0-8) = 8.$

16.1.39 This is given by $\displaystyle\int_0^1\int_0^3 xy^2\sqrt{1-x^2}\,dy\,dx = 9\int_0^1 x\sqrt{1-x^2}\,dx = -3(1-x^2)^{3/2}\Big|_0^1 = 3.$

16.1.40 Integrate first with respect to x. $\displaystyle\iint\limits_R y\cos xy\,dA = \int_0^{\pi/3}\int_0^1 y\cos xy\,dx\,dy = \int_0^{\pi/3}\sin xy\Big|_0^1\,dy =$ $\displaystyle\int_0^{\pi/3}\sin y\,dy = -\cos y\Big|_0^{\pi/3} = -(1/2-1) = 1/2.$

16.1.41 Integrate first with respect to x. $\displaystyle\iint\limits_R (y+1)e^{x(y+1)}\,dA = \int_{-1}^{1}\int_0^1(y+1)e^{x(y+1)}\,dx\,dy =$ $\displaystyle\int_{-1}^{1}e^{x(y+1)}\Big|_0^1\,dy = \int_{-1}^{1}(e^{y+1}-1)\,dy = (e^{y+1}-y)\Big|_{-1}^{1} = (e^2-1)-(1-(-1)) = e^2-3.$

16.1.42 Integrate first with respect to y. $\iint\limits_R x\,\sec^2(x\,y)\;dA = \int_0^{\pi/3}\int_0^1 x\,\sec^2(x\,y)\;dy\,dx =$
$\int_0^{\pi/3}\left(x\cdot\left(\frac{1}{x}\right)\tan(x\,y)\right)\Big|_0^1\,dx = \int_0^{\pi/3}\tan x\,dx = (\ln|\sec x\,|)\Big|_0^{\pi/3} = \ln 2.$

16.1.43 Integrate first with respect to y. $\iint\limits_R 6x^5e^{x^3y}\,dA = \int_0^2\int_0^2 6x^5e^{x^3y}\,dy\,dx = \int_0^2 6x^2e^{x^3y}\Big|_0^2\,dx =$
$\int_0^2(6x^2e^{2x^3}-6x^2)\,dx = (e^{2x^3}-2x^3)\Big|_0^2 = e^{16}-16-(1-0) = e^{16}-17.$

16.1.44 Integrate first with respect to x. $\iint\limits_R y^3\,\sin\left(xy^2\right)\;dA = \int_0^{\sqrt{\pi/2}}\int_0^2 y^3\sin\left(xy^2\right)\;dx\,dy =$
$\int_0^{\sqrt{\pi/2}}\left(-\frac{y^3}{y^2}\cos\left(x\,y^2\right)\right)\Big|_0^2\,dy = \int_0^{\sqrt{\pi/2}}\left(y-y\,\cos\left(2y^2\right)\right)\;dy = \left(\frac{y^2}{2}-\frac{1}{4}\sin\left(2y^2\right)\right)\Big|_0^{\sqrt{\pi/2}} = \frac{\pi}{4}.$

16.1.45 Integrate first with respect to y. $\iint\limits_R \frac{x}{(1+x\,y)^2}\,dA = \int_0^4\int_1^2\frac{x}{(1+x\,y)^2}\,dy\,dx.$

Let $u = 1+xy$, so that $du = x\,dy$. Then we have $\int_0^4\left(-\frac{1}{1+x\,y}\right)\Big|_1^2\,dx = \int_0^4\left(\frac{1}{1+x}-\frac{1}{1+2x}\right)\,dx =$
$\left(\ln|1+x|-\frac{1}{2}\ln|1+2x|\right)\Big|_0^4 = \ln 5-\frac{1}{2}\ln 9 = \ln\left(\frac{5}{3}\right).$

16.1.46 $\overline{f}_{\text{ave}} = \frac{1}{\text{area of }R}\iint\limits_R f(x,\,y)\;dA = \frac{1}{4}\int_0^2\int_0^2(4-x-y)\;dy\,dx = \frac{1}{4}\int_0^2\left(4y-x\,y-\frac{y^2}{2}\right)\Big|_0^2\,dx =$
$\frac{1}{4}\int_0^2(6-2x)\;dx = \frac{1}{4}\left(6x-x^2\right)\Big|_0^2 = \frac{1}{4}(8) = 2.$

16.1.47 $\overline{f}_{\text{ave}} = \frac{1}{\text{area of }R}\iint\limits_R f(x,\,y)\;dA = \frac{1}{6\,\ln 2}\int_0^6\int_0^{\ln 2}e^{-y}\,dy\,dx = \frac{1}{6\,\ln 2}\int_0^6\left(-e^{-y}\right)\Big|_0^{\ln 2}\,dx =$
$\frac{1}{6\,\ln 2}\int_0^6\left(\frac{1}{2}\right)\;dx = \frac{1}{6\,\ln 2}\left(\frac{x}{2}\right)\Big|_0^6 = \frac{1}{6\,\ln 2}(3) = \frac{1}{2\ln 2}.$

16.1.48 $\overline{f}_{\text{ave}} = \frac{1}{\text{area of }R}\iint\limits_R f(x,\,y)\;dA = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi\sin x\sin y\,dy\,dx = \frac{1}{\pi^2}\int_0^\pi(-\sin x\cos y)\Big|_0^\pi\,dx =$
$\frac{1}{\pi^2}\int_0^\pi 2\,\sin x\,dx = \frac{2}{\pi^2}(-\cos x)\Big|_0^\pi = \frac{2}{\pi^2}(2) = \frac{4}{\pi^2}.$

16.1.49 $\overline{f}_{\text{ave}} = \frac{1}{\text{area of }R}\iint\limits_R f(x,\,y)\;dA = \frac{1}{8}\int_{-2}^2\int_0^2\left(x^2+y^2\right)\;dy\,dx = \frac{1}{8}\int_{-2}^2\left(x^2y+\frac{1}{3}y^3\right)\Big|_0^2\,dx =$
$\frac{1}{8}\int_{-2}^2\left(2x^2+\frac{8}{3}\right)\;dx = \frac{1}{8}\left(\frac{2}{3}x^3+\frac{8}{3}x\right)\Big|_{-2}^2 = \frac{8}{3}.$

16.1.50 $\overline{f}_{\text{ave}} = \frac{1}{\text{area of }R}\iint\limits_R f(x,\,y)\;dA = \frac{1}{9}\int_0^3\int_0^3\left((x-3)^2+(y-3)^2\right)\;dy\,dx =$
$\frac{1}{9}\int_0^3\left(y\,(x-3)^2+\frac{1}{3}(y-3)^3\right)\Big|_0^3\,dx = \frac{1}{9}\int_0^3\left(3\,(x-3)^2+9\right)\;dx = \frac{1}{9}\left((x-3)^2+9x\right)\Big|_0^3 = 6.$

16.1.51

a. True. The region $R = \{(x,\, y)\,|\, 1 \le x \le 3,\, 4 \le y \le 6\}$ is a rectangle that has width 2 and length 2, thus is a square.

b. False. The region for $\int_4^6 \int_1^3 f(x,\, y)\, dx\, dy$ is $R = \{(x,\, y)\,|\, 1 \le x \le 3,\, 4 \le y \le 6\}$ is not equivalent to the region for $\int_4^6 \int_1^3 f(x,\, y)\, dy\, dx$ which is $R = \{(x,\, y)\,|\, 4 \le x \le 6,\, 1 \le y \le 3\}$ thus Fubini's Theorem does not apply.

c. True. The region is $R = \{(x,\, y)\,|\, 1 \le x \le 3,\, 4 \le y \le 6\}$.

16.1.52

a. $\iint\limits_R xy\, e^{-\left(x^2+y^2\right)}\, dA = \int_{-a}^{a} \int_{-b}^{b} xy\, e^{-\left(x^2+y^2\right)}\, dy\, dx$. Let $f(x,y) = xye^{-(x^2+y^2)}$. Note that $f(-x,y) = -f(x,y)$, so the integral over the portion of R in the first and fourth quadrants cancels the integral over the portion of R in the second and third quadrants, yielding a value of 0.

b. Let $f(x,\, y) = \dfrac{\sin(x-y)}{x^2+y^2+1}$. Note that $f(-x,-y) = -f(x,y)$, so the integral over the portion of R in the first quadrant cancels the integral over the portion in the third quadrant, and likewise for the second and fourth quadrants, yielding a value of 0.

16.1.53

a. total population = (total population region 1) + (total population region 2) + ... +
(total population region 9) = (population density of region 1) × (area of region 1) +
(population density of region 2) × (area of region 2) + ...
+ (population density of region 9) × (area of region 9) = $(350)\left(\frac{1}{2}\right) + (300)\left(\frac{1}{4}\right) + (150)\left(\frac{3}{4}\right) + (500)\left(\frac{1}{2}\right) + (400)\left(\frac{1}{4}\right) + (250)\left(\frac{3}{4}\right) + (250)(1) + (200)\left(\frac{1}{2}\right) + (150)\left(\frac{3}{2}\right) = 1475$.

b. The calculation above could be expressed as the sum $\sum_{k=1}^{3} f(\overline{x}_k,\, \overline{y}_k)\, \Delta A_k$ where $f(\overline{x}_k,\, \overline{y}_k)$ represents the population density for each sub-region of R with area ΔA_k. This sum can be used as an approximation for a 'continuous' function $f(x,\, y)$ that represents the population density at each point in R then $\sum_{k=1}^{n} f(\overline{x}_k,\, \overline{y}_k)\, \Delta A_k \approx \iint\limits_R f(x,\, y)\, dA$.

16.1.54 V $\approx \sum_{k=1}^{n} f(\overline{x}_k,\, \overline{y}_k)\, \Delta A_k$. Also, $\Delta A_k = \Delta x_k \Delta y_k = 5 \cdot 6 = 30$. So $\sum_{k=1}^{15} f(\overline{x}_k,\, \overline{y}_k)\, \Delta A_k = 30 \cdot (f(\overline{x}_1,\, \overline{y}_1) + f(\overline{x}_2,\, \overline{y}_2) + \cdots + f(\overline{x}_{15},\, \overline{y}_{15})) =$
$30 \cdot (1 + 1.5 + 2.0 + 2.5 + 3.0 + 1 + 1.5 + 2.0 + 2.5 + 3.0 + 0.75 + 1.25 + 1.75 + 2.25 + 2.75) = 862.5\,\text{m}^3$.

16.1.55 The area of the constant cross section of S is $A = \int_a^b f(x)\, dx$ for any value of y. The volume of S can be expressed as $\int_c^d \int_a^b f(x)\, dx\, dy = \int_c^d A\, dy = A \cdot y\Big|_c^d = A(d-c)$.

16.1.56

a. $\int_c^d \int_a^b f(x,y)\, dx\, dy = \int_c^d \int_a^b (g(x) \cdot h(y))\, dx\, dy = \int_c^d h(y) \left(\int_a^b g(x)\, dx \right) dy$, because $h(y)$ is constant relative to x.
Also, $\int_a^b g(x)\, dx$ is constant relative to y, so we have $\left(\int_a^b g(x)\, dx \right) \left(\int_c^d h(y)\, dy \right)$.

b. $\left(\int_a^b g(x)\,dx\right)^2 = \left(\int_a^b g(x)\,dx\right)\left(\int_a^b g(x)\,dx\right) = \int_a^b (g(x))^2\,dx.$

c. $\int_0^{2\pi}\int_{10}^{30}\left(\cos x\cdot e^{-4y^2}\right)dy\,dx = \left(\int_0^{2\pi}\cos x\,dx\right)\left(\int_{10}^{30}e^{-4y^2}\,dy\right) = (0)\left(\int_{10}^{30}e^{-4y^2}\,dy\right) = 0.$

16.1.57 $\int_0^a\int_0^\pi \sin(x+y)\,dx\,dy = \int_0^a (-\cos(x+y))\Big|_0^\pi dy = \int_0^a(\cos y - \cos(\pi+y))\,dy =$ $(\sin y - \sin(\pi+y))\Big|_0^a = (\sin a - \sin(\pi+a)) = 1$. Solving for a yields: $(\sin a - \sin(\pi+a)) = 1$, so $2\sin a = 1$, so $\sin a = \frac{1}{2}$. The solutions are $a = \frac{\pi}{6}$ and $a = \frac{5\pi}{6}$.

16.1.58 $\overline{f} = \frac{1}{a^2}\int_0^a\int_0^a (x+y-8)\,dx\,dy = \frac{1}{a^2}\int_0^a\left(\frac{x^2}{2}+x\,y-8x\right)\Big|_0^a dy = \frac{1}{a^2}\int_0^a\left(\frac{a^2}{2}+a\,y-8a\right)dy =$ $\frac{1}{a^2}\left(\frac{a^2}{2}y+\frac{a\,y^2}{2}-8a\,y\right)\Big|_0^a = \frac{1}{a^2}\left(\frac{a^3}{2}+\frac{a^3}{2}-8a^2\right) = a-8 = \overline{f}.$

Setting $\overline{f}=0$ and solving for a yields $a-8=0$, thus $a=8$.

16.1.59 $\overline{f} = \frac{1}{a^2}\int_0^a\int_0^a\left(4-x^2-y^2\right)dy\,dx = \frac{1}{a^2}\int_0^a\left(4y-x^2y-\frac{y^3}{3}\right)\Big|_0^a dx =$ $\frac{1}{a^2}\int_0^a\left(4a-a\,x^2-\frac{a^3}{3}\right)dx = \frac{1}{a^2}\left(4a\,x-\frac{a\,x^3}{3}-\frac{a^3}{3}x\right)\Big|_0^a = \frac{1}{a^2}\left(4a^2-\frac{2a^4}{3}\right) = 4-\frac{2a^2}{3} = \overline{f}$. Setting $\overline{f}=0$ and solving for a yields $4-\frac{2a^2}{3}=0$, so $a^2=6$. Because $a>0$, we have $a=\sqrt{6}$.

16.1.60 Consider $\int_0^b\int_0^a (6-x-3y)\,dx\,dy = \int_0^b\left(6x-x^2/2-3xy\right)\Big|_0^a dy = \int_0^b\left(6a-a^2/2-3ay\right)dy =$ $\left((6a-a^2/2)y-3ay^2/2\right)\Big|_0^b = \left(6ab-a^2b/2-3ab^2/2\right).$

Maximize $\left(6ab-a^2b/2-3ab^2/2\right)$ with the constraint $a=6-3b$. Substituting $6-3b$ for a gives $9(2b-b^2)$. This is maximized for $b=1$. Thus, the desired point is $(3,1)$ and the value of the integral is 9.

16.1.61

a. $m = \iint_R \rho(x,y)\,dA = \int_0^\pi\int_0^{\pi/2}(1+\sin x)\,dx\,dy = \int_0^\pi (x-\cos x)\Big|_0^{\pi/2} dy =$ $\int_0^\pi\left(\frac{\pi}{2}+1\right)dy = \left(\frac{\pi}{2}+1\right)y\Big|_0^\pi = \frac{\pi^2}{2}+\pi.$

b. $m = \iint_R \rho(x,y)\,dA = \int_0^{\pi/2}\int_0^\pi(1+\sin y)\,dy\,dx = \int_0^{\pi/2}(y-\cos y)\Big|_0^\pi dx =$ $\int_0^\pi(\pi+2)\,dx = (\pi+2)\,x\Big|_0^{\pi/2} = \frac{\pi^2}{2}+\pi.$

c. $m = \iint_R \rho(x,y)\,dA = \int_0^\pi\int_0^{\pi/2}(1+\sin x\,\sin y)\,dx\,dy =$ $\int_0^\pi(x-\cos x\,\sin y)\Big|_0^{\pi/2} dy = \int_0^\pi\left(\frac{\pi}{2}+\sin y\right)dy = \left(\frac{\pi}{2}y-\cos y\right)\Big|_0^\pi = \frac{\pi^2}{2}+2.$

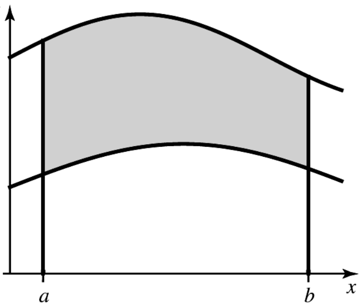
16.1.62 Let the base of the shed be in the xy-plane with corners at (0, 0), (16, 0), (16, 10), and (0, 10). Let the peak of the roof be over the point (0, 0). With these assumptions the height of the roof $h(x, y)$ over any point (x, y) in the base is the lesser of these two functions $f(x, y) = 12 - \frac{x}{4}$ and $g(x, y) = 12 - \frac{2y}{5}$. Divide the base of the shed into 40 squares each of size 2 by 2 (that is, $\triangle x = \triangle y = 2$), and produce the Riemann sum for $h(x, y)$ over the base, evaluating $h(x, y)$ at the midpoint of each square. Thus $\overline{x}_1 = 1$, $\overline{x}_2 = 3$, $\overline{x}_3 = 5$, $\overline{x}_4 = 7$, $\overline{x}_5 = 9$, $\overline{x}_6 = 11$, $\overline{x}_7 = 13$, and $\overline{x}_8 = 15$. Likewise $\overline{y}_1 = 1$, $\overline{y}_2 = 3$, $\overline{y}_3 = 5$, $\overline{y}_4 = 7$, and $\overline{y}_5 = 9$. The volume will thus be approximated by the Riemann sum $\displaystyle\sum_{j=1}^{8}\sum_{i=1}^{5} h\left(\overline{x}_i, \overline{y}_j\right) \triangle x_i \triangle y_j = 373.7 \times 4 = 1494.8$.

16.1.63 $$\iint_R \frac{\partial^2 f}{\partial x\, \partial y}\, dA = \int_0^b \int_0^a \left(\frac{\partial^2 f}{\partial x\, \partial y}\right) dx\, dy = \int_0^b \left(\frac{\partial f}{\partial y}\right)\Bigg|_0^a dy = \int_0^b \left(\frac{\partial f}{\partial y}(a, y) - \frac{\partial f}{\partial y}(0, y)\right) dy =$$

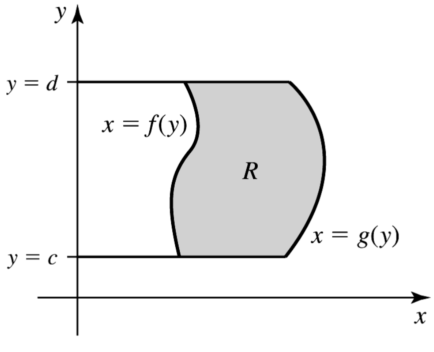
$$\left(f(a, y) - f(0, y)\right)\Bigg|_0^b = f(a, b) - f(0, b) - f(a, 0) + f(0, 0).$$

16.2 Double Integrals over General Regions

16.2.1 A region is bounded below by $y = f(x)$ and above by $y = g(x)$, on the left by $x = a$ and on the right by $x = b$. $R = \{(x, y) \mid a \le x \le b,\ f(x) \le y \le g(x)\}$.



16.2.2 A region is bounded below by $y = c$ and above by $y = d$, on the left by $x = f(y)$ and on the right by $x = g(y)$. $R = \{(x, y) \mid f(y) \le x \le g(y),\ c \le y \le d$.

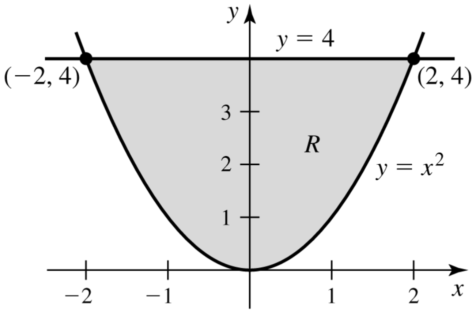


16.2.3 Integrate first with respect to x, then with respect to y. $\displaystyle\iint_R f(x, y)\, dA = \int_0^1 \int_{y-1}^{1-y} x\, y\, dx\, dy$.

16.2.4 Integrate first with respect to x, then with respect to y. $\displaystyle A = \iint_R (1)\, dA = \int_0^{17} \int_{\frac{y-3}{2}}^{\frac{y+4}{3}} (1)\, dx\, dy$.

16.2.5 $\displaystyle\int_0^1 \int_{x^2}^{\sqrt{x}} f(x, y)\, dy\, dx$.

16.2.6



16.2.7 This is a right triangle with base 2 and height 4, so the area is 4.

16.2.8 This is circular cylinder of radius 4 and height 10, so its volume is $\pi r^2 h = \pi \cdot 16 \cdot 10 = 160\pi$.

16.2.9 $\displaystyle\iint_R f(x,\,y)\;dA = \int_0^2 \int_{x^2}^{4x} f(x,\,y)\;\,dy\,dx.$

16.2.10 $\displaystyle\iint_R f(x,\,y)\;dA = \int_{-3}^{4} \int_{2x^2}^{2x+24} f(x,\,y)\;dy\,dx.$

16.2.11 $\displaystyle\int_0^1 3y^2\Big|_x^1\,dx = \int_0^1 (3-3x^2)\,dx = (3x-x^3)\Big|_0^1 = 2.$

16.2.12 $\displaystyle\int_0^1 5xy^3\Big|_0^{2x}\,dx = \int_0^1 40x^4\,dx = 8x^5\Big|_0^1 = 8.$

16.2.13 $\displaystyle\int_0^2 \int_{x^2}^{2x} x\,y\,dy\,dx = \int_0^2 \left(\frac{x\,y^2}{2}\right)\Bigg|_{x^2}^{2x} dx = \frac{1}{2}\int_0^2 \left(4x^3 - x^5\right)\,dx = \frac{1}{2}\left(x^4 - \frac{x^6}{6}\right)\Bigg|_0^2 = \frac{8}{3}\,.$

16.2.14 $\displaystyle\int_{-\pi/4}^{\pi/4} \int_{\sin x}^{\cos x} dy\,dx = \int_{-\pi/4}^{\pi/4} (y)\Big|_{\sin x}^{\cos x} dx = \int_{-\pi/4}^{\pi/4} (\cos x - \sin x)\,dx = (\sin x + \cos x)\Big|_{-\pi/4}^{\pi/4} = \sqrt{2}.$

16.2.15 $\displaystyle\int_{-2}^{2}\int_{x^2}^{8\,-\,x^2} x\,dy\,dx = \int_{-2}^{2} (x\,y)\Big|_{x^2}^{8-x^2} dx = \int_{-2}^{2} \left(8x - 2x^3\right)\,dx = \left(4x^2 - \frac{x^4}{2}\right)\Bigg|_{-2}^{2} = 0.$

16.2.16 $\displaystyle\int_0^{\ln 2}\int_{e^x}^{2} dy\,dx = \int_0^{\ln 2} (y)\Big|_{e^x}^{2} dx = \int_0^{\ln 2} (2 - e^x)\,dx = (2x - e^x)\Big|_0^{\ln 2} = 2\ln 2 - 1.$

16.2.17 $\displaystyle\int_0^1 \int_0^x 2e^{x^2}\,dy\,dx = \int_0^1 2xe^{x^2}\,dx = e^{x^2}\Big|_0^1 = e - 1.$

16.2.18 $\displaystyle\int_0^{\sqrt[3]{\pi/2}} \int_0^x y\cos x^3\,dy\,dx = \int_0^{\sqrt[3]{\pi/2}} (y^2/2)\cos x^3\Big|_0^x dx = \frac{1}{2}\int_0^{\sqrt[3]{\pi/2}} x^2 \cos x^3\,dx =$
$\displaystyle\frac{1}{6}\left(\sin x^3\right)\Big|_0^{\sqrt[3]{\pi/2}} = \frac{1}{6}.$

16.2.19 $\displaystyle\int_0^{\ln 2} \int_{e^y}^{2} \left(\frac{y}{x}\right)\,dx\,dy = \int_0^{\ln 2} (y\ln|\,x\,|)\Big|_{e^y}^{2} dy = \int_0^{\ln 2} \left(y\ln 2 - y^2\right)\,dy =$
$\displaystyle\left(\frac{\ln 2}{2}y^2 - \frac{y^3}{3}\right)\Bigg|_0^{\ln 2} = \frac{(\ln 2)^3}{6}.$

16.2.20 $\int_0^4 \int_y^{2y} (x\,y)\,dx\,dy = \int_0^4 \left(\frac{x^2y}{2}\right)\bigg|_y^{2y} dy = \int_0^4 \left(\frac{3}{2}y^3\right)\,dy = \left(\frac{3}{8}y^4\right)\bigg|_0^4 = 96.$

16.2.21 $\int_0^{\pi/2} \int_y^{\pi/2} 6\sin(2x-3y)\,dx\,dy = \int_0^{\pi/2} (-3\cos(2x-3y))\bigg|_y^{\pi/2} dy =$
$-3\int_0^{\pi/2} (\cos(\pi-3y)-\cos(-y))\,dy = (\sin(\pi-3y)-3\sin(-y))\bigg|_0^{\pi/2} = -1-(-3) = 2.$

16.2.22 $\int_0^{\pi/2} \int_0^{\cos y} e^{\sin y}\,dx\,dy = \int_0^{\pi/2} (xe^{\sin y})\bigg|_0^{\cos y} dy = \int \cos y e^{\sin y}\,dy = e^{\sin y}\bigg|_0^{\pi/2} = e-1.$

16.2.23 $\int_0^{\pi/2} y\cos y\,dy = y\sin y\bigg|_0^{\pi/2} - \int_0^{\pi/2} \sin y\,dy = \pi/2 + \left(\cos y\bigg|_0^{\pi/2}\right) = \frac{\pi}{2}-1$, where the first integral is computed via integration by parts with $u = y$ and $dv = \cos y$.

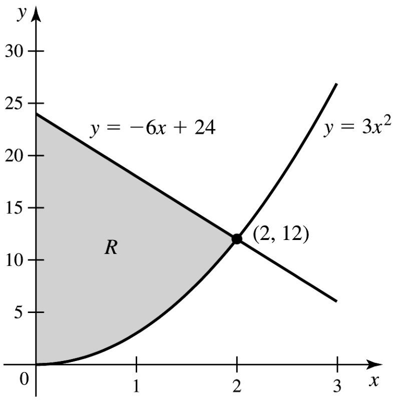
16.2.24 $\int_0^1 2x\left(\frac{\pi}{4}-\tan^{-1}x\right)\,dx = \frac{x^2\pi}{4}\bigg|_0^1 - \int_0^1 2x\tan^{-1}x\,dx$. Using Integration by Parts, (with $u = \tan^{-1}x$ and $dv = 2x\,dx$, we can deduce that $\int 2x\tan^{-1}x\,dx = -x+\tan^{-1}x + x^2\tan^{-1}x + C$, so our integral is equal to $\frac{\pi}{4} - \left(-x+\tan^{-1}x+x^2\tan^{-1}x\right)\bigg|_0^1 = 1-\frac{\pi}{4}.$

16.2.25 $R = \{(x,\,y) \mid -\sqrt{16-y^2} \le x \le \sqrt{16-y^2},\, 0 \le y \le 4\}$. We have $\int_0^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 2xy\,dx\,dy =$
$\int_0^4 (x^2\,y)\bigg|_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} dy = \int_0^4 0\,dy = 0.$

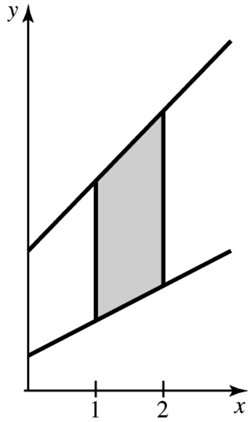
16.2.26 $\int_0^1 \int_0^x 2e^x\,dy\,dx = \int_0^1 2e^x y\bigg|_{y=0}^{y=x} dx = \int_0^1 2xe^x\,dx = 2e^x(x-1)\bigg|_0^1 = 0-(-2) = 2$, where the last integral is computed via integration by parts with $u = x$ and $dv = e^x\,dx$.

16.2.27 $\int_{\pi/2}^{\pi} \int_0^{y^2} \cos\frac{x}{y}\,dx\,dy = \int_{\pi/2}^{\pi} y\sin\frac{x}{y}\bigg|_{x=0}^{x=y^2} dy = \int_{\pi/2}^{\pi} y\sin y\,dy = (\sin y - y\cos y)\bigg|_{\pi/2}^{\pi} = \pi-1.$

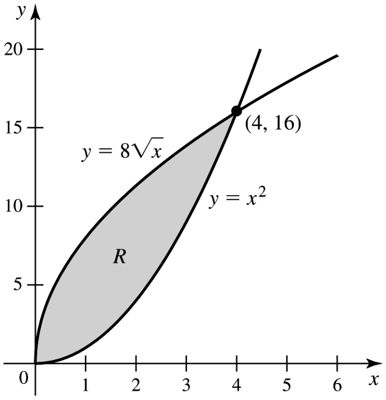
16.2.28 $\iint_R f(x,\,y)\,dA = \int_0^2 \int_{3x^2}^{-6x+24} f(x,\,y)\,dy\,dx.$



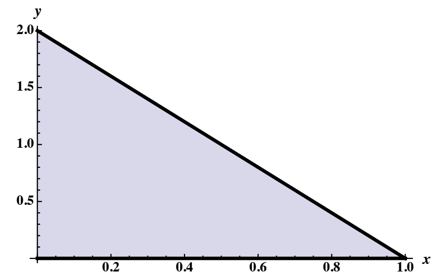
16.2.29 $\displaystyle\iint_R f(x, y)\, dA = \int_1^2 \int_{x+1}^{2x+4} f(x, y)\, dy\, dx.$



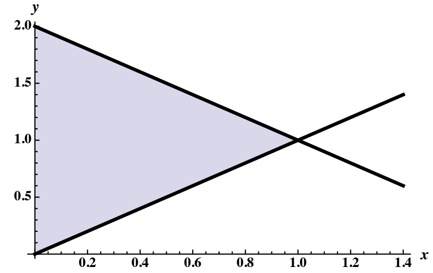
16.2.30 $\displaystyle\iint_R f(x, y)\, dA = \int_0^4 \int_{x^2}^{8\sqrt{x}} f(x, y)\, dy\, dx \quad \text{(or)} \quad = \int_0^{16} \int_{y^2/16}^{\sqrt{y}} f(x, y)\, dx\, dy.$



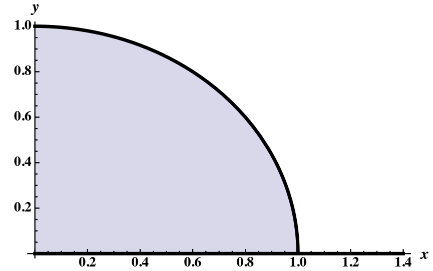
16.2.31 $\displaystyle\iint_R f(x,y)\, dA = \int_0^1 \int_0^{-2x+2} f(x,y)\, dy\, dx.$



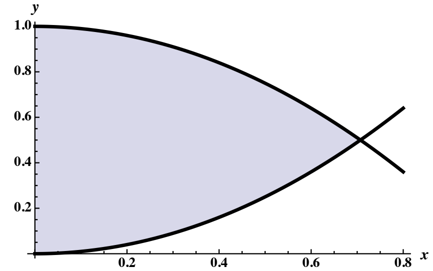
16.2.32 $\displaystyle\iint_R f(x,y)\, dA = \int_0^1 \int_x^{-x+2} f(x,y)\, dy\, dx.$



16.2.33 $\displaystyle\iint_R f(x,y)\,dA = \int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\,dy\,dx.$



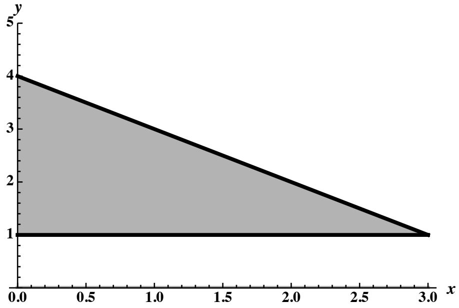
16.2.34 $\displaystyle\iint_R f(x,y)\,dA = \int_0^{1/\sqrt{2}} \int_{x^2}^{1-x^2} f(x,y)\,dy\,dx.$



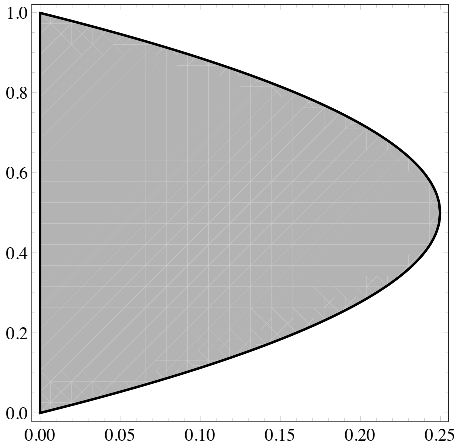
16.2.35 $\displaystyle\iint_R f(x,\,y)\,dA = \int_0^{18} \int_{y/2}^{(y+9)/3} f(x,\,y)\,dx\,dy.$

16.2.36 $\displaystyle\iint_R f(x,\,y)\,dA = \int_{-4}^{5} \int_{(5-y)/3}^{\sqrt{25-y^2}} f(x,\,y)\,dx\,dy.$

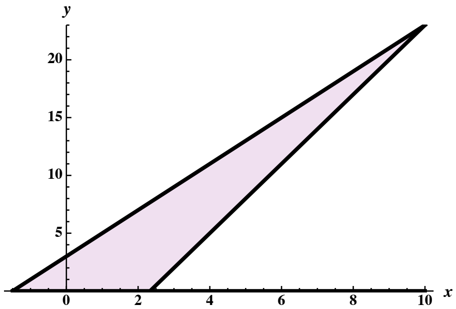
16.2.37 $\displaystyle\iint_R f(x,\,y)\,dA = \int_1^4 \int_0^{4-y} f(x,\,y)\,dx\,dy = \int_0^3 \int_1^{4-x} f(x,\,y)\,dy\,dx.$



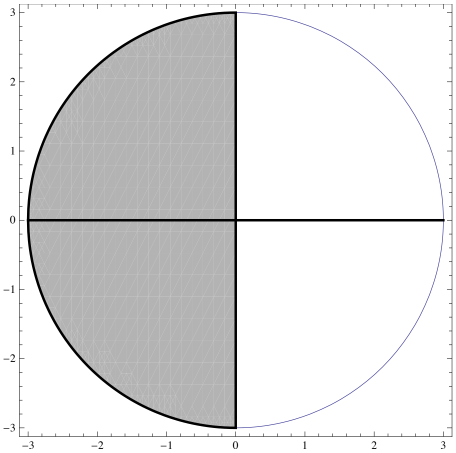
16.2.38 $\displaystyle\iint_R f(x,\, y)\, dA = \int_0^1 \int_0^{y(1-y)} f(x,\, y)\, dx\, dy.$



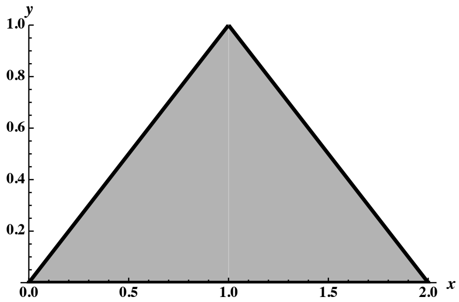
16.2.39 $\displaystyle\iint_R f(x,\, y)\, dA = \int_0^{23} \int_{(y-3)/2}^{(y+7)/3} f(x,\, y)\, dx\, dy.$



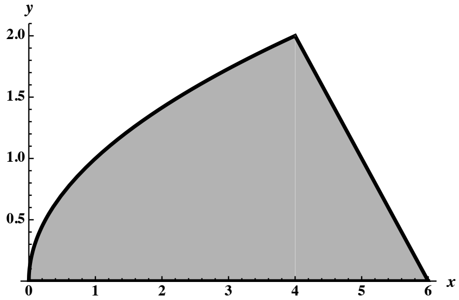
16.2.40 $\displaystyle\iint_R f(x,\, y)\, dA = \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{0} f(x,\, y)\, dx\, dy.$



16.2.41 $\displaystyle\iint_R f(x,\, y)\, dA = \int_0^1 \int_y^{2-y} f(x,\, y)\, dx\, dy.$

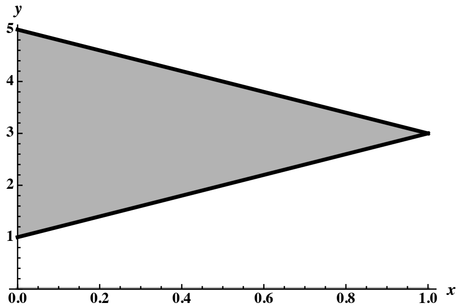


16.2.42 $\displaystyle\iint\limits_R f(x,\,y)\;dA = \int_0^2\int_{y^2}^{6-y} f(x,\,y)\;dx\,dy.$



16.2.43 $\displaystyle\iint\limits_R xy\,dA = \int_0^1\int_{2x+1}^{-2x+5} xy\,dy\,dx = \int_0^1 \left(\frac{xy^2}{2}\right)\bigg|_{2x+1}^{-2x+5} dx = \int_0^1 \left(12x-12x^2\right)\,dx =$

$\displaystyle\left(6x^2-4x^3\right)\bigg|_0^1 = 2.$

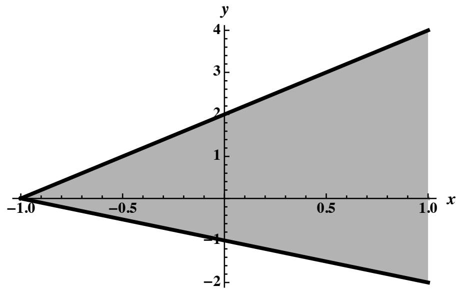


16.2.44 $\displaystyle\iint\limits_R (x+y)\;dA = \int_0^2\int_{x^2}^{8-x^2} (x+y)\;dy\,dx = \int_0^2 \left(x\,y+\frac{y^2}{2}\right)\bigg|_{x^2}^{8-x^2} dx =$

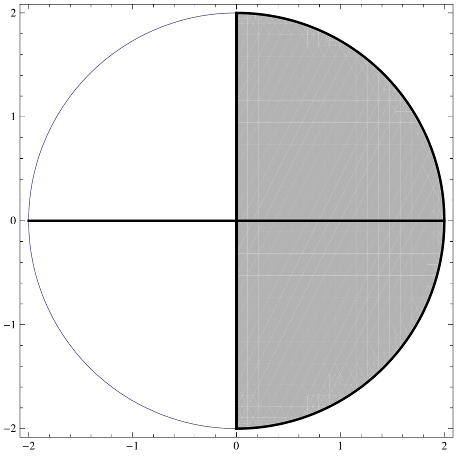
$\displaystyle\int_0^2 \left(32+8x-8x^2-2x^3\right)\,dx = \left(32x+4x^2-\frac{8}{3}x^3-\frac{x^4}{2}\right)\bigg|_0^2 = \frac{152}{3}.$

16.2.45 $\displaystyle\iint\limits_R y^2\,dA = \int_{-1}^1\int_{-x-1}^{2x+2} y^2\,dy\,dx = \int_{-1}^1 \frac{y^3}{3}\bigg|_{-x-1}^{2x+2} dx = \frac{1}{3}\int_{-1}^1 \left(8\,(x+1)^3+(x+1)^3\right)\,dx =$

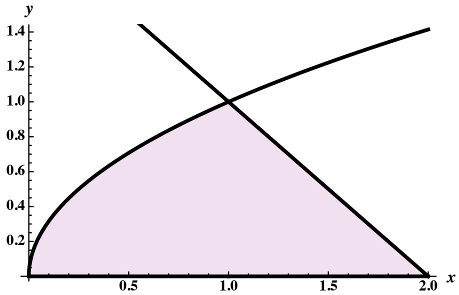
$\displaystyle 3\int_{-1}^1 (x+1)^3\;dx = \frac{3}{4}\,(x+1)^4\bigg|_{-1}^1 = 12.$



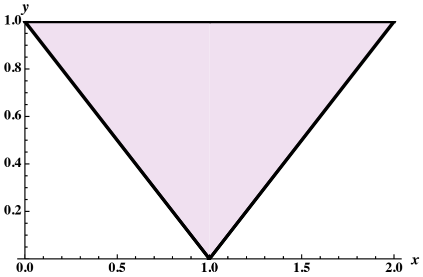
16.2.46 $\iint\limits_R x^2y\,dA = \int_{-4}^{4}\int_0^{\sqrt{16-y^2}} x^2y\,dx\,dy = \int_{-4}^{4}\left(\frac{x^3y}{3}\right)\Big|_0^{\sqrt{16-y^2}}\,dy = \frac{1}{3}\int_{-4}^{4} y\left(16-y^2\right)^{3/2}\,dx$. Let $u = 16-y^2$ so that $du = -2y\,dy$, Then the integral is equal to $\frac{1}{6}\int_0^0 u^{3/2}\,du = 0$.



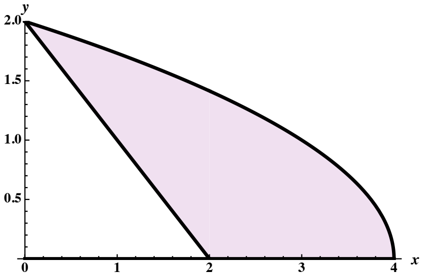
16.2.47 $\iint\limits_R 12y\,dA = \int_0^1\int_{y^2}^{2-y} 12y\,dx\,dy = \int_0^1 (12xy)\Big|_{y^2}^{2-y}\,dy = 12\int_0^1 (2y-y^2-y^3)\,dy = 12(y^2-y^3/3-y^4/4)\Big|_0^1 = 12(5/12) = 5$.



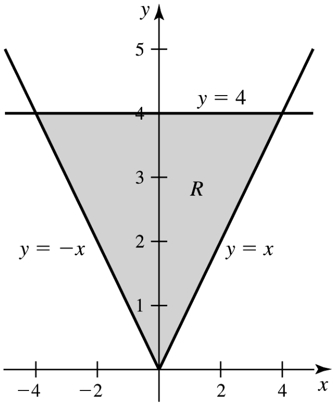
16.2.48 $\iint\limits_R y^2\,dA = \int_0^1\int_{1-y}^{y+1} y^2\,dx\,dy = \int_0^1 (xy^2)\Big|_{1-y}^{y+1}\,dy = \int_0^1 (y^3+y^2-y^2+y^3)\,dy = \int_0^1 2y^3\,dy = y^4/2\Big|_0^1 = 1/2$.



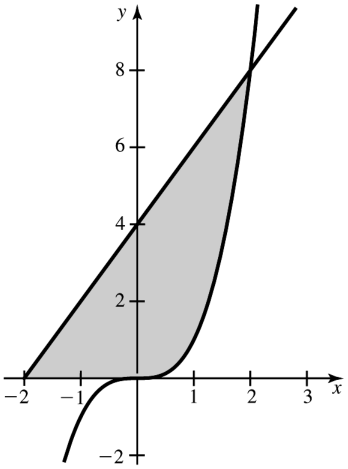
16.2.49 $\iint_R 3xy\,dA = \int_0^2 \int_{2-y}^{4-y^2} 3xy\,dx\,dy = \int_0^2 (3x^2y/2)\Big|_{2-y}^{4-y^2} dy = \int_0^2 (3(4-y^2)^2y/2 - 3(2-y)^2y/2)\,dy = \frac{3}{2}\int_0^2 (16y-8y^3+y^5-(4y-4y^2+y^3))\,dy = \frac{3}{2}\int_0^2 (y^5-9y^3+4y^2+12y)\,dy = \frac{3}{2}(y^6/6-9y^4/4+4y^3/3+6y^2)\Big|_0^2 = \frac{3}{2}(32/3-36+32/3+24) = 14.$



16.2.50 $\iint_R (x+y)\;dA \;=\; \int_0^4 \int_{-y}^{y} (x+y)\;dx\,dy \;=\; \int_0^4 \left(\frac{x^2}{2}+xy\right)\Big|_{-y}^{y} dy \;=\; \int_0^4 \left(2y^2\right)\;dy \;=\; \left(\frac{2}{3}y^3\right)\Big|_0^4 = \frac{128}{4}.$



16.2.51 $\iint_R 3x^2\,dA = \int_0^8 \int_{\frac{y-4}{2}}^{\sqrt[3]{y}} 3x^2\,dx\,dy = \int_0^8 \left(x^3\right)\Big|_{\frac{y-4}{2}}^{\sqrt[3]{y}} dy = \int_0^8 \left(y-(1/8)(y^3-12y^2+48y-64)\right)\,dy = \int_0^8 (-y^3/8+3y^2/2-5y+8)\,dy = (-y^4/32+y^3/2-5y^2/2+8y)\Big|_0^8 = 32.$



16.2.52 $\displaystyle\iint_R 8xy\,dA = \int_0^{\pi/4}\int_0^{\sec x} 8xy\,dy\,dx = \int_0^{\pi/4} 4x\sec^2 x\,dx = 4(\ln\cos x + x\tan x)\Big|_0^{\pi/4} = 4(\ln\sqrt{2}/2 + \pi/4) - 0 = \ln(1/4) + \pi = -2\ln 2 + \pi$

16.2.53 $\displaystyle\iint_R (x+y)\ dA = \int_{1/2}^{2}\int_{1/x}^{5/2-x} (x+y)\ dy\,dx = \int_{1/2}^{2}\left(xy + \frac{1}{2}y^2\right)\Bigg|_{\frac{1}{x}}^{\frac{5}{2}-x} dx = \int_{1/2}^{2}\left(\frac{17}{8} - \frac{x^2}{2} - \frac{1}{2x^2}\right) dx = \left(\frac{17}{8}x - \frac{x^3}{6} + \frac{1}{2x}\right)\Bigg|_{1/2}^{1} = \frac{9}{8}.$

16.2.54

$$\begin{aligned}\int_0^1\int_0^{y^2}\frac{y}{1+x+y^2}\,dx\,dy &= \int_0^1 y\ln(1+x+y^2)\Big|_{x=0}^{x=y^2}\\ &= \int_0^1 \left(y\ln(1+2y^2) - y\ln(1+y^2)\right)\,dy\\ &= \int_0^1 y\ln(1+2y^2)\,dy - \int_0^1 y\ln(1+y^2)\,dy\end{aligned}$$

For the first integral we let $u = 1+2y^2$ and for the second we let $u = 1+y^2$. We have

$$\begin{aligned}&\frac{1}{4}\left(2y^2+1\right)\left(\ln\left(2y^2+1\right)-1\right)\Big|_0^1 - \left(-\frac{y^2}{2} + \frac{1}{2}y^2\ln\left(y^2+1\right) + \frac{1}{2}\ln\left(y^2+1\right)\right)\Bigg|_0^1\\ &= \frac{3}{4}(\ln 3 - 1) + \frac{1}{4} - \left(\ln 2 - \frac{1}{2}\right) = \frac{3}{4}\ln 3 - \ln 2.\end{aligned}$$

16.2.55 $\displaystyle\iint_R x\sec^2 y\ dA = \int_0^{\sqrt{\pi}/2}\int_0^{x^2} x\sec^2 y\ dy\,dx = \int_0^{\sqrt{\pi}/2} x\tan y\Big|_0^{x^2} dx = \int_0^{\sqrt{\pi}/2} x\tan x^2\,dx = \left(\frac{1}{2}\ln\left|\sec x^2\right|\right)\Bigg|_0^{\sqrt{\pi}/2} = \frac{1}{2}\ln\left(\sqrt{2}\right) = \frac{1}{4}\ln 2.$

16.2.56 $\displaystyle\iint_R \frac{8xy}{1+x^2+y^2}\ dA = \int_0^2\int_0^x \frac{8xy}{1+x^2+y^2}\,dy\,dx = \int_0^2 4x\ln\left(1+x^2+y^2\right)\Big|_0^x dx =$ $\displaystyle 4\int_0^2\left(x\ln\left(1+2x^2\right) - x\ln\left(1+x^2\right)\right)\,dx$. Let $u = 1+2x^2$ and $v = 1+x^2$. Substituting gives $\displaystyle\int_1^9 \ln u\,du - 2\int_1^5 \ln v\,dv = (u\ln u - u)\Big|_{u=1}^{u=9} - 2\,(v\ln v - v)\Big|_{v=1}^{v=5} = (9\ln 9 - 8) - (10\ln 5 - 8) = 9\ln 9 - 10\ln 5.$

16.2.57 $\int_0^2 \int_{x^2}^{2x} f(x,\,y)\;dy\,dx = \int_0^4 \int_{y/2}^{\sqrt{y}} f(x,\,y)\;dx\,dy.$

16.2.58 $\int_0^3 \int_0^{6-2x} f(x,\,y)\;dy\,dx = \int_0^6 \int_0^{\frac{6-y}{2}} f(x,\,y)\;dx\,dy.$

16.2.59 $\int_{1/2}^1 \int_0^{-\ln y} f(x,\,y)\;dx\,dy = \int_0^{\ln 2} \int_{1/2}^{e^{-x}} f(x,\,y)\;dy\,dx.$

16.2.60 $\int_0^1 \int_1^{e^y} f(x,\,y)\;dx\,dy = \int_1^e \int_{\ln x}^1 f(x,\,y)\;dy\,dx.$

16.2.61 $\int_0^1 \int_0^{\cos^{-1} y} f(x,\,y)\;dx\,dy = \int_0^{\pi/2} \int_0^{\cos x} f(x,\,y)\;dy\,dx.$

16.2.62 $\int_1^e \int_0^{\ln x} f(x,\,y)\;dy\,dx = \int_0^1 \int_{e^y}^e f(x,\,y)\;dx\,dy.$

16.2.63 $\int_0^1 \int_y^1 e^{x^2}\,dx\,dy = \int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 y\,e^{x^2}\Big|_0^x dx = \int_0^1 \left(x\,e^{x^2}\right)\,dx = \left(\frac{1}{2}e^{x^2}\right)\Big|_0^1 = \frac{1}{2}(e-1).$

16.2.64 $\int_0^\pi \int_x^\pi \sin y^2\,dy\,dx = \int_0^\pi \int_0^y \sin y^2\,dx\,dy = \int_0^\pi \left(x\,\sin y^2\right)\Big|_0^y dy = \int_0^\pi y\,\sin y^2\,dy =$
$\left(-\frac{1}{2}\cos y^2\right)\Big|_0^\pi = \frac{1}{2} - \frac{1}{2}\cos \pi^2.$

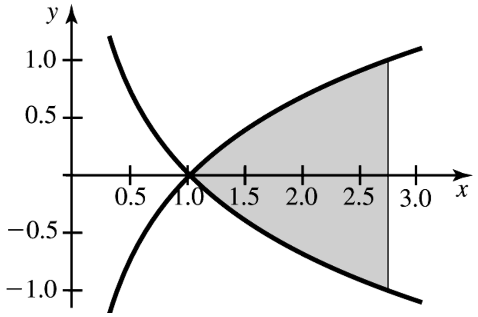
16.2.65 $\int_0^{1/2} \int_{y^2}^{1/4} y\,\cos\left(16\pi x^2\right)\,dx\,dy = \int_0^{1/4} \int_0^{\sqrt{x}} y\,\cos\left(16\pi x^2\right)\,dy\,dx =$
$\int_0^{1/4} \left(\frac{1}{2}y^2\,\cos\left(16\pi x^2\right)\right)\Big|_0^{\sqrt{x}} dx = \int_0^{1/4} \frac{1}{2}x\,\cos\left(16\pi x^2\right)\,dx$. Let $u = 16\pi x^2$ so that $du = 32\pi\,dx$. We have $\int_0^\pi \frac{\cos u}{64\pi}\,du = \left(\frac{\sin u}{64\pi}\right)\Big|_{u=0}^{u=\pi} = 0$.

16.2.66 $\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1}\,dy\,dx = \int_0^2 \int_0^{y^2} \frac{x}{y^5+1}\,dx\,dy = \int_0^2 \left(\frac{x^2}{2\left(y^5+1\right)}\right)\Big|_0^{y^2} dy = \int_0^2 \frac{y^4}{2\left(y^5+1\right)}\,dy$. Let $u = y^5+1$ so that $du = 5y^4\,dy$. Substituting gives $\int_1^{33} \frac{1}{10u}\,du = \frac{1}{10}\left(\ln|u|\right)\Big|_{u=1}^{u=33} = \frac{\ln 33}{10}$.

16.2.67 $\int_0^{\sqrt[3]{\pi}} \int_y^{\sqrt[3]{\pi}} x^4\,\cos\left(x^2 y\right)\,dx\,dy = \int_0^{\sqrt[3]{\pi}} \int_0^x x^4\,\cos\left(x^2 y\right)\,dy\,dx = \int_0^{\sqrt[3]{\pi}} \left(x^2\,\sin\left(x^2 y\right)\right)\Big|_0^x dx =$
$\int_0^{\sqrt[3]{\pi}} x^2\,\sin\left(x^3\right)\,dx = -\frac{1}{3}\left(\cos x^3\right)\Big|_{u=0}^{u=\sqrt[3]{\pi}} = \frac{2}{3}$.

16.2.68 $\int_0^2 \int_0^{4-x^2} \frac{x\,e^{2y}}{4-y}\,dy\,dx = \int_0^4 \int_0^{\sqrt{4-y}} \frac{x\,e^{2y}}{4-y}\,dx\,dy = \int_0^4 \left(\frac{x^2\,e^{2y}}{2\,(4-y)}\right)\Big|_0^{\sqrt{4-y}} dy =$
$\int_0^4 \frac{(4-y)\,e^{2y}}{2\,(4-y)}\,dy = \int_0^4 \frac{e^{2y}}{2}\,dy = \frac{1}{4}\left(e^{2y}\right)\Big|_0^4 = \frac{1}{4}\left(e^8-1\right).$

16.2.69 $\int_0^1 \int_{e^y}^e f(x,\,y)\;dx\,dy + \int_{-1}^0 \int_{e^{-y}}^e f(x,\,y)\;dx\,dy = \int_1^e \int_{-\ln x}^{\ln x} f(x,\,y)\;dy\,dx.$



16.2.70 $\int_{-4}^{0}\int_{0}^{\sqrt{16-x^2}} f(x,y)\,dy\,dx + \int_{0}^{4}\int_{0}^{4-x} f(x,y)\,dy\,dx = \int_{0}^{4}\int_{-\sqrt{16-y^2}}^{4-y} f(x,y)\,dx\,dy.$

16.2.71 $V = \int_0^1\int_0^x (2-y^2)\,dy\,dx = \int_0^1 \left(2y - \frac{1}{3}y^3\right)\Big|_0^x dx = \frac{1}{3}\int_0^1 (6x-x^3)\,dx = \frac{1}{12}\left(12x^2 - x^4\right)\Big|_0^1 = \frac{11}{12}.$

16.2.72 $2\int_0^\pi\int_0^y \sin^2 x\,dx\,dy = \int_0^\pi\int_0^y (1-\cos 2x)\,dx\,dy = \int_0^\pi \left(y - \frac{\sin 2y}{2}\right)\,dy = \left(\frac{y^2}{2} + \frac{\cos 2y}{4}\right)\Big|_0^\pi = \frac{\pi^2}{2}.$

16.2.73 $V = \int_0^4\int_0^{2-\frac{x}{2}} (8-2x-4y)\;dy\,dx = \int_0^4 \left(8y - 2xy - 2y^2\right)\Big|_0^{2-\frac{x}{2}} dx = \int_0^4 \left(8 - 4x - \frac{1}{2}x^2\right)\,dx =$ $\left(8x - 2x^2 - \frac{1}{6}x^3\right)\Big|_0^4 = \frac{32}{3},$ or $V = \int_0^2\int_0^{4-2y} (8-2x-4y)\;dx\,dy\; = \frac{32}{3}.$

16.2.74 $\int_0^1\int_0^{1-x^2} (1-y-x^2)\,dy\,dx = \int_0^1 (y - y^2/2 - x^2y)\Big|_0^{1-x^2} dx = \int_0^1 (1 - x^2 - (1/2)(1-2x^2+x^4) -$ $x^2 + x^4)\,dx = \frac{1}{2}\int_0^1 (x^4 - 2x^2 + 1)\,dx = \frac{1}{2}(x^5/5 - 2x^3/3 + x)\Big|_0^1 = \frac{4}{15}.$

16.2.75 $V = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (12+x+y)\;dy\,dx = \int_{-1}^1 \left(12y + xy + \frac{1}{2}y^2\right)\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx =$ $\int_{-1}^1 \left(24\sqrt{1-x^2} + 2x\sqrt{1-x^2}\right)\,dx = \left(12x\sqrt{1-x^2} + 12\sin^{-1}x - \frac{2}{3}\left(1-x^2\right)^{3/2}\right)\Big|_{-1}^1 =$ $12\left(\sin^{-1}(1) - \sin^{-1}(-1)\right) = 12\pi.$

16.2.76 $V = \iint_R \left(e^{x-y} - \left(-e^{x-y}\right)\right)\,dA = \int_0^1\int_0^y 2e^{x-y}\,dx\,dy = \int_0^1 \left(2e^{x-y}\right)\Big|_0^y dy = \int_0^1 2\left(1 - e^{-y}\right)\,dy =$ $2\left(y + e^{-y}\right)\Big|_0^1 = \frac{2}{e}.$

16.2.77 $V = \iint_R (8-2x-3y)\;dA = \int_0^1\int_0^{2-x} (8-2x-3y)\;dy\,dx = \int_0^1 \left((8-2x)\,y - \frac{3}{2}y^2\right)\Big|_0^{2-x} dx =$ $\int_0^1 \left(10 - 6x + \frac{1}{2}x^2\right)\,dx = \left(10x - 3x^2 + \frac{1}{6}x^3\right)\Big|_0^1 = \frac{43}{6}.$

16.2.78 $V = \int_0^1\int_0^{1-x^2} (2-y-1)\,dy\,dx = \int_0^1 (y - y^2/2)\Big|_0^{1-x^2} dx = \frac{1}{2}\int_0^1 (2 - 2x^2 - (1 - 2x^2 + x^4))\,dx =$ $\frac{1}{2}\int_0^1 (-x^4 + 1)\,dx = \frac{1}{2}(-x^5/5 + x)\Big|_0^1 = \frac{2}{5}.$

16.2.79 $V = \int_0^1 \int_0^{1-x} ((2-x^2-y^2)-(x^2+y^2))\,dy\,dx = 2\int_0^1 \int_0^{1-x} (1-x^2-y^2)\,dy\,dx = 2\int_0^1 (y - x^2y - y^3/3)\Big|_0^{1-x} dx = \frac{2}{3}\int_0^1 ((3-3x)-(3x^2-3x^3)-(1-3x+3x^2-x^3))\,dx = \frac{2}{3}\int_0^1 (4x^3-6x^2+2)\,dx = \frac{2}{3}(x^4-2x^3+2x)\Big|_0^1 = \frac{2}{3}.$

16.2.80 $V = \iint\limits_R \big((y+1)-(x^2+1)\big)\,dA = \int_{-1}^1 \int_{x^2}^1 (y-x^2)\,dy\,dx = \int_{-1}^1 \left(\frac{y^2}{2} - x^2y\right)\Big|_{x^2}^1 dx = \int_{-1}^1 \left(\left(\frac{1}{2}-x^2\right) - \left(\frac{x^4}{2}-x^4\right)\right) dx = \int_{-1}^1 \left(\frac{1}{2} - x^2 + \frac{x^4}{2}\right) dx = \left(\frac{x}{2} - \frac{x^3}{3} + \frac{x^5}{10}\right)\Big|_{-1}^1 = \frac{8}{15}.$

16.2.81 $V = \iint\limits_R (4-x-y)\ dA = \int_{-1}^1 \int_{-1}^1 (4-x-y)\ dy\,dx = \int_{-1}^1 \left((4-x)\,y - \frac{1}{2}y^2\right)\Big|_{-1}^1 dx = \int_{-1}^1 2\,(4-x)\ dx = 2\left(4x - \frac{1}{2}x^2\right)\Big|_{-1}^1 = 16.$

16.2.82 $V = \iint\limits_R (z_{\text{top}} - z_{\text{bottom}})\ dA$ where R is enclosed by the intersection of the surfaces $z = x^2+y^2$ and $z = 1-2y$. Setting $x^2+y^2 = 1-2y$ gives the circle $x^2+(y+1)^2 = 2$.
So $V = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-1-\sqrt{2-x^2}}^{-1+\sqrt{2-x^2}} \big((1-2y)-\left(x^2+y^2\right)\big)\ dy\,dx = \int_{-\sqrt{2}}^{\sqrt{2}} \left(y - y^2 - x^2y - \frac{1}{3}y^3\right)\Big|_{-1-\sqrt{2-x^2}}^{-1+\sqrt{2-x^2}} dx = \int_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{8}{3}\sqrt{2-x^2} - \frac{4}{3}x^2\sqrt{2-x^2}\right) dx = \left(\frac{5}{3}x\sqrt{2-x^2} - \frac{1}{3}x^3\sqrt{2-x^2} + 2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right)\Big|_{-\sqrt{2}}^{\sqrt{2}} = 2\pi.$

16.2.83 $V = \iint\limits_R (a\,(2-x) - a\,(x-2))\ dA = 2a\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2-x)\ dy\,dx = 2a\int_{-1}^1 \Big[(2-x)\,y\Big]\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = 4a\int_{-1}^1 (2-x)\sqrt{1-x^2}\,dx = 4a\int_{-1}^1 \left(2\sqrt{1-x^2} - x\sqrt{1-x^2}\right) dx = \left[4a\left(x\sqrt{1-x^2} + \sin^{-1} x\right) + \frac{4a}{3}\left(1-x^2\right)^{3/2}\right]\Big|_{-1}^1 = 4a\left(\sin^{-1}(1) - \sin^{-1}(-1)\right) = 4a\pi.$

16.2.84 Using symmetry, this is given by
$2\int_0^2 \int_{-\sqrt{12-3y^2}}^{\sqrt{12-3y^2}} (3x^2+y^2+1)\,dx\,dy = 2\int_0^2 \left(x^3+xy^2+x\right)\Big|_{-\sqrt{12-3y^2}}^{\sqrt{12-3y^2}} = 2\int_0^2 \left(26\sqrt{12-3y^2} - 4y^2\sqrt{12-3y^2}\right)\ dy = 2\left(\sqrt{3}\left(22\sin^{-1}\left(\frac{y}{2}\right) - \frac{1}{2}y\sqrt{4-y^2}\left(y^2-15\right)\right)\right)\Big|_0^2 = 22\sqrt{3}\pi,$
where the last integral is found either via a table of integrals or a computer algebra system.

16.2.85 $A = \iint\limits_R 1\,dA = \int_{-2}^2 \int_{x^2}^4 1\,dy\,dx = \int_{-2}^2 (y)\Big|_{x^2}^4 dy = \int_{-2}^2 \left(4-x^2\right)\ dx = \left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^2 = \frac{32}{3}.$

16.2.86 $A = \iint\limits_R 1\,dA = \int_{-1}^2 \int_{x^2}^{x+2} 1\,dy\,dx = \int_{-1}^2 (y)\Big|_{x^2}^{x+2} dx = \int_{-1}^2 \left(x+2-x^2\right)\ dx = \left(\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\right)\Big|_{-1}^2 = \frac{9}{2}\,.$

16.2.87 $A = \iint\limits_R 1\,dA = \int_0^{\ln 2}\int_0^{e^x} 1\,dy\,dx = \int_0^{\ln 2} (y)\Big|_0^{e^x} dx = \int_0^{\ln 2} e^x\,dx = (e^x)\Big|_0^{\ln 2} = 1.$

16.2.88 $A = \iint\limits_R 1\,dA = \int_0^{\pi}\int_{1-\sin x}^{1+\sin x} 1\,dy\,dx = \int_0^{\pi} (y)\Big|_{1-\sin x}^{1+\sin x} dx = \int_0^{\pi} 2\,\sin x\,dx = (-2\,\cos x)\Big|_0^{\pi} = 4.$

16.2.89 $A = \iint\limits_R 1\,dA. = \int_0^2\int_{6-x}^{5x+6} 1\,dy\,dx + \int_2^6\int_{x^2}^{5x+6} 1\,dy\,dx = \int_0^2 (y)\Big|_{6-x}^{5x+6} dx + \int_2^6 (y)\Big|_{x^2}^{5x+6} dx = \int_0^2 6x\,dx + \int_2^6 \left(5x+6-x^2\right)\,dx = \left(3x^2\right)\Big|_0^2 + \left(\frac{5}{2}x^2+6x-\frac{1}{3}x^3\right)\Big|_2^6 = \frac{140}{3}.$

16.2.90 $A = \iint\limits_R 1\,dA = \int_0^4\int_x^{2x+1} 1\,dy\,dx = \int_0^4 (y)\Big|_x^{2x+1} dx = \int_0^4 (x+1)\,dx = \left(\frac{1}{2}x^2+x\right)\Big|_0^4 = 12.$.

16.2.91

a. False. a and b must be constants or functions of y.

b. False. a and d must be constants.

c. False. Variable limits of integration are not allowed in the outermost integral.

16.2.92

a. $\int_0^4\int_0^4 (4-x-y)\,dx\,dy = \int_0^4 \left(4x-\frac{x^2}{2}-xy\right)\Big|_{x=0}^{x=4} dy = \int_0^4 (8-4y)\,dy = \left(8y-2y^2\right)\Big|_0^4 = 32-32 = 0.$

b. The integrand is positive provided that $4-x-y \geq 0$ or $y \leq 4-x$. It follows that $\int_0^4\int_0^4 |4-x-y|\,dx\,dy = 2\int_0^4\int_0^{4-x} (4-x-y)\,dy\,dx = 2\int_0^4 \left(4y-xy-\frac{y^2}{2}\right)\Big|_{y=0}^{y=4-x} dx = 2\int_0^4 \left(16-4x-(4x-x^2)-\frac{(4-x)^2}{2}\right)\,dx = 2\int_0^4 \left(8-4x+\frac{1}{2}x^2\right)\,dx = 2\left(8x-2x^2+\frac{x^3}{6}\right)\Big|_0^4 = \frac{64}{3}.$

16.2.93 $V = \iint\limits_R f\,(x,\,y)\;dA$ (orient y-axis vertically) $= \iint\limits_R \left(y_{\text{top}}-y_{\text{bottom}}\right)\;dzdx = \int_0^5\int_0^2 \left[(z+1) - \left(\frac{-z-1}{2}\right)\right]\,dz\,dx = \int_0^5\int_0^2 \left(\frac{3}{2}z+\frac{3}{2}\right)\,dz\,dx = \int_0^5 \left(\frac{3}{4}z^2+\frac{3}{2}z\right)\Big|_0^2 dx = \int_0^5 6\,dx = (6x)\Big|_0^5 = 30.$

16.2.94

a. $A = \int_{-1}^0\int_{-1-x}^{1+x} 1\,dy\,dx + \int_0^1\int_{x-1}^{1-x} 1\,dy\,dx = \int_{-1}^0 (2+2x)\,dy\,dx + \int_0^1 (2-2x)\,dx = \left(2x+x^2\right)\Big|_{-1}^0 + \left(2x-x^2\right)\Big|_0^1 = 1+1 = 2.$

b. $A = \int_{-1}^0\int_{-1-x}^{1+x} (12-3x-4y)\,dy\,dx + \int_0^1\int_{x-1}^{1-x} (12-3x-4y)\,dy\,dx = \int_{-1}^0 \left((12-3x)\,y-2y^2\right)\Big|_{-1-x}^{1+x} dx + \int_0^1 \left((12-3x)\,y-2y^2\right)\Big|_{x-1}^{1-x} dx = \int_{-1}^0 \left(24+18x-6x^2\right)\,dy\,dx + \int_0^1 \left(24-30x+6x^2\right)\,dx = \left(24x+9x^2-2x^3\right)\Big|_{-1}^0 + \left(24x-15x^2+2x^3\right)\Big|_0^1 = 13+11 = 24.$

c. $V = \int_{-1}^{0}\int_{-1-x}^{1+x}\sqrt{1-x^2}\,dy\,dx + \int_{0}^{1}\int_{x-1}^{1-x}\sqrt{1-x^2}\,dy\,dx = 2\int_{0}^{1}\int_{x-1}^{1-x}\sqrt{1-x^2}\,dy\,dx$, where the last equality is due to symmetry. Then we have $2\int_0^1 \left(y\sqrt{1-x^2}\right)\Big|_{x-1}^{1-x} dx = 2\int_0^1 (2-2x)\sqrt{1-x^2}\,dx =$ $4\int_0^1 \left(\sqrt{1-x^2} - x\sqrt{1-x^2}\right) dx = 4\left(\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + \frac{1}{3}\left(1-x^2\right)^{3/2}\right)\Big|_0^1 =$
$4\left(0 + \frac{\pi}{4} + 0 - 0 - 0 - \frac{1}{3}\right) = \pi - \frac{4}{3}$.

d. Use symmetry.

$$V = 4\int_0^1\int_0^{1-x} 6\left(1-x-y\right)\,dy\,dx = 24\int_0^1 \left(\left(1-x\right)y - \frac{1}{2}y^2\right)\Big|_0^{1-x} dx = 24\int_0^1 \frac{1}{2}\left(1-x\right)^2\,dx =$$
$$12\left(-\frac{1}{3}\left(1-x\right)^3\right)\Big|_0^1 = 4.$$

16.2.95

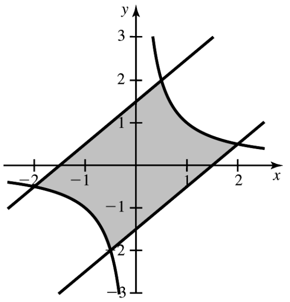
$$\overline{f} = \frac{1}{\text{Area of } R}\iint\limits_R f\left(x,\,y\right)\,dA = \frac{1}{\frac{1}{2}a^2}\int_0^a\int_0^{a-x}\left(a-x-y\right)\,dy\,dx$$
$$= \frac{2}{a^2}\int_0^2\left(\left(a-x\right)y - \frac{1}{2}y^2\right)\Big|_0^{a-x} dx = \frac{2}{a^2}\int_1^9 \frac{1}{2}\left(a-x\right)^2\,dx = \frac{1}{a^2}\left(-\frac{1}{3}\left(a-x\right)^3\right)\Big|_0^a = \frac{1}{a^2}\left(\frac{a^3}{3}\right) = \frac{a}{3}.$$

16.2.96 $\overline{f} = \frac{1}{\text{Area of } R}\iint\limits_R f\left(x,\,y\right)\,dA = \frac{1}{\pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(a^2-x^2-y^2\right)\,dy\,dx$, which by symmetry is equal to

$$\frac{4}{\pi a^2}\int_0^a\int_0^{\sqrt{a^2-x^2}}\left(a^2-x^2-y^2\right)\,dy\,dx = \frac{4}{\pi a^2}\int_0^a\left(\left(a^2-x^2\right)y - \frac{1}{3}y^3\right)\Big|_0^{\sqrt{a^2-x^2}} dx$$
$$= \frac{4}{\pi a^2}\int_0^a \frac{2}{3}\left(a^2-x^2\right)^{3/2}\,dx = \frac{8}{3\pi a^2}\left(-\frac{x}{8}\left(2x^2-5a^2\right)\sqrt{a^2-x^2} + \frac{3a^4}{8}\sin^{-1}\left(\frac{x}{a}\right)\right)\Big|_0^a = \frac{a^2}{3}.$$

16.2.97

a.



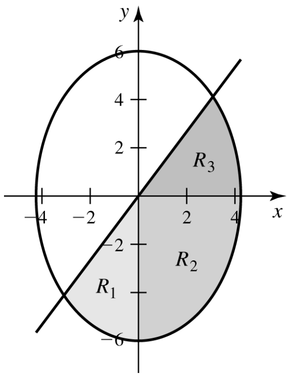
b.

$$
\begin{aligned}
A &= \iint_R 1\,dA = \iint_{R_1} 1\,dA + \iint_{R_2} 1\,dA + \iint_{R_3} 1\,dA \\
&= \int_{-2}^{-1/2} \int_{1/x}^{x+3/2} 1\,dy\,dx + \int_{-1/2}^{1/2} \int_{x-3/2}^{x+3/2} 1\,dy\,dx + \int_{1/2}^{2} \int_{x-3/2}^{1/x} 1\,dy\,dx \\
&= \int_{-2}^{-1/2} \left(x + \frac{3}{2} - \frac{1}{x} \right) dx + \int_{-1/2}^{1/2} (3)\ dx + \int_{1/2}^{2} \left(\frac{1}{x} - x + \frac{3}{2} \right) dx \\
&= \left(\frac{1}{2}x^2 + \frac{3}{2}x - \ln|x| \right) \Bigg|_{-2}^{-1/2} + (3x) \Bigg|_{-1/2}^{1/2} + \left(\ln|x| - \frac{1}{2}x^2 + \frac{3}{2}x \right) \Bigg|_{1/2}^{2} \\
&= \frac{3}{8} + 2\ln 2 + 3 + \frac{3}{8} + 2\ln 2 = \frac{15}{4} + 4\ln 2.
\end{aligned}
$$

c. $\displaystyle\iint_R xy\ dA = \int_{-2}^{-1/2} \int_{1/x}^{x+3/2} xy\,dy\,dx + \int_{-1/2}^{1/2} \int_{x-3/2}^{x+3/2} xy\,dy\,dx + \int_{1/2}^{2} \int_{x-3/2}^{1/x} xy\,dy\,dx =$
$\displaystyle\int_{-2}^{-1/2} \left(\frac{1}{2}xy^2 \right) \Bigg|_{1/x}^{x+3/2} dx + \int_{-1/2}^{1/2} \left(\frac{1}{2}xy^2 \right) \Bigg|_{x-3/2}^{x+3/2} dx + \int_{1/2}^{2} \left(\frac{1}{2}xy^2 \right) \Bigg|_{x-3/2}^{1/x} dx =$
$\displaystyle\frac{1}{2} \int_{-2}^{-1/2} \left(x^3 + 3x^2 + \frac{9}{4}x - \frac{1}{x} \right) dx + \frac{1}{2} \int_{-1/2}^{1/2} (6x^2)\ dx + \frac{1}{2} \int_{1/2}^{2} \left(\frac{1}{x} - \frac{9}{4}x + 3x^2 - x^3 \right) dx =$
$\displaystyle\frac{1}{2} \left(\frac{1}{4}x^4 + x^3 + \frac{9}{8}x^2 - \ln|x| \right) \Bigg|_{-2}^{-1/2} + (x^3) \Bigg|_{-1/2}^{1/2} + \frac{1}{2} \left(\ln|x| - \frac{9}{8}x^2 + x^3 - \frac{1}{4}x^4 \right) \Bigg|_{1/2}^{2} =$
$\displaystyle\frac{1}{2} \left(-\frac{21}{64} + 2\ln 2 \right) + \frac{1}{4} + \frac{1}{2} \left(-\frac{21}{64} + 2\ln 2 \right) = -\frac{5}{64} + 2\ln 2.$

16.2.98

a.



b. $\displaystyle A = \iint_R 1\,dA = \iint_{R_1} 1\,dA + \iint_{R_2} 1\,dA + \iint_{R_3} 1\,dA.$ Note: By symmmetry we see that the area of $(R_1 + R_3)$ is equal to the area of R_2. Thus

$$
\begin{aligned}
A &= 2\iint_{R_2} 1\,dA = 2\int_0^{3\sqrt{2}} \int_{-\sqrt{36-2x^2}}^{0} 1\,dy\,dx = 2\sqrt{2} \int_0^{3\sqrt{2}} \sqrt{18 - x^2}\,dx \\
&= 2\sqrt{2} \left(\frac{1}{2}x\sqrt{18 - x^2} + 9\sin^{-1}\left(\frac{x}{3\sqrt{2}} \right) \right) \Bigg|_0^{3\sqrt{2}} = 9\sqrt{2}\pi.
\end{aligned}
$$

c. Again by symmetry $R_1 + R_3$ will be equivalent to R_2 but with opposite sign. $xy > 0$ in Quadrant I and III and $xy < 0$ in Quadrant II and IV. By symmetry about the origin we can deduce equivalent components of surface above the xy-plane over regions R_1 and R_3 as below the xy-plane in R_2. Thus $\iint\limits_R xy\ dA = 0$.

16.2.99 $\displaystyle\int_1^{\infty}\int_0^{e^{-x}} xy\,dy\,dx = \int_1^{\infty}\left(\frac{1}{2}xy^2\right)\Big|_0^{e^{-x}}dx = \int_1^{\infty}\frac{1}{2}x\,e^{-2x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{2}x\,e^{-2x}\,dx =$ $\displaystyle\lim_{b\to\infty}\frac{1}{2}\left(e^{-2x}\left(-\frac{1}{2}x-\frac{1}{4}\right)\right)\Big|_1^b = \frac{1}{4}\lim_{b\to\infty}\left(e^{-2b}\left(-b-\frac{1}{2}\right)+e^{-2}\left(\frac{3}{2}\right)\right) = \frac{1}{4}\left(0+\frac{3}{2}e^{-2}\right) = \frac{3}{8e^2}$.

16.2.100 $\displaystyle\int_1^{\infty}\int_0^{1/x^2}\frac{2y}{x}\,dy\,dx = \int_1^{\infty}\left(\frac{y^2}{x}\right)\Big|_0^{1/x^2}dx = \int_1^{\infty}\frac{1}{x^5}\,dx = \lim_{b\to\infty}\int_1^b\frac{1}{x^5}\,dx =$ $\displaystyle\lim_{b\to\infty}\left(-\frac{1}{4x^4}\right)\Big|_1^b = \lim_{b\to\infty}\left(-\frac{1}{4b^4}+\frac{1}{4}\right) = 0+\frac{1}{4} = \frac{1}{4}$.

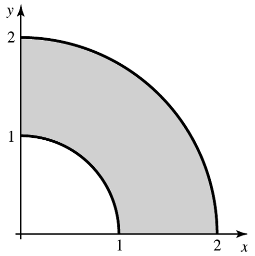
16.2.101 $\displaystyle\int_0^{\infty}\int_0^{\infty}e^{-x-y}\,dy\,dx = \int_0^{\infty}\left(\lim_{b\to\infty}\int_0^b e^{-x-y}\,dy\right)dx = \int_0^{\infty}\left(\lim_{b\to\infty}\left(-e^{-x-y}\right)\Big|_0^b\right)dx =$ $\displaystyle\int_0^{\infty}\left(\lim_{b\to\infty}\left(e^{-x}-e^{-x-b}\right)\right)dx = \int_0^{\infty}e^{-x}\,dx = \lim_{b\to\infty}\int_0^b -e^{-x}\,dx = \lim_{b\to\infty}\left(-e^{-x}\right)\Big|_0^b =$ $\displaystyle\lim_{b\to\infty}\left(1-e^{-b}\right) = 1$.

16.2.102 First compute $\displaystyle\int_{-\infty}^{\infty}\frac{1}{u^2+1}\,du$. This is equal to $\displaystyle\int_{-\infty}^0\frac{1}{u^2+1}\,du + \int_0^{\infty}\frac{1}{u^2+1}\,du = \lim_{a\to-\infty}\int_a^0\frac{1}{u^2+1}\,du + \lim_{b\to\infty}\int_0^b\frac{1}{u^2+1}\,du = \lim_{a\to-\infty}\left(\tan^{-1}u\Big|_{u=a}^{u=0}\right) +$ $\displaystyle\lim_{b\to\infty}\left(\tan^{-1}u\Big|_{u=0}^{u=b}\right) = \lim_{a\to-\infty} -\tan^{-1}a + \lim_{b\to\infty}\tan^{-1}b = \frac{\pi}{2}+\frac{\pi}{2} = \pi$

Now use this result to compute the inner integral: $\displaystyle\int_{-\infty}^{\infty}\frac{1}{(x^2+1)(y^2+1)}\,dy = \frac{1}{x^2+1}\int_{-\infty}^{\infty}\frac{1}{y^2+1}\,dy =$ $\displaystyle\frac{\pi}{x^2+1}$, so the integral is $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(x^2+1)(y^2+1)}\,dy\,dx = \int_{-\infty}^{\infty}\frac{\pi}{x^2+1}\,dx = \pi^2$.

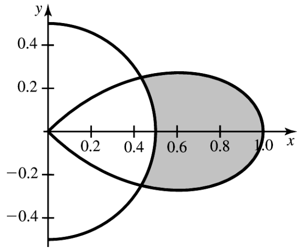
16.3 Double Integrals in Polar Coordinates

16.3.1 It is called a polar rectangle because it is analogous to a cartesian rectangle; they both have each variable constrained by constants.



16.3.2 $\displaystyle\iint\limits_R f(x,\,y)\ dA = \int_{\alpha}^{\beta}\int_a^b f(r\cos\theta,\,r\sin\theta)\ r\,dr\,d\theta$.

16.3.3 $R = \left\{ (r,\, \theta) : \dfrac{1}{2} \le r \le \cos 2\theta,\, -\dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{6} \right\}$



16.3.4 Geometrically $dx\,dy$ represents the length times width of a rectangular subsection of the region of integration. For equal partitions with respect to x and y each subsection has equivalent area. In polar coordinates each partition is a portion of a circular sector. For equal partitions with respect to r and θ the areas of the subsections are proportional to the distance from the origin, or r.

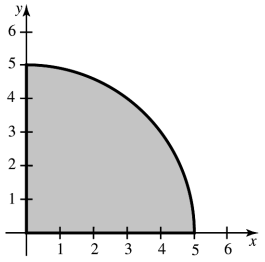
16.3.5 For $R = \left\{ (r,\, \theta) : g(\theta) \le r \le h(\theta),\, \alpha \le \theta \le \beta \right\}$, the area of R given by $\displaystyle\iint_R (1)\; dy\, dx$ converts to polar coordinates $\displaystyle\int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\, dr\, d\theta$.

16.3.6 For $R = \left\{ (r,\, \theta) : g(\theta) \le r \le h(\theta),\, \alpha \le \theta \le \beta \right\}$ the average value of a function $f(x,\, y)$ is given by

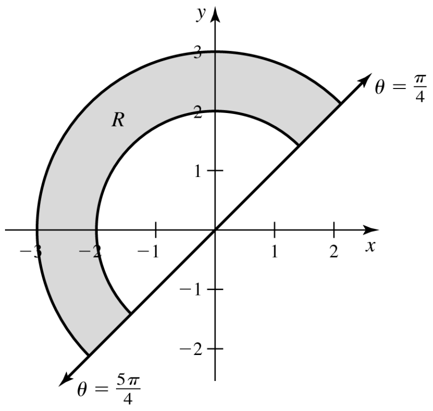
$$\overline{f} = \frac{1}{\text{area of } R} \iint_R f(x,\, y)\; dy\, dx$$

which converts to polar coordinates $\displaystyle\frac{1}{\text{area of } R} \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta,\, r\sin\theta)\; r\, dr\, d\theta$.

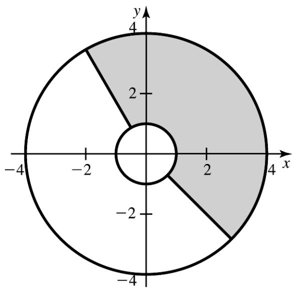
16.3.7



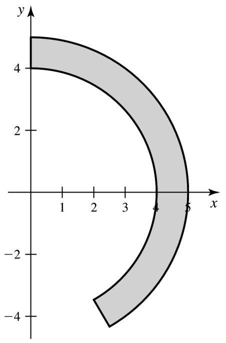
16.3.8



16.3.9



16.3.10



16.3.11 Setting $z \geq 0$ gives $x^2 + y^2 \leq 16$, and thus $0 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$. It follows that $V = \int_0^{2\pi} \int_0^4 \left(4 - \sqrt{r^2}\right) r\,dr\,d\theta = \int_0^{2\pi} \int_0^4 \left(4r - r^2\right) dr\,d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{3}r^3\right) \Big|_0^4 d\theta = \int_0^{2\pi} \frac{32}{3}\,d\theta = \frac{64\pi}{3}$.

16.3.12 Setting $z \geq 0$ gives $x^2 + y^2 \leq 4$, and thus $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. It follows that $V = \int_0^{2\pi} \int_0^2 \left(16 - 4r^2\right) r\,dr\,d\theta = \int_0^{2\pi} \int_0^2 \left(16r - 4r^3\right) dr\,d\theta = \int_0^{2\pi} \left(8r^2 - r^4\right) \Big|_0^2 d\theta = \int_0^{2\pi} 16\,d\theta = 32\pi$.

16.3.13 Setting $z \geq 0$, gives $x^2 + y^2 \leq 16$, and thus $0 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$. It follows that $V = \int_0^{2\pi} \int_0^4 \left(e^{-r^2/8} - e^{-2}\right) r\,dr\,d\theta = \int_0^{2\pi} \int_0^4 \left(r\,e^{-r^2/8} - r\,e^{-2}\right) dr\,d\theta = \int_0^{2\pi} \left(-4\,e^{-r^2/8} - \frac{1}{2}r^2\,e^{-2}\right) \Big|_0^4 d\theta = \int_0^{2\pi} \left(-12e^{-2} + 4\right) d\theta = 8\pi\left(1 - 3e^{-2}\right)$.

16.3.14 Setting $z \geq 0$ gives $x^2 + y^2 \leq 9$, and thus thus $0 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$. It follows that $V = \int_0^{2\pi} \int_0^3 \left(\frac{20}{1 + r^2} - 2\right) r\,dr\,d\theta = \int_0^{2\pi} \int_0^3 \left(\frac{20r}{1 + r^2} - 2r\right) dr\,d\theta = \int_0^{2\pi} \left(10\,\ln\left(1 + r^2\right) - r^2\right) \Big|_0^3 d\theta = \int_0^{2\pi} (10\,\ln 10 - 9)\,d\theta = 2\pi\,(10\,\ln 10 - 9)$.

16.3.15 $V = \iint\limits_R f\left(r\cos\theta,\, r\sin\theta\right) r\,dr\,d\theta = \int_0^{2\pi} \int_0^1 \left(4 - r^2\right) r\,dr\,d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{4}r^4\right) \Big|_0^1 d\theta = \int_0^{2\pi} \left(\frac{7}{4}\right) d\theta = \frac{7\pi}{2}$.

16.3.16 $V = \iint\limits_R f\left(r\cos\theta,\, r\sin\theta\right) r\,dr\,d\theta = \int_0^{2\pi} \int_0^2 \left(4 - r^2\right) r dr\,d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{4}r^4\right) \Big|_0^2 d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi$.

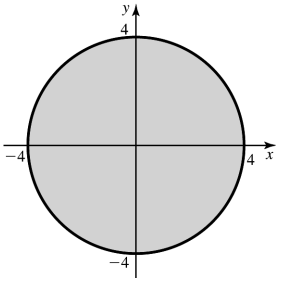
16.3.17 $V = \iint\limits_R f\left(r\cos\theta,\, r\sin\theta\right) r\,dr\,d\theta = \int_0^{2\pi} \int_1^2 \left(4 - r^2\right) r\,dr\,d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{4}r^4\right) \Big|_1^2 d\theta = \int_0^{2\pi} \frac{9}{4}\,d\theta = \frac{9\pi}{2}$.

16.3.18 $V = \iint\limits_R f\left(r\cos\theta,\, r\sin\theta\right) r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \int_1^2 \left(4 - r^2\right) r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \left(2r^2 - \frac{1}{4}r^4\right) \Big|_1^2 d\theta = \int_{-\pi/2}^{\pi/2} \frac{9}{4}\,d\theta = \frac{9\pi}{4}$.

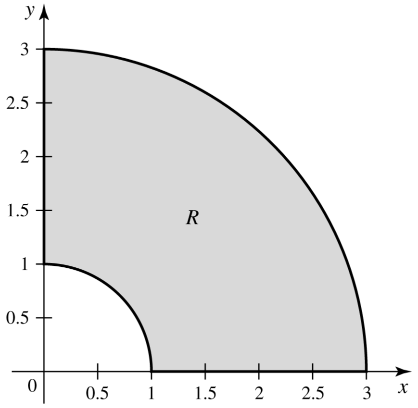
16.3.19 $V = \iint\limits_R \left(5 - \sqrt{1+x^2+y^2}\right)\, dA = \int_0^{2\pi}\int_{\sqrt{3}}^{2\sqrt{2}} \left(5-\sqrt{1+r^2}\right) r\,dr\,d\theta =$
$\int_0^{2\pi} \left(\frac{5}{2}r^2 - \frac{1}{3}\left(1+r^2\right)^{3/2}\right)\Big|_{\sqrt{3}}^{2\sqrt{2}} d\theta = \int_0^{2\pi} (20-9-(15/2-8/3))\, d\theta = \int_0^{2\pi} (37/6)\,d\theta = 37\pi/3.$

16.3.20 $V = \iint\limits_R \left(5 - \sqrt{1+x^2+y^2}\right)\, dA = \int_{-\pi/2}^{\pi}\int_{\sqrt{3}}^{\sqrt{15}} \left(5-\sqrt{1+r^2}\right) r\,dr\,d\theta =$
$\int_{-\pi/2}^{\pi} \left(\frac{5}{2}r^2 - \frac{1}{3}\left(1+r^2\right)^{3/2}\right)\Big|_{\sqrt{3}}^{\sqrt{15}} d\theta = \int_{-\pi/2}^{\pi} (75/2-64/3-(15/2-8/3))\, d\theta = \int_{-\pi/2}^{\pi} (34/3)\,d\theta =$
$\left(\frac{34}{3}\cdot\frac{3\pi}{2}\right) = 17\pi.$

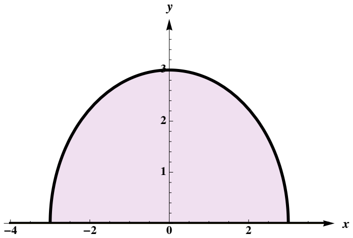
16.3.21 $\iint\limits_R \left(x^2+y^2\right)\, dA = \int_0^{2\pi}\int_0^4 \left(r^2\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^4 r^3\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{4}r^4\right)\Big|_0^4 d\theta = \int_0^{2\pi} 64\,d\theta =$
$128\pi.$



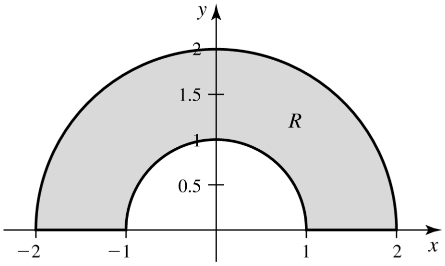
16.3.22 $\iint\limits_R (2xy)\, dA = \int_0^{\pi/2}\int_1^3 2\,(r\,\cos\theta)\,(r\,\sin\theta)\; r\,dr\,d\theta = \int_0^{\pi/2}\int_1^3 2r^3\sin\theta\,\cos\theta\,dr\,d\theta =$
$\int_0^{\pi/2} \sin\theta\,\cos\theta\,\left(\frac{1}{2}r^4\right)\Big|_1^3 d\theta = \int_0^{\pi/2} 40\,\sin\theta\,\cos\theta\,d\theta = \left(20\sin^2\,\theta\right)\Big|_{\theta=0}^{\theta=\pi/2} = 20.$



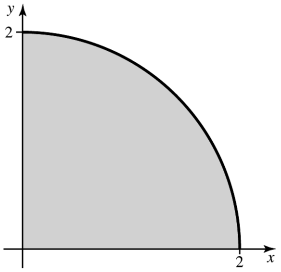
16.3.23 $\iint\limits_R (2xy)\, dA = \int_0^{\pi}\int_0^3 2\,(r\,\cos\theta)\,(r\,\sin\theta)\; r\,dr\,d\theta = \int_0^{\pi}\int_0^3 2r^3\sin\theta\,\cos\theta\,dr\,d\theta =$
$\int_0^{\pi} \cos\theta\,\sin\theta\,\left(\frac{1}{2}r^4\right)\Big|_0^3 d\theta = (81/2)\int_0^{\pi} \cos\theta\,\sin\theta\,d\theta = \left(\frac{81}{4}\sin^2\,\theta\right)\Big|_{\theta=0}^{\theta=\pi} = 0.$



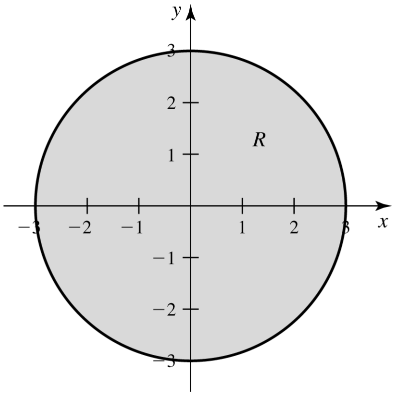
16.3.24 $\displaystyle\iint_R \frac{1}{1+x^2+y^2}\, dA = \int_0^{\pi}\int_1^2 \left(\frac{1}{1+r^2}\right) r\,dr\,d\theta = \int_0^{\pi}\int_1^2 \frac{r}{1+r^2}\,dr\,d\theta =$ $\displaystyle\int_0^{\pi}\left(\frac{1}{2}\ln\left(1+r^2\right)\right)\Big|_1^2 d\theta = \int_0^{\pi}\frac{1}{2}\left(\ln 5 - \ln 2\right)\, d\theta = \frac{\pi}{2}\,\ln\left(\frac{5}{2}\right).$



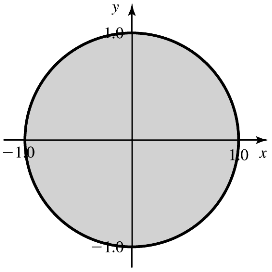
16.3.25 $\displaystyle\iint_R \frac{1}{\sqrt{16-x^2-y^2}}\, dA = \int_0^{\pi/2}\int_0^2 \frac{1}{\sqrt{16-r^2}}\, r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 \frac{r}{\sqrt{16-r^2}}\,dr\,d\theta =$ $\displaystyle\int_0^{\pi/2}\left(-\sqrt{16-r^2}\right)\Big|_0^2 d\theta = \int_0^{\pi/2}\left(-2\sqrt{3}+4\right)\, d\theta = \pi\left(2-\sqrt{3}\right).$



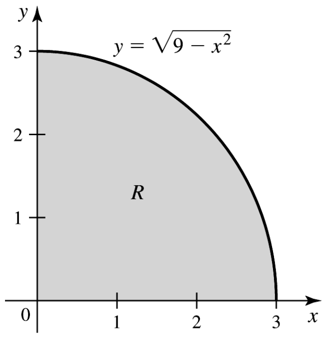
16.3.26 $\displaystyle\iint_R e^{-x^2-y^2}\, dA = \int_0^{2\pi}\int_0^3 e^{-r^2}\, r\,dr\,d\theta = \int_0^{2\pi}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^3 d\theta = \int_0^{2\pi}\left(-\frac{1}{2}e^{-9}+\frac{1}{2}\right)\, d\theta$ $= \pi\left(1-e^{-9}\right).$



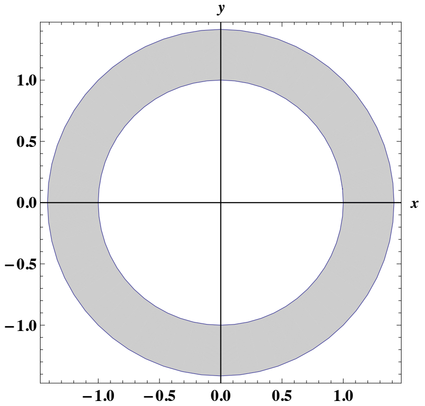
16.3.27 $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dy\,dxy = \int_0^{2\pi}\int_0^1 (r^2)^{3/2}\,r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{5}r^5\right)\Big|_0^1 d\theta = \int_0^{2\pi}\left(\frac{1}{5}\right)d\theta = \left(\frac{1}{5}\theta\right)\Big|_{\theta=0}^{\theta=2\pi} = \frac{2\pi}{5}.$



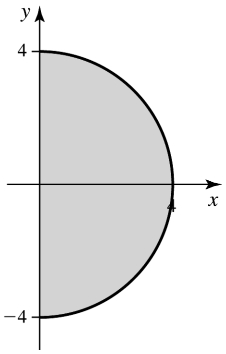
16.3.28 $\int_0^3\int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^3 \sqrt{r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\left(\frac{1}{3}r^3\right)\Big|_0^3 d\theta = \int_0^{\pi/2}(9)\,d\theta = (9\theta)\Big|_{\theta=0}^{\theta=\pi/2} = \frac{9\pi}{2}.$



16.3.29 Let $x = \cos\theta$ and $y = \sin\theta$. The we have $\int_0^{2\pi}\int_1^2 r^2\,dr\,d\theta = \int_0^{2\pi}(r^3/3)\Big|_1^2 d\theta = \frac{7}{3}\cdot 2\pi = \frac{14\pi}{3}.$



16.3.30 $\int_{-4}^{4}\int_0^{\sqrt{16-y^2}} (16-x^2-y^2)\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_0^4 (16-r^2)\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left(8r^2-\frac{1}{4}r^4\right)\Big|_0^4 d\theta = \int_{-\pi/2}^{\pi/2} 64\,d\theta = (64\theta)\Big|_{\theta=-\pi/2}^{\theta=\pi/2} = 64\pi.$



16.3.31 $V = \iint\limits_R \left(9-x^2-y^2\right)\,dA = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(9-x^2-y^2\right)\,dy\,dx =$
$\int_{-3}^{3}\left(\left(9-x^2\right)y-\frac{1}{3}y^3\right)\Big|_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\,dx = \int_{-3}^{3}\left(12\sqrt{9-x^2}-\frac{4}{3}x^2\sqrt{9-x^2}\right)\,dx =$
$\left(\frac{15}{2}x\sqrt{9-x^2}-\frac{1}{3}x^2\sqrt{9-x^2}+\frac{81}{2}\sin^{-1}\left(\frac{x}{3}\right)\right)\Big|_{-3}^{3} = \frac{81\pi}{2}.$

16.3.32 Let $x = r\cos\theta$ and $y = r\sin\theta$. $V = \iint\limits_R (2-x^2-y^2-1)\,dA = \int_0^{2\pi}\int_0^1 (1-r^2)r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\theta = \int_0^{2\pi}(r^2/2-r^4/4)\Big|_0^1\,d\theta = (1/4)\cdot 2\pi = \frac{\pi}{2}.$

16.3.33 Let $x = r\cos\theta$ and $y = r\sin\theta$. $V = \iint\limits_R ((2-x^2-y^2)-(x^2+y^2))\,dA = \int_0^{2\pi}\int_0^1 ((2-r^2)-(r^2))r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (2r-2r^3)\,dr\,d\theta = \int_0^{2\pi}(r^2-r^4/2)\Big|_0^1\,d\theta = \frac{1}{2}\cdot 2\pi = \pi.$

16.3.34 Let $x = r\cos\theta$ and $y = r\sin\theta$. $V = \iint\limits_R ((27-x^2-2y^2)-(2x^2+y^2))\,dA = \int_0^{2\pi}\int_0^3 (27-3r^2)r\,dr\,d\theta = \int_0^{2\pi}\int_0^3 (27r-3r^3)\,dr\,d\theta = \int_0^{2\pi}(27r^2/2-3r^4/4)\Big|_0^3\,d\theta = ((243/2)-(243/4))\cdot 2\pi = \frac{243\pi}{2}.$

16.3.35 Let $x = r\cos\theta$ and $y = r\sin\theta$. $V = \iint\limits_R ((4-x-y)-(x^2+y^2-x-y))\,dA = \int_0^{2\pi}\int_0^2 (4-r^2)r\,dr\,d\theta = \int_0^{2\pi}\left(2r^2-\frac{r^4}{4}\right)\Big|_0^2\,d\theta = 8\pi.$

16.3.36 Let $x = r\cos\theta$ and $y = r\sin\theta$. $V = \iint\limits_R ((3-x)-(x-3))\,dA = \int_0^{2\pi}\int_0^2 (6-2r\cos\theta)r\,dr\,d\theta = \int_0^{2\pi}\left(3r^2-\frac{2r^3}{3}\cos\theta\right)\Big|_0^2\,d\theta = \int_0^{2\pi}\left(12-\frac{16}{3}\cos\theta\right)\,d\theta = \left(12\theta-\frac{16}{3}\sin\theta\right)\Big|_0^{2\pi} = 24\pi-0 = 24\pi.$

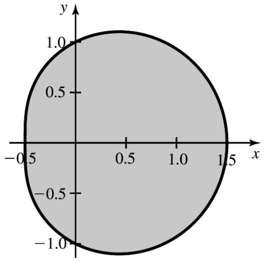
16.3.37 Let $x = r\cos\theta$ and $y = r\sin\theta$. $V = \iint\limits_R (18-x^2-3y^2-x^2+y^2)\,dA = \int_0^{2\pi}\int_0^3 (18-2r^2)r\,dr\,d\theta = \int_0^{2\pi}\int_0^3 (18r-2r^3)\,dr\,d\theta = \int_0^{2\pi}(9r^2-r^4/2)\Big|_0^3\,d\theta = \frac{81}{2}\cdot 2\pi = 81\pi.$

16.3.38 Let $x = r\cos\theta$ and $y = r\sin\theta$. $V = \iint\limits_R ((8 - x^2 + y^2) - (x^2 + 3y^2))\,dA = \int_0^{2\pi}\int_1^2 (8r - 2r^3)\,dr\,d\theta =$ $\int_0^{2\pi} \left(4r^2 - \frac{1}{2}r^4\right)\Big|_1^2 \,d\theta = 2\pi\left(16 - 8 - 4 + \frac{1}{2}\right) = 9\pi.$

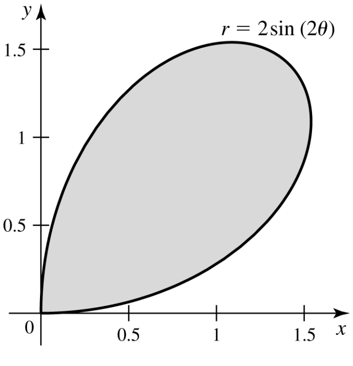
16.3.39 Let $x = r\cos\theta$ and $y = r\sin\theta$. $V = \iint\limits_R (\sqrt{8 - (x^2 + y^2)} - \sqrt{x^2 + y^2})\,dA = \int_0^{2\pi}\int_1^2 (\sqrt{8 - r^2} -$ $r)r\,dr\,d\theta = \int_0^{2\pi}\int_1^2 r\sqrt{8 - r^2}\,dr\,d\theta - \int_0^{2\pi}\int_1^2 r^2\,dr\,d\theta = \int_0^{2\pi} \frac{-1}{3}(8 - r^2)^{3/2}\Big|_1^2 \,d\theta - \int_0^{2\pi} \frac{r^3}{3}\Big|_1^2 \,d\theta =$ $\frac{-2\pi}{3}\left(8 - 7^{3/2}\right) - \frac{2\pi}{3}(7) = 2\pi\left(\frac{7\sqrt{7}}{3} - 5\right).$

16.3.40 Let $x = r\cos\theta$ and $y = r\sin\theta$.
$V = \iint\limits_R (2 - \sqrt{x^2 + y^2} - \sqrt{1 + x^2 + y^2})\,dA = \int_0^{2\pi}\int_0^{3/4} (2 - r - \sqrt{1 + r^2})r\,dr\,d\theta =$ $\int_0^{2\pi} \left(r^2 - \frac{r^3}{3} - \frac{1}{3}(1 + r^2)^{3/2}\right)\Big|_0^{3/4} \,d\theta = \int_0^{2\pi} \frac{1}{3}\left(\frac{27}{16} - \frac{27}{64} - \frac{125}{64}\right) + \frac{1}{3}\,d\theta = \frac{2\pi}{3}\left(1 - \frac{33}{48}\right) = \frac{5\pi}{24}.$

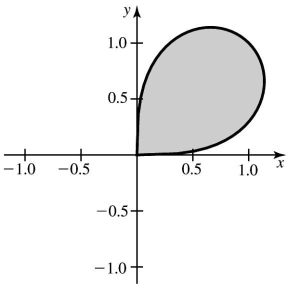
16.3.41 $\iint\limits_R g(r,\theta)\ dA = \int_0^{2\pi}\int_0^{1 + \frac{1}{2}\cos\theta} g(r,\theta)\ \ r\,dr\,d\theta.$



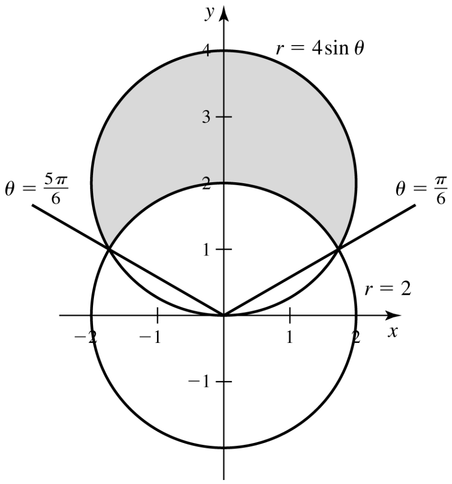
16.3.42 $\iint\limits_R g(r,\theta)\ dA = \int_0^{\pi/2}\int_0^{2\sin(2\theta)} g(r,\theta)\ \ r\,dr\,d\theta.$



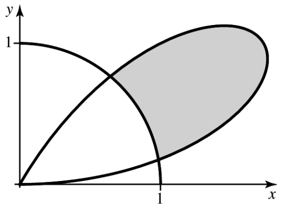
16.3.43 $\iint\limits_R g(r,\theta)\ dA = \int_0^{\pi/2}\int_0^{\sqrt{2\sin(2\theta)}} g(r,\theta)\ \ r\,dr\,d\theta.$


y
1.0
0.5
−1.0
−0.5
0.5
1.0
x
−0.5
−1.0

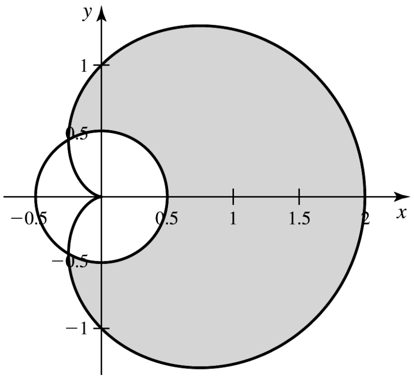
16.3.44 $\displaystyle\iint_R g(r,\theta)\;dA=\int_{\pi/6}^{5\pi/6}\int_2^{4\sin\theta} g(r,\theta)\;\;r\,dr\,d\theta.$


y
$r = 4\sin\theta$
4
3
$\theta = \frac{5\pi}{6}$
2
$\theta = \frac{\pi}{6}$
1
$r = 2$
−2
−1
1
2
x
−1

16.3.45 $\displaystyle\iint_R g(r,\theta)\;dA=\int_{\pi/18}^{5\pi/18}\int_1^{2\sin(3\theta)} g(r,\theta)\;\;r\,dr\,d\theta.$



16.3.46 $\displaystyle\iint_R g(r,\theta)\;dA=\int_{-2\pi/3}^{2\pi/3}\int_{1/2}^{1+\cos\theta} g(r,\theta)\;\;r\,dr\,d\theta.$


y
1
0.5
−0.5
0.5
1
1.5
2
x
−0.5
−1

16.3.47 $A = \iint\limits_R 1\,dA = \int_0^\pi \int_1^2 r\,dr\,d\theta = \int_0^\pi \left(\frac{1}{2}r^2\right)\Big|_1^2 d\theta = \int_0^\pi \left(\frac{3}{2}\right) d\theta = \left(\frac{3}{2}\theta\right)\Big|_{\theta=0}^{\theta=\pi} = \frac{3\pi}{2}.$

16.3.48

$$\begin{aligned} A &= \iint\limits_R 1\,dA = \int_0^{2\pi} \int_0^{2(1-\sin\theta)} r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}r^2\Big|_0^{2(1-\sin\theta)} d\theta \\ &= \frac{1}{2}\int_0^{2\pi} 4\left(1 - 2\,\sin\theta + \sin^2\,\theta\right) d\theta = 2\int_0^{2\pi} \left(\frac{3}{2} - 2\,\sin\theta - \frac{1}{2}\cos 2\theta\right) d\theta \\ &= 2\left(\frac{3}{2}\theta + 2\,\cos\theta - \frac{1}{4}\sin 2\theta\right)\Big|_{\theta=0}^{\theta=2\pi} = 6\pi. \end{aligned}$$

16.3.49 $A = \iint\limits_R 1\,dA = \int_0^\pi \int_0^{2\cos(3\theta)} r\,dr\,d\theta = \int_0^\pi \left(\frac{1}{2}r^2\right)\Big|_0^{2\cos(3\theta)} d\theta = \int_0^\pi \left(2\cos^2(3\theta)\right) d\theta =$
$\int_0^\pi (1 + \cos(6\theta))\,d\theta = \left(\theta + \frac{1}{6}\sin(6\theta)\right)\Big|_{\theta=0}^{\theta=\pi} = \pi.$

16.3.50 $A = \iint\limits_R 1\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta + \int_{\pi/2}^{3\pi/2}\int_0^1 r\,dr\,d\theta$. By symmetry this is equal to $2\left(\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta + \int_{\pi/2}^{\pi}\int_0^1 r\,dr\,d\theta\right) = \int_0^{\pi/2}\left(r^2\right)\Big|_0^{1-\cos\theta} d\theta + \int_{\pi/2}^{\pi}\left(r^2\right)\Big|_0^1 d\theta = \int_0^{\pi/2}(1-\cos\theta)^2\,d\theta +$ $\int_{\pi/2}^{\pi}(1)\,d\theta = \int_0^{\pi/2}\left(\frac{3}{2} - 2\,\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta + \int_{\pi/2}^{\pi}(1)\,d\theta = \left(\frac{3}{2}\theta - 2\sin\,\theta + \frac{1}{4}\sin 2\theta\right)\Big|_{\theta=0}^{\theta=\pi/2} + (\theta)\Big|_{\theta=0}^{\theta=\pi/2} =$ $\left(\frac{3\pi}{4} - 2\right) + \left(\frac{\pi}{2}\right) = \frac{5\pi}{4} - 2..$

16.3.51 $A = \iint\limits_R 1\,dA = \int_{-3\pi/4}^{\pi/4}\int_0^{1+\sin\theta} r\,dr\,d\theta + \int_{\pi/4}^{5\pi/4}\int_0^{1+\cos\theta} r\,dr\,d\theta$. By symmetry this is equal to $2\int_{\pi/4}^{5\pi/4}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_{\pi/4}^{5\pi/4}\left(r^2\right)\Big|_0^{1+\cos\theta} d\theta = \int_{\pi/4}^{5\pi/4}\left(1 + 2\cos\theta + \cos^2\,\theta\right) d\theta =$
$\int_{\pi/4}^{5\pi/4}\left(\frac{3}{2} + 2\,\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta = \left(\frac{3}{2}\theta + 2\sin\,\theta + \frac{1}{4}\sin 2\theta\right)\Big|_{\theta=\pi/4}^{\theta=5\pi/4} = \frac{3\pi}{2} - 2\sqrt{2}.$

16.3.52 $A = \iint\limits_R 1\,dA = \int_0^\pi \int_0^{2\theta} r\,dr\,d\theta = \int_0^\pi \left(\frac{1}{2}r^2\right)\Big|_0^{2\theta} d\theta = \int_0^\pi \left(2\theta^2\right) d\theta = \left(\frac{2}{3}\theta^3\right)\Big|_{\theta=0}^{\theta=\pi} = \frac{2\pi^3}{3}.$

16.3.53 $\overline{f} = \frac{1}{\text{area of } R}\iint\limits_R f(r,\,\theta)\;dA = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a (r)\;r\,dr\,d\theta = \frac{1}{\pi a^2}\int_0^{2\pi}\left(\frac{1}{3}r^3\right)\Big|_0^a d\theta =$
$\frac{1}{\pi a^2}\int_0^{2\pi}\left(\frac{1}{3}a^3\right) d\theta = \frac{a}{3\pi}(\theta)\Big|_{\theta=0}^{\theta=2\pi} = \frac{2a}{3}.$

16.3.54 $\overline{f} = \frac{1}{\text{area of } R}\iint\limits_R f(r,\,\theta)\;dA = \frac{1}{12\pi}\int_0^{2\pi}\int_2^4 \frac{1}{r^2}\,r\,dr\,d\theta = \frac{1}{12\pi}\int_0^{2\pi}\int_2^4 \frac{1}{r}\,dr\,d\theta =$
$\frac{1}{12\pi}\int_0^{2\pi}(\ln|r|)\Big|_2^4 d\theta = \frac{1}{12\pi}\int_0^{2\pi}(\ln\,4 - \ln\,2)\,d\theta = \frac{\ln\,2}{6}.$

16.3.55

a. False. $\iint\limits_R \left(x^2+y^2\right)\,dA = \int_0^{2\pi}\int_0^1 \left(r^2\right)\,r dr\,d\theta$.

b. True. The distance from every point on a hemisphere with radius 2 to the origin is 2.

c. True. $\int_0^1\int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\,dx\,dy$ converts to $\int_0^{\pi/2}\int_0^r r\,e^{r^2}\,dr\,d\theta$.

16.3.56 For $r=2a\cos\theta$: $A=\iint\limits_R 1\,dA=\int_0^\pi\int_0^{2a\,\cos\theta} r\,dr\,d\theta=\int_0^\pi\left(\frac{1}{2}r^2\right)\Big|_0^{2a\,\cos\theta}d\theta=\int_0^\pi 2a^2\cos^2\,\theta\,d\theta.$

$=\int_0^\pi a^2\left(1+\cos 2\theta\right)\,d\theta=a^2\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_{\theta=0}^{\theta=\pi}=\pi a^2.$

For $r=2a\sin\theta$: $A=\iint\limits_R 1\,dA=\int_0^\pi\int_0^{2a\,\sin\theta} r\,dr\,d\theta=\int_0^\pi\left(\frac{1}{2}r^2\right)\Big|_0^{2a\,\sin\theta}d\theta=\int_0^\pi 2a^2\sin^2\,\theta\,d\theta$

$=\int_0^\pi a^2\left(1-\cos 2\theta\right)\,d\theta=a^2\left(\theta-\frac{1}{2}\sin 2\theta\right)\Big|_{\theta=0}^{\theta=\pi}=\pi a^2.$

16.3.57 Paraboloid: $V=\iint\limits_R \left(4-\left(x^2+y^2\right)\right)\,dA=\int_0^{2\pi}\int_0^2\left(4-r^2\right)\,r\,dr\,d\theta=\int_0^{2\pi}\int_0^2(4r-r^3)\,dr\,d\theta=$ $\int_0^{2\pi}\left(2r^2-\frac{1}{4}r^4\right)\Big|_0^2 d\theta=\int_0^{2\pi}4\,d\theta=8\pi.$

Cone: $V=\iint\limits_R(4-\sqrt{x^2+y^2})\,dA=\int_0^{2\pi}\int_0^4(4r-r^2)\,dr\,d\theta=\int_0^{2\pi}\left(2r^2-r^3/3\right)\Big|_0^4 d\theta=\int_0^{2\pi}\frac{32}{3}\,d\theta=\frac{64\pi}{3}.$

Hyperboloid: $V=\iint\limits_R\left(5-\sqrt{1+x^2+y^2}\right)dA=\int_0^{2\pi}\int_0^{\sqrt{24}}\left(5-\sqrt{1+r^2}\right)\,r\,dr\,d\theta=$

$\int_0^{2\pi}\int_0^{\sqrt{24}}\left(5r-r\sqrt{1+r^2}\right)\,dr\,d\theta=\int_0^{2\pi}\left(5r^2/2-\frac{1}{3}\left(1+r^2\right)^{3/2}\right)\Big|_0^{\sqrt{24}}d\theta=\int_0^{2\pi}\frac{56}{3}\,d\theta=\frac{112\pi}{3}.$

The bowl that holds the most water is the hyperboloid.

16.3.58 Paraboloid: $V=\int_0^{2\pi}\int_0^{\sqrt{h}}\left(r^2\right)\,r\,dr\,d\theta=\int_0^{2\pi}\int_0^{\sqrt{h}}r^3\,dr\,d\theta=\int_0^{2\pi}\left(\frac{1}{4}r^4\right)\Big|_0^{\sqrt{h}}d\theta=\int_0^{2\pi}\frac{h^2}{4}\,d\theta=$ $\frac{\pi h^2}{2}$. Setting this equal to $\frac{176\pi}{3}$ and solving gives $h=\sqrt{\frac{352}{3}}\approx 10.83$ units.

Cone: $V=\int_0^{2\pi}\int_0^h r\cdot r\,dr\;d\theta=\int_0^{2\pi}\int_0^h r^2\,dr\;d\theta=\int_0^{2\pi}\left(\frac{1}{3}r^3\right)\Big|_0^h d\theta=\int_0^{2\pi}\frac{h^3}{3}\,d\theta=\frac{2\pi h^3}{3}$. Setting this equal to $\frac{176\pi}{3}$ and solving gives $h=\sqrt[3]{88}\approx 4.45$ units.

16.3.59

a. $z\geq 0$ implies that $x^2-y^2\geq 0$, which in turn implies that $x^2\geq y^2$. Thus $x\geq|y|$ or $x\leq-|y|$ so $R=\left\{(r,\,\theta)\,|-\frac{\pi}{4}\leq\theta\leq\frac{\pi}{4}\text{ or }\frac{3\pi}{4}\leq\theta\leq\frac{5\pi}{4}\right\}$.

b. $V=\iint\limits_R\left(x^2-y^2\right)\,dA=\int_{-\pi/4}^{\pi/4}\int_0^a\left(r^2\cos^2\,\theta-r^2\sin^2\,\theta\right)\,r\,dr\;d\theta=$

$\int_{-\pi/4}^{\pi/4}\int_0^a \left(\cos^2\theta-\sin^2\theta\right) r^3\,dr\,d\theta = \int_{-\pi/4}^{\pi/4}\left(\cos^2\theta-\sin^2\theta\right)\left(\frac{1}{4}r^4\right)\Big|_0^a d\theta = \left(\frac{1}{4}a^4\right)\int_{-\pi/4}^{\pi/4}\cos 2\theta\,d\theta =$ $\left(\frac{1}{4}a^4\right)\left(\frac{1}{2}\sin 2\theta\right)\Big|_{\theta=-\pi/4}^{\theta=\pi/4} = \frac{a^4}{4}.$

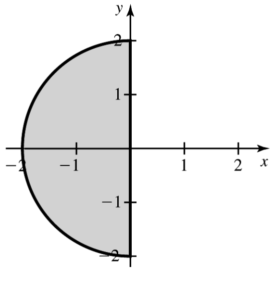
16.3.60 $2\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}r\,dr\,d\theta$ represents the volume. Let $u=a^2-r^2$ so that $du=-2r\,dr$. Then we have $2\cdot\frac{-1}{2}\int_0^{2\pi}\int_{a^2}^0 \sqrt{u}\,du\,d\theta = 2\pi\int_0^{a^2} u^{1/2}\,du = \frac{4}{3}\pi u^{3/2}\Big|_0^{a^2} = \frac{4}{3}\pi a^3.$

16.3.61 The volume is given by $\iint_R \sqrt{x^2+y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\frac{8\cos^3\theta}{3}\,d\theta =$ $\frac{8}{3}\int_{-\pi/2}^{\pi/2}\cos\theta(1-\sin^2\theta)\,d\theta = \frac{8}{3}\left(\sin\theta\Big|_{-\pi/2}^{\pi/2} - \frac{\sin^3\theta}{3}\Big|_{-\pi/2}^{\pi/2}\right) = \frac{8}{3}\left(2-\frac{2}{3}\right) = \frac{32}{9}.$

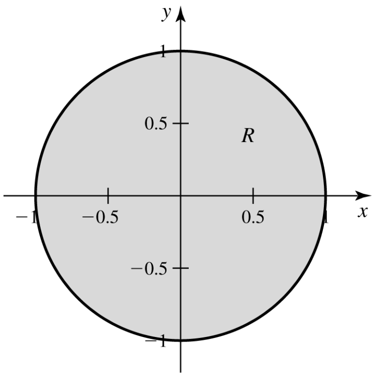
16.3.62 The volume is given by

$$\begin{aligned}\iint_R 2(x^2+y^2)\,dA &= \int_0^{\pi}\int_0^{2\sin\theta} 2r^3\,dr\,d\theta = 2\int_0^{\pi}\frac{r^4}{4}\Big|_0^{2\sin\theta} d\theta = 8\int_0^{\pi}\sin^4\theta\,d\theta\\ &= 8\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta = 2\int_0^{\pi}\left(1-2\cos 2\theta+\cos^2 2\theta\right)\,d\theta\\ &= 2\int_0^{\pi}\left(\frac{3}{2}-2\cos 2\theta+\frac{1}{2}\cos 4\theta\right)\,d\theta = \left(3\theta-2\sin 2\theta+\frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi} = 3\pi.\end{aligned}$$

16.3.63 $\iint_R \frac{1}{4+\sqrt{x^2+y^2}}\,dA = \int_{\pi/2}^{3\pi/2}\int_0^2 \frac{1}{4+r}\,r\,dr\,d\theta = \int_{\pi/2}^{3\pi/2}\int_0^2\left(1-\frac{4}{r+4}\right)\,dr\,d\theta =$ $\int_{\pi/2}^{3\pi/2}\left(r-4\,\ln|r+4|\right)\Big|_0^2 d\theta = 2\int_{\pi/2}^{3\pi/2}\left(1-2\,\ln\left(\frac{3}{2}\right)\right)\,d\theta = 2\left(\theta\left(1-2\ln\left(\frac{3}{2}\right)\right)\right)\Big|_{\theta=\pi/2}^{\theta=3\pi/2} =$ $2\pi\left(1-2\ln\left(\frac{3}{2}\right)\right).$



16.3.64 $\iint_R \frac{x-y}{x^2+y^2+1}\,dA = \int_0^{2\pi}\int_0^1 \frac{r\cos\theta-r\sin\theta}{r^2+1}\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{r^2\,(\cos\theta-\sin\theta)}{r^2+1}\,dr\,d\theta =$ $\left(\int_0^{2\pi}(\cos\theta-\sin\theta)\left(\int_0^1\left(1-\frac{1}{r^2+1}\right)\,dr\right)\,d\theta\right) = \int_0^{2\pi}(\cos\theta-\sin\theta)\left(\left(r-\tan^{-1}r\right)\Big|_0^1\right)\,d\theta =$ $\int_0^{2\pi}(\cos\theta-\sin\theta)\left(1-\tan^{-1}(1)\right)\,d\theta = \left(1-\frac{\pi}{4}\right)\int_0^{2\pi}(\cos\theta-\sin\theta)\,d\theta = \left(1-\frac{\pi}{4}\right)(\sin\theta+\cos\theta)\Big|_{\theta=0}^{2\pi} d\theta =$ $\left(1-\frac{\pi}{4}\right)\cdot 0 = 0$



16.3.65 $\displaystyle\int_0^{\pi/2}\int_1^{\infty}\frac{\cos\theta}{r^3}\,r\,dr d\theta = \lim_{b\to\infty}\int_0^{\pi/2}\int_1^{b}\frac{\cos\theta}{r^2}dr\,d\theta = \lim_{b\to\infty}\int_0^{\pi/2}\left(-\frac{\cos\theta}{r}\right)\Big|_1^b d\theta =$ $\displaystyle\lim_{b\to\infty}\int_0^{\pi/2}\cos\theta\left(1-\frac{1}{b}\right)d\theta = \lim_{b\to\infty}\left(\sin\theta\left(1-\frac{1}{b}\right)\right)\Big|_{\theta=0}^{\theta=\pi/2} = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1.$

16.3.66 $\displaystyle\iint_R \frac{dA}{(x^2+y^2)^{5/2}} = \int_0^{2\pi}\int_1^{\infty}\frac{1}{(r^2)^{5/2}}\,r\,dr\,d\theta = \lim_{b\to\infty}\int_0^{2\pi}\int_1^{b}\frac{1}{r^4}dr\,d\theta = \lim_{b\to\infty}\int_0^{2\pi}\left(-\frac{1}{3r^3}\right)\Big|_1^b d\theta =$ $\displaystyle\lim_{b\to\infty}\int_0^{2\pi}\frac{1}{3}\left(1-\frac{1}{b^3}\right)d\theta = \lim_{b\to\infty}\left(\frac{1}{3}\,\theta\left(1-\frac{1}{b^3}\right)\right)\Big|_{\theta=0}^{\theta=2\pi} = \lim_{b\to\infty}\frac{2\pi}{3}\left(1-\frac{1}{b^3}\right) = \frac{2\pi}{3}.$

16.3.67 $\displaystyle\iint_R e^{-x^2-y^2}\,dA = \int_0^{\pi/2}\int_0^{\infty}e^{-r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\left(\lim_{b\to\infty}\int_0^{b}e^{-r^2}\,r\,dr\right)d\theta =$ $\displaystyle\int_0^{\pi/2}\left(\lim_{b\to\infty}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^b\right)d\theta = \int_0^{\pi/2}\left(\lim_{b\to\infty}\frac{1}{2}\left(1-e^{-b^2}\right)\right)d\theta = \int_0^{\pi/2}\left(\frac{1}{2}\right)d\theta = \frac{\pi}{4}.$

16.3.68 $\displaystyle\iint_R \frac{1}{(1+x^2+y^2)^2}dA = \int_0^{\pi/2}\int_0^{\infty}\frac{1}{(1+r^2)^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\left(\lim_{b\to\infty}\int_0^{b}\frac{r}{(1+r^2)^2}\,dr\right)d\theta =$ $\displaystyle\int_0^{\pi/2}\left(\lim_{b\to\infty}\left(-\frac{1}{2\,(1+r^2)}\right)\Big|_0^b\right)d\theta = \int_0^{\pi/2}\left(\lim_{b\to\infty}\frac{1}{2}\left(1-\frac{1}{1+b^2}\right)\right)d\theta = \int_0^{\pi/2}(\frac{1}{2})\,d\theta = \frac{\pi}{4}.$

16.3.69

a. $\displaystyle V = \iint_R \sqrt{16-x^2-y^2}\,dA = \int_0^{\pi/4}\int_0^{4}\sqrt{16-r^2}\,r\,dr\,d\theta = \int_0^{\pi/4}\left(-\frac{1}{3}\left(16-r^2\right)^{3/2}\right)\Big|_0^4 d\theta =$ $\displaystyle\int_0^{\pi/4}\frac{64}{3}\,d\theta = \frac{16\pi}{3}$. Because this slice is $\frac{1}{8}$ of the hemispherical cake, the formula for the volume of a sphere can be used to confirm that the volume of the slice is $V = \frac{1}{8}\cdot\frac{1}{2}\cdot\frac{4}{3}\pi\cdot 4^3 = \frac{16\pi}{3}$.

b. $\displaystyle V = \iint_R (\sqrt{16-x^2-y^2}-a)\,dA = \int_0^{2\pi}\int_0^{\sqrt{16-a^2}}(\sqrt{16-r^2}-a)\,r\,dr\,d\theta = 2\pi\left(\frac{64}{3}-8a+\frac{a^3}{6}\right)$. Setting this equal to $\frac{16\pi}{3}$ and using a root finder gives $a \approx 2.78$.

16.3.70 The mass $m = \displaystyle\iint_R \rho\,(r,\,\theta)\;dA$ where $\rho\,(r,\,\theta)$ estimates the density at each point on the plate. Approximate $m = \displaystyle\iint_R \rho\,(r,\,\theta)\;dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n}\rho\,(r_i,\,\theta_j)\cdot r_i\triangle r\,\triangle\theta$. Let $\triangle r = 1$, $\triangle\theta = \frac{\pi}{4}$, then sum the upper right (clockwise outermost) corner for each subsection of region. We have $(2.0+2.1+2.2+2.3)\cdot(1)\cdot(1)\cdot\left(\frac{\pi}{4}\right)+(2.5+2.7+2.9+3.1)\cdot(2)\cdot(1)\cdot\left(\frac{\pi}{4}\right)(3.2+3.4+3.5+3.6)\cdot(2)\cdot(1)\cdot\left(\frac{\pi}{4}\right)\approx 56.6$ grams.

16.3.71 $m=\iint\limits_R \rho(r,\theta)\,dA=\int_0^\pi\int_1^4(4+r\sin\theta)\,r\,dr\,d\theta=\int_0^\pi\left(2r^2+\frac{1}{3}r^3\sin\theta\right)\Big|_1^4\,d\theta=$
$\int_0^\pi(30+21\sin\theta)\,d\theta=(30\theta-21\cos\theta)\Big|_{\theta=0}^{\theta=\pi}=30\pi+42$

16.3.72 $R=\{(r,\theta)\,|0\le r\le g(\theta),\,\alpha\le\theta\le\beta\}$, $A=\iint\limits_R 1\,dA=\int_\alpha^\beta\int_0^{g(\theta)}r\,dr\,d\theta=\int_\alpha^\beta\left(\frac{1}{2}r^2\right)\Big|_0^{g(\theta)}d\theta.$
Thus $A=\frac{1}{2}\int_\alpha^\beta(g(\theta))^2\,d\theta$.

16.3.73

a. Because e^{-x^2} is an even function, $\int_{-\infty}^0 e^{-x^2}\,dx=\int_0^\infty e^{-x^2}\,dx$. Because both of these are convergent, $\int_{-\infty}^\infty e^{-x^2}\,dx$ is convergent too.

b. $\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-x^2-y^2}\,dx\,dy=\int_0^{2\pi}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta=$
$\int_0^{2\pi}\left(\lim_{b\to\infty}\int_0^b e^{-r^2}\,r\,dr\right)d\theta=\int_0^{2\pi}\left(\lim_{b\to\infty}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^b\right)d\theta=\int_0^{2\pi}\left(\lim_{b\to\infty}\frac{1}{2}\left(1-e^{-b^2}\right)\right)d\theta$
$=\int_0^{2\pi}\frac{1}{2}\,d\theta=\pi.$
Thus $\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-x^2-y^2}\,dx\,dy=\pi$ $\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-x^2-y^2}\,dx\,dy=\int_{-\infty}^\infty\int_{-\infty}^\infty\left(e^{-x^2}\cdot e^{-y^2}\right)dx\,dy=$ $\left(\int_{-\infty}^\infty e^{-x^2}\,dx\right)\cdot\left(\int_{-\infty}^\infty e^{-y^2}\,dy\right)=\left(\int_{-\infty}^\infty e^{-x^2}\,dx\right)^2=\pi$. Thus $\int_{-\infty}^\infty e^{-x^2}\,dx=\sqrt{\pi}$. Note that the possibility $\int_{-\infty}^\infty e^{-x^2}\,dx=-\sqrt{\pi}$ is rejected because the integrand is everywhere positive and the integral of a positive function is positive.

c.
   i. $\int_0^\infty e^{-x^2}\,dx=\frac{1}{2}\int_{-\infty}^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$.

   ii. Let $u=-x^2$. Then $-\frac{du}{2}=x\,dx$, and the integral becomes $\frac{1}{2}\int_{-\infty}^0 e^u du=\left(\frac{1}{2}\right)\lim_{b\to-\infty}\int_b^0 e^u du=$ $\left(\frac{1}{2}\right)\lim_{b\to-\infty}(e^u)\Big|_{u=b}^{u=0}=\left(\frac{1}{2}\right)\lim_{b\to-\infty}\left(1-e^b\right)=\frac{1}{2}(1-0)=\frac{1}{2}$.

   iii. $\int_0^\infty x^2\,e^{-x^2}\,dx=\lim_{b\to\infty}\int_0^b x^2\,e^{-x^2}\,dx$ By parts: let $u=x$, $dv=xe^{-x^2}$, then $du=dx$, $v=-\frac{1}{2}e^{-x^2}$. We have $\lim_{b\to\infty}\left(-\frac{1}{2}x\,e^{-x^2}\Big|_{u=0}^b+\frac{1}{2}\int_0^b e^{-x^2}\,dx\right)=0+\frac{1}{2}\cdot\frac{\sqrt{\pi}}{2}=\frac{\sqrt{\pi}}{4}$ from (i).

16.3.74

a. $\iint\limits_R\frac{k}{(x^2+y^2)^p}\,dA=\int_0^{2\pi}\int_1^\infty\frac{k}{(r^2)^p}\,r\,dr\,d\theta=\int_0^{2\pi}\left(\lim_{b\to\infty}\int_1^b\frac{k\,r}{r^{2p}}\,dr\right)d\theta=$
$\int_0^{2\pi}\left(\lim_{b\to\infty}\int_1^b\frac{k}{r^{2p-1}}\,dr\right)d\theta$ which converges if $2p-1>1$ or $p>1$.

b. $\displaystyle\iint_R \frac{k}{(x^2+y^2)^p}\,dA = \int_0^{2\pi}\int_0^1 \frac{k}{(r^2)^p}\,r\,dr\,d\theta = \int_0^{2\pi}\left(\lim_{b\to 0}\int_b^1 \frac{k\,r}{r^{2p}}\,dr\right)d\theta = \int_0^{2\pi}\left(\lim_{b\to 0}\int_b^1 \frac{k}{r^{2p-1}}\,dr\right)d\theta$ which converges if $2p-1 \le 1$ or $p < 1$.

16.3.75

a.

$$\begin{aligned}\int_0^1\int_0^1 \frac{1}{(1+x^2+y^2)^2}\,dy\,dx &= \int_0^{\pi/4}\int_0^{\sec\theta} \frac{1}{(1+r^2)^2}\,r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{\csc\theta} \frac{1}{(1+r^2)^2}\,r\,dr\,d\theta\\ &= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2\theta}{1+\sec^2\theta}d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{\csc^2\theta}{1+\csc^2\theta}\,d\theta\\ &= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2\theta}{2+\tan^2\theta}\,d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{\csc^2\theta}{2+\cot^2\theta}d\theta = \frac{1}{2}\int_0^1 \frac{du}{2+u^2} + \frac{1}{2}\int_0^1 \frac{du}{2+u^2}\\ &= \int_0^1 \frac{du}{2+u^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)\Bigg|_{u=0}^{u=1} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right).\end{aligned}$$

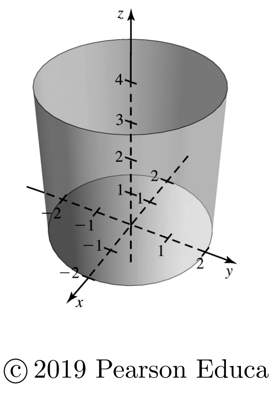
b.

$$\begin{aligned}\int_0^1\int_0^a \frac{1}{(1+x^2+y^2)^2}\,dy\,dx &= \int_0^{\tan^{-1}(a)}\int_0^{\sec\theta} \frac{1}{(1+r^2)^2}\,r\,dr\,d\theta + \int_{\tan^{-1}(a)}^{\pi/2}\int_0^{a\csc\theta} \frac{1}{(1+r^2)^2}\,r\,dr\,d\theta\\ &= \frac{1}{2}\int_0^{\tan^{-1}(a)} \frac{\sec^2\theta}{1+\sec^2\theta}d\theta + \frac{1}{2}\int_{\tan^{-1}(a)}^{\pi/2} \frac{a^2\csc^2\theta}{1+a^2\csc^2\theta}d\theta\\ &= \frac{1}{2}\int_0^{\tan^{-1}(a)} \frac{\sec^2\theta}{2+\tan^2\theta}d\theta + \frac{1}{2}\int_{\tan^{-1}(a)}^{\pi/2} \frac{\csc^2\theta}{\left(\frac{1+a^2}{a^2}\right)+\cot^2\theta}d\theta\\ &= \frac{1}{2}\int_0^1 \frac{du}{2+u^2} + \frac{1}{2}\int_0^{1/a} \frac{du}{\left(\frac{1+a^2}{a^2}\right)+u^2}\\ &= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)\Bigg|_{u=0}^{u=1} + \frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{a-u}{\sqrt{1+a^2}}\right)\Bigg|_{u=0}^{u=1/a}\\ &= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{a}{\sqrt{2}}\right) + \frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{1}{\sqrt{1+a^2}}\right).\end{aligned}$$

c. $\displaystyle\lim_{a\to\infty}\int_0^1\int_0^a \frac{1}{(1+x^2+y^2)^2}\,dy\,dx = \lim_{a\to\infty}\left[\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{a}{\sqrt{2}}\right) + \frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{1}{\sqrt{1+a^2}}\right)\right] =$ $\displaystyle\frac{1}{2\sqrt{2}}\left(\frac{\pi}{2}\right) + \frac{1}{2}(1)(0) = \frac{\pi\sqrt{2}}{8}$.

16.4 Triple Integrals

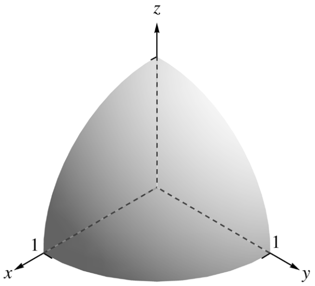
16.4.1



16.4.2 $\iiint\limits_D f(x,\,y,\,z)\;dV = \int_0^4 \int_0^6 \int_0^3 f(x,\,y,\,z)\;dx\,dy\,dz.$

16.4.3 $\iiint\limits_D f(x,\,y,\,z)\;dV = \int_{-9}^{9} \int_{-\sqrt{81-x^2}}^{\sqrt{81-x^2}} \int_{-\sqrt{81-x^2-y^2}}^{\sqrt{81-x^2-y^2}} f(x,\,y,\,z)\;dz\,dy\,dx.$

16.4.4



16.4.5 $\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2-z^2}} f(x,\,y,\,z)\;dy\,dx\,dz.$

16.4.6 $\overline{f} = \dfrac{1}{\text{volume of } D} \iiint\limits_D f(x,\,y,\,z)\;dV = \dfrac{1}{\text{volume of } D} \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} xyz\,dz\,dy\,dx.$

16.4.7 $\int_{-2}^{2} \int_3^6 \int_0^2 dx\,dy\,dz = \int_{-2}^{2} \int_3^6 (x)\Big|_0^2 dy\,dz = \int_{-2}^{2} \int_3^6 2\,dy\,dz = \int_{-2}^{2} (2y)\Big|_3^6 dz = \int_{-2}^{2} 6\,dz =$ $(6z)\Big|_{-2}^{2} = 24.$

16.4.8 $\int_{-1}^{1} \int_{-1}^{2} \int_0^1 6xyz\,dy\,dx\,dz = \int_{-1}^{1} \int_{-1}^{2} \left(3xy^2z\right)\Big|_0^1 dx\,dz = \int_{-1}^{1} \int_{-1}^{2} 3xz\,dx\,dz = \int_{-1}^{1} \left(\frac{3}{2}x^2z\right)\Big|_{-1}^{2} dz =$ $\int_{-1}^{1} \frac{9}{2}z\,dz = \left(\frac{9}{4}z^2\right)\Big|_{-1}^{1} = 0.$

16.4.9 $\int_{-2}^{2} \int_1^2 \int_1^e \frac{xy^2}{z} dz\,dx\,dy = \int_{-2}^{2} \int_1^2 xy^2 \left(xy^2\ \ln|z|\Big|_1^e\right)\,dx\,dy = \int_{-2}^{2} \int_1^2 xy^2\,dx\,dy =$ $\int_{-2}^{2} \left(\frac{1}{2}x^2y^2\right)\Big|_1^2 dy = \int_{-2}^{2} \frac{3}{2}y^2\,dy = \left(\frac{1}{2}y^3\right)\Big|_{-2}^{2} = 8.$

16.4.10 $\int_0^{\ln 4} \int_0^{\ln 3} \int_0^{\ln 2} e^{-x+y+z}\,dx\,dy\,dz = \int_0^{\ln 4} \int_0^{\ln 3} -e^{-x+y+z}\Big|_0^{\ln 2} dy\,dz = \int_0^{\ln 4} \int_0^{\ln 3} ((-1/2)(e^{y+z}) +$ $e^{y+z})\,dy\,dz = \frac{1}{2} \int_0^{\ln 4} \int_0^{\ln 3} e^{y+z}\,dy\,dz = \frac{1}{2} \int_0^{\ln 4} e^{y+z}\Big|_0^{\ln 3} dz = \frac{1}{2} \int_0^{\ln 4} (3e^z - e^z)\,dz = \int_0^{\ln 4} e^z\,dz = 4 - 1 = 3.$

16.4.11 $\int_0^{\pi/2} \int_0^1 \int_0^{\pi/2} \sin \pi x \cdot \cos y \cdot \sin 2z\,dy\,dx\,dz = \int_0^{\pi/2} \int_0^1 \sin \pi x\ \sin 2z \left((\sin y)\Big|_0^{\pi/2}\right)\,dx\,dz =$ $\int_0^{\pi/2} \int_0^1 \sin \pi x\ \sin 2z\ (1-0)\ dx\,dz = \int_0^{\pi/2} \int_0^1 \sin \pi x\ \sin 2z\,dx\,dz = \int_0^{\pi/2} \sin 2z \left(\left(-\frac{1}{\pi}\cos \pi x\right)\Big|_0^1\right)\,dz =$ $\int_0^{\pi/2} \sin 2z \left(\frac{1}{\pi} + \frac{1}{\pi}\right)\,dz = \frac{2}{\pi} \int_0^{\pi/2} \sin 2z\,dz = \frac{2}{\pi}\left(-\frac{1}{2}\cos 2z\right)\Big|_0^{\pi/2} = \frac{2}{\pi}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{2}{\pi}.$

16.4.12 $\displaystyle\int_0^2\int_1^2\int_0^1 yz\,e^x\,dx\,dz\,dy = \int_0^2\int_1^2 yz\left(\left(e^x\right)\Big|_0^1\right)dz\,dy = \int_0^2\int_1^2 yz\,(e-1)\,dz\,dy =$

$$(e-1)\int_0^2 y\left(\frac{1}{2}z^2\right)\Big|_1^2 dy = (e-1)\int_0^2 \frac{3}{2}y\,dy = \frac{3\,(e-1)}{2}\left(\frac{1}{2}y^2\right)\Big|_0^2 = 3\,(e-1).$$

16.4.13 $\displaystyle\iiint_D (xy+xz+yz)\,dV = \int_{-1}^1\int_{-2}^2\int_{-3}^3 (xy+xz+yz)\,dz\,dy\,dx =$

$$\int_{-1}^1\int_{-2}^2\left(xyz+\frac{1}{2}xz^2+\frac{1}{2}yz^2\right)\Big|_{-3}^3 dy\,dx = \int_{-1}^1\int_{-2}^2 (6xy)\,dy\,dx = \int_{-1}^1 \left(3xy^2\right)\Big|_{-2}^2 dx = \int_{-1}^1 (0)\,dx = 0.$$

16.4.14

$$\begin{aligned}\iiint_D xyze^{-x^2-y^2}\,dV &= \int_0^{\sqrt{\ln 2}}\int_0^{\sqrt{\ln 4}}\int_0^1 xyze^{-x^2-y^2}\,dz\,dy\,dx = \int_0^{\sqrt{\ln 2}}\int_0^{\sqrt{\ln 4}} \frac{1}{2}xyz^2e^{-x^2-y^2}\Big|_0^1 dy\,dx\\ &= \frac{1}{2}\int_0^{\sqrt{\ln 2}}\int_0^{\sqrt{\ln 4}} xye^{-x^2-y^2}\,dy\,dx = \frac{1}{2}\int_0^{\sqrt{\ln 2}}\int_0^{\sqrt{\ln 4}} xye^{-x^2}e^{-y^2}\,dy\,dx\\ &= \frac{1}{2}\int_0^{\sqrt{\ln 2}} xe^{-x^2}\left(-\frac{1}{2}e^{-y^2}\right)\Big|_0^{\sqrt{\ln 4}} dx = \frac{1}{2}\int_0^{\sqrt{\ln 2}} xe^{-x^2}\left(\frac{1}{2}\left(1-\frac{1}{4}\right)\right)dx\\ &= \frac{3}{16}\int_0^{\sqrt{\ln 2}} xe^{-x^2}\,dx = \frac{3}{16}\left(\left(-\frac{1}{2}e^{-x^2}\right)\Big|_0^{\sqrt{\ln 2}}\right) = \frac{3}{16}\left(\frac{1}{2}\left(1-\frac{1}{2}\right)\right) = \frac{3}{64}.\end{aligned}$$

16.4.15 $\displaystyle V = \iiint_D 1\,dV = \int_0^6\int_0^{4-2x/3}\int_0^{2-x/3-y/2} 1\,dz\,dy\,dx = \int_0^6\int_0^{4-2x/3} (z)\Big|_0^{2-x/3-y/2} dy\,dx =$

$$\int_0^6\int_0^{4-2x/3}\left(2-\frac{1}{3}x-\frac{1}{2}y\right)dy\,dx = \int_0^6\left(2y-\frac{1}{3}xy-\frac{1}{4}y^2\right)\Big|_0^{4-2x/3} dx =$$

$$\int_0^6\left(\frac{1}{9}x^2-\frac{4}{3}x+4\right)dx = \left(\frac{1}{27}x^3-\frac{2}{3}x^2+4x\right)\Big|_0^6 = 8.$$

16.4.16 $\displaystyle V = \iiint_D 1\,dV = \int_0^\pi\int_x^\pi\int_0^{\sin y} 1\,dz\,dy\,dx = \int_0^\pi\int_x^\pi (z)\Big|_0^{\sin y} dy\,dx = \int_0^\pi\int_x^\pi (\sin y)\,dy\,dx =$

$$\int_0^\pi (-\cos y)\Big|_x^\pi dx = \int_0^\pi (1+\cos x)\,dx = (x+\sin x)\Big|_0^\pi = \pi.$$

16.4.17 $\displaystyle V = \iiint_D 1\,dV = \int_{-2}^2\int_{-\sqrt{4-x^2}}^0\int_0^{-y} 1\,dz\,dy\,dx = \int_{-2}^2\int_{-\sqrt{4-x^2}}^0 (z)\Big|_0^{-y} dy\,dx =$

$$\int_{-2}^2\int_{-\sqrt{4-x^2}}^0 (-y)\,dy\,dx = \int_{-2}^2\left(-\frac{1}{2}y^2\right)\Big|_{-\sqrt{4-x^2}}^0 dx = \frac{1}{2}\int_{-2}^2\left(4-x^2\right)dx = \frac{1}{2}\left(4x-\frac{1}{3}x^3\right)\Big|_{-2}^2 = \frac{16}{3}.$$

16.4.18 $\displaystyle V = \iiint_D 1\,dV = \int_0^{1/2}\int_0^8\int_0^{2-4x} 1\,dz\,dy\,dx = \int_0^{1/2}\int_0^8 (z)\Big|_0^{2-4x} dy\,dx =$

$$\int_0^{1/2}\int_0^8 (2-4x)\,dy\,dx = \int_0^{1/2}((2-4x)\,y)\Big|_0^8 dx = \int_0^{1/2}(16-32x)\,dx = \left(16x-16x^2\right)\Big|_0^{1/2} = 4.$$

16.4.19 $\displaystyle V = \iiint_D 1\,dV = \int_0^{\ln 2}\int_0^1\int_1^{e^y} 1\,dz\,dx\,dy = \int_0^{\ln 2}\int_0^1 (z)\Big|_1^{e^y} dx\,dy = \int_0^{\ln 2}\int_0^1 (e^y-1)\,dx\,dy =$

$$\int_0^{\ln 2}(e^y-1)\,x\Big|_0^1 dy = \int_0^{\ln 2}(e^y-1)\,dy = (e^y-y)\Big|_0^{\ln 2} = 1-\ln 2.$$

16.4.20 $V = \iiint\limits_D 1\,dV = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2}^{3}\int_0^{3-y} 1\,dz\,dy\,dx = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2}^{3} (z)\Big|_0^{3-y} dy\,dx = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2}^{3} (3-y)\;dy\,dx =$ $\int_{-\sqrt{3}}^{\sqrt{3}}\left(3y-\frac{1}{2}y^2\right)\Big|_{x^2}^{3} dx = \int_{-\sqrt{3}}^{\sqrt{3}}\left(\frac{9}{2}-3x^2+\frac{1}{2}x^4\right)dx = \left(\frac{9}{2}x - x^3 + \frac{1}{10}x^5\right)\Big|_{-\sqrt{3}}^{\sqrt{3}} = \frac{24\sqrt{3}}{5}.$

16.4.21

$$\begin{aligned}V = \iiint\limits_D 1\,dV &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} 1\,dz\,dy\,dx\\ &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (z)\Big|_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} dy\,dx\\ &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(\sqrt{19-x^2-y^2}-\sqrt{1+x^2+y^2}\right)dy\,dx.\end{aligned}$$

Converting to polar coordinates gives

$$\begin{aligned}&\int_0^{2\pi}\int_0^3\left(\sqrt{19-r^2}-\sqrt{1+r^2}\right) r\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^3\left(r\sqrt{19-r^2}-r\sqrt{1+r^2}\right)dr\,d\theta\\ &= \int_0^{2\pi}\left(-\frac{1}{3}\right)\left(\left(19-r^2\right)^{3/2}+\left(1+r^2\right)^{3/2}\right)\Big|_0^3 d\theta = \left(-\frac{1}{3}\right)\int_0^{2\pi}\left(20\sqrt{10}-19\sqrt{19}-1\right)d\theta\\ &= \frac{1+19\sqrt{19}-20\sqrt{10}}{3}\int_0^{2\pi}d\theta = \frac{2\pi}{3}\left(1+19\sqrt{19}-20\sqrt{10}\right).\end{aligned}$$

16.4.22

$$\begin{aligned}V = \iiint\limits_D 1\,dV &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} 1\,dz\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (z)\Big|_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dy\,dx\\ &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(\sqrt{8-x^2-y^2}-\sqrt{x^2+y^2}\right)dy\,dx.\end{aligned}$$

Converting to polar coordinates gives

$$\begin{aligned}\int_0^{2\pi}\int_0^2\left(\sqrt{8-r^2}-\sqrt{r^2}\right)r\,dr\,d\theta &= \int_0^{2\pi}\int_0^2\left(r\sqrt{8-r^2}-r^2\right)dr\,d\theta = \int_0^{2\pi}\left(-\frac{1}{3}\left(8-r^2\right)^{3/2}-\frac{1}{3}r^3\right)\Big|_0^2 d\theta\\ &= \int_0^{2\pi}\left(\frac{-16+16\sqrt{2}}{3}\right)d\theta = \frac{32\pi}{3}\left(\sqrt{2}-1\right).\end{aligned}$$

16.4.23 $V = \iiint\limits_D 1\,dV = \iint\limits_R (9-3(x^2+z^2))\,dA = \int_0^{2\pi}\int_0^{\sqrt{3}}(9-3r^2)r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{9r^2}{2}-\frac{3r^4}{4}\right)\Big|_0^{\sqrt{3}} d\theta =$ $2\pi\left(\frac{27}{2}-\frac{27}{4}\right) = \frac{27\pi}{2}.$

16.4.24 $V = \iiint\limits_D 1\,dV = \int_0^1\int_0^2\int_{z^2}^{2-z} 1\,dx\,dy\,dz = \int_0^1\int_0^2(2-z-z^2)\,dy\,dz = 2\int_0^1(2-z-z^2)\,dz =$ $2\left(2z-\frac{z^2}{2}-\frac{z^3}{3}\right)\Big|_0^1 = 2\left(2-\frac{1}{2}-\frac{1}{3}\right) = \frac{7}{3}.$

16.4.25 $V = \iiint\limits_D 1\,dV = \int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}}\int_{x-3}^{3-x} 1\,dy\,dx\,dz = \int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}} y\Big|_{x-3}^{3-x}\,dx\,dz =$
$\int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}} (6-2x)\;dx\,dz = \int_{-1}^{1}\left(6x-x^2\right)\Big|_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}}\,dz = \int_{-1}^{1} 12\sqrt{4-4z^2}\,dx = 24\int_{-1}^{1}\sqrt{1-z^2}\,dx =$
$24\cdot\frac{\pi}{2} = 12\pi.$

16.4.26 $V = \int_0^1\int_0^2\int_0^{e^{-z}} 1\,dy\,dx\,dz = \int_0^1\int_0^2 y\Big|_0^{e^{-z}}\,dx\,dz$
$= \int_0^1\int_0^2 e^{-z}\,dx\,dz = \int_0^1 2e^{-z}\,dz = \left(-2e^{-z}\right)\Big|_0^1 = 2-\frac{2}{e}.$

16.4.27 $V = \int_0^1\int_0^{1-z^2}\int_0^{1-z} 1\,dy\,dx\,dz = \int_0^1\int_0^{1-z^2} y\Big|_0^{1-z}\,dx\,dz = \int_0^1\int_0^{1-z^2}(1-z)\,dx\,dz =$
$\int_0^1 (1-z)x\Big|_0^{1-z^2} = \int_0^1 (1-z)(1-z^2)\,dz = \int_0^1 (1-z-z^2+z^3)\,dz = \left(z-\frac{z^2}{2}-\frac{z^3}{3}+\frac{z^4}{4}\right)\Big|_0^1 = \frac{5}{12}.$

16.4.28

$$V = \int_0^1\int_0^{2-z-z^2}\int_{z^2}^{2-x-z} 1\,dy\,dx\,dz = \int_0^1\int_0^{2-z-z^2} y\Big|_{z^2}^{2-x-z}\,dx\,dz = \int_0^1\int_0^{2-z-z^2}(2-x-z-z^2)\,dx\,dz$$
$$= \int_0^1\left(2x-\frac{x^2}{2}-zx-z^2x\right)\Big|_0^{2-z-z^2}\,dz = \int_0^1\left(\frac{z^4}{2}+z^3-\frac{3z^2}{2}-2z+2\right)\,dz$$
$$= \left(\frac{z^5}{10}+\frac{z^4}{4}-\frac{z^3}{2}-z^2+2z\right)\Big|_0^1 = \frac{17}{20}.$$

16.4.29 $V = \int_0^2\int_0^4\int_z^{z+1} 1\,dy\,dz\,dx = \int_0^2\int_0^4 y\Big|_z^{z+1}\,dz\,dx = \int_0^2\int_0^4 1\,dz\,dx = \int_0^2 4\,dx = 8.$

16.4.30 $\int_0^1\int_0^x\int_0^{2\sqrt{1-x^2}} f(x,y,z)\,dz\,dy\,dz.$

16.4.31 $\int_0^1\int_y^1\int_0^{2\sqrt{1-x^2}} f(x,y,z)\,dz\,dx\,dy.$

16.4.32 $\int_0^2\int_0^{\frac{1}{2}\sqrt{4-z^2}}\int_0^x f(x,y,z)\,dy\,dx\,dz.$

16.4.33 $\int_0^1\int_0^{2\sqrt{1-x^2}}\int_0^x f(x,y,z)\,dy\,dz\,dx.$

16.4.34 $\int_0^2\int_0^{\frac{1}{2}\sqrt{4-z^2}}\int_y^{\frac{1}{2}\sqrt{4-z^2}} f(x,y,z)\,dx\,dy\,dz.$

16.4.35 $\int_0^1\int_0^{2\sqrt{1-y^2}}\int_{y^2}^{\frac{1}{2}\sqrt{4-z^2}} f(x,y,z)\,dx\,dz\,dy.$

16.4.36

i. $\int_0^1\int_x^1\int_0^{1-y^2} dz\,dy\,dx.$

ii. $\int_0^1 \int_0^y \int_0^{1-y^2} dz\, dx\, dy.$

iii. $\int_0^1 \int_0^{1-x^2} \int_x^{\sqrt{1-z}} dy\, dz\, dx.$

iv. $\int_0^1 \int_0^{\sqrt{1-z}} \int_x^{\sqrt{1-z}} dy\, dx\, dz.$

v. $\int_0^1 \int_0^{\sqrt{1-z}} \int_0^y dx\, dy\, dz.$

vi. $\int_0^1 \int_0^{1-y^2} \int_0^y dx\, dz\, dy.$

It is a good exercise to see that each of these evaluate to $\frac{1}{4}$.

16.4.37

i. $\int_0^1 \int_0^2 \int_0^{1-y} dz\, dx\, dy.$

ii. $\int_0^2 \int_0^1 \int_0^{1-z} dy\, dz\, dx.$

iii. $\int_0^1 \int_0^2 \int_0^{1-z} dy\, dx\, dz.$

iv. $\int_0^1 \int_0^{1-z} \int_0^2 dx\, dy\, dz.$

v. $\int_0^1 \int_0^{1-y} \int_0^2 dx\, dz\, dy.$

16.4.38 $\int_0^\pi \int_0^\pi \int_0^{\sin x} \sin y \; dz\, dx\, dy = \int_0^\pi \int_0^\pi \sin x \cdot \sin y\, dx\, dy = \int_0^\pi (-\cos x \cdot \sin y) \Big|_0^\pi dy =$ $\int_0^\pi 2 \sin y\, dy = (-2 \cos y) \Big|_0^\pi = 4.$

16.4.39 $\int_0^2 \int_0^4 \int_{y^2}^4 \sqrt{x}\, dz\, dx\, dy = \int_0^2 \int_0^4 z\sqrt{x} \Big|_{y^2}^4 dx\, dy = \int_0^2 \int_0^4 \left(4 - y^2\right) \sqrt{x}\, dx\, dy =$ $\int_0^2 (4 - y^2) \left(\frac{2}{3} x^{3/2}\right) \Big|_0^4 dy = \frac{16}{3} \int_0^2 (4 - y^2)\, dy = \frac{16}{3} \left(4y - \frac{1}{3} y^3\right) \Big|_0^2 = \frac{256}{9}.$

16.4.40 $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz\, dz\, dy\, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \left(xz^2\right) \Big|_0^{\sqrt{1-x^2-y^2}} dy\, dx =$ $\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x - x^3 - xy^2\right) dy\, dx$. Switching the order of integration yields $\int_0^1 \int_0^{\sqrt{1-y^2}} \left(x - x^3 - xy^2\right) dx\, dy = \int_0^1 \left(\frac{1}{2} x^2 - \frac{1}{4} x^4 - \frac{1}{2} x^2 y^2\right) \Big|_0^{\sqrt{1-y^2}} dy = \int_0^1 \left(\frac{1}{4} - \frac{1}{2} y^2 + \frac{1}{4} y^4\right) dy =$ $\left(\frac{1}{4} y - \frac{1}{6} y^3 + \frac{1}{20} y^5\right) \Big|_0^1 = \frac{2}{15}.$

16.4.41 $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dz\,dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}} (z)\Big|_0^{\sqrt{1-x^2}} dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx =$ $\int_0^1 y\sqrt{1-x^2}\Big|_0^{\sqrt{1-x^2}} dx = \int_0^1 \left(1-x^2\right)\,dx = \left(x-\frac{1}{3}x^3\right)\Big|_0^1 = \frac{2}{3}.$

16.4.42

$$\begin{aligned}\int_1^6\int_0^{4-2y/3}\int_0^{12-2y-3z}\frac{1}{y}\,dx\,dz\,dy &= \int_1^6\int_0^{4-2y/3}\left(\frac{x}{y}\right)\Big|_0^{12-2y-3z} dz\,dy\\ &= \int_1^6\int_0^{4-2y/3}\frac{12-2y-3z}{y}\,dz\,dy\\ &= \int_1^6\frac{1}{y}\left(\left(12-2y\right)z-\frac{3}{2}z^2\right)\Big|_0^{4-2y/3} dy\\ &= \int_1^6\frac{1}{y}\left(24-8y+\frac{2}{3}y^2\right)\,dy = \int_1^6\left(\frac{24}{y}-8+\frac{2}{3}y\right)\,dy\\ &= \left(24\,\ln|y|-8y+\frac{1}{3}y^2\right)\Big|_1^6 = 24\,\ln 6+\frac{85}{3}.\end{aligned}$$

16.4.43 $\int_0^3\int_0^{\sqrt{9-z^2}}\int_0^{\sqrt{1+x^2+z^2}} dy\,dx\,dz = \int_0^3\int_0^{\sqrt{9-z^2}}\sqrt{1+x^2+z^2}\,dz\,dy.$ Use polar coordinates.

$$\int_0^{\pi/2}\int_0^3\sqrt{1+r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\left(\frac{1}{3}\left(1+r^2\right)^{3/2}\right)\Big|_0^3 d\theta = \int_0^{\pi/2}\frac{1}{3}\left(10^{3/2}-1\right)\,d\theta = \frac{\pi}{6}\left(10\sqrt{10}-1\right).$$

16.4.44

$$\begin{aligned}\int_0^1\int_y^{2-y}\int_0^{2-x-y}15xy\;dz\,dx\,dy &= 15\int_0^1\int_y^{2-y}(xyz)\Big|_0^{2-x-y} dx\,dy\\ &= 15\int_0^1\int_y^{2-y}\left(2xy-x^2y-xy^2\right)\,dx\,dy\\ &= 15\int_0^1\left(x^2y-\frac{1}{3}x^3y-\frac{1}{2}x^2y^2\right)\Big|_y^{2-y} dy\\ &= 15\int_0^1\left(\left(2-y\right)^2y-\frac{1}{3}\left(2-y\right)^3y-\frac{1}{2}\left(2-y\right)^2y^2-y^3+\frac{1}{3}y^4+\frac{1}{2}y^4\right)\,dy\\ &= 15\int_0^1\left(\frac{4}{3}y-2y^2+\frac{2}{3}y^4\right)\,dy = 15\left(\frac{2}{3}y^2-\frac{2}{3}y^3+\frac{2}{15}y^5\right)\Big|_0^1 = 2.\end{aligned}$$

16.4.45 $\int_1^{\ln 8}\int_1^{\sqrt{z}}\int_{\ln y}^{\ln(2y)} e^{x+y^2-z}\,dx\,dy\,dz = \int_1^{\ln 8}\int_1^{\sqrt{z}} e^{x+y^2-z}\Big|_{\ln y}^{\ln(2y)} dy\,dz =$ $\int_1^{\ln 8}\int_1^{\sqrt{z}}\left(2ye^{y^2-z}-ye^{y^2-z}\right)\,dy\,dz = \int_1^{\ln 8}\int_1^{\sqrt{z}} ye^{y^2-z}\,dy\,dz = \int_1^{\ln 8}\left(\frac{1}{2}\,e^{y^2-z}\right)\Big|_1^{\sqrt{z}} dz =$ $\frac{1}{2}\int_1^{\ln 8}\left(1-e^{1-z}\right)\,dz = \frac{1}{2}\left(z+e^{1-z}\right)\Big|_1^{\ln 8} = \frac{1}{2}\left(\ln 8+\frac{e}{8}-(1+1)\right) = \frac{1}{2}\ln 8+\frac{e}{16}-1.$

16.4.46 $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{2-x} 4yz\;dz\,dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}}\left(2yz^2\right)\Big|_0^{2-x} dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}} 2y\left(2-x\right)^2\,dy\,dx =$ $\int_0^1 y^2\left(2-x\right)^2\Big|_0^{\sqrt{1-x^2}} dx = \int_0^1\left(1-x^2\right)\left(2-x\right)^2\,dx = \int_0^1\left(4-4x-3x^2+4x^3-x^4\right)\,dx =$ $\left(4x-2x^2-x^3+x^4-\frac{1}{5}x^5\right)\Big|_0^1 = \frac{9}{5}-0 = \frac{9}{5}.$

16.4.47 We can rewrite $\int_0^5 \int_{-1}^0 \int_0^{4x+4} dy\,dx\,dz$ as $\int_0^4 \int_{(y-4)/4}^0 \int_0^5 dz\,dx\,dy$. This is then evaluated as $\int_0^4 \int_{(y-4)/4}^0 5\,dx\,dy = \int_0^4 -\frac{5}{4}(y-4)\,dy = -\frac{5}{4}\left(\frac{1}{2}y^2 - 4y\right)\Big|_0^4 = 10$.

16.4.48 We can rewrite $\int_0^1 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} dz\,dy\,dx$ as $\int_0^1 \int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy\,dz\,dx$. This can then be evaluated as $\int_0^1 \int_0^2 2\sqrt{4-z^2}\,dz\,dx = \int_0^1 \left(z\sqrt{4-z^2} + 4\sin^{-1}\left(\frac{z}{2}\right)\right)\Big|_0^2 dx = \int_0^1 (2\pi)\,dx = (2\pi x)\Big|_0^1 = 2\pi$.

16.4.49 We can rewrite $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dy\,dz\,dx$ as $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\,dy\,dx$. This is then evaluated as $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx = \int_0^1 \left(\sqrt{1-x^2}\right)\left(\sqrt{1-x^2}\right)dx = \int_0^1 (1-x^2)\,dx = \left(x - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{2}{3}$.

16.4.50 We can rewrite $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} dy\,dz\,dx$ as $\int_0^4 \int_0^{\sqrt{16-z^2}} \int_0^{\sqrt{16-y^2-z^2}} dx\,dy\,dz$. This can then be evaluated as $\int_0^4 \int_0^{\sqrt{16-z^2}} \sqrt{16-y^2-z^2}\,dy\,dz$. Converting to polar coordinates gives $\int_0^{\pi/2} \int_0^4 \sqrt{16-r^2}\,r\,dr\,d\theta = \int_0^{\pi/2} \left(-\frac{1}{3}\left(16-r^2\right)^{3/2}\right)\Big|_0^4 d\theta = \int_0^{\pi/2} \frac{64}{3}\,d\theta = \frac{32\pi}{3}$.

16.4.51 The volume of the box is $1 \cdot 2 \cdot \frac{\pi}{2} = \pi$. The average value is therefore

$$\frac{1}{\pi}\int_0^2 \int_0^1 \int_0^{\pi/2} 8xy\cos z\,dz\,dx\,dy = \frac{1}{\pi}\int_0^2 \int_0^1 8xy\,dx\,dy = \frac{1}{\pi}\int_0^2 4x^2y\Big|_0^1 dy = \frac{1}{\pi}\cdot 2y^2\Big|_0^2 = \frac{8}{\pi}.$$

16.4.52 The average value is $\frac{1}{\text{volume of } D}\iiint_D T(x, y, z)\,dV$. This can be written as $\frac{1}{\ln 2 \cdot \ln 4 \cdot \ln 8}\int_0^{\ln 2}\int_0^{\ln 4}\int_0^{\ln 8} 128\,e^{-x-y-z}\,dz\,dy\,dx = \frac{128}{6(\ln 2)^3}\int_0^{\ln 2}\int_0^{\ln 4}\int_0^{\ln 8} e^{-x}e^{-y}e^{-z}\,dz\,dy\,dx = \frac{64}{3(\ln 2)^3}\int_0^{\ln 2}\int_0^{\ln 4} e^{-x}e^{-y}\left(-e^{-z}\right)\Big|_0^{\ln 8} dy\,dx = \frac{64}{3(\ln 2)^3}\int_0^{\ln 2}\int_0^{\ln 4} e^{-x}e^{-y}\left(1-\frac{1}{8}\right)dy\,dx = \frac{56}{3(\ln 2)^3}\int_0^{\ln 2}\int_0^{\ln 4} e^{-x}e^{-y}\,dy\,dx = \frac{56}{3(\ln 2)^3}\int_0^{\ln 2} e^{-x}\left(-e^{-y}\right)\Big|_0^{\ln 4} dx = \frac{56}{3(\ln 2)^3}\int_0^{\ln 2} e^{-x}\left(1-\frac{1}{4}\right)dx = \frac{14}{(\ln 2)^3}\int_0^{\ln 2} e^{-x}\,dx = \frac{14}{(\ln 2)^3}\left(-e^{-x}\right)\Big|_0^{\ln 2} = \frac{14}{(\ln 2)^3}\left(1-\frac{1}{2}\right) = \frac{7}{(\ln 2)^3}$.

16.4.53 The average value is given by $\frac{1}{\text{volume of } D}\iiint_D f(x, y, z)\,dV$. This can be written as

$$\frac{1}{(\pi\cdot 2^2\cdot 2)}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{2}\left(x^2+y^2+z^2\right)\,dz\,dy\,dx=\frac{1}{8\pi}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(\left(x^2+y^2\right)z+\frac{1}{3}z^3\right)\Bigg|_{0}^{2}dy\,dx$$

$$=\frac{1}{8\pi}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(2\left(x^2+y^2\right)+\frac{8}{3}\right)dy\,dx=\frac{1}{8\pi}\int_{-2}^{2}\left(2x^2y+\frac{2}{3}y^3+\frac{8}{3}y\right)\Bigg|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$

$$=\frac{1}{2\pi}\int_{-2}^{2}\left(x^2\sqrt{4-x^2}+\frac{1}{3}\left(4-x^2\right)^{3/2}+\frac{4}{3}\sqrt{4-x^2}\right)dx$$

$$=\frac{1}{2\pi}\left(-\frac{x}{4}\left(x^2-2\right)\sqrt{4-x^2}+2\sin^{-1}\left(\frac{x}{2}\right)+\frac{x}{12}\left(x^2-10\right)+2\sin^{-1}\left(\frac{x}{2}\right)+\frac{x}{6}\sqrt{4-x^2}+\frac{8}{3}\sin^{-1}\left(\frac{x}{2}\right)\right)\Bigg|_{-2}^{2}$$

$$=\frac{1}{2\pi}\left(\frac{10\pi}{3}+\frac{10\pi}{3}\right)=\frac{10}{3}.$$

16.4.54 The average value is

$$\frac{1}{\text{volume of } D}\iiint\limits_{D}f\left(x,\,y,\,z\right)\,dV=\frac{1}{\frac{1}{2}\left(\frac{4}{3}\pi\cdot 4^3\right)}\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{0}^{\sqrt{16-x^2-y^2}}z\,dz\,dy\,dx$$

$$=\frac{3}{128\pi}\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\left(\frac{1}{2}z^2\right)\Bigg|_{0}^{\sqrt{16-x^2-y^2}}dy\,dx$$

$$=\frac{3}{256\pi}\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\left(16-x^2-y^2\right)dy\,dx.$$

Converting to polar coordinates gives

$$\frac{3}{256\pi}\int_{0}^{2\pi}\int_{0}^{4}\left(16-r^2\right)r\,dr\,d\theta=\frac{3}{256\pi}\int_{0}^{2\pi}\int_{0}^{4}\left(16r-r^3\right)dr\,d\theta$$

$$=\frac{3}{256\pi}\int_{0}^{2\pi}\left(8r^2-\frac{1}{4}r^4\right)\Bigg|_{0}^{4}d\theta=\frac{3}{256\pi}\int_{0}^{2\pi}\left(128-64\right)\,d\theta$$

$$=\frac{3}{4\pi}\int_{0}^{2\pi}d\theta=\frac{3}{2}.$$

16.4.55

a. False. Only six iterations are possible.

b. False. The outermost limits of integration must be constants.

c. False. D is the intersection of two cylinders in first octant.

16.4.56 $\displaystyle\int_{1}^{4}\int_{z}^{4z}\int_{0}^{\pi^2}\frac{\sin\sqrt{yz}}{x^{3/2}}\,dy\,dx\,dz=\int_{0}^{\pi^2}\int_{1}^{4}\int_{z}^{4z}\frac{\sin\sqrt{yz}}{x^{3/2}}\,dx\,dz\,dy=$

$$\int_{0}^{\pi^2}\int_{1}^{4}\left(-\frac{2\sin\sqrt{yz}}{\sqrt{x}}\right)\Bigg|_{z}^{4z}dz\,dy=\int_{0}^{\pi^2}\int_{1}^{4}\left(-\frac{\sin\sqrt{yz}}{\sqrt{z}}+\frac{2\sin\sqrt{yz}}{\sqrt{z}}\right)dz\,dy=\int_{0}^{\pi^2}\int_{1}^{4}\left(\frac{\sin\sqrt{yz}}{\sqrt{z}}\right)dz\,dy.$$

Let $u=\sqrt{yz}$, $du=\frac{1}{2}\left(yz\right)^{-1/2}\cdot y\,dz$. Substituting gives $\displaystyle\int_{0}^{\pi^2}\int_{\sqrt{y}}^{2\sqrt{y}}\left(\frac{2}{\sqrt{y}}\sin u\right)du\,dy=$

$\displaystyle\int_{0}^{\pi^2}\left(-\frac{2}{\sqrt{y}}\cos u\right)\Bigg|_{u=\sqrt{y}}^{u=2\sqrt{y}}dy=\int_{0}^{\pi^2}\left(-\frac{2\,\cos\left(2\sqrt{y}\right)}{\sqrt{y}}+\frac{2\,\cos\left(\sqrt{y}\right)}{\sqrt{y}}\right)dy$ Let $v=\sqrt{y}$ so that $2\,dv=\dfrac{dy}{\sqrt{y}}$.

Then we have $\displaystyle 4\int_{0}^{\pi}\left(-\cos\left(2v\right)+\cos v\right)\,dv=4\left(-\frac{1}{2}\sin\left(2v\right)+\sin v\right)\Bigg|_{v=0}^{v=\pi}=0.$

16.4.57 $V = \int_{-1}^{0}\int_{-x-1}^{x+1}\int_{0}^{1-x-y} 1\,dz\,dy\,dx + \int_{0}^{1}\int_{x-1}^{-x+1}\int_{0}^{1-x-y} 1\,dz\,dy\,dx = \int_{-1}^{0}\int_{-x-1}^{x+1}(1-x-y)\,dy\,dx +$ $\int_{0}^{1}\int_{x-1}^{-x+1}(1-x-y)\,dy\,dx = \int_{-1}^{0}\left((1-x)\,y-\frac{1}{2}y^2\right)\Big|_{-x-1}^{x+1}dx + \int_{0}^{1}\left((1-x)\,y-\frac{1}{2}y^2\right)\Big|_{x-1}^{-x+1}dx =$ $\int_{-1}^{0}\left(2-2x^2\right)dx + \int_{0}^{1}\left(2-4x+2x^2\right)dx = \left(2x-\frac{2}{3}x^3\right)\Big|_{-1}^{0} + \left(2x-2x^2+\frac{2}{3}x^3\right)\Big|_{0}^{1} = \left(2-\frac{2}{3}\right)+2-2+\frac{2}{3} =$ $2.$

16.4.58 The surfaces intersect when $z_1 = z_2 \Longrightarrow \sin x = \sin y \Longrightarrow x = y$ or $x = \pi - y$. By symmetry, the volume equals 4 times the volume over the region, bounded by $y = x$ and $y = \pi - x$ from $x = 0$ to $x = \frac{\pi}{2}$ under $z = \sin y$. $V = 4\int_{0}^{\pi/2}\int_{x}^{\pi-x}\int_{0}^{\sin y} 1\,dz\,d\,y\,dx = 4\int_{0}^{\pi/2}\int_{x}^{\pi-x}\sin y\,dy\,dx = 4\int_{0}^{\pi/2}(-\cos y)\Big|_{x}^{\pi-x}dx =$ $4\int_{0}^{\pi/2}(\cos x - \cos(\pi - x))\,dx = 4\int_{0}^{\pi/2}(2\cos x)\,dx = 8\,(\sin x)\Big|_{0}^{\pi/2} = 8.$

16.4.59 $V = \int_{0}^{1}\int_{z}^{z+1}\int_{0}^{1} dx\,dy\,dz = \int_{0}^{1}\int_{z}^{z+1} 1\,dy\,dz = \int_{0}^{1} 1\,dz = 1.$

16.4.60 $V = \int_{0}^{1}\int_{z}^{2}\int_{0}^{2} dx\,dy\,dz = \int_{0}^{1}\int_{z}^{2} 2\,dy\,dz = \int_{0}^{1}(4-2z)\,dz = \left(4z - z^2\right)\Big|_{0}^{1} = 3.$

16.4.61 $V = \int_{0}^{2}\int_{0}^{4-2y}\int_{0}^{(4-z)/2} 1\,dx\,dz\,dy = \int_{0}^{2}\int_{0}^{4-2y}\left(\frac{4-z}{2}\right)dz\,dy = \int_{0}^{2}\left(2z-\frac{1}{4}z^2\right)\Big|_{0}^{4-2y}dy =$ $\int_{0}^{2}\left(4-y^2\right)dy = \left(4y-\frac{1}{3}y^3\right)\Big|_{0}^{2} = \frac{16}{3}.$

16.4.62 $V = \iiint\limits_{D} 1\,dV = \int_{0}^{1}\int_{0}^{1-z}\int_{1-y-z}^{\sqrt{(1-z)^2-y^2}} 1\,dx\,dy\,dz = \int_{0}^{1}\int_{0}^{1-z}(x)\Big|_{1-y-z}^{\sqrt{(1-z)^2-y^2}}dy\,dz =$
$\int_{0}^{1}\int_{0}^{1-z}\left(\sqrt{(1-z)^2-y^2}-1+y+z\right)dy\,dz =$
$\int_{0}^{1}\left(\frac{1}{2}y\sqrt{(1-z)^2-y^2}+\frac{1}{2}(1-z)^2\sin^{-1}\left(\frac{y}{2}\right)+\frac{1}{2}y^2-(1-z)\,y\right)\Big|_{0}^{1-z}dz =$
$\int_{0}^{1}\left(\frac{1}{2}(1-z)^2\sin^{-1}\left(\frac{1-z}{2}\right)-\frac{1}{2}(1-z)^2\right)dz.$
After a substitution, we have $\frac{1}{2}\int_{0}^{1}\left(u^2\sin^{-1}\left(\frac{u}{2}\right)-u^2\right)dz$. Integration by parts yields

$$\frac{1}{2}\left(\frac{1}{3}u^3\sin^{-1}\left(\frac{u}{2}\right)+\frac{1}{9}\left(8+u^2\right)\sqrt{4-u^2}-\frac{1}{3}u^3\right)\Big|_{u=0}^{u=1} = \frac{\pi}{12}-\frac{1}{6}.$$

16.4.63 Due to symmetry, this region is made up of eight identical pieces, one in each octant. Consider the piece in the first octant. A particle moving through this region parallel to the positive y-axis would start on the xz-plane $(y = 0)$ within the unit circle $x^2 + z^2 = 1$ and would end on the cylinder that runs parallel to the x-axis $(y^2 + z^2 = 1)$. The total volume is thus $V = 8\int_{0}^{1}\int_{0}^{\sqrt{1-z^2}}\int_{0}^{\sqrt{1-z^2}} 1\,dy\,dx\,dz =$ $8\int_{0}^{1}\int_{0}^{\sqrt{1-z^2}}\sqrt{1-z^2}\,dx\,dz = 8\int_{0}^{1}\left(1-z^2\right)dz = \frac{16}{3}.$

16.4.64 Due to symmetry, this region is made up of eight identical pieces, one in each octant. Consider the piece in the first octant. A particle moving through this region parallel to the positive y-axis would start on the xz-plane $(y = 0)$ within the unit circle $x^2 + z^2 = 1$ and would either end on the cylinder that runs parallel to the x-axis ($y^2 + z^2 = 1$) or the cylinder that runs parallel to the z-axis ($x^2 + y^2 = 1$). If the particle starts at a point on the xz-plane for which $x < z$, then $\sqrt{1-z^2} < \sqrt{1-x^2}$ and the particle ends on the cylinder $y^2 + z^2 = 1$. If the particle starts at a point on the xz-plane for which $z < x$, then $\sqrt{1-x^2} < \sqrt{1-z^2}$ and the particle ends on the cylinder $x^2 + y^2 = 1$. The total volume is thus $V = 8\left(\int_0^{\sqrt{2}/2}\int_x^{\sqrt{1-x^2}}\int_0^{\sqrt{1-z^2}} 1\,dy\,dz\,dx + \int_0^{\sqrt{2}/2}\int_z^{\sqrt{1-z^2}}\int_0^{\sqrt{1-x^2}} 1\,dy\,dx\,dz\right) =$ $8\left(\int_0^{\sqrt{2}/2}\int_x^{\sqrt{1-x^2}} \sqrt{1-z^2}\,dz\,dx + \int_0^{\sqrt{2}/2}\int_z^{\sqrt{1-z^2}} \sqrt{1-x^2}\,dx\,dz\right) =$ $8\left(\left(1 - \frac{\sqrt{2}}{2}\right) + \left(1 - \frac{\sqrt{2}}{2}\right)\right) = 8\left(2 - \sqrt{2}\right)$.

16.4.65 The volume of the cheese is $\int_0^4\int_0^4\int_0^y 1\,dz\,dy\,dx = \int_0^4\int_0^4 y\,dy\,dx = \int_0^4 \left(\frac{1}{2}y^2\right)\Big|_0^4 dx = \int_0^4 8\,dx =$ $(8x)\Big|_0^4 = 32$ $\frac{1}{2}(32) = \int_0^4\int_0^a\int_0^y 1\,dz\,dy\,dx = \int_0^4\int_0^a y\,dy\,dx =$ $\int_0^4 \left(\frac{1}{2}y^2\right)\Big|_0^a dx = \int_0^4 \left(\frac{1}{2}a^2\right) dx = \left(\left(\frac{1}{2}a^2\right)x\right)\Big|_0^4 = 2a^2$. Thus, $16 = 2a^2$, so $a = 2\sqrt{2}$.

16.4.66

a. $V = \iiint\limits_{D_1} 1\,dV = \int_0^1\int_0^z\int_0^y 1\,dx\,dy\,dz = \int_0^1\int_0^z y\,dy\,dz = \int_0^1 \frac{1}{2}z^2\,dz = \left(\frac{1}{6}z^3\right)\Big|_0^1 = \frac{1}{6}$.

b. Let $D_2 = \{(x,\, y,\, z) : 0 \le y \le x \le z \le 1\}$. Then $V = \iiint\limits_{D_2} 1\,dV = \int_0^1\int_0^z\int_0^x 1\,dy\,dx\,dz = \frac{1}{6}$, by swapping x and y from part a. Likewise all other "cousins" have volume $\frac{1}{6}$.

c. The union includes all points with $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le 1$, which is the unit cube.

16.4.67 The equation of a cone with height h and whose base is centered at the origin with radius r in the xy-plane is $z = h - \frac{h}{r}\sqrt{x^2+y^2}$. The volume is $V = \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}\int_0^{h-\frac{h}{r}\sqrt{x^2+y^2}} 1\,dz\,dy\,dx =$ $\int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \left(h - \frac{h}{r}\sqrt{x^2+y^2}\right) dy\,dx$. We switch to polar coordinates, using $x^2+y^2 = a^2$ in order to avoid confusion with the constant r. Then we have $\int_0^{2\pi}\int_0^r \left(h - \frac{h}{r}a\right) a\,da\,d\theta = \int_0^{2\pi}\int_0^r \left(ha - \frac{h}{r}a^2\right) da\,d\theta =$ $\int_0^{2\pi} \left(\frac{1}{2}ha^2 - \frac{h}{3r}a^3\right)\Big|_{a=0}^{a=r} d\theta = \int_0^{2\pi} \left(\frac{1}{2}r^2h - \frac{hr^3}{3r}\right) d\theta = \int_0^{2\pi} \frac{1}{6}r^2h\,d\theta = \frac{1}{3}\pi r^2 h$.

16.4.68 The equation of the plane through $(a,\, 0,\, 0)$, $(0,\, b,\, 0)$ and $(0,\, 0,\, c)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. The volume is $\int_0^a\int_0^{b(1-x/a)}\int_0^{c(1-x/a-y/b)} 1\,dz\,dy\,dx = \int_0^a\int_0^{b(1-x/a)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\,dx =$ $c\int_0^a \left(\left(1 - \frac{x}{a}\right)y - \frac{y^2}{2b}\right)\Big|_0^{b(1-x/a)} dx = \frac{bc}{2}\int_0^a \left(1 - \frac{x}{a}\right)^2 dx = \frac{bc}{2}\left(x - \frac{x^2}{a} + \frac{x^3}{3a^2}\right)\Big|_0^a = \frac{bc}{2}\left(a - a + \frac{a}{3}\right) =$ $\frac{abc}{6}$.

16.4.69 The equation of a sphere with radius R is $x^2 + y^2 + z^2 = R^2$. The volume is given by

$$\int_{-\sqrt{2Rh-h^2}}^{\sqrt{2Rh-h^2}}\int_{-\sqrt{2Rh-h^2-x^2}}^{\sqrt{2Rh-h^2-x^2}}\int_{R-h}^{\sqrt{R^2-x^2-y^2}} 1\,dz\,dy\,dx$$
$$=\int_{-\sqrt{2Rh-h^2}}^{\sqrt{2Rh-h^2}}\int_{-\sqrt{2Rh-h^2-x^2}}^{\sqrt{2Rh-h^2-x^2}}\left(\sqrt{R^2-x^2-y^2}-(R-h)\right)dy\,dx.$$

Converting to polar coordinates gives

$$\int_0^{2\pi}\int_0^{\sqrt{2Rh-h^2}}\left(\sqrt{R^2-r^2}-(R-h)\right)r\,dr\,d\theta=\int_0^{2\pi}\int_0^{\sqrt{2Rh-h^2}}\left(r\sqrt{R^2-r^2}-r\,(R-h)\right)dr\,d\theta$$
$$=\int_0^{2\pi}\left(\left(-\frac{1}{3}\left(R^2-r^2\right)^{3/2}-\frac{1}{2}r^2\,(R-h)\right)\right)\Big|_0^{\sqrt{2Rh-h^2}}d\theta$$
$$=\int_0^{2\pi}\left[-\frac{1}{3}\,(R-h)^3-\frac{1}{2}\left(2Rh-h^2\right)(R-h)+\frac{1}{3}R^3\right]d\theta=\int_0^{2\pi}\left(\frac{h^2}{6}\,(3R-h)\right)d\theta$$
$$=\frac{1}{3}\pi h^2\,(3R-h)\,.$$

16.4.70 There are two volumes to consider: the volume V_1 of the cylinder of radius r inside the frustum, and the volume V_2 that remains when that cylinder is removed. The first volume can be computed without calculus: $V_1=\pi r^2h$. Note that the base of the volume V_2 is the annulus centered at the origin with inner radius r and outer radius R. Polar coordinates may be used (with a as the radius to avoid confusion with r) to show that the equation of the given frustrum is $z=\dfrac{h}{R-r}\,(R-a)$. The volume V_2 is $\displaystyle\int_0^{2\pi}\int_r^R\int_0^{\frac{h}{R-r}(R-a)} a\,dz\,da\,d\theta=\int_0^{2\pi}\int_r^R\frac{ha}{R-r}\,(R-a)\,da\,d\theta=\frac{h}{R-r}\int_0^{2\pi}\int_r^R\left(Ra-a^2\right)da\,d\theta=\frac{h}{R-r}\int_0^{2\pi}\left(\frac{Ra^2}{2}-\frac{a^3}{3}\right)\Big|_{a=r}^{a=R}d\theta=\frac{h}{R-r}\int_0^{2\pi}\left(\frac{R^3}{6}-\frac{R\,r^2}{2}+\frac{r^3}{3}\right)d\theta=\frac{h}{R-r}\cdot 2\pi\cdot\left(\frac{R^3}{6}-\frac{R\,r^2}{2}+\frac{r^3}{3}\right)=\frac{h}{R-r}\cdot\frac{\pi\,R^3-3\pi\,R\,r^2+2\pi\,r^3}{3}$. The total volume V is $V=V_1+V_2=\pi r^2h+\dfrac{h}{R-r}\cdot\dfrac{\pi\,R^3-3\pi\,R\,r^2+2\pi\,r^3}{3}=\dfrac{\pi\,h}{3}\cdot\dfrac{3r^2\,(R-r)+R^3-3R\,r^2+r^3}{R-r}=\dfrac{\pi\,h}{3}\cdot\dfrac{R^3-r^3}{R-r}=\dfrac{\pi\,h}{3}\left(R^2+R\,r+r^2\right).$

16.4.71 The equation of the given ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Using symmetry, the volume V of the ellipsoid is eight times the volume of the portion of the ellipsoid in the first octant. Thus $V=8\displaystyle\int_0^a\int_0^{b\sqrt{1-x^2/a^2}}\int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}}1\,dz\,dy\,dx=8c\int_0^a\int_0^{b\sqrt{1-x^2/a^2}}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx=8c\int_0^a\left(\frac{y}{2}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}+\frac{b\left(a^2-x^2\right)}{2a^2}\sin^{-1}\left(\frac{ay}{b\sqrt{a^2-x^2}}\right)\right)\Big|_0^{b\sqrt{1-x^2/a^2}}dx=8c\int_0^a\frac{b\pi\left(a^2-x^2\right)}{4a^2}\,dx=\frac{2\pi bc}{a^2}\left(a^2x-\frac{x^3}{3}\right)\Big|_0^a=\frac{2\pi bc}{a^2}\cdot\frac{2a^3}{3}=\frac{4}{3}\pi abc$

16.4.72

a. $p\,(0.75)=\displaystyle\int_0^{0.75}0.8e^{-0.8t}\,dt=\left(-e^{-0.8t}\right)\Big|_{t=0}^{t=0.75}=0.4512.$

b. $p\,(0.75)=\displaystyle\int_0^{0.75}\int_0^{0.75}\left(0.8e^{-0.8t}\right)\left(0.1e^{-0.1s}\right)dt\,ds=\int_0^{0.75}\left(0.1e^{-0.1s}\right)\left(-e^{-0.8t}\right)\Big|_{t=0}^{t=0.75}ds=\int_0^{0.75}\left(0.1e^{-0.1s}\right)(0.4512)\,ds=(0.4512)\left(-e^{-0.1s}\right)\Big|_{s=0}^{s=0.75}=(0.4512)\,(0.0723)=0.0326.$

c.

$$\begin{aligned}
p\left(0.75\right) &= \int_0^{0.75}\int_0^{0.75}\int_0^{0.75}\left(0.8e^{-0.8t}\right)\left(0.1e^{-0.1s}\right)\left(0.05e^{-0.05u}\right)\,dt\,ds\,du\\
&= \int_0^{0.75}\int_0^{0.75}\left(0.1e^{-0.1s}\right)\left(0.05e^{-0.05u}\right)\left(-e^{-0.8t}\right)\Bigg|_{t=0}^{t=0.75}\,ds\,du\\
&= \left(0.4512\right)\int_0^{0.75}\int_0^{0.75}\left(0.1e^{-0.1s}\right)\left(0.05e^{-0.05u}\right)\,ds\,du = \left(0.4512\right)\int_0^{0.75}\left(0.05e^{-0.05u}\right)\left(-e^{-0.1s}\right)\Bigg|_{s=0}^{s=0.75}\,du\\
&= \left(0.4512\right)\left(0.0723\right)\int_0^{0.75}\left(0.05e^{-0.05u}\right)\,du\\
&= \left(0.4512\right)\left(0.0723\right)\left(-e^{-0.05u}\right)\Bigg|_{u=0}^{u=0.75} = \left(0.4512\right)\left(0.0723\right)\left(0.0368\right) = 0.0012.
\end{aligned}$$

16.4.73 The Hypervolume is given by $\int_0^1\int_0^{1-x}\int_0^{1-x-y}\int_0^{1-x-y-z} 1\,dw\,dz\,dy\,dx$. This can be computed as $\int_0^1\int_0^{1-x}\int_0^{1-x-y}\left(1-x-y-z\right)\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left(\left(1-x-y\right)z-\frac{1}{2}z^2\right)\Big|_0^{-x-y}\,dy\,dx =$ $\int_0^1\int_0^{1-x}\left(\left(1-x-y\right)^2-\frac{1}{2}\left(1-x-y\right)^2\right)dy\,dx = \frac{1}{2}\int_0^1\int_0^{1-x}\left(1-x-y\right)^2\,dy\,dx =$ $\frac{1}{2}\int_0^1[-\frac{1}{3}\left(1-x-y\right)^3]\Big|_0^{1-x}\,dx = \frac{1}{6}\int_0^1\left(1-x\right)^3\,dx = \frac{1}{6}\left(-\frac{1}{4}\left(1-x\right)^4\right)\Big|_0^1 = \frac{1}{24}$.

16.4.74 The region of integration for the integral $\int_0^1\int_x^1\int_x^y f\left(x\right)f\left(y\right)f\left(z\right)\,dz\,dy\,dx$ is the set of points $(x,\,y,\,z)$ such that $0 < x < z < y < 1$. This region is one sixth of the unit cube. By rearranging x, y, and z, the other five sixths of the unit cube may be generated. Thus the integral of $f\left(x\right)f\left(y\right)f\left(z\right)$ over the unit cube is the sum of the integrals over these six regions. Now notice that the integrand $f\left(x\right)f\left(y\right)f\left(z\right)$ does not change as x, y, and z are rearranged, so the integral over each of the six regions is the same. Thus $\int_0^1\int_0^1\int_0^1 f\left(x\right)f\left(y\right)f\left(z\right)\,dz\,dy\,dx = 6\int_0^1\int_x^1\int_x^y f\left(x\right)f\left(y\right)f\left(z\right)\,dz\,dy\,dx$. The integral of $f\left(x\right)f\left(y\right)f\left(z\right)$ over the unit cube can be calculated as follows: $\int_0^1\int_0^1\int_0^1 f\left(x\right)f\left(y\right)f\left(z\right)\,dz\,dy\,dx =$ $\int_0^1\int_0^1 f\left(x\right)f\left(y\right)\left(\int_0^1 f\left(z\right)\,dz\right)dy\,dx = \left(\int_0^1 f\left(z\right)\,dz\right)\int_0^1\int_0^1 f\left(x\right)f\left(y\right)\,dy\,dx =$ $\left(\int_0^1 f\left(z\right)\,dz\right)\int_0^1 f\left(x\right)\left(\int_0^1 f\left(y\right)\,dy\right)dx = \left(\int_0^1 f\left(z\right)\,dz\right)\left(\int_0^1 f\left(y\right)\,dy\right)\left(\int_0^1 f\left(x\right)\,dx\right)$.

Because the name of the variable is immaterial, we have $\int_0^1\int_0^1\int_0^1 f\left(x\right)f\left(y\right)f\left(z\right)\,dz\,dy\,dx =$ $\left(\int_0^1 f\left(z\right)\,dz\right)\left(\int_0^1 f\left(y\right)\,dy\right)\left(\int_0^1 f\left(x\right)\,dx\right) = \left(\int_0^1 f\left(x\right)\,dx\right)^3$.

Thus, $\int_0^1\int_x^1\int_x^y f\left(x\right)f\left(y\right)f\left(z\right)\,dz\,dy\,dx = \frac{1}{6}\int_0^1\int_0^1\int_0^1 f\left(x\right)f\left(y\right)f\left(z\right)\,dz\,dy\,dx = \frac{1}{6}\left(\int_0^1 f\left(x\right)\,dx\right)^3$.

16.5 Triple Integrals in Cylindrical and Spherical Coordinates

16.5.1 r measures the distance from the point to the z axis, θ is the angle that the segment from the point to the z-axis makes with the positive xz-plane, and z is the directed distance from the point to the xy-plane.

16.5.2 The triple $(\rho,\,\varphi,\,\theta)$ describes a point whose distance from the origin is ρ, such that the line from the origin to the point makes an angle of φ with the z-axis, and such that the projection of this line to the xy-plane makes an angle of θ with the positive x-axis.

16.5.3 A double cone (opening both upwards and downwards), where the radius is always 4 times the distance to the xy-plane.

16.5.4 A cone opening upwards making an angle of $\frac{\pi}{4}$ radian with the z-axis.

16.5.5 It approximates the volume of the cylindrical wedge formed by the changes Δr, $\Delta\theta$, and Δz.

16.5.6 It approximates the volume formed by the changes $\Delta\rho$, $\Delta\varphi$, and $\Delta\theta$.

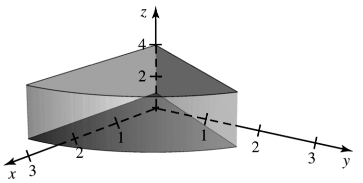
16.5.7 $\displaystyle\iiint\limits_D w(r,\,\theta,\,z)\,dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r,\,\theta)}^{H(r,\,\theta)} w(r,\,\theta,\,z)\,dz\,r\,dr\,d\theta.$

16.5.8 $\displaystyle\iiint\limits_D w(\rho,\,\varphi,\,\theta)\,dV = \int_\alpha^\beta \int_a^b \int_{g(\varphi,\,\theta)}^{h(\varphi,\,\theta)} w(\rho,\,\varphi,\,\theta)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$

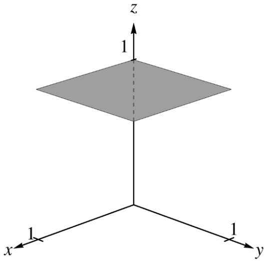
16.5.9 Cylindrical coordinates, because in cylindrical coordinates x^2+y^2 simplifies to r^2.

16.5.10 Spherical coordinates, because in spherical coordinates $x^2+y^2+z^2$ simplifies to ρ^2.

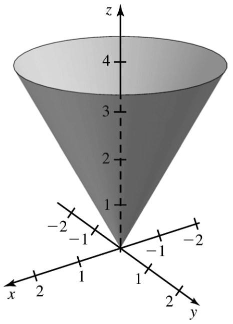
16.5.11 This is a wedge of a cylinder of radius 3 from $z=1$ to $z=4$, where the wedge angle is $\frac{\pi}{3}$.



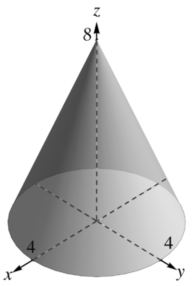
16.5.12 This is the first quadrant of the plane $z=1$ for $x,\,y\ge 0$.



16.5.13 This is the solid upward pointing cone given by $z=2r$ from $z=0$ to $z=4$.



16.5.14 This is a solid downward-pointing cone with vertex at $(0,\,0,\,8)$ between $z=8$ and $z=0$. Its intersection with the xy-plane is the circle $x^2+y^2=16$.



16.5.15 $\int_0^{2\pi}\int_0^1\int_{-1}^1 dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r\, z\Big|_{-1}^1 dr\, d\theta = \int_0^{2\pi}\int_0^1 2r\, dr\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi.$

16.5.16 This is the integral over half of the cone whose equation is $z = 9 - 3r$, $0 \le r \le 3$. Thus $\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_0^{9-3\sqrt{x^2+y^2}} dz\, dx\, dy = \int_0^{\pi}\int_0^3\int_0^{9-3r} r\, dz\, dr\, d\theta = \int_0^{\pi}\int_0^3 r\, z\Big|_0^{9-3r} dr\, d\theta =$ $\int_0^{\pi}\int_0^3 \left(9r - 3r^2\right) dr\, d\theta = \int_0^{\pi}\left(\frac{9}{2}r^2 - r^3\right)\Big|_0^3 d\theta = \int_0^{\pi}\left(\frac{81}{2} - 27\right) d\theta = \frac{27}{2}\pi.$

16.5.17 This is the integral of $\left(x^2+y^2\right)^{3/2}$ over a cylinder with radius 1 from $z = -1$ to $z = 1$. We convert to cylindrical coordinates, so the integrand becomes r^3 Because $x^2 + y^2 = r^2$. The integral is thus $\int_0^{2\pi}\int_0^1\int_{-1}^1 r^3\, dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 2\, r^4\, dr\, d\theta = \frac{2}{5}\int_0^{2\pi} r^5\Big|_0^1 = \frac{4\pi}{5}.$

16.5.18 Converting to cylindrical coordinates, the integrand becomes $\frac{1}{1+r^2}$; the region of integration is the half-cylinder shown, so the integral is

$$\int_0^{\pi}\int_0^3\int_0^2 \frac{1}{1+r^2}\, dz\, r\, dr\, d\theta = \int_0^{\pi}\int_0^3 \frac{2r}{1+r^2}\, dr\, d\theta = \int_0^{\pi} \ln\left(1+r^2\right)\Big|_0^3 = \int_0^{\pi} \ln(10)\, d\theta = \pi\, \ln(10)\,.$$

16.5.19 The region of integration is a wedge from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$ with radius 1, between $z = 0$ and $z = 4$. This can be seen by noting that the limits of integration for y range from x to the boundary of the unit circle (y-coordinate $\sqrt{1-x^2}$) and that $x = \frac{\sqrt{2}}{2}$corresponds to $\frac{\pi}{4}$. Thus the integral is $\int_{\pi/4}^{\pi/2}\int_0^1\int_0^4 e^{-r^2}\, dz\, r\, dr\, d\theta =$ $\int_{\pi/4}^{\pi/2}\int_0^1 4r\, e^{-r^2}\, dr\, d\theta = -2\int_0^{\pi} e^{-r^2}\Big|_0^1 = -2\int_0^{\pi}\left(e^{-1} - 1\right) d\theta = \frac{\pi}{2}\left(1 - e^{-1}\right).$

16.5.20 The region of integration is the volume between the cone $x^2 + y^2 = z^2$ and the plane $z = 4$. To see this, note that x and y vary over the disk $x^2 + y^2 = 16$ and that z varies from the boundary of the cone to 4. Converting to cylindrical coordinates, we then have $\int_0^{2\pi}\int_0^4\int_r^4 1\, dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^4 r\,(4-r)\, dr\, d\theta =$ $\int_0^{2\pi}\left(2r^2 - \frac{1}{3}r^3\right)\Big|_0^4 = \int_0^{2\pi} \frac{32}{3}\, d\theta = \frac{64\pi}{3}.$

16.5.21 The region of integration is below the cone $x^2 + y^2 = z^2$ in the first octant, so the integral is $\int_0^{\pi/2}\int_0^3\int_0^r \frac{1}{r}\, dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_0^3 r\, dr\, d\theta = \int_0^{\pi/2} \frac{9}{2}\, d\theta = \frac{9\pi}{4}\,.$

16.5.22 The region of integration is a wedge from $\theta = 0$ to $\theta = \frac{\pi}{6}$ with radius 1, between $z = -1$ and $z = 1$. The integral is then $\int_0^{\pi/6}\int_0^1\int_{-1}^1 r\, dz\, r\, dr\, d\theta = \int_0^{\pi/6}\int_0^1\int_{-1}^1 r^2\, dz\, dr\, d\theta = \int_0^{\pi/6} \frac{2}{3}\, d\theta = \frac{\pi}{9}\,.$

16.5.23 $\int_0^{2\pi}\int_0^4\int_0^{10}\left(1+\frac{z}{2}\right)\,dz\,r\,dr\,d\theta=\int_0^{2\pi}\int_0^4\left(z+\frac{z^2}{4}\right)\bigg|_0^{10}r\,dr\,d\theta=\int_0^{2\pi}\int_0^4 35r\,dr\,d\theta=560\pi.$

16.5.24 $\int_0^{2\pi}\int_0^3\int_0^2 5e^{-r^2}\,dz\,r\,dr\,d\theta=-5\int_0^{2\pi}\int_0^3\left(-2re^{-r^2}\right)dr\,d\theta=10\pi\left(1-e^{-9}\right).$

16.5.25 $\int_0^{2\pi}\int_0^6\int_0^{6-r}(7-z)\,dz\,r\,dr\,d\theta=\int_0^{2\pi}\int_0^6\left(7z-\frac{1}{2}z^2\right)\bigg|_0^{6-r}r\,dr\,d\theta=$ $\int_0^{2\pi}\int_0^4\left(7r\,(6-r)-\frac{r}{2}\,(6-r)^2\right)dr\,d\theta=396\pi.$

16.5.26 $\int_0^{2\pi}\int_0^3\int_0^{9-r^2}\left(1+\frac{z}{9}\right)dz\,r\,dr\,d\theta=\int_0^{2\pi}\int_0^3\left(z+\frac{z^2}{18}\right)\bigg|_0^{9-r^2}r\,dr\,d\theta=$ $\int_0^{2\pi}\int_0^3\left(r\left(9-r^2\right)-\frac{r}{18}\left(9-r^2\right)^2\right)dr\,d\theta=54\pi.$

16.5.27 The base of both surfaces in the xy-plane is the area bounded by the unit circle $r=1$. The mass of the solid bounded by the xy-plane and $z=4-4r$ is given by $\int_0^{2\pi}\int_0^1\int_0^{4-4r}(10-2z)\,dz\,r\,dr\,d\theta=$ $\int_0^{2\pi}\int_0^1\left(10z-z^2\right)\bigg|_0^{4-4r}r\,dr\,d\theta=\int_0^{2\pi}\int_0^1\left(10r\,(4-4r)-r\,(4-4r)^2\right)dr\,d\theta=\frac{32\pi}{3}$. The mass of the solid bounded by the xy-plane and $z=4-4r^2$ is given by $\int_0^{2\pi}\int_0^1\int_0^{4-4r^2}(10-2z)\,dz\,r\,d\,rd\theta=$ $\int_0^{2\pi}\int_0^1\left(10z-z^2\right)\bigg|_0^{4-4r^2}r\,dr\,d\theta=\int_0^{2\pi}\int_0^1\left(10r\left(4-4r^2\right)-r\left(4-4r^2\right)^2\right)dr\,d\theta=\frac{44\pi}{3}$. Thus the mass of the solid bounded by the paraboloid is larger. This can also be seen by noting that for $r\le 1$, $r^2\le r$ so that $4-4r^2\ge 4-4r$, so that the cone is contained in the paraboloid.

16.5.28 By the argument given above, again the solid bounded by the paraboloid will have greater mass. For the solid bounded by the cone, the mass is

$$\int_0^{2\pi}\int_0^1\int_0^{4-4r}\left(\frac{8}{\pi}e^{-z}\right)dz\,r\,dr\,d\theta=\frac{8}{\pi}\int_0^{2\pi}\int_0^1 -r\left(e^{4r-4}-1\right)dr\,d\theta=5-e^{-4}.$$

The mass of the solid bounded by the xy-plane and $z=4-4r^2$ is $\int_0^{2\pi}\int_0^1\int_0^{4-4r^2}\left(\frac{8}{\pi}e^{-z}\right)dz\,r\,dr\,d\theta=$ $\frac{8}{\pi}\int_0^{2\pi}\int_0^1 -r\left(e^{4r^2-4}-1\right)dr\,d\theta=6+2e^{-4}.$

16.5.29 The base of this solid in the xy-plane is given by $3=\sqrt{1+x^2+y^2}$ so is the area bounded by the circle $x^2+y^2=8$ or $r=2\sqrt{2}$. Thus $V=\int_0^{2\pi}\int_0^{2\sqrt{2}}\int_0^{3-\sqrt{1+r^2}}1\,dz\,r\,dr\,d\theta=$ $\int_0^{2\pi}\int_0^{2\sqrt{2}}\left(3r-r\sqrt{1+r^2}\right)dr\,d\theta=2\pi\left(\frac{3r^2}{2}-\frac{(1+r^2)^{3/2}}{3}\right)\bigg|_0^{2\sqrt{2}}=2\pi\left(12-9-\left(0-\frac{1}{3}\right)\right)=\frac{20\pi}{3}.$

16.5.30 This solid sits over the circle $r=5$ in the xy-plane, so the volume is $\int_0^{2\pi}\int_0^5\int_{r^2}^{25}1\,dz\,r\,dr\,d\theta=$ $\int_0^{2\pi}\int_0^5\left(25r-r^3\right)dr\,d\theta=\frac{625\pi}{2}.$

16.5.31 This solid sits over the circle $r=5$ in the xy-plane, so the volume is $\int_0^{2\pi}\int_0^5\int_{\sqrt{4+r^2}}^{\sqrt{29}}1\,dz\,r\,d\,rd\theta=$ $\int_0^{2\pi}\int_0^5\left(r\sqrt{29}-r\sqrt{4+r^2}\right)dr\,d\theta=\frac{\pi\left(17\sqrt{29}+16\right)}{3}.$

16.5.32 The base is swept out completely as θ ranges from 0 to π, so the volume is

$$\int_0^\pi \int_0^{2\cos\theta} \int_0^4 1\,dz\,r\,dr\,d\theta = \int_0^{2\pi} \int_0^{2\cos\theta} 4r\,dr\,d\theta = 4\pi,$$

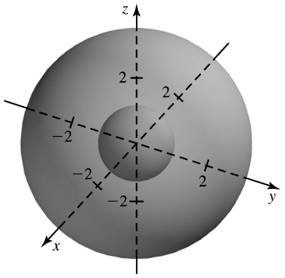
which is in fact the volume of a right circular cylinder of height 4 whose base is a unit circle.

16.5.33 The first octant is determined by $0 \le \theta \le \frac{\pi}{2}$, $0 \le z$, and the condition $z = x$ in cylindrical coordinates becomes $z = r\cos\theta$, so the volume of the solid is

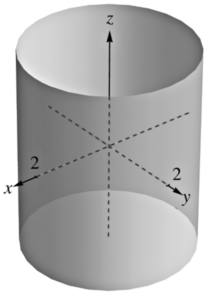
$$\int_0^{\pi/2} \int_0^1 \int_0^{r\cos\theta} 1\,dz\,r\,dr\,d\theta = \int_0^{\pi/2} \int_0^1 r^2\cos\theta\,dr\,d\theta = \frac{1}{3}.$$

16.5.34 The volume is $\displaystyle\int_0^{2\pi} \int_1^2 \int_0^{4-r(\cos\theta+\sin\theta)} 1\,dz\,r\,dr\,d\theta = \int_0^{2\pi} \int_1^2 \left(4r - r^2(\cos\theta+\sin\theta)\right) dr\,d\theta =$ $\displaystyle\int_0^{2\pi} \left(2r^2 - \frac{1}{3}r^3(\cos\theta+\sin\theta)\right)\Big|_1^2 d\theta = \int_0^{2\pi} \left(6 - \frac{7}{3}(\cos\theta+\sin\theta)\right) d\theta = 12\pi$

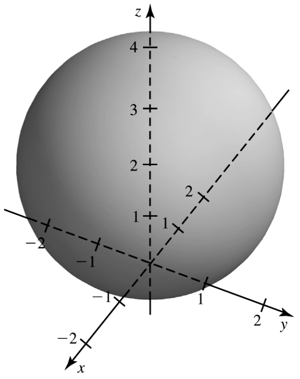
16.5.35 This is a spherical shell centered at the origin with outer radius 3 and inner radius 1.



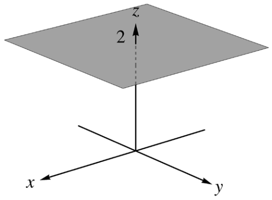
16.5.36 Because $\rho = 2\csc\varphi$, $\rho\sin\varphi = 2$. Now $x^2+y^2 = \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta = \rho^2\sin^2\varphi = 4$. Thus any point on the surface satisfies $x^2+y^2 = 4$, so that the surface is a cylinder of radius 2 oriented along the z-axis.



16.5.37 This is a sphere of radius 2 centered at (0, 0, 2). To see this, note that $\rho = 4\cos\varphi$ implies that $\rho^2 = 4\rho\cos\varphi$. Converting to rectangular coordinates gives $x^2+y^2+z^2 = 4z$, and completing the square yields $x^2+y^2+(z-2)^2 = 4$.



16.5.38 This is the plane $z = 2$, Because $\rho = 2\sec\varphi$ implies that $z = \rho\cos\varphi = 2$.



16.5.39

a. For Seattle, we have $\rho = 3960$, $\varphi = \dfrac{(90-47.6)\pi}{180} \approx 0.74$, and $\theta = \dfrac{-122.3\pi}{180} \approx -2.13$. So the coordinates are approximately $(3960, 0.74, -2.13)$. Then note that

$$u = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) = (-1426.85, -2257.05, 2924.28).$$

b. For Rome, we have $\rho = 3960$, $\varphi = \dfrac{(90-41.9)\pi}{180} \approx 0.84$, and $\theta = \dfrac{12.5\pi}{180} \approx 0.22$. So the coordinates are approximately $(3960, 0.84, 0.22)$. Then note that

$$u = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) = (2877.61, 637.95, 2644.62).$$

c. The distance is given by $\rho\cos^{-1}\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) \approx 5666$ miles.

16.5.40 For Los Angeles, we have $\rho = 3960$, $\varphi = \dfrac{(90-34.05)\pi}{180} \approx 0.98$, and $\theta = \dfrac{-118.24\pi}{180} \approx -2.06$. Then

$$u = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) = (-1552.48, -2890.52, 2217.27).$$

For New York City, we have $\rho = 3960$, $\varphi = \dfrac{(90-40.71)\pi}{180} \approx 0.86$, and $\theta = \dfrac{-74.01\pi}{180} \approx -1.29$. Then

$$v = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) = (826.89, -2885.62, 2582.83).$$

The distance between them is given by $\rho\cos^{-1}\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = 2446$ miles.

16.5.41 $\displaystyle\iiint\limits_D \left(x^2+y^2+z^2\right)^{5/2} dV = \int_0^1\int_0^{2\pi}\int_0^{\pi} \rho^5\,\rho^2\sin\varphi\;d\varphi\,d\theta\,d\rho = \int_0^1\int_0^{2\pi}\int_0^{\pi} \rho^7\sin\varphi\;d\varphi\,d\theta\,d\rho =$ $\displaystyle\int_0^1\int_0^{2\pi} 2\rho^7\,d\theta\,d\rho = 4\pi\int_0^1 \rho^7\,d\rho = \frac{\pi}{2}.$

16.5.42 $\iiint\limits_D e^{-\left(x^2+y^2+z^2\right)^{3/2}} dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{-\rho^3}\,\rho^2\sin\varphi\; d\rho\, d\varphi\, d\theta = -\dfrac{1}{3}\int_0^{2\pi}\int_0^{\pi} e^{-\rho^3}\Big|_0^1 \sin\varphi\; d\varphi\, d\theta =$
$-\dfrac{\left(e^{-1}-1\right)}{3}\int_0^{2\pi}\int_0^{\pi}\sin\varphi\, d\varphi\, d\theta = -\dfrac{4\pi}{3}\left(e^{-1}-1\right).$

16.5.43

$$\begin{aligned}\iiint\limits_D \left(x^2+y^2+z^2\right)^{-3/2} dV &= \int_0^{2\pi}\int_0^{\pi}\int_1^2 \rho^{-3}\,\rho^2\sin\varphi\; d\rho\, d\varphi\, d\theta\\ &= \int_0^{2\pi}\int_0^{\pi}\int_1^2 \rho^{-1}\sin\varphi\, d\rho\, d\varphi\, d\theta = \ln 2\int_0^{2\pi}\int_0^{\pi}\sin\varphi\, d\varphi\, d\theta = 4\pi\ln 2.\end{aligned}$$

16.5.44 $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\,\sec\varphi}\rho^2\sin\varphi\; d\rho\, d\varphi\, d\theta = \frac{64}{3}\int_0^{2\pi}\int_0^{\pi/3}\sec^3\varphi\sin\varphi\, d\varphi\, d\theta = 64\pi.$

16.5.45 $\displaystyle\int_0^{\pi}\int_0^{\pi/6}\int_{2\,\sec\varphi}^{4}\rho^2\sin\varphi\; d\rho\, d\varphi\, d\theta = \frac{1}{3}\int_0^{\pi}\int_0^{\pi/6}\left(64-8\sec^3\varphi\right)\sin\varphi\; d\varphi\, d\theta = \left(\frac{188}{9}-\frac{32}{3}\sqrt{3}\right)\pi.$

16.5.46

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi/4}\int_1^{2\,\sec\varphi}\left(\rho^{-3}\right)\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta &= \int_0^{2\pi}\int_0^{\pi/4}\sin\varphi\ln\left(2\,\sec\varphi\right)\; d\varphi\, d\theta\\ &= 2\pi\left(\ln\,2\left(1-\frac{3\sqrt{2}}{4}\right)+1-\frac{\sqrt{2}}{2}\right).\end{aligned}$$

16.5.47 $\displaystyle\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{2\csc\varphi}\rho^2\sin\varphi\; d\rho\, d\varphi\, d\theta = \frac{8}{3}\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\csc^3\varphi\sin\varphi\; d\varphi\, d\theta = \frac{32}{9}\pi\sqrt{3}.$

16.5.48 A ball of radius a around the origin has volume given by

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\rho^2\sin\varphi\; d\rho\, d\varphi\, d\theta = \frac{a^3}{3}\int_0^{2\pi}\int_0^{\pi/4}\sin\varphi\; d\varphi\, d\theta = \frac{4}{3}a^3.$$

16.5.49 The two spheres intersect where $2\cos\varphi = 1$, i.e. when $\varphi = \frac{\pi}{3}$. That circle is then at $z = \frac{1}{2}$ (again looking at $\cos\varphi = \frac{1}{2}$ and using the fact that the lower sphere has radius 1). The volume in question is the sum of the volumes above and below that circle of intersection; because the circle is at height $\frac{1}{2}$ and both spheres have radius 1, the volumes above and below are identical. Thus we need only compute the upper volume and double it:

$$2\int_0^{\pi/3}\int_{(\sec\varphi)/2}^{1}\int_0^{2\pi}\rho^2\sin\varphi\, d\theta\, d\rho\, d\varphi = \frac{4\pi}{3}\int_0^{\pi/3}\left(1-\frac{\sec^3\varphi}{8}\right)\sin\varphi\, d\theta = \frac{5}{12}\pi.$$

16.5.50

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\int_0^{1+\cos\varphi}\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta &= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\left(1+\cos\varphi\right)^3\sin\varphi\, d\varphi\, d\theta\\ &= -\frac{1}{12}\int_0^{2\pi}\left(1+\cos\varphi\right)^4\Big|_{\varphi=0}^{\varphi=\pi} d\theta = \frac{4}{3}\int_0^{2\pi} d\theta = \frac{8}{3}\pi.\end{aligned}$$

16.5.51 The portion of the sphere is determined by $\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}$, so the integral is

$$\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{4\;\cos\varphi}\rho^2\sin\varphi\; d\rho\, d\varphi\, d\theta = \frac{64}{3}\int_0^{\pi/3}\int_{\pi/4}^{\pi/2}\cos^3\varphi\;\sin\,\varphi d\varphi\, d\theta = \frac{8}{3}\pi.$$

16.5.52 $\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_{\csc\varphi}^{2\,\csc\varphi} \rho^2\sin\varphi\ d\rho\,d\varphi\,d\theta = \frac{7}{3}\int_0^{2\pi}\int_{\pi/6}^{\pi/3} \csc^2\ \varphi\,d\varphi\,d\theta = -\frac{7}{3}\int_0^{2\pi} \cot\varphi\Big|_{\varphi=\pi/6}^{\varphi=\pi/3} d\theta =$
$-\frac{7}{3}\int_0^{2\pi}\left(\frac{1}{\sqrt{3}}-\sqrt{3}\right)d\theta = \frac{28}{9}\pi\sqrt{3}.$

16.5.53 The value of φ corresponding to the circle of intersection of the sphere ρ and the plane $z = 2\sqrt{3}$ is found by noting that $\cos\varphi = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$, so that $\varphi = \frac{\pi}{6}$. Similarly, the intersection with the plane $z = 2$ is where $\cos\varphi = \frac{2}{4} = \frac{1}{2}$ so that $\varphi = \frac{\pi}{3}$. We can find the volume of the required region by determining the volume of the regions above $z = 2$ and $z = 2\sqrt{3}$ and subtracting. The volume of the region above the plane $z = 2$ is

$$\int_0^{2\pi}\int_0^{\pi/3}\int_{2\sec\varphi}^{4} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3}\left(64-8\sec^3\varphi\right)\sin\ \varphi\,d\varphi\,d\theta = \frac{40}{3}\pi,$$

and the volume of the region above $z = 2\sqrt{3}$ is

$$\int_0^{2\pi}\int_0^{\pi/6}\int_{2\sqrt{3}\ \sec\varphi}^{4} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3}\left(64-24\sqrt{3}\sec^3\varphi\right)\sin\ \varphi\,d\varphi\,d\theta = \frac{128}{3}\pi - 24\sqrt{3}\pi,$$

so that the volume of the region between the two planes is $\left(24\sqrt{3}-\frac{88}{3}\right)\pi$.

16.5.54 This cone makes an angle of $\frac{\pi}{4}$ with the z-axis. We thus want

$$\int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\varphi}^{2\ \sec\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{7}{3}\int_0^{2\pi}\int_0^{\pi/3} \sec^3\varphi\ \sin\ \varphi\,d\varphi\,d\theta = \frac{7}{3}\pi.$$

A simpler method of arriving at this result is to recall that the volume of a cone is $\frac{1}{3}Ah$, where A is the area of the base and h is the height, so in this case the volume of the larger cone is $\frac{1}{3}\cdot 4\pi\cdot 2$ and of the smaller $\frac{1}{3}\cdot\pi\cdot 1$, so the difference is as above.

16.5.55

a. True. In either set of coordinates, any value of θ may be chosen.

b. True. Note that $r = z$ if and only if $\rho\sin\varphi = \rho\cos\varphi$, which happens if and only if $\varphi = \frac{\pi}{4}$.

16.5.56 $\rho^2 = \sec(2\varphi)$ implies that $1 = \rho^2\cos(2\varphi) = \rho^2\left(\cos^2\varphi - \sin^2\varphi\right) = z^2 - x^2 - y^2$ which is a hyperboloid of two sheets.

16.5.57 $\rho^2 = -\sec(2\varphi)$ implies that $-1 = \rho^2\cos(2\varphi) = \rho^2\left(\cos^2\varphi - \sin^2\varphi\right) = z^2 - x^2 - y^2$ so that $x^2 + y^2 - z^2 = 1$, which is a hyperboloid of one sheet. Because $\frac{\pi}{4} < \varphi < \frac{3\pi}{4}$, we only have the upper sheet.

16.5.58 $\int_0^{2\pi}\int_0^{\pi}\int_0^{4}(1+\rho)\ \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{3}\rho^3+\frac{1}{4}\rho^4\right)\Big|_0^4 \sin\varphi\,d\varphi\,d\theta =$
$\int_0^{2\pi}\int_0^{\pi}\frac{256}{3}\sin\varphi\,d\varphi\,d\theta$
$= \frac{1024}{3}\pi.$

16.5.59 $\int_0^{2\pi}\int_0^{\pi}\int_0^{8} 2e^{-\rho^3}\ \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = -\frac{2}{3}\int_0^{2\pi}\int_0^{\pi}\ e^{-\rho^3}\Big|_0^8 \sin\ \varphi\,d\varphi\,d\theta =$
$-\frac{2}{3}\left(e^{-512}-1\right)\int_0^{2\pi}\int_0^{\pi/3}\sin\varphi\,d\varphi\,d\theta = \frac{8}{3}\left(1-e^{-512}\right)\pi.$

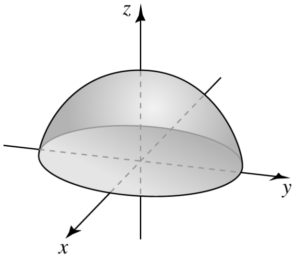
16.5.60 The mass is

$$\begin{aligned}\int_0^{2\pi}\int_0^{4\sqrt{3}}\int_{r/\sqrt{3}}^{4}(5-z)r\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^{4\sqrt{3}}(5z-z^2/2)r\Big|_{r/\sqrt{3}}^{4}\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^{4\sqrt{3}}((20-8)-(5r/\sqrt{3}-r^2/6))r\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^{4\sqrt{3}}(12r-5r^2/\sqrt{3}+r^3/6)\,dr\,d\theta\\ &= \int_0^{2\pi}(6r^2-5r^3/3\sqrt{3}+r^4/24)\Big|_0^{4\sqrt{3}}\,d\theta\\ &= 2\pi(288-320+96)=128\pi.\end{aligned}$$

16.5.61 Note that because of the absolute value sign, the mass is symmetric around the xy-plane, so we can compute the mass for $0 \le z \le 1$ and double it. Using cylindrical coordinates, the mass is then $2\int_0^1\int_0^{2\pi}\int_0^2 (2-z)\,(4-r)\,r\,dr\,d\theta\,dz = 2\int_0^1\int_0^{2\pi}\int_0^2 (2-z)\left(4r-r^2\right)\,dr\,d\theta\,dz =$ $2\int_0^1\int_0^{2\pi}(2-z)\left(2r^2-\frac{1}{3}r^3\right)\Big|_0^2 d\theta\,dz = \frac{64\pi}{3}\int_0^1 (2-z)\,dz = 32\pi.$

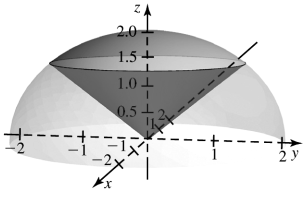
16.5.62 The cylinder $r=1$ intersects the sphere $\rho=5$ along the circle $x^2+y^2=1$, $z=2\sqrt{6}$. Note also that for any point (r,θ,z) on the sphere $r^2+z^2=x^2+y^2+z^2=25$. We have

$$\begin{aligned}\int_0^{2\pi}\int_1^5\int_0^{\sqrt{25-r^2}} f\,(r,\theta,z)\,r\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^{2\sqrt{6}}\int_1^{\sqrt{25-z^2}} f\,(r,\,\theta,z)\,r\,dr\,dz\,d\theta\\ &= \int_1^5\int_0^{\sqrt{25-r^2}}\int_0^{2\pi} f\,(r,\,\theta,\,z)\,r\,d\theta\,dz\,dr.\end{aligned}$$

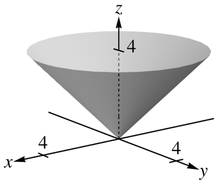


16.5.63

$$\begin{aligned}&\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} f\,(r,\theta,z)\,r\,dz\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^{\sqrt{2}}\int_0^{z} f\,(r,\theta,z)\,r\,dr\,dz\,d\theta + \int_0^{2\pi}\int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-z^2}} f\,(r,\,\theta,\,z)\,r\,dr\,dz\,d\theta\\ &= \int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}}\int_0^{2\pi} f\,(r,\,\theta,\,z)\,r\,d\theta\,dz\,dr.\end{aligned}$$

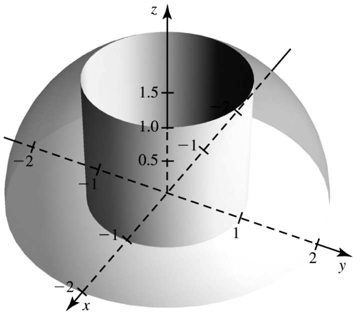


16.5.64 The region of integration is a cone with vertex at the origin making an angle of $\frac{\pi}{4}$ with the positive z-axis, from $z = 0$ to $z = 4$.



We have $\displaystyle\int_0^{\pi/4}\int_0^{2\pi}\int_0^{4\sec\varphi} g\left(\rho,\,\varphi,\,\theta\right)\,\rho^2\sin\,\varphi\,d\rho\,d\theta\,d\varphi = \int_0^{\pi/4}\int_0^{4\sec\varphi}\int_0^{2\pi} g\left(\rho,\,\varphi,\,\theta\right)\,\rho^2\sin\,\varphi\,d\theta\,d\rho\,d\varphi.$

16.5.65 The region of integration is the solid between the upper half-sphere of radius 2 centered at the origin and a cylinder of radius 1 oriented along the z-axis.



We have $\displaystyle\int_{\pi/6}^{\pi/2}\int_0^{2\pi}\int_{\csc\varphi}^{2} g\left(\rho,\,\varphi,\,\theta\right)\;\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = \int_{\pi/6}^{\pi/2}\int_{\csc\varphi}^{2}\int_0^{2\pi} g\left(\rho,\,\varphi,\,\theta\right)\;\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi.$

16.5.66 Using spherical coordinates, this volume is

$$\int_{\pi/4}^{\pi/2}\int_0^1\int_0^{2\pi}\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{2\pi}{3}\int_{\pi/4}^{\pi/2}\sin\,\varphi\,d\varphi = \frac{2\pi}{3}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{3}\pi.$$

16.5.67 This region is symmetric about the xy-plane, so we compute the volume of the region inside the solid cylinder for $z \ge 0$ that is below the cone $\varphi = \frac{\pi}{3}$ and double it. Use spherical coordinates. For a given value of φ, we have $0 \le r \le 2\csc\,\varphi$, so the volume is

$$2\int_{\pi/3}^{\pi/2}\int_0^{2\csc\varphi}\int_0^{2\pi}\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{32\pi}{3}\int_{\pi/3}^{\pi/2}\csc^3\,\varphi\,\sin\varphi\,d\varphi = \frac{32}{9}\pi\sqrt{3}.$$

16.5.68 This region is symmetric about the xy-plane, so we compute the volume of the region inside the upper half-sphere of radius 2 and below the cone $\varphi = \frac{\pi}{3}$ and double it. The volume is thus

$$2\int_{\pi/3}^{\pi/2}\int_0^2\int_0^{2\pi}\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{32\pi}{3}\int_{\pi/3}^{\pi/2}\sin\varphi\,d\varphi = \frac{16\pi}{3}.$$

16.5.69 Use cylindrical coordinates. Note that $x+y \ge 0$ for $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$, so this is the range of integration for θ.

$$\begin{aligned}\int_0^1\int_{-\pi/4}^{3\pi/4}\int_0^{r\,\cos\theta+r\,\sin\theta} r\,dz\,d\theta\,dr &= \int_0^1\int_{-\pi/4}^{3\pi/4} r^2\left(\cos\theta+\sin\theta\right)d\theta\,dr\\ &= \int_0^1 r^2\left(\sin\theta-\cos\theta\right)\Big|_{\theta=-\pi/4}^{\theta=3\pi/4} = 2\sqrt{2}\int_0^1 r^2\,dr = \frac{2\sqrt{2}}{3}.\end{aligned}$$

16.5.70 Use cylindrical coordinates. The region is traced out for $0 \le \theta \le \pi$; however, it is symmetric around the xz-plane, so compute the volume for $0 \le \theta \le \frac{\pi}{2}$ to avoid negative values of r, and double it. $2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{4-r\cos\theta} r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{2\cos\theta} r\left(4-r\cos\theta\right)\,dr\,d\theta =$ $2\int_0^{\pi/2}\int\left(2r^2-\frac{1}{3}r^3\cos\theta\right)\Big|_0^{2\cos\theta}\,d\theta = 2\int_0^1\left(8\cos^2\theta-\frac{8}{3}\cos^4\theta\right)\,d\theta = 3\pi.$

16.5.71 The planes $z = x-2$ and $z = 2-x$ intersect when $x = 2$, which is at the boundary of the cardioid, so we can simply integrate between those two planes. Thus the volume is $\int_0^{2\pi}\int_0^{1+\cos\theta}\int_{r\ \cos\theta-2}^{2-r\ \cos\theta} r\,dz\,dr\,d\theta =$ $\int_0^{2\pi}\int_0^{1+\cos\theta} r\left(4-2r\ \cos\theta\right)\,d\theta\,dr = \frac{7\pi}{2}.$

16.5.72 Use cylindrical coordinates. $\int_1^2\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r\,d\theta\,dz\,dr = 2\pi\int_1^2 r\sqrt{4-r^2}\,dr = 2\pi\sqrt{3}.$

16.5.73 $\int_0^{2\pi}\int_0^2\int_0^8 r\left(1-0.05e^{-0.01r^2}\right)\,dz\,dr\,d\theta \approx 95.60362.$

16.5.74

a. $\int_1^\infty\int_0^\pi\int_0^{2\pi}\frac{2\cdot 10^{-4}}{\rho^4}\rho^2\sin\varphi\,d\theta\,d\varphi\,d\rho = 4\pi\cdot 10^{-4}\int_1^\infty\int_0^\pi \rho^{-2}\sin\varphi\,d\varphi d\rho = 8\pi\cdot 10^{-4}\int_1^\infty\rho^{-2}\,d\rho =$ $\lim_{b\to\infty}\left(8\pi\cdot 10^{-4}\int_1^b\rho^{-2}\,d\rho\right) = 8\pi\cdot 10^{-4}\lim_{b\to\infty}\left(-\rho^{-1}\right)\Big|_1^b = 8\pi\cdot 10^{-4}\lim_{b\to\infty}\left(\frac{-1}{b}+1\right) = 8\pi\cdot 10^{-4}.$

b. $2\cdot 10^{-4}\int_0^\infty\int_0^\pi\int_0^{2\pi} e^{-0.01\rho^3}\rho^2\ \sin\ \varphi\,d\theta\,d\varphi\,d\rho = 4\pi\cdot 10^{-4}\int_0^\infty\int_0^\pi \rho^2\ e^{-0.01\rho^3}\sin\varphi\,d\varphi\,d\rho =$ $\lim_{b\to\infty}\left(-8\pi\cdot 10^{-4}\int_0^b\rho^2\ e^{-0.01\rho^3}\,d\rho\right) = \lim_{b\to\infty}\left(-8\pi\cdot 10^{-4}\left(-\frac{100}{3}\ e^{-0.01\rho^3}\right)\Big|_0^b\right) =$ $\lim_{b\to\infty}\left(\frac{8\pi}{3}\cdot 10^{-6}\left(1-e^{-0.01b^3}\right)\right) = \frac{8\pi}{3}\cdot 10^{-6}$

16.5.75

a. With $x = \cos\varphi$ we have $\sin\varphi = \sqrt{1-x^2}$ and $dx = -\sin\varphi\,d\varphi$ so that $d\varphi = -\frac{1}{\sqrt{1-x^2}}\,dx$. The limits of integration for φ become 1 to -1, so the integral then becomes

$$\begin{aligned}
F\left(d\right) &= -\frac{GMm}{4\pi}\int_0^{2\pi}\int_{-1}^1\frac{\left(d-Rx\right)\sqrt{1-x^2}}{\left(R^2+d^2-2Rdx\right)^{3/2}\left(-\sqrt{1-x^2}\right)}\,dx\,d\theta\\
&= \frac{GMm}{4\pi}\int_{-1}^1\int_0^{2\pi}\frac{d-Rx}{\left(R^2+d^2-2Rdx\right)^{3/2}}d\theta\,dx\\
&= \frac{GMm}{2}\int_{-1}^1\left[\frac{d}{\left(R^2+d^2-2Rdx\right)^{3/2}}-\frac{Rx}{\left(R^2+d^2-2Rdx\right)^{3/2}}\right]\,dx\\
&= \frac{GMm}{2}\left(\frac{1}{R\sqrt{R^2+d^2-2Rdx}}-\frac{R^2+d^2-Rdx}{Rd^2\sqrt{R^2+d^2-2Rdx}}\right)\Bigg|_{-1}^1\\
&= \frac{GMm}{2}\frac{Rdx-R^2}{Rd^2\sqrt{R^2+d^2-2Rdx}}\Bigg|_{-1}^1.
\end{aligned}$$

Assuming $d > R$, this simplifies to $F\left(d\right) = \frac{GMm}{2}\left(\frac{Rd-R^2}{Rd^2(d-R)}+\frac{Rd+R^2}{Rd^2(d+R)}\right) = \frac{GMm}{2}\left(\frac{1}{d^2}+\frac{1}{d^2}\right) = \frac{GMm}{d^2}.$

b. Suppose $d < R$. Then we have

$$F(d) = \tfrac{GMm}{2}\tfrac{Rdx-R^2}{Rd^2\sqrt{R^2+d^2-2Rdx}}\Big|_{-1}^{1} = \tfrac{GMm}{2}\left(\tfrac{Rd-R^2}{Rd^2(R-d)} + \tfrac{Rd+R^2}{Rd^2(R+d)}\right) = \tfrac{GMm}{2}\left(-\tfrac{1}{d^2}+\tfrac{1}{d^2}\right) = 0.$$

16.5.76 Use cylindrical coordinates with the origin in the center of one of the tank ends, the z-axis parallel to the long axis of the tank and the y-axis parallel to the surface of the water such that the positive x-axis points through the water. The lines from the origin to the point where the water meets the tank make angles of $\pm\frac{\pi}{3}$ with the x-axis Because each of these angles has cosine $\frac{1}{2}$ (the height of the water is one half the radius). Thus the volume of water (in cubic feet) is

$$\int_{-\pi/3}^{\pi/3}\int_{(\sec\theta)/2}^{1}\int_{0}^{2} r\,dz\,dr\,d\theta = \int_{-\pi/3}^{\pi/3}\left(1-\frac{1}{4}\sec^2\theta\right)d\theta = \frac{2\pi}{3}-\frac{\sqrt{3}}{2}.$$

This integral is perhaps just as easy to work in Cartesian coordinates: the area of the region of one end of the tank that contains water is simply

$$\int_{1/2}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1)\;dy\,dx = \frac{\pi}{3}-\frac{\sqrt{3}}{4},$$

which gets multiplied by 2 for the length of the tank.

16.5.77 Assume the base of the cone lies in the xy-plane and that the center of the base is at the origin, with the vertex of the cone on the positive z-axis. Then the equation of the cone in cylindrical coordinates (a,θ,z) is $z = h-\frac{h}{r}a$ where h and r are the given constants. Thus the volume is $\displaystyle\int_0^r\int_0^{h-\frac{h}{r}a}\int_0^{2\pi} a\,d\theta\,dz\,da =$ $\displaystyle 2\pi\int_0^r a\left(h-\frac{h}{r}a\right)da = 2\pi\left(\frac{hr^2}{2}-\frac{hr^2}{3}\right) = \frac{1}{3}\pi hr^2.$

16.5.78 In spherical coordinates, $0 \le \varphi \le \cos^{-1}\left(\frac{R-h}{R}\right)$. $\displaystyle\int_0^{\cos^{-1}((R-h)/R)}\int_{(R-h)\sec\varphi}^{R}\int_0^{2\pi}\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi =$ $\displaystyle\frac{2}{3}\int_0^{\cos^{-1}((R-h)/R)}\left(R^3-(R-h)^3\sec^3\varphi\right)\sin\varphi\,d\varphi = \frac{1}{3}\pi h^2(3R-h).$

16.5.79 Using similar triangles, if the frustum is extended to a complete cone, the height of the cone is $\frac{Rh}{R-r}$. Thus the equation of the cone (in cylindrical coordinates(a,θ,z) is $z = \frac{Rh}{R-r}-\frac{h}{R-r}a$ so that $a = R-z\frac{R-r}{h}$ and the volume of the frustum is $\displaystyle\int_0^h\int_0^{R-z\frac{R-r}{h}}\int_0^{2\pi} a\,d\theta\,da\,dz = \pi\int_0^r\left(R-z\frac{R-r}{h}\right)dz = \frac{\pi}{3}\left(R^2+rR+r^2\right)h.$

16.5.80 The easiest way to do this problem is using Cartesian coordinates with a change of variable. The volume of the ellipsoid is 8 times the volume of the first-octant portion, so we have $\displaystyle 8\int_0^1\int_0^{\sqrt{1-x^2/a^2}}\int_0^{\sqrt{1-x^2/a^2-y^2/b^2}} 1\,dz\,dy\,dx$. Now make a change of variables $x = au$, $y = bv$, $z = cw$ so that $dx = a\,du$, $dy = b\,dv$, $dz = c\,dw$ to get $\displaystyle 8\int_0^1\int_0^{\sqrt{1-u^2}}\int_0^{\sqrt{1-u^2-v^2}} abc\;dw\,dv\,du =$ $\displaystyle 8abc\int_0^1\int_0^{\sqrt{1-u^2}}\int_0^{\sqrt{1-u^2-v^2}} dw\,dv\,du$, which is just abc times the first-octant volume of a sphere of radius 1, so is equal to $\frac{4}{3}\pi abc$.

16.5.81 The two spheres are $x^2+y^2+z^2 = R^2$, $x^2+y^2+(z-r)^2 = r^2$. The equation of the second sphere simplifies to $x^2+y^2+z^2-2zr = 0$, so the two spheres meet when $R^2-2zr = 0$ or $z = \dfrac{R^2}{2r}$. This is the plane of intersection of the spheres. The volume in question now consists of two spherical caps, one on either side of this plane. The upper one is a spherical cap of the sphere of radius R with height $R-\dfrac{R^2}{2r}=\dfrac{2Rr-R^2}{2r}$, so by problem 78 has volume $\dfrac{\pi}{3}\left(\dfrac{2Rr-R^2}{2r}\right)^2\left(3R-\dfrac{2Rr-R^2}{2r}\right)$. The lower one is a spherical cap of the sphere of radius r with height $\dfrac{R^2}{2r}$, so again by problem 78 it has volume $\dfrac{\pi}{3}\dfrac{R^4}{4r^2}\left(3r-\dfrac{R^2}{2r}\right)$. Adding these two and simplifying gives for the volume $\dfrac{\pi R^3\,(8r-3R)}{12r}$.

16.6 Integrals for Mass Calculations

16.6.1 By definition, the system will balance when the pivot is located at the center of mass of the two people.

16.6.2 Its mass is $1\cdot 50+2\cdot 50=150\,\text{g}$. Its center of mass is given by $\dfrac{1}{150}\left(\displaystyle\int_0^{50}x\,dx+\int_{50}^{100}2x\,dx\right)=\dfrac{1}{150}\,(1250+7500)=\dfrac{175}{3}$. So the center of mass is $\dfrac{175}{3}$ cm from the less dense end of the rod.

16.6.3 Integrate the density function over the region to find the mass; then integrate x times the density function and divide by the mass to get the y-coordinate and integrate y times the density function and divide by the mass to get the x-coordinate.

16.6.4 Because to compute M_x, we need to compute a weighted average of distances from the x-axis, which is y.

16.6.5 Integrate the density function over the region to find the mass. To find M_{yz}, integrate x times the density function over the region; the x-coordinate is then $\dfrac{M_{yz}}{M}$. Similarly for the other coordinates.

16.6.6 Because to compute M_{xz} we are finding a weighted sum of distances from the xz-plane, which is y.

16.6.7 The center of mass is $\dfrac{1}{13}\,(10\cdot 3+3\,(-1))=\dfrac{27}{13}$.

16.6.8 The center of mass is $\dfrac{1}{8+4+1}\,(8\cdot 2+4\cdot(-4)+1\cdot 0)=0$.

16.6.9 The mass is $\displaystyle\int_0^{\pi}(1+\sin x)\;dx=\pi+2$, and the center of mass is then $\dfrac{1}{\pi+2}\displaystyle\int_0^{\pi}x\,(1+\sin x)\;dx=\dfrac{1}{\pi+2}\left(\dfrac{1}{2}\pi^2+\pi\right)=\dfrac{\pi}{2}$

16.6.10 The mass is $\displaystyle\int_0^1\left(1+x^3\right)\,dx=\dfrac{5}{4}$, so the center of mass is $\dfrac{1}{5/4}\displaystyle\int_0^1x\left(1+x^3\right)\,dx=\dfrac{4}{5}\left(\dfrac{1}{2}+\dfrac{1}{5}\right)=\dfrac{14}{25}$.

16.6.11 The mass is $\displaystyle\int_0^4\left(2-\dfrac{x^2}{16}\right)\,dx=8-\dfrac{4}{3}=\dfrac{20}{3}$, so the center of mass is $\dfrac{1}{20/3}\displaystyle\int_0^4x\left(2-\dfrac{x^2}{16}\right)dx=\dfrac{3}{20}\,(16-4)=\dfrac{9}{5}$.

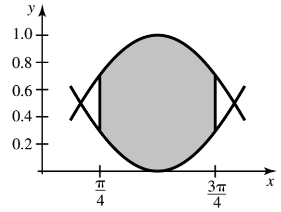
16.6.12 The mass is $\displaystyle\int_0^{\pi}(2+\cos x)\;dx=2\pi$, so the center of mass is $\dfrac{1}{2\pi}\displaystyle\int_0^{\pi}x\,(2+\cos x)\;dx=\dfrac{\pi^2-2}{2\pi}$.

16.6.13 The mass is $1\cdot 2+\displaystyle\int_2^4(1+x)\;dx=10$ so that the center of mass is

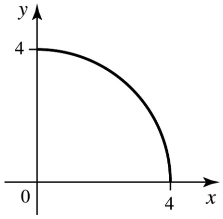
$$\frac{1}{10}\left(\int_0^2x\,dx+\int_2^4x\,(1+x)\;dx\right)=\frac{8}{3}.$$

16.6.14 The mass is $\int_0^1 x^2\,dx + \int_1^2 x\,(2-x)\,dx = 1$, so the center of mass is $\int_0^1 x^3\,dx + \int_1^2 x^2\,(2-x)\,dx = \frac{7}{6}$.

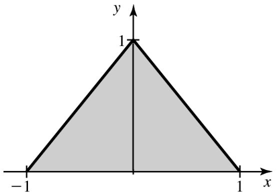
16.6.15 The region is symmetric with respect to $x = \frac{\pi}{2}$ and $y = \frac{1}{2}$, so its center of mass is at $\left(\frac{\pi}{2}, \frac{1}{2}\right)$.



16.6.16 Assume density 1. Clearly $M_x = M_y$, so we need only compute one of them. The mass is one quarter the area of the circle, or 4π. Then using polar coordinates we have $\overline{x} = \frac{M_y}{M} = \frac{1}{4\pi}\int_0^4 \int_0^{\pi/2} (r\,\sin\theta)\,r\,d\theta\,dr = \frac{1}{4\pi}\int_0^4 r^2\,dr = \frac{16}{3\pi}$, so that $\overline{y} = \frac{16}{3\pi}$ as well.

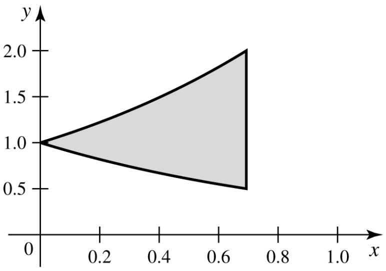


16.6.17 Assume density 1. By symmetry, $\overline{x} = 0$. The mass of the region is 1, because the two triangles together make a unit square. Thus $\overline{y} = \frac{M_x}{M} = \int_{-1}^0 \int_0^{1+x} y\,dy\,dx + \int_0^1 \int_0^{1-x} y\,dy\,dx = \frac{1}{3}$



16.6.18 Assume density 1. The mass is $\int_0^{\ln 2}\int_{e^{-x}}^{e^x} 1\,dy\,dx = \int_0^{\ln 2}\left(e^x - e^{-x}\right)dx = \left(e^x + e^{-x}\right)\Big|_0^{\ln 2} = \frac{1}{2}$. Then $\overline{x} = \frac{M_y}{M} = \frac{1}{1/2}\int_0^{\ln 2}\int_{e^{-x}}^{e^x} x\,dy\,dx = 2\int_0^{\ln 2} x\left(e^x - e^{-x}\right)dx = 5\ln(2) - 3$. Also, $\overline{y} = \frac{M_x}{M} = \frac{1}{1/2}\int_0^{\ln 2}\int_{e^{-x}}^{e^x} y\,dy\,dx$

$= \int_0^{\ln 2}\left(e^{2x} - e^{-2x}\right)dx = \frac{9}{8}$, so the center of mass is $\left(5\,\ln(2) - 3, \frac{9}{8}\right)$.



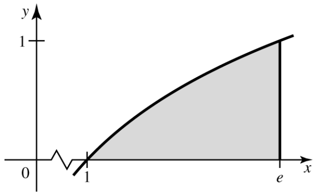
16.6.19 Assume density 1. The mass is $\int_1^e \int_0^{\ln x} 1\,dy\,dx = \int_1^e \ln x\,dx = 1$. Then

$$\overline{x} = \frac{M_y}{M} = \frac{1}{1}\int_1^e \int_0^{\ln x} x\,dy\,dx = \int_1^e x\ln x\,dx = \frac{1}{4}\left(e^2+1\right)$$

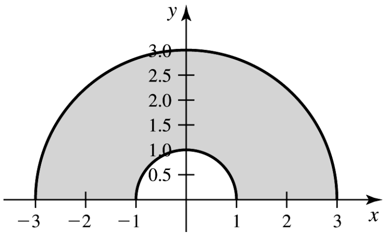
and

$$\overline{y} = \frac{M_x}{M} = \frac{1}{1}\int_1^e \int_0^{\ln x} x\,dy\,dx = \frac{1}{2}\int_1^e (\ln x)^2\,\,dx = \frac{1}{2}e - 1,$$

so that the center of mass is $\left(\frac{1}{4}\left(e^2+1\right), \frac{1}{2}e-1\right)$.



16.6.20 Assume density 1. The mass is $\int_0^{\pi}\int_1^3 r\,dr\,d\theta = 4\pi$. Then clearly $\overline{x} = 0$ by symmetry, and $\overline{y} = \frac{M_x}{M} = \frac{1}{4\pi}\int_0^{\pi}\int_1^3 r^2\sin\theta\,dr\,d\theta = \frac{13}{6\pi}\int_0^{\pi}\sin\,\theta\,\,d\theta = \frac{13}{3\pi}$, so the center of mass is $\left(0, \frac{13}{3\pi}\right)$.



16.6.21 The mass is $\int_0^4\int_0^2 \left(1+\frac{x}{2}\right)\,dy\,dx = 2\int_0^4 \left(1+\frac{x}{2}\right)\,dx = 16$. Then

$$\overline{x} = \frac{M_y}{M} = \frac{1}{16}\int_0^4\int_0^2 \left(x+\frac{x^2}{2}\right)\,dy\,dx = \frac{1}{8}\int_0^4\left(x+\frac{x^2}{2}\right)\,dx = \frac{7}{3},$$

and

$$\overline{y} = \frac{M_x}{M} = \frac{1}{16}\int_0^4\int_0^2 y\left(1+\frac{x}{2}\right)\,dy\,dx = \frac{1}{8}\int_0^4\left(1+\frac{x}{2}\right)\,dx = 1.$$

Thus, the center of mass is at $\left(\frac{7}{3}, 1\right)$. The density of the plate increases as you move toward the right.

16.6.22 The mass is $\int_0^5 \int_0^1 2e^{-y/2}\,dx\,dy = 2\int_0^5 2e^{-y/2}\;dy = 4 - 4e^{-5/2}$. So we have

$$\overline{x} = \frac{M_y}{M} = \frac{1}{4-4e^{-5/2}} \int_0^5 \int_0^1 2xe^{-y/2}\,dx\,dy = \frac{1}{4-4e^{-5/2}} \int_0^5 e^{-y/2}\,dy = \frac{1}{2}.$$

$$\overline{y} = \frac{M_x}{M} = \frac{1}{4-4e^{-5/2}} \int_0^5 \int_0^1 2ye^{-y/2}\,dx\,dy = \frac{1}{4-4e^{-5/2}} \int_0^5 2ye^{-5y/2}\,dy = \frac{2e^{5/2}-7}{e^{5/2}-1},$$

so that the center of mass is at $\left(\frac{1}{2}, \frac{2e^{5/2}-7}{e^{5/2}-1}\right)$. The density of the plate decreases as you move up the plate.

16.6.23 The mass is $\int_0^4 \int_0^{4-x} (1+x+y)\;dy\,dx = \int_0^4 \left(4 - x + x\,(4-x) + \frac{(4-x)^2}{2}\right)\,dx = \frac{88}{3}$. By symmetry, $\overline{x} = \overline{y}$ (both the region and the density function are symmetric around $x = y$), and $\overline{x} = \frac{M_y}{M} = \frac{1}{88/3} \int_0^4 \int_0^{4-x} \left(x + x^2 + xy\right)\,dy\,dx = \frac{3}{88} \int_0^4 \left(x\,(4-x) + x^2\,(4-x) + x\frac{(4-x)^2}{2}\right)\,dx = \frac{16}{11}$, so the center of mass is $\left(\frac{16}{11}, \frac{16}{11}\right)$. The density of the plate increases as you move right and/or up.

16.6.24 The mass is $\int_0^\pi \int_0^2 \left(r + \frac{1}{2}r^2\,\sin\theta\right)\,dr\,d\theta = \frac{6\pi+8}{3}$. The density function does not depend on x, and the region is symmetric about the y-axis, so $\overline{x} = 0$. Then

$$\overline{y} = \frac{M_x}{M} = \frac{1}{(6\pi+8)/3} \int_0^\pi \int_0^2 r\,\sin\theta\,\left(1 + \frac{1}{2}r\,\sin\theta\right) r\,dr\,d\theta = \frac{3\pi+16}{6\pi+8},$$

so the center of mass is $\left(0, \frac{3\pi+16}{6\pi+8}\right)$. The density increases as you move up the plate.

16.6.25 The mass is $\int_{-3}^3 \int_0^{\sqrt{9-x^2}/3} (1+y)\;dy\,dx = \int_{-3}^3 \left(\frac{\sqrt{9-x^2}}{3} + \frac{9-x^2}{18}\right)\,dx = \frac{3}{2}\pi + 2 = \frac{3\pi+4}{2}$. The density function does not depend on x, and the region is symmetric about the y-axis, so $\overline{x} = 0$. Then $\overline{y} = \frac{M_x}{M} = \frac{1}{(3\pi+4)/2} \int_{-3}^3 \int_0^{\sqrt{9-x^2}/3} \left(y + y^2\right)\,dy\,dx = \frac{3\pi+16}{12\pi+16}$, so the center of mass is $\left(0, \frac{3\pi+16}{12\pi+16}\right)$. The density increases as you move up the plate.

16.6.26 The mass is $\int_0^{\pi/2} \int_0^2 r\left(1+r^2\right)\,dr\,d\theta = 3\pi$. Because both the density function and the region are symmetric around $x = y$, $\overline{x} = \overline{y}$. We have $\overline{x} = \frac{M_y}{M} = \frac{1}{3\pi} \int_0^{\pi/2} \int_0^2 r^2\left(1+r^2\right)\cos\theta\,dr\,d\theta = \frac{1}{3\pi} \int_0^{\pi/2} \frac{136}{15}\,\cos\theta\,d\theta$
$= \frac{136}{45\pi}$, so that the center of mass is $\left(\frac{136}{45\pi}, \frac{136}{45\pi}\right)$. The density of the plate increases as you move away from the origin.

16.6.27 Assuming density 1, the mass is the volume of a half-sphere of radius 4, which is $\frac{1}{2} \cdot \frac{4}{3}\pi \cdot 4^3 = \frac{128\pi}{3}$. By symmetry, $\overline{x} = \overline{y} = 0$. Also, $\overline{z} = \frac{1}{(128\pi)/3} \int_0^4 \int_0^{2\pi} \int_0^{\pi/2} \rho\,\cos\varphi \cdot \rho^2\,\sin\varphi\,d\varphi\,d\theta\,d\rho =$
$\frac{3}{128\pi} \int_0^4 \int_0^{2\pi} \int_0^{\pi/2} \rho^3\,\cos\varphi\,\sin\varphi\,d\varphi\,d\theta\,d\rho = \frac{3}{128} \int_0^4 \rho^3\,d\rho = \frac{3}{2}$, so the center of mass is at $\left(0, 0, \frac{3}{2}\right)$.

16.6.28 Assuming density 1, the mass is $\int_0^{2\pi}\int_0^5\int_{r^2}^{25} r\,dz\,dr\,d\theta = 2500\pi$. By symmetry, $\overline{x}=\overline{y}=0$, and $\overline{z}=\frac{1}{2500\pi}\int_0^{2\pi}\int_0^5\int_{r^2}^{25} r\,z\,dz\,dr\,d\theta = \frac{1}{2500\pi}\cdot\frac{312500\pi}{7}=\frac{125}{7}$, so that the center of mass is at $\left(0,0,\frac{125}{7}\right)$.

16.6.29 Assuming density 1, the mass is the volume of a pyramid with height 1 and base area $\frac{1}{2}$, so the volume is $\frac{1}{6}$. The region is symmetric with respect to the line $x=y=z$, $\overline{x}=\overline{y}=\overline{z}$, and $\overline{z}=\frac{1}{1/6}\int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz\,dy\,dx = 3\int_0^1\int_0^{1-x}(1-x-y)^2\,dy\,dx=\frac{1}{4}$, so the center of mass is $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$.

16.6.30 Assuming density 1, the mass is the volume of a cone with base area 256π and height 16, so is $\frac{4096\pi}{3}$. By symmetry, $\overline{x}=\overline{y}=0$, and $\overline{z}=\frac{1}{(4096\pi)/3}\int_0^{16}\int_0^{16-r}\int_0^{2\pi} rz\,d\theta\,dz\,dr=\frac{3}{4096}\int_0^{16} r(16-r)^2\,dr=4$, so the center of mass is $(0,\,0,\,4)$.

16.6.31 Assuming density 1, the mass is

$$\int_0^1\int_0^{2\pi}\int_0^{1-r\,\sin\theta} r\,dz\,d\theta\,dr=\int_0^1\int_0^{2\pi} r\,(1-r\,\sin\theta)\,d\theta\,dr=\pi.$$

The region is symmetric around the yz-plane, so $\overline{x}=0$. We have

$$\overline{y}=\frac{1}{\pi}\int_0^1\int_0^{2\pi}\int_0^{1-r\sin\theta} r^2\,\sin\theta\,dz\,d\theta\,dr=\frac{1}{\pi}\int_0^1\int_0^{2\pi} r^2\,(1-r\,\sin\theta)\sin\theta\,d\theta\,dr=-\frac{1}{4}.$$

$$\overline{z}=\frac{1}{\pi}\int_0^1\int_0^{2\pi}\int_0^{1-r\sin\theta} rz\,dz\,d\theta\,dr=\frac{1}{2\pi}\int_0^1\int_0^{2\pi} r\,(1-r\,\sin\theta)^2\,d\theta\,dr=\frac{5}{8},$$

so that the center of mass is $\left(0,\,-\frac{1}{4},\,\frac{5}{8}\right)$.

16.6.32 Assuming density 1, the mass is $\int_0^2\int_0^{2\pi}\int_0^{2\sqrt{4-r^2}} r\,dz\,d\theta\,dr = 4\pi\int_0^2 r\sqrt{4-r^2}\,dr=\frac{32\pi}{3}$. By symmetry, $\overline{x}=\overline{y}=0$, and $\overline{z}=\frac{1}{(32\pi/3)}\int_0^2\int_0^{2\pi}\int_0^{2\sqrt{4-r^2}} rz\,dz\,d\theta\,dr=\frac{3}{8}\int_0^2 r\left(4-r^2\right)\,dr=\frac{3}{2}$, so that the center of mass is $\left(0,\,0,\,\frac{3}{2}\right)$.

16.6.33 The mass is $\int_0^4\int_0^1\int_0^1\left(1+\frac{x}{2}\right)dz\,dy\,dx=\int_0^4\left(1+\frac{x}{2}\right)dx=8$. By symmetry, Because the density depends only on x, $\overline{y}=\overline{z}=\frac{1}{2}$, while $\overline{x}=\frac{1}{8}\int_0^4\int_0^1\int_0^1\left(x+\frac{x^2}{2}\right)dz\,dy\,dx=\frac{1}{8}\int_0^4\left(x+\frac{x^2}{2}\right)dx=\frac{7}{3}$, so the center of mass is $\left(\frac{7}{3},\frac{1}{2},\frac{1}{2}\right)$.

16.6.34 The mass is

$$\int_0^4\int_0^{\sqrt{4-z}}\int_0^{2\pi} r\,(5-z)\,d\theta\,dr\,dz=2\pi\int_0^4\int_0^{\sqrt{4-z}} r\,(5-z)\,dr\,dz=\pi\int_0^4 (5-z)\,(4-z)\,dz=\frac{88\pi}{3}.$$

By symmetry, because the density depends only on z, $\overline{x}=\overline{y}=0$, while

$$\begin{aligned}\overline{z}&=\frac{1}{(88\pi)/3}\int_0^4\int_0^{\sqrt{4-z}}\int_0^{2\pi} rz\,(5-z)\,d\theta\,dr\,dz=\frac{3}{44}\int_0^4\int_0^{\sqrt{4-z}} rz\,(5-z)\,dr\,dz\\&=\frac{3}{88}\int_0^4 (5-z)\,(4-z)\,dz=\frac{12}{11},\end{aligned}$$

so the center of mass is $\left(0,\ 0,\ \dfrac{12}{11}\right)$.

16.6.35 The mass of the sphere is given by $\displaystyle\int_0^6\int_0^{\pi/2}\int_0^{2\pi}\left(1+\frac{\rho}{4}\right)\rho^2\,\sin\varphi\,d\theta\,d\varphi\,d\rho = 2\pi\int_0^6\left(1+\frac{\rho}{4}\right)\rho^2\,d\rho =$ 306π. By symmetry, because the density function depends only on ρ, $\overline{x}=\overline{y}=0$. Also, $\displaystyle\overline{z}=\frac{1}{306\pi}\int_0^6\int_0^{\pi/2}\int_0^{2\pi}\left(1+\frac{\rho}{4}\right)\rho^2\,\sin\varphi\cdot\rho\,\cos\varphi\,d\theta\,d\varphi\,d\rho = \frac{1}{153}\int_0^6\int_0^{\pi/2}\rho^3\left(1+\frac{\rho}{4}\right)\sin\varphi\,\cos\varphi\,d\varphi\,d\rho =$ $\displaystyle\frac{1}{306}\int_0^6\rho^3\left(1+\frac{\rho}{4}\right)\,d\rho=\frac{198}{85}$, so the center of mass is $\left(0,\ 0,\ \dfrac{198}{85}\right)$.

16.6.36 The mass of the cube is given by

$$\int_0^1\int_0^1\int_0^1 (2+x+y+z)\;dz\,dy\,dx = \int_0^1\int_0^1\left(\frac{5}{2}+x+y\right)\,dy\,dx = \int_0^1 (3+x)\;dx = \frac{7}{2}.$$

Both the region and the density function are symmetric with respect to the line $x=y=z$, so $\overline{x}=\overline{y}=\overline{z}$, and

$$\begin{aligned}\overline{x}&=\frac{1}{7/2}\int_0^1\int_0^1\int_0^1\left(2x+x^2+xy+xz\right)\,dz\,dy\,dx=\frac{2}{7}\int_0^1\int_0^1\left(\frac{5}{2}x+x^2+xy\right)\,dy\,dx\\&=\frac{2}{7}\int_0^1\left(3x+x^2\right)\,dx=\frac{11}{21},\end{aligned}$$

so that the center of mass is $\left(\dfrac{11}{21},\dfrac{11}{21},\dfrac{11}{21}\right)$.

16.6.37 The mass is $\displaystyle\int_0^1\int_0^4\int_0^x (2+y)\;dz\,dy\,dx=\int_0^1\int_0^4 (2x+xy)\;dy\,dx=\int_0^1 16x\,dx=8$. Then

$$\overline{x}=\frac{1}{8}\int_0^1\int_0^4\int_0^x (2x+xy)\;dz\,dy\,dx=\frac{1}{8}\int_0^1\int_0^4 x^2\,(y+2)\,dy\,dx=2\int_0^1 x^2\,dx=\frac{2}{3}.$$

$$\overline{y}=\frac{1}{8}\int_0^1\int_0^4\int_0^x\left(2y+y^2\right)\,dz\,dy\,dx=\frac{1}{8}\int_0^1\int_0^4 x\left(y+2y^2\right)dy\,dx=\frac{14}{3}\int_0^1 x\,dx=\frac{7}{3}.$$

$$\overline{z}=\frac{1}{8}\int_0^1\int_0^4\int_0^x (2z+yz)\;dz\,dy\,dx=\frac{1}{8}\int_0^1\int_0^4\left(x^2+\frac{x^2y}{2}\right)dy\,dx=\frac{1}{8}\int_0^1 8x^2\,dx=\frac{1}{3},$$

so that the center of mass is $\left(\dfrac{2}{3},\dfrac{7}{3},\dfrac{1}{3}\right)$.

16.6.38 The mass is $\displaystyle\int_0^9\int_0^{9-r}\int_0^{2\pi} r\,(1+z)\;d\theta\,dz\,dr\,=2\pi\int_0^9 r\left(9-r+\frac{(9-r)^2}{2}\right)\,dr\;=\;\frac{3159\pi}{4}$. By symmetry and the fact that the density depends only on z, we have $\overline{x}=\overline{y}=0$. We also have $\displaystyle\overline{z}=\frac{1}{(3159\pi)/4}\int_0^9\int_0^{9-r}\int_0^{2\pi} r\left(z+z^2\right)\,d\theta\,dz\,dr=\frac{8}{3159}\int_0^9 r\left(\frac{(9-r)^2}{2}+\frac{(9-r)^3}{3}\right)\,dr=\frac{207}{65}$, so that the center of mass is $\left(0,0,\dfrac{207}{65}\right)$.

16.6.39

a. False. It has a center of mass with a y-coordinate of zero.

b. True. Because every point is balanced by the corresponding point on the other side of the origin.

c. False. For example, the annulus $1 \le r \le 3$ has center of mass at the origin.

d. False. For example, the solid resulting from revolving the annulus in part (c) about the x-axis is connected, but its center of mass is at the origin.

16.6.40 The mass of the rod is $\int_0^L 2e^{-x/3}\,dx = 6\left(1-e^{-L/3}\right)$, so its center of mass is given by

$$\frac{1}{6\left(1-e^{-L/3}\right)}\int_0^L 2x\,e^{-x/3}\,dx = \frac{18-18e^{-L/3}-6Le^{-L/3}}{6\left(1-e^{-L/3}\right)},$$

so that as $L\to\infty$, the center of mass approaches
$\lim_{L\to\infty}\dfrac{18-18e^{-L/3}-6Le^{-L/3}}{6\left(1-e^{-L/3}\right)} = 3$, because all terms involving L approach 0 in the limit.

16.6.41 The mass of the rod is $\int_0^L \frac{10}{1+x^2}\,dx = 10\tan^{-1}(L)$, so its center of mass is

$$\frac{1}{\tan^{-1}(L)}\int_0^L \frac{10x}{1+x^2}\,dx = \frac{\ln\left(1+L^2\right)}{\tan^{-1}(L)}.$$

As $L\to\infty$, the numerator grows without bound while the denominator approaches $\frac{\pi}{2}$, so $\overline{x}\to\infty$ as $L\to\infty$.

16.6.42 The mass of the plate is $\int_0^L\int_{-e^{-x}}^{e^{-x}} dy\,dx = \int_0^L 2e^{-x}\,dx = 2-2e^{-L}$, so that

$$\lim_{L\to\infty}\overline{x} = \lim_{L\to\infty}\frac{1}{2-2e^{-L}}\int_0^L\int_{-e^{-x}}^{e^{-x}} x\,dy\,dx = \lim_{L\to\infty}\frac{1}{2-2e^{-L}}\int_0^L 2x\,e^{-x}\,dx = \lim_{L\to\infty}\frac{2-2e^{-L}(L+1)}{2-2e^{-L}} = 1.$$

Also, $\lim_{L\to\infty}\overline{y} = \lim_{L\to\infty}\frac{1}{2-2e^{-L}}\int_0^L\int_{-e^{-x}}^{e^{-x}} y\,dy\,dx = \lim_{L\to\infty} 0 = 0$. Thus as $L\to\infty$, the center of mass approaches (1, 0).

16.6.43 The mass of the plate, assuming density 1, is $8+4=12$ Because the area of the rectangle is 8 and the two triangles together to form a 2×2 square. By symmetry, $\overline{x}=0$. Compute $\overline{y}$ by computing M_x for each of the three pieces. The moments around the x-axis are equal for the two triangles. So we need only compute M_x for one of the triangles. This is $M_x = \int_2^4\int_0^{4-x} y\,dy\,dx = \int_2^4 \frac{1}{2}(4-x)^2\,dx = \frac{4}{3}$

The moment around the x-axis for the rectangle is

$$M_x = \int_{-2}^{2}\int_0^2 y\,dy\,dx = 8,$$

so that $\overline{y} = \frac{1}{12}\left(\frac{4}{3}+8+\frac{4}{3}\right) = \frac{8}{9}$. Thus the center of mass is at $\left(0,\ \frac{8}{9}\right)$.

16.6.44 Assuming density 1, the mass of the plate is the area of the outer rectangle (48) less the area of the inner, missing, rectangle (4), so the mass is 44. By symmetry, $\overline{x}=0$. To compute $\overline{y}$, compute the moment around the x-axis by computing the moment for the outer rectangle and that for the missing rectangle, and subtract one from the other. Thus $\overline{y} = \frac{1}{44}\left(\int_{-4}^{4}\int_{-4}^{2} y\,dy\,dx - \int_{-2}^{2}\int_{-1}^{0} y\,dy\,dx\right) = \frac{1}{44}(-48+2) = -\frac{23}{22}$. Thus the center of mass is at $\left(0,\ -\frac{23}{22}\right)$.

16.6.45 The mass of the region (assuming density 1) is $\frac{1}{2}\pi r^2 = 2\pi$, by symmetry, $\overline{x}=0$, and $\overline{y} = \frac{1}{2\pi}\int_0^{\pi}\int_0^2 r^2\sin\theta\,dr\,d\theta = \frac{1}{2\pi}\int_0^{\pi}\frac{8}{3}\sin\theta\,d\theta = \frac{8}{3\pi}$, so the center of mass is at $\left(0,\ \frac{8}{3\pi}\right)$.

16.6.46 The mass of the region is π; by symmetry, $\overline{x} = \overline{y}$, and

$$\overline{x} = \frac{1}{\pi}\int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta = \frac{8}{3\pi}\int_0^{\pi/2}\cos\theta\,d\theta = \frac{8}{3\pi},$$

thus the center of mass is at $\left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)$.

16.6.47 The mass of the cardioid is $\int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}(1+\cos\theta)^2\,d\theta = \frac{3\pi}{2}$. By symmetry, $\overline{y} = 0$, and $\overline{x} = \frac{1}{(3\pi)/2}\int_0^{2\pi}\int_0^{1+\cos\theta} r^2\cos\theta\,dr\,d\theta = \frac{2}{9\pi}\int_0^{2\pi}(1+\cos\theta)^3\cos\theta\,d\theta = \frac{5}{6}$, so that the center of mass is at $\left(\frac{5}{6}, 0\right)$.

16.6.48 The mass of the cardioid is $\int_0^{2\pi}\int_0^{3-3\cos\theta} r\,dr\,d\theta = \frac{9}{2}\int_0^{2\pi}(1-\cos\theta)^2\,d\theta = \frac{27\pi}{2}$. By symmetry, $\overline{y} = 0$, and $\overline{x} = \frac{1}{(27\pi)/2}\int_0^{2\pi}\int_0^{3-3\cos\theta} r^2\cos\theta\,dr\,d\theta = \frac{2}{3\pi}\int_0^{2\pi}(1-\cos\theta)^3\cos\theta\,d\theta = -\frac{5}{2}$, so that the center of mass is at $\left(-\frac{5}{2}, 0\right)$.

16.6.49 The mass of the leaf is $\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\sin^2 2\theta\,d\theta = \frac{\pi}{8}$. By symmetry, $\overline{x} = \overline{y}$, and $\overline{x} = \frac{1}{\pi/8}\int_0^{\pi/2}\int_0^{\sin 2\theta} r^2\cos\theta\,dr\,d\theta = \frac{8}{3\pi}\int_0^{\pi/2}\sin^3 2\theta\,\cos\theta\,d\theta = \frac{128}{105\pi}$, so that the center of mass is at $\left(\frac{128}{105\pi}, \frac{128}{105\pi}\right)$.

16.6.50 The mass of the limacon is $\int_0^{2\pi}\int_0^{2+\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}(2+\cos\theta)^2\,d\theta = \frac{9\pi}{2}$. By symmetry, $\overline{y} = 0$, and $\overline{x} = \frac{1}{(9\pi)/2}\int_0^{2\pi}\int_0^{2+\cos\theta} r^2\cos\theta\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}(2+\cos\theta)^3\cos\theta\,d\theta = \frac{17}{18}$, so that the center of mass is at $\left(\frac{17}{18}, 0\right)$.

16.6.51 Assume the origin is at the midpoint of the diameter with the y-axis pointing up. Assume the density is one. The mass is πr (the length of the wire). The moment is $\int_0^{\pi}(r\sin\theta)\,d\theta = 2r$. Therefore $\overline{y} = \frac{2r}{\pi}$, while $\overline{x} = 0$ by symmetry.

16.6.52 The line $y = b$ intersects the parabola at $x = \pm\sqrt{b/a}$, so the mass of the plate (assuming density 1) is

$$\int_{-\sqrt{b/a}}^{\sqrt{b/a}}\int_{ax^2}^{b} 1\,dy\,dx = \int_{-\sqrt{b/a}}^{\sqrt{b/a}}\left(b - ax^2\right)\,dx = \frac{4b}{3}\sqrt{b/a}.$$

By symmetry, $\overline{x} = 0$, and

$$\overline{y} = \frac{1}{\left(\frac{4b}{3}\sqrt{b/a}\right)}\int_{-\sqrt{b/a}}^{\sqrt{b/a}}\int_{ax^2}^{b} y\,dy\,dx = \frac{3}{8b}\sqrt{\frac{a}{b}}\int_{-\sqrt{b/a}}^{\sqrt{b/a}}\left(b^2 - a^2x^4\right)\,dx = \frac{3b}{5},$$

so that the center of mass is at $\left(0, \frac{3b}{5}\right)$ and is independent of a.

16.6.53 The mass of the region, assuming density 1, is $a^2 - \frac{\pi}{4}a^2 = \frac{4-\pi}{4}a^2$. By symmetry, $\overline{x} = \overline{y}$, and $\overline{x} = \frac{1}{a^2\left(4-\pi\right)/4}\int_0^a\int_{\sqrt{a^2-x^2}}^a x\,dy\,dx = \frac{4}{a^2\left(4-\pi\right)}\int_0^a x\left(a-\sqrt{a^2-x^2}\right)\,dx = \frac{2a}{3\left(4-\pi\right)}$, so that the center of mass is at $\left(\frac{2a}{3\left(4-\pi\right)}, \frac{2a}{3\left(4-\pi\right)}\right)$.

16.6.54 Clearly the center of mass is at the center of the box - halfway between each opposite pair of faces. The easiest way to see this is to place the origin at the center of the box; then (for example)

$$\overline{x} = \frac{1}{m}\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} x\,dx\,dy\,dz = \frac{1}{m}\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\frac{1}{2}x^2\bigg|_{-a/2}^{a/2}\,dy\,dz = 0.$$

16.6.55 Place the origin at the vertex of the cone and let the z-axis be the axis of the cone. Then the cone has equation (in cylindrical coordinates (a, θ, z)) $z = \frac{h}{r}a$. The mass of the cone is $\frac{1}{3}\pi r^2 h$, and by symmetry $\overline{x} = \overline{y} = 0$. Now $\overline{z} = \frac{1}{\pi r^2 h/3}\int_0^r\int_0^{(h/r)a}\int_0^{2\pi} az\,d\theta\,dz\,da = \frac{3h}{r^4}\int_0^r a^3\,da = \frac{3h}{4}$, so that the center of mass is one quarter of the way from the base to the vertex.

16.6.56 Place the origin at the center of the hemisphere. The sphere has mass $\frac{2}{3}\pi a^3$, so (in spherical coordinates)

$$\overline{z} = \frac{1}{\left(2\pi a^3\right)/3}\int_0^a\int_0^{\pi/2}\int_0^{2\pi}\rho^3\sin\varphi\,\cos\varphi\,d\theta\,d\varphi\,d\rho = \frac{3}{a^3}\int_0^a\int_0^{\pi/2}\rho^3\sin\varphi\,\cos\varphi\,d\varphi\,d\rho = \frac{3}{8}a,$$

so that the center of mass is $\frac{3}{8}$ of the way from the origin to the top of the hemisphere.

16.6.57 Place the origin at the middle of the base of the triangle. Then the y-coordinate of the center of mass can be determined. If h is the height of the triangle, its area is $\frac{bh}{2}$, $\overline{x} = 0$ by symmetry, and $\overline{y} = \frac{1}{\left(bh\right)/2}\int_0^h\int_{b(y-h)/(2h)}^{-b(y-h)/(2h)} y\,dx\,dy = \frac{2}{bh}\int_0^h\frac{-b\left(y-h\right)}{h}\,dy = -\frac{2}{h^2}\int_0^h\left(y^2-hy\right)\,dy = \frac{h}{3}$, so that the center of mass is $\frac{1}{3}$ of the way from the base to the vertex.

16.6.58 The tetrahedron has height a, and the base has area $\frac{a^2}{2}$, so the volume of the tetrahedron is $\frac{a^3}{6}$. By symmetry, $\overline{x} = \overline{y} = \overline{z}$, and $\overline{x} = \frac{1}{a^3/6}\int_0^a\int_0^{a(1-x/a)}\int_0^{a(1-x/a-y/a)} x\,dz\,dy\,dx$. Make the change of variable $u = \frac{x}{a}.$, $v = \frac{y}{a}$, $w = \frac{z}{a}$; then $dx = a\,du$, $du = a\,dv$, $dz = a\,dw$ and then

$$\overline{x} = \frac{1}{a^3/6}\int_0^a\int_0^{1-u}\int_0^{1-u-v} a^4 u\,dw\,dv\,du = 6a\int_0^1\int_0^{1-u} u\left(1-u-v\right)\,dv\,du = \frac{a}{4},$$

so the center of mass is $\left(\frac{a}{4}, \frac{a}{4}, \frac{a}{4}\right)$.

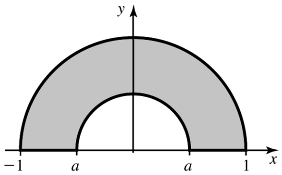
16.6.59 Place the origin at the center of the ellipse with the circular base of radius r in the xy-plane, so that the top of the ellipsoid is on the positive z-axis, at $(0, 0, a)$. Use cylindrical coordinates (ρ, θ, z); then the equation for the top half of the ellipsoid is $z = a\sqrt{1-\frac{\rho^2}{r^2}}$. We know (Problem 80, Section 13.5) that the volume of the top half of this ellipsoid is $\frac{2\pi r^2 a}{3}$, so $\overline{z} = \frac{1}{\left(2\pi r^2 a\right)/3}\int_0^r\int_0^{a\sqrt{1-\rho^2/r^2}}\int_0^{2\pi}\rho z\,d\theta\,dz\,d\rho = \frac{3a}{2r^2}\int_0^r\rho\left(1-\frac{\rho^2}{r^2}\right)\,d\rho = \frac{3}{8}a$, so the center of mass is $\frac{3}{8}$ of the way from the base to the top of the ellipsoid.

16.6.60 The area of the country is (adding up the large rectangle and the two small squares) $48+2\cdot 4=56$. $\overline{x}=0$ by symmetry. To compute $\overline{y}$, we subtract the moment of the missing small rectangle around the x-axis from that of the large rectangle. But the large rectangle is symmetric around the x-axis, so its moment is zero and thus $\overline{y}=\frac{1}{56}\left(0-\int_{-4}^{-2}\int_{-2}^{2} y\,dx\,dy\right)=-\frac{1}{14}\int_{-4}^{-2} y\,dy=\frac{3}{7}$, so that the geographical center is at $\left(0,\frac{3}{7}\right)$. The population center is a discrete computation. Count population in thousands: $\overline{x}=\frac{10\cdot(-2)+15\cdot 2+20\cdot 2+5\cdot 4+15\cdot(-2)}{10+15+20+5+15}=\frac{40}{65}=\frac{8}{13}\overline{y}=\frac{10\cdot 2+15\cdot 3+20\cdot 0+5\cdot(-4)+15\cdot(-2)}{10+15+20+5+15}=\frac{15}{65}=\frac{3}{13}$, so that the population center is at $\left(\frac{8}{13},\frac{3}{13}\right)$.

16.6.61

a. The mass of the plate is the difference of the area of the two semicircles, so is $\frac{1}{2}\pi\left(1-a^2\right)$. The y-coordinate of the center of mass is then

$$\overline{y}=\frac{1}{\pi\left(1-a^2\right)/2}\int_a^1\int_0^\pi r^2\sin\theta\,d\theta\,dr=\frac{4}{3\pi\left(1-a^2\right)}\int_a^1 r^2\,dr=\frac{4\left(1-a^3\right)}{3\pi\left(1-a^2\right)}=\frac{4(a^2+a+1)}{3(a+1)\pi}.$$



b. The center of mass always has x coordinate 0, so it lies on the edge of the plate exactly when $\frac{4(a^2+a+1)}{3(a+1)\pi}=a$ or 1. Solving for equality with 1 gives no solutions in the range $0\le a\le 1$. Solving for equality with a gives $a=-\frac{1}{2}\left(1-\sqrt{\frac{3\left(\pi+4\right)}{3\pi-4}}\right)\approx 0.49366$ while the other solution is outside of the range $0\le a\le 1$.

16.6.62

a. The mass of the solid is the difference of the volumes of the two hemispheres, so is $\frac{2}{3}\pi\left(1-a^3\right)$. The z-coordinate of the center of mass is then $\overline{z}=\frac{1}{2\pi\left(1-a^3\right)/3}\int_0^1\int_0^{\pi/2}\int_0^{2\pi} r^3\cos\varphi\sin\varphi\,d\theta\,d\varphi\,dr=\frac{3}{2\left(1-a^3\right)}\int_0^1 r^3\,dr=\frac{3\left(1-a^4\right)}{8\left(1-a^3\right)}$.

b. The center of mass always has x and y-coordinates 0, so it lies on the edge of the plate exactly when $\frac{3\left(1-a^4\right)}{8\left(1-a^3\right)}=a$ or 1. Solving for equality with 1 gives no solutions in the range $0\le a\le 1$; solving for equality with a gives $a=\frac{\left(1450+450\sqrt{11}\right)^{2/3}-5\left(1450+450\sqrt{11}\right)^{1/3}-50}{15\left(1450+450\sqrt{11}\right)^{1/3}}\approx 0.38936$.

16.6.63 Place the origin at the center of the bottom of the soda can. If the height of soda in the can is h, for $0\le h\le 12$, the mass of the can is $16\pi h+\frac{16}{1000}\pi\left(12-h\right)=\frac{6}{125}\pi\left(333h+4\right)$. To compute $\overline{z}$, compute the moments around the x-axis separately for the soda and the air:

$$\begin{aligned}
\overline{z} &= \frac{1}{6\pi\left(333h+4\right)/125}\int_0^4\int_0^{2\pi}\left(\frac{1}{1000}\int_h^{12} zr\,dz + \int_0^h zr\,dz\right)d\theta\,dr\\
&= \frac{125}{6\pi\left(333h+4\right)}\int_0^4\int_0^{2\pi}\left(\frac{1}{2000}\left(144-h^2\right)r+\frac{1}{2}h^2r\right)d\theta\,dr\\
&= \frac{125}{6\pi\left(333h+4\right)}\cdot\frac{\pi\left(144+999h^2\right)}{125} = \frac{3}{2}\cdot\frac{111h^2+16}{333h+4}.
\end{aligned}$$

The center of mass is at its lowest point when the derivative of this function is zero, i.e. when $\frac{333h}{333h+4} - \frac{999\left(16+111h^2\right)}{2\left(333h+4\right)^2} = 0$. Placing over a common denominator, setting the numerator to zero and solving gives $h = \frac{40\sqrt{10}-4}{333} \approx 0.3678$ cm.

16.6.64

a. There are two cases: either $0 < a \le b$ or $0 < b \le a$. In either case, the area of the triangle is $\frac{1}{2}bh$.

Case 1: $0 < a \le b$.

$$\overline{x} = \frac{1}{bh/2}\left(\int_0^a\int_0^{(h/a)x} x\,dy\,dx + \int_a^b\int_0^{(h/(a-b))(x-b)} x\,dy\,dx\right) =$$

$$\frac{2}{bh}\left(\frac{h}{a}\int_0^a x^2\,dx + \frac{h}{a-b}\int_a^b\left(x^2-bx\right)\,dx\right) = \frac{2}{bh}\left(\frac{h}{3a}a^3+\frac{h\left(b^2+ab-2a^2\right)}{6}\right) = \frac{a+b}{3}.$$

$$\overline{y} = \frac{1}{bh/2}\left(\int_0^a\int_0^{(h/a)x} y\,dy\,dx + \int_a^b\int_0^{(h/(a-b))(x-b)} y\,dy\,dx\right) =$$

$$\frac{2}{bh}\left(\frac{h^2}{2a^2}\int_0^a x^2\,dx + \frac{h^2}{2\left(a-b\right)^2}\int_a^b\left(x-b\right)^2\,dx\right) = \frac{2}{bh}\left(\frac{ah^2}{6}-\frac{h^2\left(a-b\right)}{6}\right) = \frac{2}{bh}\cdot\frac{bh^2}{6} = \frac{h}{3}.$$

Case 2: $0 < b \le a$.

$$\overline{x} = \frac{1}{bh/2}\left(\int_0^b\int_0^{(h/a)x} x\,dy\,dx + \int_b^a\int_{(h/(a-b))(x-b)}^{(h/a)x} x\,dy\,dx\right) =$$

$$\frac{2}{bh}\left(\frac{h}{a}\int_0^b x^2\,dx + \int_b^a x\left(\frac{h}{a}x-\frac{h}{a-b}\left(x-b\right)\right)\,dx\right) = \frac{2}{bh}\left(\frac{hb^3}{3a}+\frac{\left(a^2+ab-2b^2\right)bh}{6a}\right) = \frac{a+b}{3}.$$

$$\overline{y} = \frac{1}{bh/2}\left(\int_0^b\int_0^{(h/a)x} y\,dy\,dx + \int_a^b\int_{(h/(a-b))(x-b)}^{(h/a)x} y\,dy\,dx\right) =$$

$$\frac{1}{bh}\left(\frac{h^2}{a^2}\int_0^b x^2\,dx + \int_b^a \frac{h^2}{a^2}x^2-\frac{h^2}{\left(a-b\right)^2}\left(x-b\right)^2\,dx\right) = \frac{1}{bh}\left(\frac{h^2b^3}{3a^2}+\frac{h^2b\left(a^2-b^2\right)}{3a^2}\right) = \frac{h}{3}.$$

In either case, the centroid is at $\left(\frac{a+b}{3}, \frac{h}{3}\right)$.

b. Because each median bisects the triangle, the centroid must lie on each median Because the triangle will be balanced with respect to the axis determined by the median. Because this is true of each median, the centroid must be at the intersection of the medians.

16.6.65

a. Place the origin at Q, with the circles to the right of the y-axis. Then using polar coordinates $(a,\,\theta)$, the equation of the circles are $a = 2R\cos\theta$ and $a = 2r\cos\theta$ for $0 \le \theta \le \pi$. The mass of the earring is $\pi\left(R^2 - r^2\right)$, and $\overline{y} = 0$.

$$\overline{x} = \frac{1}{\pi\left(R^2-r^2\right)}\int_0^{\pi}\int_{2r\,\cos\theta}^{2R\,\cos\theta} a^2\,\cos\theta\,da\,d\theta = \frac{8\left(R^3-r^3\right)}{3\pi\left(R^2-r^2\right)}\int_0^{\pi}\cos^4\theta\,d\theta = \frac{R^2+Rr+r^2}{R+r}.$$

With the origin instead at the center of the large circle, the equations of the circles are $x^2+y^2=R^2$ and $\left(x-(R-r)\right)^2+y^2=r^2$. Then the moment around the y-axis of the large circle is zero, so to compute $\overline{x}$ for the earring, we compute $\overline{x} = \dfrac{1}{\pi\left(R^2-r^2\right)}\left(0-\displaystyle\int_{R-2r}^{R}\int_{-\sqrt{r^2-(x-(R-r))^2}}^{\sqrt{r^2-(x-(R-r))^2}} x\,dy\,dx\right) =$ $-\dfrac{1}{\pi\left(R^2-r^2\right)}\displaystyle\int_{R-2r}^{R} x\sqrt{r^2-\left(x-(R-r)\right)^2}\,dx = -\dfrac{r^2}{R+r}$.

b. With the origin at the center of the large circle, point P is $(R-2r,\,0)$, so we want $R-2r = -\dfrac{r^2}{R+r}$. Multiplying both sides by $R+r$, dividing by r^2 and letting $x = \dfrac{R}{r}$, we find that $x = \dfrac{1}{x-1}$ or $x^2-x-1=0$, which has roots $\dfrac{1\pm\sqrt{5}}{2}$. Because $R,\,r>0$, it must be the positive value, so $x = \dfrac{1+\sqrt{5}}{2}$ satisfies the condition.

16.7 Change of Variables in Multiple Integrals

16.7.1 It is the square with vertices (0, 0), (2, 0), (2, 2), and (0, 2).

16.7.2 The Jacobian is $J(u,\,v) = \begin{vmatrix}\dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v}\\ \dfrac{\partial h}{\partial u} & \dfrac{\partial h}{\partial v}\end{vmatrix}$.

16.7.3 The Jacobian is $\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix} = -2$, so $\displaystyle\iint\limits_R f(x,\,y)\,dA = \iint\limits_S f(u+v,\,u-v)\,|J(u,\,v)|\,dA =$ $\displaystyle\iint\limits_S 2f(u+v,\,u-v)\,dA = \int_0^1\int_0^1 2f(u+v,\,u-v)\,dv\,du.$

16.7.4 It is the cube of side length $\dfrac{1}{2}$ in the first octant with one vertex at the origin.

16.7.5 $u = \dfrac{x}{2}$, so $0 \le u \le 1$ means $0 \le \dfrac{x}{2} \le 1$ or $0 \le x \le 2$. $v = 2y$, so $0 \le v \le 1$ means $0 \le 2y \le 1$, or $0 \le y \le \dfrac{1}{2}$. Thus the image is the region $\left\{(x,y)\,:\,0\le x\le 2,\,0\le y\le\dfrac{1}{2}\right\}$, which is a rectangle in the first quadrant with vertices (0, 0), (2, 0), $\left(2,\,\dfrac{1}{2}\right)$, $\left(0,\,\dfrac{1}{2}\right)$.

16.7.6 $0 \le u \le 1 \Rightarrow 0 \le -x \le 1 \Rightarrow 0 \ge x \ge -1$, and $0 \le v \le 1 \Rightarrow 0 \le -y \le 1 \Rightarrow 0 \ge y \ge -1$, so we get a unit square in the third quadrant with one vertex at the origin.

16.7.7 Solving for u and v gives $u = x+y$ and $v = x-y$. The region is thus the square bounded by the lines $x+y=0$, $x+y=1$, $x-y=0$, and $x-y=1$.

16.7.8 Solving for u and v gives $u = \dfrac{y}{2}$ and $v = x - y$. The region is thus the parallelogram bounded by the lines $y = 0$, $y = 2$, $x - y = 0$, and $x - y = 1$.

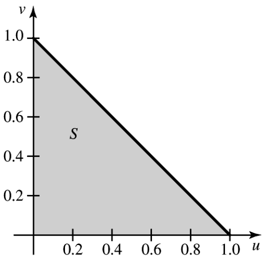
16.7.9 From (0, 0) to (1, 0), $x = u^2$ and $y = 0$, so this traces out the segment from (0, 0) to (1, 0). Similarly, from (0, 1) to (0, 0), $x = -v^2$ and $y = 0$, so this traces out the segment from $(-1, 0)$ to (0, 0). From (1, 0) to (1, 1), $(x, y) = \left(1 - v^2, 2v\right)$, so that (x, y) satisfies the equation $y^2 = 4 - 4x$. From (1, 1) to (0, 1), $(x, y) = \left(u^2 - 1, 2u\right)$, so that (x, y) satisfies the equation $y^2 = 4 + 4x$. So the result is the region enclosed by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$.

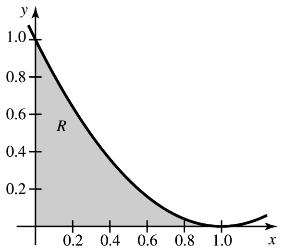
16.7.10 This is the result of interchanging x and y in the previous exercise, so the result is the region enclosed by the y-axis and the parabolas $x^2 = 4 - 4y$ and $x^2 = 4 + 4y$.

16.7.11 As (u, v) goes from (0, 0) to (1, 0), (x, y) goes from (0, 0) to (1, 0) along the x-axis. As (u, v) goes from (1, 0) to (1, 1), (x, y) traces out the upper half of the unit circle. As (u, v) goes from (1, 1) to (0, 1), (x, y) goes from $(-1, 0)$ to (0, 0) along the x-axis. Finally, as (u, v) goes from (0, 1) to (0, 0), (x, y) is stationary at the origin. Thus the region swept out is the upper half of the unit circle.

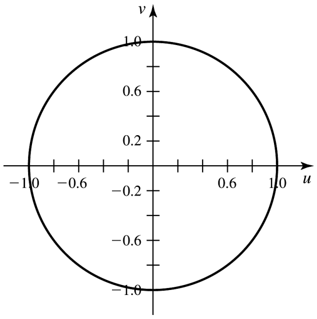
16.7.12 As (u, v) goes from (0, 0) to (1, 0), (x, y) is stationary at the origin. As (u, v) goes from (1, 0) to (1, 1), (x, y) traces out the segment from (0, 0) to $(0, -1)$. As (u, v) goes from (1, 1) to (0, 1), (x, y) traces out the right half of the unit circle. Finally, as (u, v) goes from (0, 1) to (0, 0), (x, y) traces out the segment from (1, 0) to (0, 0). The result is the right half of the unit circle.

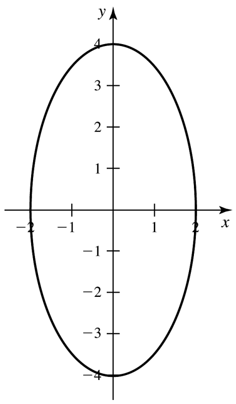
16.7.13 $R = \left\{(x, y) : \sqrt{y} \le 1 - x,\ x \ge 0,\ y \ge 0\right\} = \left\{(x, y) : y \le (1 - x)^2,\ x \ge 0,\ y \ge 0\right\}$.



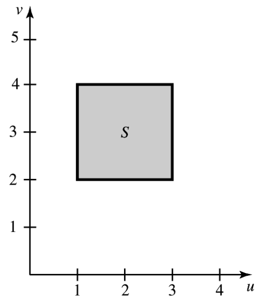


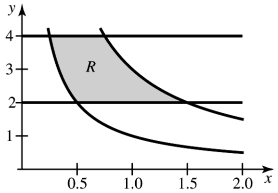
16.7.14 $R = \left\{(x, y) : \left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{4}\right)^2 \le 1\right\} = \left\{(x, y) : 4x^2 + y^2 \le 16\right\}$.



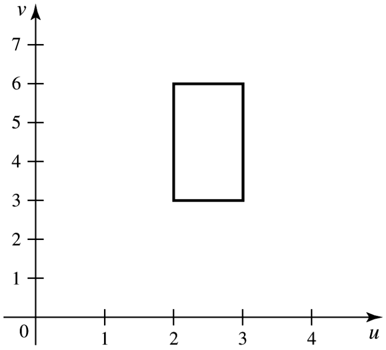


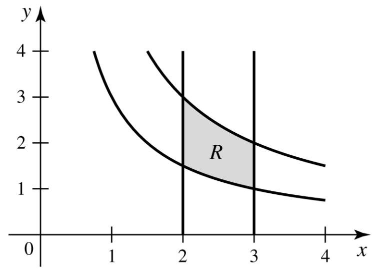
16.7.15 $R = \{(x, y) : 1 \le xy \le 3,\ 2 \le y \le 4\} = \left\{(x, y) : 2 \le y \le 4,\ \dfrac{1}{y} \le x \le \dfrac{3}{y}\right\}$.





16.7.16 $R = \{(x,\, y) : 2 \le x \le 3,\, 3 \le xy \le 6\} = \left\{(x,\, y) : 2 \le x \le 3,\, \dfrac{3}{x} \le y \le \dfrac{6}{x}\right\}.$





16.7.17 $J = \begin{vmatrix} 3 & 0 \\ 0 & -3 \end{vmatrix} = -9.$

16.7.18 $J = \begin{vmatrix} 0 & 4 \\ -2 & 0 \end{vmatrix} = 8.$

16.7.19 $J = \begin{vmatrix} 2v & 2u \\ 2u & -2v \end{vmatrix} = -4\left(u^2 + v^2\right).$

16.7.20 $J = \begin{vmatrix} \cos(\pi v) & -u\pi \sin(\pi v) \\ \sin(\pi v) & u\pi \cos(\pi v) \end{vmatrix} = u\pi.$

16.7.21 $J = \begin{vmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \end{vmatrix} = -1.$

16.7.22 $J = \begin{vmatrix} v^{-1} & -uv^{-2} \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}.$

16.7.23 Add the two equations to get $3x = u + v$, so $x = \dfrac{u+v}{3}$, and $y = u - \dfrac{u+v}{3} = \dfrac{2u - v}{3}$. The Jacobian is then $J = \begin{vmatrix} \dfrac{1}{3} & \dfrac{1}{3} \\ \dfrac{2}{3} & -\dfrac{1}{3} \end{vmatrix} = -\dfrac{1}{3}$

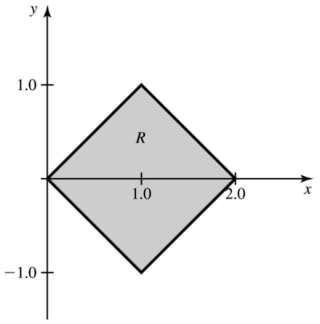
16.7.24 $x = v$, so $y = \dfrac{u}{v}$. The Jacobian is then $J = \begin{vmatrix} 0 & 1 \\ v^{-1} & -uv^{-2} \end{vmatrix} = -\dfrac{1}{v}.$

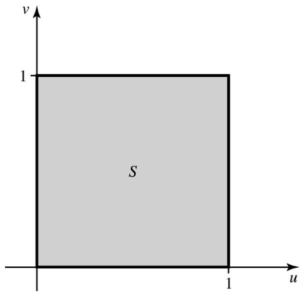
16.7.25 Add twice the second equation to the first to get $-y = u + 2v$ so that $y = -u - 2v$, and then $x = -u - 3v$. The Jacobian is $J = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = -1$.

16.7.26 Subtract twice the second equation from the first to get $u - 2v = -5x$ so that $x = \dfrac{2v - u}{5}$, and then $2y = v - 3x = v - 3\dfrac{2v - u}{5} = \dfrac{3u - v}{5}$, so that $y = \dfrac{3u - v}{10}$. The Jacobian is $J = \begin{vmatrix} -\dfrac{1}{5} & \dfrac{2}{5} \\ \dfrac{3}{10} & -\dfrac{1}{10} \end{vmatrix} = -\dfrac{1}{10}$.

16.7.27

a.





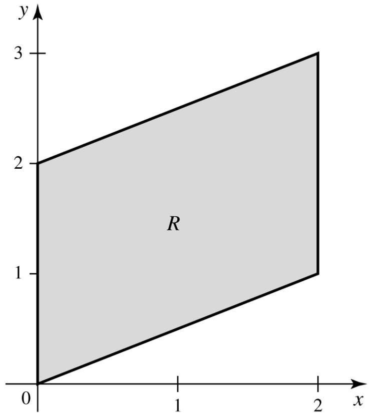
b. The region S is the first-quadrant unit square with one vertex at the origin, so the limits of integration are $0 \le u,\ v \le 1$.

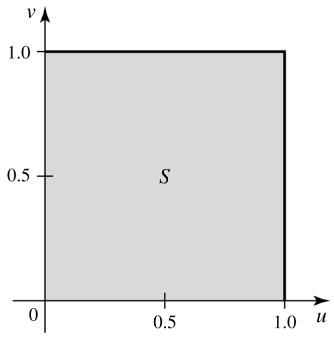
c. The Jacobian of the transformation is $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$.

d. $\displaystyle\iint_R xy\ dA = \int_0^1 \int_0^1 (u + v) \cdot (u - v)\,|-2|\ dv\,du = 2\int_0^1 \int_0^1 \left(u^2 - v^2\right)\ dv\,du = 2\int_0^1 \left(u^2 - \frac{1}{3}\right) du = 0.$

16.7.28

a. $S = \{(u,\ v) : 0 \le 2u \le 2,\ 2u \le 4v + 2u \le 2u + 4\} = \{(u,\ v) : 0 \le u \le 1,\ u \le 2v + u \le u + 2\} = \{(u,\ v) : 0 \le u \le 1,\ 0 \le v \le 1\}$.





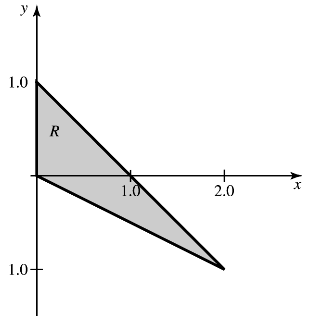
b. From the above computation, the new integration limits are $0 \le u,\ v \le 1$.

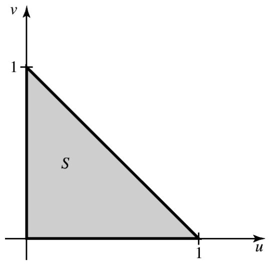
c. The Jacobian is $\begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8$.

d. $\displaystyle\iint_R x^2y\; dA = 64\int_0^1\int_0^1 u^2\,(2v+u)\;dv\,du = 64\int_0^1 \left(u^2+u^3\right)du = \frac{112}{3}$.

16.7.29

a. $S = \{(u,\,v) : 0 \le 2u \le 2,\ -u \le v-u \le 1-2u\} = \{(u,\,v) : 0 \le u \le 1,\ 0 \le v \le 1-u\}$





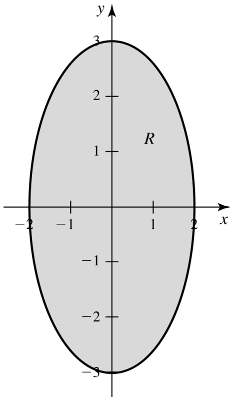
b. From the above, the new limits of integration are $0 \le u \le 1,\ 0 \le v \le 1-u$.

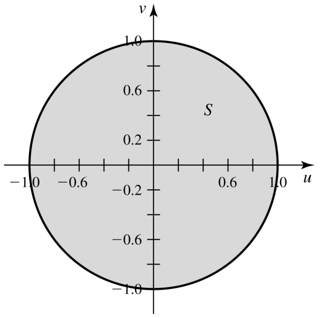
c. The Jacobian is $\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = 2$.

d. $\displaystyle\iint_R x^2\sqrt{x+2y}\;dA = 8\int_0^1\int_0^{1-u} u^2\sqrt{2v}\,dv\,du = \frac{16\sqrt{2}}{3}\int_0^1 u^2\,(1-u)^{3/2}\;\,du = \frac{256\sqrt{2}}{945}$.

16.7.30

a. Under the given transformation, $9x^2+4y^2=36$ becomes $9\,(2u)^2+4\,(3v)^2 = 36u^2+36v^2 = 36$, which is the unit circle $u^2+v^2=1$.





b. The limits of integration over the unit circle are $-1 \le u \le 1,\ -\sqrt{1-u^2} \le v \le \sqrt{1-u^2}$.

c. The Jacobian is $\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$.

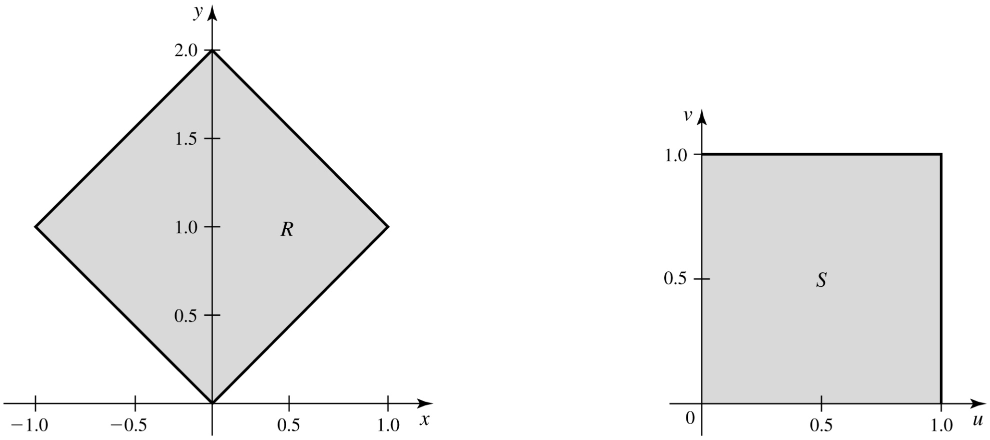
d. $\displaystyle\iint_R xy\,dA = 36\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} uv\,dv\,du = 0.$

16.7.31 Use the substitution $y = v$, $x = u + v$. Then $S = \{(u,\,v) : 0 \le v \le 1,\, v \le u + v \le v + 2\} = \{(u,\,v) : 0 \le v \le 1,\, 0 \le u \le 2\}$.

The Jacobian of this transformation is $J\,(u,\,v) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$ so that

$$\int_0^1\int_y^{y+2}\sqrt{x-y}\,dx\,dy = \int_0^1\int_0^2\sqrt{u}\,du\,dv = \frac{2}{3}2^{3/2} = \frac{4\sqrt{2}}{3}.$$

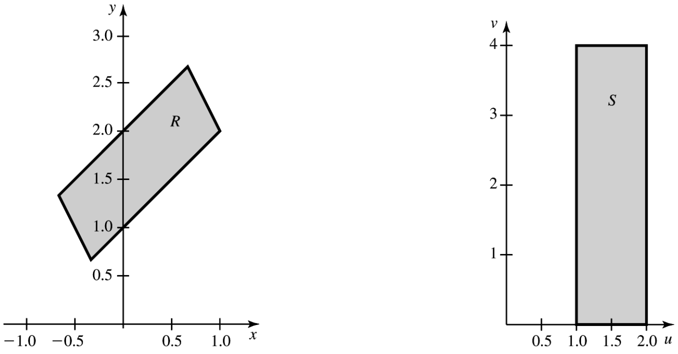
16.7.32 Use the substitution $x = u - v$, $y = u + v$. Then S becomes the unit square in the first quadrant with one vertex at the origin, as can be seen by tracing each edge.



The Jacobian of this transformation is $J\,(u,\,v) = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$ so that

$$\iint_R \sqrt{y^2 - x^2}\;dA = 2\int_0^1\int_0^1\sqrt{(u+v)^2 - (u-v)^2}\;dv\,du = 2\int_0^1\int_0^1\sqrt{4uv}\,dv\,du = \frac{16}{9}.$$

16.7.33 The points of intersection of the given lines are $\left(-\frac{1}{3},\,\frac{2}{3}\right)$, $\left(-\frac{2}{3},\,\frac{4}{3}\right)$, $\left(\frac{2}{3},\,\frac{8}{3}\right)$, and $(1,\,2)$. Setting $u = y - x$, $v = y + 2x$ sends these points into the rectangle with vertices $(1,\,0)$, $(2,\,0)$, $(2,\,4)$, and $(1,\,4)$.

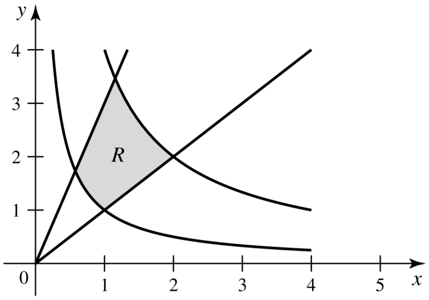


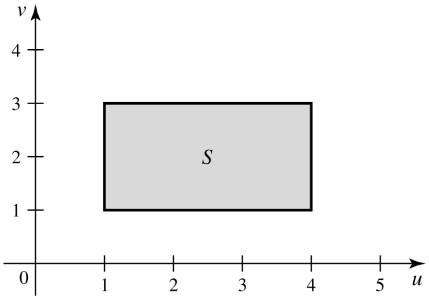
So use the transformation (solving for x, y) $x = \dfrac{v-u}{3}$, $y = \dfrac{v+2u}{3}$, $J(u, v) = \begin{vmatrix} -\dfrac{1}{3} & \dfrac{1}{3} \\ \dfrac{2}{3} & \dfrac{1}{3} \end{vmatrix} = -\dfrac{1}{3}$, so that

$$\iint_R \left(\frac{y-x}{y+2x+1}\right)^4 dA = \frac{1}{3}\int_1^2 \int_0^4 \left(\frac{u}{v+1}\right)^4 dv\,du = \frac{1}{3}\int_1^2 \frac{124}{375}u^4\,du = \frac{3844}{5625}.$$

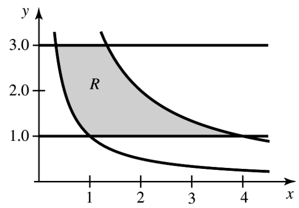
16.7.34 Use the transformation $u = xy$, $v = \dfrac{y}{x}$, so that $x = \sqrt{\dfrac{u}{v}}$, $y = \sqrt{uv}$. The Jacobian is

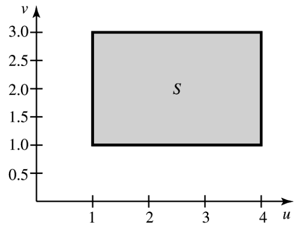
$\begin{vmatrix} \dfrac{u^{-1/2}v^{-1/2}}{2} & -\dfrac{u^{-1/2}v^{-3/2}}{2} \\ \dfrac{u^{-1/2}v^{1/2}}{2} & \dfrac{u^{1/2}v^{-1/2}}{2} \end{vmatrix} = \dfrac{1}{2v}$, and the new region is the square with vertices (1, 1), (1, 3), (4, 1), and (4, 3), so that $\displaystyle\iint_R e^{xy}\,dA = \int_1^4 \int_1^3 e^u \frac{1}{2v}\,dv\,du = \frac{\ln 3}{2}\int_1^4 e^u\,du = \frac{\ln 3}{2}\left(e^4 - e\right)$.





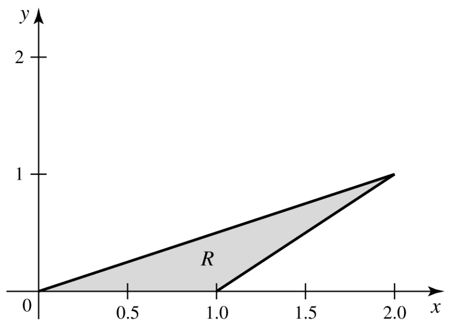
16.7.35 Use the transformation $u = xy$, $v = y$, so that $x = \dfrac{u}{v}$, $y = v$. The new region is the square with vertices (1, 1), (4, 1), (4, 3), and (1, 3). The Jacobian of the transformation is $\begin{vmatrix} \dfrac{1}{v} & -\dfrac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$.

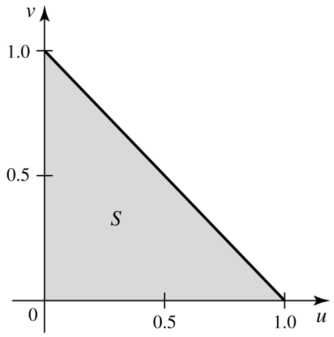




Thus $\displaystyle\iint_R xy\,dA = \int_1^4 \int_1^3 \frac{u}{v}\,dv\,du = \int_1^4 u\ln 3\,du = \frac{15\ln 3}{2}$.

16.7.36 R is the interior of the triangle with vertices (0, 0), (1, 0), and (2, 1). Looking at the form of the integrand, try $u = x - 2y$, $v = y$ so that $x = u + 2v$, $y = v$. Then S is the triangle bounded by (0, 0), (1, 0), and (0, 1).





The Jacobian of this transformation is $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$, and the integral is equal to $\displaystyle\iint_R (x-y)\sqrt{x-2y}\,dA = \int_0^1 \int_0^{1-u} (u+v)\sqrt{u}\,dv\,du = \frac{4}{21}$.

16.7.37 $J(u,\,v,\,w) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2.$

16.7.38 $J(u,\,v,\,w) = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -4.$

16.7.39 $J(u,\,v,\,w) = \begin{vmatrix} 0 & w & v \\ w & 0 & u \\ 2u & -2v & 0 \end{vmatrix} = 2w\left(u^2 - v^2\right).$

16.7.40 Solving for x, y, and z gives $x = \dfrac{u+v+w}{2}$, $y = \dfrac{-u+v+w}{2}$, $z = \dfrac{u-v+w}{2}$, so that $J(u,\,v,\,w) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \dfrac{1}{2}$.

16.7.41 Let $u = y - x$, $v = z - y$, $w = z$; then $x = w - v - u$, $y = w - v$, $z = w$ and the Jacobian is $J(u,\,v,\,w) = \begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$. The new region is clearly $0 \le u \le 2$, $0 \le v \le 1$, $0 \le w \le 3$, so we obtain $\displaystyle\iiint_D xy\,dV = \int_0^2 \int_0^1 \int_0^3 (w-v-u)(w-v)\ dw\,dv\,du = 5$.

16.7.42 Let $u = y - 2x$, $v = z - 3y$, $w = z - 4x$. Then $x = -\dfrac{3}{2}u - \dfrac{1}{2}v + \dfrac{1}{2}w$, $y = -2u - v + w$, $z = -6u - 2v + 3w$, so the Jacobian is $J(u,\,v,\,w) = \begin{vmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & -1 & 1 \\ -6 & -2 & 3 \end{vmatrix} = \dfrac{1}{2}$, and the new region of integration is $0 \le u \le 1$, $0 \le v \le 1$, $0 \le w \le 3$. Thus $\displaystyle\iiint_D dV = \int_0^3 \int_0^1 \int_0^1 \frac{1}{2}\,du\,dv\,dw = \frac{3}{2}$.

16.7.43 Using the given change of variables, the Jacobian is $\begin{vmatrix} 4\cos v & -4u\sin v & 0 \\ 2\sin v & 2u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = 8u$, and the new range of integration is $0 \le u \le 1$, $0 \le v \le 2\pi$, $0 \le w \le 16 - 16u^2$. Thus

$$\iiint\limits_D z\,dV = \int_0^1 \int_0^{2\pi} \int_0^{16-16u^2} 8uw\,dw\,dv\,du = \int_0^1 2048\pi\,u\left(u^2-1\right)^2\,du = \frac{1024\pi}{3}.$$

(Note that this change of variables is essentially cylindrical coordinates, with an adjustment for the fact that we are integrating over a paraboloid with differently sized axes.)

16.7.44 Using the given change of variables, the Jacobian is $\begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$. In uvw-coordinates, the equation becomes $u^2+v^2+w^2=1$, so the integral is half the volume of the unit sphere, or $\frac{2\pi}{3}$. Multiplying by the Jacobian gives 4π.

16.7.45

a. True. This is because $g\left(u,\,v\right)$ and $h\left(u,\,v\right)$ are of the form $au+bv$ so their partial derivatives are constants.

b. True. This is because the transformation maps lines to lines.

c. True. It simply halves lengths and reflects in the x-axis.

16.7.46 Expand the determinant about the third row. $J\left(r,\,\theta,\,\mathrm{Z}\right) = \begin{vmatrix} \cos\theta & -r\,\sin\theta & 0 \\ \sin\theta & r\,\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$.

16.7.47 Expand the determinant about the third row. $J\left(\rho,\varphi,\theta\right) = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$
$= \cos\varphi\left(\rho^2\sin\varphi\,\cos\varphi\cos^2\,\theta + \rho^2\sin\varphi\,\cos\varphi\sin^2\theta\right) + \rho\,\sin\varphi\left(\rho\,\sin^2\varphi\,\cos^2\,\theta + \rho\,\sin^2\varphi\,\sin^2\,\theta\right) =$
$\rho^2\sin\varphi\,\cos^2\,\varphi + \rho^2\sin^3\varphi = \rho^2\sin\varphi$.

16.7.48 Under T, the ellipse becomes $u^2+v^2=1$, the unit circle, with area π. The Jacobian of the transformation is ab, so the ellipse area is πab.

16.7.49 This integral is four times the integral over the first quadrant. This is $4ab\displaystyle\int_0^1\int_0^{\sqrt{1-u^2}} abuv\,dv\,du = 2a^2b^2\int_0^1 u\left(1-u^2\right)du = \frac{a^2b^2}{2}$.

16.7.50 The area of R is given by Exercise 48 $(\frac{\pi ab}{2})$, and $\overline{x}=0$ by symmetry. To find $\overline{y}$, we compute using the same change of variables (where R^+ is the upper half) $\displaystyle\frac{1}{(\pi ab)/2}\int_{-1}^1\int_0^{\sqrt{1-u^2}} ab^2v\,dv\,du = \frac{b}{\pi}\int_{-1}^1\left(1-u^2\right)\,du = \frac{4b}{3\pi}$.

16.7.51 The distance of a point from the origin is $\sqrt{x^2+y^2}=\sqrt{a^2u^2+b^2v^2}$, so the average squared distance is $\dfrac{1}{\pi ab}\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\left(a^2u^2+b^2v^2\right)\,dv\,du=\dfrac{a^2+b^2}{4}$.

16.7.52 The distance of a point in the upper half of R and the x-axis is simply the point's y-coordinate, so the average distance is $\dfrac{1}{(\pi ab)/2}\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-u^2}} ab^2v\,dv\,du=\dfrac{4b}{3\pi}$.

16.7.53 Under the given transformation, the equation becomes $u^2+v^2+w^2=1$, the unit sphere. The Jacobian of the transformation is abc, so the volume of D is abc times the volume of the unit sphere, or $\dfrac{4}{3}\pi abc$.

16.7.54 This integral is eight times the integral over the first octant, so it is
$$8abc\int_0^1\int_0^{\sqrt{1-u^2}}\int_0^{\sqrt{1-u^2-v^2}} abcuvw\,dw\,dv\,du=\frac{a^2b^2c^2}{6}.$$

16.7.55 The mass of the upper half is $\dfrac{2\pi abc}{3}$ by Problem 53, and $\overline{x}=\overline{y}=0$ by symmetry.
$$\overline{z}=\frac{1}{(2\pi abc)/3}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_0^{\sqrt{1-u^2-v^2}} abc^2w\ dw\,dv\,du=\frac{3c}{2\pi}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\left(1-u^2-v^2\right)\,dv\,du=\frac{3c}{8}.$$

16.7.56 The distance of a point from the origin is $\sqrt{x^2+y^2+z^2}=\sqrt{a^2u^2+b^2v^2+c^2w^2}$, so the average squared distance is $\dfrac{3}{4\pi abc}\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}}\left(a^2u^2+b^2v^2+c^2w^2\right)abc\,dw\,dv\,du$

To evaluate this integral, switch to cylindrical coordinates; we obtain
$$\frac{3}{2\pi}\int_0^1\int_0^{2\pi}\int_0^{\sqrt{1-r^2}} r\left(a^2r^2\cos^2\theta+b^2r^2\sin^2\theta+c^2z^2\right)dz\,d\theta\,dr=$$
$$\frac{1}{2\pi}\int_0^1\int_0^{2\pi} r\sqrt{1-r^2}\left(c^2-c^2r^2+3b^2r^2+3a^2r^2\cos^2\theta-3b^2r^2\cos^2\theta\right)\,d\theta\,dr=\frac{a^2+b^2+c^2}{5}.$$

16.7.57

a. The line $u=a$ is the set $\{(a,\,v)\}$, which maps under T to $\left\{\left(a^2-v^2,\,2av\right)\right\}$. But points of this form satisfy the equation $a^2-x=\dfrac{1}{4a^2}y^2$, or $x=-\dfrac{1}{4a^2}y^2+a^2$, which is a parabola opening in the negative x direction. The vertex of the parabola $x=Ay^2+By+C$ is at $\left(C-\dfrac{B^2}{4A},\,-\dfrac{B}{2A}\right)$, which for this parabola is $\left(a^2,\,0\right)$, which lies on the positive x-axis.

b. The line $v=b$ is the set $\{(u,\,b)\}$, which maps under T to $\left\{\left(u^2-b^2,\,2ub\right)\right\}$. But points of this form satisfy the equation $b^2+x=\dfrac{1}{4b^2}y^2$, or $x=\dfrac{1}{4b^2}y^2-b^2$, which is a parabola opening in the positive x direction. The vertex of the parabola $x=Ay^2+By+C$ is at $\left(C-\dfrac{B^2}{4A},\,-\dfrac{B}{2A}\right)$, which for this parabola is $\left(-b^2,\,0\right)$, which lies on the negative x-axis.

c. $J(u,\,v)=\begin{vmatrix}2u & -2v\\ 2v & 2u\end{vmatrix}=4\left(u^2+v^2\right)$.

d. Use the transformation $x=v^2-u^2$, $y=2uv$. The Jacobian of this transformation is $\begin{vmatrix}-2u & 2v\\ 2v & 2u\end{vmatrix}=-4\left(u^2+v^2\right)$. $x=4-\dfrac{1}{16}y^2$ corresponds to the lines $u=\pm2$ (by an analysis similar to part (a)), while

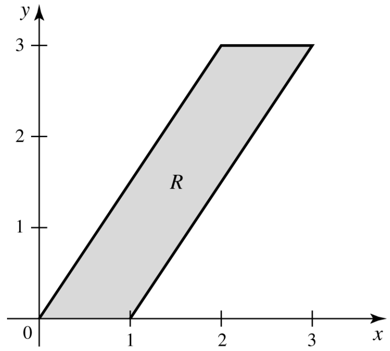
$x = \frac{1}{4}y^2 - 1$ corresponds to the line $v = \pm 1$, so that the rectangle with vertices $(-2, -1)$, $(-2, 1)$, $(2, 1)$, $(2, -1)$ is mapped to the region bounded by the parabolas. However, note that the area of that rectangle to the left of the v-axis and the area to the right of the v-axis are each mapped onto that region (that is, the map is $2 : 1$; it is a simple computation to see that (a, b) and $(-a, -b)$ map to the same xy point). Thus to determine the area of the original region, we want to integrate over only (say) the right half of the rectangle. So the area is $\int_0^2 \int_{-1}^1 4\left(u^2+v^2\right)\,dv\,du = \frac{80}{3}$.

e. As in part (d), use the transformation $x = v^2 - u^2$, $y = 2uv$, with Jacobian $-4\left(u^2+v^2\right)$. The correspondences are: $x = \frac{y^2}{4} - 1 \Leftrightarrow u = 1x = \frac{y^2}{64} - 16 \Leftrightarrow u = 4x = 9 - \frac{y^2}{36} \Leftrightarrow v = \pm 3x = 4 - \frac{y^2}{16} \Leftrightarrow v = \pm 2$. Because we are looking at the positive portion of the bounded piece, the new range of integration is $1 \le u \le 4$, $2 \le v \le 3$. Thus the area is $\int_1^4 \int_2^3 4\left(u^2+v^2\right)\,dv\,du = 160$.

f. This simply reverses the roles of x and y in parts (a) and (b). Thus lines $u = a$ in the uv plane map to parabolas in the xy plane that open in the negative y direction with vertices on the positive y-axis, while lines $v = b$ in the uv plane map to parabolas in the xy plane that open in the positive y direction with vertices on the positive y-axis.

16.7.58

a. Here is the effect of the shear transformation with $a = 1$, $b = 2$, $c = 3$ ($x = u + 2v$; $y = 3v$):



The y coordinates are all multiplied by c, while the x coordinates are gotten by expanding by a and then adding b times the y coordinate. This has the effect of pushing the square further and further to the right as the y coordinate increases.

b. $J(u, v) = \begin{vmatrix} a & b \\ 0 & c \end{vmatrix} = ac$

c. R is a parallelogram with vertices $(0, 0)$, $(a, 0)$, $(a + b, c)$, and (b, c), so it has base a and height c, so has area ac (which is exactly what we would expect because the Jacobian is ac).

d. By symmetry, $\overline{y} = \frac{c}{2}$ and $\overline{x} = \frac{a+b}{2}$.

e. The analogous shear transformation is $x = au$, $y = bu + cv$.

16.7.59

a. The z coordinate remains the same, while the other coordinates are stretched by an amount depending on all three coordinates. The result is thus a parallelepiped with its base in the xy-plane.

b. $J(u,\,v,\,w) = \begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{vmatrix} = ad$

c. D has a height of 1 Because S and the z coordinate remains unchanged. Its base is in the xy plane, and is the parallelogram that is the result of the shear transformation $x = au + by$, $y = dv$ (set $w = 0$ in the original equations to see this). The area of this parallelogram, from Problem 58, is ad, so the total volume is ad (again what we would expect from the Jacobian).

d. By symmetry, the center of mass is $\left(\dfrac{a+b+c}{2},\, \dfrac{d+e}{2},\, \dfrac{1}{2}\right)$.

16.7.60

a. $J(u,\,v) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

b. R is the parallelogram with vertices $(0,\,0)$, $(a,\,c)$, $(b,\,d)$, and $(a+b,\,c+d)$. Compute its area by thinking of $(a,\,c)$ and $(b,\,d)$ as vectors in 3-space with zero z coordinate; then the area of the parallelogram is the cross-product $\left|\left\langle a,\,c,\,0\right\rangle \times \left\langle b,\,d,\,0\right\rangle\right| = \left|\left\langle 0,\,0,\,ad-bc\right\rangle\right| = |ad-bc| = |J(u,\,v)|$

c. Let $P = (q,\,r)$ and let ℓ be the line given by $(q+et,\,r+ft)$; $0 \le t \le 1$ so that $Q = (q+e,\,r+f)$. Then $T(P) = (aq+br,\,cq+dr)$, while $T(Q) = (aq+ae+br+bf,\,cq+ce+dr+df)$. Then
$T((q+et,\,r+ft)) = (aq+br,\,(ae+bf)\,t, cq+dr+(ce+df)\,t) =$
$(aq+br,\,cq+dr) + (ae+bf,\,ce+df)\,t = T(P) + (ae+bf,\,ce+df)\,t$. Now, when $t = 1$, this is $T(Q)$, so that ℓ maps to a line from $T(P)$ to $T(Q)$.

d. By moving and reflecting the x and y axes if necessary, we may assume that the vertices of the parallelogram are $(0,\,0)$, $(r,\,0)$, $(s,\,t)$, and $(r+s,\,t)$ for $r,\,s,\,t > 0$. The area of this parallelogram is then rt (base times height). Then the vertices of S are $(0,\,0)$, $(ar,\,cr)$, $(as+bt,\,cs+dt)$, and $(ar+as+bt,\,cr+cs+dt)$. Again regarding $(ar,\,cr)$ and $(as+bt,\,cs+dt)$ as vectors in 3-space, the area of the parallelogram is $\left|\left\langle ar, cr, 0\right\rangle \times \left\langle as+bt, cs+dt, 0\right\rangle\right| = \left|\left\langle 0, 0, ar\,(cs+dt) - cr\,(as+bt)\right\rangle\right| =$
$\left|\left\langle 0, 0, (ad-bc)\,rt\right\rangle\right| = |J(u,v)|\,rt = |J(u,v)|\,\text{area}\,(S)$.

16.7.61

a. This is just the definition of the transformation T, which is given by $x = g(u,\,v)$, $y = h(u,\,v)$, so that the image of $(x,\,y)$ is $T((x,\,y)) = (g(u,\,v),\,h(u,\,v))$. Apply this to the coordinates of the points O, P, Q.

b. The Taylor expansions of g and h at $(0,\,0)$ are

$$g(u,\,v) = g(0,\,0) + u\,g_u(0,\,0) + v\,g_v(0,\,0) + \text{ terms involving higher derivatives}$$

$$rh(u,\,v) = h(0,\,0) + u\,h_u(0,\,0) + v\,h_v(0,\,0) + \text{terms involving higher derivatives.}$$

Substituting the two points $(\triangle u,\,0)$ and $(0,\,\triangle v)$ into these equations and considering only the terms up through the first derivative gives the desired result.

c. From part (b),

$$P' \approx (g(0,\,0) + g_u(0,\,0)\,\triangle u,\, h(0,\,0) + h_u(0,\,0)\,\triangle u)$$
$$Q' \approx (g(0,\,0) + g_v(0,\,0)\,\triangle v,\, h(0,\,0) + h_v(0,\,0)\,\triangle v),$$

so that $\overrightarrow{O'P'} \approx \triangle u\left\langle g_u(0,\,0),\, h_u(0,\,0)\right\rangle$ and $\overrightarrow{O'Q'} \approx \triangle v\left\langle g_v(0,\,0),\, h_v(0,\,0)\right\rangle$. The area of the parallelogram determined by $\overrightarrow{O'P'}$ and $\overrightarrow{O'Q'}$ is the magnitude of the cross product of these vectors (considered as vectors in 3-space with zero z coordinate), so the area of the resulting region is approximately

$$\begin{aligned}&\left|\triangle u\left\langle g_u(0,0), h_u(0,0), 0\right\rangle \times \triangle v\left\langle g_v(0,0), h_v(0,0), 0\right\rangle\right|\\&= \triangle u \triangle v\left|\left\langle g_u(0,0), h_u(0,0), 0\right\rangle \times \left\langle g_v(0,0), h_v(0,0), 0\right\rangle\right|\\&= \triangle u\triangle v\left|g_u(0,0)\, h_v(0,0) - g_v(0,0)\, h_u(0,0)\right|\\&= \left|J(u, v)\right| \triangle u \triangle v.\end{aligned}$$

d. Because the area of R is $\triangle u \triangle v$, the ratio is approximately $|J(u, v)|$.

16.7.62

a. Let $n_1 = \langle at, bt, 0\rangle$, $\mathrm{n}_2 = \langle ct, 0, dt\rangle$, $\mathrm{n}_3 = \langle 0, et, ft\rangle$ for $t \in R$. Then n_1 is normal to the first pair, n_2 to the second pair, and n_3 to the third pair.

b. The triple scalar product of any three vectors is the volume of the parallelepiped that they determine. This volume is zero if and only if they are coplanar.

c. The triple scalar product is $\langle at, bt, 0\rangle \cdot (\langle ct, 0, dt\rangle \times \langle 0, et, ft\rangle) = \langle at, bt, 0\rangle \cdot t^2\langle -de, cf, ce\rangle = t^3(-ade - bcf)$, so the normal vectors are coplanar if and only if $ade + bcf = 0$.

d. If the three vectors are coplanar, then the cross product of any two of them is perpendicular to the plane they determine. Thus, for example, from part (c), $N = t(de, cf, -ce)$ is normal to the plane. So any line in the direction of N is parallel to all six planes. By choosing the line appropriately, we can ensure that it does not lie in any of the six planes, and thus it does not intersect any of them. Thus the six planes do not form a bounded region if $ade + bcf = 0$.

e. The Jacobian is $\begin{vmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & f\end{vmatrix} = -ade - bcf$. The Jacobian is zero if R is unbounded.

Chapter Sixteen Review

1

a. False. For example, if $g(x, y) = 2$, then $\int_c^d \int_a^b 2\, dx\, dy = 2(b-a)(d-c)$ while

$$\left(\int_c^d 2\, dy\right)\left(\int_a^b 2\, dx\right) = 4(b-a)(d-c).$$

b. True. The first set is the set whose φ coordinate is $\frac{\pi}{2}$; φ is the angle the line to the point makes with the z-axis, so this is the set of points in the xy-plane.

c. False. The integrand doesn't change.

d. False. For example, it maps the standard unit square into the square with vertices $(0, 0)$, $(0, -1)$, $(1, -1)$, $(1, 0)$.

2

$$\begin{aligned}\int_1^2 \int_1^4 \frac{xy}{(x^2+y^2)^2}\, dx\, dy &= \frac{1}{2}\int_1^2 \int_1^4 \frac{(2x)\, y}{(x^2+y^2)^2}\, dx\, dy = -\frac{1}{2}\int_1^2 \left.\frac{y}{x^2+y^2}\right|_1^4 dy\\&= -\frac{1}{2}\int_1^2 \left(\frac{y}{y^2+16} - \frac{y}{y^2+1}\right) dy = \frac{1}{4}\ln 17 - \frac{3}{4}\ln 2.\end{aligned}$$

3 $\displaystyle\int_1^3\int_1^{e^x}\frac{x}{y}\,dy\,dx=\int_1^3 x\,\ln y\Big|_1^{e^x}dx=\int_1^3 x^2dx=\frac{26}{3}.$

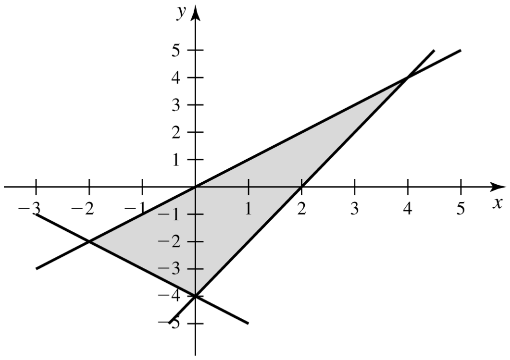
4 $\displaystyle\int_1^2\int_0^{\ln x}x^3e^y\,dy\,dx=\int_1^2 x^3e^y\Big|_0^{\ln x}dx=\int_1^2\left(x^4-x^3\right)dx=\frac{49}{20}.$

5 $\displaystyle\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}}f\left(x,\,y\right)\,dx\,dy.$

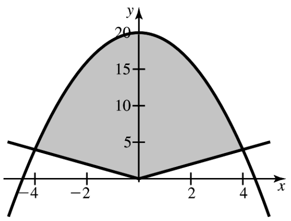
6 $\displaystyle\int_{-1}^1\int_0^{x+1}f\left(x,\,y\right)\,dy\,dx.$

7 $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}f\left(x,\,y\right)\,dy\,dx.$

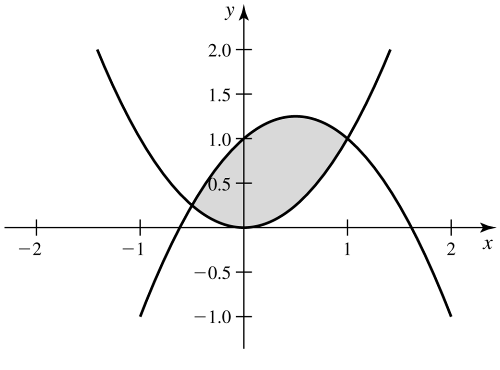
8 $\displaystyle\int_{-2}^0\int_{-x-4}^{x}1\,dy\,dx+\int_0^4\int_{2x-4}^{x}1\,dy\,dx=\int_{-2}^0\left(2x+4\right)\,dx+\int_0^4\left(4-x\right)\,dx=12.$



9 $\displaystyle 2\int_0^4\int_x^{20-x^2}1\,dy\,dx=2\int_0^4\left(20-x^2-x\right)\,dx=\frac{304}{3}.$



10 $\displaystyle\int_{-1/2}^1\int_{x^2}^{1+x-x^2}1\,dy\,dx=\int_{-1/2}^1\left(1+x-2x^2\right)\,dx=\frac{9}{8}.$



11 $\displaystyle\int_1^2\int_0^{x^{3/2}} \frac{2y}{\sqrt{x^4+1}}\,dy\,dx = \int_1^2 \frac{x^3}{\sqrt{x^4+1}}\,dx = \frac{\sqrt{17}-\sqrt{2}}{2}.$

12 $\displaystyle\int_1^4\int_0^{\sqrt{x}} x^{-1/2}e^y dy\,dx = \int_1^4 x^{-1/2}\left(e^{x^{1/2}}-1\right)dx = 2e^2-2e-2.$

13 $\displaystyle\int_0^{\pi}\int_0^{4\,\sin\theta} r^2\left(\cos\theta+\sin\theta\right)\,dr\,d\theta = \frac{64}{3}\int_0^{\pi}\sin^3\theta\,\left(\cos\theta+\sin\theta\right)\,d\theta = 8\pi.$

14 $\displaystyle\int_0^2\int_0^x \left(x^2+y^2\right)\,dy\,dx = \int_0^2 \frac{4}{3}x^3\,dx = \frac{16}{3}.$

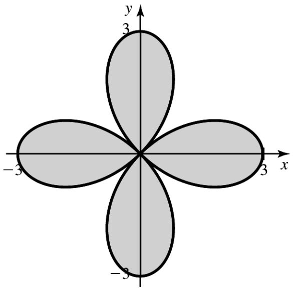
15 $\displaystyle\int_0^1\int_{y^{1/3}}^1 x^{10}\cos\left(\pi x^4 y\right)\,dx\,dy = \int_0^1\int_0^{x^3} x^{10}\cos\left(\pi x^4 y\right)\,dy\,dx = \int_0^1 \frac{x^{10}}{\pi x^4}\,\sin\left(\pi x^7\right)\,dx =$
$\displaystyle\frac{1}{\pi}\int_0^1 x^6\,\sin\left(\pi x^7\right)\,dx = \frac{2}{7\pi^2}.$

16 $\displaystyle\int_0^2\int_{y^2}^4 x^8 y\sqrt{1+x^4y^2}\,dx\,dy = \int_0^4\int_0^{\sqrt{x}} x^8 y\sqrt{1+x^4y^2}\,dy\,dx = \frac{1}{3}\int_0^4 x^4\left(-1+\left(1+x^5\right)^{3/2}\right)\,dx =$
$\displaystyle\frac{420250}{3}\sqrt{41}-\frac{5122}{25}.$

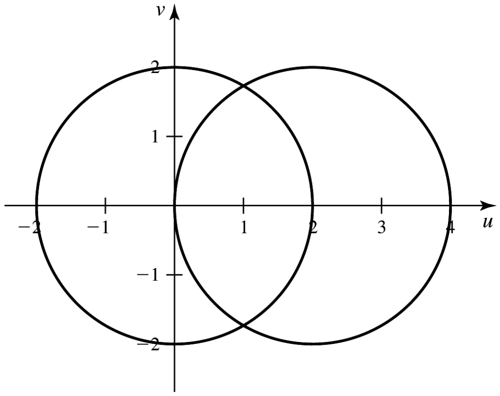
17 $\displaystyle\int_0^1\int_0^{\pi/2} 3\left(r\,\cos\theta\right)^2\,r\,\sin\theta\,r\,d\theta\,dr = \int_0^1\int_0^{\pi/2} 3r^4\cos^2\theta\,\sin\theta\,d\theta\,dr = \int_0^1 r^4\,dr = \frac{1}{5}.$

18 $\displaystyle\int_1^4\int_0^{\pi} \frac{1}{\left(1+r^2\right)^2}\,r\,d\theta\,dr = \pi\int_1^4 \frac{r}{\left(1+r^2\right)^2}\,dr = \frac{15}{68}\pi.$

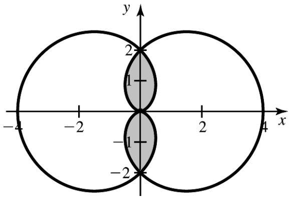
19 The area is four times the area of one leaf, so is $\displaystyle 4\int_{-\pi/4}^{\pi/4}\int_0^{3\,\cos 2\theta} r\,dr\,d\theta = 18\int_{-\pi/4}^{\pi/4}\cos^2 2\theta\,d\theta = \frac{9\pi}{2}.$



20 The area is twice the area of the region above the x-axis. The two circles intersect above the x axis at $\cos\theta = \dfrac{1}{2}$, or $\theta = \dfrac{\pi}{3}$. Thus the area of the region is $\displaystyle 2\left(\int_0^{\pi/3}\int_0^2 r\,dr\,d\theta + \int_{\pi/3}^{\pi/2}\int_0^{4\,\cos\theta} r\,dr\,d\theta\right) =$
$\displaystyle 2\left(\int_0^{\pi/3} 2\,d\theta + \int_{\pi/3}^{\pi/2} 8\cos^2\theta\,d\theta\right) = \frac{4\pi}{3}+\frac{4\pi}{3}-2\sqrt{3} = \frac{8\pi}{3}-2\sqrt{3}.$



21 The area is four times the area bounded by the cardioid $2 - 2\cos\theta$ and the y-axis for positive x (i.e. for $0 \leq \theta \leq \frac{\pi}{2}$), Because all four portions of the area are congruent. Thus it is $4\int_0^{\pi/2}\int_0^{2-2\cos\theta} r\,dr\,d\theta = 8\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta = 6\pi - 16$.



22 The area of the disk is $\pi \cdot 4^2 = 16\pi$. Using polar coordinates, the average value is then $\frac{1}{16\pi}\int_0^4\int_0^{2\pi} r\sqrt{16-r^2}\,d\theta\,dr = \frac{1}{8}\int_0^4 r\sqrt{16-r^2}\,dr = \frac{8}{3}$.

23 The volume of the cone is $\frac{1}{3}$ times the area of the base times the height. The base (at $z = 8$) is a circle of radius 4, so the volume is $\frac{1}{3}\pi \cdot 4^2 \cdot 8 = \frac{128\pi}{3}$. Use cylindrical coordinates to integrate; then the distance to the z-axis is r and the average is $\frac{1}{(128\pi)/3}\int_0^8\int_0^{z/2}\int_0^{2\pi} r^2\,d\theta\,dr\,dz = \frac{1}{64}\int_0^8 \frac{z^3}{8}\,dz = 2$.

24 $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} f(x,\,y,\,z)\;dz\,dy\,dx$.

25 $\int_0^1\int_{2y}^2\int_0^{\sqrt{z^2-4y^2}/2} f(x,y,z)\,dx\,dz\,dy$.

26 $\int_0^2\int_y^2\int_0^{9-x^2} f(x,\,y,\,z)\;dz\,dx\,dy$.

27 $\int_0^1\int_{-z}^z\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx\,dz = 2\int_0^1\int_{-z}^z \sqrt{1-x^2}\,dx\,dz = 2\int_0^1\left(z\sqrt{1-z^2} + \arcsin(z)\right)dz = \pi - \frac{4}{3}$.

28 $\int_0^\pi\int_0^y\int_0^{\sin x} dz\,dx\,dy = \int_0^\pi\int_0^y \sin x\,dx\,dy = \int_0^\pi (1-\cos y)\;dy = \pi$.

29 The region in the xy-plane can be restated as $0 \le x \le 2$, $0 \le y \le \dfrac{x}{2}$. Reordering the integral gives $\displaystyle\int_1^9 \int_0^2 \int_0^{x/2} \frac{4\,\sin\left(x^2\right)}{\sqrt{z}}\,dy\,dx\,dz = 2\int_1^9 \int_0^2 \frac{x\,\sin\left(x^2\right)}{\sqrt{z}}\,dx\,dz = (1-\cos 4)\int_1^9 z^{-1/2}\,dz = 4 - 4\cos 4$.

30 This is the integral of a function over the right half of the ellipse $\dfrac{x^2}{2} + y^2 = 2$. Change the order of integration of x and y so that $-\sqrt{2} \le y \le \sqrt{2}$, $0 \le x \le \sqrt{4-2y^2}$. We have $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{4-2y^2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy =$ $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{4-2y^2}} \left(8 - 2x^2 - 4y^2\right)\,dx\,dy = \int_{-\sqrt{2}}^{\sqrt{2}} \left(\left(8-4y^2\right)\sqrt{4-2y^2} - \frac{2}{3}\left(4-2y^2\right)^{3/2}\right)dy = 4\pi\sqrt{2}$.

31 Reorder to integrate with respect to x last. Because $0 \le y \le 2$, we have $0 \le x \le 4$. The integral is

$$\begin{aligned}\int_0^4 \int_{\sqrt{x}}^2 \int_0^{y^{1/3}} yz^5\left(1+x+y^2+z^6\right)^2\,dz\,dy\,dx &= \frac{1}{18}\int_0^4 \int_{\sqrt{x}}^2 \left(7y^7 + 9y^5 + 9xy^5 + 3y^3 + 6xy^3 + 3x^2y^3\right)\,dy\,dx\\ &= \frac{1}{18}\int_0^4 \left(332 - \frac{25}{8}x^4 - 3x^3 + \frac{45}{4}x^2 + 120x\right)\,dx = \frac{848}{9}.\end{aligned}$$

32 $\displaystyle\int_1^2 \int_0^1 \left(12 - x^2 - 2y^2\right)\,dy\,dx = \int_1^2 \left(12y - x^2y - \frac{2}{3}y^3\right)\Big|_0^1\,dx = \int_1^2 \left(\frac{34}{3} - x^2\right)dx =$ $\displaystyle\left(\frac{34}{3}x - \frac{x^3}{3}\right)\Big|_1^2 = 9$.

33 This is given by $\displaystyle\int_0^1 \int_0^{1-x^2} 2(1-y-x^2)\,dy\,dx = \int_0^1 (2y - y^2 - 2x^2y)\Big|_0^{1-x^2}\,dx = \int_0^1 (2 - 2x^2 - (1-x^2)^2 -$ $\displaystyle 2x^2(1-x^2))\,dx = \int_0^1 (1 - 2x^2 + x^4)\,dx = \left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right)\Big|_0^1 = \frac{8}{15}$.

34 $\displaystyle\int_0^1 \int_0^{3-3x} \int_0^2 1\,dz\,dy\,dx = \int_0^1 (6-6x)\,dx = 3$.

35 $\displaystyle\int_0^2 \int_0^{\pi} \int_0^{r\,\sin\theta} r\,dz\,d\theta\,dr = \int_0^2 \int_0^{\pi} r^2\,\sin\theta\,d\theta\,dr = \int_0^2 2r^2\,dr = \frac{16}{3}$.

36 Note that $2 - x^2 - (1+y^2) = 1 - (x^2+y^2)$. Note also that the projection of the intersection of the surfaces to the plane $z = 0$ is the unit circle. We use cylindrical coordinates to obtain $\displaystyle\int_0^{2\pi} \int_0^1 (1-r^2)r\,dr\,d\theta =$ $\displaystyle\int_0^{2\pi} \int_0^1 (r - r^3)\,dr\,d\theta = 2\pi\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}$.

37 Look at the intersection of the cylinders from the positive z-axis. The vertical sides of the region lie on the cylinder $x^2 + y^2 = 4$, and the top and bottom lie on $x^2 + z^2 = 4$. Thus the region is $\left\{(x,\,y,\,z) : x^2 + y^2 \le 4,\ -\sqrt{4-x^2} \le z \le \sqrt{4-x^2}\right\}$. Thus the volume is $\displaystyle\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1\,dz\,dy\,dx =$ $\displaystyle\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2\sqrt{4-x^2}\,dy\,dx = \int_{-2}^2 4\left(4-x^2\right)\,dx = \frac{128}{3}$.

38 $\displaystyle\int_0^1 \int_0^x \int_0^y 1\,dz\,dy\,dx = \int_0^1 \int_0^x y\,dy\,dx = \int_0^1 \frac{x^2}{2}\,dx = \frac{1}{6}$.

39 Rewrite the integral as $\int_0^{1/2}\int_{\sin^{-1}x}^{\sin^{-1}2x} 1\,dy\,dx$ and then change the order of integration. This results in the integral breaking up into two integrals, and we obtain

$$\int_0^{\pi/6}\int_{(\sin y)/2}^{\sin y} 1\,dx\,dy + \int_{\pi/6}^{\pi/2}\int_{(\sin y)/2}^{1/2} 1\,dx\,dy = \int_0^{\pi/6}\frac{\sin y}{2}\,dy + \int_{\pi/6}^{\pi/2}\left(\frac{1}{2}-\frac{\sin y}{2}\right)dy = \frac{1-\sqrt{3}}{2}+\frac{\pi}{6}.$$

40 The plane of the slanted surface of the tetrahedron is $6x+3y+2z=6$, so the possibilities are

$$\begin{aligned}\int_0^1\int_0^{2-2x}\int_0^{(6-6x-3y)/2} dz\,dy\,dx &= \int_0^1\int_0^{3-3x}\int_0^{(6-6x-2z)/3} dy\,dz\,dx\\ &= \int_0^2\int_0^{(2-y)/2}\int_0^{(6-6x-3y)/2} dz\,dx\,dy\\ &= \int_0^2\int_0^{2-2x}\int_0^{(6-3y-2z)/6} dx\,dz\,dy\\ &= \int_0^3\int_0^{(3-z)/3}\int_0^{(6-6x-2z)/3} dy\,dx\,dz\\ &= \int_0^3\int_0^{2(3-z)/3}\int_0^{(6-3y-2z)/6} dx\,dy\,dz.\end{aligned}$$

41

$$\begin{aligned}\int_0^{\pi/4}\int_0^{\sec\theta} r^3\ dr\,d\theta &= \int_0^1\int_0^x (x^2+y^2)\,dy\,dx = \int_0^1\left(x^2y+\frac{1}{3}y^3\right)\Big|_0^x dx\\ &= \int_0^1\left(\frac{4}{3}x^3\right)dx = \left(\frac{1}{3}x^4\right)\Big|_0^1 = \frac{1}{3}.\end{aligned}$$

42 Use cylindrical coordinates. The volume of the paraboloid is $\int_0^2\int_0^{2\pi}\int_0^{4-r^2} r\,dz\,d\theta\,dr = 2\pi\int_0^2\left(4r-r^3\right)\ dr = 8\pi$. The distance of a point from the origin is $\sqrt{r^2+z^2}$, so the average squared distance is $\frac{1}{8\pi}\int_0^2\int_0^{2\pi}\int_0^{4-r^2} r\left(r^2+z^2\right)\ dz\,d\theta\,dr = \frac{1}{4}\int_0^2\left(\frac{1}{3}r\left(4-r^2\right)^3+r^3\left(4-r^2\right)\right)dr = 4$.

43 The volume of the prism is $\int_0^1\int_0^{3-3x}\int_0^2 1\,dz\,dy\,dx = \int_0^1 (6-6x)\ dx = 3$. Thus the average x coordinate is $\frac{1}{3}\int_0^1\int_0^{3-3x}\int_0^2 x\,dz\,dy\,dx = \frac{1}{3}\int_0^1 2x\left(3-3x\right)\ dx = \frac{1}{3}$.

44 This is a quarter of a cylinder of radius 3 and length 3 oriented along the z-axis, so $\int_0^3\int_0^{\pi/2}\int_0^3 r^3\cdot r\,dz\,d\theta\,dr = \frac{3\pi}{2}\int_0^3 r^4\,dr = \frac{729\pi}{10}$.

45 Use cylindrical coordinates. We have $\int_{-2}^2\int_{-\pi/2}^{\pi/2}\int_0^1 \frac{1}{(1+r^2)^2} r\,dr\,d\theta\,dz$. This can be written as $(2-(-2))\cdot\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)\int_0^1\frac{r}{(1+r^2)^2}\,dr = 4\pi\int_1^2\frac{1}{2}\left(\frac{1}{u^2}\right)du = 2\pi\left(-\frac{1}{u}\right)\Big|_1^2 = \pi$.

46 $\sqrt{9-x^2-y^2}=\sqrt{1+x^2+y^2}$ for $x^2+y^2=4$. We have

$$\begin{aligned}\int_0^{2\pi}\int_0^2\int_{\sqrt{1+r^2}}^{\sqrt{9-r^2}} dz\,r\,dr\,d\theta &= 2\pi\int_0^2\left(r\sqrt{9-r^2}-r\sqrt{1+r^2}\right)\,dr\\ &= -\frac{2\pi}{3}\left((9-r^2)^{3/2}+(1+r^2)^{3/2}\right)\Big|_0^2\\ &= \frac{56\pi}{3}-\frac{20\sqrt{5}\pi}{3}.\end{aligned}$$

47 $\displaystyle\int_0^4\int_0^\pi\int_0^{2\cos\theta} r\,dr\,d\theta\,dz = 2\int_0^4\int_0^\pi \cos^2\theta\,d\theta\,dz = 4\pi.$

48 $\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/2}\cos^3\varphi\,\sin\varphi\,d\varphi\,d\theta = \frac{4\pi}{3}.$

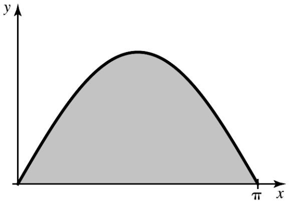
49 $\displaystyle\int_0^{\pi}\int_0^{\pi/4}\int_{2\sec\varphi}^{4\sec\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{56}{3}\int_0^{\pi}\int_0^{\pi/4}\sec^3\varphi\,\sin\varphi\,d\varphi\,d\theta = \frac{28\pi}{3}.$

50 $\displaystyle\int_0^{\pi}\int_0^{(1-\cos\varphi)/2}\int_0^{2\pi}\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{\pi}{12}\int_0^{\pi}(1-\cos\varphi)^3\,\sin\varphi\,d\rho\,d\varphi = \frac{\pi}{3}.$

51 $\displaystyle\int_0^{\pi/2}\int_0^{4\sin 2\varphi}\int_0^{2\pi}\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{128\pi}{3}\int_0^{\pi/2}\sin^3 2\varphi\,\sin\varphi\,d\varphi = \frac{2048\pi}{105}.$

52 $\displaystyle\int_0^{\pi/4}\int_0^{4\cos\varphi}\int_0^{2\pi}\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{128\pi}{3}\int_0^{\pi/4}\cos^3\varphi\,\sin\varphi\,d\varphi = 8\pi.$

53

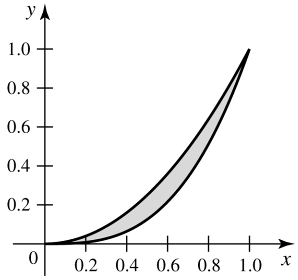


The mass of the plate is $\displaystyle\int_0^{\pi}\sin x\,dx = -\cos\pi+\cos 0 = 2$. By symmetry, $\overline{x}=\dfrac{\pi}{2}$,

$$\overline{y}=\frac{1}{2}\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx = \frac{1}{4}\int_0^4\sin^2 x\,dx = \frac{\pi}{8}.$$

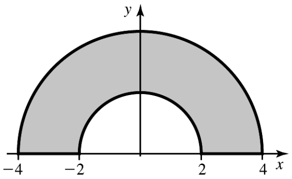
The center of mass is $\left(\dfrac{\pi}{2},\dfrac{\pi}{8}\right)$.

54



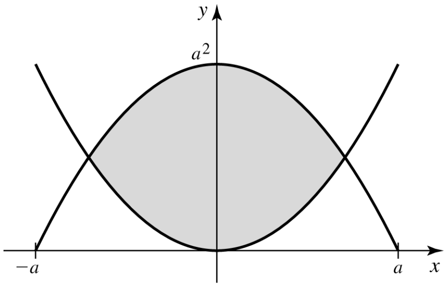
The mass of the plate is $\int_0^1 \int_{x^3}^{x^2} 1\, dy\, dx = \int_0^1 \left(x^2 - x^3\right)\, dx = \frac{1}{12}$, so that $\overline{x} = \frac{1}{1/12} \int_0^1 \int_{x^3}^{x^2} x\, dy\, dx = 12 \int_0^1 \left(x^3 - x^4\right)\, dx = \frac{3}{5} \overline{y} = \frac{1}{1/12} \int_0^1 \int_{x^3}^{x^2} y\, dy\, dx = 6 \int_0^1 \left(x^4 - x^6\right)\, dx = \frac{12}{35}$. The center of mass is at $\left(\frac{3}{5}, \frac{12}{35}\right)$.

55



By symmetry, $\overline{x} = 0$. The mass of the region is $8\pi - 2\pi = 6\pi$, so $\overline{y} = \dfrac{1}{6\pi}\displaystyle\int_0^{\pi}\int_2^4 r^2 \sin\theta\,dr\,d\theta = \dfrac{28}{9\pi}\int_0^{\pi} \sin\theta\,d\theta = \dfrac{56}{9\pi}$. The center of mass is at $\left(0, \dfrac{56}{9\pi}\right)$.

56



By symmetry, $\overline{x} = 0$. The two parabolas intersect where $x^2 = a^2 - x^2$, so where $x = \pm\dfrac{a}{\sqrt{2}}$. The mass is then

$$\int_{-a/\sqrt{2}}^{a/\sqrt{2}}\int_{x^2}^{a^2-x^2} 1\,dy\,dx = \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \left(a^2 - 2x^2\right) dx = \frac{2}{3}a^3\sqrt{2}$$

so that $\overline{y} = \dfrac{1}{2a^3\sqrt{2}/3}\displaystyle\int_{-a/\sqrt{2}}^{a/\sqrt{2}}\int_{x^2}^{a^2-x^2} y\,dy\,dx = \dfrac{3}{4a^3\sqrt{2}}\int_{-a/\sqrt{2}}^{a/\sqrt{2}} \left(\left(a^2 - x^2\right)^2 - x^4\right) dx = \dfrac{1}{2}a^2$. The center of mass is at $\left(0, \dfrac{1}{2}a^2\right)$.

57 By symmetry, $\overline{x} = \overline{y} = 0$. The mass (using cylindrical coordinates) is

$$\int_0^6\int_{r^2}^{36}\int_0^{2\pi} r\,d\theta\,dz\,dr = 2\pi\int_0^6 r\left(36 - r^2\right) dr = 648\pi,$$

so $\overline{z} = \dfrac{1}{648\pi}\displaystyle\int_0^6\int_{r^2}^{36}\int_0^{2\pi} rz\,d\theta\,dz\,dr = \dfrac{1}{648}\int_0^6 r\left(1296 - r^4\right) dr = 24$. The center of mass is $(0, 0, 24)$.

58 The mass is $\dfrac{1}{3}$ times the area of the base (4) times the height (4), so is $\dfrac{16}{3}$. The plane intersects the x-axis at $x = 4$, the y-axis at $y = 2$, and the z-axis at $z = 4$, so by symmetry, $\overline{x} = \overline{z}$. Then $\overline{x} = \dfrac{1}{16/3}\displaystyle\int_0^2\int_0^{4-2y}\int_0^{4-x-2y} x\,dz\,dx\,dy = \dfrac{3}{16}\int_0^2\int_0^{4-2y}\left(4x - x^2 - 2xy\right) dx\,dy = \dfrac{1}{32}\int_0^2 (4-2y)^3\,dy = 1$. $\overline{y} = \dfrac{1}{16/3}\displaystyle\int_0^2\int_0^{4-2y}\int_0^{4-x-2y} y\,dz\,dx\,dy = \dfrac{3}{16}\int_0^2\int_0^{4-2y}\left(4y - xy - 2y^2\right) dx\,dy = \dfrac{3}{16}\int_0^2\left(8y - 8y^2 + 2y^3\right) dy = \dfrac{1}{2}$, so the center of mass is $\left(1, \dfrac{1}{2}, 1\right)$.

59 Use spherical coordinates. The mass is $\int_0^{\pi/2}\int_0^{16}\int_0^{2\pi}\left(1+\frac{\rho}{4}\right)\rho^2\,\sin\varphi\,d\theta\,d\rho\,d\varphi =$ $2\pi\int_0^{\pi/2}\int_0^{16}\left(\rho^2+\frac{\rho^3}{4}\right)\,\sin\varphi\,d\rho\,d\varphi = \frac{32768\pi}{3}\int_0^{\pi/2}\sin\varphi\,d\varphi = \frac{32768\pi}{3}$. By symmetry of the region and of the density function around the z axis, we have $\overline{x}=\overline{y}=0$.

$$\begin{aligned}\overline{z} &= \frac{1}{32768\pi/3}\int_0^{\pi/2}\int_0^{16}\int_0^{2\pi}\rho\cos\varphi\left(1+\frac{\rho}{4}\right)\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi\\ &= \frac{3}{32768}\int_0^{\pi/2}\int_0^{16}\left(\rho^3+\frac{\rho^4}{4}\right)\,\cos\varphi\,\sin\varphi\,d\rho\,d\varphi = \frac{3}{32768}\int_0^{\pi/2}\frac{16384\cdot 21}{5}\cos\varphi\,\sin\varphi\,d\varphi = \frac{63}{10},\end{aligned}$$

so the center of mass is $\left(0,0,\frac{63}{10}\right)$.

60 The mass is $\int_0^2\int_0^2\int_0^2(1+x+y+z)\,dz\,dy\,dx = 32$. By symmetry, $\overline{x}=\overline{y}=\overline{z}$, and $\overline{x}=\frac{1}{32}\int_0^2\int_0^2\int_0^2 x\,(1+x+y+z)\,dz\,dy\,dx = \frac{13}{12}$, so the center of mass is $\left(\frac{13}{12},\frac{13}{12},\frac{13}{12}\right)$.

61 Place the vertex at the origin with the paraboloid opening upwards along the positive z-axis. Then the equation of the paraboloid is $z=\frac{h}{R^2}r^2$. Its mass is $\int_0^R\int_0^{2\pi}\int_{hr^2/R^2}^h r\,dz\,d\theta\,dr = 2\pi\int_0^R\left(r-\frac{h}{R^2}r^2\right)dr =$ $\frac{1}{2}\pi hR^2$. Then $\overline{z}=\frac{1}{\pi hR^2/2}\int_0^R\int_0^{2\pi}\int_{hr^2/R^2}^h rz\,dz\,d\theta\,dr = \frac{2}{hR^2}\int_0^R rh^2\frac{R^4-r^4}{R^4}dr = \frac{2}{3}h$, so that the center of mass is $\frac{1}{3}$ of the way from the base to the vertex.

62 Place one vertex at the origin and another at $(s,\,0)$ so that the third is at $\left(\frac{s}{2},\,\frac{s\sqrt{3}}{2}\right)$. Then the equations of the two sides of the triangle to that third vertex are $y=x\sqrt{3}$ and $y=-\sqrt{3}\,(x-s)$. The area of the triangle is $\frac{s^2\sqrt{3}}{4}$ (one half the base times the height), so $\overline{y}=\frac{1}{s^2\sqrt{3}/4}\int_0^{s/2}\int_0^{\sqrt{3}x}y\,dy\,dx+\int_{s/2}^s\int_0^{-\sqrt{3}(x-s)}y\,dy\,dx =$ $\frac{2}{s^2\sqrt{3}}\left(\int_0^{s/2}3x^2\,dx+\int_{s/2}^s 3\,(x-s)^2\,dx\right) = \frac{6}{s^2\sqrt{3}}\left(\frac{s^3}{24}+\frac{s^3}{24}\right) = \frac{\sqrt{3}}{6}s$, which is one third the height of the triangle.

63 The area of the sector is 1/8 the area of the entire circle, so it is $\frac{1}{8}\pi a^2$. We can calculate $M_y=\iint_R x\,dA =$ $\int_0^{\pi/4}\int_0^a r^2\cos\theta\,dr\,d\theta = \int_0^{\pi/4}\left.\frac{r^3}{3}\right|_0^a\cos\theta\,d\theta = \frac{a^3}{3}\left.(\sin\theta)\right|_0^{\pi/4} = \frac{a^3\sqrt{2}}{6}$. Thus $\overline{x}=\frac{M_y}{m}=\frac{4\sqrt{2}a}{3\pi}$. Also, we have $M_y=\iint_R y\,dA = \int_0^{\pi/4}\int_0^a r^2\sin\theta\,dr\,d\theta = \int_0^{\pi/4}\left.\frac{r^3}{3}\right|_0^a\sin\theta\,d\theta = \frac{a^3}{3}\left.(-\cos\theta)\right|_0^{\pi/4} = \frac{a^3(2-\sqrt{2})}{6}$. Thus $\overline{y}=\frac{M_x}{m}=\frac{4a(2-\sqrt{2})}{3\pi}$.

64 By symmetry, we have $\overline{x}=\overline{y}=0$. To calculate the mass, note that $x^2+y^2+z^2=a^2$ and $x^2+y^2=z^2$, so $2z^2=a^2$, and $z=\frac{a}{\sqrt{2}}$. The mass is given by $\int_0^{2\pi}\int_0^{a/\sqrt{2}}\int_r^{\sqrt{a^2-r^2}}r\,dz\,dr\,d\theta$. This is equal to $2\pi\int_0^{a/\sqrt{2}}\left(\sqrt{a^2-r^2}-r\right)r\,dr = 2\pi\left.\left(-\frac{1}{3}(a^2-r^2)^{3/2}-\frac{r^3}{3}\right)\right|_r^{a/\sqrt{2}} = \frac{\pi a^3(2-\sqrt{2})}{3}$. A similar calculation shows that $M_{xy}=\int_0^{2\pi}\int_0^{a/\sqrt{2}}\int_r^{\sqrt{a^2-r^2}}z\cdot r\,dz\,dr\,d\theta = \frac{a^4\pi}{8}$. We therefore have $\overline{z}=\frac{M_{xy}}{m}=\frac{3a(2+\sqrt{2})}{16}$.

65 The equation of the cone in cylindrical coordinates is $z = 2 - \frac{1}{2}r$. The volume of the cone is one third the area of the base times the height, or $\frac{32\pi}{3}$.

a. The volume of a slice of $\frac{\pi}{4}$ radians is $\int_0^4 \int_0^{\pi/4} \int_0^{2-r/2} r\,dz\,d\theta\,dr = \frac{\pi}{4}\int_0^4 r\left(2 - \frac{r}{2}\right)\,dr = \frac{4\pi}{3}$. The volume of the wedge is in fact one eighth the volume of the entire cone ($\frac{\pi}{4}$ radians is an eighth-circle).

b. The volume of a slice of Q radians is $\int_0^4 \int_0^{Q} \int_0^{2-r/2} r\,dz\,d\theta\,dr = Q\int_0^4 r\left(2 - \frac{r}{2}\right)\,dr = \frac{16}{3}Q$. Geometrically, Q radians is $\frac{Q}{2\pi}$ of a circle, and indeed the volume of the slice is $\frac{Q}{2\pi}$ times the volume of the cone.

66 Compute the volume of empty space, using spherical coordinates with the center of the tank at the origin. The water level is at $\varphi = \frac{\pi}{3}$ (Pythagorean theorem), so the volume of the empty spherical cap is $\int_0^{\pi/3} \int_{\sec\varphi/2}^{1} \int_0^{2\pi} \rho^2 \sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{2\pi}{3}\int_0^{\pi/3}\left(1 - \frac{\sec^3\varphi}{8}\right)\sin\varphi\,d\varphi = \frac{5\pi}{24}$ cubic feet. The total volume of the sphere is $\frac{4\pi}{3}$ cubic feet.

a. The volume of water is $\frac{4\pi}{3} - \frac{5\pi}{24} = \frac{27\pi}{24} = \frac{9\pi}{8}$ cubic feet and it weighs $1.125 \cdot 62.5 \cdot \pi \approx 220.893$ pounds.

b. The amount of water that must be added to fill the tank is the volume of empty space, or $\frac{5\pi}{24}$ cubic feet.

67 The transformation just switches the coordinates, so the image is again the unit square.

68 $T = \{(x, y) : 0 \le v \le 1,\ 0 \le u \le 1\} = \{(x, y) : 0 \le -x \le 1,\ 0 \le y \le 1\} =$
$\{(x, y) : -1 \le x \le 0,\ 0 \le y \le 1\}$ so that T is the unit square in the second quadrant with one vertex at the origin.

69 As (u, v) goes from $(0, 0)$ to $(1, 0)$, (x, y) goes from $(0, 0)$ to $(3, 1)$; as (u, v) goes from $(1, 0)$ to $(1, 1)$, (x, y) goes from $(3, 1)$ to $(4, 4)$; as (u, v) goes from $(1, 1)$ to $(0, 1)$, (x, y) goes from $(4, 4)$ to $(1, 3)$. Thus the image of S is the parallelogram with vertices $(0, 0)$, $(3, 1)$, $(4, 4)$, and $(1, 3)$.

70 The transformation leaves the x coordinate alone and stretches and translates the y coordinate. So the new region is the rectangle with vertices $(0, 2)$, $(1, 2)$, $(1, 4)$, $(0, 4)$.

71 $J(u, v) = \begin{vmatrix} 4 & -1 \\ -2 & 3 \end{vmatrix} = 10.$

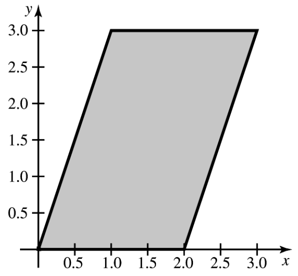
72 $J(u, v) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$

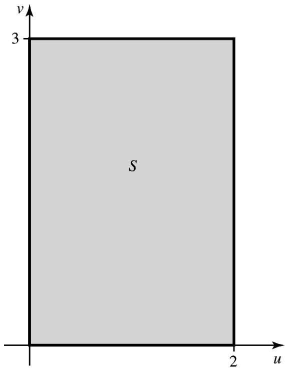
73 $J(u, v) = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6.$

74 $J(u, v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4\left(u^2 + v^2\right).$

75

a.


y
3.0
2.5
2.0
1.5
1.0
0.5
0.5
1.0
1.5
2.0
2.5
3.0
x



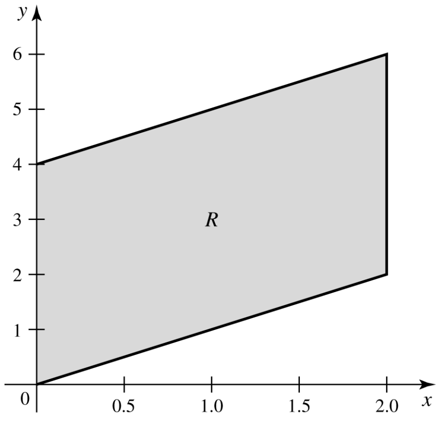
b. $S = \left\{(u,\, v) : \dfrac{v}{3} \le u + \dfrac{v}{3} \le \dfrac{v+6}{3},\, 0 \le v \le 3\right\} = \{(u,\, v) : 0 \le u \le 2,\, 0 \le v \le 3\}$.

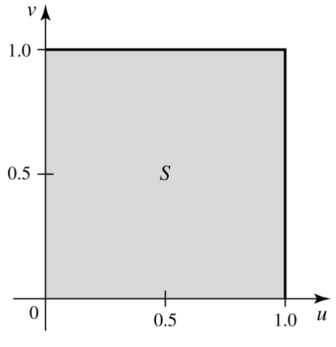
c. $J\,(u,\, v) = \begin{vmatrix} 1 & \dfrac{1}{3} \\ 0 & 1 \end{vmatrix} = 1$

d. $\displaystyle\iint\limits_R xy^2\, dA = \int_0^2 \int_0^3 \left(u + \frac{v}{3}\right) v^2\, dv\, du = \frac{63}{2}$.

76

a.


y
6
5
4
3
2
1
0
R
0.5
1.0
1.5
2.0
x


v
1.0
0.5
S
0
0.5
1.0
u

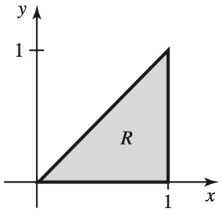
b. $S = \{(u,\, v) : 0 \le 2u \le 2,\, 2u \le 4v + 2u \le 2u + 4\} = \{(u,\, v) : 0 \le u \le 1,\, 0 \le v \le 1\}$.

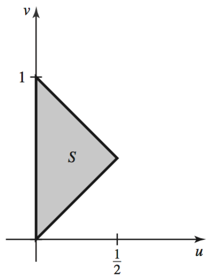
c. $J\,(u,\, v) = \begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8$.

d. $\displaystyle\iint\limits_R 3xy^2\, dA = \int_0^1 \int_0^1 24\,(2u)\,(4v + 2u)^2\; dv\, du = 192 \int_0^1 \int_0^1 u\,(2v + u)^2\; dv\, du = 304$.

77

a.





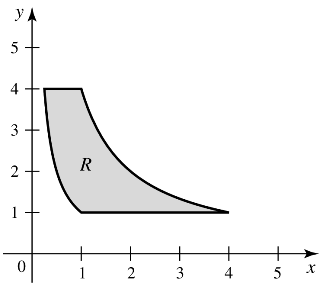
b. $S = \left\{(u,\, v) : u \le v \le 1-u, 0 \le u \le \frac{1}{2}\right\}$.

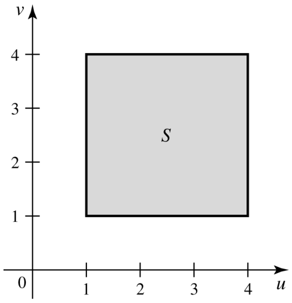
c. $J\,(u,\, v) = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$.

d. The given integral is equal to $\int_0^{1/2}\int_u^{1-u} (2u+1)(2u)^9 \cdot 2\, dv\, du = 1024\int_0^{1/2} (2u+1)u^9\,(1-u-u)\; du =$ $1024\int_0^{1/2}(u^9 - 4u^{11})\, du = 1024\left(\frac{u^{10}}{10} - \frac{4u^{12}}{12}\right) = \frac{1}{60}$.

78

a.



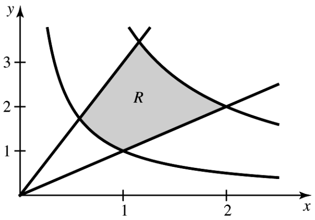


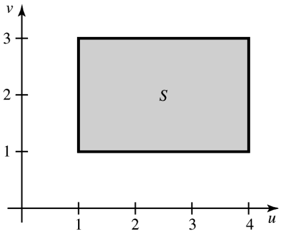
b. $xy = 1$ becomes $u = 1$ while $xy = 4$ becomes $u = 4$; $y = 1$ becomes $v = 1$; $y = 4$ becomes $v = 4$. So $S = \{(u,\, v) : 1 \le u \le 4,\ 1 \le v \le 4\}$.

c. $J\,(u,\, v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ 0 & 1 \end{vmatrix} = v^{-1}$.

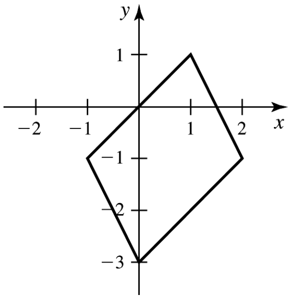
d. $\iint\limits_R xy^2\, dA = \int_1^4\int_1^4 \frac{u}{v}\, v^2\, v^{-1}\, dv\, du = \int_1^4\int_1^4 u\, dv\, du = \frac{45}{2}$.

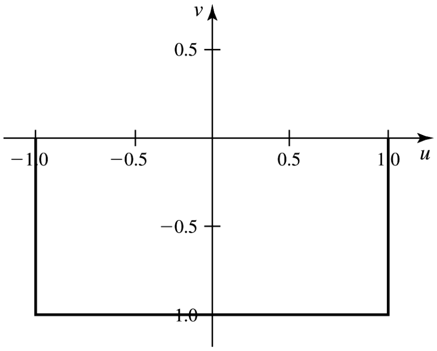
79 Use the transformation $u = xy$; $v = \dfrac{y}{x}$ so that $x = \sqrt{\dfrac{u}{v}}$, $y = \sqrt{uv}$. Then $xy = u$ and $\dfrac{y}{x} = v$, so the new region S is $\{(u,\, v) : 1 \le u \le 4,\ 1 \le v \le 3\}$. The Jacobian of this transformation is $J\,(u,\, v) =$ $\begin{vmatrix} \dfrac{u^{-1/2}v^{-1/2}}{2} & -\dfrac{u^{1/2}v^{-3/2}}{2} \\ \dfrac{v^{1/2}u^{-1/2}}{2} & \dfrac{u^{1/2}v^{-1/2}}{2} \end{vmatrix} = \dfrac{1}{2v}$ so that $\iint\limits_R y^4\, dA = \dfrac{1}{2}\int_1^4\int_1^3 u^2 v\, dv\, du = 42$.





80 Let $x = u - v$, $y = u + 2v$; then $y = x$ becomes $v = 0$, $y = x - 3$ becomes $v = -1$, $y = -2x + 3$ becomes $u = 1$, and $y = -2x - 3$ becomes $u = -1$. So the new region of integration is $S = \{(u,\, v) : -1 \le u \le 1,\, -1 \le v \le 0\}$. The Jacobian is $J\,(u,\, v) = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 3$, so that $\displaystyle\iint\limits_R \left(y^2 + xy - 2x^2\right)\, dA$ $\displaystyle = 3\int_{-1}^{1}\int_{-1}^{0} \left((u+2v)^2 + (u-v)\,(u+2v) - 2\,(u-v)^2\right)\, dv\, du = 3\int_{-1}^{1}\int_{-1}^{0} 9uv\, dv\, du = 0.$





81 Use the transformation $u = x + 2y$, $v = x - z$, $w = 2y - z$; solving for x, y, z gives $x = \dfrac{u + v - w}{2}$, $y = \dfrac{u - v + w}{4}$, $z = \dfrac{u - v - w}{2}$.

The new range of integration becomes $1 \le u \le 2$, $0 \le v \le 2$, $0 \le w \le 3$. The Jacobian is $J\,(u,\, v,\, w) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \dfrac{1}{4}$ so that $\displaystyle\iiint\limits_D yz\, dV = \frac{1}{32}\int_1^2\int_0^2\int_0^3 (u - v + w)\,(u - v - w)\; dw\, dv\, du = -\frac{7}{16}.$

82 Use the transformation $u = y - 2x$, $v = z - 3y$, $w = z - 4x$; solving for x, y, z gives $x = \dfrac{-3u - v + w}{2}$, $y = -2u - v + w$, $z = -6u - 2v + 3w$. The new range of integration becomes $0 \le u \le 1$, $0 \le v \le 1$, $0 \le w \le 3$. The Jacobian is $J\,(u,\, v,\, w) = \begin{vmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & -1 & 1 \\ -6 & -2 & 3 \end{vmatrix} = \dfrac{1}{2}$ so that

$$\iiint\limits_D x\, dV = \frac{1}{4}\int_0^1\int_0^1\int_0^3 (-3u - v + w)\, dw\, dv\, du = -\frac{3}{8}.$$

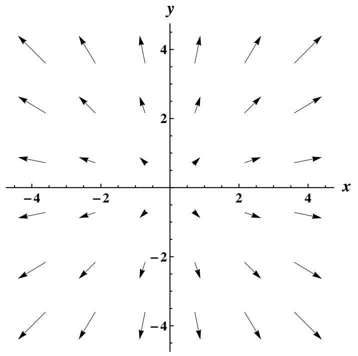
Chapter 17

Vector Calculus

17.1 Vector Fields

17.1.1 A vector field describes the motion of the air as a vector at each point in the room.

17.1.2



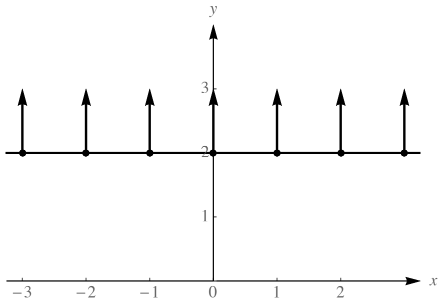
17.1.3 At selected points (a, b), plot the vector $\langle f(a, b), g(a, b)\rangle$.

17.1.4 The gradient of a function at a point is a vector describing the direction in which the value of the function is increasing most rapidly. The collection of these vectors over all points is a vector field.

17.1.5 The gradient field gives, at each point, the direction in which the temperature is increasing most rapidly and the amount of increase.

17.1.6 The length of $\mathbf{F}$ is $\left|\dfrac{\sqrt{2}\,\langle x, y\rangle}{\sqrt{x^2+y^2}}\right| = \dfrac{\sqrt{2}}{\sqrt{x^2+y^2}}\,|\langle x, y\rangle| = \dfrac{\sqrt{2}}{\sqrt{x^2+y^2}}\sqrt{x^2+y^2} = \sqrt{2}$.

17.1.7



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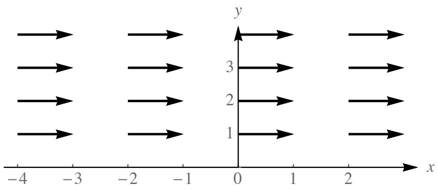

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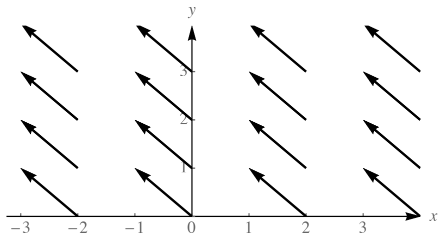

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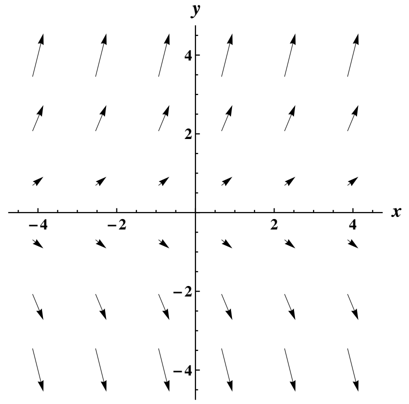

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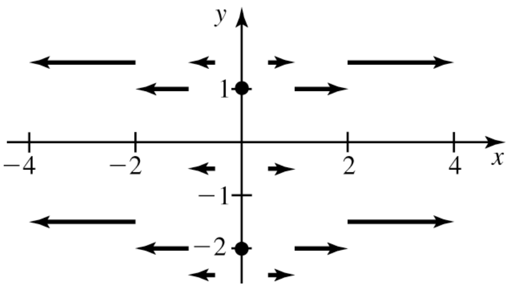

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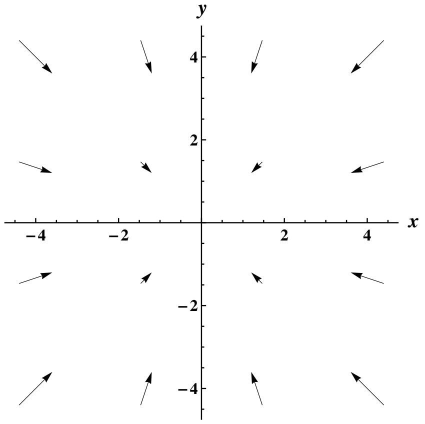

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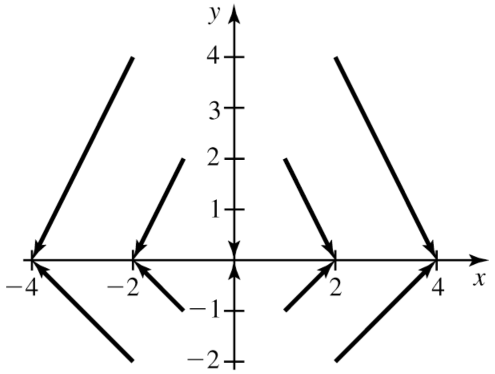

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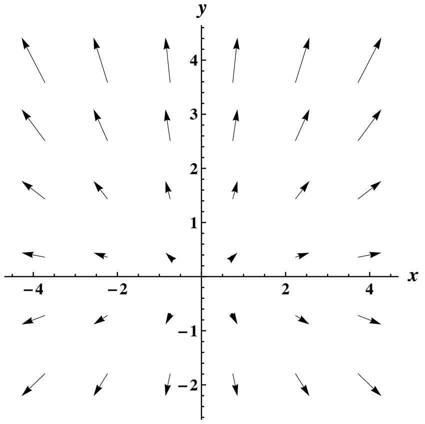

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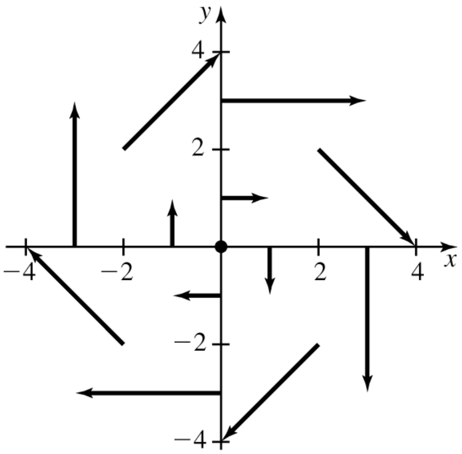

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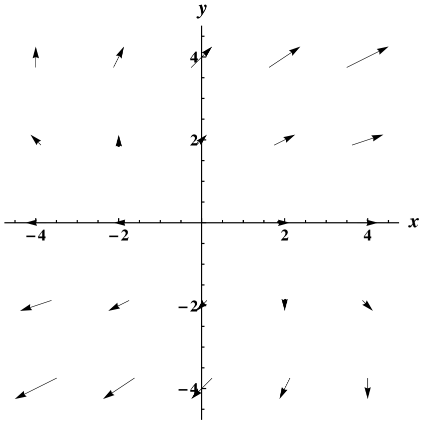
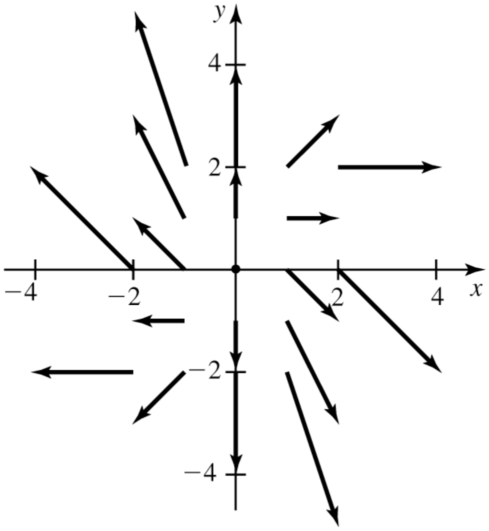


17.1.17



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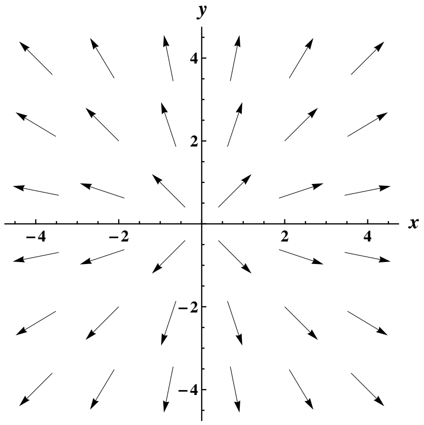

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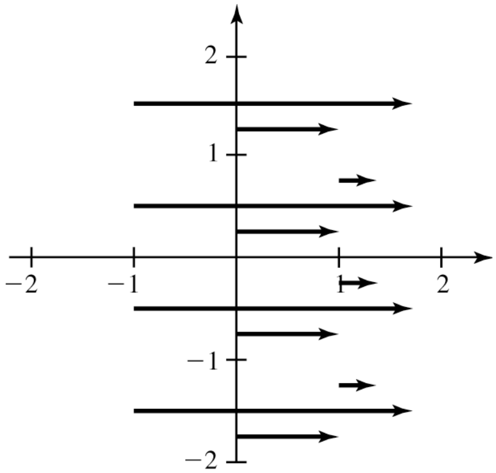
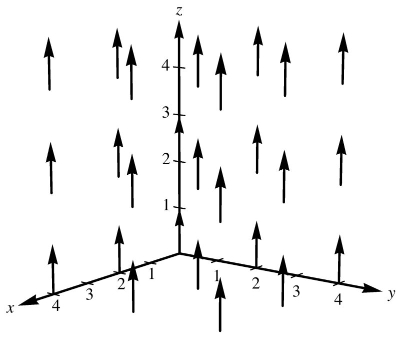
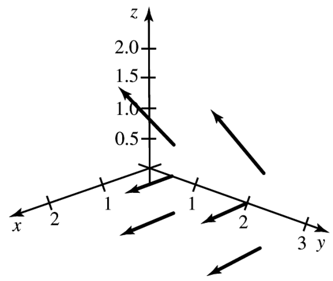


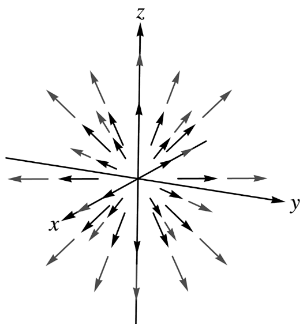
17.1.20



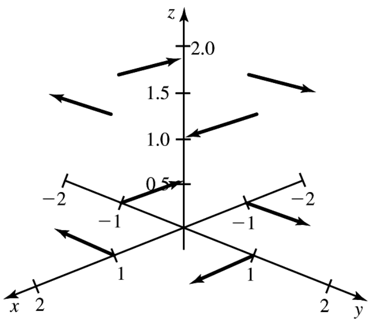
17.1.21



17.1.22



17.1.23



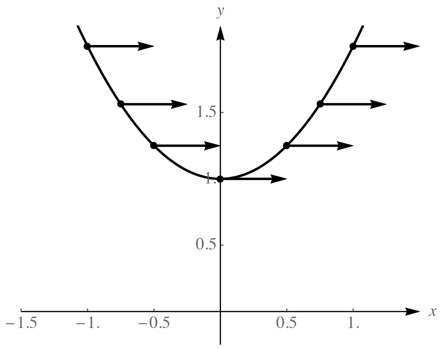
17.1.24 (a) corresponds to (D), since (D) has zero x component, and the y component increases as y does. (b) corresponds to (C), since the x component appears to be zero along the line $y = x$. (c) corresponds to (B) since the y component is zero on the x-axis and the x component is zero on the y axis. Finally, (d) corresponds to (A) since the x component is zero on the x-axis and the y component is zero on the y-axis.

17.1.25

a. Let $g(x, y) = y - x^2$. Then $\nabla g = \langle -2x, 1\rangle$. This is orthogonal to $\mathbf{F}$ when $\nabla g \cdot \mathbf{F} = 0$, or when $\frac{1}{2}(-2x) + 0 = 0$, or $x = 0$. So the point on C is $(0, 1)$

b. ∇g is never parallel to $\mathbf{F}$, because no nonzero multiple of $\left\langle \frac{1}{2}, 0\right\rangle$ can have a second component of 1.

c.

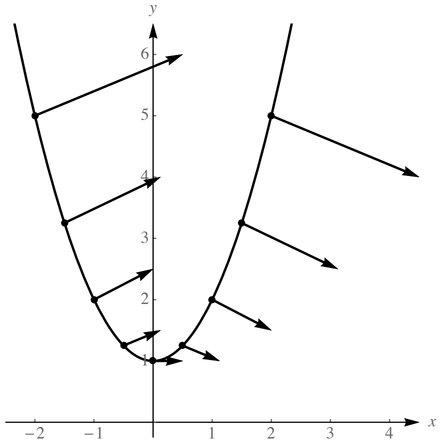


17.1.26

a. Let $g(x, y) = y - x^2$. Then $\nabla g = \langle -2x, 1\rangle$. This is perpendicular to $\mathbf{F}$ when $\nabla g \cdot \mathbf{F} = 0$, or $\frac{y}{2}(-2x) + \frac{-x}{2}(1) = 0$, or $-xy - \frac{x}{2} = 0$, so for $x = 0$ and $y = -\frac{1}{2}$. However, on our curve, $y \geq 1$, so we must have $x = 0$, in which case $y = 1$. So $\mathbf{F}$ is tangent to C at $(0, 1)$.

b. If $\mathbf{F} = k\nabla g$, then $y = -4kx$ and $x = -2k$. Substituting into the equation for C, we have $2x^2 - x^2 = 1$, so $x = \pm 1$ and $y = 2$. So $\mathbf{F}$ is normal to C at the points $(\pm 1, 2)$.

c.

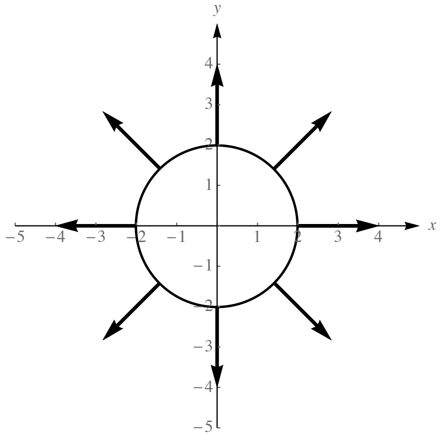


17.1.27

a. Let $g(x,y) = x^2+y^2$. Then $\nabla g = \langle 2x, 2y\rangle$. This is perpendicular to $\mathbf{F}$ when $\nabla g \cdot \mathbf{F} = 0$, or $2x^2+2y^2 = 0$, which doesn't occur on the circle C.

b. If $\mathbf{F} = k\nabla g$, then $\langle x, y\rangle = k\,\langle 2x, 2y\rangle$, which occurs for all points on the circle.

c.

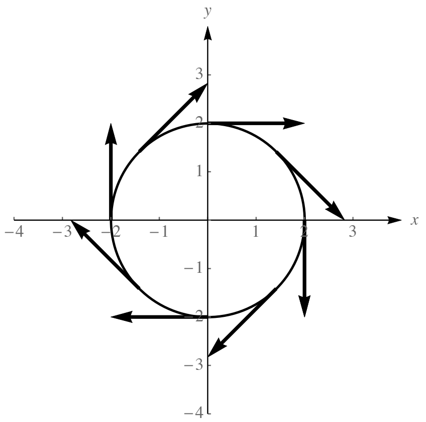


17.1.28

a. Let $g(x,y) = x^2+y^2$. Then $\nabla g = \langle 2x, 2y\rangle$. This is perpendicular to $\mathbf{F}$ when $\nabla g \cdot \mathbf{F} = 0$, or $2xy-2xy = 0$, which occurs for all points on the circle.

b. If $\mathbf{F} = k\nabla g$, then $\langle y, -x\rangle = k\,\langle 2x, 2y\rangle$, or $y = 2kx$ and $x = -2ky$. But then $y = 2kx = 2k(-2ky) = -4k^2y$, which doesn't occur. There are no points where $\mathbf{F}$ is normal to the curve.

c.

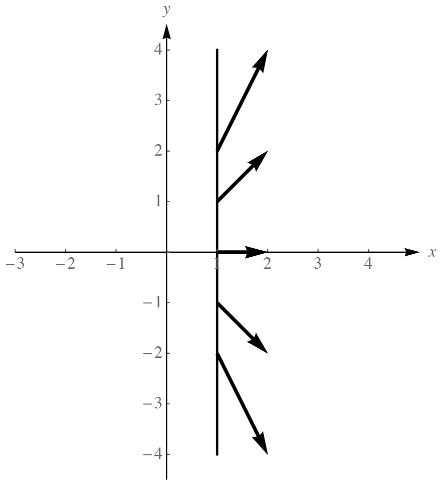


17.1.29

a. Let $g(x, y) = x$. Then $\nabla g = \langle 1, 0\rangle$. This is perpendicular to $\mathbf{F}$ for no points on C.

b. If $\mathbf{F} = k\nabla g$, then $y = 0$, so the point on C is $(1, 0)$.

c.

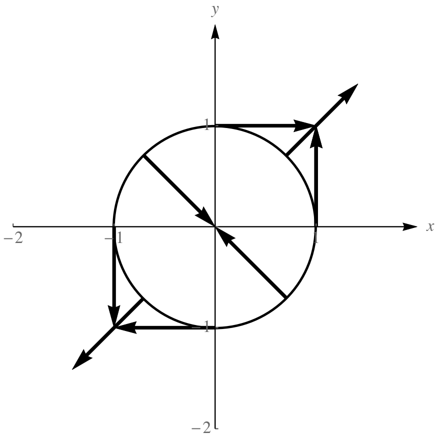


17.1.30

a. Let $g(x, y) = x^2 + y^2$. Then $\nabla g = \langle 2x, 2y\rangle$. This is perpendicular to $\mathbf{F}$ if $2xy + 2yx = 4xy = 0$, which occurs if $x = 0$ or $y = 0$. So the corresponding points on C are $(\pm 1, 0)$ and $(0, \pm 1)$.

b. If $\mathbf{F} = k\nabla g$, then $\langle y, x\rangle = k\langle 2x, 2y\rangle$, so $y = 2kx$ and $x = 2ky$. Then $y = 2kx = 2k(2ky)$, so $4k^2 = 1$ and $k = \pm\dfrac{1}{2}$. Then either $x = y$ or $x = -y$, so the points are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

c.



17.1.31 For example, $\mathbf{F} = \left\langle -y,\, x \right\rangle$ or $\mathbf{F} = \left\langle -1,\, 1 \right\rangle$.

17.1.32 For example, $\mathbf{F} = \left\langle y,\, 0 \right\rangle$.

17.1.33 For example, $\mathbf{F} = \frac{1}{\sqrt{x^2+y^2}} \left\langle x, y \right\rangle = \frac{\mathbf{r}}{|\mathbf{r}|}$, $\mathbf{F}\left(0, 0\right) = \mathbf{0}$.

17.1.34 For example, $\mathbf{F} = \left\langle -y,\, x \right\rangle$. The magnitude is $\sqrt{x^2+y^2}$.

17.1.35 $\nabla\varphi = \left\langle \varphi_x, \varphi_y \right\rangle = \left\langle 2xy - y^2, x^2 - 2xy \right\rangle$.

17.1.36 $\nabla\varphi = \left\langle \varphi_x, \varphi_y \right\rangle = \left\langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \right\rangle = \frac{1}{2\sqrt{xy}} \left\langle y, x \right\rangle$.

17.1.37 $\nabla\varphi = \left\langle \varphi_x, \varphi_y \right\rangle = \left\langle 1/y, -x/y^2 \right\rangle$.

17.1.38 $\nabla\varphi = \left\langle \frac{1}{1+(x^2/y^2)} \cdot \frac{1}{y}, \frac{1}{1+(x^2/y^2)} \cdot \frac{-x}{y^2} \right\rangle = \left\langle \frac{y}{y^2+x^2}, \frac{-x}{y^2+x^2} \right\rangle = \frac{1}{x^2+y^2} \left\langle y, -x \right\rangle$.

17.1.39 $\nabla\varphi = \left\langle x, y, z \right\rangle = \mathbf{r}$.

17.1.40 $\nabla\varphi = \left\langle \frac{2x}{1+x^2+y^2+z^2}, \frac{2y}{1+x^2+y^2+z^2}, \frac{2z}{1+x^2+y^2+z^2} \right\rangle = \frac{2\,\mathbf{r}}{|\mathbf{r}|^2+1}$.

17.1.41 $\nabla\varphi = -\left(x^2+y^2+z^2\right)^{-3/2} \left\langle x,\, y,\, z \right\rangle = -\frac{\mathbf{r}}{|\mathbf{r}|^3}$.

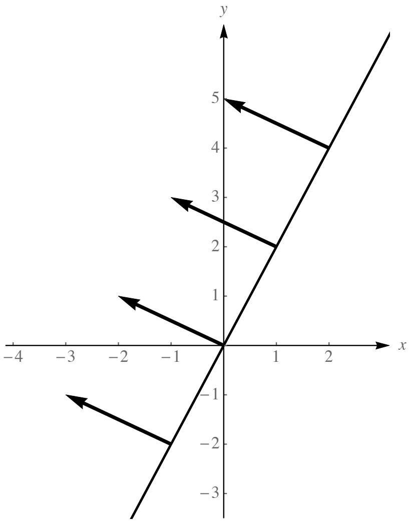
17.1.42 $\nabla\varphi = \left\langle e^{-z}\cos\left(x+y\right),\, e^{-z}\cos\left(x+y\right),\, -e^{-z}\sin\left(x+y\right) \right\rangle$.

17.1.43

a. $\mathbf{F} = \nabla\varphi = \langle -2, 1 \rangle$.

b. $\mathbf{F}(-1,-2) = \mathbf{F}(0,0) = \mathbf{F}(1,2) = \mathbf{F}(2,4) = \langle -2, 1 \rangle$.

c.

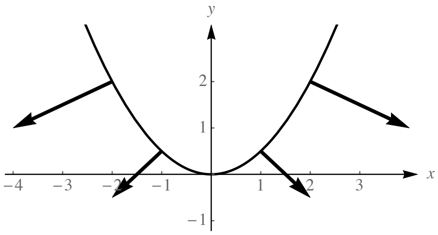


17.1.44

a. $\mathbf{F} = \nabla\varphi = \langle x, -1\rangle$.

b. $\mathbf{F}(-2,2) = \langle -2,-1\rangle$. $\mathbf{F}(-1,1/2) = \langle -1,-1\rangle$, $\mathbf{F}(1,1/2) = \langle 1,-1\rangle$, $\mathbf{F}(2,2) = \langle 2,-1\rangle$.

c.

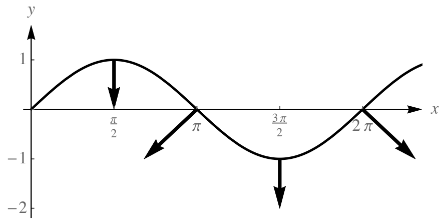


17.1.45

a. $\mathbf{F} = \nabla\varphi = \langle \cos x, -1\rangle$.

b. $\mathbf{F}(\pi/2,1) = \langle 0,-1\rangle$. $\mathbf{F}(\pi,0) = \langle -1,-1\rangle$, $\mathbf{F}(3\pi/2,-1) = \langle 0,-1\rangle$, $\mathbf{F}(2\pi,0) = \langle 1,-1\rangle$.

c.

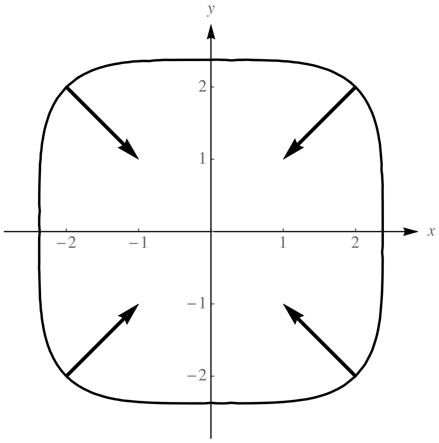


17.1.46

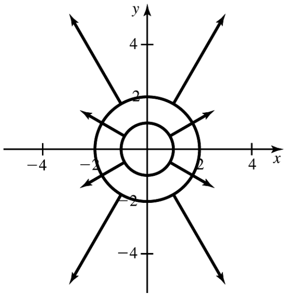
a. $\mathbf{F} = \nabla\varphi = \left\langle -x^3/8, -y^3/8\right\rangle$.

b. $\mathbf{F}(2,2) = \langle -1,-1\rangle$. $\mathbf{F}(-2,2) = \langle 1,-1\rangle$, $\mathbf{F}(-2,-2) = \langle 1,1\rangle$, $\mathbf{F}(2,-2) = \langle -1,1\rangle$.

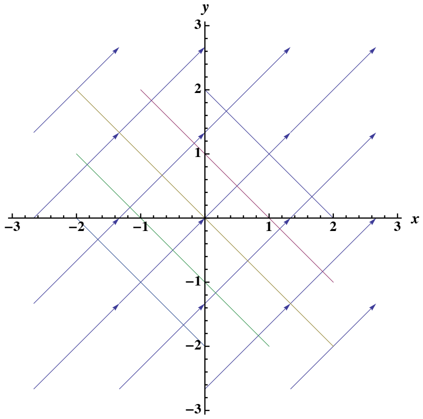
c.



17.1.47 The gradient field is $\langle \varphi_x, \varphi_y \rangle = \langle 2x,\, 2y \rangle$.



17.1.48 The gradient field is $\langle \varphi_x, \varphi_y \rangle = \langle 1, 1 \rangle$.



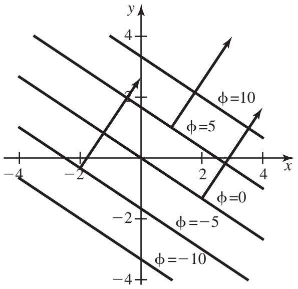
17.1.49

a. The gradient field is $\langle 2,\, 3 \rangle$.

b. The equipotential curve at (1, 1) is $2x + 3y = 5$, which is a line of slope $-\frac{2}{3}$ so has a tangent vector at $(x,\, y)$ parallel to $\langle 1,\, -\frac{2}{3} \rangle$. But $\langle 2,\, 3 \rangle \cdot \langle 1,\, -\frac{2}{3} \rangle = 0$, so the gradient field is normal to the equipotential line through (1, 1).

c. The equipotential curve at any point is a line of slope $-\frac{2}{3}$ and thus has a tangent vector at $(x,\, y)$ parallel to $\langle x,\, -\frac{2}{3}y \rangle$. The same argument as in part (b) shows that this is normal to the gradient field.

d.



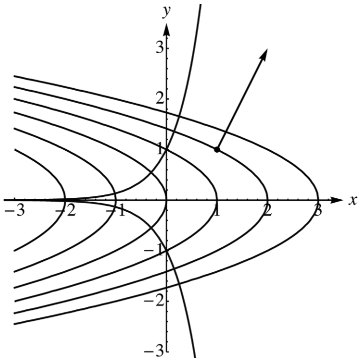
17.1.50

a. The gradient field is $\langle 1,\, 2y\rangle$.

b. At (1, 1), the tangent vector is parallel to $\langle -\varphi_y(1,\,1),\, \varphi_x(1,\,1)\rangle = \langle -2,\, 1\rangle$, which is normal to the gradient at (1, 1) (which is $\langle 1,\, 2\rangle$).

c. At $(x,\, y)$, the tangent vector is parallel to $\langle -2y,\, 1\rangle$, and $\langle 1,\, 2y\rangle \cdot \langle -2y,\, 1\rangle = 0$, so the gradient is everywhere normal to the equipotential curves.

d.



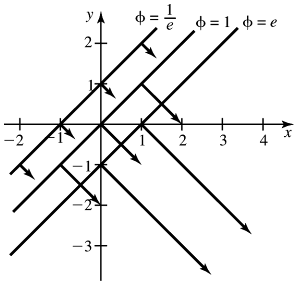
17.1.51

a. The gradient field is $\langle e^{x-y},\, -e^{x-y}\rangle$.

b. At (1, 1), the tangent vector is parallel to $\langle -\varphi_y(1,\,1),\, \varphi_x(1,\,1)\rangle = \langle 1,\, 1\rangle$, which is normal to the gradient $\langle 1,\, -1\rangle$ at (1, 1).

c. At $(x,\, y)$, the tangent vector is parallel to $\langle e^{x-y},\, e^{x-y}\rangle$, and $\langle e^{x-y},\, -e^{x-y}\rangle \cdot \langle e^{x-y},\, e^{x-y}\rangle = 0$, so the gradient is everywhere normal to the equipotential curves.

d.



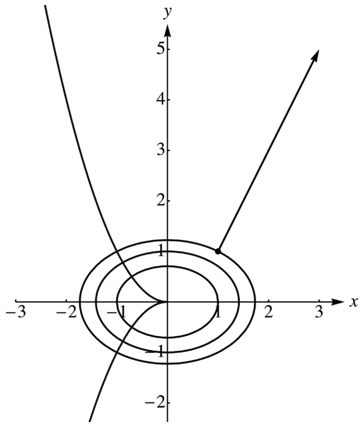
17.1.52

a. The gradient field is $\langle 2x, 4y\rangle$.

b. At (1, 1), the tangent vector is parallel to $\langle -\varphi_y(1, 1), \varphi_x(1, 1)\rangle = \langle -4, 2\rangle$, which is normal to the gradient $\langle 2, 4\rangle$ at (1, 1).

c. At (x, y), the tangent vector is parallel to $\langle -4y, 2x\rangle$, which is normal to the gradient field.

d.



17.1.53

a. True. $(\varphi_1)_x = (\varphi_2)_x = 3x^2$, and $(\varphi_1)_y = (\varphi_2)_y = 1$.

b. False. It is constant in magnitude (magnitude 1) but not direction.

c. True. For example, it points outwards along the line $y = x$ but horizontally along the line $x = 0$.

17.1.54

a. $V(x, y) = k\left(x^2+y^2\right)^{-1/2}$, so $\mathbf{E} = -\nabla V = -\langle V_x, V_y\rangle = k\left(x^2+y^2\right)^{-3/2}\langle x, y\rangle$.

b. From the above formula, the field is a varying multiple of $\langle x, y\rangle$, which is a radial field pointing away from the origin. The radial component of $\mathbf{E}$ is thus $|\mathbf{E}| = k\left(x^2+y^2\right)^{-3/2}\left|\langle x, y\rangle\right| = k\left(x^2+y^2\right)^{-3/2}\left(x^2+y^2\right)^{1/2} = \frac{k}{r^2}$.

c. The equipotential curves are curves of the form $\frac{k}{\sqrt{x^2+y^2}} = C$ so that $\sqrt{x^2+y^2} = \frac{k}{C}$ and the equipotential curves are circles. Thus the tangent vectors to the equipotential curves are proportional to $\left|\langle -y, x\rangle\right|$ and thus are normal to $\mathbf{E}$, which is proportional to $\left|\langle x, y\rangle\right|$.

17.1.55

a. $V_x = \frac{c\sqrt{x^2+y^2}}{r_0}\cdot\left(-\frac{1}{2}r_0\left(x^2+y^2\right)\right)$ and similarly, $V_y = -\frac{cy}{x^2+y^2}$, so that $\mathbf{E} = -\nabla V = \frac{c}{x^2+y^2}\langle x, y\rangle = \frac{c}{|\mathbf{r}|^2}\mathbf{r} = \frac{c}{|\mathbf{r}|}\cdot\frac{\mathbf{r}}{|\mathbf{r}|}$.

b. From the above formula, the field is a varying multiple of $\langle x, y\rangle$, which is a radial field pointing away from the origin. The radial component of $\mathbf{E}$ is thus $|\mathbf{E}| = \frac{c}{x^2+y^2}\sqrt{x^2+y^2} = \frac{c}{\sqrt{x^2+y^2}}$.

c. The equipotential curves are curves of the form $c\ln\left(\dfrac{r_0}{\sqrt{x^2+y^2}}\right)=K$, so are solutions to $\frac{r_0}{\sqrt{x^2+y^2}}=e^{cK}=C$ and thus of the form $\sqrt{x^2+y^2}=K_0$ for some constant K_0. Hence the equipotential curves are circles, so have tangent vectors proportional to $\langle -y,\, x\rangle$; these are clearly normal to $\mathbf{E}$, which is proportional to $\langle x,\, y\rangle$.

17.1.56

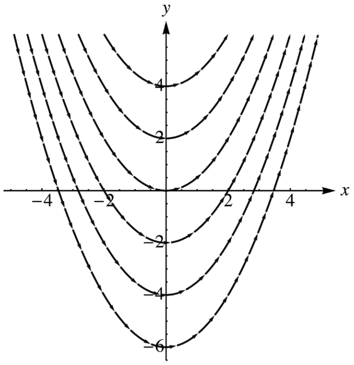
a. $U_x=\left(GMm\left(x^2+y^2+z^2\right)^{-1/2}\right)_x=$
$-\frac{1}{2}GMm\left(x^2+y^2+z^2\right)^{-3/2}(2x)=-GMmx\left(x^2+y^2+z^2\right)^{-3/2}$ and similarly for U_y and U_z. Thus $\mathbf{F}=-\nabla U=GMm\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle$.

b. From the above formula the field is a varying multiple of $\langle x,y,z\rangle$, which is a radial field pointing away from the origin. The radial component of $\mathbf{F}$ is thus $|\mathbf{F}|=GMm\left(x^2+y^2+z^2\right)^{-3/2}\sqrt{x^2+y^2+z^2}=\dfrac{GMm}{r^2}$.

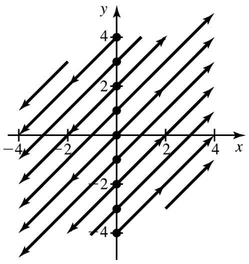
c. The equipotential surfaces are solutions to $GMm\sqrt{x^2+y^2+z^2}=K$ and so are spheres. The tangent plane at (x_0,y_0,z_0) is $U_x(x_0,y_0,z_0)(x-x_0)+U_y(x_0,y_0,z_0)(y-y_0)+U_z(x_0,y_0,z_0)(z-z_0)$ and so a normal to the plane is $\langle U_x,U_y,U_z\rangle$, which is proportional to $\mathbf{F}$.

17.1.57 The flow curve $y(x)$ of the vector field $\mathbf{F}$ at (x,y) is defined to be a continuous curve through (x,y) that is aligned with the vector field, i.e. whose tangent at (x,y) is given by $\mathbf{F}(x,y)=\langle f(x,y),g(x,y)\rangle$. The slope of the tangent line is then $\frac{g(x,y)}{f(x,y)}$, so this is $y'(x)$.

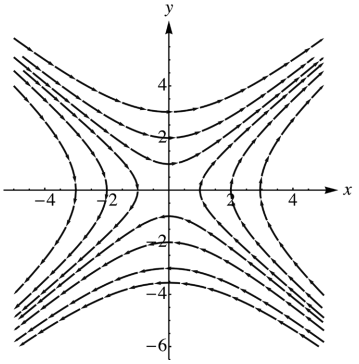
17.1.58 The streamlines satisfy $y'(x)=x$, so that $y(x)=\frac{1}{2}x^2+C$.



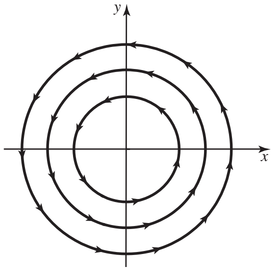
17.1.59 The streamlines satisfy $y'(x)=1$, so that $y(x)=x+C$.



17.1.60 The streamlines satisfy $y'(x)=\dfrac{x}{y}$. But also $\dfrac{d}{dx}\left(y^2\right)=2yy'(x)$, so that $y'(x)=\dfrac{\frac{d}{dx}\left(y^2\right)}{2y}$ and thus $\dfrac{d}{dx}\left(y^2\right)=2x$. Thus $y^2=x^2+C$ and the streamlines are the hyperbolas $x^2-y^2=K$.



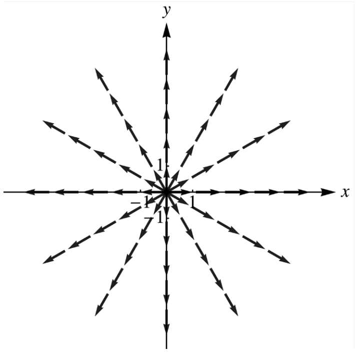
17.1.61 The streamlines satisfy $y'(x) = -\dfrac{x}{y}$. Because $\dfrac{d}{dx}\left(y^2\right) = 2yy'(x)$, we have $y'(x) = \dfrac{\frac{d}{dx}\left(y^2\right)}{2y}$ and thus $\dfrac{d}{dx}\left(y^2\right) = -2x$. Thus $y^2 = -x^2 + C$ and the streamlines are the circles $x^2 + y^2 = C$.



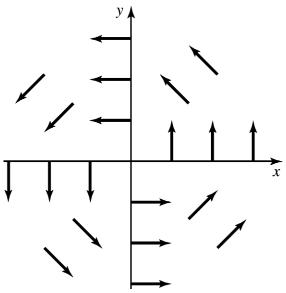
17.1.62 $\mathbf{u}_r$ is the unit vector forming an angle of θ with $\mathbf{i}$, so it is of unit length and proportional to $\left\langle \cos\theta, \sin\theta \right\rangle$. But $\cos^2(\theta) + \sin^2(\theta) = 1$, so this is in fact the unit vector. Thus $\mathbf{u}_r = \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j}$. Similarly, $\mathbf{u}_\theta$ forms an angle of $\theta + \frac{\pi}{2}$ with $\mathbf{i}$, so it is the unit vector $\left\langle \cos\left(\theta + \frac{\pi}{2}\right), \sin\left(\theta + \frac{\pi}{2}\right) \right\rangle = \left\langle -\sin(\theta), \cos(\theta) \right\rangle$. The other two formulas can be found by solving these for $\mathbf{i}$, $\mathbf{j}$ as linear equations. For example, $\mathbf{u}_r = \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j}, \mathbf{u}_\theta = -\sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j}$. Multiply the first equation by $\sin(\theta)$ and the second by $\cos(\theta)$ and add to obtain $\mathbf{u}_r \sin(\theta) + \mathbf{u}_\theta \cos(\theta) = \sin^2(\theta)\,\mathbf{j} + \cos^2(\theta)\,\mathbf{j} = \mathbf{j}$.

17.1.63 For $\theta = 0$, $\mathbf{u}_r$ is coincident with $\mathbf{i}$, and $\mathbf{u}_\theta$ with $\mathbf{j}$. From the formula $\mathbf{u}_r = \cos(0)\,\mathbf{i} + \sin(0)\,\mathbf{j} = \mathbf{i}$, $\mathbf{u}_\theta = -\sin(0)\,\mathbf{i} + \cos(0)\,\mathbf{j} = \mathbf{j}$. For $\theta = \frac{\pi}{2}$, the picture implies that we should have $\mathbf{u}_r = \mathbf{j}$, $\mathbf{u}_\theta = -\mathbf{i}$. From the formulas, $\mathbf{u}_r = \cos\left(\frac{\pi}{2}\right)\mathbf{i} + \sin\left(\frac{\pi}{2}\right)\mathbf{j} = \mathbf{j}$, $\mathbf{u}_\theta = -\sin\left(\frac{\pi}{2}\right)\mathbf{i} + \cos\left(\frac{\pi}{2}\right)\mathbf{j} = -\mathbf{i}$. For $\theta = \pi$, $\mathbf{u}_r$ is coincident with $-\mathbf{i}$, and $\mathbf{u}_\theta$ with $-\mathbf{j}$. From the formula $\mathbf{u}_r = \cos(\pi)\,\mathbf{i} + \sin(\pi)\,\mathbf{j} = -\mathbf{i}$, $\mathbf{u}_\theta = -\sin(\pi)\,\mathbf{i} + \cos(\pi)\,\mathbf{j} = -\mathbf{j}$. For $\theta = \frac{3\pi}{2}$, the picture implies that we should have $\mathbf{u}_r = -\mathbf{j}$, $\mathbf{u}_\theta = \mathbf{i}$. From the formulas, $\mathbf{u}_r = \cos\left(\frac{3\pi}{2}\right)\mathbf{i} + \sin\left(\frac{3\pi}{2}\right)\mathbf{j} = -\mathbf{j}$, $\mathbf{u}_\theta = -\sin\left(\frac{3\pi}{2}\right)\mathbf{i} + \cos\left(\frac{3\pi}{2}\right)\mathbf{j} = \mathbf{i}$.

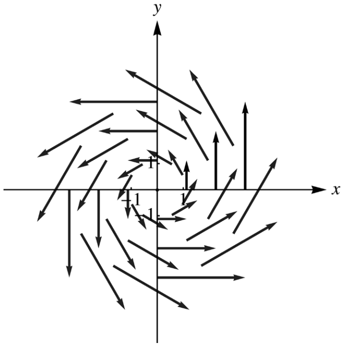
17.1.64 $\mathbf{F}(r, \theta) = \mathbf{u}_r = \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j} = \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\mathbf{j} = \dfrac{1}{\sqrt{x^2+y^2}}\left(x\mathbf{i} + y\mathbf{j}\right)$.



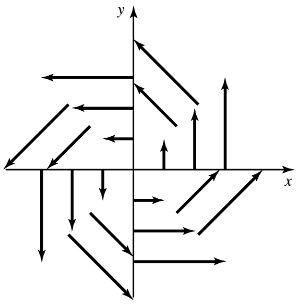
17.1.65 $\mathbf{F}(r,\,\theta) = \mathbf{u}_\theta = -\sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j} = -\dfrac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{j} = \dfrac{1}{\sqrt{x^2+y^2}}\left(-y\mathbf{i} + x\mathbf{j}\right)$.



17.1.66 $\mathbf{F}(r,\,\theta) = r\,\mathbf{u}_\theta = \sqrt{x^2+y^2}\left(-\sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j}\right) = \sqrt{x^2+y^2}\dfrac{1}{\sqrt{x^2+y^2}}\left(-y\mathbf{i} + x\mathbf{j}\right) = \left\langle -y,\, x\right\rangle$.



17.1.67 $\mathbf{F}(x,y) = -y\left(\mathbf{u}_r\cos(\theta) - \mathbf{u}_\theta\sin(\theta)\right) + x\left(\mathbf{u}_r\sin(\theta) + \mathbf{u}_\theta\cos(\theta)\right) =$
$-\,r\sin(\theta)\left(\cos(\theta)\,\mathbf{u}_r - \sin(\theta)\,\mathbf{u}_\theta\right) + r\cos(\theta)\left(\sin(\theta)\cos(\theta)\,\mathbf{u}_r + \cos(\theta)\,\mathbf{u}_\theta\right) = r\left(\sin^2(\theta) + \cos^2(\theta)\right)\mathbf{u}_\theta = r\mathbf{u}_\theta.$



17.2 Line Integrals

17.2.1 A single-variable integral integrates along a segment while a line integral integrates along an arbitrary curve.

17.2.2 $|\mathbf{r}'(t)| = \sqrt{\left(\mathbf{r}'_x\right)^2 + \left(\mathbf{r}'_y\right)^2} = \sqrt{1+4t^2}$

17.2.3 $\mathbf{r}'(t) = \langle -\sin t, 1\rangle$, so $|\mathbf{r}'(t)| = \sqrt{1+\sin^2}$. The line integral is $\displaystyle\int_{\pi/2}^{\pi} \frac{\cos t\sqrt{1+\sin^2 t}}{t}\,dt.$

17.2.4 $\mathbf{r}(t) = \langle \sin \pi t, t\rangle$ for $0 \le t \le 3$.

17.2.5 A direction vector for the line is $\langle 5-1, 4-2, 0-3\rangle = \langle 4, 2, -3\rangle$. The line segment is given by $\mathbf{r}(t) = \langle 1+4t, 2+2t, 3-3t\rangle$ for $0 \le t \le 1$.

17.2.6 $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \le t \le \pi/2$.

17.2.7 $\mathbf{r}(t) = \left\langle t^2+1, t\right\rangle$ for $2 \le t \le 4$.

17.2.8 $\mathbf{F} = \langle \cos t - \sin t, \sin t - \cos t\rangle$.

17.2.9

a. $\mathbf{F}\cdot\mathbf{T} = \left\langle t, 2t^3\right\rangle \cdot \left\langle 1, 3t^2\right\rangle = t + 6t^5$. The integral is given by $\displaystyle\int_0^2 (t+6t^5)\,dt$.

b. $\displaystyle\int_0^2 (t+6t^5)\,dt = \left(\frac{t^2}{2} + t^6\right)\Big|_0^2 = 2 + 64 = 66.$

17.2.10

a. Let $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ and $\mathbf{F} = \langle f, g\rangle = \langle 1, x\rangle = \langle 1, \cos t\rangle$. Then

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} (f(t)y'(t) - g(t)x'(t))\,dt = \int_0^{2\pi} (\cos t + \cos t \sin t)\,dt.$$

b. $\displaystyle\int_0^{2\pi} (\cos t + \cos t\sin t)\,dt = \left(\sin t + \frac{\sin^2 t}{2}\right)\Big|_0^{2\pi} = 0.$

17.2.11 $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ and $\displaystyle\int_C f\,dx + g\,dy + h\,dz$.

17.2.12 By reversing the orientation of C, we obtain the curve $-C$ from B to A. The value of the integral with the opposite orientation is the opposite of the value of the integral, so $\displaystyle\int_{-C} f\,ds = -10$.

17.2.13 $\displaystyle\int_{C_1} f\,ds + \int_{C_2} f\,ds = \int_C f\,ds = 10$, so $\displaystyle\int_{C_2} f\,ds = 7$.

17.2.14

a. Positive.

b. 0.

c. 0.

d. Negative.

e. 0.

f. Positive.

g. Negative.

h. 0.

17.2.15 Take the line integral of $\mathbf{F}\cdot\mathbf{T}$ along the curve using arc length parameterization.

17.2.16 The flux measures the degree to which the vector field is outwards normal to the curve C as C is traversed with a particular orientation.

17.2.17 $\mathbf{r}(s)$ is an arc length parameterization, so we have $\int_C xy\,ds = \int_0^{2\pi} \cos(s)\sin(s)\;ds = 0$.

17.2.18 With $\mathbf{r}(s) = \langle \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \rangle$, $|\mathbf{r}'(t)| = 1$, so that we have $\int_C \left(x^2 - 2y^2\right)\;ds = \int_0^4 \left(\frac{s^2}{2} - 2\frac{s^2}{2}\right)\;ds = -\int_0^4 \frac{s^2}{2}\;ds = -\frac{s^3}{6}\Big|_0^4 = -\frac{32}{3}$.

17.2.19 $|\mathbf{r}'(t)| = |\langle 3,4\rangle| = \sqrt{9+16} = 5$, so $\int_C (2x+y)\,ds = \int_0^2 (6t+4t)5\,dt = 25t^2\Big|_0^2 = 100$.

17.2.20 $|\mathbf{r}'(t)| = \left|\langle 3t^2, 4\rangle\right| = \sqrt{9t^4+16}$, so $\int_C x\,ds = \int_0^1 t^3\sqrt{9t^4+16}\,dt = \frac{1}{36}\int_{16}^{25} u^{1/2}\,du = \frac{u^{3/2}}{54}\Big|_{16}^{25} = \frac{125-64}{54} = \frac{61}{54}$.

17.2.21 $|\mathbf{r}'(t)| = |\langle -2\sin t, 2\cos t\rangle| = 2$, so $\int_C xy^3\,ds = \int_0^{\pi/2} 32\cos t\sin^3 t\,dt = 32\int_0^1 u^3\,du = 8u^4\Big|_0^1 = 8$.

17.2.22 $|\mathbf{r}'(t)| = |\langle \cos t, 1\rangle| = \sqrt{\cos^2 t + 1}$, so

$$\int_C 3x\cos y\,ds = \int_0^{\pi/2} 3\sin t\cos t\sqrt{\cos^2 t+1}\,dt = -\frac{3}{2}\int_2^1 \sqrt{u}\,du = u^{3/2}\Big|_1^2 = 2^{3/2} - 1 = 2\sqrt{2} - 1.$$

17.2.23 $|\mathbf{r}'(t)| = |\langle -3\sin t, 3\cos t, 4\rangle| = \sqrt{25} = 5$, so

$$\int_C (y-z)\,ds = \int_0^{2\pi} 5(3\sin t - 4t)\,dt = 5(-3\cos t - 2t^2)\Big|_0^{2\pi} = 5(-3 - 8\pi^2 + 3) = -40\pi^2.$$

17.2.24 $|\mathbf{r}'(t)| = \sqrt{0 + 9\sin^2 t + 9\cos^2 t} = 3$, so $\int_C (x - y + 2z)\;ds = 3\int_0^{2\pi} (1 - 3\cos(t) + 6\sin t)\;dt = 6\pi$.

17.2.25 $\mathbf{r}(t) = \langle 4\cos t, 4\sin t\rangle$, $0 \le t \le 2\pi$. $|\mathbf{r}'(t)| = \sqrt{(-4\sin t)^2 + (4\cos t)^2} = 4$, so

$$\int_C \left(x^2+y^2\right)\,ds = \int_0^{2\pi} 4\left(16\cos^2 t + 16\sin^2 t\right)\;dt = \int_0^{2\pi} 64\,dt = 128\pi.$$

17.2.26 $\mathbf{r}(t) = \langle 5t, 5t\rangle$, $0 \le t \le 1$. $|\mathbf{r}'(t)| = \sqrt{(5)^2 + (5)^2} = 5\sqrt{2}$, so

$$\int_C \left(x^2+y^2\right)\;ds = \int_0^1 50t^2\cdot 5\sqrt{2}\,dt = 250\sqrt{2}\int_0^1 t^2\,dt = \frac{250\sqrt{2}}{3}.$$

17.2.27 $\mathbf{r}(t) = \langle t, t\rangle$, $1 \le t \le 10$. $|\mathbf{r}'(t)| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$, so

$$\int_C \frac{x}{x^2+y^2}\,ds = \int_1^{10} \frac{t}{t^2+t^2}\cdot\sqrt{2}\,dt = \frac{\sqrt{2}}{2}\int_1^{10} \frac{1}{t}\,dt = \frac{\sqrt{2}}{2}\ln 10.$$

17.2.28] $\mathbf{r}(t) = \langle t, t^2\rangle$, $0 \le t \le 1$. $|\mathbf{r}'(t)| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1+4t^2}$, so

$$\int_C (xy)^{1/3}\;ds = \int_0^1 \left(t^3\right)^{1/3}\;\sqrt{1+4t^2}\,dt = \int_0^1 t\,\sqrt{1+4t^2}\,dt = \frac{5\sqrt{5}-1}{12}.$$

17.2.29 $\mathbf{r}\,(t) = \langle 2\cos t, 4\sin t\rangle$, $0 \le t \le \pi/2$.
$|\mathbf{r}'\,(t)| = \sqrt{4\sin^2 t + 16\cos^2 t} = \sqrt{4 + 12\cos^2 t} = 2\sqrt{1 + 3\cos^2 t}$.

$$\int_C xy\,ds = \int_0^{\pi/2} 16\sin t\cos t\sqrt{1 + 3\cos^2 t}\,dt.$$

Let $u = (1 + 3\cos^2 t)\,dt$ so that $du = -6\cos t\sin t\,dt$. Substituting gives

$$\frac{8}{3}\int_1^4 u^{1/2}\,du = \frac{16}{9}\left(u^{3/2}\right)\Big|_1^4 = \frac{16}{9}(8-1) = \frac{112}{9}.$$

17.2.30 $\mathbf{r}_1\,(t) = \langle t-1, t\rangle$, $\mathbf{r}_2\,(t) = \langle t, 1-t\rangle$, $0 \le t \le 1$. $|\mathbf{r}_1'\,(t)| = \sqrt{2}$, $|\mathbf{r}_2'\,(t)| = \sqrt{2}$.

$$\int_C (2x - 3y)\;ds = \int_0^1 (2\,(t-1) - 3t)\;\sqrt{2}\,dt + \int_0^1 (2t - 3\,(1-t))\;\sqrt{2}\,dt = \sqrt{2}\int_0^1 (4t-5)\;dt = -3\sqrt{2}..$$

17.2.31 $|\mathbf{r}'\,(t)| = \sqrt{4\sin^2 t + 0 + 4\cos^2 t} = 2$, so

$$\int_C (x+y+z)\;ds = 2\int_0^{\pi} (2\cos t + 2\sin t)\;dt = 4\,(\sin t - \cos t)\Big|_0^{\pi} = 8.$$

17.2.32 Let $\mathbf{r}\,(t) = \langle 2t+1, 2t+4, 2t+1\rangle$, $0 \le t \le 1$. Then $|\mathbf{r}'\,(t)| = \sqrt{12} = 2\sqrt{3}$, so $\displaystyle\int_C \frac{xy}{z}\,ds = 2\sqrt{3}\int_0^1 \frac{(2t+1)\,(2t+4)}{(2t+1)}\,dt = 2\sqrt{3}\int_0^1 (2t+4)\;dt = 10\sqrt{3}$.

17.2.33 For the first segment, $\mathbf{r}(t) = \langle 3t, 2t, 6t\rangle$ for $0 \le t \le 1$, so $\mathbf{r}'(t) = |\langle 3,2,6\rangle| = 7$, so $\displaystyle\int_C dz\,ds = \int_0^1 7(18)t^2\,dt = 42t^3\Big|_0^1 = 42$.

For the second segment, $\mathbf{r}(t) = \langle 3+4t, 2+7t, 6+4t\rangle$ for $0 \le t \le 1$, so $\mathbf{r}'(t) = \langle 4,7,4\rangle$, and $|\mathbf{r}'(t)| = |\langle 4,7,4\rangle| = 9$, so $\displaystyle\int_C xz\,ds = \int_0^1 9(3+4t)(6+4t)\,dt = 18\int_0^1 (8t^2+18t+9)\,dt = 18\left(\frac{8}{3}t^3 + 9t^2 + 9t\right)\Big|_0^1 = 372$.

Therefore, for the union of the two segments, we have $\displaystyle\int_C xz\,ds = 42 + 372 = 414$.

17.2.34 $|\mathbf{r}'\,(t)| = \sqrt{9} = 3$, so $\displaystyle\int_C xe^{yz}\,ds = 3\int_0^2 t\,e^{-4t^2}\,dt = -\frac{3}{8}e^{-4t^2}\Big|_0^2 = \frac{3}{8}\left(1 - \frac{1}{e^{16}}\right)$.

17.2.35 Parameterize C by $\mathbf{r}(t) = \langle t, 2t^2\rangle$ for $0 \le t \le 3$. Then $|\mathbf{r}'\,(t)| = \sqrt{1+16t^2}$ and $\displaystyle\int_C \rho\,ds = \int_0^3 \left(1+2t^3\right)\;\sqrt{1+16t^2}\,dt \approx 409.5$.

17.2.36 $\mathbf{r}'(\theta) = \langle -\sin\theta, \cos\theta\rangle$, so that$|\mathbf{r}'\,(\theta)| = 1$ and $\displaystyle\int_C \rho\,ds = \int_0^{\pi}\left(\frac{2\theta}{\pi} + 1\right)\,d\theta = 2\pi$.

17.2.37 Let $\mathbf{r}\,(t) = \langle t+1, 4t+1\rangle$, $0 \le t \le 1$. Then $|\mathbf{r}'\,(t)| = \sqrt{17}$ and $\displaystyle\int_C (x+2y)\;ds =$
$\displaystyle\int_0^1 ((t+1) + 2\,(4t+1))\cdot\sqrt{17}\,dt = \sqrt{17}\int_0^1 (9t+3)\;dt = \frac{15}{2}\sqrt{17}$. The length of the line segment is $\sqrt{17}$, so the average value is $\dfrac{15}{2}$.

17.2.38 Let $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$. Then $|\mathbf{r}'(t)| = 1$ and $\int_C (x\,e^y)\,ds = \int_0^{2\pi} \cos t\, e^{\sin t}\,dt = 0$. Thus, the average value is 0.

17.2.39 The length of the curve is the line integral of 1 along the curve. Then $\mathbf{r}'(t)$ simplifies to $\langle 5\cos(t/4), -5\sin(t/4), \frac{1}{2}\rangle$, and then

$$|\mathbf{r}'(t)| = \sqrt{25 + \frac{1}{4}} = \frac{1}{2}\sqrt{101}$$

so that the arc length is $\frac{1}{2}\int_0^2 \sqrt{101}\,dt = \sqrt{101}$.

17.2.40 The length of the curve is the line integral of 1 along the curve.
$|\mathbf{r}'(t)| = \sqrt{900\cos^2(t) + 1600\cos^2 t + 2500\sin^2 t} = 50$ so that $\int_C 1\,ds = 50\int_0^{2\pi} dt = 100\pi$.

17.2.41 $\mathbf{r}'(t) = \langle 4, 2t\rangle$, and $\mathbf{F}\cdot\mathbf{r}'(t) = \langle 4t, t^2\rangle \cdot \langle 4, 2t\rangle = 16t + 2t^3$, so that

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^1 \left(16t + 2t^3\right)\,dt = \frac{17}{2}.$$

17.2.42 $\mathbf{r}'(t) = \langle -4\sin t, 4\cos t\rangle$, so $\mathbf{F}\cdot\mathbf{r}'(t) = \langle -4\,\sin t, 4\cos t\rangle \cdot \langle -4\,\sin t, 4\cos t\rangle = 16$. Then $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{\pi} \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{\pi} 16\,dt = 16\pi$.

17.2.43 Let $\mathbf{r}(t) = \langle 4t+1, 9t+1\rangle$, $0 \le t \le 1$; then $\mathbf{r}'(t) = \langle 4, 9\rangle$ and $\mathbf{F}\cdot\mathbf{r}'(t) = \langle 9t+1, 4t+1\rangle\cdot\langle 4, 9\rangle = 72t+13$. Then $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 (72t+13)\,dt = 49$.

17.2.44 Let $\mathbf{r}(t) = \langle t, t^2\rangle$, $0 \le t \le 1$; then $\mathbf{r}'(t) = \langle 1, 2t\rangle$ and $\mathbf{F}\cdot\mathbf{r}'(t) = \langle -t^2, t\rangle \cdot \langle 1, 2t\rangle = t^2$. Then $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 t^2\,dt = \frac{1}{3}$.

17.2.45 $\mathbf{r}'(t) = \langle 2t, 6t\rangle$; then $\mathbf{F}\cdot\mathbf{r}'(t) = \left(10t^4\right)^{-3/2}\langle t^2, 3t^2\rangle\cdot\langle 2t, 6t\rangle = \frac{20t^3}{10\sqrt{10}\,t^6} = \frac{2}{\sqrt{10}}t^{-3}$ and $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \frac{2}{\sqrt{10}}\int_1^2 t^{-3}\,dt = \frac{3}{40}\sqrt{10}$.

17.2.46 $\mathbf{r}'(t) = \langle 1, 4\rangle$, so that $\mathbf{F}\cdot\mathbf{r}'(t) = \frac{1}{17t^2}\langle t, 4t\rangle\cdot\langle 1, 4\rangle = \frac{1}{t}$. Then $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_1^{10} \frac{1}{t}\,dt = \ln(10)$.

17.2.47

a. For C_1, the vector field points “against” the curve for most of its length, and with larger magnitude, so the integral is negative.

b. For C_2, the vector field points with the curve for its entire length, so the integral is positive.

17.2.48

a. For C_1, the vector field points against the curve for its entire length, so the integral over C_1 is negative.

b. For C_2, the vector field points with the curve, so the integral over C_2 is positive.

17.2.49 $\mathbf{r}_1(t) = \langle 1-t, 2-2t\rangle$, $\mathbf{r}_2(t) = \langle 0, 4t\rangle$, $0 \le t \le 1$. Then $\mathbf{r}_1'(t) = \langle -1, -2\rangle$, $\mathbf{r}_2'(t) = \langle 0, 4\rangle$, so that $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^1 \langle 2-2t, 1-t\rangle\cdot\langle -1, -2\rangle\,dt + \int_0^1 \langle 4t, 0\rangle\cdot\langle 0, 4\rangle\,dt = \int_0^1 0\,dt = 0$.

17.2.50 $\mathbf{r}_1(t) = \langle t-1, 8t\rangle$, $\mathbf{r}_2(t) = \langle 2t, 8\rangle$, $0 \le t \le 1$. Then $\mathbf{r}_1'(t) = \langle 1, 8\rangle$, $\mathbf{r}_2'(t) = \langle 2, 0\rangle$, so that $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^1 \langle t-1, 8t\rangle\cdot\langle 1,8\rangle\,dt + \int_0^1 \langle 2t, 8\rangle\cdot\langle 2,0\rangle\,dt = \int_0^1 (69t-1)\,dt = \frac{67}{2}$.

17.2.51 $\mathbf{r}(t) = \langle 2t, 8t^2\rangle$, $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 2, 16t\rangle$, so $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^1 \langle 8t^2, 2t\rangle\cdot\langle 2, 16t\rangle\,dt = \int_0^1 48t^2\,dt = 16$.

17.2.52 $\mathbf{r}(t) = \langle 2t+1, 8-4t\rangle$, $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 2, -4\rangle$, so $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^1 \langle 8-4t, -1-2t\rangle\cdot\langle 2,-4\rangle\,dt = \int_0^1 20\,dt = 20$.

17.2.53 $\mathbf{r}'(t) = \langle -4\sin t, 4\cos t, -4\sin t\rangle$. Thus we have $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt =$
$\displaystyle\int_0^{2\pi} \langle 4\cos t, 4\sin t, 4\cos t\rangle\cdot\langle -4\sin t, 4\cos t, -4\sin t\rangle\,dt = \int_0^{2\pi} -16\sin t\cos(t)\,dt = 0$.

17.2.54 $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, \tfrac{1}{2\pi}\rangle$, so $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'t\,dt = \int_0^{2\pi} \langle -2\sin t, 2\cos t, \frac{t}{2\pi}\rangle\cdot\langle -2\sin t, 2\cos t, \frac{1}{2\pi}\rangle\,dt = \int_0^{2\pi}\left(4 + \frac{t}{4\pi^2}\right)dt = \frac{1}{2} + 8\pi$.

17.2.55 Let $\mathbf{r}(t) = \langle t+1, t+1, t+1\rangle$, $0 \le t \le 9$, so that $\mathbf{r}'(t) = \langle 1,1,1\rangle$. Then $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^9 \frac{1}{\left(3(t+1)^2\right)^{3/2}}\langle t+1, t+1, t+1\rangle\cdot\langle 1,1,1\rangle\,dt = \frac{1}{\sqrt{3}}\int_0^9 \frac{1}{(t+1)^3}(t+1)\,dt = \frac{1}{\sqrt{3}}\int_0^9 \frac{1}{(t+1)^2}\,dt = \frac{3}{10}\sqrt{3}$.

17.2.56 Let $\mathbf{r}(t) = \langle 7t+1, 3t+1, t+1\rangle$, $0 \le t \le 1$, so that $\mathbf{r}'(t) = \langle 7, 3, 1\rangle$. Then $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^1 \frac{1}{(7t+1)^2 + (3t+1)^2 + (t+1)^2}\langle 7t+1, 3t+1, t+1\rangle\cdot\langle 7,3,1\rangle\,dt = \int_0^1 \frac{59t+11}{(7t+1)^2+(3t+1)^2+(t+1)^2}\,dt = \ln(2) + \frac{1}{2}\ln(7) = \ln\left(2\sqrt{7}\right)$.

17.2.57

a. Looking at the vector field, it appears that the vector field points counterclockwise just as much as it points clockwise at the boundary of the region, so we would expect the circulation to be zero.

b. $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, so $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi} \langle 2(\sin t - \cos t), 2\cos t\rangle\cdot\langle -2\sin(t), 2\cos t\rangle\,dt = 4\int_0^{2\pi}\left(\cos^2 t + \sin t\cos t - \sin^2 t\right)dt = 0$.

17.2.58

a. The circulation appears to be negative.

b. On C, $\mathbf{F} = \dfrac{\langle 4\sin t, -4\cos t\rangle}{\sqrt{16\cos^2 t + 16\sin^2 t}} = \langle \sin t, -\cos t\rangle$, so $\mathbf{F}\cdot\mathbf{r}'(t) = \langle \sin t, -\cos t\rangle\cdot\langle -2\sin t, 4\cos t\rangle = -2\sin^2 t - 4\cos^2 t = -2 - 2\cos^2 t$. So the circulation is $\displaystyle\int_0^{2\pi}(-2-2\cos^2 t)\,dt = \int_0^{2\pi}(-2-1-\cos 2t)\,dt = \int_0^{2\pi}(-3-\cos 2t)\,dt = -6\pi$.

17.2.59

a. Looking at the vector field, the inward-pointing vectors (in quadrants II and IV) appear larger than the outward-pointing vectors (in quadrants I and III). Thus we would expect the flux to be negative.

b. $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, so that $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} (2(\sin t - \cos t)\cdot 2\cos t - 2\cos t\cdot(-2\sin t))\,dt = -4\int_0^{2\pi}\left(\cos^2 t - 2\sin t\cos t\right)\,dt = -4\pi.$

17.2.60

a. The flux appears to be 0.

b. We have $\mathbf{r}(t) = \langle 2\cos t, 4\sin t\rangle$, and on C, $\mathbf{F} = \dfrac{\langle 4\sin t, -4\cos t\rangle}{\sqrt{16\cos^2 t + 16\sin^2 t}} = \langle \sin t, -\cos t\rangle$, so the flux is $\int_0^{2\pi}(4\sin t\cos t - (-\cos t)(-2\sin t))\,dt = \int_0^{2\pi} 2\sin t\cos t\,dt = \sin^2 t\Big|_0^{2\pi} = 0.$

17.2.61

a. True. This is the definition of an arc length parameterization.

b. True. Let $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$. Then $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$, and $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}\langle \sin t, \cos t\rangle\cdot\langle -\sin t, \cos t\rangle\,dt = \int_0^{2\pi}\left(\cos^2 t - \sin^2 t\right)\,dt = 0$. $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi}(\sin t\cdot\cos t - \cos t(-\sin t))\,dt = 2\int_0^{2\pi}\sin t\cos t\,dt = 0.$

c. True.

d. True. It is the line integral $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$.

17.2.62 The work done on either path is simply $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt$.

a. Here $\mathbf{r}(t) = \langle -t, 50\rangle$, for $-100 \le t \le 100$, so $\mathbf{r}'(t) = \langle -1, 0\rangle$ and $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_{-100}^{100} -150\,dt = 30000.$

b. Here $\mathbf{r}(t) = \langle 100\cos t, 100\sin t\rangle$ and $\mathbf{r}'(t) = \langle -100\sin t, 100\cos t\rangle$, so we have $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{\pi} -15000\sin t\,dt = 30000$. The same amount of work is done along each path.

17.2.63

a. For the first path, the work done is $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_{-100}^{100} -141\,dt = 28200$, and for the second path $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{\pi}(-14100\sin t - 5000\cos(t))\,dt = 28200,$
so again they are equal.

b. For the first path, the work done is $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_{-100}^{100} -141\,dt = 28200$, while for the second path, $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{\pi}(-14100\sin t - 5000\cos t)\,dt = 28200$, so the amount of work is still equal along the two paths.

17.2.64

a. Let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le 2\pi$ so that $|\mathbf{r}'(t)| = 1$. Then $\int_C f\,ds = \int_0^{2\pi} (\cos t + 2\sin(t))\,dt = 0$.

b. Let $\mathbf{r}(t) = \langle \cos t, -\sin t \rangle$ for $0 \le t \le 2\pi$ so that $|\mathbf{r}'(t)| = 1$. Note that this parameterization traces out the unit circle, but clockwise. Then $\int_C f\,ds = \int_0^{2\pi} (\cos t - 2\sin(t))\,dt = 0$.

c. The two integrals are equal.

17.2.65

a. Let $\mathbf{r}(t) = \langle t, t^2 \rangle$ for $0 \le t \le 1$ so that $|\mathbf{r}'(t)| = \sqrt{4t^2+1}$, and $\int_C f\,ds = \int_0^1 t\sqrt{4t^2+1}\,dt = \frac{5\sqrt{5}-1}{12}$.

b. $\mathbf{r}(t) = \langle 1-t, (1-t)^2 \rangle$ for $0 \le t \le 1$. Then $|\mathbf{r}'(t)| = \sqrt{(-1)^2 + (-2(1-t))^2} = \sqrt{4t^2 - 8t + 5}$ and $\int_C f\,ds = \int_0^1 (1-t)\sqrt{4t^2-8t+5}\,dt = \frac{5\sqrt{5}-1}{12}$.

c. The two integrals are equal.

17.2.66 Parameterize C$_1$ by $\mathbf{r}(t) = \langle 1-t, t \rangle$ for $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle -1, 1 \rangle$, and $\int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 ((-t)(-1) + (1-t)(1))\,dt = \int_0^1 1\,dt = 1$.

Parameterize C_2 by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le \frac{\pi}{2}$. Then $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$, and $\int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{\pi/2} ((-\sin t)(-\sin t) + (\cos t)(\cos t))\,dt = \int_0^{\pi/2} 1\,dt = \frac{\pi}{2}$.

Finally, parameterize C_3 by the two paths $\mathrm{r}_1(t) = \langle 1-t, 0 \rangle$ and $\mathrm{r}_2(t) = \langle 0, t \rangle$ for $0 \le t \le 1$, so that $r_1'(t) = \langle -1, 0 \rangle$ and $r_{\mathbf{2}}'(t) = \langle 0, 1 \rangle$. Then $\int_{C_3} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 ((0)(-1) + (1-t)(0))\,dt + \int_0^1 ((-t)(0) + (0)(1))\,dt = 0$. None of the three path integrals are equal.

17.2.67 Using the same parameterizations as for the previous problem, we have $\int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 ((t)(-1) + (1-t)(1))\,dt = \int_0^1 (1-2t)\,dt = 0$, $\int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{\pi/2} ((\sin t)(-\sin t) + (\cos t)(\cos t))\,dt = \int_0^{\pi/2} \left(\cos^2 t - \sin^2 t\right)\,dt = 0$, $\int_{C_3} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 ((0)(-1) + (1-t)(0))\,dt + \int_0^1 (t\cdot 0 + 0\cdot 1)\,dt = 0$. All three are equal to zero.

17.2.68 We have $dx = dt$, $dy = 2dt$, and $dz = 2t\,dt$. It follows that $\int_C x^2\,dx + dy + y\,dz = \int_0^3 (t^2 + 2 + 4t^2)\,dt = \int_0^3 (5t^2+2)\,dt = \left(\frac{5}{3}t^3 + 2t\right)\Big|_0^3 = 51$.

17.2.69 We have $dx = 2\,dt$, $dy = \cos t\,dt$, and $dz = -\sin t\,dt$. It follows that $\int_C x^3 y\,dx + xz\,dy + (x+y)^2\,dz = \int_0^{4\pi} (16t^3 \sin t + 2t\cos^2 t - (2t + \sin t)^2 \sin t)\,dt = 8\pi(48 + 7\pi - 128\pi^2)$, where the integral was calculated with a computer algebra system.

17.2.70 $\mathbf{r}(t) = \left\langle 3t^2, t \right\rangle$ for $1 \le t \le 3$. Therefore, $\int_C \frac{x^2}{y^4}\,ds = \int_1^3 \frac{9t^4}{t^4}\sqrt{36t^2+1}\,dt \approx 216.8$, where the integral was calculated with a computer algebra system.

17.2.71 $\mathbf{r}(t) = \langle 4\sin t, 4\cos t\rangle$, for $0 \le t \le \pi/2$. Then $\displaystyle\int_C \frac{y}{\sqrt{x^2+y^2}}\,dx - \frac{x}{\sqrt{x^2+y^2}}\,dy = \int_0^{\pi/2} (4\cos^2 t + 4\sin^2 t)\,dt = 4\cdot\frac{\pi}{2} = 2\pi$.

17.2.72 $\mathbf{r}(t) = \langle 1+2t, 2+6t, 4+9t\rangle$ for $0 \le t \le 1$. So $\displaystyle\int_C (x+y)\,dx + (x-y)\,dy + x\,dz = \int_0^1 2(3+8t) + 6(-1-4t) + 9(1+2t))\,dt = \int_0^1 (9+10t)\,dt = 14$.

17.2.73

a. $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, so the flux along the quarter circle is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{\pi/2} \left((2\sin t)(2\cos t) - (2\cos t)(-2\sin t)\right)\,dt = \int_0^{\pi/2} 8\sin t\cos t\,dt = 4$.

b. With $\mathbf{r}'(t)$ as above, the flux is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{\pi/2}^{\pi} (8\sin t\,\cos t)\,dt = -4$.

c. Both the normal vectors and the vector field F in the third quadrant are the negatives of their values in the first quadrant, so their dot product is the same. Thus the flux is identical.

d. Both the normal vectors and the vector field F in the fourth quadrant are the negatives of their values in the second quadrant, so their dot product is the same. Thus the flux is identical.

e. The total flux is $4 - 4 + 4 - 4 = 0$.

17.2.74 Letting $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ and $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$ as usual, we have $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi} \left(-b\sin^2 t + c\cos^2 t\right)\,dt = \pi(c-b)$, so that the circulation is zero only for $b = c$.

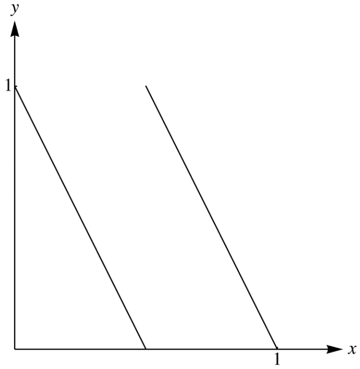
17.2.75 Let $\mathbf{r}(t) = \langle r\cos t, r\sin t\rangle$ for a circle of radius r, so that $\mathbf{r}'(t) = \langle -r\sin t, r\cos t\rangle$. Then $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi} \left((ar\cos t + br\sin t)(-r\sin t) + (cr\cos t + dr\sin t)(r\cos t)\right)\,dt = r^2\int_0^{2\pi}\left(-a\sin t\cos t - b\sin^2 t + c\cos^2 t + d\sin t\cos t\right)\,dt = \pi(c-b)\,r^2$, so that the circulation is zero provided $b = c$.

17.2.76 Let $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$; then the flux is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} (a\cos t\cos t - d\sin t(-\sin t))\,dt = (a+d)\pi$, so the flux is zero if $a = -d$.

17.2.77 Let $\mathbf{r}(t) = \langle r\cos t, r\sin t\rangle$ for a circle of radius r; then the flux is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \left((ar\cos t + br\sin t)(r\cos t) - (cr\cos t + dr\sin t)(-r\sin t)\right)\,dt = r^2\int_0^{2\pi}\left(a\cos^2 t + b\sin t\cos t + c\sin t\cos t + d\sin^2 t\right)\,dt = r^2(a+d)\pi$, so the flux is zero provided that $a = -d$.

17.2.78

a.



b. The gradient is $-50\mathbf{i} - 25\mathbf{j}$.

c. $\mathbf{F} = 50\mathbf{i} + 25\mathbf{j}$.

d. Parameterize the boundary C by $\mathbf{r}(t) = \langle 1, t\rangle$ for $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 0, 1\rangle$ and $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 (50\cdot 1 - 25\cdot 0)\,dt = 50$.

e. Parameterize the boundary C by $\mathbf{r}(t) = \langle 1-t, 1\rangle$ for $0 \le t \le 1$ (note: we do not use $\langle t, 1\rangle$ because we need counterclockwise orientation). Then $\mathbf{r}'(t) = \langle -1, 0\rangle$ and $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 (50\cdot 0 - 25\cdot(-1))\,dt = 25$.

17.2.79 $\mathbf{r}'(t) = \langle 1,1,1\rangle$ and $|\mathbf{r}\,(t)| = t\sqrt{3}$, so the work is $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_1^a \frac{3t}{(t\sqrt{3})^p}dt = 3^{1-p/2}\int_1^a t^{1-p}\,dt$

a. For $p = 2$, we have $\int_1^a \frac{1}{t}dt = \ln a$.

b. The work is not finite.

c. For $p = 4$, we have $\frac{1}{3}\int_1^a t^{-3}\,dt = -\frac{1}{6t^2}\Big|_1^a = \frac{1}{6} - \frac{1}{6a^2}$.

d. As $a \to \infty$, the work approaches $\frac{1}{6}$.

e. For the general $p > 1$, the analysis above shows that the integral is (for $p \ne 2$) $3^{1-p/2}\frac{1}{2-p}t^{2-p}\Big|_1^a = \frac{3^{1-p/2}}{2-p}\left(-1 + a^{2-p}\right)$ while for $p = 2$ the integral is (from part (a)) $\ln a$.

e. This approaches a limit only for $p > 2$ (when $2 - p < 0$). This limit is $\frac{3^{1-p/2}}{p-2}$.

17.2.80

a. $\int_C f\,dx = \int_a^b (f(t)x'(t) + 0(y'(t)))\,dt = \int_a^b f(t)x'(t)\,dt$.

b. $\int_C g\,dy = \int_a^b (0(x'(t)) + g(t)(y'(t)))\,dt = \int_a^b g(t)y'(t)\,dt$.

c. $\int_C xy\,dx = \int_0^1 300t^2\,dt = 100.$

d. $\int_C xy\,dy = \int_{-1}^1 t^3\,dt = 0.$

17.2.81 We use four line segment parameterizations for the rectangle, all for $0 \le t \le 1$: $\mathbf{r}_1(t) = \langle at, 0\rangle$, so $\mathbf{r}_1'(t) = \langle a, 0\rangle$. $\mathbf{r}_2(t) = \langle a, bt\rangle$, so $\mathbf{r}_2'(t) = \langle 0, b\rangle$. $\mathbf{r}_3(t) = \langle a - at, b\rangle$, so $\mathbf{r}_3'(t) = \langle -a, 0\rangle$. $\mathbf{r}_4(t) = \langle 0, b - bt\rangle$, so $\mathbf{r}_4'(t) = \langle 0, -b\rangle$. Then

$$\int_C x\,dy = \int_0^1 (at \cdot 0 + a \cdot b + (a - at)\cdot 0 + 0 \cdot (-b))\,dt = \int_0^1 ab\,dt = ab.$$

17.2.82 Parameterize C by $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$, so that $\mathbf{r}'(t) = \langle -a\sin t, a\cos t\rangle$. Then

$$-\int_C y\,dx = -\int_0^{2\pi} (a\sin t \cdot (-a\sin t))\,dt = a^2\int_0^{2\pi} \sin^2 t\,dt = \pi a^2.$$

17.3 Conservative Vector Fields

17.3.1 A simple curve has no self-intersections; the initial and terminal points of a closed curve are identical.

17.3.2 A region is connected, roughly speaking, if it consists of one piece. A simply connected region has the property that every closed loop can be contracted to a point.

17.3.3 If $\mathbf{F} = \langle f, g\rangle$ is a vector field in $\mathbb{R}^2$ and if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, then $\mathbf{F}$ is conservative.

17.3.4 If $\mathbf{F} = \langle f, g, h\rangle$ is a vector field in $\mathbb{R}^3$ and if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}$ and $\frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$, then $\mathbf{F}$ is conservative.

17.3.5 Integrate f with respect to x to get an answer where the "constant" is actually a function of y. Take the partial with respect to y and equate with g to compute the constant.

17.3.6 The integral is $\varphi(B) - \varphi(A)$ where $\nabla\varphi =$F.

17.3.7 The integral is zero.

17.3.8

- There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$.
- $\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in R and all smooth oriented curves C from A to B (path independence).
- $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ on all simple smooth closed oriented curves C in R.

17.3.9 Conservative, because $1_y = 1_x = 0$.

17.3.10 Conservative, because $x_y = y_x = 0$.

17.3.11 Not Conservative, because $-y_y = -1$, but $x_x = 1$.

17.3.12 Not Conservative. $-y_y = -1$, but $(x + y)_x = 1$.

17.3.13 Conservative. $\frac{\partial}{\partial y}e^{-x}\cos(y) = -e^{-x}\sin(y) = \frac{\partial}{\partial x}e^{-x}\sin(y)$.

17.3.14 Not Conservative. $\frac{\partial}{\partial y}\left(2x^3+xy^2\right)=2xy$, and $\frac{\partial}{\partial x}\left(2y^3-x^2y\right)=-2xy$.

17.3.15 Conservative. $f_y=z\cos xz=g_x$, $f_z=yz(-x\sin xz)+(\cos xz)y=-xyz\sin xz+y\cos xz=h_x$, and $g_z=x\cos xz=h_y$.

17.3.16 Not Conservative. Note that $f_z=-ye^{x-z}$ while $h_x=ye^{x-z}$.

17.3.17 Conservative. $\frac{\partial}{\partial y}(x)=\frac{\partial}{\partial x}(y)=0$. A potential function is $\frac{x^2+y^2}{2}$.

17.3.18 Conservative. $\frac{\partial}{\partial y}(-y)=\frac{\partial}{\partial x}(-x)=-1$. A potential function is $-xy$.

17.3.19 Not Conservative. $\frac{\partial}{\partial y}\left(x^3-xy\right)=-x$, and $\frac{\partial}{\partial x}\left(\frac{x^2}{2}+y\right)=x$.

17.3.20 Conserative. $\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)=\frac{-2xy}{\left(x^2+y^2\right)^2}=\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)$. Integrating $\frac{x}{x^2+y^2}$ with respect to x, we see that a potential function is $\frac{1}{2}\ln\left(x^2+y^2\right)$.

17.3.21 Conservative. $\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right)=\frac{-xy}{\left(x^2+y^2\right)^{3/2}}=\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$. Integrating $\frac{x}{\sqrt{x^2+y^2}}$ with respect to x, we obtain a potential function of $\sqrt{x^2+y^2}$.

17.3.22 Not Conservative. $f_z=0$ while $h_x=1$.

17.3.23 Conservative, because the mixed partials are pairwise equal. A potential function is $xz+y$.

17.3.24 Conservative, because the mixed partials are pairwise equal. A potential function is xyz.

17.3.25 Not Conservative. $g_z=e^z$ but $h_y=-e^z$.

17.3.26 Not Conservative. $g_z=-1$ and $h_y=1$.

17.3.27 Conservative, because the mixed partials are pairwise equal. To find the potential function, integrate $y+z$ with respect to x to get $x\left(y+z\right)+f\left(y,z\right)$; differentiating with respect to y gives $x+z=x+f_y\left(y,z\right)$ so that $f_y\left(y,z\right)=z$ and$f\left(y,z\right)=yz$. Thus a potential function is $xy+yz+xz$.

17.3.28 Conservative, because the mixed partials are pairwise equal. A potential function is $\frac{1}{2}\ln\left(x^2+y^2+z^2\right)$.

17.3.29 Conservative, because the mixed partials are pairwise equal. A potential function is $\sqrt{x^2+y^2+z^2}$.

17.3.30 Conservative, because the mixed partials are pairwise equal (and zero). A potential function is $\frac{1}{4}x^4+y^2-\frac{1}{4}z^4$.

17.3.31

a. $\nabla\varphi=\langle y,x\rangle$, so $\int_C \nabla\varphi\cdot\mathbf{r}'\left(t\right)\,dt=\int_0^\pi\langle\sin t,\cos t\rangle\cdot\langle-\sin t,\cos t\rangle\,dt=\int_0^\pi\cos\left(2t\right)\,dt=0$.

b. Because $\nabla\varphi$ is obviously conservative, the integral is simply $\varphi\left(\cos\left(\pi\right),\sin\left(\pi\right)\right)-\varphi\left(\cos\left(0\right),\sin\left(0\right)\right)=0$.

17.3.32

a. $\nabla\varphi = \langle 1,3\rangle$, so $\int_C \nabla\varphi\cdot\mathbf{r}'(t)\ dt = \int_0^2 \langle 1,3\rangle\cdot\langle -1,1\rangle\,dt = \int_0^2 2\,dt = 4$.

b. The integral is $\varphi(0,2)-\varphi(2,0) = 6-2 = 4$. (Note that $\varphi(0,2)$ is φ evaluated at the point where $t=2$).

17.3.33

a. $\nabla\varphi = \langle x,y,z\rangle$, so $\int_C \nabla\varphi\cdot\mathbf{r}'(t)\ dt = \int_0^{2\pi} \langle \cos t, \sin t, \frac{t}{\pi}\rangle\cdot\langle -\sin t, \cos t, \frac{1}{\pi}\rangle\,dt = \int_0^{2\pi}\left(\frac{t}{\pi^2}\right)\ dt = 2$.

b. The integral is $\varphi(1,0,2)-\varphi(1,0,0) = \frac{5}{2}-\frac{1}{2} = 2$.

17.3.34

a. $\nabla\varphi = \langle y+z, x+z, x+y\rangle$, so $\int_C \nabla\varphi\cdot\mathbf{r}'(t)\ dt = \int_0^4 \langle 5t,4t,3t\rangle\cdot\langle 1,2,3\rangle\,dt = \int_0^4 22t\,dt = 176$.

b. The integral is $\varphi(4,8,12)-\varphi(0,0,0) = 176-0 = 176$.

17.3.35 $\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(\mathbf{r}(2)) - \varphi(\mathbf{r}(1)) = \varphi(3,6)-\varphi(1,2) = 10-7 = 3$.

17.3.36 $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \varphi(6,4)-\varphi(1,2) = 20-7 = 13$.

17.3.37 $\int_C f\,dx + g\,dy = \varphi(1,2)-\varphi(6,4)+\varphi(3,6)-\varphi(1,2) = 7-20+10-7 = -10$.

17.3.38 $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$.

17.3.39 Write $\varphi(x,y) = x^2+y^2$. Then using the Fundamental Theorem, this integral is equal to $\varphi(3,4)-\varphi(0,1) = 25-1 = 24$.

17.3.40 Write $\varphi(x,y,z) = x+y+z$. Then using the Fundamental Theorem, this integral is equal to $\varphi(3,0,7)-\varphi(1,-1,2) = 10-2 = 8$.

17.3.41 Write $\varphi(x,y) = e^{-x}\cos(y)$. Then using the Fundamental Theorem, this integral is equal to $\varphi(\ln 2, 2\pi) - \varphi(0,0) = e^{-\ln 2}\cos(2\pi) - e^0\cos(0) = \frac{1}{2}-1 = -\frac{1}{2}$.

17.3.42 Write $\varphi(x,y,z) = 1+x^2yz$. Then using the Fundamental Theorem, this integral is equal to $\varphi(\cos(8\pi), \sin(8\pi), 4\pi) - \varphi(\cos(0), \sin(0), 0) = \varphi(1,0,4\pi)-\varphi(1,0,0) = 1-1 = 0$.

17.3.43 Let $\varphi(x,y,z) = x\cos(z-2y)$. The given integral is equal to $\varphi(\pi^2,\pi,\pi)-\varphi(0,0,0) = -\pi^2-0 = -\pi^2$.

17.3.44 Let $\varphi(x,y) = ye^x$. The given integral is equal to $\varphi(4,9)-\varphi(0,1) = 9e^4-1$.

17.3.45 Parameterize C by $\mathbf{r}(t) = \langle 4\cos t, 4\sin t\rangle$, $0\le t\le 2\pi$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle 4\cos t, 4\sin t\rangle\cdot\langle -4\sin t, 4\cos t\rangle\,dt = \int_0^{2\pi} 0\,dt = 0$.

17.3.46 Parameterize C by $\mathbf{r}(t) = \langle 8\cos t, 8\sin t\rangle$, $0\le t\le 2\pi$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle 8\sin t, 8\cos t\rangle\cdot\langle -8\sin t, 8\cos t\rangle\,dt = 8\int_0^{2\pi}\left(\cos^2 t - \sin^2(t)\right)\ dt = 0$.

17.3.47 Parameterize C by three paths, all for $0 \le t \le 1$: $\mathbf{r}_1(t) = \langle t, t-1\rangle$, so $\mathbf{r}_1'(t) = \langle 1, 1\rangle$. $\mathbf{r}_2(t) = \langle 1-t, t\rangle$, so $\mathbf{r}_2'(t) = \langle -1, 1\rangle$. $\mathbf{r}_3(t) = \langle 0, 1-2t\rangle$, so $\mathbf{r}_3'(t) = \langle 0, -2\rangle$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle t, t-1\rangle \cdot \langle 1, 1\rangle\, dt + \int_0^1 \langle 1-t, t\rangle \cdot \langle -1, 1\rangle\, dt + \int_0^1 \langle 0, 1-2t\rangle \cdot \langle 0, -2\rangle\, dt = \int_0^1 (t + (t-1) + (t-1) + t - 2(1-2t))\, dt = \int_0^1 (8t-4)\, dt = 0$.

17.3.48 Parameterize C by $\mathbf{r}(t) = \langle 3\cos t, 3\sin t\rangle$, $0 \le t \le 2\pi$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 3\sin t, -3\cos t\rangle \cdot \langle -3\sin t, 3\cos t\rangle\, dt = -9\int_0^{2\pi} 1\, dt = -18\pi$. This integral is not zero because the vector field $\langle y, -x\rangle$ is not conservative: $\frac{\partial}{\partial y}(y) = 1$, while $\frac{\partial}{\partial x}(-x) = -1$.

17.3.49 Using the given parameterization,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle \cos t, \sin t, 2\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt = \int_0^{2\pi} 0\, dt = 0.$$

17.3.50 Using the given parameterization,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle \sin t - \cos t, 0, \cos t - \sin t\rangle \cdot \langle -\sin t, \cos t, -\sin t\rangle\, dt = \int_0^{2\pi} 0\, dt = 0.$$

17.3.51 $\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = 2 - 7 = -5$.

17.3.52 $\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = 9 - 0 = 9$.

17.3.53 $\mathbf{F}$ is a conservative vector field; a potential function can be found by integrating x^2 with respect to y to obtain $x^2y + f(x,z)$; differentiate with respect to z to get $f_z(x,z) = 2xz$, so that $f(x,z) = xz^2 + g(x)$. Thus the potential function is $x^2y + xz^2 + g(x)$; differentiating with respect to x gives $2xy + z^2 + g_x(x) = 2xy + z^2$, so that $g_x(x) = 0$ and we may take $g(x) = 0$. So if $\varphi = x^2y + xz^2$, then $\nabla\varphi = \mathbf{F}$ and thus the integral is zero because both sine and cosine, and thus φ, have the same values at the two endpoints of C.

17.3.54 If $x = x(t)$, $y = y(t)$, then $\oint_C e^{-x}(\cos y\, dx + \sin y\, dy) = \oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle e^{-x}\cos y, e^{-x}\sin y\rangle$. This is a conservative vector field, because $\frac{\partial}{\partial y}\left(e^{-x}\cos y\right) = -e^{-x}\sin y = \frac{\partial}{\partial x}\left(e^{-x}\sin y\right)$, so that the integral around the closed curve is zero.

17.3.55 The value of the integral is $\sin xy\Big|_{(0,0)}^{(2,\pi/4)} = \sin\pi/2 - \sin 0 = 1$.

17.3.56 Let $\varphi(x,y) = \frac{x^4}{4} + \frac{y^4}{4}$. The value of the integral is $\varphi(2,0) - \varphi(1,1) = 4 - \frac{1}{2} = \frac{7}{2}$.

17.3.57

a. False. Parametrize the curve by $x = 4\cos t + 1$, $y = 4\sin t$, for $0 \le t \le 2\pi$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle -4\sin t, 4\cos t + 1\rangle \cdot \langle -4\sin t, 4\cos t\rangle\, dt = \int_0^{2\pi} (16\sin^2 t + 16\cos^2 t + 4\cos t)\, dt = 32\pi \ne 0$.

b. True. This is because $\mathbf{F}$ is conservative.

c. True. If the vector field is $\langle a, b\rangle$, then a potential function is $ax + by$.

d. True. This is because $\dfrac{\partial}{\partial y} f(x) = \dfrac{\partial}{\partial x} g(y) = 0$.

e. True. This follows from the definitions.

17.3.58 $\displaystyle\oint_C ds$ is the length of the curve C, which is 2π. The other two integrals are zero, because they are the same as integrating the conservative vector fields $\langle 1,0,0\rangle$, and $\langle 0,1,0\rangle$ respectively around a closed curve.

17.3.59 This is a conservative vector field with potential function $\varphi(x,y) = \dfrac{1}{2}x^2 + 2y$, so the work is $\varphi(2,4) - \varphi(0,0) = 10$.

17.3.60 This is a conservative vector field with potential function $\varphi(x,y) = \dfrac{1}{2}\left(x^2+y^2\right)$, so the work is $\varphi(3,-6) - \varphi(1,1) = \dfrac{45}{2} - 1 = \dfrac{43}{2}$.

17.3.61 This is a conservative vector field with potential function $\varphi(x,y,z) = \dfrac{1}{2}\left(x^2+y^2+z^2\right)$, so the work is $\varphi(2,4,6) - \varphi(1,2,1) = 28 - 3 = 25$.

17.3.62 This is not a conservative vector field, because for example $\dfrac{\partial}{\partial z}\left(e^{x+y}\right) = 0$, while $\dfrac{\partial}{\partial x}\left(ze^{x+y}\right) = ze^{x+y}$. Parameterize C by $\langle -t, 2t, -4t\rangle$, $0 \le t \le 1$; then $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle e^t, e^t, -4te^t\rangle \cdot \langle -1, 2, -4\rangle\, dt = \int_0^1 \left(e^t + 16te^t\right) dt = e + 15$.

17.3.63

a. Negative.

b. Positive.

c. No. The value of the integral is not zero, so **F** is not conservative.

17.3.64 No.

17.3.65 $\mathbf{F} = \langle a,b,c\rangle$ is a conservative force field with potential function $\varphi(x,y,z) = ax + by + cz$, so the work done is $\varphi(B) - \varphi(A) = \mathbf{F}\cdot B - \mathbf{F}\cdot A = \mathbf{F}\cdot(B-A) = \mathbf{F}\cdot\overrightarrow{AB}$.

17.3.66

a. The acceleration is the time derivative of velocity, so Newton's second law says that $m\dfrac{d\mathbf{v}}{dt} = m\mathbf{a} = \mathbf{F} = -\nabla\varphi$

b. By the product rule, $\dfrac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \dfrac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \mathbf{v}\cdot\dfrac{d\mathbf{v}}{dt} = 2\dfrac{d\mathbf{v}}{dt}\cdot\mathbf{v}$, and the desired equation follows.

c. Multiplying part (a) by $\mathbf{v} = \mathbf{r}'$ and using (b), we have (letting C be a path from A to B) $m\dfrac{d\mathbf{v}}{dt}\cdot\mathbf{v} = -\nabla\varphi\cdot\mathbf{r}' = \dfrac{1}{2}m\dfrac{d}{dt}(\mathbf{v}\cdot\mathbf{v})$.

Thus, $\displaystyle\int_C -\nabla\varphi\cdot\mathbf{r}' = \frac{1}{2}m\int_A^B \frac{d}{dt}(\mathbf{v}(t)\cdot\mathbf{v}(t))\, dt$, so $\varphi(A) - \varphi(B) = \dfrac{1}{2}m\left(|\mathbf{v}|\right)^2\Big|_A^B$. The last equality follows because the integrand is a conservative vector field. Thus $\dfrac{1}{2}m|\mathbf{v}(B)|^2 - \dfrac{1}{2}m|\mathbf{v}(A)|^2 = \varphi(A) - \varphi(B)$, so $\dfrac{1}{2}m|\mathbf{v}(B)|^2 + \varphi(B) = \dfrac{1}{2}m|\mathbf{v}(A)|^2 + \varphi(A)$.

17.3.67

a. Away from the origin (where the denominator of the force field equation is undefined), the force field is conservative because, for example,

$$\frac{\partial}{\partial y} GMm \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} = GMm \frac{2xy}{\left(x^2+y^2+z^2\right)^{5/2}} = \frac{\partial}{\partial x} GMm \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}.$$

b. A potential function for the force field is $\varphi(x,y,z) = GMm\left(x^2+y^2+z^2\right)^{-1/2} = GMm\frac{1}{|\mathbf{r}|}$.

c. The work done in moving the point from A to B, because the force field is conservative, is $\varphi(B) - \varphi(A) = GMm\left(\frac{1}{|B|} - \frac{1}{|A|}\right) = GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$.

d. Because the field is conservative, the work done does not depend on the path.

17.3.68 This vector field is $\mathbf{F} = \langle x,y,z\rangle\left(x^2+y^2+z^2\right)^{-p/2}$, so away from the origin is conservative with potential function $\varphi(x,y,z) = \frac{1}{2-p}\left(x^2+y^2+z^2\right)^{1-p/2}$ as long as $p \neq 2$. When $p = 2$, the potential function is $\varphi = \frac{1}{2}\ln(x^2+y^2+z^2)$. The field is conservative at the origin if it is defined and if its potential function is defined, i.e. if both $-\frac{p}{2}$ and $1-\frac{p}{2}$ are nonnegative, which happens only if $p \le 0$.

17.3.69

a. This field is

$$\mathbf{F} = \langle -y, x\rangle\left(x^2+y^2\right)^{-p/2},$$

and we have

$$\begin{aligned}\frac{\partial}{\partial y}\left(-y\left(x^2+y^2\right)^{-p/2}\right) &= -\left(x^2+y^2\right)^{-p/2} + py^2\frac{\left(x^2+y^2\right)^{-p/2}}{x^2+y^2} \\ &= -\left(x^2+y^2\right)^{-p/2} + py^2\left(x^2+y^2\right)^{-1-p/2} = \frac{-x^2+(p-1)y^2}{(x^2+y^2)^{1+p/2}}.\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x}\left(x\left(x^2+y^2\right)^{-p/2}\right) &= \left(x^2+y^2\right)^{-p/2} - px^2\frac{\left(x^2+y^2\right)^{-p/2}}{x^2+y^2} \\ &= \left(x^2+y^2\right)^{-p/2} - px^2\left(x^2+y^2\right)^{-1-p/2} = \frac{-(p-1)x^2+y^2}{(x^2+y^2)^{1+p/2}}.\end{aligned}$$

For the force field to be conservative, these two would have to be equal. However, their difference is $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2\left(x^2+y^2\right)^{-p/2} - p\left(x^2+y^2\right)\left(x^2+y^2\right)^{-1-p/2} = 2\left(x^2+y^2\right)^{-p/2} - p\left(x^2+y^2\right)^{-p/2} = (2-p)\left(x^2+y^2\right)^{-p/2}$ which is in general nonzero.

b. From the above formula, if $p = 2$, then the mixed partials are equal, so that $\mathbf{F}$ is conservative.

c. For $p = 2$, $\mathbf{F} = \frac{1}{x^2+y^2}\langle -y, x\rangle$. Integrating the x component of F with respect to x gives $\varphi = \tan^{-1}\left(\frac{y}{x}\right)$.

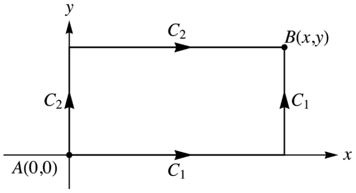
17.3.70

a. Because $\frac{\partial}{\partial y}(ax+by) = b$ and $\frac{\partial}{\partial x}(cx+dy) = c$, the field is conservative when $b = c$.

b. Because $\frac{\partial}{\partial y}\left(ax^2 - by^2\right) = -2by$ and $\frac{\partial}{\partial x}(cxy) = cy$, the field is conservative when $c = -2b$.

17.3.71

a.



b. Parameterize C_1 by two paths: $\mathrm{r}_1(t) = \langle t, 0\rangle,\ 0 \le t \le x$, and $\mathrm{r}_2(t) = \langle x, t\rangle,\ 0 \le t \le y$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle 2t, -t\rangle\cdot\langle 1,0\rangle\,dt + \int_0^y \langle 2x - t, -x + 2t\rangle\cdot\langle 0,1\rangle\,dt = \int_0^x 2t\,dt + \int_0^y (2t - x)\,dt = x^2 + y^2 - xy$.

c. Parameterize C_2 by the paths $\mathrm{r}_1(t) = \langle 0, t\rangle,\ 0 \le t \le y$, and $\mathrm{r}_2(t) = \langle t, y\rangle,\ 0 \le t \le x$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^y \langle -t, 2t\rangle\cdot\langle 0,1\rangle\,dt + \int_0^x \langle 2t - y, -t + 2y\rangle\cdot\langle 1,0\rangle\,dt = \int_0^y 2t\,dt + \int_0^x (2t - y)\,dt = x^2 + y^2 - xy$.

17.3.72 Using problem 71, we have the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, and $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle 0, -t\rangle\cdot\langle 1,0\rangle\,dt + \int_0^y \langle -t, -x\rangle\cdot\langle 0,1\rangle\,dt = \int_0^x 0\,dt + \int_0^y (-x)\,dt = -xy$.

17.3.73 Using problem 71 and the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, we have $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle t, 0\rangle\cdot\langle 1,0\rangle\,dt + \int_0^y \langle x, t\rangle\cdot\langle 0,1\rangle\,dt = \int_0^x t\,dt + \int_0^y t\,dt = \frac{1}{2}\left(x^2 + y^2\right)$.

17.3.74 Using problem 71 and the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, note that $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \langle x, y\rangle\left(x^2 + y^2\right)^{-1/2}$, and $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle 1,0\rangle\cdot\langle 1,0\rangle\,dt + \int_0^y \left\langle \frac{x}{\sqrt{x^2+t^2}}, \frac{t}{\sqrt{x^2+t^2}}\right\rangle\cdot\langle 0,1\rangle\,dt = \int_0^x 1\,dt + \int_0^y \frac{t}{\sqrt{x^2+t^2}}\,dt = x + \sqrt{x^2+y^2} - x = \sqrt{x^2+y^2}$.

17.3.75 Using problem 71 and the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, we have $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle 2t^3, 0\rangle\cdot\langle 1,0\rangle\,dt + \int_0^y \langle 2x^3 + xt^2, 2t^3 + x^2t\rangle\cdot\langle 0,1\rangle\,dt = \int_0^x 2t^3\,dt + \int_0^y \left(2t^3 + x^2t\right)\,dt = \frac{1}{2}\left(x^4 + x^2y^2 + y^4\right)$.

17.4 Green's Theorem

17.4.1 As with the Fundamental Theorem of Calculus, it allows evaluation of the integral of a derivative by looking at the value of the underlying function on the boundary of a region (or, in the case of the Fundamental Theorem, an interval).

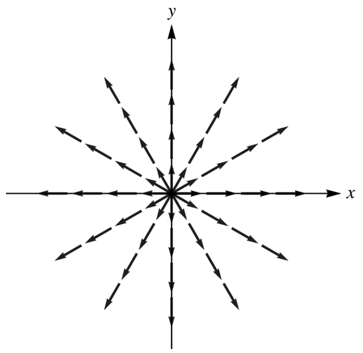
17.4.2 The line integral for flux corresponds to the double integral of the divergence; the line integral for circulation to the double integral of the curl.

17.4.3 The area is $\frac{1}{2}\oint_C (x\,dy - y\,dx)$ where C is the boundary of the region.

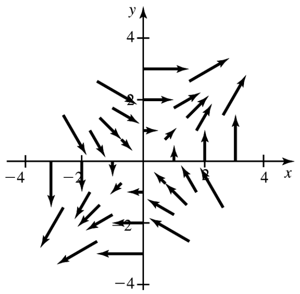
17.4.4 Because the curl being zero is an equivalent condition to the field being conservative.

17.4.5 Because the flux is the integrand in Green's theorem, so the integral vanishes.

17.4.6 A conservative vector field such as $\langle x, y\rangle$ will have zero curl:



17.4.7



17.4.8 A conservative and a source-free field each have functions (a potential function in the case of a conservative field; a stream function in the case of a source-free field) that closely reflect the vector field. The properties of the partials of these functions are such that the curl (or divergence, for a source-free field) vanish.

17.4.9

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0 - 0 = 0$.

b. The two-dimensional divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 1 + 1 = 2$.

c. The vector field is irrotational because its curl is zero.

d. The vector field is not source free because the divergence is not zero.

17.4.10

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = -1 - 1 = -2$.

b. The two-dimensional divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 0 + 0 = 0$.

c. The vector field is not irrotational because its curl is not zero.

d. The vector field is source free because the divergence is zero.

17.4.11

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = -3 - 1 = -4$.

b. The two-dimensional divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 0 + 0 = 0$.

c. The vector field is not irrotational because its curl is not zero.

d. The vector field is source free because the divergence is zero.

17.4.12

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = -2y - 2x$

b. The two-dimensional divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 2x + 2y - 2x - 2y = 0$.

c. The vector field is not irrotational because its curl is not zero.

d. The vector field is source free because the divergence is zero.

17.4.13

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = y^2 + 4x^3 - 4x^3 = y^2$.

b. The two-dimensional divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 12x^2y + 2xy$.

c. The vector field is not irrotational because its curl is not zero.

d. The vector field is not source free because the divergence is not zero.

17.4.14

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 12y - 1$.

b. The two-dimensional divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 12x^2 + 12x$.

c. The vector field is not irrotational because its curl is not zero.

d. The vector field is not source free because the divergence is not zero.

17.4.15

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 1 - 0 = 1$, so $\mathbf{F}$ is not irrotational.

b. $\mathbf{r}_1(t) = \left\langle t, t^2 \right\rangle$ for $0 \le t \le 1$ and $\mathbf{r}_2(t) = \langle 1-t, 1-t \rangle$ for $0 \le t \le 1$.

c. $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t^2 + 1)\,dt + \int_0^1 (t-2)\,dt = \left(\frac{2t^3}{3} + t\right)\Big|_0^1 + \left(\frac{t^2}{2} - 2t\right)\Big|_0^1 = \frac{5}{3} - \frac{3}{2} = \frac{1}{6}$.
$\displaystyle\int_0^1 \int_{x^2}^x 1\,dy\,dy = \int_0^1 (x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{6}$.

d. The two-dimensional divergence is $0 + 0 = 0$, $\displaystyle\iint_R 0\,dA = 0$.

17.4.16

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0 - 0 = 0$.

b. The two-dimensional divergence is $1 + 0 = 1$. The vector field is not source free.

c. $\mathbf{r}_1(t) = \left\langle t, 2t^2 - 2t \right\rangle$ for $0 \le t \le 1$ and $\mathbf{r}_2(t) = \langle 1 - t, 0 \rangle$ for $0 \le t \le 1$.

d. $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \oint_{C_1} \mathbf{F}\cdot\mathbf{n}\, ds + \oint_{C_2} \mathbf{F}\cdot\mathbf{n}\, ds = \int_0^1 (4t^2 - 2t - 1)\, dt + \int_0^1 dt = \left(\frac{4}{3}t^3 - t^2 - t\right)\Big|_0^1 + 1 = -\frac{2}{3} + 1 = \frac{1}{3}$.
$\displaystyle\int_0^1 \int_{2x^2-2x}^0 dy\, dx = 2\int_0^1 (x - x^2)\, dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$.

17.4.17

a. The curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = -2 - 2 = -4$.

b. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \langle 0, -2t\rangle \cdot \langle 1, 0\rangle\, dt + \int_\pi^0 \langle 2\sin t, -2t\rangle \cdot \langle 1, \cos t\rangle\, dt = \int_\pi^0 (2\sin t - 2t\cos t)\, dt = -8$.
$\displaystyle\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R (-4)\, dA = -4\int_0^\pi \sin x\, dx = -8$.

17.4.18

a. The curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 3 + 3 = 6$.

b. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle 0, 3t\rangle \cdot \langle 1, 0\rangle\, dt + \int_0^1 \langle -6t, 3 - 3t\rangle \cdot \langle -1, 2\rangle\, dt + \int_0^1 \langle -3\,(2 - 2t)\,, 0\rangle \cdot \langle 0, -2\rangle\, dt =$
$\displaystyle\int_0^1 (6t + 6 - 6t)\, dt = 6$.
$\displaystyle\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R 6\, dA = 6\int_0^1 (2 - 2x)\, dx = 6$.

17.4.19

a. The curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 2x + 2x = 4x$.

b. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 \langle 0, t^2\rangle \cdot \langle 1, 0\rangle\, dt + \int_2^0 \langle -2t^2(2-t), t^2\rangle \cdot \langle 1, 2 - 2t\rangle\, dt = \int_2^0 \left(-2t^2\,(2 - t) + t^2\,(2 - 2t)\right) dt =$
$\displaystyle\int_2^0 \left(-4t^2 + 2t^3 + 2t^2 - 2t^3\right) dt = \int_2^0 -2t^2\, dt = -\frac{2t^3}{3}\Big|_2^0 = \frac{16}{3}$.
$\displaystyle\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R 4x\, dA = \int_0^2 \int_0^{2x-x^2} 4x\, dy\, dx = \int_0^2 (8x^2 - 4x^3)\, dx = \left(\frac{8x^3}{3} - x^4\right)\Big|_0^2 =$
$\displaystyle\frac{64}{3} - 16 = \frac{16}{3}$.

17.4.20

a. The curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 2x$.

b. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 0, \sin^2 t + \cos^2 t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt = \int_0^{2\pi} \cos t\, dt = 0.$

$\displaystyle\iint\limits_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint\limits_R (2x)\; dA = \int_0^{2\pi}\int_0^1 2r\cos\theta\, r\, dr\, d\theta = 0.$

17.4.21 Parameterize the boundary by $x = 5\cos t$, $y = 5\sin t$, $0 \le t \le 2\pi$. Then $dx = -5\sin t\, dt$, $dy = 5\cos t\, dt$, and the area is $\displaystyle\frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2}\int_0^{2\pi} ((5\,\cos t)\cdot(5\cos t) - (5\sin t)\cdot(-5\sin t))\; dt = \frac{25}{2}\int_0^{2\pi} 1\, dt = 25\pi.$

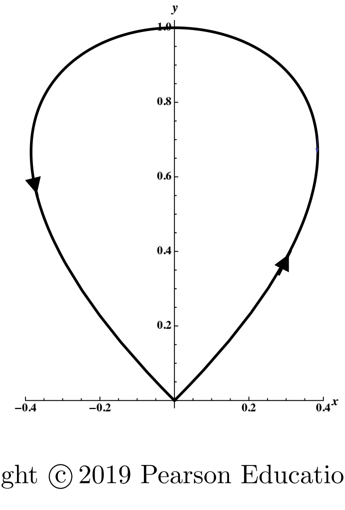
17.4.22 Parameterize the boundary by $x = 6\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$. Then $dx = -6\sin t\, dt$, $dy = 4\cos t\, dt$, and the area is $\displaystyle\frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2}\int_0^{2\pi} ((6\,\cos t)\cdot(4\cos t) - (4\sin t)\cdot(-6\sin t))\; dt = 12\int_0^{2\pi} 1\, dt = 24\pi.$

17.4.23 Parameterize the boundary by $x = 4\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$. Then $dx = -4\sin t\, dt$, $dy = 4\cos t\, dt$, and the area is $\displaystyle\frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2}\int_0^{2\pi} ((4\,\cos t)\cdot(4\cos t) - (4\sin t)\cdot(-4\sin t))\; dt = 8\int_0^{2\pi} 1\, dt = 16\pi.$

17.4.24 Note that C_1 can be parameterized by $x = -\frac{\sqrt{2}}{2}(1-t) + \frac{\sqrt{2}}{2}t$ and $y = \frac{\sqrt{2}}{2}$, $0 \le t \le 1$ while C_2 can be parametrized by $x = \cos t$ and $y = \sin t$ for $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$. For C_1 we have $dy = 0$ and $dx = \sqrt{2}\, dt$. Thus $\displaystyle\frac{1}{2}\oint_{C_1} x\, dy - y\, dx = \frac{1}{2}\int_0^1 -\frac{\sqrt{2}}{2}\cdot\sqrt{2}\, dt = -\frac{1}{2}$. For C_2, we have $dx = -\sin t\, dt$ and $dy = \cos t\, dt$, and we have $\displaystyle\frac{1}{2}\oint_{C_2} x\, dy - y\, dx = \frac{1}{2}\int_{-\pi/4}^{\pi/4} dt = \frac{\pi}{4}$. Thus the area is $\frac{\pi}{4} - \frac{1}{2}$. As a quick check, note that the region could be thought of as a quarter circle of radius one minus a triangle with area $\frac{1}{2}$.

17.4.25 Traverse the first path from -2 to 2, then the second path back from 2 to -2. The area is then $\displaystyle\frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2}\int_{-2}^{2} \left(t\cdot(4\,t) - \left(2t^2\right)\cdot 1\right)\; dt + \frac{1}{2}\int_2^{-2} \left(t\cdot(-2\,t) - \left(12 - t^2\right)\cdot 1\right)\; dt = \frac{1}{2}\int_{-2}^{2} 2t^2\, dt + \frac{1}{2}\int_2^{-2} \left(-2t^2 - 12 + t^2\right)\; dt = 32.$

17.4.26



We have $dx = \left(1-t^2+t\cdot(-2t)\right)\,dt = \left(1-3t^2\right)\,dt$ and $dy=-2t\,dt$. We parameterize the curve from $t=1$ to $t=-1$ so that we traverse the region counterclockwise; then the area is $\frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_1^{-1}\left(t\left(1-t^2\right)(-2t)-\left(1-t^2\right)\left(1-3t^2\right)\right)\,dt = \frac{8}{15}$.

17.4.27

a. The divergence is $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y} = 1+1=2$.

b. $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = 4\int_0^{2\pi}\left(\cos t\,(\cos t)-\sin t\,(-\sin t)\right)\,dt = 8\pi$.

$\iint\limits_R\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right)\,dA = \iint\limits_R (2)\ dA = 2\cdot 4\pi = 8\pi$.

17.4.28

a. The divergence is $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y} = 1-3=-2$.

b. Parameterize the triangle by the three paths: $\mathbf{r}_1(t)=\langle t,2t\rangle$, $\mathbf{r}_2(t)=\langle 1-t,2\rangle$, and $\mathbf{r}_3(t)=\langle 0,2-2t\rangle$, all for $0\le t\le 1$. Then $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 (2t+3(2t))\,dt + \int_0^1 (0-6)\,dt + \int_0^1 (0+0)\,dt = \int_0^1 (8t-6)\,dt = \left(4t^2-6t\right)\Big|_0^1 = -2$.

$\iint\limits_R\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right)\,dA = \int_0^1\int_{2x}^2 (-2)\,dy\,dx = \int_0^1(-4+4x)\,dx = \left(-4x+2x^2\right)\Big|_0^1 = -2$.

17.4.29

a. The divergence is $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y} = 2y+0=2y$.

b. Parameterize the region by $\mathbf{r}_1(t)=\langle t,0\rangle$ for $0\le t\le 2$ and $\mathbf{r}_2(t)=\langle t,2t-t^2\rangle$, for $0\le t\le 2$ traversed backwards (from $t=2$ to $t=0$.) Then $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^2 (0-t^2)\,dt + \int_2^0 \left(2t(2t-t^2)(2-2t)-t^2\right)\,dt = \int_0^2(-8t^2+12t^3-4t^4)\,dt = \left(\frac{-8t^3}{3}+3t^4-\frac{4t^5}{5}\right)\Big|_0^2 = \frac{16}{15}$.

$\iint\limits_R\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right)\,dA = \int_0^2\int_0^{2x-x^2} 2y\,dy\,dx = \int_0^2 (2x-x^2)^2\,dx = \int_0^2(4x^2-4x^3+x^4)\,dx = \left(\frac{4x^3}{3}-x^4+\frac{x^5}{5}\right)\Big|_0^2 = \frac{32}{3}-16+\frac{32}{5} = \frac{16}{15}$.

17.4.30

a. The divergence is $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y} = 2x$.

b. $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi}\left(\left(\sin^2 t+\cos^2 t\right)\cdot\cos t + 0\cdot\sin t\right)\,dt = \int_0^{2\pi}\cos t\,dt = 0$.

$\iint\limits_R\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right)\,dA = \iint\limits_R 2x\,dA = \int_0^{2\pi}\int_0^1 2\cos\theta\cdot r\,dr\,d\theta = 0$.

17.4.31 $\oint_C \langle 3y+1, 4x^2+3\rangle \cdot d\mathbf{r} = \int_0^2 \int_0^4 (8x-3)\,dx\,dy = \int_0^2 (4x^2-3x)\Big|_0^4 dx = \int_0^2 52\,dx = 104.$

17.4.32 $\oint_C \langle \sin y, x\rangle \cdot d\mathbf{r} = \int_0^{\pi/2} \int_0^x (1-\cos y)\,dy\,dx = \int_0^{\pi/2} (y-\sin y)\Big|_0^x dx = \int_0^{\pi/2} (x-\sin x)\,dx = \left(\frac{x^2}{2}+\cos x\right)\Big|_0^{\pi/2} = \frac{\pi^2}{8}-1.$

17.4.33 $\oint_C xe^y\,dx + x\,dy = \int_0^2 \int_0^{x^2} (1-xe^y)\,dy\,dx = \int_0^2 (y-xe^y)\Big|_0^{x^2} dx = \int_0^2 \left(x^2+x-xe^{x^2}\right) dx = \left(\frac{x^3}{3}+\frac{x^2}{2}-\frac{1}{2}e^{x^2}\right)\Big|_0^2 = \frac{8}{3}+2-\frac{1}{2}e^4+\frac{1}{2} = \frac{31-3e^4}{6}.$

17.4.34 $\oint_C \frac{1}{1+y^2}\,dx + y\,dy = \int_0^1 \int_0^x \left(\frac{2y}{(1+y^2)^2}\right) dy\,dx = \int_0^1 \left(-\frac{1}{1+y^2}\right)\Big|_0^x dx = \int_0^1 \left(1-\frac{1}{1+x^2}\right) dx = \left(x-\tan^{-1}x\right)\Big|_0^1 = 1-\frac{\pi}{4} = \frac{4-\pi}{4}.$

17.4.35 The line integral, using the flux form of Green's theorem, is equal to

$$\iint_R \left(\frac{\partial}{\partial x}\left(2x+e^{y^2}\right)+\frac{\partial}{\partial y}\left(4y^2+e^{x^2}\right)\right) dA = \iint_R (2+8y)\;dA = \int_0^1\int_0^1 (2+8y)\;dx\,dy$$
$$= \int_0^1 (2+8y)\;dy = 6.$$

17.4.36 Using the flux form of Green's theorem, the integral is equal to

$$\iint_R \left(\frac{\partial}{\partial x}(2x-3y)+\frac{\partial}{\partial y}(3x+4y)\right) dA = \iint_R 6\,dA = 6\times \text{area of } R = 6\pi.$$

17.4.37 Using the flux form of Green's theorem, the integral is equal to

$$\iint_R \left(\frac{\partial}{\partial x}(0)+\frac{\partial}{\partial y}(xy)\right) dA = \int_0^2\int_0^{4-2x} x\,dy\,dx = \int_0^2 \left(4x-2x^2\right) dx = \frac{8}{3}.$$

17.4.38 Using the flux form of Green's theorem, the integral is equal to (note the leading minus sign to correct for the orientation)

$$-\iint_R \left(\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}\left(2y^2\right)\right) dA = -\int_{-1}^1\int_0^{\sqrt{1-x^2}} (2x+4y)\;dy\,dx$$
$$= -\int_{-1}^1 \left(2xy+2y^2\right)\Big|_{y=0}^{y=\sqrt{1-x^2}} dx = -\int_{-1}^1 \left(2x\sqrt{1-x^2}+2-2x^2\right) dx = -\frac{8}{3}.$$

17.4.39 Using the circulation form of Green's theorem, the integral is equal to

$$\iint_R \left(\frac{\partial}{\partial x}\left(4x+y^3\right)-\frac{\partial}{\partial y}\left(x^2+y^2\right)\right) dA = \iint_R (4-2y)\;dA = \int_0^\pi\int_0^{\sin(x)} (4-2y)\;dy\,dx = 8-\frac{\pi}{2}.$$

17.4.40 Using the flux form of Green's theorem, the integral is equal to

$$\iint_R \left(\frac{\partial}{\partial x}\left(e^{x-y}\right)+\frac{\partial}{\partial y}\left(e^{y-x}\right)\right) dA = \iint_R \left(e^{x-y}+e^{y-x}\right) dA = \int_0^1\int_0^x \left(e^{x-y}+e^{y-x}\right) dy\,dx$$
$$= \int_0^1 \left(-e^{x-y}+e^{y-x}\right)\Big|_{y=0}^{y=x} dx = \int_0^1 \left(e^x-e^{-x}\right) dx = e+e^{-1}-2.$$

17.4.41

a. Using Green's theorem, the circulation is $\displaystyle\iint\limits_R \left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right) dA = \iint\limits_R 0\, dA = 0,$

b. Using Green's theorem, the flux is $\displaystyle\iint\limits_R \left(\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y)\right) dA = \iint\limits_R 2\, dA = 2 \cdot \text{area of}\, R = 2 \cdot \frac{1}{2}(4\pi - \pi) = 3\pi.$

17.4.42

a. Using Green's theorem, the circulation is $\displaystyle\iint\limits_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right) dA = \iint\limits_R 2\, dA = 2 \cdot \text{area of} R = 2 \cdot (9\pi - \pi) = 16\pi.$

b. Using Green's theorem, the flux is $\displaystyle\iint\limits_R \left(\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)\right) dA = 0.$

17.4.43

a. Using Green's theorem, the circulation is $\displaystyle\iint\limits_R \left(\frac{\partial}{\partial x}(x-4y) - \frac{\partial}{\partial y}(2x+y)\right) dA = \iint\limits_R (1-1)\ dA = 0.$

b. Using Green's theorem, the flux is $\displaystyle\iint\limits_R \left(\frac{\partial}{\partial x}(2x+y) + \frac{\partial}{\partial y}(x-4y)\right) dA = \iint\limits_R (-2)\ dA = -2 \cdot$ area of $R = -2 \cdot \dfrac{1}{4}(16\pi - \pi) = -\dfrac{15}{2}\pi.$

17.4.44

a. Using Green's theorem, the circulation is $\displaystyle\iint\limits_R \left(\frac{\partial}{\partial x}(-x+2y) - \frac{\partial}{\partial y}(x-y)\right) dA = \iint\limits_R (0)\ dA = 0.$

b. Using Green's theorem, the flux is $\displaystyle\iint\limits_R \left(\frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(-x+2y)\right) dA = \iint\limits_R 3\, dA = 3 \cdot \text{area of}\, R = 6.$

17.4.45

a. Because **F** is conservative, the circulation on the boundary of R is zero.

b. $\mathbf{F} = \left(x^2+y^2\right)^{-1/2} \langle x, y\rangle$, so the flux is $\displaystyle\iint\limits_R \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right) dA = \iint\limits_R \left(x^2+y^2\right)^{-1/2}\ dA = \int_0^\pi \int_1^3 \frac{1}{r} r\, dr\, d\theta = \int_0^\pi \int_1^3 1\, dr\, d\theta = 2\pi.$

17.4.46

a. The circulation is $\displaystyle\iint\limits_R \left(\frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) - \frac{\partial}{\partial y}\left(\ln\left(x^2+y^2\right)\right)\right) dA = \iint\limits_R \left(\frac{-y}{x^2+y^2} - \frac{2y}{x^2+y^2}\right) dA = -3\iint\limits_R \frac{y}{x^2+y^2}\, dA = -3\int_0^{2\pi}\int_1^2 \frac{r\sin\theta}{r^2} r\, dr\, d\theta = -3\int_0^{2\pi}\int_1^2 2\sin\theta\, dr\, d\theta = -3\int_0^{2\pi} \sin\theta\, d\theta = 0.$

b. The flux is $\iint\limits_R \left(\frac{\partial}{\partial x}\left(\ln\left(x^2+y^2\right)\right)+\frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)\right)\,dA = \iint\limits_R \frac{3x}{x^2+y^2}\,dA =$ $\int_0^{2\pi}\int_1^2 \frac{3r\cos\theta}{r^2} r\,dr\,d\theta = \int_0^{2\pi}\int_1^2 3\cos\theta\,dr\,d\theta = \int_0^{2\pi} 3\cos(\theta)\,d\theta = 0.$

17.4.47 Note that the region is the area between $x = y^2$ and$x = 2 - y^2$, which intersect at $y = \pm 1$.

a. The circulation is

$$\begin{aligned}
\iint\limits_R \left(\frac{\partial}{\partial x}\left(x^2-y\right)-\frac{\partial}{\partial y}\left(x+y^2\right)\right)\,dA &= \iint\limits_R (2x-2y)\,dA \\
&= \int_{-1}^{1}\int_{y^2}^{2-y^2} (2x-2y)\,dx\,dy \\
&= \int_{-1}^{1} \left(x^2-2xy\right)\Big|_{y^2}^{2-y^2}\,dy \\
&= \int_{-1}^{1} \left((2-y^2)^2-(4y-2y^3)-(y^4-2y^3)\right)\,dy \\
&= \int_{-1}^{1} (4-4y-4y^2+4y^3)\,dy \\
&= \left(4y-2y^2-\frac{4y^3}{3}+y^4\right)\Big|_{-1}^{1} = \frac{16}{3}.
\end{aligned}$$

b. The flux is $\iint\limits_R \left(\frac{\partial}{\partial x}\left(x+y^2\right)+\frac{\partial}{\partial y}\left(x^2-y\right)\right)\,dA = \iint\limits_R 0\,dA = 0.$

17.4.48

a. The circulation is

$$\begin{aligned}
&\iint\limits_R \left(\frac{\partial}{\partial x}(-\sin x)-\frac{\partial}{\partial y}(y\cos x)\right)\,dA = 2\iint\limits_R (-\cos x)\,dA \\
&= 2\int_0^{\pi/2}\int_0^{\pi/2} (-\cos x)\,dy\,dx = \int_0^{\pi/2} (-\pi\cos x)\,dx = -\pi.
\end{aligned}$$

b. The flux is

$$\begin{aligned}
&\iint\limits_R \left(\frac{\partial}{\partial x}(y\cos x)+\frac{\partial}{\partial y}(-\sin x)\right)\,dA = \iint\limits_R (-y\sin x)\,dA \\
&= \int_0^{\pi/2}\int_0^{\pi/2} (-y\sin x)\,dy\,dx = -\frac{1}{8}\pi^2.
\end{aligned}$$

17.4.49

a. True. This is the definition of work along a path.

b. False. Divergence corresponds to flux, so if the divergence is zero throughout a region, the flux is zero across the boundary.

c. True. This follows from Green's theorem.

17.4.50 Let $\mathbf{F} = \langle 1, 0\rangle$. By Green's theorem, $\oint_C 1\,dx = \oint_C 1\,dx + 0\,dy = \iint_R \left(\frac{\partial 0}{\partial x} - \frac{\partial 1}{\partial y}\right) dA = \iint_R 0\,dA =$ 0. Similarly, if we let $\mathbf{F} = \langle 0, 1\rangle$, we have $\oint_C 1\,dy = \oint_C 0\,dx + 1\,dy = \iint_R \left(\frac{\partial 1}{\partial x} - \frac{\partial 0}{\partial y}\right) dA = \iint_R 0\,dA = 0$.

17.4.51 Because $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 0$, the integral is zero (because $\mathbf{F}$ is conservative.)

17.4.52 Let $f(x, y) = 0$, $g(x, y) = xy^2 + y^4$; then

$$\iint_R \left(2xy + 4y^3\right) dA = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA = \oint_C (0\,dy - g\,dx) = -\oint_C \left(xy^2 + y^4\right) dx$$

by Green's theorem, where C is the boundary of the triangle. To evaluate this line integral, parameterize C by three paths, all for $0 \le t \le 1$: $\mathbf{r}_1(t) = \langle t, 0\rangle$, so that $\mathbf{r}_1'(t) = \langle 1, 0\rangle$. $\mathbf{r}_2(t) = \langle 1 - t, t\rangle$, so that $\mathbf{r}_2'(t) = \langle -1, 1\rangle$. $\mathbf{r}_3(t) = \langle 0, 1 - t\rangle$, so that $\mathbf{r}_3'(t) = \langle 0, -1\rangle$. Then $-\oint_C \left(xy^2 + y^4\right) dx = -\int_0^1 \left(t\left(0^2\right) - 0^4\right) dt - \int_0^1 t^2 (1 - t)(-1)\,dt - \int_0^1 \left(0\,(1 - t)^2 + t^4 \cdot 0\right) dt = \int_0^1 \left(t^4 - t^3 + t^2\right) dt = \frac{17}{60}$.

17.4.53 By Green's theorem,

$$\begin{aligned}\oint_C xy^2\,dx + \left(x^2y + 2x\right) dy &= \iint_R \left(\frac{\partial}{\partial x}\left(x^2y + 2x\right) - \frac{\partial}{\partial y}\left(xy^2\right)\right) dA \\ &= \iint_R (2xy + 2 - 2xy)\,dA = \iint_R 2\,dA = 2 \cdot \text{area of } A.\end{aligned}$$

17.4.54 Using the circulation form of Green's theorem, the integral is

$$\oint_C ay\,dx + bx\,dy = \iint_R (b - a)\,dA = (b - a) \cdot \text{area of } A.$$

17.4.55

a. The divergence is $\frac{\partial}{\partial x}(4) + \frac{\partial}{\partial y}(2) = 0$.

b. $\psi = 4y - 2x$.

17.4.56

a. The divergence is $\frac{\partial}{\partial x}\left(y^2\right) + \frac{\partial}{\partial y}\left(x^2\right) = 0 + 0 = 0$.

b. $\psi = \frac{1}{3}\left(y^3 - x^3\right)$.

17.4.57

a. The divergence is $\frac{\partial}{\partial x}\left(-e^{-x}\sin y\right) + \frac{\partial}{\partial y}\left(e^{-x}\cos y\right) = e^{-x}\sin y - e^{-x}\sin y = 0$.

b. $\psi = e^{-x}\cos y$.

17.4.58

a. The divergence is $\frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}(-2xy) = 2x - 2x = 0$.

b. $\psi = x^2 y$.

17.4.59

a. The curl and divergence are curl $\mathbf{F} = \dfrac{\partial}{\partial x}\left(-e^x \sin y\right) - \dfrac{\partial}{\partial y}\left(e^x \cos y\right) = -e^x \sin y + e^x \sin y = 0$. div $\mathbf{F} = \frac{\partial}{\partial x}\left(e^x \cos y\right) + \frac{\partial}{\partial y}\left(-e^x \sin y\right) = e^x \cos y - e^x \cos y = 0$.

b. $\varphi(x, y) = e^x \cos(y)$. $\psi(x, y) = e^x \sin(y)$.

c. $\varphi_{xx} + \varphi_{yy} = \dfrac{\partial^2}{\partial x^2}\left(e^x \cos y\right) + \dfrac{\partial^2}{\partial y^2}\left(e^x \cos y\right) = e^x \cos y - e^x \cos y = 0$. $\psi_{xx} + \psi_{yy} = \frac{\partial^2}{\partial x^2}\left(e^x \sin y\right) + \frac{\partial^2}{\partial y^2}\left(e^x \sin y\right) = e^x \sin y - e^x \sin y = 0$.

17.4.60

a. The curl and divergence are curl $\mathbf{F} = \dfrac{\partial}{\partial x}\left(y^3 - 3x^2 y\right) - \dfrac{\partial}{\partial y}\left(x^3 - 3xy^2\right) = -6xy + 6xy = 0$.
div $\mathbf{F} = \dfrac{\partial}{\partial x}\left(x^3 - 3xy^2\right) + \dfrac{\partial}{\partial y}\left(y^3 - 3x^2 y\right) = 3x^2 - 3y^2 + 3y^2 - 3x^2 = 0$

b. $\varphi(x, y) = \dfrac{1}{4}\left(x^4 + y^4\right) - \dfrac{3}{2}x^2 y^2$. $\psi(x, y) = x^3 y - xy^3$.

c.

$$\begin{aligned}\varphi_{xx} + \varphi_{yy} &= \frac{\partial^2}{\partial x^2}\left(\frac{1}{4}\left(x^4 + y^4\right) - \frac{3}{2}x^2 y^2\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{4}\left(x^4 + y^4\right) - \frac{3}{2}x^2 y^2\right)\\ &= \frac{\partial}{\partial x}\left(x^3 - 3xy^2\right) + \frac{\partial}{\partial y}\left(y^3 - 3x^2 y\right) = 3x^2 - 3y^2 + 3y^2 - 3x^2 = 0\end{aligned}$$

$$\begin{aligned}\psi_{xx} + \psi_{yy} &= \frac{\partial^2}{\partial x^2}\left(x^3 y - xy^3\right) + \frac{\partial^2}{\partial y^2}\left(x^3 y - xy^3\right)\\ &= \frac{\partial}{\partial x}\left(3x^2 y - y^3\right) + \frac{\partial}{\partial y}\left(x^3 - 3xy^2\right) = 6xy - 6xy = 0.\end{aligned}$$

17.4.61

a. The curl and divergence are curl $\mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}\ln\left(x^2 + y^2\right)\right) - \dfrac{\partial}{\partial y}\left(\tan^{-1}\left(\dfrac{y}{x}\right)\right) = 0$.
div $\mathbf{F} = \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}\ln\left(x^2 + y^2\right)\right) = 0$.

b. $\varphi(x, y) = x\,\tan^{-1}\left(\dfrac{y}{x}\right) - y + \dfrac{1}{2}y\,\ln\left(x^2 + y^2\right)$. $\psi(x, y) = \int -\frac{1}{2}\ln\left(x^2 + y^2\right)\,dy =$
$y\tan^{-1}\left(\frac{y}{x}\right) - \frac{x}{2}\ln\left(x^2 + y^2\right) + x$.

c.

$$\begin{aligned}\varphi_{xx} + \varphi_{yy} &= \frac{\partial^2}{\partial x^2}\left(x\tan^{-1}\left(\frac{y}{x}\right) - y + \frac{1}{2}y\,\ln\left(x^2 + y^2\right)\right) + \frac{\partial^2}{\partial y^2}\left(x\tan^{-1}\left(\frac{y}{x}\right) - y + \frac{1}{2}y\,\ln\left(x^2 + y^2\right)\right)\\ &= \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right) - \frac{xy}{x^2 + y^2} + \frac{xy}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x^2}{x^2 + y^2} - 1 + \frac{1}{2}\ln\left(x^2 + y^2\right) + \frac{y^2}{x^2 + y^2}\right)\\ &= \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}\ln\left(x^2 + y^2\right)\right) = 0.\end{aligned}$$

$$\begin{aligned}\psi_{xx}+\psi_{yy} &= \frac{\partial^2}{\partial x^2}\left(y\tan^{-1}\left(\frac{y}{x}\right)-\frac{x}{2}\ln\left(x^2+y^2\right)+x\right)+\frac{\partial^2}{\partial y^2}\left(y\tan^{-1}\left(\frac{y}{x}\right)-\frac{x}{2}\ln\left(x^2+y^2\right)+x\right)\\ &= \frac{\partial}{\partial x}\left(-\frac{y^2}{x^2+y^2}-\frac{x^2}{x^2+y^2}-\frac{1}{2}\ln\left(x^2+y^2\right)+1\right)+\frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)+\frac{xy}{x^2+y^2}-\frac{xy}{x^2+y^2}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right)+\frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)=0.\end{aligned}$$

17.4.62

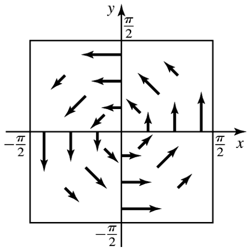
a. The curl and divergence are $\operatorname{curl}\mathbf{F}=\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)=0$, and $\operatorname{div}\mathbf{F}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)=0$.

b. $\varphi(x,y)=\frac{1}{2}\ln\left(x^2+y^2\right)$. $\psi(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$.

c. $\varphi_{xx}+\varphi_{yy}=\frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right)=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}+\frac{x^2-y^2}{\left(x^2+y^2\right)^2}=0.$
$\psi_{xx}+\psi_{yy}=\frac{\partial^2}{\partial x^2}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)+\frac{\partial^2}{\partial y^2}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)=\frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)=\frac{2xy}{\left(x^2+y^2\right)^2}-\frac{2xy}{\left(x^2+y^2\right)^2}=0.$

17.4.63

a. The velocity field is $\langle -4\cos x\sin y, 4\sin x\cos y\rangle$.



b. The field is source-free if its divergence is zero.

$$\operatorname{div}\mathbf{F}=\frac{\partial}{\partial x}(-4\cos x\sin y)+\frac{\partial}{\partial y}(4\sin x\cos y)=4\sin x\sin y-4\sin x\sin y=0,$$

so the field is source-free.

c. The field is irrotational if its curl is zero.

$$\operatorname{curl}\mathbf{F}=\frac{\partial}{\partial x}(4\sin x\cos y)-\frac{\partial}{\partial y}(-4\cos x\sin y)=4\cos x\cos y+4\cos x\cos y\ =8\cos x\cos y,$$

so the field is not irrotational.

d. Since the field is source-free, it has zero flux across the boundary.

e. The circulation around the boundary of the rectangle is (by Green's theorem) given by

$$\iint\limits_R 8\cos x\cos y\,dA=\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}8\cos x\cos y\,dy\,dx=\int_{-\pi/2}^{\pi/2}16\cos x\,dx=32.$$

17.4.64 If $f(x)$ is continuous, then the circulation form of Green's theorem says that

$$\oint_C \frac{f(x)}{c}dy=\frac{1}{c}\iint\limits_R \frac{df}{dx}\,dA.$$

The right side of this equation evaluates to

$$\frac{1}{c}\iint\limits_R \frac{df}{dx}\,dA=\frac{1}{c}\int_a^b\int_0^c\frac{df}{dx}dy\,dx=\int_a^b\frac{df}{dx}\,dx.$$

To evaluate the left side, parameterize the boundary of R with four paths, each for $0\le t\le 1$: $\mathbf{r}_1(t)=\langle a+(b-a)t,0\rangle$, so $\mathbf{r}_1'(t)=\langle b-a,0\rangle$. $\mathbf{r}_2(t)=\langle b,ct\rangle$, so $\mathbf{r}_2'(t)=\langle 0,c\rangle$. $\mathbf{r}_3(t)=\langle b+(a-b)t,c\rangle$, so $\mathbf{r}_3'(t)=\langle a-b,0\rangle$. $\mathbf{r}_4(t)=\langle a,c-ct\rangle$, so $\mathbf{r}_4'(t)=\langle 0,-c\rangle$. Then we evaluate $\mathbf{F}\cdot\mathbf{r}_i$ for each i and add:

$$\frac{1}{c}\oint_C f(x)\,dy=\frac{1}{c}\int_0^1\left(f(a+(b-a)t)\cdot 0+f(b)\cdot c+f(b+(a-b)t)\cdot 0+f(a)\cdot(-c)\right)dt=f(b)-f(a).$$

17.4.65 If $f(x)$ is continuous, then the flux form of Green's theorem says that

$$\oint_C \frac{f(x)}{c}dx=\frac{1}{c}\iint\limits_R \frac{df}{dx}\,dA.$$

The right side of this equation evaluates to

$$\frac{1}{c}\iint\limits_R \frac{df}{dx}\,dA=\frac{1}{c}\int_a^b\int_0^c\frac{df}{dx}dy\,dx=\int_a^b\frac{df}{dx}\,dx.$$

To evaluate the left side, parameterize the boundary of R with four paths, each for $0\le t\le 1$: $\mathbf{r}_1(t)=\langle a+(b-a)t,0\rangle$, so $\mathbf{r}_1'(t)=\langle b-a,0\rangle$. $\mathbf{r}_2(t)=\langle b,ct\rangle$, so $\mathbf{r}_2'(t)=\langle 0,c\rangle$. $\mathbf{r}_3(t)=\langle b+(a-b)t,c\rangle$, so $\mathbf{r}_3'(t)=\langle a-b,0\rangle$. $\mathbf{r}_4(t)=\langle a,c-ct\rangle$, so $\mathbf{r}_4'(t)=\langle 0,-c\rangle$. Then we evaluate $\mathbf{F}\cdot\mathbf{r}_i$ for each i and add:

$$\oint_C \frac{f(x)}{c}\,dy=\frac{1}{c}\int_0^1\left(0+f(b)\cdot c+0+f(a)\cdot(-c)\right)dt=f(b)-f(a)$$

so that

$$\int_a^b\frac{df}{dx}\,dx=f(b)-f(a).$$

17.4.66

a. The curl is $\dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2+y^2}\right)-\dfrac{\partial}{\partial y}\left(\dfrac{-y}{x^2+y^2}\right)=0.$

b. Take a line integral around the unit circle, parameterized as $\langle\cos t,\sin t\rangle$. The circulation is then $\displaystyle\oint_C \frac{-y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy=\int_0^{2\pi}\left(-\sin t(-\sin t)+\cos t\cos t\right)\,dt=\int_0^{2\pi}1\,dt=2\pi.$

c. The vector field is not defined everywhere in R; specifically, it is undefined at the origin.

17.4.67

a. The divergence is $\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)=0$.

b. Take a line integral around the unit circle, parameterized as $\langle \cos t, \sin t\rangle$. The flux is then

$$\oint_C \frac{x}{x^2+y^2}dy+\frac{y}{x^2+y^2}dx=\int_0^{2\pi}(\cos t\cos t-\sin t\,(-\sin t))\,dt=\int_0^{2\pi}1\,dt=2\pi.$$

c. The vector field is not defined everywhere in R; specifically, it is undefined at the origin.

17.4.68

a. Green's theorem does not apply to a region including the origin because **F** is not defined at the origin.

b. $\iint\limits_R\left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right)dA=\iint\limits_R\left(x^2+y^2\right)^{-1/2}dA=\int_0^{2\pi}\int_0^1\frac{1}{r}r\,dr\,d\theta=2\pi.$

c. $\oint_C\frac{x}{\sqrt{x^2+y^2}}dy-\frac{y}{\sqrt{x^2+y^2}}dx=\int_0^{2\pi}(\cos t\,(\cos t)-\sin t\,(-\sin t))\,dt=\int_0^{2\pi}1\,dt=2\pi.$

d. They do agree. Because Green's theorem does not apply, there is no particular reason why they should.

17.4.69 Because ψ is a stream function, $d\psi=\psi_x\,dx+\psi_y\,dy$, so the flux integral is

$$\int_C\mathbf{F}\cdot\mathbf{n}\,ds=\int_C f\,dy-g\,dx=\int_C\psi_y\,dy-\psi_x\,dx=\int_C d\psi=\psi(B)-\psi(A),$$

so that the integral is independent of the path.

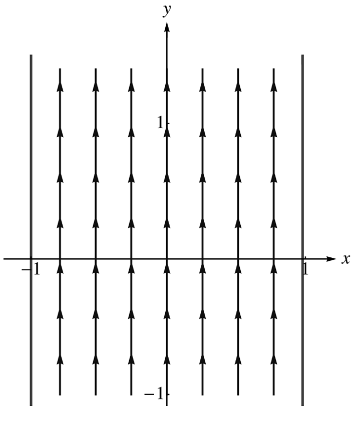
17.4.70 Showing that **F** is tangent to the level curves of the stream function is the same as showing that **F** is normal to the gradient of the stream function. But that gradient is $\langle\psi_x,\psi_y\rangle$, and

$$\mathbf{F}\cdot\langle\psi_x\,,\,\psi_y\rangle=\langle-\psi_y,\psi_x\rangle\cdot\langle\psi_x\,,\psi_y\rangle=0.$$

17.4.71 Showing that the level curves of φ and ψ are orthogonal is equivalent to showing that the gradients of φ and ψ are orthogonal. But $\nabla\varphi\cdot\nabla\psi=\langle f,g\rangle\cdot\langle -g,f\rangle=0$.

17.4.72

a. The stream function is found by taking $-\int\left(1-x^2\right)dx=\frac{1}{3}x^3-x$. A plot together with some streamlines is



b. The curl of F is $\frac{\partial}{\partial x}\left(1-x^2\right)=-2x$, so the curl on $x=0$ is 0; on $x=\frac{1}{4}$ it is $-\frac{1}{2}$; on $x=\frac{1}{2}$, it is -1, and on $x=1$, it is -2.

c. The circulation is (by Green's theorem) $\iint_R (-2x)\,dA=\int_{-5}^{5}\int_{-1}^{1}(-2x)\,dx\,dy=\int_{-5}^{5}0\,dy=0$.

d. The curl is positive for negative x and negative for positive x. These cancel, giving a net circulation of zero. This can easily be seen from the picture - any circulation resulting from the top boundary ($y=1$) is cancelled by the circulation in the opposite direction resulting from the bottom boundary.

17.5 Divergence and Curl

17.5.1 The divergence is $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}$.

17.5.2 The divergence measures the expansion or contraction of the vector field at each point.

17.5.3 It means that the field has no sources or sinks.

17.5.4 The curl is $\nabla\times$F$=\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\mathbf{i}+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\mathbf{j}+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\mathbf{k}$.

17.5.5 The curl indicates the axis and speed of rotation of a vector field at each point.

17.5.6 It means that the vector field is irrotational.

17.5.7 $\nabla\cdot(\nabla\times\mathbf{F})=0$; see Theorem 17.10.

17.5.8 Here $\mathbf{u}$ is a potential function, so $\nabla\times\nabla\mathbf{u}$ is the curl of a conservative vector field, which is 0.

17.5.9 $\frac{\partial}{\partial x}(2x)+\frac{\partial}{\partial y}(4y)+\frac{\partial}{\partial z}(-3z)=3$.

17.5.10 $\frac{\partial}{\partial x}(-2y)+\frac{\partial}{\partial y}(3x)+\frac{\partial}{\partial z}(z)=1$.

17.5.11 $\frac{\partial}{\partial x}(12x)+\frac{\partial}{\partial y}(-6y)+\frac{\partial}{\partial z}(-6z)=0$.

17.5.12 $\frac{\partial}{\partial x}\left(x^2yz\right)+\frac{\partial}{\partial y}\left(-xy^2z\right)+\frac{\partial}{\partial z}\left(-xyz^2\right)=2xyz-2xyz-2xyz=-2xyz$.

17.5.13 $\frac{\partial}{\partial x}\left(x^2-y^2\right)+\frac{\partial}{\partial y}\left(y^2-z^2\right)+\frac{\partial}{\partial z}\left(z^2-x^2\right)=2x+2y+2z$.

17.5.14 $\frac{\partial}{\partial x}\left(e^{y-x}\right)+\frac{\partial}{\partial y}\left(e^{z-y}\right)+\frac{\partial}{\partial z}\left(e^{x-z}\right)=-\left(e^{y-x}+e^{z-y}+e^{x-z}\right)$.

17.5.15 $\frac{\partial}{\partial x}\left(\frac{x}{1+x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{1+x^2+y^2}\right)+\frac{\partial}{\partial z}\left(\frac{z}{1+x^2+y^2}\right)=\frac{x^2+y^2+3}{\left(1+x^2+y^2\right)^2}$.

17.5.16 $\frac{\partial}{\partial x}(yz\sin(x))+\frac{\partial}{\partial y}(xz\cos(y))+\frac{\partial}{\partial z}(xy\cos(z))=yz\cos(x)-xz\sin(y)-xy\sin(z)$.

17.5.17

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right)+\frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right)\\
&=\frac{1}{\left(x^2+y^2+z^2\right)^2}\left(\left(z^2+y^2-x^2\right)+\left(x^2+z^2-y^2\right)+\left(x^2+y^2-z^2\right)\right)\\
&=\frac{1}{\left(x^2+y^2+z^2\right)^2}\left(x^2+y^2+z^2\right)=\frac{1}{|\mathbf{r}|^2}.
\end{aligned}$$

17.5.18

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}\right)+\frac{\partial}{\partial z}\left(\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\right)\\
&=\frac{1}{\left(x^2+y^2+z^2\right)^{5/2}}\left(\left(z^2+y^2-2x^2\right)+\left(x^2+z^2-2y^2\right)+\left(x^2+y^2-2z^2\right)\right)=0.
\end{aligned}$$

17.5.19

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{x}{\left(x^2+y^2+z^2\right)^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\left(x^2+y^2+z^2\right)^2}\right)+\frac{\partial}{\partial z}\left(\frac{z}{\left(x^2+y^2+z^2\right)^2}\right)\\
&=\frac{1}{\left(x^2+y^2+z^2\right)^3}\left(\left(z^2+y^2-3x^2\right)+\left(x^2+z^2-3y^2\right)+\left(x^2+y^2-3z^2\right)\right)\\
&=\frac{-1}{\left(x^2+y^2+z^2\right)^3}\left(x^2+y^2+z^2\right)=\frac{-1}{|\mathbf{r}|^4}.
\end{aligned}$$

17.5.20

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)\right)+\frac{\partial}{\partial y}\left(y\left(x^2+y^2+z^2\right)\right)+\frac{\partial}{\partial z}\left(z\left(x^2+y^2+z^2\right)\right)\\
&=\left(3x^2+y^2+z^2\right)+\left(x^2+3y^2+z^2\right)+\left(x^2+y^2+3z^2\right)=5\,|\mathbf{r}|^2.
\end{aligned}$$

17.5.21

a. At both P and Q, the arrows going away from the point are larger in both number and magnitude than those going in, so we would expect the divergence to be positive at both points.

b. The divergence is $\frac{\partial}{\partial \boldsymbol{x}}(x)+\frac{\partial}{\partial y}(x+y)=1+1=2$, so is positive everywhere.

c. The arrows all point roughly away from the origin, so we the flux is outward everywhere.

d. The net flux across C should be positive.

17.5.22

a. At P, the divergence should be positive, while at Q, the larger arrows point in towards Q, so the divergence should be negative.

b. The divergence is $\dfrac{\partial}{\partial \boldsymbol{x}}(x) + \dfrac{\partial}{\partial y}\left(y^2\right) = 1 + 2y$; at $P = (-1, 1)$, this is 3, while at $Q = (-1, -1)$, it is -1.

c. The flux is outward above the line $y = -1$ (approximately); below this line, the flux is inward across C.

d. The size of the arrows pointing outward at the top of the circle seems to roughly equal those pointing inward at the bottom, so the remaining outward-pointing arrows result in a net positive flux across C.

17.5.23

a. The axis of rotation is $\langle 1, 0, 0\rangle$, the x-axis. $\boldsymbol{\nabla} \times \mathbf{F} = \boldsymbol{\nabla} \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ x & y & z \end{vmatrix} = \boldsymbol{\nabla} \times (-z\mathbf{j} + y\mathbf{k}) = (1+1)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = 2\mathbf{i}$. It is in the same direction as the axis of rotation.

b. The magnitude of the curl is $|2\mathbf{i}| = 2$.

17.5.24

a. The axis of rotation is $\langle 1, -1, 0\rangle$. $\boldsymbol{\nabla} \times \mathbf{F} = \boldsymbol{\nabla} \times \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 1 & -1 & 0 \\ x & y & z \end{vmatrix} = \boldsymbol{\nabla} \times (-z\mathrm{i} - z\mathrm{j} + (x+y)\,\mathrm{k}) = (1+1)\,\mathrm{i} + (-1-1)\,\mathrm{j} + (0-0)\,\mathrm{k} = 2\langle 1, -1, 0\rangle$
and the curl is in the same direction as the axis of rotation.

b. The magnitude of the curl is $2\,|\langle 1, -1, 0\rangle| = 2\sqrt{2}$

17.5.25

a. The axis of rotation is $\langle 1, -1, 1\rangle$.

$\boldsymbol{\nabla} \times \mathbf{F} = \boldsymbol{\nabla} \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix} = \boldsymbol{\nabla} \times (-(y+z)\,\mathbf{i} + (x-z)\,\mathbf{j} + (x+y)\,\mathbf{k}) = (1+1)\,\mathbf{i} + (-1-1)\,\mathbf{j} + (1+1)\,\mathbf{k} = 2\langle 1, -1, 1\rangle$, and the curl is in the same direction as the axis of rotation.

b. The magnitude of the curl is $2\,|\langle 1, -1, 1\rangle| = 2\sqrt{3}$.

17.5.26

a. The axis of rotation is $\langle 1, -2, -3\rangle$.

$\boldsymbol{\nabla} \times \mathbf{F} = \boldsymbol{\nabla} \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -3 \\ x & y & z \end{vmatrix} = \boldsymbol{\nabla} \times ((3y - 2z)\,\mathbf{i} + (-3x - z)\,\mathbf{j} + (2x + y)\,\mathbf{k}) = (1+1)\,\mathbf{i} + (-2-2)\,\mathbf{j} + (-3-3)\,\mathbf{k} = 2\langle 1, -2, -3\rangle$, and the curl is in the same direction as the axis of rotation.

b. The magnitude of the curl is $2\,|\langle 1, -2, -3\rangle| = 2\sqrt{14}$.

17.5.27 $\boldsymbol{\nabla} \times \langle x^2 - y^2, xy, z\rangle = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (y + 2y)\,\mathbf{k} = 3y\mathbf{k}$.

17.5.28 $\nabla \times \langle 0, z^2 - y^2, -yz\rangle = (-z - 2z)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = -3z\mathbf{i}$.

17.5.29 $\nabla \times \langle x^2 - z^2, 1, 2xz\rangle = (0-0)\,\mathbf{i} + (-2z - 2z)\,\mathbf{j} + (0-0)\,\mathbf{k} = -4z\mathbf{j}$.

17.5.30 $\nabla \times \langle x, y, z\rangle = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$.

17.5.31

$$\begin{aligned}&\nabla \times \frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\langle x, y, z\rangle\\ &= \frac{1}{\left(x^2+y^2+z^2\right)^{5/2}}\left((-3yz+3yz)\,\mathbf{i} + (-3xz+3xz)\,\mathbf{j} + (-3xy+3xy)\,\mathbf{k}\right) = \mathbf{0}.\end{aligned}$$

17.5.32

$$\begin{aligned}&\nabla \times \frac{1}{\left(x^2+y^2+z^2\right)^{1/2}}\langle x, y, z\rangle\\ &= \frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\left((-yz+yz)\,\mathbf{i} + (-xz+xz)\,\mathbf{j} + (-xy+xy)\,\mathbf{k}\right) = \mathbf{0}.\end{aligned}$$

17.5.33

$$\begin{aligned}&\nabla \times \langle z^2\sin(y), xz^2\cos y, 2xz\sin y\rangle\\ &= (2xz\cos y - 2xz\cos(y))\,\mathbf{i} + (2z\sin y - 2z\sin y)\,\mathbf{j} + \left(z^2\cos y - z^2\cos y\right)\mathbf{k} = \mathbf{0}.\end{aligned}$$

17.5.34

$$\begin{aligned}\nabla \times \langle 3xz^3e^{y^2}, 2xz^3e^{y^2}, 3xz^2e^{y^2}\rangle &= \left(6xyz^2e^{y^2} - 6xz^2e^{y^2}\right)\mathbf{i} + \left(9xz^2e^{y^2} - 3z^2e^{y^2}\right)\mathbf{j} + \left(2z^3e^{y^2} - 6xyz^3e^{y^2}\right)\mathbf{k}\\ &= z^2e^{y^2}\left((6xy-6x)\,\mathbf{i} + (9x-3)\,\mathbf{j} + (2z-6xyz)\,\mathbf{k}\right).\end{aligned}$$

17.5.35 Simply compute it:

$$\begin{aligned}&\left\langle \frac{\partial}{\partial x}\left(\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\right), \frac{\partial}{\partial y}\left(\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\right), \frac{\partial}{\partial z}\left(\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\right)\right\rangle\\ &= \left\langle \frac{-3x}{\left(x^2+y^2+z^2\right)^{5/2}}, \frac{-3y}{\left(x^2+y^2+z^2\right)^{5/2}}, \frac{-3z}{\left(x^2+y^2+z^2\right)^{5/2}}\right\rangle = \frac{-3\mathbf{r}}{|\mathbf{r}|^5}.\end{aligned}$$

17.5.36

$$\begin{aligned}&\left\langle \frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2+z^2}\right), \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2+z^2}\right), \frac{\partial}{\partial z}\left(\frac{1}{x^2+y^2+z^2}\right)\right\rangle\\ &= \left\langle \frac{-2x}{\left(x^2+y^2+z^2\right)^2}, \frac{-2y}{\left(x^2+y^2+z^2\right)^2}, \frac{-2z}{\left(x^2+y^2+z^2\right)^2}\right\rangle = \frac{-2\mathbf{r}}{|\mathbf{r}|^4}.\end{aligned}$$

17.5.37 $\nabla\left(\frac{1}{|\mathbf{r}|^2}\right) = \frac{-2\mathbf{r}}{|\mathbf{r}|^4}$, from Problem 36; applying Theorem 17.8 we have

$$\nabla\cdot\nabla\left(\frac{1}{|\mathbf{r}|^2}\right) = -2\nabla\cdot\frac{\mathbf{r}}{|\mathbf{r}|^4} = -2\frac{3-4}{|\mathbf{r}|^4} = \frac{2}{|\mathbf{r}|^4}.$$

17.5.38

$$\begin{aligned}\nabla(\ln|\mathbf{r}|) &= \nabla\left(\ln\left(\sqrt{x^2+y^2+z^2}\right)\right) = \frac{1}{2}\nabla\left(\ln\left(x^2+y^2+z^2\right)\right)\\ &= \frac{1}{2\left(x^2+y^2+z^2\right)}\langle 2x, 2y, 2z\rangle = \frac{\mathbf{r}}{|\mathbf{r}|^2}.\end{aligned}$$

17.5.39

a. False. For example, $\mathbf{F} = \langle y, z, x\rangle$ has zero divergence yet is not constant.

b. False. For example, $\mathbf{F} = \langle x, y, z\rangle$ is a counterexample.

c. False. For example, consider the vector field $\langle 0, 1 - x^2\rangle$.

d. False. For example, $\mathbf{F} = \langle x, 0, 0\rangle$ has divergence 1.

e. False. For example, the curl of $\langle z, -z, y\rangle$ is $\langle 2, 1, 0\rangle$.

17.5.40

a. $(\mathbf{F} \cdot \boldsymbol{\nabla})\, u = \left(\langle f, g, h\rangle \cdot \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle\right) u = \left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}\right) u = f\frac{\partial u}{\partial x} + g\frac{\partial u}{\partial y} + h\frac{\partial u}{\partial z}.$

b. Because $\mathbf{F} = \langle 1, 1, 1\rangle$, $\mathbf{F}\cdot\boldsymbol{\nabla}\left(xy^2z^3\right) = \frac{\partial}{\partial x}\left(xy^2z^3\right) + \frac{\partial}{\partial y}\left(xy^2z^3\right) + \frac{\partial}{\partial z}\left(xy^2z^3\right) = y^2z^3 + 2xyz^3 + 3xy^2z^2.$

17.5.41

a. No; divergence is a concept that applies to vector fields.

b. No; the gradient applies to functions.

c. Yes; this is the divergence of the gradient and is thus a scalar function.

d. No, because $\boldsymbol{\nabla} \cdot \varphi$ does not make sense (part (a)).

e. No; curl applies to vector fields, $\boldsymbol{\nabla} \times \varphi$ does not make sense.

f. No, because $\boldsymbol{\nabla}\cdot$F is a function, so that applying $\boldsymbol{\nabla}\cdot$ to it does not make sense.

g. Yes, this is the curl of a vector field and is thus a vector field.

h. No, because $\boldsymbol{\nabla}\cdot$F is a function, not a vector field.

i. Yes; this is the curl of the curl of a vector field and is thus a vector field.

17.5.42 Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$; then $\mathbf{F} = \mathbf{a} \times \mathbf{r} = (a_2z - a_3y)\,\mathbf{i} + (a_3x - a_1z)\,\mathbf{j} + (a_1y - a_2x)\,\mathbf{k}$, so that $\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial}{\partial x}(a_2z - a_3y) + \frac{\partial}{\partial y}(a_3x - a_1z) + \frac{\partial}{\partial z}(a_1y - a_2x) = 0.$

17.5.43

a. $\langle 0, 1, 0\rangle \times \langle 0, 1, 1\rangle = \langle 1, 0, 0\rangle$, so $\mathbf{F}$ points in the positive x-direction. $\langle 0, 1, 0\rangle \times \langle 1, 1, 0\rangle = \langle 0, 0, -1\rangle$, so $\mathbf{F}$ points in the negative z-direction. $\langle 0, 1, 0\rangle \times \langle 0, 1, -1\rangle = \langle -1, 0, 0\rangle$, so $\mathbf{F}$ points in the negative x-direction. $\langle 0, 1, 0\rangle \times \langle -1, 1, 0\rangle = \langle 0, 0, 1\rangle$, so $\mathbf{F}$ points in the positive z-direction.

b. Note that these vectors circle the y-axis in the counterclockwise direction looking along $\mathbf{a}$ from head to tail.

17.5.44 Note that $\mathbf{a} \times \mathbf{r} = \langle 0, 1, 0\rangle \times \langle x, y, z\rangle = \langle z, 0, -x\rangle$ is a rotational field whose vectors circle the y-axis in the counterclockwise direction looking along $\mathbf{a}$ from head to tail.

17.5.45 Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$; then $\mathbf{F} = \mathbf{a} \times \mathbf{r} = (a_2z - a_3y)\,\mathbf{i} + (a_3x - a_1z)\,\mathbf{j} + (a_1y - a_2x)\,\mathbf{k}$, so that $\boldsymbol{\nabla} \times \mathbf{F} = \frac{\partial}{\partial x}(a_1 + a_1)\,\mathbf{i} + \frac{\partial}{\partial y}(a_2 + a_2)\,\mathbf{j} + \frac{\partial}{\partial z}(a_3 + a_3)\,\mathbf{k} = 2\mathbf{a}.$

17.5.46 The field switches from inward-pointing to outward-pointing at points where it is tangent to the circle $x^2 + y^2 = 2$, i.e. where it is orthogonal to the normal to the circle. The normal to the circle at (x, y) is a multiple of (x, y), so we want to find x, y so that $\langle x, y\rangle \cdot \langle x^2, y\rangle = x^3 + y^2 = 0$ with $x^2 + y^2 = 2$. Thus $x^3 - x^2 + 2 = 0$. The solutions are $x = -1$ and $y = \pm 1$.

17.5.47 $\operatorname{div}\mathbf{F} = 2x + 2xyz + 2x = 2x\,(yz+2)$; this function clearly achieves its maximum magnitude at $(-1,1,1)$, $(-1,-1,-1)$, $(1,1,1)$, and $(1,-1,-1)$, where its magnitude is 6.

17.5.48 For $\mathbf{F} = \langle z, 0, -y\rangle$, $\operatorname{curl}\mathbf{F} = \langle -1, 1, 0\rangle$.

a. $\operatorname{scal}_{\langle 1,0,0\rangle}\langle -1,1,0\rangle = \langle -1,1,0\rangle \cdot \langle 1,0,0\rangle = -1$.

b. $\operatorname{scal}_{\langle 1/\sqrt{3},-1/\sqrt{3},1/\sqrt{3}\rangle}\langle -1,1,0\rangle = \langle -1,1,0\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle = -\frac{2}{\sqrt{3}}$.

c. We have $\operatorname{scal}_{\mathbf{n}}\langle -1,1,0\rangle = \langle -1,1,0\rangle \cdot \mathbf{n} = \sqrt{2}\cos\theta$, where θ is the angle between $\langle -1,1,0\rangle$ and $\mathbf{n}$. This is maximized when $\mathbf{n}$ points in the same direction as $\operatorname{curl}\mathbf{F}$. So $\mathbf{n} = \frac{\langle -1,1,0\rangle}{|\langle -1,1,0\rangle|} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\rangle$, in which case $\operatorname{scal}_{\mathbf{n}}\langle -1,1,0\rangle = \sqrt{2}$.

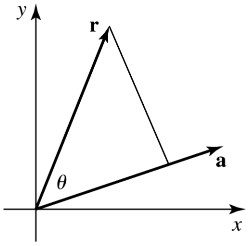
17.5.49 $\operatorname{curl}\mathbf{F} = \langle 0+2, 0+1, 0-1\rangle = \langle 2,1,-1\rangle$. If $\mathbf{n} = \langle a,b,c\rangle$, then $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = 0$ when $2a+b-c=0$ so that $c = 2a+b$; thus all such vectors are of the form $\langle a, b, 2a+b\rangle$, where a, b are real numbers.

17.5.50 $\mathbf{F} = \langle z,0,0\rangle$, or $\mathbf{F} = \langle 0,0,-x\rangle$, so it is not unique.

17.5.51 $\mathbf{F} = \frac{1}{2}\langle y^2+z^2, 0, 0\rangle$ or $\mathbf{F} = \langle 0, -xy, -xz\rangle$, so it is not unique.

17.5.52

a. Looking at the picture, it is clear that the distance from P to $\mathbf{a}$ is $|\mathbf{r}|$.



b. The velocity field is $\mathbf{a}\times\mathbf{r}$, so the speed, which is the magnitude of velocity, is $|\mathbf{a}\times\mathbf{r}|$. Now $|\mathbf{a}\times\mathbf{r}| = |\mathbf{a}|\cdot|\mathbf{r}|\cdot\sin(\theta)\cdot\mathbf{n} = R\,|\mathbf{a}|\,\mathbf{n}$, where $\mathbf{n}$ is a vector normal to the plane determined by $\mathbf{a}$ and $\mathbf{r}$. Thus the motion of the particle is always perpendicular to this plane, so it rotates about the axis $\mathbf{a}$. It is moving at a speed $R\,|\mathbf{a}|$ around a circle of radius R, so its angular speed is $\frac{R\,|\mathbf{a}|}{R} = |\mathbf{a}|$.

c. Because $|\nabla\times\mathbf{v}| = 2\,|\mathbf{a}|$, it follows from part (b) that $\omega = |\mathbf{a}| = \frac{1}{2}\,|\nabla\times\mathbf{v}|$.

17.5.53 The curl of this vector field is $\langle 0,1,0\rangle$. The component of the curl along some unit vector $\mathbf{n}$ is $(\nabla\times\mathbf{F})\cdot\mathbf{n}$.

a. $\langle 0,1,0\rangle\cdot\langle 1,0,0\rangle = 0$, so the wheel does not spin.

b. $\langle 0,1,0\rangle\cdot\langle 0,1,0\rangle = 1$, so the wheel spins clockwise (looking towards positive y).

c. $\langle 0,1,0\rangle\cdot\langle 0,0,1\rangle = 0$, so the wheel does not spin.

17.5.54 The curl of the vector field is $\nabla\times\mathbf{v} = \langle -2,0,2\rangle$.

a. The wheel is placed with its axis in the direction $\langle 0,0,1\rangle$, so the component of velocity in that direction is $\langle -2,0,2\rangle\cdot\langle 0,0,1\rangle = 2$, and $\omega = \frac{1}{2}\cdot 2 = 1$.

b. The wheel is placed with its axis in the direction $\langle 0,1,0\rangle$, so the component of velocity in that direction is $\langle -2,0,2\rangle \cdot \langle 0,1,0\rangle = 0$, and the wheel does not turn.

c. The wheel is placed with its axis in the direction $\langle 1,0,0\rangle$, so the component of velocity in that direction is $\langle -2,0,2\rangle \cdot \langle 1,0,0\rangle = -2$, and $\omega = \dfrac{1}{2}\cdot|-2| = 1$.

17.5.55 The curl of the vector field is $\nabla\times$v$=\langle -20,0,0\rangle$. Because the wheel is placed with its axis normal to the plane $x+y+z=1$, its axis must point in the direction $\langle 1,1,1\rangle$ (with unit vector $\dfrac{1}{\sqrt{3}}\langle 1,1,1\rangle$). Thus, the component of velocity along that direction is $\dfrac{1}{\sqrt{3}}\langle -20,0,0\rangle\cdot\langle 1,1,1\rangle = \dfrac{-20}{\sqrt{3}}$ and then ω is the absolute value of one half of that amount, or $\omega = \dfrac{10}{\sqrt{3}}$ or $\dfrac{5}{\pi\sqrt{3}} \approx 0.9189$ revolutions per time unit.

17.5.56

$$\mathbf{F} = -100k\nabla e^{-\sqrt{x^2+y^2+z^2}} = \frac{100ke^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\langle x,y,z\rangle.$$

Looking at the x component, its contribution to the divergence is

$$100k\frac{\partial}{\partial x}\left[\frac{xe^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right] = -100k\frac{\left(x^2\sqrt{x^2+y^2+z^2}-y^2-z^2\right)e^{-\sqrt{x^2+y^2+z^2}}}{\left(x^2+y^2+z^2\right)^{3/2}}$$

and similarly for the y and z components. Thus the divergence is the sum of these three terms, which is

$$\begin{aligned}&-100k\frac{e^{-\sqrt{x^2+y^2+z^2}}}{\left(x^2+y^2+z^2\right)^{3/2}}\left(\left(x^2+y^2+z^2\right)^{3/2}-2\left(x^2+y^2+z^2\right)\right)\\ &=100k\frac{e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\left(2-\sqrt{x^2+y^2+z^2}\right).\end{aligned}$$

17.5.57 $\mathbf{F} = -100k\nabla e^{-x^2+y^2+z^2} = -200ke^{-x^2+y^2+z^2}\langle -x,y,z\rangle$, so the divergence is

$$\begin{aligned}&-200k\left(\frac{\partial}{\partial x}\left(-xe^{-x^2+y^2+z^2}\right)+\frac{\partial}{\partial y}\left(ye^{-x^2+y^2+z^2}\right)+\frac{\partial}{\partial z}\left(ze^{-x^2+y^2+z^2}\right)\right)\\ &=-200k\Big(-e^{-x^2+y^2+z^2}+2x^2e^{-x^2+y^2+z^2}+e^{-x^2+y^2+z^2}+2y^2e^{-x^2+y^2+z^2}+e^{-x^2+y^2+z^2}+2z^2e^{-x^2+y^2+z^2}\Big)\\ &=-200k\left(e^{-x^2+y^2+z^2}+2\left(x^2+y^2+z^2\right)e^{-x^2+y^2+z^2}\right)\\ &=-200k\left(2x^2+2y^2+2z^2+1\right)e^{-x^2+y^2+z^2}.\end{aligned}$$

17.5.58 $\mathbf{F} = -100k\nabla\left(1+\sqrt{1+x^2+y^2+z^2}\right) = -100k\left(x^2+y^2+z^2\right)^{-1/2}\langle x,y,z\rangle$, and thus the divergence is

$$\nabla\cdot\mathbf{F} = \frac{-200k}{\sqrt{x^2+y^2+z^2}}.$$

17.5.59

a. $\mathbf{F} = -\nabla\varphi = -GMm\Big\langle\dfrac{\partial}{\partial x}\left[\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right], \dfrac{\partial}{\partial y}\left[\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right], \dfrac{\partial}{\partial z}\left[\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right]\Big\rangle =$
$-GMm\left(x^2+y^2+z^2\right)^{-3/2}\langle -x,-y,-z\rangle = GMm\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle = GMm\dfrac{\mathbf{r}}{|\mathbf{r}|^3}.$

b. $\dfrac{\partial}{\partial y}x\left(x^2+y^2+z^2\right)^{-3/2}=-3xy\left(x^2+y^2+z^2\right)^{-5/2}$. Applying this pattern in computing the curl gives $\nabla\times\mathbf{F}=GMm\left(x^2+y^2+z^2\right)^{-5/2}\left((-3yz+3yz)\,\mathbf{i}+(-3xz+3xz)\,\mathbf{j}+(-3xy+3xy)\,\mathbf{k}\right)=\mathbf{0}$, so the field is irrotational.

17.5.60 Note: this is identical to the previous problem except for the constant.

a. $\mathbf{F}=-\nabla\varphi=-\dfrac{q}{4\pi\epsilon_0}\left\langle\dfrac{\partial}{\partial x}\left[\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right],\dfrac{\partial}{\partial y}\left[\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right],\dfrac{\partial}{\partial z}\left[\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right]\right\rangle=$ $-\dfrac{q}{4\pi\epsilon_0}\left(x^2+y^2+z^2\right)^{-3/2}\langle -x,-y,-z\rangle=\dfrac{q}{4\pi\epsilon_0}\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle=\dfrac{q}{4\pi\epsilon_0}\dfrac{\mathbf{r}}{|\mathbf{r}|^3}$.

b. $\dfrac{\partial}{\partial y}x\left(x^2+y^2+z^2\right)^{-3/2}=-3xy\left(x^2+y^2+z^2\right)^{-5/2}$. Applying this pattern in computing the curl gives $\nabla\times\mathbf{F}=\dfrac{q}{4\pi\epsilon_0}\left(x^2+y^2+z^2\right)^{-5/2}\left((-3yz+3yz)\,\mathbf{i}+(-3xz+3xz)\,\mathbf{j}+(-3xy+3xy)\,\mathbf{k}\right)=\mathbf{0}$, so the field is irrotational.

17.5.61 Using Exercise 40, we have

$$\rho\left(\langle\frac{\partial u}{\partial t},\frac{\partial v}{\partial t},\frac{\partial w}{\partial t}\rangle+\left(u\frac{\partial}{\partial t}+v\frac{\partial}{\partial t}+w\frac{\partial}{\partial t}\right)\langle u,v,w\rangle\right)=-\langle\frac{\partial p}{\partial t},\frac{\partial p}{\partial t},\frac{\partial p}{\partial t}\rangle+\mu\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial t^2}\right)\langle u,v,w\rangle,$$

so that

$$\begin{aligned}\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)&=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\\ \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)&=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\\ \rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)&=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right).\end{aligned}$$

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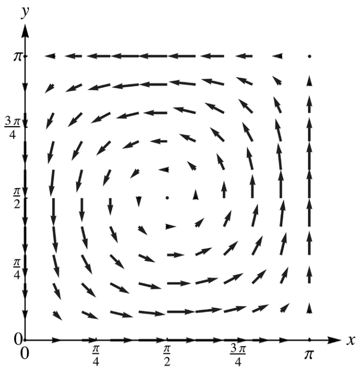
17.5.62

a. $\nabla\times\langle 2,-3y,5z\rangle=\mathbf{0}$ and $\nabla\times\langle y,x-z,-y\rangle=\mathbf{0}$ so they are both irrotational.

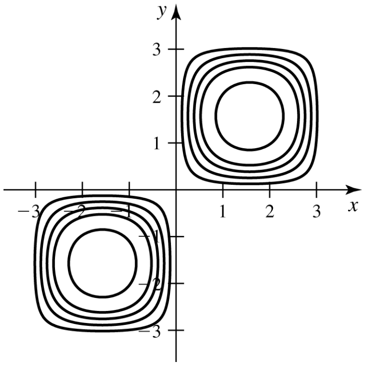
b. If ψ is defined as stated, then $\nabla^2\psi=\dfrac{\partial^2}{\partial x^2}\psi+\dfrac{\partial^2}{\partial y^2}\psi+\dfrac{\partial^2}{\partial z^2}\psi=-\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}=\dfrac{\partial u}{\partial y}-\dfrac{\partial v}{\partial x}$ while $\zeta=\mathrm{k}\cdot\nabla\times\langle u,v,0\rangle=\mathrm{k}\cdot\left(-\dfrac{\partial v}{\partial z}\mathrm{i}+\dfrac{\partial u}{\partial z}\mathrm{j}+\left(\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\right)\mathrm{k}\right)=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}$ so that $\nabla^2\psi=-\zeta$ as desired.

c. $u=\dfrac{\partial\psi}{\partial y}=\sin(x)\cos(y)$ and $v=-\dfrac{\partial\psi}{\partial x}=-\cos(x)\sin(y)$. The velocity field looks like



d. The vorticity function is $\zeta = -\boldsymbol{\nabla}^2\psi = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = \sin(x)\sin(y) + \sin(x)\,sin(y) = 2\sin(x)\sin(y)$

The diagram shows level curves for ζ at $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, 1 and $\frac{3}{2}$ (from outer to inner).

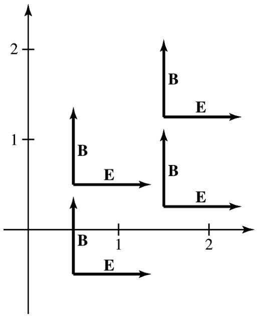


Using implicit differentiation (or from looking at the diagram), ζ achieves its maximum at $x = y = \dfrac{\pi}{2}$, where it has value 2, and its minimum on the boundary, where it is zero.

17.5.63

a. We have $\boldsymbol{\nabla} \times \mathbf{B} = -\dfrac{\partial}{\partial z}(A\sin(kz - \omega t))\,\mathbf{i} + 0\mathbf{j} + \dfrac{\partial}{\partial x}(A\sin(kz - \omega t))\,\mathbf{k} = -Ak\cos(kz - \omega t)\,\mathbf{i}$.
$C\dfrac{\partial \mathbf{E}}{\partial t} = C\dfrac{\partial}{\partial t}(A\sin(kz - \omega t)\,\mathbf{i}) = -\omega\, CA\cos(kz - \omega t)\,\mathbf{i}$ so that the two are equal when $k = \omega\, C$, or $\omega = \dfrac{k}{C}$.

b.



17.5.64 Let $\mathbf{V} = \langle xy, -\frac{1}{2}y^2, 0\rangle$ and $\mathbf{W} = \langle 0, \frac{1}{2}y^2, 0\rangle$. Then $\boldsymbol{\nabla} \cdot \mathbf{V} = 0$ and $\boldsymbol{\nabla} \times \mathbf{W} = \mathbf{0}$.

17.5.65 Let $\mathbf{F} = \langle f, g, h\rangle$ and $\mathbf{G} = \langle k, m, n\rangle$. Then

a. $\boldsymbol{\nabla} \cdot (\mathbf{F} + \mathbf{G}) = \boldsymbol{\nabla} \cdot \langle f + k, g + m, h + n\rangle = \dfrac{\partial}{\partial x}(f + k) + \dfrac{\partial}{\partial y}(g + m) + \dfrac{\partial}{\partial z}(h + n) = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z} + \dfrac{\partial k}{\partial x} + \dfrac{\partial m}{\partial y} + \dfrac{\partial n}{\partial z} = \boldsymbol{\nabla} \cdot \mathbf{F} + \boldsymbol{\nabla} \cdot \mathbf{G}$.

b.

$$\begin{aligned}
&\boldsymbol{\nabla} \times (\mathbf{F} + \mathbf{G})\\
&= \left(\frac{\partial}{\partial y}(h + n) - \frac{\partial}{\partial z}(g + m)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(f + k) - \frac{\partial}{\partial x}(h + n)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(g + m) - \frac{\partial}{\partial y}(f + k)\right)\mathbf{k}\\
&= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} + \left(\frac{\partial n}{\partial y} - \frac{\partial m}{\partial z}\right)\mathbf{i} + \left(\frac{\partial k}{\partial z} - \frac{\partial n}{\partial x}\right)\mathbf{j} + \left(\frac{\partial m}{\partial x} - \frac{\partial k}{\partial y}\right)\mathbf{k}\\
&= \boldsymbol{\nabla} \times \mathbf{F} + \boldsymbol{\nabla} \times \mathbf{G}.
\end{aligned}$$

c. $\boldsymbol{\nabla} \cdot (c\mathbf{F}) = \frac{\partial}{\partial x}(cf) + \frac{\partial}{\partial y}(cg) + \frac{\partial}{\partial z}(ch) = c\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) = c\,(\boldsymbol{\nabla} \cdot \mathbf{F})$.

d. $\boldsymbol{\nabla} \times (c\mathbf{F}) = \left(\frac{\partial}{\partial y}(ch) - \frac{\partial}{\partial z}(cg)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(cf) - \frac{\partial}{\partial x}(ch)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(cg) - \frac{\partial}{\partial y}(cf)\right)\mathbf{k} =$
$c\left[\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}\right] = c\,(\boldsymbol{\nabla} \times \mathbf{F})$.

17.5.66 The statement is not true. The conditions imply that $\mathbf{F} - \mathbf{G}$ is irrotational and source-free, but this can happen with nonconstant vector fields. For example, if $\mathbf{F} = \langle x^2 - y^2, -2xy\rangle$ and $\mathbf{G} = \langle 2xy, x^2 - y^2\rangle$, then $\mathbf{F} - \mathbf{G}$ is irrotational and source-free (i.e. has zero curl and zero divergence). Clearly the two vector fields do not differ by a constant.

17.5.67 $\boldsymbol{\nabla} \cdot (\varphi\mathbf{F}) = \boldsymbol{\nabla} \cdot \langle \varphi f, \varphi g, \varphi h\rangle = \frac{\partial}{\partial x}(\varphi f) + \frac{\partial}{\partial y}(\varphi g) + \frac{\partial}{\partial z}(\varphi h) = \varphi\frac{\partial f}{\partial x} + f\frac{\partial \varphi}{\partial x} + \varphi\frac{\partial g}{\partial y} + g\frac{\partial \varphi}{\partial y} + \varphi\frac{\partial h}{\partial z} + h\frac{\partial \varphi}{\partial z} = \varphi\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) + \langle f, g, h\rangle\langle\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\rangle = \varphi\boldsymbol{\nabla} \cdot \mathbf{F} + \boldsymbol{\nabla\varphi} \cdot \mathbf{F}$

17.5.68 $\boldsymbol{\nabla} \times (\varphi\mathbf{F}) = \boldsymbol{\nabla} \times \langle \varphi f, \varphi g, \varphi h\rangle = \langle\frac{\partial}{\partial y}(\varphi h) - \frac{\partial}{\partial z}(\varphi g), \frac{\partial}{\partial z}(\varphi f) - \frac{\partial}{\partial x}(\varphi h), \frac{\partial}{\partial x}(\varphi g) - \frac{\partial}{\partial y}(\varphi f)\rangle = \langle\varphi\frac{\partial h}{\partial y} + h\frac{\partial \varphi}{\partial y} - \varphi\frac{\partial g}{\partial z} - g\frac{\partial \varphi}{\partial z}, \varphi\frac{\partial f}{\partial z} + f\frac{\partial \varphi}{\partial z} - \varphi\frac{\partial h}{\partial x} - h\frac{\partial \varphi}{\partial x}, \varphi\frac{\partial g}{\partial x} + g\frac{\partial \varphi}{\partial x} - \varphi\frac{\partial f}{\partial y} - f\frac{\partial \varphi}{\partial y}\rangle = \langle\varphi\frac{\partial h}{\partial y} - \varphi\frac{\partial g}{\partial zy}, \varphi\frac{\partial f}{\partial z} - \varphi\frac{\partial h}{\partial x}, \varphi\frac{\partial g}{\partial x} - \varphi\frac{\partial f}{\partial y}\rangle + \langle h\frac{\partial \varphi}{\partial y} - g\frac{\partial \varphi}{\partial z}, f\frac{\partial \varphi}{\partial z} - h\frac{\partial \varphi}{\partial x}, g\frac{\partial \varphi}{\partial x} - f\frac{\partial \varphi}{\partial y}\rangle = \varphi\boldsymbol{\nabla} \times \mathbf{F} + \boldsymbol{\nabla\varphi} \times \mathbf{F}$

17.5.69 If $\mathbf{F} = \langle f, g, h\rangle$ and $\mathbf{G} = \langle k, m, n\rangle$, then

$$\begin{aligned}
\mathbf{G} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) - \mathbf{F} \cdot (\boldsymbol{\nabla} \times \mathbf{G}) &= \langle k, m, n\rangle \cdot \langle\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\rangle \\
&\quad - \langle f, g, h\rangle \cdot \langle\frac{\partial n}{\partial y} - \frac{\partial m}{\partial z}, \frac{\partial k}{\partial z} - \frac{\partial n}{\partial x}, \frac{\partial m}{\partial x} - \frac{\partial k}{\partial y}\rangle \\
&= k\frac{\partial h}{\partial y} - k\frac{\partial g}{\partial z} + m\frac{\partial f}{\partial z} - m\frac{\partial h}{\partial x} + n\frac{\partial g}{\partial x} - n\frac{\partial f}{\partial y} \\
&\quad - f\frac{\partial n}{\partial y} + f\frac{\partial m}{\partial z} - g\frac{\partial k}{\partial z} + g\frac{\partial n}{\partial x} - h\frac{\partial m}{\partial x} + h\frac{\partial k}{\partial y} \\
&= n\frac{\partial g}{\partial x} + g\frac{\partial n}{\partial x} - h\frac{\partial m}{\partial x} - m\frac{\partial h}{\partial x} + h\frac{\partial k}{\partial y} + k\frac{\partial h}{\partial y} - f\frac{\partial n}{\partial y} - n\frac{\partial f}{\partial y} \\
&\quad + f\frac{\partial m}{\partial z} + m\frac{\partial f}{\partial z} - g\frac{\partial k}{\partial z} - k\frac{\partial g}{\partial z} \\
&= \frac{\partial}{\partial x}(gn - hm) + \frac{\partial}{\partial y}(hk - fn) + \frac{\partial}{\partial z}(fm - gk) = \boldsymbol{\nabla} \cdot (\mathbf{F} \times \mathbf{G}).
\end{aligned}$$

17.5.70 First, $(\mathbf{G} \cdot \boldsymbol{\nabla})\,\mathbf{F} = \left(k\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)\langle f, g, h\rangle = \langle k\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z}, k\frac{\partial g}{\partial x} + m\frac{\partial g}{\partial y} + n\frac{\partial g}{\partial z}, k\frac{\partial h}{\partial x} + m\frac{\partial h}{\partial y} + n\frac{\partial h}{\partial z}\rangle$ and similarly for $(\mathbf{F} \cdot \boldsymbol{\nabla})\,\mathbf{G}$. Next, $\mathbf{G}\,(\boldsymbol{\nabla} \cdot \mathbf{F}) = \langle k, m, n\rangle\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) = \langle k\frac{\partial f}{\partial x} + k\frac{\partial g}{\partial y} +$

$k\frac{\partial h}{\partial z}, m\frac{\partial f}{\partial x}+m\frac{\partial g}{\partial y}+m\frac{\partial h}{\partial z}, n\frac{\partial f}{\partial x}+n\frac{\partial g}{\partial y}+n\frac{\partial h}{\partial z}\rangle$ and similarly for $\mathbf{F}\left(\boldsymbol{\nabla}\cdot\mathbf{G}\right)$. Thus

$$\begin{aligned}
&\left(\mathbf{G}\cdot\boldsymbol{\nabla}\right)\mathbf{F}-\mathbf{G}\left(\boldsymbol{\nabla}\cdot\mathbf{F}\right)-\left(\mathbf{F}\cdot\boldsymbol{\nabla}\right)\mathbf{G}+\mathbf{F}\left(\boldsymbol{\nabla}\cdot\mathbf{G}\right)\\
&=\langle k\frac{\partial f}{\partial x}+m\frac{\partial f}{\partial y}+n\frac{\partial f}{\partial z}, k\frac{\partial g}{\partial x}+m\frac{\partial g}{\partial y}+n\frac{\partial g}{\partial z}, k\frac{\partial h}{\partial x}+m\frac{\partial h}{\partial y}+n\frac{\partial h}{\partial z}\rangle\\
&\quad-\langle k\frac{\partial f}{\partial x}+k\frac{\partial g}{\partial y}+k\frac{\partial h}{\partial z}, m\frac{\partial f}{\partial x}+m\frac{\partial g}{\partial y}+m\frac{\partial h}{\partial z}, n\frac{\partial f}{\partial x}+n\frac{\partial g}{\partial y}+n\frac{\partial h}{\partial z}\rangle\\
&\quad-\langle f\frac{\partial k}{\partial x}+g\frac{\partial k}{\partial y}+h\frac{\partial k}{\partial z}, f\frac{\partial m}{\partial x}+g\frac{\partial m}{\partial y}+h\frac{\partial m}{\partial z}, f\frac{\partial n}{\partial x}+g\frac{\partial n}{\partial y}+h\frac{\partial n}{\partial z}\rangle\\
&\quad+\langle f\frac{\partial k}{\partial x}+f\frac{\partial m}{\partial y}+f\frac{\partial n}{\partial z}, g\frac{\partial k}{\partial x}+g\frac{\partial m}{\partial y}+g\frac{\partial n}{\partial z}, h\frac{\partial k}{\partial x}+h\frac{\partial m}{\partial y}+h\frac{\partial n}{\partial z}\rangle\\
&=\langle\frac{\partial}{\partial y}\left(fm-gk\right)-\frac{\partial}{\partial z}\left(hk-fn\right), \frac{\partial}{\partial z}\left(gn-hm\right)-\frac{\partial}{\partial x}\left(fm-gk\right), \frac{\partial}{\partial x}\left(hk-fn\right)-\frac{\partial}{\partial y}\left(gn-hm\right).
\end{aligned}$$

But $\mathbf{F}\times\mathbf{G}=\langle gn-hm, hk-fn, fm-gk\rangle$, so the above expression is indeed equal to $\boldsymbol{\nabla}\times\left(\mathbf{F}\times\mathbf{G}\right)$.

17.5.71 Use the values of$\left(\mathbf{G}\cdot\boldsymbol{\nabla}\right)\mathbf{F}$ and $\left(\mathbf{F}\cdot\boldsymbol{\nabla}\right)\mathbf{G}$ from the previous problem. Then $\mathbf{G}\times\left(\boldsymbol{\nabla}\times\mathbf{F}\right)=\langle k,m,n\rangle\times\langle\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}, \frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}, \frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\rangle=\langle m\frac{\partial g}{\partial x}-m\frac{\partial f}{\partial y}-n\frac{\partial f}{\partial z}+n\frac{\partial h}{\partial x}, n\frac{\partial h}{\partial y}-n\frac{\partial g}{\partial z}-k\frac{\partial g}{\partial x}+k\frac{\partial f}{\partial y}, k\frac{\partial f}{\partial z}-k\frac{\partial h}{\partial x}-m\frac{\partial h}{\partial y}+m\frac{\partial g}{\partial z}\rangle$ and similarly for $\mathbf{F}\times\left(\boldsymbol{\nabla}\times\mathbf{G}\right)$.

Thus

$$\begin{aligned}
&\left(\mathbf{G}\cdot\boldsymbol{\nabla}\right)\mathbf{F}+\left(\mathbf{F}\cdot\boldsymbol{\nabla}\right)\mathbf{G}+\mathbf{G}\times\left(\boldsymbol{\nabla}\times\mathbf{F}\right)+\mathbf{F}\times\left(\boldsymbol{\nabla}\times\mathbf{G}\right)\\
&=\langle k\frac{\partial f}{\partial x}+m\frac{\partial f}{\partial y}+n\frac{\partial f}{\partial z}, k\frac{\partial g}{\partial x}+m\frac{\partial g}{\partial y}+n\frac{\partial g}{\partial z}, k\frac{\partial h}{\partial x}+m\frac{\partial h}{\partial y}+n\frac{\partial h}{\partial z}\rangle\\
&\quad+\langle f\frac{\partial k}{\partial x}+g\frac{\partial k}{\partial y}+h\frac{\partial k}{\partial z}, f\frac{\partial m}{\partial x}+g\frac{\partial m}{\partial y}+h\frac{\partial m}{\partial z}, f\frac{\partial n}{\partial x}+g\frac{\partial n}{\partial y}+h\frac{\partial n}{\partial z}\rangle\\
&\quad+\langle m\frac{\partial g}{\partial x}-m\frac{\partial f}{\partial y}-n\frac{\partial f}{\partial z}+n\frac{\partial h}{\partial x}, n\frac{\partial h}{\partial y}-n\frac{\partial g}{\partial z}-k\frac{\partial g}{\partial x}+k\frac{\partial f}{\partial y}, k\frac{\partial f}{\partial z}-k\frac{\partial h}{\partial x}-m\frac{\partial h}{\partial y}+m\frac{\partial g}{\partial z}\rangle\\
&\quad+\langle g\frac{\partial m}{\partial x}-g\frac{\partial k}{\partial y}-h\frac{\partial k}{\partial z}+h\frac{\partial n}{\partial x}, h\frac{\partial n}{\partial y}-h\frac{\partial m}{\partial z}-f\frac{\partial m}{\partial x}+f\frac{\partial k}{\partial y}, f\frac{\partial k}{\partial z}-f\frac{\partial n}{\partial x}-g\frac{\partial n}{\partial y}+g\frac{\partial m}{\partial z}\rangle\\
&=\langle\frac{\partial}{\partial x}\left(fk+gm+hn\right), \frac{\partial}{\partial y}\left(fk+gm+hn\right), \frac{\partial}{\partial z}\left(fk+gm+hn\right)\rangle=\nabla\left(\mathbf{F}\cdot\mathbf{G}\right).
\end{aligned}$$

17.5.72

$$\begin{aligned}
\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\mathbf{F}\right)&=\boldsymbol{\nabla}\times\langle\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}, \frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}, \frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\rangle\\
&=\langle\frac{\partial^2 g}{\partial y\partial x}-\frac{\partial^2 f}{\partial y^2}-\frac{\partial^2 f}{\partial z^2}+\frac{\partial^2 h}{\partial z\partial x}, \frac{\partial^2 h}{\partial z\partial y}-\frac{\partial^2 g}{\partial z^2}-\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 f}{\partial x\partial y}, \frac{\partial^2 f}{\partial x\partial z}-\frac{\partial^2 h}{\partial x^2}-\frac{\partial^2 h}{\partial^2 y}-\frac{\partial^2 g}{\partial y\partial z}\rangle
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{F}\right)-\left(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\right)\mathbf{F}&=\boldsymbol{\nabla}\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\langle f,g,h\rangle\\
&=\langle\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 g}{\partial x\partial y}+\frac{\partial^2 h}{\partial x\partial z}, \frac{\partial^2 f}{\partial y\partial x}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 h}{\partial y\partial z}, \frac{\partial^2 f}{\partial z\partial x}+\frac{\partial^2 g}{\partial z\partial y}+\frac{\partial^2 h}{\partial z^2}\rangle\\
&\quad-\langle\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}, \frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}\rangle.
\end{aligned}$$

The two expressions are equal after cancellations and noting that mixed partials are equal.

17.5.73

$$\begin{aligned}
&\nabla \cdot \frac{\langle x,y,z\rangle}{\left(x^2+y^2+z^2\right)^{p/2}}\\
&= \frac{(1-p)\,x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{1+p/2}} + \frac{x^2+(1-p)\,y^2+z^2}{\left(x^2+y^2+z^2\right)^{1+p/2}} + \frac{x^2+y^2+(1-p)\,z^2}{\left(x^2+y^2+z^2\right)^{1+p/2}}\\
&= \frac{(3-p)\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{1+p/2}} = \frac{3-p}{\left(x^2+y^2+z^2\right)^{p/2}} = \frac{3-p}{|\mathbf{r}|^p}
\end{aligned}$$

17.5.74 $\boldsymbol{\nabla}\left(\dfrac{1}{|r|^p}\right) = \boldsymbol{\nabla}\left(\dfrac{1}{\left(x^2+y^2+z^2\right)^{p/2}}\right) = -\dfrac{p}{\left(x^2+y^2+z^2\right)^{1+p/2}}\langle x,y,z\rangle = -\dfrac{p\,\mathbf{r}}{|\mathbf{r}|^{p+2}}$.

17.5.75 $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\left(\dfrac{1}{|r|^p}\right) = \boldsymbol{\nabla}\cdot\left(-\dfrac{p\,\mathbf{r}}{|\mathbf{r}|^{p+2}}\right)$ by Exercise 72, and then by Exercise 71, $\boldsymbol{\nabla}\cdot\left(-\dfrac{p\,\mathbf{r}}{|\mathbf{r}|^{p+2}}\right) = -p\,\boldsymbol{\nabla}\cdot\dfrac{\mathbf{r}}{|\mathbf{r}|^{p+2}} = \dfrac{-p\,(3-(p+2))}{|\mathbf{r}|^{p+2}} = \dfrac{p\,(p-1)}{|\mathbf{r}|^{p+2}}$.

17.6 Surface Integrals

17.6.1 $\mathbf{r}\,(u,v) = \langle a\cos u, a\sin u, v\rangle$ where $0\le u\le 2\pi$; $0\le v\le h$.

17.6.2 $\mathbf{r}\,(u,v) = \left\langle \dfrac{av}{h}\cos u, \dfrac{av}{h}\sin u, v\right\rangle$ where $0\le u\le 2\pi$; $0\le v\le h$.

17.6.3 $\mathbf{r}\,(u,v) = \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$ where $0\le u\le \pi$; $0\le v\le 2\pi$.

17.6.4 A cone of height h and radius a has equation $a^2z^2 = h^2\left(x^2+y^2\right)$; thus $z_x = \dfrac{h^2x}{a^2z}$ and similarly for z_y, so compute

$$\begin{aligned}
\iint\limits_S f\,(x,y,z)\;dS &= \iint\limits_R f\left(x,y,\frac{h^2}{a^2}\sqrt{x^2+y^2}\right)\sqrt{\left(\frac{h^2x}{a^2z}\right)^2+\left(\frac{h^2y}{a^2z}\right)^2+1}\,dA\\
&= \iint\limits_R f\left(x,y,\frac{h^2}{a^2}\sqrt{x^2+y^2}\right)\sqrt{1+\frac{h^2}{a^2}}\,dA.
\end{aligned}$$

17.6.5 Use the parametric description from problem 3 and compute

$$\int_0^{\pi}\int_0^{2\pi} a^2 f\,(a\sin u\cos v, a\sin u\sin v, a\cos u)\sin u\,du\,dv.$$

17.6.6 It means that we can make a consistent choice of normal vectors such that when you walk along the surface, the direction of the normal vectors does not change discontinuously.

17.6.7 The usual orientation of a closed surface is that the normal vectors point outwards.

17.6.8 Because the vector field is vertical, the same amount of materials goes through the surface as through its projection on the xy-plane.

17.6.9 $\left\langle u, v, \dfrac{1}{3}\,(16-2u+4v)\right\rangle$, $|u|<\infty$, $|v|<\infty$.

17.6.10 $\langle 4\sin u\cos v, 4\sin u\sin v, 4\cos v\rangle$, $0\le u\le \dfrac{\pi}{4}$, $0\le v\le 2\pi$.

17.6.11 $\langle v\cos u, v\sin u, v\rangle$, $0 \le u \le 2\pi$, $2 \le v \le 8$.

17.6.12 $\left\langle \frac{v}{2}\cos u, \frac{v}{2}\sin u, v\right\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4$.

17.6.13 $\langle 3\cos u, 3\sin u, v\rangle$, $0 \le u \le \dfrac{\pi}{2}$, $0 \le v \le 3$.

17.6.14 $\langle v, 6\cos u, 6\sin u\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 9$.

17.6.15 The segment of the plane $z = 2x + 3y - 1$ above $[1,3] \times [2,4]$.

17.6.16 The segment of the plane $z = 2 - y$ above $[0,2] \times [0,4]$.

17.6.17 The portion of the cone $z^2 = 16x^2 + 16y^2$ of height 12 and radius 3, where $y \ge 0$.

17.6.18 The cylinder $y^2 + z^2 = 36$ of radius 6 whose axis is the x-axis, for $0 \le x \le 2$.

17.6.19 Using the standard parametric description of the cylinder, we have $\mathbf{r}(u,v) = \langle 4\cos u, 4\sin u, v\rangle$ for $0 \le v \le 7$, $0 \le u \le \pi$. Then $|\mathbf{t}_u \times \mathbf{t}_v| = 4$ and the area is $\displaystyle\iint_S 1\,dS = \iint_R 4\,dA = \int_0^\pi \int_0^7 4\,dv\,du = 28\pi$.

17.6.20 The plane has the parametric description $\mathbf{r}(u,v) = \langle u, v, 3 - u - 3v\rangle$, $0 \le v \le 1$, $0 \le u \le 3 - 3v$. Then $\mathbf{t}_u \times \mathbf{t}_v = \langle 1,0,-1\rangle \times \langle 0,1,-3\rangle = \langle 1,3,1\rangle$, so that $|\mathbf{t}_u \times \mathbf{t}_v| = \sqrt{11}$. Then

$$\iint_S 1\,dS = \sqrt{11}\iint_R 1\,dA = \sqrt{11}\int_0^1 \int_0^{3-3v} 1\,du\,dv = \sqrt{11}\int_0^1 (3-3v)\,dv = \frac{3\sqrt{11}}{2}.$$

17.6.21 The plane has parametric description $\mathbf{r}(u,v) = \langle u, v, 10 - u - v\rangle$, for $-2 \le u \le 2$, $-2 \le v \le 2$. Then $\mathbf{t}_u \times \mathbf{t}_v = \langle 1,0,-1\rangle \times \langle 0,1,-1\rangle = \langle 1,1,1\rangle$, so that $\displaystyle\iint_S 1\,dS = \sqrt{3}\iint_R 1\,dA = 16\sqrt{3}$.

17.6.22 Using the standard parametric description of the sphere with $0 \le u \le \dfrac{\pi}{2}$, $0 \le v \le 2\pi$, we have

$$|\mathrm{t}_u \times \mathrm{t}_v| = a^2 \sin u$$

so that

$$\iint_S 1\,dS = \iint_R 100\,\sin u\,dA = \int_0^{\pi/2}\int_0^{2\pi} 100\sin u\,dv\,du = 200\pi\int_0^{\pi/2} \sin u\,du = 200\pi.$$

17.6.23 Parameterize the cone by $\mathbf{r}(u,v) = \left\langle \frac{r}{h}v\cos u, \frac{r}{h}v\sin u, v\right\rangle$, for $0 \le v \le h$, $0 \le u \le 2\pi$; then $\mathbf{t}_u \times \mathbf{t}_v = \left\langle -\frac{r}{h}v\sin u, \frac{r}{h}v\cos u, 0\right\rangle \times \left\langle \frac{r}{h}\cos u, \frac{r}{h}\sin u, 1\right\rangle$ and $|\mathrm{t}_u \times \mathrm{t}_v| = \frac{r}{h^2}v\sqrt{h^2 + r^2}$. Then

$$\begin{aligned}\iint_S 1\,dS &= \frac{r\sqrt{h^2+r^2}}{h^2}\iint_R v\,dA = \frac{r\sqrt{h^2+r^2}}{h^2}\int_0^{2\pi}\int_0^h v\,dv\,du \\ &= \frac{r\sqrt{h^2+r^2}}{h^2}\int_0^{2\pi} \frac{1}{2}h^2\,du = \frac{2\pi r\sqrt{h^2+r^2}}{2} = \pi r\sqrt{h^2+r^2}.\end{aligned}$$

17.6.24 Using the standard parameterization of the sphere for $1 \le 2\cos u \le 2$, or $0 \le u \le \dfrac{\pi}{3}$, and $0 \le v \le 2\pi$, we obtain

$$\iint_S 1\,dS = \iint_R 4\,\sin u\,dA = 4\int_0^{2\pi}\int_0^{\pi/3} \sin u\,du\,dv = 4\int_0^{2\pi} (-\cos u)\,\Big|_{u=0}^{u=\pi/3} dv = 4\int_0^{2\pi} \frac{1}{2}\,dv = 4\pi.$$

17.6.25 Using the standard parameterization of the sphere for $0 \leq u \leq \frac{\pi}{2}$, $0 \leq v \leq 2\pi$, we obtain

$$\iint\limits_S \left(x^2+y^2\right)\, dS = \int_0^{2\pi}\int_0^{\pi/2} 36\sin^2 u \cdot 36 \sin u\, du\, dv = 1296\int_0^{2\pi}\int_0^{\pi/2} \sin^3 u\, du\, dv$$
$$= 1296\int_0^{2\pi} \frac{2}{3}\, dv = 1728\pi.$$

17.6.26 Use the standard parameterization of the cylinder, for $0 \leq u \leq 2\pi$, $0 \leq v \leq 3$; then

$$\iint\limits_S y\, dS = \int_0^{2\pi}\int_0^3 3\sin u \cdot 3\, dv\, du = 27\int_0^{2\pi} \sin u\, du = 0.$$

17.6.27 Use the standard parameterization, for $0 \leq u \leq 2\pi$, $0 \leq v \leq 3$; then

$$\iint\limits_S x\, dS = \int_0^{2\pi}\int_0^3 \cos u \cdot 1\, dv\, du = 3\int_0^{2\pi} \cos u\, du = 0.$$

17.6.28 Using the standard parameterization (with $u = \varphi$ and $v = \theta$) for $0 \leq u \leq \frac{\pi}{2}$ and $0 \leq v \leq \frac{\pi}{2}$, we have

$$\iint\limits_R \cos u\, dA = \int_0^{\pi/2}\int_0^{\pi/2} \cos u \cdot \sin u\, du\, dv = \frac{\pi}{4}.$$

17.6.29 $z_x = 2$ and $z_y = 2$, so $\sqrt{z_x^2+z_y^2+1} = \sqrt{9} = 3$.

$$\int_0^2\int_0^{2x} 3\, dy\, dx = \int_0^2 6x\, dx = 3x^2\Big|_0^2 = 12.$$

17.6.30 $z_x = 1$ and $z_y = 3$, so $\sqrt{1+9+1} = \sqrt{11}$.

$$\int_0^{2\pi}\int_1^2 \sqrt{11} r\, dr\, d\theta = 2\pi\sqrt{11}\left(\frac{r^2}{2}\right)\Big|_1^2 = 3\pi\sqrt{11}.$$

17.6.31 $2z\, dz = 8x\, dx$, so $z_x = \frac{4x}{z}$; similarly, $z_y = \frac{4y}{z}$. Thus $\sqrt{z_x^2+z_y^2+1} = \sqrt{\frac{16x^2+16y^2+z^2}{z^2}} = \sqrt{\frac{20\left(x^2+y^2\right)}{4\left(x^2+y^2\right)}} = \sqrt{5}$. Further, this cone sits over $x^2+y^2=4$. Then $\iint\limits_S 1\, dS = \iint\limits_R \sqrt{5}\, dA = 4\pi\sqrt{5}$.

17.6.32 $z_x = x$ and $z_y = 0$, so that $\iint\limits_S 1\, dS = \int_0^4\int_{-1}^1 \sqrt{x^2+1}\, dx\, dy = 4\sqrt{2} + 4\ln(1+\sqrt{2})$.

17.6.33 $dz = 4x\, dx$ so that $z_x = 4x$ and similarly $z_y = 4y$. The paraboloid sits over $x^2+y^2=4$ Thus $\iint\limits_S 1\, dS = \iint\limits_R \sqrt{16x^2+16y^2+1}\, dA = \int_0^{2\pi}\int_0^2 r\sqrt{16r^2+1}\, dr\, d\theta = \frac{\left(65\sqrt{65}-1\right)\pi}{24}$.

17.6.34 We have $z_x = 2x$, $z_y = -2y$, so

$$\iint\limits_S 1\, dS = \iint\limits_R \sqrt{4x^2+4y^2+1}\, dA = \int_0^{\pi/2}\int_0^{\sqrt{2}} r\sqrt{4r^2+1}\, dr\, d\theta = \frac{13\pi}{12}.$$

17.6.35 $z_x = z_y = -1$, so $\displaystyle\iint_S xy\,dS = \sqrt{3}\iint_R xy\,dA = \sqrt{3}\int_0^2\int_0^{2-x} xy\,dy\,dx = \frac{2\sqrt{3}}{3}$.

17.6.36 $z_x = 2x$, $z_y = 2y$, and the paraboloid sits over $x^2+y^2=4$, so $\displaystyle\iint_S \left(x^2+y^2\right)\,dS =$ $\displaystyle\iint_R \left(x^2+y^2\right)\sqrt{4x^2+4y^2+1}\,dA = \int_0^{2\pi}\int_0^1 r^3\sqrt{4r^2+1}\,dr\,d\theta = 2\pi\int_0^1 r^3\sqrt{4r^2+1}\,dr$. Let $u = 4r^2+1$ so that $du = 8r\,dr$. Substituting gives

$$\begin{aligned}
2\pi\cdot\frac{1}{8}\int_1^5 \frac{u-1}{4}\sqrt{u}\,du &= \frac{\pi}{16}\int_1^5 (u^{3/2}-u^{1/2})\,du\\
&= \frac{\pi}{16}\left(\frac{2u^{5/2}}{5}-\frac{2u^{3/2}}{3}\right)\Bigg|_1^5\\
&= \frac{\pi}{16}\left(10\sqrt{5}-\frac{10\sqrt{5}}{3}-\left(\frac{2}{5}-\frac{2}{3}\right)\right)\\
&= \frac{\pi}{16}\left(\frac{100\sqrt{5}}{15}+\frac{4}{15}\right) = \frac{\pi(1+25\sqrt{5})}{60}.
\end{aligned}$$

17.6.37 $x^2+y^2+z^2 = 25$, so $z_x = --\dfrac{x}{z}$ and $z_y = -\dfrac{y}{z}$. Then $\displaystyle\iint_S \left(25-x^2-y^2\right)\,dS =$ $\displaystyle\iint_R \left(25-x^2-y^2\right)\sqrt{\frac{x^2+y^2+z^2}{z^2}}\,dA = 5\iint_R \sqrt{25-x^2-y^2}\,dA = 5\int_0^{2\pi}\int_0^5 r\sqrt{25-r^2}\,dr\,d\theta = \frac{1250\pi}{3}$.

17.6.38 $z_x = -1$, $z_y = -2$, and the limits of integration are $0 \le y \le 4$, $0 \le x \le 8-2y$. Then $\displaystyle\iint_S e^z\,dS =$ $\displaystyle\iint_R e^{8-x-2y}\sqrt{6}\,dA = \sqrt{6}\int_0^4\int_0^{8-2y} e^{8-x-2y}\,dx\,dy = \frac{\sqrt{6}\left(e^8-9\right)}{2}$.

17.6.39 $\displaystyle\iint_S 1\,dS = \int_0^1\int_0^1 \sqrt{1+4+4}\,dx\,dy = 3$.

$\displaystyle\iint_S e^{2x+y+z-3}\,dS = 3\int_0^1\int_0^1 e^{2x+y+4-2x-2y-3}\,dx\,dy = 3\int_0^1\int_0^1 e^{1-y}\,dx\,dy = 3(e-1)$. So the average temperature is $\dfrac{3(e-1)}{3} = e-1$.

17.6.40 $z_x = -2x$, $z_y = -2y$. The paraboloid sits over $x^2+y^2=4$. Thus the area of the paraboloid is

$$\iint_S 1\,dS = \iint_R \sqrt{4x^2+4y^2+1}\,dA = \int_0^{2\pi}\int_0^2 r\sqrt{4r^2+1}\,dr\,d\theta \approx 36.1769,$$

and the integral of the square of the distance from the origin is

$$\begin{aligned}
\iint_S \left(x^2+y^2+z^2\right)\,dS &= \iint_R \left(x^2+y^2+\left(4-x^2-y^2\right)^2\right)\sqrt{4x^2+4y^2+1}\,dA\\
&= \int_0^{2\pi}\int_0^2 r\left(r^2+\left(4-r^2\right)^2\right)\sqrt{4r^2+1}\,dr\,d\theta \approx 227.18.
\end{aligned}$$

Thus the average squared distance is approximately 6.2797.

17.6.41 $z_x = -\dfrac{x}{z}$ and $z_y = -\dfrac{y}{z}$, so that $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{\dfrac{x^2+y^2+z^2}{z^2}} = \left(1 - x^2 - y^2\right)^{-1/2}$. The area of the sphere is $\dfrac{1}{8} \cdot 4\pi = \dfrac{\pi}{2}$, and the integral of the function is

$$\iint\limits_S xyz\, dS = \iint\limits_R xy \left(1 - x^2 - y^2\right)^{1/2} \left(1 - x^2 - y^2\right)^{-1/2}\, dA = \int_0^{\pi/2} \int_0^1 r^3 \sin(\theta) \cos(\theta)\; dr\, d\theta = \frac{1}{8},$$

so that the average value is $\dfrac{1}{4\pi}$.

17.6.42 $z_x = \dfrac{x}{z}$ and $z_y = \dfrac{y}{z}$ so that $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{2}$. The cone sits over $x^2 + y^2 = 4$. The area of the cone is $\displaystyle\iint\limits_S 1\, dS = \sqrt{2} \iint\limits_R dA = 4\pi\sqrt{2}$, and the integral of the temperature function is $\displaystyle\iint\limits_S (100 - 25z)\; dS = \sqrt{2} \iint\limits_R \left(100 - 25\sqrt{x^2+y^2}\right) dA = \sqrt{2} \int_0^{2\pi} \int_0^2 r\,(100 - 25r)\; dr\, d\theta = \frac{800\pi\sqrt{2}}{3}$ so that the average temperature is $\dfrac{200}{3}$.

17.6.43 $z_x = z_y = -1$, so the normal vector is $\langle 1, 1, 1\rangle$, which points in the positive z-direction. Then $\displaystyle\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_R (0 \cdot 1 + 0 \cdot 1 - 1 \cdot 1)\; dA = \int_0^4 \int_0^{4-x} (-1)\; dy\, dx = -8.$

17.6.44 $z_x = -2$, $z_y = -5$, so the normal vector is $\langle 2, 5, 1\rangle$, which points in the positive z-direction.

$$\begin{aligned}\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_R (x \cdot 2 + y \cdot 5 + z \cdot 1)\; dA &= \int_0^2 \int_0^{(10-5y)/2} (2x + 5y + (10 - 2x - 5y))\; dx\, dy \\ &= \int_0^2 \int_0^{(10-5y)/2} 10\, dx\, dy = 50.\end{aligned}$$

17.6.45 We have $z = \sqrt{x^2+y^2}$; then $\mathbf{n} = \left\langle -\dfrac{x}{z}, -\dfrac{y}{z}, 1\right\rangle$, which points upwards.

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_R \left(\left(-\frac{x}{z}\right) \cdot x + \left(-\frac{y}{z}\right) \cdot y + 1 \cdot z\right) dA = \iint\limits_R \left(z - \frac{x^2+y^2}{z}\right) dA = \iint\limits_R (z - z)\; dA = 0.$$

17.6.46 $z_x = 0$, $z_y = -\sin(y)$, so an upward-pointing normal is $\langle 0, \sin y, 1\rangle$. We have $\displaystyle\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS =$

$$\iint\limits_R (2 \sin y \cos y + xy)\; dA = \int_0^4 \int_{-\pi}^{\pi} (2 \sin y\, \cos y + xy)\; dy\, dx = 0.$$

17.6.47 An outward-pointing normal is $\dfrac{\mathbf{r}}{|\mathbf{r}|}$. The sphere has radius a, so the vector field is in fact $\dfrac{\mathbf{r}}{|\mathbf{r}|^3}$.

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_S \frac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}\, dS = \iint\limits_S \frac{1}{|\mathbf{r}|^2} dS = \iint\limits_S \frac{1}{a^2} dS = \frac{1}{a^2} \iint\limits_S 1\, dS = \frac{1}{a^2} 4\pi\, a^2 = 4\pi.$$

17.6.48 The parametric form is $\langle u, u^2, v\rangle$ for $0 \le u \le 1$, $0 \le v \le 4$. We have $\mathbf{t}_u = \langle 1, 2u, 0\rangle$ and $\mathbf{t}_v = \langle 0, 0, 1\rangle$, so that $\mathbf{t}_u \times \mathbf{t}_v = \langle 2u, -1, 0\rangle$; since we want normal vectors to point in the positive y direction, we choose $\langle -2u, 1, 0\rangle$ for the normal vector. Then $\displaystyle\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_R \left\langle -u^2, u, 1\right\rangle \cdot \langle -2u, 1, 0\rangle\; dA =$

$$\iint\limits_R \left(u + 2u^3\right)\; dA = \int_0^4 \int_0^1 \left(u + 2u^3\right)\; du\, dv = 4.$$

17.6.49

a. True. The formula in Theorem 17.12 gives $\displaystyle\iint_S f(x,y,z)\,dS = \iint_R f(x,y,10)\sqrt{0+0+1}\,dA$.

b. False. The formula in Theorem 17.12 gives

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,x)\sqrt{1+0+1}\,dA = \sqrt{2}\iint_R f(x,y,x)\,dA.$$

c. True. Substituting $2u$ for u and $\sqrt{v}$ for v in the first parameterization gives $\langle\sqrt{v}\cos 2u, \sqrt{v}\sin 2u, v\rangle$, $0 \le 2u \le \pi$, $0 \le \sqrt{v} \le 2$. Simplifying the bounds conditions gives the second parameterization.

d. True. The standard parameterization is $\langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$ for $0 \le u \le \pi$, $0 \le v \le 2\pi$. Then $\mathbf{t}_u \times \mathbf{t}_v = \left\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\right\rangle$, and it is easily seen that these are outward-pointing vectors, by considering various ranges for u and v.

17.6.50 $\nabla \ln|\mathbf{r}| = \nabla \ln\sqrt{x^2+y^2+z^2} = \dfrac{1}{|\mathbf{r}|^2}\langle x,y,z\rangle = \dfrac{1}{a^2}\langle x,y,z\rangle$ on the sphere of radius a; using the explicit description for the sphere, we have $\mathbf{n} = \left\langle \dfrac{x}{z}, \dfrac{y}{z}, 1\right\rangle$, so that

$$\iint_S \nabla\ln|\mathbf{r}|\cdot\mathbf{n}\,dS = \iint_R \frac{1}{a^2}\langle x,y,z\rangle\cdot\left\langle\frac{x}{z},\frac{y}{z},1\right\rangle dA = \frac{1}{a^2}\iint_R\left(\frac{x^2+y^2}{z}+z\right)dA =$$

$$\frac{1}{a^2}\iint_R\left(\frac{x^2+y^2+z^2}{z}\right)dA = \frac{1}{a^2}\iint_R\frac{a^2}{z}\,dA = \iint_R\frac{1}{z}\,dA = \int_0^{2\pi}\int_0^a\frac{1}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta = 2\pi a.$$

17.6.51 Parameterize the surface by $\langle 2\cos u, 2\sin u, v\rangle$ for $0 \le u \le 2\pi$, $0 \le v \le 8$. Then the normal vector has magnitude 2, and

$$\iint_S |\mathbf{r}|\,dS = \iint_R \sqrt{x^2+y^2+z^2}\,dA = 2\int_0^{2\pi}\int_0^8\sqrt{4+v^2}\,dv\,du = 8\pi\left(4\sqrt{17}+\ln\left(4+\sqrt{17}\right)\right).$$

17.6.52 $z_x = 0, z_y = -1$, so

$$\iint_S xyz\,dS = \iint_R xyz\sqrt{2}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^2 r\cdot r\cos\theta\cdot r\sin\theta\cdot(6-r\sin\theta)\,dr\,d\theta =$$

$$\sqrt{2}\int_0^{2\pi}\int_0^2 r^3\cos\theta\,\sin\theta\,(6-r\sin\theta)\,dr\,d\theta = 0.$$

17.6.53 The normal vector is $\langle x,0,z\rangle$, so

$$\iint_S \frac{1}{\sqrt{x^2+z^2}}\langle x,0,z\rangle\cdot\langle x,0,z\rangle\,dS = \iint_R\sqrt{x^2+z^2}\,dA = \int_{-2}^2\int_0^{2\pi} a\,dA = 8\pi a.$$

17.6.54 The two curves intersect where $x^2+y^2 = 16-x^2-y^2$, so on the plane $z = 2\sqrt{2}$. The projection on the xy-plane of the circle of intersection is $x^2+y^2=8$. The total surface area of the sphere, of radius 4, is 64π. Finally, the outward normals to the sphere are $\left\langle\dfrac{x}{z},\dfrac{y}{z},1\right\rangle$.

a. From part (b) below and the fact that the total surface area is 64π, we get an answer of $64\pi - 32\pi + 16\pi\sqrt{2} = 16\pi\left(2+\sqrt{2}\right)$.

b.

$$\iint_S 1\,dS = \iint_R \sqrt{\frac{x^2}{z^2}+\frac{y^2}{z^2}+1}\,dA = \iint_R \frac{4}{z}\,dA = \iint_R \frac{4}{\sqrt{16-x^2-y^2}}\,dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{8}} \frac{4r}{\sqrt{16-r^2}}\,dr\,d\theta = 16\pi\left(2-\sqrt{2}\right).$$

c. For the cone $z^2 = x^2+y^2$, we have $\sqrt{z_x^2+z_y^2+1} = \sqrt{2}$, so

$$\iint_S 1\,dS = \iint_R \sqrt{2}\,dA = \sqrt{2}\cdot\pi\cdot\left(\sqrt{8}\right)^2 = 8\pi\sqrt{2}.$$

17.6.55 The surface of the cylinder inside the sphere is defined parametrically by $\langle 1+\cos u, \sin u, v\rangle$ where $0 \le u \le 2\pi$ and also (because the z-coordinate must stay inside the sphere of radius 2),

$$0 \le v \le \sqrt{4-(1+\cos u)^2-\sin^2 u},$$

or

$$0 \le v \le \sqrt{2-2\cos u}.$$

The normal is $\langle \cos u, \sin u, 0\rangle$, which has magnitude 1, so we have

$$\iint_S 1\,dS = \iint_R 1\,dA = \int_0^{2\pi}\int_0^{\sqrt{2-2\cos u}} 1\,dv\,du = \int_0^{2\pi}\sqrt{2-2\cos u}\,du = 8.$$

17.6.56 We have $z = -\frac{c}{a}x + \left(-\frac{c}{b}\right)y + c$, so that $z_x = -\frac{c}{a}$, $z_y = -\frac{c}{b}$, and an upward-pointing normal is $\left\langle \frac{c}{a}, \frac{c}{b}, 1\right\rangle$. Then $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \left(\frac{c}{a}x+\frac{c}{b}y+z\right)dA = \iint_R \left(\frac{c}{a}x+\frac{c}{b}y+\left(-\frac{c}{a}x+\left(-\frac{c}{b}\right)y+c\right)\right)dA = \iint_R c\,dA$, which is c times the area of A. Recall that if the vector field is vertical, then the flux is equal to the area of the base. As c increases, the slope of the plane gets closer to vertical, so that the x and y components of the vector field $\langle x,y,z\rangle$ contribute more to the flux; also, the values of z get larger. Thus the flux increases as c does.

17.6.57

a. Using the standard parameterization, $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \left(\frac{-x^2}{z}+\frac{-y^2}{z}+z\right)dA = \iint_R 0\,dA = 0$, because $x^2+y^2=z^2$, so that the flux is zero. This is due to the fact that the field $\mathbf{F}$ is aligned with the cone at all points on the cone.

b. $2z\,dz = \left(\frac{2x}{a^2}\right)dx$ so that $z_x = \frac{x}{a^2z}$. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \left(\frac{-x^2}{a^2z}+\frac{-y^2}{a^2z}+z\right)dA = \iint_R \left(\frac{-a^2z^2}{a^2z}+z\right)dA = 0,$$

so the flux is again zero. This is because the flow is a radial flow, so is always tangent to this surface.

17.6.58 Parameterize the cone by $\mathbf{r}(u,v) = \left\langle \frac{a}{h}v\cos u, \frac{a}{h}v\sin u, v\right\rangle$for $0 \le v \le h$, $0 \le u \le 2\pi$; then $\mathbf{t}_u \times \mathbf{t}_v = \left\langle -\frac{a}{h}v\sin u, \frac{a}{h}v\cos u, 0\right\rangle \times \left\langle \frac{a}{h}\cos u, \frac{a}{h}\sin u, 1\right\rangle$ and $|\mathbf{t}_u \times \mathbf{t}_v| = \frac{a}{h^2}v\sqrt{h^2+a^2}$. Then

$$\begin{aligned}\iint_S 1\,dS &= \frac{a\sqrt{h^2+a^2}}{h^2}\iint_R v\,dA = \frac{a\sqrt{h^2+a^2}}{h^2}\int_0^{2\pi}\int_0^h v\,dv\,du\\ &= \frac{a\sqrt{h^2+a^2}}{h^2}\int_0^{2\pi}\frac{1}{2}h^2\,du = \frac{2\pi a\sqrt{h^2+a^2}}{2} = \pi a\sqrt{h^2+a^2}.\end{aligned}$$

17.6.59 Because the cap has height h, the circle at the boundary of the cap has radius $\sqrt{a^2-(a-h)^2} = \sqrt{2ah-h^2}$, so that the equation of the base of the region is $x^2+y^2 = 2ah-h^2$. The outward normals to the sphere are $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$. Thus $\iint_S 1\,dS = \iint_R \sqrt{\frac{x^2}{z^2}+\frac{y^2}{z^2}+1}\,dA = \iint_R \sqrt{\frac{a^2-z^2}{z^2}+1}\,dA = \iint_R \frac{a}{z}\,dA = \iint_R \frac{a}{\sqrt{a^2-x^2-y^2}}\,dA = \int_0^{2\pi}\int_0^{\sqrt{2ah-h^2}} \frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta = 2\pi ah$. (See also problem 54(b)).

17.6.60 Using a parametric description, we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_R \frac{1}{(x^2+y^2+z^2)^{p/2}}\langle x,y,z\rangle\cdot(\mathbf{t}_u\times\mathbf{t}_v)\,dA\\ &= \frac{1}{a^p}\iint_R \langle a\sin u\cos v, a\sin u\sin v, a\,\cos u\rangle\cdot\left\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\right\rangle\,dA\\ &= \frac{1}{a^p}\iint_R \left(a^3\sin^3 u\,\cos^2 v + a^3\sin^3 u\sin^2 v + a^3\cos^2 u\sin u\right)\,dA\\ &= a^{3-p}\int_0^{\pi}\int_0^{2\pi}\left(\sin^3 u + \cos^2 u\sin u\right)\,dv\,du\\ &= a^{3-p}\int_0^{\pi}\int_0^{2\pi}\sin u\,dv\,du = \frac{4\pi}{a^{p-3}}.\end{aligned}$$

Using an explicit description, compute the flux on the upper half hemisphere and double it. There, for $z \ge 0$, we have $z_x = -\frac{x}{z}$, $z_y = -\frac{y}{z}$, so that

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_R \frac{1}{(x^2+y^2+z^2)^{p/2}}\left(\frac{x^2}{z}+\frac{y^2}{z}+z\right)\,dA\\ &= \frac{1}{a^p}\iint_R \frac{a^2}{z}\,dA = a^{2-p}\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = \frac{2\pi}{a^{p-3}}.\end{aligned}$$

After doubling, we get the same answer.

17.6.61 $\mathbf{F} = -\boldsymbol{\nabla} T = -\langle T_x, T_y, T_z\rangle = \langle 100e^{-x-y}, 100e^{-x-y}, 0\rangle$. Thus the flow is parallel to the two sides where $z = \pm 1$ so that the flux is zero there. We thus need only compute the flux on the remaining four sides. Parameterize the sides as

$$\begin{array}{ll} S_1 : \langle -1, y, z\rangle & \mathbf{t}_y \times \mathbf{t}_z = \langle 1,0,0\rangle\\ S_2 : \langle 1, y, z\rangle & \mathbf{t}_y \times \mathbf{t}_z = \langle 1,0,0\rangle\\ S_3 : \langle x, -1, z\rangle & \mathbf{t}_x \times \mathbf{t}_z = \langle 0,-1,0\rangle\\ S_4 : \langle x, 1, z\rangle & \mathbf{t}_x \times \mathbf{t}_z = \langle 0,-1,0\rangle \end{array}$$

for $-1 \le x, y, z \le 1$.

We are looking for the outward flux, so we must choose outward normals, which are (respectively) $\langle -1,0,0\rangle$, $\langle 1,0,0\rangle$,$\langle 0,-1,0\rangle$, and $\langle 0,1,0\rangle$. Then

$$\iint\limits_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS_1 = \iint\limits_R -100e^{-x-y}\,dA = -100\int_{-1}^{1}\int_{-1}^{1} e^{1-y}\,dz\,dy = -200e^2+200$$

$$\iint\limits_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS_2 = \iint\limits_R -100e^{-x-y}\,dA = 100\int_{-1}^{1}\int_{-1}^{1} e^{-1-y}\,dz\,dy = -200e^{-2}+200$$

$$\iint\limits_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS_3 = \iint\limits_R -100e^{-x-y}\,dA = -100\int_{-1}^{1}\int_{-1}^{1} e^{-x+1}dz\,dx = -200e^2+200$$

$$\iint\limits_{S_4} \mathbf{F}\cdot\mathbf{n}\,dS_4 = \iint\limits_R -100e^{-x-y}\,dA = 100\int_{-1}^{1}\int_{-1}^{1} e^{-x-1}dz\,dx = -200e^{-2}+200$$

so that the total flux is $-400\left(e^2+e^{-2}-2\right) = -400\left(e-\frac{1}{e}\right)^2$.

17.6.62 $\mathbf{F} = -\boldsymbol{\nabla} T = -\langle T_x, T_y, T_z\rangle = \left\langle 200xe^{-x^2-y^2-z^2}, 200ye^{-x^2-y^2-z^2}, 200ze^{-x^2-y^2-z^2}\right\rangle$. Thus integrating over the top half of the sphere gives

$$\begin{aligned}\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint\limits_R \left\langle 200xe^{-x^2-y^2-z^2}, 200ye^{-x^2-y^2-z^2}, 200ze^{-x^2-y^2-z^2}\right\rangle\cdot\left\langle \frac{x}{z},\frac{y}{z},1\right\rangle dA\\ &= 200e^{-a^2}\iint\limits_R \left(\frac{x^2}{z}+\frac{y^2}{z}+z\right)dA = 200a^2e^{-a^2}\iint\limits_R \frac{1}{z}dA\\ &= 200a^2e^{-a^2}\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = 400\pi a^3e^{-a^2},\end{aligned}$$

and because the vector field is symmetric, the answer is $800\pi a^3e^{-a^2}$.

17.6.63 $\mathbf{F} = -\boldsymbol{\nabla} T = \dfrac{2}{x^2+y^2+z^2}\langle x,y,z\rangle$. Thus integrating on the top half of the sphere gives $\displaystyle\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_R \left(\frac{x^2}{z}+\frac{y^2}{z}+z\right)dA = 2\iint\limits_R \frac{1}{z}dA = 2\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = 4\pi a$, and because the vector field is symmetric, the answer is $2\cdot 4\pi a = 8\pi a$.

17.6.64

a. $\displaystyle\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_R \langle x,y,0\rangle\cdot\langle x,y,0\rangle\,dA = \iint\limits_R \left(x^2+y^2\right)dA = a^2\int_0^{2\pi}\int_0^a r\,dr\,d\theta = \pi a^4.$

b. On the cylinder, this field is $\left(x^2+y^2\right)^{-p/2} = a^{-p}$ times as large as the field in part (a), so the flux is $4\pi La^{2-p}$.

c. As $a\to\infty$, this converges for $2-p\le 0$, or $p\ge 2$.

d. As $L\to\infty$, the flux never converges.

17.6.65

a. From problem 60, the outward flux across a sphere of radius b is $\dfrac{4\pi}{b^{p-3}}$, so the total flux across the concentric spheres when $p=0$ is $4\pi b^3 - 4\pi a^3 = 4\pi\left(b^3-a^3\right)$.

b. For $p = 3$, the flux across the sphere of radius b is 4π, so the net flux is zero across S.

17.6.66 By symmetry, $\overline{x} = \overline{y} = 0$. Since the shell has constant density, we assume the density is 1; then its mass is $2\pi a^2$, and $M_{xy} = \iint\limits_S z\,dS = \iint\limits_R z\sqrt{\left(\frac{-x}{z}\right)^2 + \left(\frac{-y}{z}\right)^2 + 1}\,dA = \iint\limits_R z\cdot\frac{a}{z}\,dA = a\int_0^{2\pi}\int_0^a r\,dr\,d\theta =$ $\pi\,a^3$ so that $\overline{z} = \frac{a}{2}$.

17.6.67 The cone is rotationally symmetric around the z axis, so $\overline{x} = \overline{y} = 0$. Parameterize the cone by $\left\langle \frac{r}{h}v\cos u, \frac{r}{h}v\sin u, v\right\rangle$. Then from problem 23, the surface area of the cone is $\pi r\sqrt{h^2+r^2}$, so its mass is $\rho\pi r\sqrt{h^2+r^2}$. Using the parameterization from that problem, $|\mathbf{t}_u \times \mathbf{t}_v| = \frac{r}{h^2}v\,\sqrt{h^2+r^2}$, so that $M_{xy} = \rho\iint\limits_S z\,dS = \rho\frac{r\sqrt{h^2+r^2}}{h^2}\iint\limits_R vz\,dA = \rho\frac{r\sqrt{h^2+r^2}}{h^2}\int_0^{2\pi}\int_0^h v^2\,dv\,du = \rho\frac{r\sqrt{h^2+r^2}}{h^2}\int_0^{2\pi}\frac{1}{3}h^3\,du =$ $\rho\frac{2\pi rh\sqrt{h^2+r^2}}{3}$ so that $\overline{z} = \frac{M_{xy}}{m} = \rho\frac{2\pi rh\sqrt{h^2+r^2}}{3}\cdot\frac{1}{\rho\pi r\sqrt{h^2+r^2}} = \frac{2h}{3}$.

17.6.68 Assume the shell has density 1. Then the mass of the shell, which is half the area of the entire cylinder, is πah. Further, by symmetry, $\overline{x} = \overline{y} = 0$. Use the parameterization $\langle a\cos u, v, a\sin u\rangle$; then $|\mathbf{t}_u \times \mathbf{t}_v| = a$ and $M_{xy} = \iint\limits_S z\,dS = a\iint\limits_R z\,dA = a\int_{-h/2}^{h/2}\int_0^{\pi} a\sin u\,du\,dv = 2a^2h$ and $\overline{z} = \frac{2}{\pi}a$.

17.6.69 Using the standard parameterization, the mass of the shell is

$$m = \iint\limits_S (1+z)\ dS = a\iint\limits_R (1+z)\ dA = a\int_0^{2\pi}\int_0^2 (1+v)\ dv\,du = 8\pi a.$$

The density does not depend on either x or y, and the cylinder is symmetric about the z axis, so $\overline{x} = \overline{y} = 0$. Then

$$M_{xy} = \iint\limits_S z\,(1+z)\ dS = a\iint\limits_R z\,(1+z)\ dA = a\int_0^{2\pi}\int_0^2 v\ (1+v)\ dv\,du = \frac{28\pi a}{3}.$$

Then $\overline{z} = \frac{7}{6}$.

17.6.70 $\mathbf{t}_u = \langle a\cos u\cos v, a\cos u\sin v, -a\sin u\rangle$ and $\mathbf{t}_v = \langle -a\sin u\sin v, a\sin u\cos v, 0\rangle$, and then $\mathbf{t}_u \times \mathbf{t}_v =$ $\left\langle a^2\sin^2 u\cos v, a^2\sin^2\ u\sin v, a^2\cos u\sin u\right\rangle$ so that $|\mathbf{t}_u \times \mathbf{t}_v| = a^2\left|\left\langle \sin^2 u\cos v, \sin^2 u\sin v, \cos u\sin u\right\rangle\right| =$ $a^2\sqrt{\sin^4 u\cos^2 v + \sin^4 u\sin^2 v + \cos^2 u\sin^2 u} = a^2\sqrt{\sin^4 u + \sin^2 u\cos^2 u} = a^2\sin u$.

17.6.71 The explicit formula $z = g\,(x, y)$ becomes, on regarding x and y as parameters, the parametric form $\langle x, y, g\,(x, y)\rangle$, and now $\mathbf{t}_x = \langle 1, 0, z_x\rangle$ and $\mathbf{t}_y = \langle 0, 1, z_y\rangle$. Then $\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x, -z_y, 1\rangle$, so that $|\mathbf{t}_x \times \mathbf{t}_y| = \sqrt{z_x^2 + z_y^2 + 1}$. Now the formula $\iint\limits_S f\,(x, y, z)\ dS = \iint\limits_R f\,(x, y, g\,(x, y))\sqrt{z_x^2 + z_y^2 + 1}\,dA$ follows from the definition of the surface integral for parameterized surfaces.

17.6.72

a. Each point on the graph of f on $[a, b]$, say $f\,(x)$ becomes, in the surface of revolution, a circle of radius $f\,(x)$ with center on the x-axis. Letting $u = x$, that circle is then parameterized by $\langle f\,(u)\cos v, f\,(u)\sin v\rangle$ for $0 \le v \le 2\pi$, so the entire surface is parameterized by
$\langle u, f\,(u)\cos v, f\,(u)\sin v\rangle$, $a \le u \le b$, $0 \le v \le 2\pi$.

b. $\mathbf{t}_u = \langle 1, f'(u)\cos v, f'(u)\sin v\rangle$ and $\mathbf{t}_v = \langle 0, -f(u)\sin v, f(u)\cos v\rangle$, and then

$$\mathbf{t}_u \times \mathbf{t}_v = \langle f'(u)\, f(u), -f(u)\cos v, -f(u)\sin v\rangle$$

and

$$|\mathbf{t}_u \times \mathbf{t}_v| = \sqrt{f'(u)^2\, f(u)^2 + f(u)^2\left(\sin^2 v + \cos^2 v\right)} = f(u)\sqrt{f'(u)^2 + 1}.$$

We have

$$\iint\limits_S f(x,y,z)\, dS = \iint\limits_R f(u)\sqrt{f'(u)^2+1}\, dA = \int_a^b \int_0^{2\pi} f(u)\sqrt{f'(u)^2+1}\, dv\, du$$
$$= 2\pi \int_a^b f(u)\sqrt{f'(u)^2+1}\, du.$$

c. The area of the surface is $2\pi \int_1^2 x^3\sqrt{9x^4+1}\, dx = \frac{\pi}{27}\left(145^{3/2} - 10^{3/2}\right)$.

d. The area of the surface is

$$2\pi \int_3^4 \left(25-x^2\right)^{1/2}\sqrt{\left(-x\left(25-x^2\right)^{-1/2}\right)^2 + 1}dx = 2\pi \int_3^4 \left(25-x^2\right)^{1/2}\sqrt{\frac{x^2}{25-x^2}+1}\, dx$$
$$= 2\pi \int_3^4 \left(25-x^2\right)^{1/2}\sqrt{\frac{25}{25-x^2}}\, dx = 2\pi\int_3^4 5\, dx = 10\pi.$$

17.6.73 We have $z = s(x,y)$, so a normal vector is $\langle -z_x, -z_y, 1\rangle$. Since we are interested in the downward flux, we choose a downward-pointing normal, which is $\langle s_x(x,y), s_y(x,y), -1\rangle$. Then $\iint\limits_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint\limits_R \langle 0,0,-1\rangle \cdot \langle \mathrm{s}_x(x,y), s_y(x,y), -1\rangle\, dA = \iint\limits_R 1\, dA$, which is the area of R. Since the vector field is constant and pointed downwards vertically, everything that goes through the surface is matched by something going through R.

17.6.74

a. Imagine the torus as built by starting with a circle of radius R in the xy plane, centered at the origin. From each point on this circle, parameterize as $\langle R\cos v, R\sin v, 0\rangle$, we can reach a circle of points on the surface by making it the center of a circle of radius r, parameterized by u. This second circle is drawn in a vertical plane that includes the z-axis. Each point of this second circle is thus in the plane determined by $\langle R\cos v, R\sin v, 0\rangle$ and the z-axis; its z-coordinate will be $r\sin u$ and its x and y-coordinates will then be (from its center) $r\cos v\cos u$ and $r\sin v\cos u$. Thus the set of points on the torus can be parameterized by the sum of these vectors, which is $\langle R\cos v, R\sin v, 0\rangle + \langle r\cos v\cos u, r\sin v\cos u, r\sin u\rangle = \langle (R+r\cos u)\cos v, (R+r\cos u)\sin v, r\sin u\rangle$

b. From the parameterization above, we have

$$\begin{aligned}\mathbf{t}_u &= \langle -r\sin u\cos v, -r\sin u\sin v, r\cos u\rangle\\ \mathbf{t}_v &= \langle -(R+r\cos u)\sin v, (R+r\cos u)\cos v, 0\rangle\end{aligned}$$

so that

$$\mathbf{t}_u \times \mathbf{t}_v = (R+r\cos u)\langle -r\cos u\cos v, -r\cos u\sin v, -r\sin u\rangle$$

and $|\mathbf{t}_u\times\mathbf{t}_v| = (R+r\cos u)\sqrt{r^2\cos^2 u\cos^2 v + r^2\cos^2 u\sin^2 v + r^2\sin^2 u} = r(R+r\cos u)$, so that the area of the torus is $\iint\limits_S 1\, dS = \int_0^{2\pi}\int_0^{2\pi} r(R+r\cos u)\, du\, dv = 4\pi^2 Rr$.

17.6.75 The goal is to start with the surface area formula $A = \iint\limits_R \sqrt{z_x^2 + z_y^2 + 1}\, dA$ of Theorem 17.12 (where we have set $f(x, y, g(x, y)) = 1$) and derive the surface area formula $A = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx$ of section 6.6. Because S is generated by revolving the graph of f about the x-axis, we can use symmetry and take $R = \{(x, y) : a \le x \le b,\ 0 \le y \le f(x)\}$. The resulting surface area is then one-quarter of the desired surface area. The key observation is that the surface generated is given by $z^2 = f(x)^2 - y^2$ over the region R. It follows that $2zz_x = 2ff'$, or $z_x = \frac{ff'}{2}$ and $2zz_y = -2y$, or $z_y = \frac{-y}{z}$. Substituting, we have that the surface area is $A = \iint\limits_R \sqrt{z_x^2 + z_y^2 + 1}\, dA = 4\int_a^b \int_0^{f(x)} \sqrt{\left(\frac{ff'}{z}\right)^2 + \left(\frac{-y}{z}\right)^2 + 1}\, dy\, dx =$ $4\int_a^b \int_0^{f(x)} \sqrt{\frac{f^2f'2 + y^2 + z^2}{z^2}}\, dy\, dx = 4\int_a^b \int_0^{f(x)} \sqrt{\frac{f^2f'^2 + f^2}{f^2 - y^2}}\, dy\, dx$, because $z^2 = f^2 - y^2$.

Continuing, we have

$$\begin{aligned} A &= 4\int_a^b \int_0^{f(x)} \frac{f(x)\sqrt{1 + f'(x)^2}}{\sqrt{f(x)^2 - y}}\, dy\, dx = 4\int_a^b f(x)\sqrt{1 + f'(x)^2} \int_0^{f(x)} \frac{1}{\sqrt{f(x)^2 - y}}\, dy\, dx \\ &= 4\int_a^b f(x)\sqrt{1 + f'(x)^2} \left(\sin^{-1} \frac{y}{f(x)}\Bigg|_0^{f(x)} \right) dx = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx. \end{aligned}$$

17.7 Stokes' Theorem

17.7.1 It measures the circulation of the vector field **F** along the closed curve C.

17.7.2 It measures the accumulated rotation of the vector field **F** over the surface S.

17.7.3 It says that the circulation of a vector field along a closed curve is equal to the net circulation of the field over a surface whose boundary is that curve, so that either can be calculated from the other.

17.7.4 This is the fundamental theorem of line integrals - the integral of any conservative vector field around a closed curve is zero.

17.7.5 The line integral is $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_R \langle \sin t, -\cos t, 10\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dA = \int_0^{2\pi} (-1)\ dt = -2\pi$. For the surface integral, use the standard parameterization of the sphere; then

$$\mathbf{n} = \langle \sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u\rangle$$

and $\boldsymbol{\nabla} \times \mathbf{F} = \langle 0, 0, -2\rangle$ so that $\iint\limits_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^{\pi/2} (-2\cos u\ \sin u)\ du\, dv = -2\pi$.

17.7.6 The line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_R \langle 0, -2\cos t, 2\sin t\rangle \cdot \langle -2\ \sin t, 2\cos t, 0\rangle\, dA = \int_0^{2\pi} \left(-4\ \cos^2 t\right)\ dt = -4\pi.$$

$\boldsymbol{\nabla} \times \mathbf{F} = \langle 1, 0, -1\rangle$. The outward normal to the sphere is $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$, so the surface integral is

$$\iint\limits_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint\limits_R \langle 1, 0, -1\rangle \cdot \left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle\, dA = \int_0^{2\pi} \int_0^2 \left(\frac{r^2\cos\theta}{\sqrt{4 - r^2}} - r\right) dr\, d\theta = -4\pi.$$

17.7.7 The line integral is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 2\sqrt{2}\cos t, 2\sqrt{2}\sin t, 0\rangle \cdot \left\langle -2\sqrt{2}\sin t, 2\sqrt{2}\cos t, 0\right\rangle\, dt = \int_0^{2\pi} 0\, dt = 0.$$

For the surface integral, we have $\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{0}$ so that the surface integral is also zero.

17.7.8 The boundary of the region is the intersection of the sphere with the plane $z = 12$, which has the equation $x^2 + y^2 = 25$ and $z = 12$. Thus the line integral is $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 24, -4\cdot 5\cos t, 3\cdot 5\sin t\rangle \cdot \langle -5\sin t, 5\,\cos t, 0\rangle\, dt = \int_0^{2\pi} \left(-120\sin t - 100\cos^2 t\right)\, dt = -100\pi$.

The surface sits over $x^2 + y^2 = 25$ and the normal to the sphere is $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$. $\boldsymbol{\nabla}\times\mathbf{F} = \langle 3, 2, -4\rangle$, so the surface integral is $\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS = \int_0^{2\pi}\int_0^5 r\left(\frac{3r\cos\theta}{\sqrt{169 - r^2}} + \frac{2r\sin\theta}{\sqrt{169 - r^2}} - 4\right)\, dr\, d\theta = -100\pi.$

17.7.9 The boundary of the region is the intersection of the sphere with the plane $z = \sqrt{7}$, which has the equation $x^2 + y^2 = 9$ and $z = \sqrt{7}$. Then the line integral is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 3\sin t - \sqrt{7}, \sqrt{7} - 3\cos t, 3\cos t - 3\,\sin t\rangle \cdot \langle -3\sin t, 3\cos t, 0\rangle\, dt$$

$$= \int_0^{2\pi} \left(-9\sin^2 t + 3\sqrt{7}\sin t + 3\sqrt{7}\cos t - 9\cos^2 t\right)\, dt = \int_0^{2\pi} \left(-9 + 3\sqrt{7}\sin t + 3\sqrt{7}\,\cos t\right)\, dt = -18\pi.$$

The surface sits over $x^2 + y^2 = 9$ and the normal to the sphere is $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$. $\boldsymbol{\nabla}\times\mathbf{F} = \langle -2, -2, -2\rangle$, so the surface integral is

$$\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS = -2\int_0^{2\pi}\int_0^3 r\left(\frac{r\cos\theta}{\sqrt{16 - r^2}} + \frac{r\sin\theta}{\sqrt{16 - r^2}} + 1\right)\, dr\, d\theta = -18\pi.$$

17.7.10 The boundary of the region can be parameterized by $\langle 4\cos t, 4\sin t, 6 - 4\sin t\rangle$, so the line integral is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle -4\sin\ \mathrm{t}, 4\sin t - 4\cos t - 6, 4\sin t - 4\cos t\rangle \cdot \langle -4\sin t, 4\cos t, -4\,\cos t\rangle\, dt$$

$$= \int_0^{2\pi} \left(16\sin^2 t + 16\sin t\,\cos t - 16\cos^2 t - 24\cos t - 16\sin t\,\cos t + 16\cos^2 t\right)\, dt$$

$$= \int_0^{2\pi} \left(16\sin^2 t - 24\cos t\right)\, dt = 16\pi.$$

$\boldsymbol{\nabla}\times\mathbf{F} = \langle 2, 1, 0\rangle$, and an outward pointing normal to the plane is $\langle 0, 1, 1\rangle$, so the surface integral is $\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint_R 1\, dA = \int_0^{2\pi}\int_0^4 r\, dr\, d\theta = 16\pi.$

17.7.11 $\boldsymbol{\nabla}\times\mathbf{F} = \langle 1, -1, -2\rangle$; for S take the disk $x^2 + y^2 \le 12$ with upward-oriented normal vector $\langle 0, 0, 1\rangle$. Then $\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint_R (-2)\, dA = -24\pi.$

17.7.12 $\boldsymbol{\nabla}\times\mathbf{F} = \langle -1 - x, 0, z - 1\rangle$; for S take the region $x^2 + \frac{y^2}{4} \le 1$ in the plane $z = 1$, with normal vector $\langle 0, 0, 1\rangle$; then $\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint_R (z - 1)\, dA = 0$ because $z = 1$ on the region of integration.

17.7.13 $\boldsymbol{\nabla}\times\mathbf{F} = \langle 0,-4z,0\rangle$. For S take the plane in the first octant, which sits over $0\le x\le 4$, $0\le y\le 4-x$. The upward-pointing normal to this plane is $\langle 1,1,1\rangle$. Then $\iint\limits_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\,dS = \iint\limits_R (-4z)\;dA = \int_0^4\int_0^{4-x} (-4)\,(4-x-y)\;dy\,dx = -\dfrac{128}{3}$.

17.7.14 $\boldsymbol{\nabla}\times\mathbf{F} = \langle 2y-2z,0,2y-2x\rangle$. Take S to be the square bounded by C, with normal $\langle 0,0,1\rangle$. Then $\iint\limits_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\,dS = \iint\limits_R (2y-2x)\;dA = \int_{-1}^1\int_{-1}^1 (2y-2x)\;dy\,dx = 0$.

17.7.15 $\boldsymbol{\nabla}\times\mathbf{F} = \langle 2z,-1,-2y\rangle$. Take S to be the disk $\langle 3r\cos t, 4r\cos t, 5r\sin t\rangle$ for $0\le r\le 1$, $0\le t\le 2\pi$. $\mathbf{t}_r\times\,\mathbf{t}_t = \langle 20r,-15r,0\rangle$ is a normal vector. Then $\iint\limits_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\,dS = \iint\limits_R \langle 2z,-1,-2y\rangle\cdot\langle 20r,-15r,0\rangle\,dA = \int_0^{2\pi}\int_0^1 ((10r\ \sin t)\cdot 20r + 15r)\;dr\,dt = 15\pi$.

17.7.16 $\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{0}$, so the surface integral is zero.

17.7.17 The boundary of the surface is the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$, found by setting $z=0$. Parameterize the path by $\mathbf{r}(t) = \langle 2\cos t, 3\sin t, 0\rangle$; then $\oint \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}\cdot\mathbf{r}'\,(t)\;dt = \int_0^{2\pi}(-4\cos t\sin t + 9\cos t\ \sin t)\;dt = \int_0^{2\pi} 5\cos t\sin t\,dt = 0$.

17.7.18 The boundary of the surface is on the plane $x=0$, and it is the circle $y^2+z^2=9$. Parameterize the circle by $\mathbf{r}(t) = \langle 0, 3\cos t, 3\sin t\rangle$; then $\mathbf{r}'(t) = \langle 0,-3\sin t, 3\cos t\rangle$. $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{1}{\sqrt{x^2+y^2+z^2}}\langle x,y,z\rangle$, and $\oint \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}\cdot\mathbf{r}'\,(t)\;dt = \dfrac{1}{3}\int_0^{2\pi}(-9\sin t\cos t + 9\ \sin t\cos t)\;dt = 0$.

17.7.19 The boundary of the surface is the intersection of the plane $x=3$ with the sphere $x^2+y^2+z^2=25$, so is the circle $y^2+z^2=16$ at $x=3$. Parametrize the circle with $x=3$, $y=4\cos t$ and $z=4\sin t$. We have $\mathbf{r}'(t) = \langle 0,-4\sin t, 4\cos t\rangle$, so $\oint \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle 8\cos t, -4\sin t, 3-4\cos t-4\sin t\rangle\cdot\langle 0,-4\sin t,4\cos t\rangle\,dt = \int_0^{2\pi}\left(16\sin^2 t + 12\cos t - 16\cos^2 t - 16\sin t\cos t\right)\;dt = 0$.

17.7.20 The boundary of the surface is given in the problem: $\mathbf{r}(t) = \langle\cos t, 2\sin t, \sqrt{3}\cos t.\rangle$; so $\mathbf{r}'\,(t) = \langle -\sin t, 2\cos\ t, -\sqrt{3}\sin t\rangle$ and the integral is

$$\begin{aligned}\oint \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}\langle\cos t + 2\sin t, 2\ \sin t + \sqrt{3}\cos t, \left(1+\sqrt{3}\right)\cos t\rangle\cdot\langle -\sin t, 2\cos t, -\sqrt{3}\ \sin t\rangle\,dt\\ &= \int_0^{2\pi}\left(-\sin t\cos t - 2\ \sin^2 t + 4\sin t\cos t + 2\sqrt{3}\cos^2 t - \left(3+\sqrt{3}\right)\cos t\sin t\right)\;dt\\ &= \int_0^{2\pi}\left(-\sqrt{3}\sin t\cos t - 2\ \sin^2 t + 2\sqrt{3}\cos^2 t\right)\;dt = 2\pi\left(\sqrt{3}-1\right).\end{aligned}$$

17.7.21 C is given by $\mathbf{r}(t) = \left\langle\cos t, \sin t, \cos^2 t\right\rangle$. Then

$$\begin{aligned}\mathbf{F}\cdot\mathbf{r}'(t) &= \left\langle\sin t, \cos^2 t - \cos t, -\sin t\right\rangle\cdot\langle -\sin t, \cos t, -2\sin t\cos t\rangle\\ &= -\sin^2 t + \cos^3 t - \cos^2 t + 2\sin^2 t\cos t = -1 + \cos^3 t + 2\sin^2 t\cos t.\end{aligned}$$

We have $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-1 + \cos^3 t + 2\sin^2 t \cos t)\, dt = -2\pi + \int_0^{2\pi} \cos t(2\sin^2 t + (1 - \sin^2 t))\, dt = -2\pi + \int_0^{2\pi} \cos t(\sin^2 t + 1)\, dt$. Let $u = \sin t$ so that $du = \cos t\, dt$. Then we have

$$-2\pi + \int_0^0 (u^2 + 1)\, du = -2\pi.$$

17.7.22 C is given by $\mathbf{r}(t) = \left\langle \frac{1}{2}\cos t, \frac{1}{2}\sin t, \frac{1}{2}\cos^2 t + \frac{1}{4}\sin^2 t \right\rangle = \left\langle \frac{1}{2}\cos t, \frac{1}{2}\sin t, \frac{1}{4}\cos^2 t + \frac{1}{4} \right\rangle$. Then $\mathbf{F} \cdot \mathbf{r}'(t) = \left\langle 2\cos t, -2\cos^2 t - 2, 2\sin t \right\rangle \cdot \left\langle -\frac{1}{2}\sin t, \frac{1}{2}\cos t, -\frac{1}{2}\cos t \sin t \right\rangle = -\sin t \cos t - 2\cos t$. We have

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (-\sin t \cos t - 2\cos t)\ dt \\ &= -\int_0^{2\pi} \cos t\,(\sin t + 2)\ dt.\end{aligned}$$

A substitution of $u = \sin t + 2$ reveals that the value of this integral is 0.

17.7.23 C is given by $\mathbf{r}(t) = \left\langle 2 + 2\cos t, 2\sin t, 2\sqrt{2 + 2\cos t} \right\rangle$. Then

$$\begin{aligned}\mathbf{F} \cdot \mathbf{r}'(t) &= \left\langle 2\sin t, 1, 2\sqrt{2 + 2\cos t} \right\rangle \cdot . \left\langle -2\sin t, 2\cos t, \frac{-2\sin t}{\sqrt{2 + 2\cos t}} \right\rangle \\ &= -4\sin^2 t + 2\cos t - 4\sin t.\end{aligned}$$

We have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-4\sin^2 t + 2\cos t - 4\sin t)\, dt = \int_0^{2\pi} (-4\sin^2 t)\, dt = -4\pi.$$

17.7.24 C is given by $\mathbf{r}(t) = \left\langle 2\cos t, \sin t, 4 - 3\sin^2 t \right\rangle$. Then

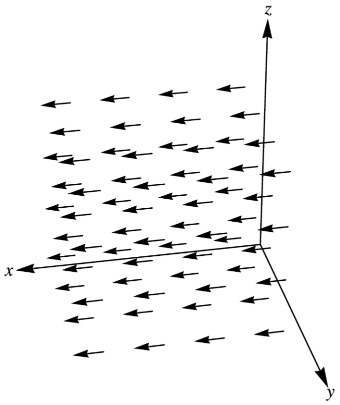
$$\begin{aligned}\mathbf{F} \cdot \mathbf{r}'(t) &= \left\langle e^{2\cos t}, \frac{1}{4 - 3\sin^2 t}, \sin t \right\rangle \cdot \langle -2\sin t, \cos t, -6\sin t \cos t \rangle \\ &= -2\sin t e^{2\cos t} + \frac{\cos t}{4 - 3\sin^2 t} - 6\sin^2 t \cos t.\end{aligned}$$

We have

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left(-2(\sin t)e^{2\cos t} + \frac{\cos t}{4 - 3\sin^2 t} - 6\sin^2 t \cos t \right) dt \\ &= \int_0^{2\pi} -2\sin t e^{2\cos t}\, dt + \int_0^{2\pi} \frac{\cos t}{4 - 3\sin^2 t}\, dt - 6\int_0^{2\pi} \sin^2 t \cos t\, dt = 0 + 0 - 0 = 0,\end{aligned}$$

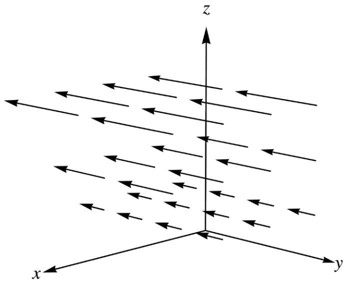
where the integrals are all evaluated either via the substitution $u = \sin t$ or $u = \cos t$.

17.7.25 $\boldsymbol{\nabla} \times \mathbf{v} = \langle 1, 0, 0 \rangle$. The curl looks like:



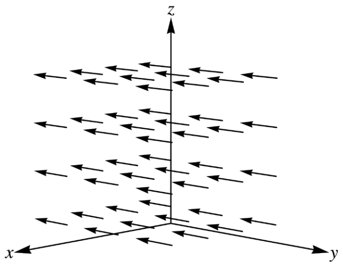
This means that the maximum rotation of the field is in the direction$\langle 1, 0, 0\rangle$. The rotation is counterclockwise looking in the negative x direction.

17.7.26 $\nabla \times \mathbf{v} = \langle 0, -2z, 0\rangle$. The curl looks like:



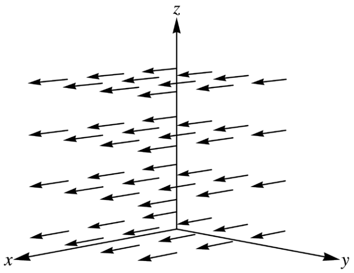
This means that the maximum rotation of the field is in the direction of the$y$-axis, and the amount of rotation, clockwise or counterclockwise, increases the further from the z-axis one gets. It rotates counterclockwise (viewed from the positive z-axis) for $z < 0$, and clockwise for $z > 0$.

17.7.27 $\nabla \times \mathbf{v} = \langle 0, -2, 0\rangle$. The curl looks like:



This means that the maximum rotation of the field is in the direction of the y-axis. It is constant at all points, and is clockwise viewed from the positive y-axis.

17.7.28 $\nabla \times \mathbf{v} = \langle 2, 0, 0\rangle$. The curl looks like:



The maximum rotation of the vector field is in the direction of the x-axis; it is constant at all points, and is counterclockwise viewed from the positive x-axis.

17.7.29

a. False. This is a rotation field with axis of rotation $\langle 1, 1, 2\rangle$, but $\langle 0, 1, -1\rangle \cdot \langle 1, 1, 2\rangle \neq 0$, so the paddle wheel axis is not perpendicular to the axis of rotation.

b. False. It relates the curl of $\mathbf{F}$, not its flux.

c. True. This is because it is conservative: it is the gradient of $ax + F(x) + by + G(y) + cz + H(z)$, where F, G, H are the antiderivatives of f, g, h, respectively.

d. True. See Theorem 14.14.

17.7.30 This is a conservative vector field, with $\varphi = x^2 - y^2 + z^2$, so the integral $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C.

17.7.31 This is a conservative vector field, so the integral around any closed curve is zero.

17.7.32 This is a conservative vector field with $\varphi = x^3y + y^2z^2$, so the integral around any closed curve is zero.

17.7.33 This is a conservative vector field with $\varphi = xy^2z^3$, so the integral around any closed curve is zero.

17.7.34 The surface S is $\langle r\cos\varphi\cos t, r\sin t, r\sin\varphi\cos t\rangle$; computing the normal vector gives $\mathbf{t}_r \times \ \mathbf{t}_t = \langle -r\sin\varphi, 0, r\cos\varphi\rangle$. Thus the surface area is $\iint\limits_S 1\, dS = \int_0^1 \int_0^{2\pi} |\mathbf{t}_r \times \mathbf{t}_t|\, dt\, dr = \int_0^1 \int_0^{2\pi} r\, dt\, dr = \pi$. This makes sense because the surface is simply the unit circle inclined at the angle φ to the xy-plane.

17.7.35 $\mathbf{r}'(t) = \langle -\cos\varphi\sin t, \cos t, -\sin\varphi\sin t\rangle$, so that $|\mathbf{r}'(t)| = 1$. Then the length of C is $\int_C 1\, ds = \int_0^{2\pi} 1\, dt = 2\pi$, again as expected because C is just an inclined unit circle.

17.7.36 By Stokes' theorem, $\oint \mathbf{F} \cdot d\mathbf{r} = \iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, $\nabla \times \mathbf{F} = \langle 0, 0, 2\rangle$; using the normal $\mathbf{t}_r \times \mathbf{t}_t = \langle -r\sin\varphi, 0, r\cos\varphi\rangle$ we have $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_0^1 \int_0^{2\pi} 2r\cos\varphi\, dt\, dr = 2\pi\cos\varphi$. This is maximal for $\varphi = 0$, when it is 2π.

17.7.37 We have

$$\begin{aligned}\oint \mathbf{F} \cdot d\mathbf{r} &= \int_C \langle -y, -z, x\rangle \cdot \langle -\cos\varphi\sin t, \cos t, -\sin\varphi\, \sin t\rangle\, dt \\ &= \int_0^{2\pi} \left(\cos\varphi\sin^2 t - \sin\varphi\cos^2 t + \cos\varphi\sin\varphi\, \cos t\sin t\right)\, dt = \pi\left(\cos\varphi - \sin\varphi\right).\end{aligned}$$

This is maximal for $\varphi = 0$, when it is π.

17.7.38 $\nabla \times (\mathbf{a} \times \mathbf{r}) = \langle 2a_1, 2a_2, 2a_3\rangle$. Thus

$$\begin{aligned}\oint \mathbf{F} \cdot d\mathbf{r} &= \iint\limits_S \langle 2a_1, 2a_2, 2a_3\rangle \cdot \langle -r\sin\varphi, 0, r\cos\varphi\rangle\, dS \\ &= 2\int_0^{2\pi} \int_0^1 \left(-a_1 r\sin\varphi + a_3 r\cos\varphi\right)\, dr\, dt = 2\pi\left(a_3\cos\varphi - a_1\sin\varphi\right).\end{aligned}$$

This is a maximum when its derivative vanishes, i.e. when $a_3\cos\varphi - a_1\sin\varphi = 0$. Now, $\langle a_1, a_2, a_3\rangle$ points in the direction of the normal if their cross-product is zero, i.e. if $\langle a_1, a_2, a_3\rangle \times \langle -r\sin\varphi, 0, r\cos\varphi\rangle = \langle a_2\, r\cos\varphi, -a_3\, r\sin\varphi - a_1\, r\cos\varphi, a_2, r\sin\varphi\rangle = 0$. This happens when $a_2 = 0$ and $a_3\sin\varphi + a_1\cos\varphi = 0$.

17.7.39 $\nabla \times \mathbf{F} = \langle 3, 0, 0\rangle$. To evaluate the circulation around C, we instead (using Stokes' theorem) evaluate $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ for the surface of the disk of which C is a boundary. Note that $\mathbf{n} = \langle 1, 1, 1\rangle$, so that $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint\limits_R 3\, dA = 3 \cdot \text{area of } A = 48\pi$. From this calculation, it is clear that the result depended on the radius of the circle, because that affects the area of A, but not on the center of the circle.

17.7.40

a. $\nabla \times \mathbf{F} = \langle 1,1,0 \rangle$, so the integrand of the surface integral is just $\dfrac{x+y}{z}$; by symmetry, the integral vanishes on each level curve, so it vanishes altogether.

b. On the boundary of S, we have $z = 0$, so that $\mathbf{F} = \langle 0,0,2y+x \rangle$, and thus $\mathbf{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$ so that the dot product and thus the line integral is zero.

17.7.41

a. The normal vectors point toward the z-axis on the curved surface of S and in the direction of $\langle 0,1,0 \rangle$ on the flat surface of S.

b. To evaluate the integral, we must add up the integrals on each of the surfaces. $\nabla \times \mathbf{F} = \langle 1,1,1 \rangle$. Let S_1 be the surface in the xz-plane, parameterized by $\langle x,0,z \rangle$ for $-2 \le x \le 2$,$x^2 \le z \le 4$; then the normal to S_1 is $\mathbf{t}_x \times \mathbf{t}_z = \langle 0,1,0 \rangle$, so that the integral over S_1 is

$$\iint\limits_{S_1} \langle 1,1,1 \rangle \cdot \langle 0,-1,0 \rangle \, dS = \int_{-2}^{2} \int_{x^2}^{4} (-1) \, dy \, dx = -\int_{-2}^{2} \left(4 - x^2\right) \, dx = -\frac{32}{3}.$$

S_2 is the half of the paraboloid for $y \ge 0$, parameterized as $\langle r\cos u, r\sin u, r^2 \rangle$, $0 \le r \le 2, -\pi \le u \le 0$. The normal to S_2 is $\mathbf{t}_r \times \mathbf{t}_u = \langle -2r^2\cos u, -2r^2 \sin u, r \rangle$. The integral over S_2 is

$$\iint\limits_{S_1} \langle 1,1,1 \rangle \cdot \langle -2r^2\cos u, -2r^2 \sin u, r \rangle \, dS = \int_0^2 \int_{-\pi}^{0} \left(-2r^2 (\cos u + \sin u) + r\right) \, du \, dr = \frac{32}{3} + 2\pi.$$

Thus the total is 2π.

c. The line integral is the sum of two line integrals:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}_2$$

where $C_1 = \langle t,0,4 \rangle$ for $-2 \le t \le 2$ and$C_2 = \langle 2\cos t, 2\sin t, 4 \rangle$ for $-\pi \le t \le 0$. Then

$$\begin{aligned}
\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 &= \int_{-2}^{2} \langle 2z+y, 2x+z, 2y+x \rangle \cdot \langle 1,0,0 \rangle \, dt = \int_{2}^{-2} 8 \, dt = -32 \\
\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 &= \int_{-\pi}^{0} \langle 2z+y, 2x+z, 2y+x \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt \\
&= \int_{-\pi}^{0} \left(-16\sin t - 4\sin^2 t + 8\cos t + 8\cos^2 t\right) \, dt = 32 + 2\pi,
\end{aligned}$$

so the total line integral is 2π.

17.7.42 We have, from Stokes' theorem and Ampere's Law, $\iint\limits_S (\nabla \times \mathbf{B}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{B} \cdot d\mathbf{r} = \mu I = \mu \iint\limits_S \mathbf{J} \cdot \mathbf{n} \, dS$ Thus we have

$$\iint\limits_S ((\nabla \times \mathbf{B}) - \mu \mathbf{J}) \cdot \mathbf{n} \, dS = 0$$

for all surfaces S bounded by any given closed curve C. It is clear that given the freedom to choose C and S, that it follows that the integrand is identically zero, i.e. that for any surface S,$((\nabla \times \mathbf{B}) - \mu \mathbf{J}) \cdot$n$= 0$. From this it is easy to see that we must have $\nabla \times \mathbf{B} = \mu \mathbf{J}$, since we are free to make the normal vector point in any direction at any given point by choosing S appropriately.

17.7.43 The boundary of the region is the circle C: $x^2+y^2=1$ for$\mathbf{r}z=0$. With the usual parameterization, we have

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \langle \cos t-\sin t, \sin t, -\cos t\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt\\ &= \int_0^{2\pi} \left(-\sin t\cos t+\sin^2 t+\sin t\cos t\right)\, dt = \int_0^{2\pi} \sin^2 t\, dt = \pi.\end{aligned}$$

So the integral is independent of a.

17.7.44

a. $\nabla\times\mathbf{F} = \left\langle \frac{\partial}{\partial y}(ay) - \frac{\partial}{\partial z}(cx), \frac{\partial}{\partial z}(bz) - \frac{\partial}{\partial x}(ay), \frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(bz)\right\rangle = \langle a,b,c\rangle$.

b. The area of R is $\iint\limits_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint\limits_S \mathbf{n}\cdot\mathbf{n}\, dS = \iint\limits_R |\mathbf{n}|^2\, dA =$ area of R because $|\mathbf{n}|=1$, so that $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS =$ area of R.

c. $\mathbf{r}'(t) = \langle 5\cos t, -13\sin t, 12\cos t\rangle$, so that $\mathbf{r}(t)\times\mathbf{r}'(t) = \langle 156, 0, -65\rangle$ and thus, because $\mathbf{r}(t)\times\mathbf{r}'(t)$ is constant, it points in a constant direction, so that $\mathbf{r}$ must lie in a plane.

d. By parts (b) and (c), we have a normal vector $\langle 156, 0, -65\rangle$; its magnitude is 169, so we take $\mathbf{F} = \left\langle 0, -\frac{5x}{13}, \frac{12y}{13}\right\rangle$; then the area of R is

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \left\langle 0, -\frac{25}{13}\sin t, 12\cos t\right\rangle \cdot \langle 5\cos t, -13\sin t, 12\cos t\rangle\, dt\\ &= \int_0^{2\pi} \left(25\sin^2 t + 144\cos^2 t\right)\, dt = 169\pi.\end{aligned}$$

17.7.45

a. The boundary of this surface is the circle $x^2+y^2=1$ at $z=0$, so we choose instead the surface of the disk bounded by that circle. $\nabla\times\mathbf{F} = \langle 2x, 0, -2z\rangle$, which is $\langle 2x, 0, 0\rangle$ at $z=0$, and the normal to the disk is $\langle 0,0,1\rangle$. Thus, the integral is equal to $\iint\limits_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint\limits_S 0\, dS = 0$.

b. With the usual parameterization of the boundary circle (and remembering that $z=0$), we have $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 0,0,1\rangle\cdot\langle -\sin t, \cos t, 0\rangle\, dt = 0$.

17.7.46

a. Let C be the circle $x^2+y^2=a^2$. Parameterize the circle in the usual way; then $\mathbf{F} = \frac{1}{a^p}\langle x,y,0\rangle$ and $\mathbf{r}(t) = \langle a\cos t, a\sin t, 0\rangle$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{a^p}\int_0^{2\pi} \left(-a^2\sin t\cos t + a^2\sin t\cos t\right)\, dt = 0$.

b. Stokes' Theorem will apply when the vector field is defined throughout the disk of radius a, which happens only when $p\le 0$. In that case, $\nabla\times\mathbf{F} = a^{-p}\langle 0,0,0\rangle$, so that the surface integral is zero.

17.7.47

a. $\nabla\times\mathbf{F} = \left\langle 0-0, 0-0, \frac{y^2-x^2}{\left(x^2+y^2\right)^2} - \frac{y^2-x^2}{\left(x^2+y^2\right)^2}\right\rangle = \mathbf{0}$.

b. Let C be the unit circle with the usual parameterization; then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle -\sin t, \cos t, 0\rangle \cdot \langle -\sin t, \cos t, 0\rangle \, dt = 2\pi$.

c. The theorem does not apply because the vector field is not defined at the origin, which is inside the curve C. For example, the limit of the y-coordinate is different depending on the direction.

17.7.48

a. The circumference of the disk is $2\pi R$, so the average circulation is $\frac{1}{2\pi R}\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\,dS$

b. As R becomes small, because $\mathbf{F}$ and thus $\boldsymbol{\nabla}\times\mathbf{F}$ are continuous, $\boldsymbol{\nabla}\times\mathbf{F}$ can be made arbitrarily close to$(\boldsymbol{\nabla}\times\mathbf{F})_P$ everywhere on S by taking R small enough. Approximately, then, $(\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n} \approx (\boldsymbol{\nabla}\times\mathbf{F})_P \cdot$n, so that

$$\frac{1}{2\pi R}\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\,dS \approx \frac{1}{2\pi R}\iint_S (\boldsymbol{\nabla}\times\mathbf{F})_P\cdot\mathbf{n}\,dS = (\boldsymbol{\nabla}\times\mathbf{F})_P\cdot\mathbf{n}\frac{1}{2\pi R}\iint_S 1\,dS = (\boldsymbol{\nabla}\times\mathbf{F})_P\cdot\mathbf{n}.$$

As R becomes smaller, the goodness of the approximation of $\boldsymbol{\nabla}\times\mathbf{F}$ becomes better, so the value of the integral does as well.

17.7.49 By the chain rule, $\frac{df}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}$ and similarly for g, h, so

$$\begin{aligned} M_y &= \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + hz_{xy} + z_x\left(\frac{\partial h}{\partial y} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial y}\right) = f_y + f_z z_y + hz_{xy} + z_x\left(h_y + h_z z_y\right) \\ N_x &= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + hz_{yx} + z_y\left(\frac{\partial h}{\partial x} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial x}\right) = g_x + g_z z_x + hz_{yx} + z_y\left(h_x + h_z z_x\right). \end{aligned}$$

17.7.50 One argument is quite simple: a closed surface has a closed (empty!) boundary, so the integral of $\mathbf{F}$ over that boundary is zero. Alternatively, choose any closed curve C dividing the surface into two pieces. On one half, the outward-pointing normals give a counterclockwise orientation to the boundary (viewed from above); on the other half, they give a clockwise orientation. Thus the integral $\iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\,dS = \oint_C \mathbf{F}\cdot d\mathbf{r} + \oint_{-C}\mathbf{F}\cdot d\mathbf{r}$, which is zero.

17.7.51 Let $\mathbf{F} = \langle 0, g(y,z), h(y,z)\rangle$ be a vector field in the yz-plane; for a region R in that plane, with boundary C, the normal is $\langle 1,0,0\rangle$. Now, $\boldsymbol{\nabla}\times\mathbf{F} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, 0, 0\right\rangle$, so by Stokes' theorem, $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dA$.

17.8 Divergence Theorem

17.8.1 The surface integral measures the flow across the boundary.

17.8.2 The volume integral measures the net expansion or contraction of the vector field in the region.

17.8.3 The Divergence Theorem says that the flow across the boundary equals the net expansion or contraction of the field within the solid, so that either can be computed from the other.

17.8.4 Since $\nabla\cdot\langle 2z + y, -x, -2x\rangle = 0$, the net flux is zero.

17.8.5 Since $\nabla \cdot \langle x, y, z\rangle = 3$, the Divergence theorem says that the net outward flux is equal to

$$\iiint\limits_D 3\, dV = 3 \cdot \text{volume of}\, S = 32\pi.$$

17.8.6 From Example 4 (or Exercise 71 in section 14.5), the divergence is zero.

17.8.7 The outward fluxes must be equal, since by the Divergence theorem the net flux, which is the difference of the two, is zero.

17.8.8 Outward, since it is equal to the integral of div $\mathbf{F}$ over the cube.

17.8.9 For the volume integral, $\boldsymbol{\nabla} \cdot \mathbf{F} = 9$, so that $\displaystyle\iiint\limits_D \boldsymbol{\nabla} \cdot \mathbf{F}\, dV = \iiint\limits_D 9\, dV = 9 \cdot \frac{4}{3} 2^3 \pi = 96\pi.$

For the surface integral, with the usual parameterization of the sphere,

$$\begin{aligned} \mathbf{F} \cdot \mathrm{n} &= \langle 2a \sin u \cos v, 3a \sin u \sin v, 4a\, \cos u\rangle \cdot \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u\rangle \\ &= 2a^3 \sin^3 u \cos^2 v + 3a^3 \sin^3 u\, \sin^2 v + 4a^3 \cos^2 u \sin u \end{aligned}$$

and here $a = 2$, so that

$$\begin{aligned} \iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS &= 8 \iint\limits_R \left(2 \sin^3 u \cos^2 v + 3\, \sin^3 u \sin^2 v + 4 \cos^2 u \sin u\right)\, dA \\ &= 8 \int_0^{2\pi} \int_0^{\pi} \left(2 \sin^3 u\, \cos^2 v + 3 \sin^3 u \sin^2 v + 4 \cos^2 u \sin u\right)\, du\, dv = 96\pi. \end{aligned}$$

17.8.10 For the volume integral, $\boldsymbol{\nabla} \cdot \mathbf{F} = -3$, so that $\displaystyle\iiint\limits_D (-3)\; dV = -3 \cdot \text{volume of} D = -24$

For the surface integral, we have six surfaces:

$$\begin{array}{ll} S_1 : x = -1 & \mathbf{n} = \langle -1, 0, 0\rangle \\ S_2 : x = 1 & \mathbf{n} = \langle 1, 0, 0\rangle \\ S_3 : y = -1 & \mathbf{n} = \langle 0, -1, 0\rangle \\ S_4 : y = 1 & \mathbf{n} = \langle 0, 1, 0\rangle \\ S_5 : z = -1 & \mathbf{n} = \langle 0, 0, -1\rangle \\ S_6 : z = 1 & \mathbf{n} = \langle 0, 0, 1\rangle \end{array}$$

and on each of those surfaces a simple computation shows that we have $\mathbf{F} \cdot \mathbf{n} = -1$. Thus

$$\sum_{i=1}^{6} \iiint\limits_{S_i} \mathbf{F} \cdot \mathbf{n}\, dS_i = \sum_{i=1}^{6} \left(-1 \cdot \text{area of}\, S_i\right) = -24.$$

17.8.11 For the volume integral, $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$, so the volume integral is zero. For the surface integral, the boundary ellipsoid can be parameterized by $\langle 2 \sin u \cos v, 2\sqrt{2} \sin u \sin v, 2\sqrt{3} \cos u\rangle$, and $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v = \langle 4\sqrt{6} \sin^2 u \cos v, 4\sqrt{3} \sin^2 u \sin v, 4\sqrt{2} \sin u\, \cos u\rangle$ so that

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} = {}& \langle 2\sqrt{3} \cos u - 2\sqrt{2} \sin u \sin v, 2 \sin u \cos v, -2 \sin u \cos v\rangle \cdot \\ & \langle 4\sqrt{6} \sin^2 u \cos v, 4\sqrt{3} \sin^2 u \sin v, 4\sqrt{2} \sin u \cos u\rangle = -8 \sin^2 u \cos v \left(-2\sqrt{2} \cos u + \sqrt{3} \sin u \sin v\right) \end{aligned}$$

and then $\displaystyle\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^{\pi} \left(-8 \sin^2 u \cos v\, \left(-2\sqrt{2}\, \cos u + \sqrt{3} \sin u \sin v\right)\right)\, du\, dv = 0.$

17.8.12 For the volume integral, $\nabla \cdot \mathbf{F} = 2\,(x+y+z)$, so that

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = 2\int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} (x+y+z)\ dz\, dy\, dx = 0.$$

For the surface integral, we have six surfaces:

$S_1 : x = -1$ $\mathbf{n} = \langle -1, 0, 0\rangle$
$S_2 : x = 1$ $\mathbf{n} = \langle 1, 0, 0\rangle$
$S_3 : y = -2$ $\mathbf{n} = \langle 0, -1, 0\rangle$
$S_4 : y = 2$ $\mathbf{n} = \langle 0, 1, 0\rangle$
$S_5 : z = -3$ $\mathbf{n} = \langle 0, 0, -1\rangle$
$S_6 : z = 3$ $\mathbf{n} = \langle 0, 0, 1\rangle$

A short computation shows that for S_1: $\mathbf{F}\cdot\mathbf{n} = -1$, for S_2: $\mathbf{F}\cdot\mathbf{n} = 1$, for $S_{3:}$ $\mathbf{F}\cdot\mathbf{n} = -4$, for S_4: $\mathbf{F}\cdot\mathbf{n} = 4$, for S_5: $\mathbf{F}\cdot\mathbf{n} = -9$, and for S_6: $\mathbf{F}\cdot\mathbf{n} = 9$. Thus, the surface integral is zero.

17.8.13 $\nabla \cdot \mathbf{F} = 0$, so by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla\cdot$F is zero.

17.8.14 $\nabla \cdot \mathbf{F} = 0$, so by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla\cdot$F is zero.

17.8.15 $\nabla \cdot \mathbf{F} = 0$, so by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla\cdot$F is zero.

17.8.16 If $\mathbf{a} = \langle a, b, c\rangle$, then $\mathbf{a}\times\mathbf{r}$ is the field $\mathbf{F}$ in Exercise 15, so the net outward flux is zero.

17.8.17 By the Divergence theorem, we can compute the integral of $\nabla\cdot$F over the ball of radius $\sqrt{6}\cdot \nabla\cdot\mathbf{F} = 2$, so the volume integral is

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = 2 \cdot \text{volume of sphere of radius } \sqrt{6} = 2\cdot\frac{4}{3}\pi\cdot 6\sqrt{6} = 16\pi\sqrt{6}.$$

17.8.18 $\nabla \cdot \mathbf{F} = 2x$, so by the Divergence theorem, the outward flux is

$$\iiint_D 2x\, dV = \int_0^1\int_0^1\int_0^1 2x\, dx\, dy\, dz = 1.$$

17.8.19 $\nabla \cdot \mathbf{F} = 4$, so by the Divergence theorem, the outward flux is 4 times the volume of the tetrahedron, which is (by the formula for the volume of a pyramid), $\frac{1}{3}$ times the area of the base times the height, or $\frac{1}{6}$. So the outward flux is $\frac{2}{3}$.

17.8.20 $\nabla \cdot \mathbf{F} = 2\,(x+y+z)$, so by the Divergence theorem, the outward flux is

$$\iiint_D 2\,(x+y+z)\ dV = 2\int_0^5\int_0^{2\pi}\int_0^{\pi} r\,(5\sin u\cos v + 5\sin u\ \sin v + 5\cos u)\ du\, dv\, dr = 0.$$

17.8.21 $\nabla \cdot \mathbf{F} = -4$, so by the Divergence theorem, the outward flux is -4 times the volume of the sphere, so is $-\dfrac{128}{3}\pi$.

17.8.22 $\nabla \cdot \mathbf{F} = 0$, so the outward flux is zero by the Divergence theorem.

17.8.23 $\nabla \cdot \mathbf{F} = 3$, so the outward flux is 3 times the volume of the paraboloid, which is

$$\int_0^2 \int_0^{2\pi} r\left(4 - r^2\right)\, d\theta\, dr = 8\pi,$$

so the outward flux is 24π.

17.8.24 $\nabla \cdot \mathbf{F} = 3$, so the outward flux is 3 times the volume of the cone. The area of the base of the cone is 16π, so the outward flux is $3 \cdot \frac{1}{3} \cdot 16\pi \cdot 4 = 64\pi$.

17.8.25 $\nabla \cdot \mathbf{F} = -3$, so the outward flux across the boundary of D is the outward flux across the sphere of radius 4 minus that across the sphere of radius 3, which is $-3 \cdot \frac{4}{3}\pi\left(4^3 - 2^3\right) = -224\pi$.

17.8.26 $\nabla \cdot \mathbf{F} = 4\,|\mathbf{r}|$, so the outward flux across a sphere of radius r is

$$\iiint\limits_D 4\sqrt{x^2+y^2+z^2}\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} 4\rho\, \rho^2 \sin u\, d\rho\, du\, dv = 4\pi r^4.$$

Thus the net outward flux across the boundary of the given region is 60π.

17.8.27 $\nabla \cdot \mathbf{F} = \frac{2}{|\mathbf{r}|}$, so the outward flux across a sphere of radius r is

$$\iiint\limits_D \frac{2}{\sqrt{x^2+y^2+z^2}}\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \frac{2}{\rho}\, \rho^2 \sin u\, d\rho\, du\, dv = 4\pi r^2.$$

Thus the net outward flux across the boundary of the given region is 12π.

17.8.28 $\nabla \cdot \mathbf{F} = 0$, so the net outward flux is zero.

17.8.29 $\nabla \cdot \mathbf{F} = 2\,(x - y + z)$. The net outward flux is thus the difference in the outward flux across the two planes, so is

$$\begin{aligned}
&\iiint\limits_D 2\,(x-y+z)\; dV \\
&= 2\left(\int_0^4 \int_0^{4-x} \int_0^{4-x-y} (x-y+z)\; dz\, dy\, dx - \int_0^2 \int_0^{2-x} \int_0^{2-x-y} (x-y+z)\; dz\, dy\, dx\right) \\
&= 20.
\end{aligned}$$

17.8.30 $\nabla \cdot \mathbf{F} = 6$, so the net outward flux is 6 times the difference in the volumes of the cylinders, so is $6 \cdot (4\pi - \pi) \cdot 8 = 144\pi$.

17.8.31

a. False. For example, $\mathbf{F} = \langle y, 0, 0\rangle$ has $\nabla \cdot \mathbf{F} = 0$ at all points of the unit sphere, but the normal to the unit sphere, $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$, is not perpendicular to $\mathbf{F}$ at all points.

b. False. For example, any rotation field has $\nabla \cdot \mathbf{F} = 0$, so that $\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = 0$ by the Divergence theorem, but $\mathbf{F}$ is not in general constant.

c. True. This is because it is bounded by $\iiint\limits_D 1\, dV$.

17.8.32 $\nabla \cdot \mathbf{F} = 3$; we compute the outward flux from the Divergence theorem as (where S_1 is the upper hemisphere of radius a.)

$$\iiint_{S_1} \nabla \cdot \mathbf{F}\, dV = 3 \iiint_{S_1} 1\, dV = 3 \int_0^{2\pi} \int_0^{\pi/2} \int_0^a r^2 \sin u\, dr\, du\, dv = 2\pi a^3.$$

The the outward flux over the whole sphere is thus $4\pi a^3$.

17.8.33 Because $\nabla \cdot \mathbf{F} = 0$, the outward flux is zero from the Divergence theorem.

17.8.34 Because $\nabla \cdot \mathbf{F} = 0$, the outward flux is zero from the Divergence theorem.

17.8.35 $\nabla \cdot \mathbf{F} = 3 \sin y$, so the outward flux is

$$\iiint_S 3 \sin y\, dV = \int_0^{\pi/2} \int_0^1 \int_0^x 3 \sin y\, dz\, dx\, dy = \frac{3}{2}.$$

17.8.36

a. We have $\mathbf{F} \cdot \mathbf{n} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p} \cdot \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{|\mathbf{r}|^2}{|\mathbf{r}|^{p+1}} = |\mathbf{r}|^{1-p}$. Thus the surface integral is $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = a^{1-p} \cdot$ area of sphere $= a^{1-p} \cdot 4\pi a^2 = 4\pi a^{3-p}$.

b. The conditions of the Divergence Theorem require that $\mathbf{F}$ be defined and have continuous partials everywhere inside the sphere; in particular, this must hold at the origin. Thus we must have $p \le -2$. Then the volume integral is

$$\iiint_S \frac{3-p}{|\mathbf{r}|^p}\, dV = (3-p) \int_0^a \int_0^{2\pi} \int_0^\pi r^{-p} \cdot r^2 \sin u\, du\, dv\, dr = 4\pi a^{3-p}.$$

17.8.37

a. Either use Exercise 36(a), or compute $\mathbf{F} \cdot \mathbf{n} = \dfrac{\mathbf{r}}{|\mathbf{r}|} \cdot \dfrac{\mathbf{r}}{|\mathbf{r}|} = 1$, so the surface integral is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \text{area of sphere} = 4\pi a^2.$$

b. $\nabla \cdot \mathbf{F} = 2\,|\mathbf{r}|^{-1}$, so if D is the shell between the spheres of radius ϵ and a, the volume integral is

$$\iiint_S 2\,|\mathbf{r}|^{-1}\ dV = 2 \int_\epsilon^a \int_0^{2\pi} \int_0^\pi r \sin u\, du\, dv\, dr = 4\pi\left(a^2 - \epsilon^2\right)$$

and $\lim_{\epsilon \to 0} 4\pi\left(a^2 - \epsilon^2\right) = 4\pi a^2$.

17.8.38

a. $\dfrac{\partial}{\partial x}\varphi = \dfrac{x}{x^2 + y^2 + z^2}$, so that $\nabla\varphi = \dfrac{1}{|\mathbf{r}|^2}\langle x, y, z\rangle = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$.

b. $\mathbf{n} = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$, so that $\mathbf{F} \cdot \mathbf{n} = \dfrac{1}{|\mathbf{r}|^2}$. Then $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \frac{1}{a^2} \int_0^{2\pi} \int_0^\pi a^3 \sin u\, du\, dv = 4\pi a$.

c. By Exercise 36, $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{1}{|\mathbf{r}|^2}$.

d. If D is the shell between the spheres of radius ϵ and$a$, the volume integral is

$$\iiint\limits_D |\mathbf{r}|^{-2}\, dV = 2\int_\epsilon^a \int_0^{2\pi} \int_0^\pi \frac{1}{r^2}\, r\sin u\, du\, dv\, dr = 4\pi\,(a-\epsilon)$$

and $\lim\limits_{\epsilon\to 0} 4\pi\,(a-\epsilon) = 4\pi a$.

17.8.39

a. By Exercise 36, the flux of $\mathbf{E}$ across a sphere of radius a is $\dfrac{Q}{4\pi\epsilon_0}4\pi a^{3-3} = \dfrac{Q}{\epsilon_0}$.

b. The net outward flux across S is the difference of the fluxes across the inner and outer spheres; but by part (a), these are equal, so the net flux across S is zero.

c. The left-hand side is the flux across the boundary of D, while the right-hand side is the sum of the charge densities at each point of D. The statement says that the flux across the boundary, up to multiplication by a constant, is the sum of the charge densities in the region.

d. By the Divergence theorem, and using part (c),

$$\frac{1}{\epsilon_0}\iiint\limits_D q\,(x,y,z)\; dV = \iint\limits_S \mathrm{E}\cdot\mathbf{n}\, dS = \iiint\limits_D \boldsymbol{\nabla}\cdot\mathrm{E}dV$$

and because this holds for all regions D, we conclude that $\boldsymbol{\nabla}\cdot\mathbf{E} = \dfrac{q\,(x,y,z)}{\epsilon_0}$.

e. $\nabla^2\varphi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\varphi = \boldsymbol{\nabla}\cdot\mathbf{E} = \dfrac{q\,(x,y,z)}{\epsilon_0}$.

17.8.40

a. By Exercise 36, the flux of $\mathbf{F}$ across a sphere of radius a is $4\pi GMa^{3-p} = 4\pi GM$.

b. Since the outward flux across a sphere, from part (a), is independent of the radius of the sphere, the outward flux across the spheres of radii a and b are equal, so their difference, which is the net flux across the spherical shell bounded by them, is zero.

c. The left hand side is the flux across the boundary of D, while the right-hand side is the sum of the mass density inside D. The statement says that the flux across the boundary is determined by (is a constant multiple of) the sum of the mass density inside D.

d. By the Divergence theorem, and using part (c),

$$4\pi G\iiint\limits_D \rho\,(x,y,z)\; dV = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint\limits_D \boldsymbol{\nabla}\cdot\mathbf{F}dV$$

and because this holds over all regions D, we have $\boldsymbol{\nabla}\cdot\mathbf{F} = 4\pi G\rho\,(x,y,z)$.

e. $\boldsymbol{\nabla}^2\varphi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\varphi = \boldsymbol{\nabla}\cdot\mathbf{F} = 4\pi G\rho\,(x,y,z)$.

17.8.41 $\mathbf{F} = -\nabla\mathbf{T} = \langle -1,-2,-1\rangle$, so that $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$ and the heat flux is zero.

17.8.42 $\mathbf{F} = -\nabla\mathbf{T} = \langle -2x,-2y,-2z\rangle$, so that $\boldsymbol{\nabla}\cdot\mathbf{F} = -6$, and the heat flux is -6 times the volume of the region, or -6.

17.8.43 $\mathbf{F} = -\nabla\mathbf{T} = \langle 0,0,e^{-z}\rangle$; then $\boldsymbol{\nabla}\cdot\mathbf{F} = -e^{-z}$. The heat flux is then

$$\int_0^1\int_0^1\int_0^1 -e^{-z}\, dx\, dy\, dz = e^{-1}-1.$$

17.8.44 From Exercise 42, $\nabla\cdot\mathbf{F}=-6$, so the heat flux is -6 times the volume of the sphere, or -8π.

17.8.45 $\mathbf{F}=-\nabla\mathbf{T}=\langle 200xe^{-x^2-y^2-z^2},200ye^{-x^2-y^2-z^2},200ze^{-x^2-y^2-z^2}\rangle$. Then

$$\nabla\cdot\mathbf{F}=200e^{-x^2-y^2-z^2}\left(3-2x^2-2y^2-2z^2\right)$$

so that

$$\iiint\limits_D\nabla\cdot\mathbf{F}\,dV=200\int_0^{2\pi}\int_0^{\pi}\int_0^{a}e^{-r^2}\left(3-2r^2\right)r^2\sin u\,dr\,du\,dv=800\pi a^3e^{-a^2}.$$

17.8.46

a. By Exercise 36, the net flux across a sphere of radius a centered at the origin is $4\pi a^{3-p}$, which is independent of a only if $p=3$.

b. In the general case, we have $\nabla\cdot\mathbf{F}=\dfrac{3-p}{|\mathbf{r}|^p}$, so

$$\begin{aligned}\iiint\limits_D\nabla\cdot\mathbf{F}\,dV&=\int_a^b\int_{\varphi_1}^{\varphi_2}\int_{\theta_1}^{\theta_2}\frac{3-p}{r^p}r^2\sin u\,du\,dv\,dr\\&=\left(a^{3-p}-b^{3-p}\right)\left(\varphi_1\cos\theta_1-\varphi_2\cos\theta_1-\varphi_1\cos\theta_2+\varphi_2\cos\theta_2\right)\\&=\left(a^{3-p}-b^{3-p}\right)\left(\varphi_1-\varphi_2\right)\left(\cos\theta_1-\cos\theta_2\right),\end{aligned}$$

and these are in general zero only if $3-p=0$.

17.8.47

a. $\varphi_x(x,y,z)=G'(\rho)\,\rho_x=G'(\rho)\cdot\dfrac{x}{\sqrt{x^2+y^2+z^2}}=G'(\rho)\,\dfrac{x}{\rho}$, so that $\nabla\varphi=\mathbf{F}=G'(\rho)\,\dfrac{\mathbf{r}}{\rho}$.

b. The normal to the sphere of radius a is $\left\langle\frac{x}{z},\frac{y}{z},1\right\rangle$, so on that sphere (where $\rho=a$)

$$\mathbf{F}\cdot\mathbf{n}=G'(a)\,\frac{\frac{x^2}{z}+\frac{y^2}{z}+z}{a}=G'(a)\,\frac{\frac{a^2-z^2}{z}+z}{a}=G'(a)\,\frac{a}{z},$$

and then the surface integral over the upper hemisphere is

$$\iint\limits_S\mathbf{F}\cdot\mathbf{n}\,dS=aG'(a)\int_0^a\int_0^{2\pi}\frac{r}{\sqrt{a^2-r^2}}\,d\theta\,dr=2\pi a^2G'(a),$$

so the total surface integral is twice that, or $4\pi a^2G'(a)$.

c. By the Chain Rule,

$$\frac{\partial}{\partial x}G'(\rho)\,\frac{x}{\rho}=G''(\rho)\,\rho_x\frac{x}{\rho}+G'(\rho)\,\frac{y^2+z^2}{\rho^3}$$

so that (noting that $\rho_x=\frac{x}{\rho}$)

$$\nabla\cdot\mathbf{F}=G''(\rho)\left(\frac{x^2+y^2+z^2}{\rho^2}\right)+G'(\rho)\,\frac{2\left(x^2+y^2+z^2\right)}{\rho^3}=G''(\rho)+\frac{2\,G'(\rho)}{\rho}.$$

d. By the Divergence theorem, the flux is also given by

$$\begin{aligned}\iiint\limits_D\nabla\cdot\mathbf{F}\,dV&=\int_0^a\int_0^{\pi}\int_0^{2\pi}\rho^2\sin u\left(G''(\rho)+\frac{2\,G'(\rho)}{\rho}\right)dv\,du\,d\rho\\&=2\pi\int_0^a\int_0^{\pi}\sin u\left(\rho^2\,G''(\rho)+2\rho\,G'(\rho)\right)\,du\,d\rho\\&=2\pi\int_0^a\left(-\cos u\right)\left(\rho^2\,G''(\rho)+2\rho\,G'(\rho)\right)\Big|_{u=0}^{u=\pi}d\rho\\&=4\pi\int_0^a\left(\rho^2\,G''(\rho)+2\rho\,G'(\rho)\right)\,d\rho.\end{aligned}$$

It remains to evaluate this integral. Using integration by parts, we have

$$\begin{aligned} 4\pi\left[\int_0^a \left(\rho^2\, G''(\rho) + 2\rho\, G'(\rho)\right)\, d\rho\right] &= 4\pi\left[\rho^2\, G'(\rho)\left.\right|_{\rho=0}^{\rho=a} - \int_0^a 2\rho G'(\rho)\, d\rho\right] \\ &= 4\pi\left[a^2 G'(a) - \int_0^a 2\rho\, G'(\rho)\, d\rho\right] \end{aligned}$$

and the remaining integrals cancel, giving $4\pi a^2 G'(a)$ as the final result.

17.8.48

a. Rearrange the given equation and integrate over D to get

$$\iiint\limits_D u\,\boldsymbol{\nabla}\cdot\mathbf{F}\, dS = \iiint\limits_D \boldsymbol{\nabla}\cdot(u\,\mathbf{F})\, dS - \iiint\limits_D \mathbf{F}\cdot\boldsymbol{\nabla} u\, dS.$$

By the Divergence theorem, the first term on the right side is equal to $\iint\limits_S u\,\mathbf{F}\cdot\mathbf{n}\, dS$ where S is the boundary of D. The result follows.

b. If you set $\mathbf{F} = \langle f(x), 0, 0\rangle$ and$u(x,y,z) = g(x)$, then

$$\begin{array}{ll} \boldsymbol{\nabla}\cdot\mathbf{F} = f'(x) & u\boldsymbol{\nabla}\cdot\mathbf{F} = f'(x)\, g(x) \\ u\mathbf{F} = \langle f(x)\, g(x), 0, 0\rangle & \nabla\cdot(u\mathbf{F}) = (fg)'(x) \\ \nabla u = \langle g'(x), 0, 0\rangle & \mathbf{F}\cdot\boldsymbol{\nabla} u = f(x)\, g'(x) \end{array}$$

so that

$$\iiint\limits_D f'(x)\, g(x)\, dV = \iiint\limits_D (fg)'(x)\, dV - \iiint\limits_D f(x)\, g'(x)\, dV = f(x)\, g(x) - \iiint\limits_D f(x)\, g'(x)\, dV,$$

which is the usual rule for integration by parts.

c. One approach is to set $u = 1$ and $\mathbf{F} = \frac{1}{2}\langle x^2z^2, x^2y^2, y^2z^2\rangle$. Gauss' formula then gives

$$\iiint\limits_D \left(x^2y + y^2z + z^2x\right) = \frac{1}{2}\iint\limits_S \langle x^2z^2, x^2y^2, y^2z^2\rangle\, dS.$$

The integral on the right is computed by integrating over each face of the cube; on faces where x is constant, the normal is either $\langle 1,0,0\rangle$ or $\langle -1,0,0\rangle$ depending on whether x is 1 or 0; similarly for y and z. Considering x first, when $x = 0$, this surface integral becomes 0 on that face (the integrand is x^2z^2 at $x = 0$), while for $x = 1$ it becomes z^2. In that case, the integral is $\frac{1}{3}$. This holds for each dimension, so the total integral on the right side is $3\cdot\frac{1}{3} = 1$, and thus the integral on the left is $\frac{1}{2}$.

17.8.49 Suppose F= $\langle f, g\rangle$ where$f = f(x,y)$, $g = g(x,y)$, and suppose$u = u(x,y)$. Then $\boldsymbol{\nabla}\cdot\mathbf{F} = f_x + g_y$, and $\nabla u = \langle u_x, u_y\rangle$. Then we have for this case

$$\begin{aligned} &\iiint\limits_D u\,\boldsymbol{\nabla}\cdot\mathbf{F}\, dS = \iint\limits_R u\,(f_x + g_y)\, dA \\ &\iint\limits_S u\mathbf{F}\cdot\mathbf{n}\, dS = \oint_C u\mathbf{F}\cdot\mathbf{n}\, ds \\ &\iiint\limits_D \mathbf{F}\cdot\boldsymbol{\nabla} u\, dS = \iint\limits_R (f\, u_x + g\, u_y)\, dA, \end{aligned}$$

and the result follows. Setting $u = 1$ then gives $\iint\limits_R (f_x + g_y)\, dA = \oint_C \mathbf{F}\cdot\mathbf{n}\, ds$, which is the flux form of Green's Theorem.

17.8.50 Apply the Divergence Theorem to the vector field $\mathbf{F} = u\boldsymbol{\nabla} u$. By the product rule (Thm. 14.11), we have $\boldsymbol{\nabla}\cdot(v\boldsymbol{\nabla} v) = \boldsymbol{\nabla} u\cdot\boldsymbol{\nabla} v + u\boldsymbol{\nabla}^2 v$, so the Divergence theorem says that

$$\iiint\limits_D \left(u\boldsymbol{\nabla}^2 v + \boldsymbol{\nabla} u\cdot\boldsymbol{\nabla} v\right) dV = \iiint\limits_D (\boldsymbol{\nabla}\cdot(v\boldsymbol{\nabla} v))\, dV = \iint\limits_S u\boldsymbol{\nabla} v\cdot\mathbf{n}\, dS.$$

17.8.51 From Exercise 50, we have both

$$\iiint\limits_D \left(u\boldsymbol{\nabla}^2 v + \boldsymbol{\nabla} u\cdot\boldsymbol{\nabla} v\right) dV = \iint\limits_S u\boldsymbol{\nabla} v\cdot\mathbf{n}\, dS$$
$$\iiint\limits_D \left(v\boldsymbol{\nabla}^2 u + \boldsymbol{\nabla} v\cdot\boldsymbol{\nabla} u\right) dV = \iint\limits_S v\boldsymbol{\nabla} u\cdot\mathbf{n}\, dS,$$

where the second formula is obtained by switching u and v in the first formula. Subtracting the second from the first, and using the fact that the dot product is commutative and integrals are linear, we have the desired result:

$$\iiint\limits_D \left(u\boldsymbol{\nabla}^2 v - v\boldsymbol{\nabla}^2 u\right) dV = \iint\limits_S (u\boldsymbol{\nabla} v - v\boldsymbol{\nabla} u)\cdot\mathbf{n}\, dS.$$

17.8.52 A computation shows that $\boldsymbol{\nabla}\varphi = \frac{-p\mathbf{r}}{|\mathbf{r}|^{p+2}}$. Thus the potential field $\boldsymbol{\nabla}\varphi = 0$ for $p = 0$, so certainly $\boldsymbol{\nabla}^2\varphi = 0$ as well. Otherwise, for $p = 1$, we have $\boldsymbol{\nabla}\varphi = \frac{-\mathbf{r}}{|\mathbf{r}|^3}$; this is an inverse square field, and we have seen many times (e.g. Exercise 36(b)) that the divergence of such a field is zero. Thus $\boldsymbol{\nabla}^2\varphi = 0$ for $p = 1$ as well.

17.8.53 The Divergence theorem applied to the field $\boldsymbol{\nabla}\varphi$ says that

$$\iiint\limits_D \left(\boldsymbol{\nabla}^2\varphi\right) dV = \iint\limits_S \boldsymbol{\nabla}\varphi\cdot\mathbf{n}\, dS$$

and if φ is harmonic, the left side is zero.

17.8.54 Apply Green's First Identity (Exercise 49) to u and u to give

$$\iiint\limits_D \left(u\boldsymbol{\nabla}^2 u + \boldsymbol{\nabla} u\cdot\boldsymbol{\nabla} u\right) dV = \iint\limits_S u\boldsymbol{\nabla} u\cdot\mathbf{n}\, dS.$$

Because $\boldsymbol{\nabla}^2 u = 0$ and $\boldsymbol{\nabla} u\cdot\boldsymbol{\nabla} u = |\boldsymbol{\nabla} u|^2$, the result follows.

17.8.55 If $\mathbf{T}$ is a vector field $\langle t, u, v\rangle$, then by $\iint \mathbf{T}$, we mean $\left\langle \iint t, \iint u, \iint v\right\rangle$.

a. Let $\mathbf{F} = \langle f, g, h\rangle$ and suppose $\mathbf{n} = \langle n_1, n_2, n_3\rangle$. Then

$$\mathbf{n}\times\mathbf{F} = \langle n_2 h - n_3 g, n_3 f - n_1 h, n_1 g - n_2 f\rangle$$
$$\boldsymbol{\nabla}\times\mathbf{F} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right\rangle$$

Considering first the $\mathbf{i}$ component of these vectors, note that for the vector field $\mathbf{F}_1 = \langle 0, h, -g\rangle$, the Divergence theorem says that

$$\iint\limits_S (n_2 h - n_3 g)\, dS = \iint\limits_S \langle 0, h, -g\rangle\cdot\langle n_1, n_2, n_3\rangle\, dS = \iint\limits_S \mathbf{F}_1\cdot\mathbf{n}\, dS = \iiint\limits_D (\boldsymbol{\nabla}\cdot\mathbf{F}_1)\, dR$$
$$= \iint\limits_D \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dA$$

and similarly for the second and third components.

b. Similarly to part (a), note that

$$\mathbf{n} \times \boldsymbol{\nabla}\varphi = \mathbf{n} \times \langle \varphi_x, \varphi_y, \varphi_z \rangle = \langle n_2\varphi_z - n_3\varphi_y, n_3\varphi_x - n_1\varphi_z, n_1\varphi_y - n_2\varphi_x \rangle.$$

Looking first at the x component of this vector, we have

$$n_2\varphi_z - n_3\varphi_y = \langle 0, \varphi_z, -\varphi_y \rangle \cdot \langle n_1, n_2, n_3 \rangle = (\nabla \times \langle \varphi, 0, 0 \rangle) \cdot \langle n_1, n_2, n_3 \rangle$$

so that Stokes' theorem says that, writing $\mathbf{F} = \langle \varphi, 0, 0 \rangle$,

$$\iint_S (\nabla \times \langle \varphi, 0, 0 \rangle) \cdot \langle n_1, n_2, n_3 \rangle \, dS = \iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \langle \varphi, 0, 0 \rangle \cdot d\mathbf{r} = \oint_C \varphi \, d\mathbf{r}.$$

and similarly for the second and third components.

Chapter Seventeen Review

1

a. False. The curl is $\dfrac{\partial}{\partial x}(x) - \dfrac{\partial}{\partial y}(-y) = 2$.

b. True. The curl of a conservative vector field is zero.

c. False. For example, $\langle -y, x \rangle$ and $\langle 0, 2x \rangle$ both have curl 2.

d. False. For example, $\langle x, 0, 0 \rangle$ and $\langle 0, y, 0 \rangle$ both have divergence 1.

e. True. By the Divergence theorem, the integral is equal to $\displaystyle\iiint_D \boldsymbol{\nabla} \cdot \mathbf{F} \, dV = \iiint_D 3 \, dV$.

2

a. Choice F, because this is a radial vector field whose magnitude increases with distance from the origin.

b. Choice E, because this is a rotational field.

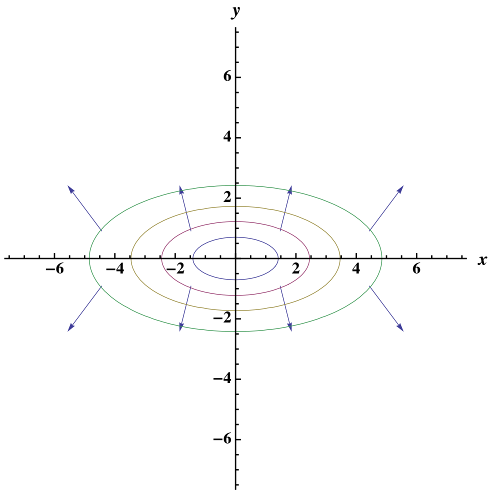
c. Choice D, because this is also a radial field, but the magnitude is constant.

d. Choice C, because this field is vertical along the line $y = x$.

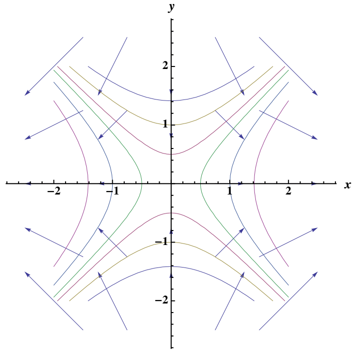
e. Choice B, because the magnitude decreases rapidly away from the origin.

f. Choice A, because this field is periodic.

3 $\nabla\varphi = \langle 2x, 8y \rangle$.



4 $\nabla\varphi = \langle x, -y\rangle$.



5 $\nabla\varphi = -\dfrac{\mathbf{r}}{|\mathbf{r}|^3}$

6 $\nabla\varphi = -e^{-x^2-y^2-z^2}\langle x, y, z\rangle$.

7 $\langle x, y\rangle$ is an outward normal; fo**r** (x, y) on the circle, $|\langle x, y\rangle| = \sqrt{x^2+y^2} = 2$, so the unit outward normal is $\dfrac{1}{2}\langle x, y\rangle$.

8 We have $|\mathbf{r}'(t)| = 5$, so the line integral is

$$\int_C \left(x^2 - 2xy + y^2\right)\, ds = 5\int_0^\pi \left(25\cos^2 t - 10\cos t\sin t + 25\sin^2 t\right)\, dt = 25\int_0^\pi (5 - 2\sin t\cos t)\, dt \; = 125\pi.$$

9 Here $|\mathbf{r}'(t)| = \sqrt{4+9+36} = \sqrt{49} = 7$, so

$$\int_C y\, e^{-xz}\, ds = 7\int_0^2 3te^{12t^2}\, dt = \frac{21}{24}e^{12t^2}\Big|_0^2 = \frac{7}{8}\left(e^{48}-1\right).$$

10 Parameterize the line by $\langle -3t, 1+6t, 2-3t\rangle$ for $0 \le t \le 1$. Then $|\mathbf{r}'(t)| = \sqrt{9+36+9} = \sqrt{54} = 3\sqrt{6}$, so

$$\int_C \left(xz - y^2\right)\, ds = 3\sqrt{6}\int_0^1 \left(3t\,(3t-2) - (6t+1)^2\right)\, dt = -57\sqrt{6}.$$

11 For the first parameterization we have $|\mathbf{r}'(t)| = 2$, so

$$\oint_C (x - 2y + 3z)\, ds = 2\int_0^{2\pi} (2\cos t - 4\,\sin t)\, dt = 0.$$

For the second parameterization we have $|\mathbf{r}'(t)| = \sqrt{16t^2\sin^2(t^2) + 16t^2\cos^2(t^2)} = 4t$, so

$$\oint_C (x - 2y + 3z)\, ds = \int_0^{\sqrt{2\pi}} \left(8t\cos t^2 - 16t\sin t^2\right)\, dt = 0.$$

12

a. Parameterize the path from P to Q by $\langle 1-t, t, 0\rangle$ for $0 \le t \le 1$; then the work done is $\int_C \langle 1, 2y, -4z\rangle \cdot d\mathbf{r} = \int_0^1 \langle 1, 2t, 0\rangle \cdot \langle -1, 1, 0\rangle\, dt = \int_0^1 (2t-1)\, dt = 0$.

b. Parameterize the path from P to O by $\langle 1-t, 0, 0\rangle$, and the path from O to Q by $\langle 0, t, 0\rangle$, both for $0 \le t \le 1$. Then the work done is $\int_C \langle 1, 2y, -4z\rangle \cdot d\mathbf{r} = \int_0^1 \left(\langle 1, 0, 0\rangle \cdot \langle -1, 0, 0\rangle + \langle 1, 2t, 0\rangle \cdot \langle 0, 1, 0\rangle\right) dt = \int_0^1 (2t-1)\, dt = 0$.

c. Parameterize the quarter circle by $\langle \cos t, \sin t, 0\rangle$ for $0 \le t \le \dfrac{\pi}{2}$; then the work done is $\displaystyle\int_C \langle 1, 2y, -4z\rangle \cdot d\mathbf{r} = \int_0^{\pi/2} \langle 1, 2\sin t, 0\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt = \int_0^{\pi/2} (2\cos t\, \sin t - \sin t)\, dt = 0.$

d. Yes, the work is independent of the path; this is a conservative vector field with potential function $x + y^2 - 2z^2$.

13 Parameterize the first path by $\mathbf{r}_1(t) = \langle 0, t, 0\rangle$, and the second by $\mathbf{r}_1(t) = \langle 0, 1, 4t\rangle$, both for $0 \le t \le 1$. Then $\mathbf{r}_1'(t) = \langle 0, 1, 0\rangle$ and $\mathbf{r}_2'(t) = \langle 0, 0, 4\rangle$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \big(\langle -t, 0, 0\rangle \cdot \langle 0, 1, 0\rangle + \langle -1, 4t, 0\rangle \cdot \langle 0, 0, 4\rangle\big)\, dt = 0.$$

14 $\mathbf{r}'(t) = \langle 2t, 6t, -2t\rangle$, so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^2 \frac{1}{(11t^4)^{3/2}} \langle t^2, 3t^2, -t^2\rangle \cdot \langle 2t, 6t, -2t\rangle\, dt = 11^{-3/2} \int_1^2 t^{-6} \cdot 22t^3\, dt = 11^{-3/2} \int_1^2 22t^{-3}\, dt = \frac{3}{44}\sqrt{11}.$$

15 The circulation is

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^{2\pi} \langle 2\sin t - 2\cos t, 2\sin t\rangle \cdot \big\langle -2\sin t, 2\cos t\big\rangle\, dt = \int_0^{2\pi} \big(-4\sin^2 t + 8\sin t\cos t\big)\, dt = -4\pi.$$

The outward flux is

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{n}\, ds &= \int_0^{2\pi} ((2\sin t - 2\cos t)(2\cos t) - (2\sin t)(-2\, \sin t))\, dt \\ &= 4\int_0^{2\pi} \big(\sin t\, \cos t - \cos^2 t + \sin^2 t\big)\, dt = 0.\end{aligned}$$

16 The circulation is

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^{2\pi} \langle 2\cos t, 2\sin t\rangle \cdot \langle -2\, \sin t, 2\cos t\rangle\, dt = 0.$$

The outward flux is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} ((2\cos t)(2\cos t) - (2\sin t)(-2\sin t))\, dt = 8\pi.$$

17 The circulation is

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \frac{1}{4}\int_0^{2\pi} \langle 2\cos t, 2\sin t\rangle \cdot \langle -2\, \sin t, 2\cos t\rangle\, dt = 0.$$

The outward flux is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \frac{1}{4}\int_0^{2\pi} ((2\cos t)(2\, \cos t) - (2\sin t)(-2\sin t))\, dt = 2\pi.$$

18 The circulation is

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^{2\pi} \langle 2\cos t - 2\sin t, 2\cos t\rangle \cdot \big\langle -2\sin t, 2\cos t\big\rangle\, dt = 4\int_0^{2\pi} (-\sin t\cos t + 1)\, dt = 4\pi.$$

The outward flux is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} ((2\cos t - 2\sin t)(2\cos t) - (2\cos t)(-2\, \sin t))\, dt = 4\int_0^{2\pi} \big(\cos^2 t\big)\, dt = 4\pi.$$

19 The normal to the plane $x = 0$ is $\langle 1, 0, 0\rangle$, so the flux is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{-1/2}^{1/2} \int_{-L}^{L} v_0 \left(L^2 - y^2\right) dy\, dz = \frac{4}{3} v_0 L^3..$$

20 A potential function is xy^2.

21 A potential function is $xy + yz^2$.

22 A potential function is $e^x \cos y$.

23 A potential function is xye^z.

24

a. $\mathbf{F} = \langle 2xy, x^2\rangle$, so

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^3 \langle 2\left(9 - t^2\right) t, \left(9 - t^2\right)^2\rangle \cdot \langle -2t, 1\rangle\, dt \\ &= \int_0^3 \left(-4t^2\left(9 - t^2\right) + \left(9 - t^2\right)^2\right) dt = \int_0^3 \left(9 - 5t^2\right)\left(9 - t^2\right) dt = 0\end{aligned}$$

b. Because $\mathbf{F} = \boldsymbol{\nabla}\varphi$, where $\varphi(x, y) = x^2 y$, we have $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(3) - \varphi(0) = 0 - 0 = 0$.

25

a. $\mathbf{F} = \langle yz, xz, xy\rangle$, so

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \langle \frac{t}{\pi} \sin t, \frac{t}{\pi} \cos t, \sin t \cos t\rangle \cdot \langle -\sin t, \cos t, \frac{1}{\pi}\rangle\, dt \\ &= \int_0^\pi \left(\frac{t}{\pi}\left(\cos^2 t - \sin^2 t\right) + \frac{1}{\pi} \sin t \cos t\right) dt = 0.\end{aligned}$$

b. $\mathbf{F} = \boldsymbol{\nabla}(xyz) = \boldsymbol{\nabla}\varphi$, so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(\cos \pi \sin \pi) - \varphi\left(\cos 0 \sin 0 \cdot \frac{0}{\pi}\right) = 0 - 0 = 0.$$

26

a. Parameterize C by the four paths $\mathbf{r}_1(t) = \langle -1 + 2t, -1\rangle$, $\mathbf{r}_2(t) = \langle 1, -1 + 2t\rangle$, $\mathbf{r}_3(t) = \langle 1 - 2t, 1\rangle$, $\mathbf{r}_4(t) = \langle -1, 1 - 2t\rangle$, for $0 \le t \le 1$. Then $\mathbf{r}_1'(t) = \langle 2, 0\rangle$, $\mathbf{r}_2'(t) = \langle 0, 2\rangle$, $\mathbf{r}_3'(t) = \langle -2, 0\rangle$, $\mathbf{r}_4'(t) = \langle 0, -2\rangle$, and

$$\begin{aligned}&\int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^1 \left(\langle -1 + 2t, 1\rangle \cdot \langle 2, 0\rangle + \langle 1, 1 - 2t\rangle \cdot \langle 0, 2\rangle + \langle 1 - 2t, -1\rangle \cdot \langle -2, 0\rangle + \langle -1, 2t - 1\rangle \cdot \langle 0, -2\rangle\right) dt \\ &= \int_0^1 (4t - 2 + 2 - 4t + 4t - 2 + 2 - 4t)\, dt = \int_0^1 0\, dt = 0.\end{aligned}$$

b. $\mathbf{F} = \boldsymbol{\nabla}\varphi$, where $\varphi(x, y) = \frac{1}{2}\left(x^2 - y^2\right)$, so the integral around any closed curve is zero.

27

a. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \left\langle \sin t, 4, -\cos t\right\rangle \cdot \left\langle -\sin t, \cos t, 0\right\rangle dt = \int_0^{2\pi} \left(-\sin^2 t + 4\cos t\right)\, dt = -\pi.$

b. The vector field is not conservative, since for example $\frac{\partial}{\partial y}(y) \neq \frac{\partial}{\partial z}(x)$.

28 For $p \neq 2$, $\mathbf{F} = \nabla\varphi$ where $\varphi = \frac{-1}{(p-2)\,|\mathbf{r}|^{p-2}}$, while for $p = 2$, $\varphi = \frac{1}{2}\ln\left(|\mathbf{r}|^2\right)$, as can be seen by taking the gradient. Thus $\mathbf{F}$ is conservative on all of $\mathbb{R}^2$ for $p < 0$.

29 By the circulation form of Green's Theorem,

$$\oint_C xy^2\,dx + x^2y\,dy = \iint_R \left(\frac{\partial}{\partial x}\left(x^2y\right) - \frac{\partial}{\partial y}\left(xy^2\right)\right) dA = \iint_R \left(2xy - 2xy\right)\, dA = 0.$$

30 By the circulation form of Green's Theorem,

$$\oint_C \left(-3y + x^{3/2}\right) dx + \left(x - y^{2/3}\right) dy = \iint_R \left(\frac{\partial}{\partial x}\left(x - y^{2/3}\right) - \frac{\partial}{\partial y}\left(-3y + x^{3/2}\right)\right) dA = \iint_R 4\,dA = 4\pi.$$

31 By the circulation form of Green's Theorem,

$$\begin{aligned}\oint_C \left(x^3 + xy\right)\, dy + \left(2y^2 - 2x^2y\right)\, dx &= \iint_R \left(\frac{\partial}{\partial x}\left(x^3 + xy\right) - \frac{\partial}{\partial y}\left(2y^2 - 2x^2y\right)\right) dA \\ &= \iint_R \left(3x^2 + y - 4y + 2x^2\right)\, dA = \int_{-1}^{1}\int_{-1}^{1} \left(5x^2 - 3y\right)\, dy\,dx = \frac{20}{3}.\end{aligned}$$

32 By the flux form of Green's Theorem, $\oint_C 3x^3\,dy - 3y^3\,dx = \iint_R \left(9x^2 + 9y^2\right)\, dA = 9\int_0^4\int_0^{2\pi} r^3\,d\theta\,dr =$ 1152π. Because the orientation is clockwise, the answer is -1152π.

33 The ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$; parameterize it by $\mathbf{r}(t) = \langle x, y\rangle = \langle 4\cos t, 2\sin t\rangle$, $0 \leq t \leq 2\pi$. Then the area of the region is

$$\frac{1}{2}\oint_C \left(\left(4\cos t\right)\left(2\cos t\right) - \left(2\sin t\right)\left(-4\sin t\right)\right)\, dt = \int_0^{2\pi} 4\left(\cos^2 t + \sin^2 t\right)\, dt = 8\pi.$$

34 $dx = -3\cos^2 t\sin t\,dt$, and$dy = 3\sin^2 t\cos t\,dt$, so the area of the hypocycloid is

$$\begin{aligned}\frac{1}{2}\oint_C x\,dy - y\,dx &= \frac{1}{2}\int_0^{2\pi} \left(\left(\cos^3\, t\right)\left(3\sin^2 t\cos t\right) - \left(\sin^3\, t\right)\left(-3\cos^2 t\sin t\right)\right)\, dt \\ &= \frac{3}{2}\int_0^{2\pi} \left(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right)\, dt = \frac{3\pi}{8}.\end{aligned}$$

35

a. $\mathbf{F} = \left(x^2 + y^2\right)^{-1/2}\langle x, y\rangle$, so the circulation is

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_R \left(\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right) dA \\ &= \iint_R \left(\frac{-xy}{\sqrt{x^2+y^2}} - \frac{-xy}{\sqrt{x^2+y^2}}\right) dA = 0.\end{aligned}$$

b. The flux is

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right)dA$$

$$= \iint_R \left(\frac{y^2}{\left(x^2+y^2\right)^{3/2}}+\frac{x^2}{\left(x^2+y^2\right)^{3/2}}\right)dA = \iint_R \left(\frac{1}{\sqrt{x^2+y^2}}\right)dA = \int_0^{\pi}\int_1^3 1\,dr\,d\theta = 2\pi.$$

36

a. The circulation is $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(\frac{\partial}{\partial x}(x\cos y)-\frac{\partial}{\partial y}(-\sin y)\right)dA = \int_0^{\pi/2}\int_0^{\pi/2} 2\,\cos y\,dy\,dx = \pi.$

b. The flux is $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \left(\frac{\partial}{\partial x}(-\sin y)+\frac{\partial}{\partial y}(x\cos y)\right)dA = \int_0^{\pi/2}\int_0^{\pi/2}(-x\sin y)\;dy\,dx = -\frac{1}{8}\pi^2.$

37

a. For $\mathbf{F}$ to be conservative, we must have $\frac{\partial}{\partial y}(ax+by)=\frac{\partial}{\partial x}(cx+dy)$, or $b=c$.

b. For $\mathbf{F}$ to be source-free, we must have $\frac{\partial}{\partial x}(ax+by)=-\frac{\partial}{\partial y}(cx+dy)$, or $a=-d$.

c. $\mathbf{F}$ is both conservative and source-free if $b=c$ and $a=-d$, i.e. if $\mathbf{F}=\langle ax+by, bx-ay\rangle$.

38 The divergence is $\frac{\partial}{\partial x}(yz)+\frac{\partial}{\partial y}(xz)+\frac{\partial}{\partial z}(xy)=0$. The curl is

$$\langle\frac{\partial}{\partial y}(xy)-\frac{\partial}{\partial z}(xz),\frac{\partial}{\partial z}(yz)-\frac{\partial}{\partial x}(xy),\frac{\partial}{\partial x}(xz)-\frac{\partial}{\partial y}(yz)\rangle=\mathbf{0}.$$

The field is both source-free and irrotational.

39 The divergence is $4\,|\mathbf{r}|$. The curl is

$$\langle\frac{\partial}{\partial y}(z\;|\mathbf{r}|)-\frac{\partial}{\partial z}(y\;|\mathbf{r}|),\frac{\partial}{\partial z}(x\;|\mathbf{r}|)-\frac{\partial}{\partial x}(z\;|\mathbf{r}|),\frac{\partial}{\partial x}(y\;|\mathbf{r}|)-\frac{\partial}{\partial y}(x\;|\mathbf{r}|)\rangle=\mathbf{0}.$$

The field is irrotational but not source-free.

40 The divergence is $\frac{\partial}{\partial x}(\sin xy)+\frac{\partial}{\partial y}(\cos yz)+\frac{\partial}{\partial z}(\sin xz) = y\cos xy - z\sin yz + x\,\cos xz$. The curl is $\langle y\sin yz, -z\cos xz, -x\cos xy\rangle$. The field is neither irrotational nor source-free.

41 The divergence is $\frac{\partial}{\partial x}\left(2xy+z^4\right)+\frac{\partial}{\partial y}\left(x^2\right)+\frac{\partial}{\partial z}\left(4xz^3\right)=2y+12xz^2$. The curl is

$$\langle\frac{\partial}{\partial y}\left(4xz^3\right)-\frac{\partial}{\partial z}\left(x^2\right),\frac{\partial}{\partial z}\left(2xy+z^4\right)-\frac{\partial}{\partial x}\left(4xz^3\right),\frac{\partial}{\partial x}\left(x^2\right)-\frac{\partial}{\partial y}\left(2xy+z^4\right)\rangle=\mathbf{0},$$

so the field is irrotational but not source-free.

42 $|\mathbf{r}|^4=\left(x^2+y^2+z^2\right)^2$, so

$$\boldsymbol{\nabla}\left(x^2+y^2+z^2\right)^{-2}=\left\langle -4x\left(x^2+y^2+z^2\right)^{-3},-4y\left(x^2+y^2+z^2\right)^{-3},-4z\left(x^2+y^2+z^2\right)^{-3}\right\rangle=-\frac{4\,\mathbf{r}}{|\mathbf{r}|^6}.$$

Now

$$\frac{\partial}{\partial x}\left(-4x\left(x^2+y^2+z^2\right)^{-3}\right)=-4\left(y^2+z^2-5x^2\right)\left(x^2+y^2+z^2\right)^{-4},$$

so that

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\left|\mathbf{r}\right|^{-4}=-4\left(x^2+y^2+z^2\right)^{-4}\left(y^2+z^2-5x^2+x^2+z^2-5y^2+x^2+y^2-5z^2\right)=\frac{12}{\left|\mathbf{r}\right|^6}.$$

43

a. The curl is

$$\langle\frac{\partial}{\partial y}(-y)-\frac{\partial}{\partial z}(x),\frac{\partial}{\partial z}(z)-\frac{\partial}{\partial x}(-y),\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(z)\rangle=\langle -1,1,1\rangle.$$

So the scalar component in the direction of $\langle 1,0,0\rangle$ is$\langle -1,1,1\rangle\cdot\langle 1,0,0\rangle=-1$,

b. The scalar component in the direction of $\langle 0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\rangle$ is$\langle 0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\rangle\cdot\langle 1,0,0\rangle=0$.

c. The scalar component of the curl is a maximum in the direction of the curl, i.e. in the direction $\langle -1,1,1\rangle$, whose unit vector is $\frac{1}{\sqrt{3}}\langle -1,1,1\rangle$.

44 The curl of the vector field is $\boldsymbol{\nabla}\times\mathbf{F}=\langle 0,0,2\rangle$, and the component of the curl along a unit vector $\mathbf{n}$ is thus $\langle 0,0,2\rangle\cdot\mathbf{n}$.

a. It does not spin, because $\langle 0,0,2\rangle\cdot\langle 1,0,0\rangle=0$.

b. The scalar component of the curl in the direction $\langle 0,0,1\rangle$ is 2.

c. It spins the fastest when the paddle wheel is aligned with $\langle 0,0,1\rangle$.

45 Parameterize the sphere by $\langle 3\sin u\cos v,3\sin u\sin v,3\cos u\rangle$, $0\le u\le\frac{\pi}{2}$; $0\le v\le 2\pi$. Then $|\mathbf{n}|=9\sin u$, so

$$\iint\limits_S 1\,dS=\iint\limits_R 9\sin u\,dA=\int_0^{2\pi}\int_0^{\pi/2}9\sin u\,du\,dv=18\pi.$$

46 Parameterize the surface by $\langle v\cos u,v\sin u,v\rangle$ for $2\le v\le 4, 0\le u\le 2\pi$. Then

$$\iint\limits_S 1\,dS=\iint\limits_R\sqrt{2}\,v\,dA=\sqrt{2}\int_2^4\int_0^{2\pi}v\,du\,dv=12\pi\sqrt{2}.$$

47 The volume element is $\sqrt{1^2+1^2+1^2}=\sqrt{3}$, so the area is

$$\iint\limits_S 1\,dS=\sqrt{3}\iint\limits_R 1\,dA=\sqrt{3}\int_{-1}^1\int_{-1}^1 1\,dx\,dy=4\sqrt{3}.$$

48 The volume element is $\sqrt{z_x^2+z_y^2+1}=\sqrt{2\left(x^2+y^2\right)+1}$, so the integral is

$$\iint\limits_S 1\,dS=\iint\limits_R\sqrt{2\left(x^2+y^2\right)+1}\,dA=\int_0^2\int_0^{2\pi}r\sqrt{2r^2+1}\,d\theta\,dr=\frac{26}{3}\pi.$$

49 The volume element for $z=2-x-y$ is $\sqrt{3}$, so the integral is

$$\iint\limits_S(1+yz)\,dS=\sqrt{3}\iint\limits_R(1+yz)\,dA=\sqrt{3}\int_0^2\int_0^{2-x}(1+y(2-x-y))\,dy\,dx=\frac{8\sqrt{3}}{3}.$$

50 The normal to the curved surface of the cylinder at (x,y,z) is $\langle 0,y,z\rangle$, so

$$\iint\limits_S \langle 0,y,z\rangle \cdot \mathbf{n}\,dS = a\iint\limits_R \left(y^2+z^2\right)\,dA = a^2\cdot \text{area of}\,R = 32\pi a^3.$$

51 Parameterize the curved surface using spherical coordinates, so that $|\mathbf{n}| = 4\sin u$; then for the curved surface we have

$$\iint\limits_S (x-y+z)\,dS = 8\int_0^{2\pi}\int_0^{\pi/2} (\sin u\cos v - \sin u\sin v + \cos u)\sin u\,du\,dv = 8\pi.$$

For the planar surface, $\mathbf{n} = \langle 0,-1,0\rangle$ so that $|\mathbf{n}| = 1$ and

$$\iint\limits_S (x-y+z)\,dS = \iint\limits_R (x-y+z)\,dA = \int_0^2\int_0^{2\pi} r\,(\cos\theta - \sin\theta)\,d\theta\,dr = 0$$

and the total integral is thus 8π.

52 The normal to the cylinder is $\langle x,y,0\rangle$ with magnitude 1, so

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_R \langle x,y,z\rangle\cdot\langle x,y,0\rangle\,dA = \iint\limits_R \left(x^2+y^2\right)\,dA = \iint\limits_R 1\,dA = 32\pi.$$

53 $\mathbf{F} = \left(x^2+y^2+z^2\right)^{-1/2}\langle x,y,z\rangle$; using spherical coordinates to parameterize the sphere gives

$$\begin{aligned}\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint\limits_R \frac{1}{a}\langle a\sin u\cos v, a\sin u\sin v, a\ \cos u\rangle\cdot\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\sin u\cos u\rangle\,dA\\ &= \iint\limits_R a^2\sin u\,dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin u\,du\,dv = 4\pi a^2.\end{aligned}$$

54

a. Using the explicit description, we have $\sqrt{z_z^2+z_y^2+1} = \sqrt{4\left(x^2+y^2\right)+1}$, so

$$\iint\limits_S 1\,dS = \iint\limits_R \sqrt{4\left(x^2+y^2\right)+1}\,dA = \int_0^2\int_0^{2\pi} r\sqrt{4r^2+1}\,d\theta\,dr = \frac{\pi}{6}\left(17\sqrt{17}-1\right).$$

b. Using the given parametric description, we have $|\mathbf{n}| = v\sqrt{1+4v^2}$, so

$$\iint\limits_S 1\,dS = \iint\limits_R v\sqrt{1+4v^2}\,dA = \int_0^{2\pi}\int_0^2 v\sqrt{1+4v^2}\,dv\,d\theta = \frac{\pi}{6}\left(17\sqrt{17}-1\right).$$

c. Using the given parametric description, we have

$$|\mathbf{t}_u\times\mathbf{t}_v| = \left|\langle -\sqrt{v}\,\sin u, \sqrt{v}\cos u, 0\rangle \times \left\langle -\frac{1}{2}v^{-1/2}\cos u, \frac{1}{2}v^{-1/2}\sin u, 1\right\rangle\right| = \frac{1}{2}\sqrt{4v+1}$$

so that

$$\iint\limits_S 1\,dS = \frac{1}{2}\iint\limits_R \sqrt{4v+1}\,dA = \frac{1}{2}\int_0^{2\pi}\int_0^4 \sqrt{4v+1}\,dv\,d\theta = \frac{\pi}{6}\left(17\sqrt{17}-1\right).$$

55

a. The base of S is the surface where $z = 0$, or the circle $x^2 + y^2 = a^2$. Similarly, the base of the paraboloid is found by setting $z = 0$; simplifying gives again $x^2 + y^2 = a^2$. The high point of the hemisphere (maximum z-coordinate) occurs when $x = y = 0$; then $z = a$. Similarly, the high point on the paraboloid also occurs when $x = y = 0$ and again this gives $z = a$.

b. The graph of the paraboloid is inside that of the hemisphere everywhere, so we would expect it to have smaller surface area. We know that the surface area of the hemisphere is $4\pi a^2 \cdot \dfrac{1}{2} = 2\pi a^2$. For the paraboloid, we have $z_x = -\dfrac{2x}{a}$, $z_y = -\dfrac{2y}{a}$, so that $|\mathbf{n}| = \sqrt{\dfrac{4\left(x^2+y^2\right)}{a^2} + 1}$, so

$$\iint\limits_S 1\,dS = \frac{1}{a}\iint\limits_R \sqrt{4\left(x^2+y^2\right)+a^2}\,dA = \frac{1}{a}\int_0^a \int_0^{2\pi} r\sqrt{4r^2+a^2}\,d\theta\,dr = \frac{\left(5\sqrt{5}-1\right)}{6}\pi a^2,$$

which is in fact smaller than the area of the hemisphere.

c. $\mathbf{n} = \langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\rangle$, so

$$\begin{aligned}\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint\limits_R \langle a\sin u\cos v, a\sin u\sin v, a\,\cos u\rangle\cdot\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\rangle\,dA\\ &= a^3\int_0^{2\pi}\int_0^{\pi/2}\left(\sin^3 u\,\cos^2 v + \sin^3 u\sin^2 v + \cos^2 u\sin u\right)\,du\,dv\\ &= a^3\int_0^{2\pi}\int_0^{\pi/2}\left(\sin^3 u + \cos^2 u\sin u\right)\,du\,dv = a^3\int_0^{2\pi}\int_0^{\pi/2}\sin u\,du\,dv = 2\pi a^3.\end{aligned}$$

d. For the paraboloid, the parameterization is $\left\langle v\cos u, v\sin u, a - \dfrac{v^2}{a}\right\rangle$, $0 \le v \le a$, $0 \le \mathrm{u} \le 2\pi$, and

$$\mathbf{n} = \langle -v\sin u, v\cos u, 0\rangle \times \left\langle \cos u, \sin u, -\frac{2v}{a}\right\rangle = \left\langle -\frac{1}{a}2v^2\cos u, -\frac{1}{a}2v^2\sin u, -v\right\rangle,$$

so that the outward-pointing normal is $\left\langle \dfrac{2v^2}{a}\cos u, \dfrac{2v^2}{a}\sin u, v\right\rangle$ and

$$\begin{aligned}\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint\limits_R \left\langle v\cos u, v\sin u, a - \frac{v^2}{a}\right\rangle\cdot\left\langle \frac{2v^2}{a}\cos u, \frac{2v^2}{a}\sin u, v\right\rangle\,dA\\ &= \int_0^a\int_0^{2\pi}\left(\frac{2}{a}v^3 + av - \frac{v^3}{a}\right)\,du\,dv = \frac{3}{2}\pi a^3.\end{aligned}$$

56

a. For the given $\mathbf{r}$, we have $\dfrac{\left(a\cos u\sin v\right)^2}{a^2} + \dfrac{\left(b\,\sin u\sin v\right)^2}{b^2} + \dfrac{\left(c\cos v\right)^2}{c^2} = \cos^2 u\sin^2 v + \sin^2 u\sin^2 v + \cos^2 v = \sin^2 v + \cos^2 v = 1$

b. The normal vector is determined by $\mathbf{n} = \mathbf{t}_u\times\,\mathbf{t}_v = \langle -bc\cos u\sin^2 v, -ac\sin u\,\sin 2v, -ab\sin v\cos v\rangle$, so that the outward pointing normal is the negative of this vector. Then

$$\begin{aligned}\iint\limits_S 1\,dS &= \int_0^{2\pi}\int_0^{\pi}\left|\langle bc\cos u\sin^2 v, ac\sin u\sin^2 v, ab\,\sin v\cos v\rangle\right|\,dv\,du\\ &= \int_0^{2\pi}\int_0^{\pi}\sqrt{b^2c^2\cos^2 u\,\sin^4 v + a^2c^2\sin^2 u\sin^4 v + a^2b^2\sin^2 v\,\cos^2 v}\,dv\,du.\end{aligned}$$

57 Let S be the disk in the xy-plane with radius 2 centered at the origin with upward-pointing normal vector $\mathbf{k}$. Then $\boldsymbol{\nabla}\times\mathbf{F}=\langle x-y,x-y,0\rangle$, so

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S(\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\,dS=\iint_R\langle x-y,x-y,0\rangle\cdot\langle 0,0,1\rangle\;dA=\iint_R 0\,dA=0.$$

58 $\mathbf{F}=\langle u^2-v^2,u,2v\,(6-2u-v)\rangle$, $\mathbf{r}\,(u,v)=\langle u,v,6-2u-v\rangle$, $\mathbf{t}_u\times\,\mathbf{t}_v=\langle 2,1,1\rangle$

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^3\int_0^{6-2u}\left(2u^2-4v^2+12v-4uv\right)\,dv\,du=\frac{39\sqrt{6}}{2}.$$

59 The boundary of this region is in the xy-plane, found by setting $z=0$, so it is $x^2+y^2=99$, the circle of radius $\sqrt{99}$ about the origin. Parameterize the circle in the usual way; then

$$\begin{aligned}\iint\limits_S(\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\,dS&=\oint_C\mathbf{F}\cdot d\mathbf{r}=\oint_C\langle 0,\,\sqrt{99}\cos t,\sqrt{99}\sin t\rangle\cdot\left\langle-\sqrt{99}\sin t,\sqrt{99}\cos t,0\right\rangle\,dt\\&=\int_0^{2\pi}99\cos^2 t\,dt=99\pi.\end{aligned}$$

60 The boundary of this region is the circle $x^2+z^2=4$ for $y=0$; parameterizing it in the usual way as $\langle 2\cos t,0,2\sin t\rangle$ gives

$$\begin{aligned}\iint\limits_S(\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\,dS&=\oint_C\mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\left\langle 4\cos^2 t-4\sin^2 t,0,4\sin t\cos t\right\rangle\cdot\langle-2\sin t,0,2\cos t\rangle\,dt\\&=8\int_0^{2\pi}\left(\left(\sin^2 t-\cos^2 t\right)\,\sin t+\sin t\cos^2 t\right)\,dt=8\int_0^{2\pi}\sin^3 t\,dt=0.\end{aligned}$$

61 By Stokes' theorem, the circulation around a closed curve C can be found by choosing a surface S of which C is the boundary; then

$$\oint_C\mathbf{F}\cdot d\mathbf{r}=\iint\limits_S(\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\,dS.$$

But for $\mathbf{F}=\boldsymbol{\nabla}\left(10-x^2+y^2+z^2\right)$, $\boldsymbol{\nabla}\times\mathbf{F}=\mathbf{0}$, so the right-hand side is zero.

62 We have $\boldsymbol{\nabla}\cdot\mathbf{F}=-3$ so that

$$\iint\limits_S\mathbf{F}\cdot\mathbf{n}\,dS=\iiint\limits_D\boldsymbol{\nabla}\cdot\mathbf{F}\,dV=-3\cdot\text{volume of cube}=-3.$$

63 $\boldsymbol{\nabla}\cdot\mathbf{F}=x^2+y^2+z^2$, so

$$\iint\limits_S\mathbf{F}\cdot\mathbf{n}\,dS=\iiint\limits_D\left(x^2+y^2+z^2\right)\,dV=\int_0^3\int_0^{2\pi}\int_0^{\pi}r^2\cdot r^2\sin u\,du\,dv\,dr=\frac{972}{5}\pi.$$

64 $\boldsymbol{\nabla}\cdot\mathbf{F}=2\,(x+y+z)$, so

$$\iint\limits_S\mathbf{F}\cdot\mathbf{n}\,dS=\iiint\limits_D 2\,(x+y+z)\,dV=\int_0^8\int_0^2\int_0^{2\pi}2r\,(r\cos\theta+r\sin\theta+t)\,d\theta\,dr\,dt=256\pi.$$

65 $\boldsymbol{\nabla}\cdot\mathbf{F}=3\left(x^2+y^2\right)$, so the outward flux across the boundary S of a hemisphere D of radius a is $\displaystyle\iint\limits_S\mathbf{F}\cdot\mathbf{n}\,dS=\iiint\limits_D 3\left(x^2+y^2\right)\,dV=\int_0^a\int_0^{2\pi}\int_0^{\pi/2}3r^2\sin^2 u\cdot r^2\sin u\,du\,dv\,dr=\frac{4}{5}\pi a^5$, so that the net flux across the region bounded by the hemispheres of radii 1 and 2 is $\dfrac{4}{5}\pi\,(32-1)=\dfrac{124}{5}\pi$.

66 $\nabla \cdot \mathbf{F} = 0$, so the flux is zero across any surface that bounds a region where F is defined and differentiable; the given region does not include zero, so is one of these. Thus the net outward flux is zero.

67 Using the Divergence theorem, $\nabla \cdot \mathbf{F} = 2x + \sin y + 2y - 2\sin y + 2z + \sin y = 2\,(x+y+z)$, so that

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint\limits_D 2\,(x+y+z)\; dV = \int_0^4 \int_0^1 \int_0^{1-x} 2\,(x+y+z)\; dy\, dx\, dz = \frac{32}{3}.$$

68

a. The normal vectors point outwards everywhere on S; that is, on the curved surface, they point upwards, and on the flat surface they point in the direction of negative x.

b. Parameterize C by two paths: $\mathbf{r}_1\,(t) = \langle a\cos t, a\sin t, 0\rangle$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ and $\mathbf{r}_2\,(t) = \langle 0, a-2t, 0\rangle$ for $0 \le t \le a$. Then $\mathbf{r}_1'\,(t) = \langle -a\sin t, a\cos t, 0\rangle$ and $\mathbf{r}_2'\,(t) = \langle 0, -2, 0\rangle$. So

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{-\pi/2}^{\pi/2} \langle -a\sin t, a\cos t, a\sin t - 2a\cos t\rangle \cdot \langle -a\sin t, a\cos t, 0\rangle\, dt$$

$$+ \int_0^a \langle 2t-a, 0, a-2t\rangle \cdot \langle 0, -2, 0\rangle dt = \int_{-\pi/2}^{\pi/2} a^2\, dt = \pi a^2.$$

c. $\nabla \times \mathbf{F} = \langle 2, 4, 2\rangle$. For the curved portion of S, using spherical coordinates, the normal vector is $\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\rangle$, and for the flat portion, the normal vector is $\langle -1, 0, 0\rangle$. Then $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint\limits_{S_1} \left(2a^2\, \sin^2 u\cos v + 4a^2\sin^2 u\sin v + 2a^2\, \cos u\sin u\right)\, dS + \iint\limits_{S_2} (-2)\; dS =$
$2a^2 \int_{-\pi/2}^{\pi/2} \int_0^{\pi/2} \left(\sin^2 u\, \cos v + 2\sin^2 u\sin v + \sin u\cos u\right)\, du\, dv - \pi a^2 = 2\pi a^2 - \pi a^2 = \pi a^2.$